Representations of Operator Algebras

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The following thesis is divided into two main chapters. In Chapter 2 we study isometric representations of product systems of correspondences over the semigroup $\mathbb{N}^k$ which are minimal dilations of finite dimensional, fully coisometric representations. We show the existence of a unique minimal cyclic coinvariant subspace for all such representations. The compression of the representation to this subspace is shown to be a complete unitary invariant. For a certain class of graph algebras the nonself-adjoint WOT-closed algebra generated by these representations is shown to contain the projection onto the minimal cyclic coinvariant subspace. This class includes free semigroup algebras. This result extends to a class of higher-rank graph algebras which includes higher-rank graphs with a single vertex.

In chapter 3 we move onto semicrossed product algebras. Let $\mathcal{S}$ be the semigroup $\mathcal{S} = \sum_{i=1}^{\oplus k} \mathcal{S}_i$, where for each $i \in I$, $\mathcal{S}_i$ is a countable subsemigroup of the additive semigroup $\mathbb{R}_+$ containing 0. We consider representations of $\mathcal{S}$ as contractions $\{T_s\}_{s \in \mathcal{S}}$ on a Hilbert space with the Nica-covariance property: $T_s^* T_t = T_t T_s^*$ whenever $t \wedge s = 0$. We show that all such representations have a unique minimal isometric Nica-covariant dilation.

This result is used to help analyse the nonself-adjoint semicrossed product algebras formed from Nica-covariant representations of the action of $\mathcal{S}$ on an operator algebra $\mathcal{A}$ by completely contractive endomorphisms. We conclude by calculating the C^*-envelope of the isometric nonself-adjoint semicrossed product algebra (in the sense of Kakariadis and Katsoulis).

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Dedication

For their unerring support I dedicate this work to my parents, Anne Hanley and Tom Fuller, and my “Canadian parents”, Theresa Farrell and Paddy Fuller.

Table of Contents

Chapter 1

Introduction

The following thesis is based on two papers I wrote whilst a student at the University of Waterloo. Chapter 2 consists of [28]. This work was developed throughout 2009 and the first half of 2010. It was published in January 2011 in the Journal of Functional Analysis. Chapter 3 can be found in [29]. This work was carried out in 2011 and is currently submitted for peer review.

Chapter 2 makes no reference to Chapter 3 and, in turn, Chapter 3 makes no reference to Chapter 2. This has lead to an admittedly disjointed thesis with a rather vague title. That is not to say that there is no connection between the two projects. On the contrary, both papers deal with the trials and tribulations of working with representations and operator algebras graded by semigroups other than the nonnegative integers. In this introduction I will present the work from the prespective of multivariate dilation theory as well as give some more of the motivation for the work, other than that contained in the individual chapters.

In 1953 Béla Szőkefalvi-Nagy [70] proved that every contraction on a Hilbert space has a unique minimal isometric dilation. That is, if T is a bounded linear operator on a Hilbert space $\mathcal{H}$ of operator norm no more than 1, then there is an isometry V on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ the isometry V has the form

$$V = \begin{bmatrix} T & 0 \\ * & * \end{bmatrix},$$

so that

$$\mathrm{Proj}(\mathcal{H}) V^n|_{\mathcal{H}} = T^n.$$

An isometric dilation V of T is called minimal if $\mathcal{K}$ is the smallest space containing $\mathcal{H}$ which is reducing for V. Sz.-Nagy's theorem also says that all minimal isometric dilations are unitarily equivalent, i.e. there is a unique minimal isometric dilation. Sz.-Nagy's theorem has had a large impact on operator theory ever since.

From another perspective, Sz.-Nagy's theorem tells us about representations of semigroups. Every representation of the nonnegative integers by contractive (isometric) operators is uniquely determined by a single contraction (isometry). Sz.-Nagy's theorem translates as the statement: Every contractive representation of the nonnegative integers can be dilated to a unique minimal isometric representation. There are many classes of operator algebras which have a natural grading by the nonnegative integers, thus Sz.-Nagy's theorem (or generalisations of it) is widely applicable.

Two examples of algebras that are discussed in this thesis which have representations graded by the nonnegative integers are semicrossed products of C^*-algebras by the nonnegative integers and graph algebras. Thus both lend themselves to a generalisation of Sz.-Nagy's theorem. In fact, the study of both can be united in the study of C^*-correspondences and Paul Muhly and Baruch Solel's generalisation of Sz.-Nagy's theorem [48] applies to both classes of algebras.

There are however other algebras where a grading by the nonnegative integers does not readily apply, such as semicrossed products by other semigroups and higher-rank graph algebras. In these instances dilation theory for representations of other semigroups is needed. The problem of when and how contractive representations of semigroups have isometric dilations is a problem central to this thesis.

It was shown by Tsuyoshi Andô [1] in 1963 that every contractive representation of $\mathbb{Z}_+^2$ has an isometric dilation. In this case, however, we no longer have any guarantee of uniqueness of minimal isometric dilations. Things get worse in the case of $\mathbb{Z}_+^3$, where Stephen Parrott [52] and Nicolas Varopoulos [73] have provided examples of contractive representations which do not have isometric dilations. These counter-examples are challenging and humbling for operator theorists. If we are dealing with an algebra which does not fit into a framework where Sz.-Nagy's theorem applies, how can we relate contractive and isometric representations? How can we relate the different algebras formed by these representations?

All is not lost. Parrott and Varopoulos warn us that we must proceed with caution, but they do not tell us that we can not proceed. There are a number of dilation theorems which apply to semigroups other than the nonnegative integers. I will highlight here two such results that are relevant to this thesis.

Any contractive representation of $\mathbb{Z}_+^k$ is uniquely determined by k commuting contractions, and k commuting contractions define a contractive representation of $\mathbb{Z}_+^k$. As Parrott and Varopoulos have shown, when $k > 2$ we can not necessarily dilate k commuting contractions $T_1, \ldots, T_k$ to k commuting isometries $V_1, \ldots, V_k$. If, however we put extra conditions on our contractions there are circumstances when we can dilate.

Theorem 1. *Let $T_1, \ldots, T_k$ be k commuting coisometries. Then there are k commuting isometries $V_1, \ldots, V_k$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ which dilate $T_1, \ldots, T_k$, i.e.*

$$\mathit{Proj}(\mathcal{H}) p(V_1, \ldots, V_k)|_{\mathcal{H}} = p(T_1, \ldots, T_k)$$

for any polynomial p in k commuting variables.

Further, all minimal dilations are unitarily equivalent (i.e. there is a unique minimal dilation), and $V_1, \dots, V_k$ are unitary when minimal.

Theorem 1, and it's generalisation to representations of product systems due to Orr Shalit [61], is a key factor in the analysis in Chapter 2. Without the fact that we have a *unique* minimal isometric dilation we would not be able to begin an in depth study of finitely correlated representations. As stated here in Theorem 1, the condition that our contractive representation must be made of coisometries is very strong. When this result is applied to representations of product systems the requirement that the representation is coisometric is a lot less restrictive. For example, a coisometry on a finite dimensional space must also be an isometry; a coisometric representation of a C^*-correspondences on a finite dimensional space need not be isometric. Thus speaking of minimal isometric dilations of finite dimensional coisometric representations makes sense.

The following corollary is also important.

Corollary 1. *Let $T_1, \dots, T_k$ be k commuting coisometries and let $V_1, \dots, V_k$ be their unique minimal isometric dilation. Then the unitary $V_1 V_2 \dots V_k$ is the unique minimal isometric dilation of $T_1 T_2 \dots T_k$ in the sense of Sz.-Nagy.*

Corollary 1 relates the isometric dilation of the k coisometries $T_1, \dots, T_k$ with the isometric dilation of the single coisometry $T_1 \dots T_k$. This allows one to move between the multivariate case of representations of $\mathbb{Z}_+^k$ and the more familiar single variable case of representations of $\mathbb{Z}_+$. Corollary 1 is a simplification of Theorem 2.3.12. In Chapter 2 we study representations over $\mathbb{Z}_+$ first, then we pass to $\mathbb{Z}_+^k$ using this theorem.

It may be helpful to the reader to explain some of the thought processes that went into writing Chapter 2. As presented here the structure is: single variable case; multivariate case; examples. Most would agree that this is the most logical way to present the material but it is the reverse order of how it was discovered.

Ken Davidson, David Kribs and Miron Shpigel [17] studied finitely correlated free semigroup algebras in 2001. These are the WOT-closed unital algebras generated by a row-isometry $[S_1, \dots, S_n]$ which is the minimal isometric dilation of a row-contraction $[A_1, \dots, A_n]$ on a finite dimensional space (here Arthur Frazho [27], John Bunce [9] and Gelu Popescu's [57] generalisation of Sz.-Nagy's theorem to row-contractions applies). The study of commuting row-isometries (in the guise of representations of single vertex 2-graphs) is the natural multivariate analogue of studying single row-isometries. Inspired by work by Ken Davidson, David Kribs, Stephen Power and Dilian Yang, particularly the papers [20, 19], I was curious to see to what extent the work of Davidson, Kribs and Shpigel on finitely correlated representations could be translated to this multivariate case. Armed with Baruch Solel's [68] and Ken Davidson, Stephen Power and Dilian Yang's [20] generalisation of Andô's theorem and my discovery of a row-contractive version of Corollary 1 the main results of Chapter 2 were proved in this setting first.

Theorems are all well and good, but they do not amount to much without interesting examples. Examples in this setting exist and several are given in section 2.4.2, but they are not easy to come by and I failed to find a broad class outside of the atomic representations already studied by Davidson, Power and Yang [19]. Thus, one motivation for generalising my results was to encompass more examples, particularly graphs with more than one vertex. Moving to the full generality of product systems of C^*-correspondences was inspired by Baruch Solel [68, 69] and Orr Shalit's [61, 62] success in generalising multivariate dilation theorems to this setting. This was justified early on by my success in translating Corollary 1 into the language of product system dilations.

The final part written was section 2.2. The earlier versions of Corollary 1 allowed me to use the results of Davidson, Kribs and Shpigel directly. In the new setting of C^*-correspondences there had been no work done on finitely correlated representations, so all of that needed to be completed in order to pass to the product system case.

In Chapter 3 we deal with semicrossed product algebras. Here too dilations are of vital importance. The following theorem, which is proved for some more general semigroups in Theorems 3.2.4 and 3.2.5, is the key dilation result in Chapter 3.

Theorem 2 (Brehmer 1961 [8])**.** *Let $T_1, \dots, T_k$ be commuting contractions on a Hilbert space $\mathcal{H}$ which satisfy $T_i^*T_j = T_jT_i^*$ when $i \neq j$. Then there are k commuting isometries $V_1, \dots, V_k$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ which dilate $T_1, \dots, T_k$, i.e.*

$$\mathit{Proj}(\mathcal{H})p(V_1, \dots, V_k)|_{\mathcal{H}} = p(T_1, \dots, T_k)$$

for any polynomial p in k commuting variables.

Further, if $\mathcal{K}$ is chosen to be minimal then there is a unique dilation $V_1, \dots, V_k$ satisfying $V_i^*V_j = V_jV_i^*$ when $i \neq j$ (up to unitary equivalence).

The condition that $V_i^*V_j = V_jV_i^*$ when $i \neq j$ is known as *doubly commuting* or *Nica-covariant*. This condition is not merely imposed to fit in with dilation theory; it arises naturally by itself. The condition of Nica-covariance arose in the study of semigroup crossed-product C^*-algebras, beginning with Alexandru Nica [51]. A main motivation for why this type of representation has been studied is that the most natural isometric representation of many semigroups has this property: the left regular representation.

In the C^*-algebra literature, actions of many semigroups of endomorphisms on a C^*-algebras have been studied. In the nonself-adjoint literature there has been some study of semicrossed products by semigroups other that the nonnegative integers, e.g. by Benton Duncan [24] and

Kuen-Shan Ling and Paul Muhly [44], but there has been considerably less than in the C^*-algebra cases. Inspired by reading papers by Marcelo Laca, Gerard Murphy, Alexandru Nica, Iain Raeburn and others (see e.g. [41, 42, 50, 51]) Chapter 3 is a modest attempt at broadening the classes of semigroups dealt with in the nonself-adjoint literature.

Chapters 2 and 3 come with their own introductions which motivate the study therein and try to place the work in context with the wider literature. Behind the scenes the problems of multivariate dilation theory underpin much of the work. It can be easy to get lost in the auxiliary problems and notation of product systems and covariant representations, but one has to be always mindful of Varopoulos and Parrott's examples. Further, it is often worthwhile and helpful to consider what theorems say in the simplest case of representing a semigroup by contractions without all of the extra structure.

Chapter 2

Finitely correlated representations of product systems of C^*-correspondences over $\mathbb{N}^k$

2.1 Introduction

A C^*-correspondence over a C^*-algebra $\mathcal{A}$ is a Hilbert bimodule with an $\mathcal{A}$-valued inner product. The C^*-algebras of representations of C^*-correspondences were first studied by Pimsner [56]. In a series of papers beginning with [48], Muhly and Solel studied representations of C^*-correspondences and their algebras. Remarkably they managed to achieve many results from single operator theory in this very general setting. In [48] they include a dilation theorem which supersedes the classical Sz.-Nagy [70] dilation theorem for contractions and the Frazho-Bunce-Popescu [27, 9, 57] dilation theorem for row-contractions. In [49] a Wold decomposition is presented as well as a Beurling-type theorem.

Product systems of C^*-correspondences over the semigroup $\mathbb{R}_+$ were introduced by Arveson in [5]. The study of product systems over discrete semigroups began with Fowler's work in [26], where the generalised Cuntz-Pimsner algebra associated to a product system was introduced. In recent years there have been several papers considering product systems of C^*-correspondences over discrete semigroups, e.g. [61, 62, 64, 65, 68, 69]. There has been work on dilation results for representations of product systems generalizing dilation results for commuting contractions. For example, Solel [68] shows the existence of a dilation for contractive representations of product systems over $\mathbb{N}^2$. This result is analogous to the well-known Ando's theorem for two commuting contractions [1]. Solel [69] gives necessary and sufficient conditions for a contractive representation of a product system over $\mathbb{N}^k$ to have what is known as a *regular* dilation. This result is analogous to a theorem of Brehmer [8]. Skalski and Zacharias [65] have presented a Wold decomposition for representations of product systems over $\mathbb{N}^k$.

The generalised Cuntz-Pimsner C^*-algebras associated to product systems over the semigroup $\mathbb{N}^k$ are not in general GCR, i.e. they can be NGCR. A theorem due to Glimm [30, Theorem 2] tells us that NGCR C^*-algebras do not have smooth duals, i.e. there is no countable family of Borel functions on the space of unitary equivalence classes of irreducible representations which separates points. It follows that trying to classify all irreducible representations up to unitary equivalence of a generalised Cuntz-Pimsner algebra would be a fruitless task. However, in this chapter we find a complete unitary invariant for a certain class of representations: *finitely correlated representations*.

An isometric representation of a product system of C^*-correspondences is finitely correlated if it is the minimal isometric dilation of a finite dimensional representation. We show the existence of a unique minimal cyclic coinvariant subspace for finitely correlated, isometric, fully coisometric representations of product systems over the semigroup $\mathbb{N}^k$. The compression of the representation to this minimal subspace will be the complete unitary invariant. This result generalises the work of Davidson, Kribs and Shpigel [17] for the minimal isometric dilation $[S_1, \ldots, S_n]$ of a finite dimensional row-contraction. Indeed, studying row-contractions is equivalent to studying representations of the C^*-correspondence $\mathbb{C}^n$ over the C^*-algebra $\mathbb{C}$. In [17], it is shown that the projection onto the minimal coinvariant subspace is contained in the WOT-closed algebra generated by the S_i's. This is an important invariant for free semigroup algebras [16]. We are able to establish this in a number of interesting special cases.

Finitely correlated representations were first introduced by Bratteli and Jorgensen [6] via finitely correlated states on $\mathcal{O}_n$. When ω is a finitely correlated state on $\mathcal{O}_n$, the GNS construction on ω will give a representation π of $\mathcal{O}_n$ with the property that $[\pi(s_1), \ldots, \pi(s_n)]$ is a finitely correlated row isometry, where $s_1, \ldots, s_n$ are pairwise orthogonal isometries generating $\mathcal{O}_n$. This relates [17] with [6]. Similarly, following the work of Skalski and Zacharias [66], we will define what it means for a state on the Cuntz-Pimsner algebra $\mathcal{O}_\Lambda$ for finite k-graph Λ to be finitely correlated. Finitely correlated states will give rise to finitely correlated representations of the product system associated to Λ.

In [18] Davidson and Pitts classified atomic representations of $\mathcal{O}_n$, which include as a special case the permutation representations studied by Bratteli and Jorgensen [7]. If $s_1, \ldots, s_n$ are pairwise orthogonal isometries which generate $\mathcal{O}_n$ then a representation π of $\mathcal{O}_n$ on a Hilbert space $\mathcal{H}$ is atomic if there is an orthonormal basis for $\mathcal{H}$ which is permuted by each $\pi(s_i)$ up to multiplication by scalars in $\mathbb{T} \cup \{0\}$. There exist finitely correlated atomic representations of $\mathcal{O}_n$ [18]. Atomic representations have been a used in the study of other objects. In [15] Davidson and Katsoulis show that the C^*-envelope of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ is $\mathcal{O}_n \times_\varphi \mathbb{Z}$, where $\mathfrak{A}_n$ is the noncommutative disc algebra. Finitely correlated atomic representations of $\mathcal{O}_n$ are used as a tool to get to this result, see [15, Theorem 4.4]. For a general C^*-correspondence or product system of C^*-correspondences it is not clear what it could mean for a representation to be atomic. Thus the finitely correlated representations presented in this chapter are possibly the nearest analogy to finitely correlated atomic representations. In [19, 22] atomic representations of single vertex k-graphs have been classified.

In section 2 we study finitely correlated representations of C^*-correspondences. To this end we follow the same program of attack as [17]. Many of the proofs follow the same line of argument as the corresponding proofs in [17]. When this is the case it is remarked upon. Lemma 2.2.12 corresponds to [17, Lemma 4.1], and is the key technical tool to our analysis in this section. It should be noted that Lemma 2.2.12 does not just generalise [17, Lemma 4.1], but the proof presented here greatly simplifies the argument in [17]. The main results of this section are summarised in Theorem 2.2.27 and Corollary 2.2.28.

Every graph can be associated with a C^*-correspondence. Thus results on representations of C^*-correspondences also apply to graph algebras. In Section 2.4.1 we apply our results to nonself-adjoint graph algebras. The study of nonself-adjoint graph algebras has received attention in several papers in recent years, e.g. [10, 32, 34, 37, 38, 67]. We strengthen our results from Section 2.2 for the case of an algebra of a finite graph with the *strong double-cycle property*, i.e. for finite graphs where every vertex has a path to a vertex which lies on two distinct minimal cycles. We show that the nonself-adjoint WOT-closed algebra generated by a finitely correlated, isometric, fully coisometric representation of such a graph contains the projection onto its unique minimal cyclic coinvariant subspace. Aided by the work of Kribs and Power [37] and Muhly and Solel [49] on the algebras of directed graphs we use the same method of proof as in [17] to prove this result. This includes the case studied in [17].

In Section 2.3 we prove the prove the main results of this chapter (Theorem 2.3.19 and Corollary 2.3.21) by generalising the results of section 2 to product systems of C^*-correspondences over $\mathbb{N}^k$. Our main tool in this section is Theorem 2.3.12. A representation of a product system of C^*-correspondences provides a representation for each C^*-correspondence in the product system. An isometric dilation of a contractive representation of a product system of C^*-correspondences gives an isometric dilation of each of the representations of the individual C^*-correspondences. Theorem 2.3.12 tells us that if we have a *minimal* isometric dilation of a fully coisometric representation of a product system over $\mathbb{N}^k$, then the dilations of the corresponding representations of certain individual C^*-correspondences in the product system will also be minimal. This allows us to deduce the existence of a unique minimal cyclic coinvariant subspace for finitely correlated, isometric, fully coisometric representations of product systems from the C^*-correspondence case. In fact, we will show in Theorem 2.3.19 that the unique minimal cyclic coinvariant subspace for a representation of a product system will be the same unique minimal cyclic coinvariant subspace for a certain C^*-correspondence.

Higher-rank graph algebras were introduced by Kumjian and Pask in [40]. A k-graph is, roughly speaking, a set of vertices with k sets of directed edges (k colours), together with a commutation rule between paths of different colours. In the last decade there has been a lot of study on the C^*-algebras generated by representations of higher-rank graphs. In more recent years there has been some study on their nonself-adjoint counterparts, see e.g. [39, 58]. The case of algebras of higher-rank graphs with a single vertex has proved to be rather interesting. Their study was begun by Kribs and Power [39]. Further study has been carried out by Davidson, Power and Yang [58, 20, 19, 21, 22, 74].

A k-graph can be associated with a product system of C^*-correspondences over the discrete semigroup $\mathbb{N}^k$. Thus results on product systems of C^*-correspondences over $\mathbb{N}^k$ apply to higher-rank graph algebras. In Section 2.4.2 we remark that since certain 1-graphs contained in a k-graph Λ share the same unique minimal cyclic coinvariant subspace for a finitely correlated representation, if Λ contains a 1-graph with the strong double-cycle property, then the WOT-closed algebra generated by a finitely correlated, isometric, fully coisometric representation will contain the projection onto its minimal cyclic coinvariant subspace. A k-graph with only one vertex satisfies this condition.

In [17] the case of non-fully coisometric, finitely correlated row isometries are also studied. The case of finitely correlated representations of product systems of C^*-correspondences which are not fully coisometric are not studied in this thesis. The reason for this is because, unlike the Frazho-Bunce-Popescu dilation used in [17], dilations of representations of product systems need not be unique if they are not fully coisometric. See section 2.3.2 for a discussion of dilation theorems for representations of product systems of C^*-correspondences over $\mathbb{N}^k$.

2.2 C^*-Correspondences

2.2.1 Preliminaries and Notation

Most of the background on C^*-correspondences needed in this thesis can be found in the works of Muhly and Solel [48, 49]. Provided here is a brief summary of the necessary definitions.

Let E be a right module over a C^*-algebra $\mathcal{A}$. An $\mathcal{A}$-valued inner product on E is a map $\langle\cdot,\cdot\rangle : E \times E \to \mathcal{A}$ which is conjugate linear in the first variable, linear in the second variable and satisfies

1. $\langle \xi, \eta a\rangle = \langle \xi, \eta\rangle a$
2. $\langle \xi, \eta\rangle^* = \langle \eta, \xi\rangle$ and
3. $\langle \xi, \xi\rangle \geq 0$ where $\langle \xi, \xi\rangle = 0$ if and only if $\xi = 0$,

for $\xi, \eta \in E$ and $a \in \mathcal{A}$. We can define a norm on E by setting $\|\xi\| = \|\langle \xi, \xi\rangle\|^{\frac{1}{2}}$. If E is complete with respect to this norm then it is called a *Hilbert C^*-module*. We denote by $\mathcal{L}(E)$ the space of all *adjointable* bounded linear functions from E to E, i.e. the bounded operators on E with a (necessarily unique) adjoint with respect to the inner product on E. The adjointable operators on a Hilbert C^*-module form a C^*-algebra. For $\xi, \eta \in E$ define $\xi\eta^* \in \mathcal{L}(E)$ by

$$\xi\eta^*(\zeta) = \xi\langle\zeta, \eta\rangle$$

for each $\zeta \in E$. Denote by $\mathcal{K}(E)$ the closed linear span of $\{\xi\eta^* : \xi, \eta \in E\}$. The space $\mathcal{K}(E)$ forms a C^*-subalgebra of $\mathcal{L}(E)$ referred to as the *compact operators on* E. More on Hilbert C^*-modules can be found in [43].

If there is a homomorphism φ from $\mathcal{A}$ to $\mathcal{L}(E)$, then the Hilbert C^*-module E, together with the left action on E defined by φ, is a C^**-correspondence over* $\mathcal{A}$. If E and F are two C^*-correspondences over $\mathcal{A}$ we will write φ_E and φ_F to describe the left action of $\mathcal{A}$ on E and F respectively. With that said, when there is little chance of confusion we will write $a\xi$ in place of $\varphi(a)\xi$.

Suppose E and F are two C^*-correspondences over a C^*-algebra $\mathcal{A}$. We define the following $\mathcal{A}$-valued inner product on the algebraic tensor product $E \otimes_A F$, of E and F: for ξ_1, ξ_2 in E and η_1, η_2 in F we let

$$\langle\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \langle\eta_1, \varphi_F(\langle\xi_1, \xi_2\rangle)\eta_2\rangle.$$

Taking the Hausdorff completion of $E \otimes_A F$ with respect to this inner product gives us the *interior tensor product* of E and F denoted $E \otimes F$. This is the only tensor product of C^*-correspondences that we will use in this thesis so we will omit the word "interior" and merely say we are taking the tensor product of C^*-correspondences. When taking the tensor product of a C^*-correspondence E with itself we will write E^2 in place of $E \otimes E$, and similarly we will write E^n in place of the n-fold tensor product of E with itself. We will also set $E^0 = \mathcal{A}$.

The Fock space $\mathcal{F}(E)$ is defined to be the C^*-correspondence

$$\mathcal{F}(E) = \sum_{n\geq 0}^{\oplus} E^n.$$

The left action of $\mathcal{A}$ on $\mathcal{F}(E)$ is denote by φ_∞ and defined by

$$\varphi_\infty(a)\xi_1 \otimes \ldots \otimes \xi_n = (a\xi_1) \otimes \ldots \otimes \xi_n.$$

We define *creation operators* T_ξ in $\mathcal{L}(\mathcal{F}(E))$ for $\xi \in E$ by

$$T_\xi(\xi_1 \otimes \ldots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \ldots \otimes \xi_n \in E^{n+1}$$

for $\xi_1 \otimes \ldots \otimes \xi_n \in E^n$. The norm closed algebra in $\mathcal{L}(\mathcal{F}(E))$ generated by

$$\{T_\xi, \varphi_\infty(a) : \xi \in E, a \in \mathcal{A}\}$$

is denoted by $\mathcal{T}_+(E)$ and called the *tensor algebra over* E. The C^*-algebra generated by $\mathcal{T}_+(E)$ is denoted $\mathcal{T}(E)$ and called the *Toeplitz algebra over* E.

A *completely contractive covariant representation* (A, σ) of a C^*-correspondence E over $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a completely contractive linear map A from E to $\mathcal{B}(\mathcal{H})$ and a unital, non-degenerate representation σ of $\mathcal{A}$ on $\mathcal{H}$ which satisfy the following covariant property:

$$A(a\xi b) = \sigma(a)A(\xi)\sigma(b)$$

for $a, b \in \mathcal{A}$ and $\xi \in E$. We will abbreviate *completely contractive covariant representation* to merely *representation*, as these will be the only representations of C^*-correspondences we will consider. A representation (A, σ) is called *isometric* if it satisfies

$$A(\xi)^* A(\eta) = \sigma(\langle \xi, \eta \rangle).$$

Why this is called isometric will become clear presently.

If σ is a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and E is a C^*-correspondence over $\mathcal{A}$, then we can form a Hilbert space $E \otimes_\sigma \mathcal{H}$ by taking the algebraic tensor product of E and $\mathcal{H}$ and taking the Hausdorff completion with respect to the inner product defined by

$$\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle) h_2 \rangle$$

for $\xi_1, \xi_2 \in E$ and $h_1, h_2 \in \mathcal{H}$. We will write $E \otimes \mathcal{H}$ in place of $E \otimes_\sigma \mathcal{H}$ when it is understood which representation we are talking about. We can *induce* σ to a representation σ^E of $\mathcal{L}(E)$ on $E \otimes \mathcal{H}$. This is defined by

$$\sigma^E(T)(\xi \otimes h) = (T\xi) \otimes h$$

for $T \in \mathcal{L}(E), \xi \in E$ and $h \in \mathcal{H}$. In particular we can induce σ to $\sigma^{\mathcal{F}(E)}$. We define an isometric representation (V, ρ) of E on $\mathcal{F}(E) \otimes \mathcal{H}$ by

$$\rho(a) = \sigma^{\mathcal{F}(E)} \circ \varphi_\infty(a)$$

for each $a \in \mathcal{A}$ and

$$V(\xi) = \sigma^{\mathcal{F}(E)}(T_\xi)$$

for each $\xi \in E$. We call (V, ρ) the representation of E *induced by* σ.

If (A, σ) is a representation of E on $\mathcal{H}$, then we define the operator $\tilde{A}$ from $E \otimes_\sigma \mathcal{H}$ to $\mathcal{H}$ by

$$\tilde{A}(\xi \otimes h) = A(\xi)h.$$

This operator was introduced by Muhly and Solel in [48], where they show that $\tilde{A}$ is a contraction. Furthermore, they show that $\tilde{A}$ is an isometry if and only if (A, σ) is an isometric representation.

A representation is called *fully coisometric* when $\tilde{A}$ is a coisometry. We write $\tilde{A}_n$ for the operator from $E^n \otimes_\sigma \mathcal{H}$ to $\mathcal{H}$ defined by

$$\tilde{A}_n(\xi_1 \otimes \ldots \otimes \xi_n \otimes h) = A(\xi_1) \ldots A(\xi_n)h.$$

Note also that $\sigma(a)\tilde{A} = \tilde{A}\sigma^E(\varphi(a))$.

If σ is a representation of $\mathcal{A}$ on $\mathcal{H}$ and X is in the commutant of $\sigma(\mathcal{A})$, then we can define a bounded operator $I \otimes X$ on $E \otimes \mathcal{H}$ by

$$(I \otimes X)(\xi \otimes h) = \xi \otimes Xh.$$

It is readily verifiable that $I \otimes X$ is a bounded operator and that $\|I \otimes X\| \leq \|X\|$. In particular if $\mathcal{M}$ is a subspace of $\mathcal{H}$ with $P_{\mathcal{M}} \in \sigma(\mathcal{A})'$ then $I \otimes P_{\mathcal{M}}$ is a projection in $\mathcal{B}(E \otimes \mathcal{H})$. Thus $E \otimes \mathcal{H}$ decomposes into a direct sum $E \otimes \mathcal{H} = (E \otimes \mathcal{M}) \oplus (E \otimes \mathcal{M}^\perp)$.

Let (S, ρ) be a representation of a C^*-correspondence E on a Hilbert space $\mathcal{H}$. We denote by I be the identity in $\mathcal{B}(H)$. We call the weak-operator topology closed algebra

$$\mathfrak{S} = \mathrm{Alg}\{I,\ S(\xi),\ \rho(a) : \xi \in E,\ a \in \overline{\mathcal{A}\}}^{\mathrm{WOT}}$$

the *unital* WOT*-closed algebra generated by the representation* (S, ρ)*.*

2.2.2 Minimal Isometric Dilations

Definition 2.2.1. Let E be a C^*-correspondence over a C^*-algebra $\mathcal{A}$ and let (A, σ) be a representation of E on a Hilbert space $\mathcal{V}$. A representation (S, ρ) of E on $\mathcal{H}$ is a *dilation* of (A, σ) if $\mathcal{V} \subseteq \mathcal{H}$ and

1. $\mathcal{V}$ reduces ρ and $\rho(a)|_{\mathcal{V}} = \sigma(a)$ for all $a \in \mathcal{A}$.
2. $\mathcal{V}^\perp$ is invariant under $S(\xi)$ for all $\xi \in E$
3. $P_{\mathcal{V}} S(\xi)|_{\mathcal{V}} = A(\xi)$ for all $\xi \in E$.

A dilation (S, ρ) of (A, σ) is an *isometric dilation* if (S, ρ) is an isometric representation. A dilation (S, ρ) of (A, σ) on $\mathcal{H}$ is called *minimal* if $\mathcal{H}$ is the smallest reducing subspace for $\mathrm{Alg}\{\sigma(a), S(\xi)\}$ which contains $\mathcal{V}$.

Theorem 2.2.2 (Muhly and Solel [48])**.** *If (A, σ) is a contractive representation of a C^*-correspondence E on a Hilbert space $\mathcal{V}$, then (A, σ) has an isometric dilation (S, ρ). Further, we can choose (S, ρ) to be minimal; and the minimal isometric dilation of (A, σ) is unique up to a unitary equivalence which fixes $\mathcal{V}$.*

The following lemma uses a standard argument in dilation theory.

Lemma 2.2.3. *If (A,σ) is a representation of a C^*-correspondence E on a Hilbert space $\mathcal{V}$ and (S,ρ) is its minimal isometric dilation on $\mathcal{H}$, then (S,ρ) is fully coisometric if and only if (A,σ) is fully coisometric.*

Proof. Clearly, if $\tilde{S}\tilde{S}^* = I_{\mathcal{H}}$ then for $v \in \mathcal{V}$, $\tilde{A}\tilde{A}^* v = P_{\mathcal{V}}\tilde{S}\tilde{S}^* v = P_{\mathcal{V}} v = v$, and so $\tilde{A}$ is a coisometry.

Conversely, suppose that $\tilde{A}$ is a coisometry. Let $\mathcal{M} = (I - \tilde{S}\tilde{S}^*)\mathcal{H}$. It is not hard to see that $\mathcal{M}$ is a $\mathfrak{S}^*$-invariant subspace, where $\mathfrak{S}$ is the unital WOT-closed algebra generated by the representation (S,ρ). Also since $\tilde{A}\tilde{A}^* = I_{\mathcal{V}}$ we have that $P_{\mathcal{V}}\tilde{S}\tilde{S}^*|_{\mathcal{V}} = I_{\mathcal{V}}$, hence $\mathcal{M}$ is a $\mathfrak{S}^*$-invariant space orthogonal to $\mathcal{V}$. But, since our dilation is minimal the only $\mathfrak{S}^*$-invariant subspace orthogonal to $\mathcal{V}$ is the zero space. Therefore $\mathcal{M} = \{0\}$. $\square$

The following two results have been proved in [17] for the case when $E = \mathbb{C}^n$ (where $2 \le n \le \infty$) and $\mathcal{A} = \mathbb{C}$. We follow much the same line of proof as found there.

Lemma 2.2.4. *Let (A,σ) be a representation of a C^*-correspondence E on a Hilbert space $\mathcal{V}$, and let (S,ρ) be the unique minimal isometric dilation of (A,σ) on a Hilbert space $\mathcal{H}$. Let $\mathcal{W} = (\mathcal{V} + \tilde{S}(E \otimes \mathcal{V})) \ominus \mathcal{V}$. Then $\mathcal{W}$ is a ρ-reducing subspace and $\mathcal{V}^\perp$ is isometrically isomorphic to $\mathcal{F}(E) \otimes \mathcal{W}$. Furthermore, the representation of E obtained by restricting (S,ρ) to $\mathcal{V}^\perp$ is the representation induced by $\rho(\cdot)|_{\mathcal{W}}$.*

Proof. First note that $\mathcal{W}$ is ρ-reducing. This follows since $\mathcal{V}$ is ρ-reducing and hence so is $\mathcal{V}^\perp$ and $\rho(a)S(\xi)\mathcal{V} = S(a\xi)\mathcal{V}$ for each $a \in \mathcal{A}$ and $\xi \in E$.

The subspace $\mathcal{V}^\perp$ is invariant under $S(\xi)$ for each $\xi \in E$. So for any n and $\xi_1, \dots, \xi_n \in E$, the space $S(\xi_1)\dots S(\xi_n)\mathcal{W}$ is orthogonal to $\mathcal{V}$. It follows that if $n \ge 1$, then $S(\xi_1)\dots S(\xi_n)\mathcal{W}$ is orthogonal to $S(\xi)\mathcal{V}$ for all $\xi \in E$. Therefore $S(\xi_1)\dots S(\xi_n)\mathcal{W}$ is orthogonal to $\mathcal{V} + \tilde{S}(E \otimes \mathcal{V})$, which contains $\mathcal{W}$.

Also note that if $\eta_1, \dots, \eta_m$ are in E, with $m < n$ and w_1 and w_2 in $\mathcal{W}$ then

$$\begin{aligned}&\langle S(\xi_1)\dots S(\xi_n)w_1, S(\eta_1)\dots S(\eta_m)w_2\rangle \\ &\qquad = \langle \rho(\langle \eta_1 \otimes \dots \otimes \eta_m, \xi_1 \otimes \dots \otimes \xi_m\rangle) S(\xi_{m+1})\dots S(\xi_n)w_1, w_2\rangle = 0.\end{aligned}$$

By minimality we have that

$$\begin{aligned}\mathcal{V}^\perp &= \sum_{n\ge 0}^{\oplus} \sum_{\xi_1,\dots,\xi_n \in E} S(\xi_1)\dots S(\xi_n)\mathcal{W} \\ &= \sum_{n \ge 0}^{\oplus} \tilde{S}^n(E^n \otimes \mathcal{W}) \\ &\simeq \mathcal{F}(E) \otimes \mathcal{W}.\end{aligned}$$

$\square$

Remark 2.2.5. When (A, σ) is a fully coisometric representation of a C^*-correspondence E on a Hilbert space $\mathcal{V}$, we have that $\mathcal{V} = \tilde{S}\tilde{S}^*\mathcal{V} = \tilde{S}\tilde{A}^*\mathcal{V} \subseteq \tilde{S}(E \otimes \mathcal{V})$. Hence, when (A, σ) is fully coisometric the space $\mathcal{W}$ in Lemma 2.2.4 is simply $\mathcal{W} = \tilde{S}(E \otimes \mathcal{V}) \ominus \mathcal{V}$.

Lemma 2.2.6. *Let (A, σ) be a representation of a C^*-correspondence E on a Hilbert space $\mathcal{V}$, and let (S, ρ) be the unique minimal isometric dilation of (A, σ) on a Hilbert space $\mathcal{H}$. Let $\mathfrak{A}$ be the* WOT*-closed unital algebra generated by (A, σ) and let $\mathfrak{S}$ be the* WOT*-closed unital algebra generated by (S, ρ). Suppose $\mathcal{V}_1$ is an $\mathfrak{A}^*$-invariant subspace of $\mathcal{V}$. Then $\mathcal{H}_1 = \mathfrak{S}[\mathcal{V}_1]$ reduces $\mathfrak{S}$.*

If $\mathcal{V}_1$ and $\mathcal{V}_2$ are orthogonal $\mathfrak{A}^*$-invariant subspaces, then $\mathcal{H}_j = \mathfrak{S}[\mathcal{V}_j]$ for $j = 1, 2$ are mutually orthogonal.

If $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{H}_j \cap \mathcal{V} = \mathcal{V}_j$ for $j = 1, 2$.

Proof. Note that for any $a \in \mathcal{A}$, $\rho(a)\mathcal{V}_1 = \sigma(a)\mathcal{V}_1 \subseteq \mathcal{V}_1$. Also for any $\xi \in E$, $S(\xi)^*\mathcal{V}_1 = A(\xi)^*\mathcal{V}_1 \subseteq \mathcal{V}_1$. Hence $\mathcal{V}_1$ is $\mathfrak{S}^*$-invariant. Now $\mathcal{H}_1$ is spanned by vectors of the form $S(\xi_1) \dots S(\xi_n)v$, where $\xi_1, \dots, \xi_n \in E$ and $v \in \mathcal{V}_1$. If $n \geq 2$ then for any $\xi \in E$,

$$S(\xi)^*S(\xi_1) \dots S(\xi_n)v = S(\langle \xi, \xi_1 \rangle \xi_2) \dots S(\xi_n)v \in \mathcal{H}_1.$$

If $n = 1$ we have $S(\xi)^*S(\xi_1)v = \rho(\langle \xi, \xi_1 \rangle)v = \sigma(\langle \xi, \xi_1 \rangle)v \in \mathcal{H}_1$. Hence $\mathcal{H}_1$ reduces $\mathfrak{S}$.

Now suppose $\mathcal{V}_1$ and $\mathcal{V}_2$ are orthogonal $\mathfrak{A}^*$-invariant subspaces. Take $v_1 \in \mathcal{V}_1$, $v_2 \in \mathcal{V}_2$ and $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m$ be in E. Suppose $n \geq m$ then

$$\begin{aligned}&\langle S(\xi_1) \dots S(\xi_n)v_1, S(\eta_1) \dots S(\eta_m)v_2 \rangle \\ &\qquad = \langle v_1, S(\xi_n)^* \dots S(\xi_{m+1})^* \rho(\langle \xi_m, \eta_m \rangle \dots \langle \xi_1, \eta_1 \rangle)v_2 \rangle = 0.\end{aligned}$$

It follows that $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal.

If $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ then, since $\mathcal{H}_1$ contains $\mathcal{V}_1$ and is orthogonal to $\mathcal{V}_2$, $\mathcal{H}_1 \cap \mathcal{V} = \mathcal{V}_1$. Finally, $\mathcal{H}_1 \oplus \mathcal{H}_2$ is an $\mathfrak{S}$-reducing subspace containing $\mathcal{V}$, so it is all of $\mathcal{H}$ by the minimality of the dilation. □

Given an isometric representation (S, ρ) of a C^*-correspondence E on $\mathcal{H}$ with corresponding unital WOT-closed algebra $\mathfrak{S}$ which is the minimal isometric dilation of a representation (A, σ) on $\mathcal{V} \subseteq \mathcal{H}$, Lemma 2.2.6 shows that $\mathfrak{S}^*$-invariant subspaces of $\mathcal{V}$ give rise to $\mathfrak{S}$-reducing subspaces of $\mathcal{H}$. In Corollary 2.2.8 we give a weak converse of this: that $\mathfrak{S}$-reducing subspaces in $\mathcal{H}$ are uniquely determined by their projections onto $\mathcal{V}$. This follows from the following more general result.

Lemma 2.2.7. *Let (A, σ) be a representation of a C^*-correspondence E on a Hilbert space $\mathcal{V}$, and let (S, ρ) be the unique minimal isometric dilation of (A, σ) on a Hilbert space $\mathcal{H}$. Let $\mathfrak{S}$*

be the unital WOT*-closed algebra generated by* (S,ρ)*. Suppose* B *is a normal operator in* $\mathcal{B}(\mathcal{H})$ *such that the range of* B *is contained in* $\mathcal{V}^\perp$ *and* B *is in* $C^*(S(E),\rho(\mathcal{A}))'$*, the commutant of the* C^**-algebra generated by* $S(E)$ *and* $\rho(\mathcal{A})$*. Then* $B=0$*.*

Proof. Suppose that B is non-zero. Take any δ such that $0<\delta<\|B\|$ and let D_δ be the open disc of radius δ about 0. Let Q be the spectral projection $Q=E_B(\mathrm{spec}(B)\backslash D_\delta)$, where $\mathrm{spec}(B)$ denotes the spectrum of B. Then $Q\in W^*(B)\subseteq C^*(S(E),\rho(\mathcal{A}))'$ and $Q\mathcal{H}$ is orthogonal to $\mathcal{V}$. In particular $Q\mathcal{H}$ is a non-zero $\mathfrak{S}^*$-invariant space orthogonal to $\mathcal{V}$. But no such space can exist since our dilation is minimal. □

Corollary 2.2.8. *Suppose* $\mathcal{M}$ *and* $\mathcal{N}$ *are two* $\mathfrak{S}$*-reducing subspaces of* $\mathcal{H}$ *and the compressions of* $P_\mathcal{M}$ *and* $P_\mathcal{N}$ *to* $\mathcal{V}$ *are equal, i.e.* $P_\mathcal{V}P_\mathcal{M}P_\mathcal{V}=P_\mathcal{V}P_\mathcal{N}P_\mathcal{V}$*. Then* $\mathcal{M}=\mathcal{N}$*.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathfrak{S}$-reducing subspaces with $P_\mathcal{V}P_\mathcal{M}P_\mathcal{V}=P_\mathcal{V}P_\mathcal{N}P_\mathcal{V}$. Elements of the form $S(\xi_1)\dots S(\xi_n)v$, with $v\in\mathcal{V}$ and $\xi_1,\dots,\xi_n\in E$, span a dense subset of $\mathcal{H}$ and

$$\begin{aligned}(P_\mathcal{M}-P_\mathcal{N})S(\xi_1)\dots S(\xi_n)v &= S(\xi_1)\dots S(\xi_n)(P_\mathcal{M}-P_\mathcal{N})v\\ &= S(\xi_1)\dots S(\xi_n)P_{\mathcal{V}^\perp}(P_\mathcal{M}-P_\mathcal{N})v\in\mathcal{V}^\perp.\end{aligned}$$

It follows that the range of $P_\mathcal{M}-P_\mathcal{N}$ lies in $\mathcal{V}^\perp$. Hence, by Lemma 2.2.7 $P_\mathcal{M}-P_\mathcal{N}=0$ and $\mathcal{M}=\mathcal{N}$. □

2.2.3 Finitely Correlated Representations

Definition 2.2.9. An isometric representation (S,ρ) of a C^*-correspondence E on a Hilbert space $\mathcal{H}$ is called *finitely correlated* if (S,ρ) is the minimal isometric dilation of a representation (A,σ) on a non-zero finite dimensional Hilbert space $\mathcal{V}\subseteq\mathcal{H}$.

In particular, if $\mathfrak{S}$ is the unital WOT-closed algebra generated by (S,ρ), then (S,ρ) is finitely correlated if there is a finite dimensional $\mathfrak{S}^*$-invariant subspace $\mathcal{V}$ of $\mathcal{H}$ such that (S,ρ) is the minimal isometric dilation of the representation $(P_\mathcal{V}S(\cdot)|_\mathcal{V},\rho(\cdot)|_\mathcal{V})$.

Remark 2.2.10. It should be noted that not all C^*-algebras can be represented non-trivially on a finite dimensional Hilbert spaces, e.g. if $\mathcal{A}$ is a properly infinite C^*-algebra then there are no non-zero finite dimensional representations of $\mathcal{A}$ since $\mathcal{A}$ contains isometries with pairwise orthogonal ranges. Likewise, any simple infinite dimensional C^*-algebra has no finite dimensional representations.

In this section we are concerned with finitely correlated fully coisometric representations. If we assume that a C^*-correspondence E over a C^*-algebra $\mathcal{A}$ has a fully coisometric representation then we are assuming that there are non-zero representations of $\mathcal{A}$ on finite-dimensional Hilbert spaces. Under this assumption there are still a wide range of C^*-correspondences which can be studied, e.g. the following example and the C^*-correspondences associated to graphs in section 2.4.

Example 2.2.11. The case when $\mathcal{A} = \mathbb{C}$ and $E = \mathbb{C}^n$ has been studied previously in [17]. A representation of E on a finite dimensional space $\mathcal{V}$ is simply a row-contraction $A = [A_1, \ldots, A_n]$ from $\mathcal{V}^{(n)}$ to $\mathcal{V}$. The representation is fully-coisometric when A is *defect free*, i.e.

$$\sum_{i=1}^{n} A_i A_i^* = I_{\mathcal{V}}.$$

The dilation of A will be the Frazho-Bunce-Popescu dilation of A to a row-isometry $S = [S_1, \ldots, S_n]$. The dilation S will be defect free as A is. These representations can alternatively be viewed as representations of a graph with 1 vertex and n edges, see §2.4.1.

Let (S, ρ) be a fully coisometric, finitely correlated representation on $\mathcal{H}$ of the C^*-correspondence E over the C^*-algebra $\mathcal{A}$, and let $\mathfrak{S}$ be the unital WOT-closed algebra generated by (S, ρ). A key tool in the analysis in [17] is that every non-zero $\mathfrak{S}^*$-invariant subspace of $\mathcal{H}$ has non-trivial intersection with $\mathcal{V}$ ([17, Lemma 4.1]), for the case $\mathcal{A} = \mathbb{C}$ and $E = \mathbb{C}^n$. The main idea of the proof is that, because the representation is fully coisometric and the unit ball in $\mathcal{V}$ is compact, one can "pull back" any non-zero element of $\mathcal{H}$ with elements in $\mathfrak{S}^*$ to $\mathcal{V}$, without the norm going to zero. However, the proof in [17] that the norm does not go to zero is quite complicated. We prove the analogous result for more general C^*-correspondences than those studied in [17] below. The proof presented below simplifies the approach in [17] by "pulling back" not in $\mathcal{H}$ but in $\mathcal{F}(E) \otimes \mathcal{H}$, making use of Muhly and Solel's $\tilde{\ }$operators.

Lemma 2.2.12. *Let (S, ρ) be a finitely correlated, fully coisometric representation of a C^*-correspondence E on $\mathcal{H}$. Let $\mathfrak{S}$ be the unital* WOT*-closed algebra generated by (S, ρ) and let $\mathcal{V}$ be a finite dimensional $\mathfrak{S}^*$-invariant subspace of $\mathcal{H}$ such that (S, ρ) is the minimal isometric dilation of the representation $(P_{\mathcal{V}} S(\cdot)|_{\mathcal{V}}, \rho(\cdot)|_{\mathcal{V}})$.*

If $\mathcal{M}$ is a non-zero, $\mathfrak{S}^*$-invariant subspace of $\mathcal{H}$, then the subspace $\mathcal{M} \cap \mathcal{V}$ is non-trivial.

Proof. Let $\mu = \|P_{\mathcal{V}} P_{\mathcal{M}}\|$. If $\mu = 1$ then for each n there is a unit vector $h_n \in \mathcal{M}$ such that $\|P_{\mathcal{V}^\perp} h_n\| < \frac{1}{n}$. Let $v_n = P_{\mathcal{V}} h_n$. We have that $(v_n)_n$ is a sequence in the unit ball of $\mathcal{V}$ therefore it has a convergent subsequence $(v_{n_i})_{n_i}$. Let v_0 be the limit of $(v_{n_i})_{n_i}$. We have then that

$$\|h_{n_i} - v_0\| \le \|h_{n_i} - v_{n_i}\| + \|v_{n_i} - v_0\| \to 0,$$

as $n_i \to \infty$ and so the subsequence $(h_{n_i})_{n_i}$ converges to v_0. Therefore v_0 is a non-zero vector in $\mathcal{M} \cap \mathcal{V}$. Thus showing that $\mu = 1$ will prove the lemma.

Let h be a unit vector in $\mathcal{M}$. Since our dilation is minimal there is a sequence $(k_n)_n$ converging to h where each k_n is of the form

$$k_n = \sum_{i=1}^{N_n} S(\xi_{n,i,1}) \dots S(\xi_{n,i,w_{n,i}}) v_{n,i}$$

with $\xi_{n,i,j} \in E$ and $v_{n,i} \in \mathcal{V}$. Without loss of generality we can assume that $\|k_n\| = 1$ for each n.

If we let $M_n = \max\{w_{n,i} : 1 \leq i \leq N_n\}$ for each n, then for any $\xi_1, \dots, \xi_{M_n} \in E$ we have

$$S(\xi_1)^* \dots S(\xi_{M_n})^* k_n \in \mathcal{V}.$$

It follows that $\tilde{S}_{M_n}^* k_n \in E^{M_n} \otimes \mathcal{V}$. Note that $\tilde{S}_{M_n}$ is a coisometry so $\|\tilde{S}_{M_n}^* k_n\| = 1$. We also have that $\tilde{S}_{M_n}^* h \in E^{M_n} \otimes \mathcal{M}$ and $\|\tilde{S}_{M_n}^* h\| = 1$.

Let $u_n = \tilde{S}_{M_n}^* k_n$ and $h_n = \tilde{S}_{M_n}^* h$. We have that

$$\|u_n - h_n\| \to 0$$

as $n \to 0$. If $\mu < 1$ we can choose $\varepsilon > 0$ such that $1 - \varepsilon \geq \mu$ and take n large enough so that

$$\|h_n - u_n\|^2 = \|h_n\|^2 + \|u_n\|^2 - 2Re\langle h_n, u_n \rangle < 2\varepsilon.$$

It follows that

$$\begin{aligned} 1 - \varepsilon &< Re\langle h_n, u_n \rangle \\ &\leq \|(I_{E_{M_n}} \otimes P_{\mathcal{V}}) h_n\| \|u_n\|, \end{aligned}$$

with last inequality being the Cauchy-Schwarz inequality. So our choice of ε tells us that $\mu < \|(I_{E_{M_n}} \otimes P_{\mathcal{V}} P_{\mathcal{M}})\| \leq \|P_{\mathcal{V}} P_{\mathcal{M}}\|$. This is a contradiction. Thus $\mu = 1$. □

Proposition 2.2.13. *Let (A, σ) be a representation of a C^*-correspondence E on a finite dimensional Hilbert space $\mathcal{V}$, and let (S, ρ) be the unique minimal isometric dilation of (A, σ) on a Hilbert space $\mathcal{H}$. Let $\mathfrak{A}$ be the unital algebra generated by the representation (A, σ) and let $\mathfrak{S}$ be the unital* WOT*-closed algebra generated by (S, ρ).*

If $\mathcal{V}_1$ is an $\mathfrak{A}^*$-invariant subspace of $\mathcal{V}$ and $\mathcal{H}_1 = \mathfrak{S}[\mathcal{V}_1]$. Then $\mathcal{H}_1 \cap \mathcal{V} = \mathfrak{A}[\mathcal{V}_1]$.

Proof. If $w \in \mathcal{V} \ominus \mathfrak{A}[\mathcal{V}_1]$ then $\mathfrak{A}^* w$ is an $\mathfrak{A}^*$-invariant space orthogonal to $\mathcal{V}_1$, hence by Lemma 2.2.6 $\mathfrak{S}[\mathfrak{A}^* w] \subseteq \mathcal{H}_1^\perp$. Therefore $\mathcal{H}_1 \cap \mathcal{V} \subseteq \mathfrak{A}[\mathcal{V}_1]$.

If $w \in \mathcal{H}_1^\perp \cap \mathcal{V}$ then for any $A \in \mathfrak{A}$ and $v \in \mathcal{V}_1$ then we have that $0 = \langle A^* w, v \rangle = \langle w, Av \rangle$. Hence $\mathfrak{A}[\mathcal{V}_1] \subseteq \mathcal{H}_1 \cap \mathcal{V}$. □

Corollary 2.2.14. *If $\mathcal{M}$ is a $\mathfrak{S}$-reducing subspace then $\mathcal{M} = \mathfrak{S}[\mathcal{M} \cap \mathcal{V}]$.*

Proof. $\mathcal{M}$ is a $\mathfrak{S}$-reducing subspace and, by Lemma 2.2.6, $\mathfrak{S}[\mathcal{M} \cap \mathcal{V}]$ is a $\mathfrak{S}$-reducing subspace. Hence $\mathcal{M} \ominus \mathfrak{S}[\mathcal{M} \cap \mathcal{V}]$ is $\mathfrak{S}$-reducing. If $\mathcal{M} \ominus \mathfrak{S}[\mathcal{M} \cap \mathcal{V}]$ is non-zero then by Lemma 2.2.12, $\mathcal{M} \ominus \mathfrak{S}[\mathcal{M} \cap \mathcal{V}]$ has non-zero intersection with $\mathcal{V}$. This yields a contradiction as the intersection will be orthogonal to $\mathcal{M} \cap \mathcal{V}$. $\square$

Corollary 2.2.15. *If $\mathfrak{A} = \mathcal{B}(\mathcal{V})$ then every $\mathfrak{S}^*$-invariant subspace of $\mathcal{H}$ contains $\mathcal{V}$.*

Proof. Suppose $\mathcal{M}$ is a non-zero $\mathfrak{S}$-reducing subspace. Then $\mathcal{M} \cap \mathcal{V}$ is a non-zero $\mathfrak{S}^*$-invariant, and hence $\mathfrak{A}^*$-invariant, subspace of $\mathcal{V}$. Hence $\mathcal{M} \cap \mathcal{V} = \mathcal{V}$. $\square$

Corollary 2.2.16. *If $\mathcal{V}_1$ and $\mathcal{V}_2$ are minimal $\mathfrak{A}^*$-invariant subspaces of $\mathcal{V}$ such that $\mathfrak{S}[\mathcal{V}_1] = \mathfrak{S}[\mathcal{V}_2]$, then $\mathcal{V}_1 = \mathcal{V}_2$.*

Proof. Let $\mathcal{H}' = \mathfrak{S}[\mathcal{V}_1] = \mathfrak{S}[\mathcal{V}_2]$. Define representations (B, σ_1) and (C, σ_2) of E on $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively by

$$B(\xi) = P_{\mathcal{V}_1} A(\xi)|_{\mathcal{V}_1} \text{ and } C(\xi) = P_{\mathcal{V}_2} A(\xi)|_{\mathcal{V}_2}$$

for all $\xi \in E$, and

$$\sigma_i(a) = \sigma(a)|_{\mathcal{V}_i}$$

for all $a \in \mathcal{A}$, $i = 1, 2$. The representations (B, σ_1) and (C, σ_2) share a unique minimal isometric dilation $(S(\cdot)|_{\mathcal{H}'}, \sigma(\cdot)|_{\mathcal{H}'})$. By Corollary 2.2.15, any $\mathfrak{S}^*$-invariant subspace of $\mathcal{H}'$ contains both $\mathcal{V}_1$ and $\mathcal{V}_2$. In particular $\mathcal{V}_1 \subseteq \mathcal{V}_2$ and $\mathcal{V}_2 \subseteq \mathcal{V}_1$. Hence $\mathcal{V}_1 = \mathcal{V}_2$. $\square$

Definition 2.2.17. Let E be a C^*-correspondence over a C^*-algebra $\mathcal{A}$. When (A, σ) is a representation of E on $\mathcal{H}$, we denote by Φ_A the completely positive map from $\sigma(\mathcal{A})'$ to $\sigma(\mathcal{A})'$ defined by

$$\Phi_A(X) = \tilde{A}(I \otimes X)\tilde{A}^*$$

for every X in $\sigma_A(\mathcal{A})'$.

Remark 2.2.18. For any $a \in \mathcal{A}$ and $X \in \sigma(\mathcal{A})'$ we have

$$\begin{aligned} \sigma(a)\Phi_A(X) &= \sigma(a)\tilde{A}(I \otimes X)\tilde{A}^* = \tilde{A}\sigma^E(a)(I \otimes X)\tilde{A}^* \\ &= \tilde{A}(I \otimes X)\sigma^E(a)\tilde{A}^* = \tilde{A}(I \otimes X)\tilde{A}^*\sigma(a). \end{aligned}$$

So Φ_A maps from $\sigma(\mathcal{A})'$ to $\sigma(\mathcal{A})'$ as claimed.

In [49] isometric representations that are not necessarily fully coisometric are studied. It is shown there that the corresponding Φ_A function for an isometric representation (A, σ) will be an endomorphism of $\sigma(\mathcal{A})'$. It is also shown that the fixed point set of Φ_A is the commutant of $\mathfrak{A}$, where $\mathfrak{A}$ is the algebra generated by the representation. In our setting, when (A, σ) is a finite dimensional, fully coisometric representation on a Hilbert space $\mathcal{V}$, we get that the commutant of $\mathfrak{A}$ is fixed by Φ_A (Lemma 2.2.19). Later, in Lemma 2.2.26, when we compress (A, σ) to a certain $\mathfrak{A}^*$-invariant subspace $\hat{\mathcal{V}} \subseteq \mathcal{V}$, we will get that the fixed point set of the corresponding $\Phi_{\hat{A}}$ map for the compressed representation $(\hat{A}, \hat{\sigma}) := (P_{\hat{\mathcal{V}}} A(\cdot)|_{\hat{\mathcal{V}}}, \sigma(\cdot)|_{\hat{\mathcal{V}}})$, is the commutant of the compression of $\mathfrak{A}$ to $\hat{\mathcal{V}}$.

The map Φ_A is a generalisation of the map Φ introduced in Section 4 of [17]. Indeed Lemma 2.2.20 and Lemma 2.2.21 are direct analogues of [17, Lemma 5.10] and [17, Lemma 5.11] respectively. We follow the same line of proof as in [17] when proving these results.

Lemma 2.2.19. *Let (A, σ) be a fully coisometric representation of a C^*-correspondence E over a C^*-algebra $\mathcal{A}$ on a finite dimensional Hilbert space $\mathcal{V}$. Let $\mathfrak{A}$ be the unital algebra generated by the representation (A, σ) and let Φ_A be the map from $\sigma(\mathcal{A})'$ to $\sigma(\mathcal{A})'$ defined in Definition 2.2.17.*

Then if X is in the commutant of $\mathfrak{A}$, X is a fixed point of Φ_A.

Proof. Suppose that $X \in \mathfrak{A}'$. Then for any $\xi \in E$ and $v \in \mathcal{V}$ we have

$$\begin{aligned} X\tilde{A}(\xi \otimes v) &= XA(\xi)v = A(\xi)Xv \\ &= \tilde{A}(\xi \otimes Xv) = \tilde{A}(I \otimes X)(\xi \otimes v). \end{aligned}$$

Hence $X\tilde{A} = \tilde{A}(I \otimes X)$. Multiplying on the right by $\tilde{A}^*$ gives $X = \Phi_A(X)$. □

Lemma 2.2.20. *Let (A, σ) be a fully coisometric representation of a C^*-correspondence E over a C^*-algebra $\mathcal{A}$ on a finite dimensional Hilbert space $\mathcal{V}$. Let $\mathfrak{A}$ be the unital algebra generated by the representation (A, σ) and let Φ_A be the map from $\sigma(\mathcal{A})'$ to $\sigma(\mathcal{A})'$ defined in Definition 2.2.17.*

Suppose there is an $X \in \sigma(\mathcal{A})'$ which is non-scalar and $\Phi_A(X) = X$. Then $\mathcal{V}$ has two pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces.

Proof. Since Φ_A is unital and self-adjoint there is a positive, non-scalar $X \in \sigma(\mathcal{A})'$ such that $\Phi_A(X) = X$. Assume $\|X\| = 1$. Note that, as $X \in \sigma(\mathcal{A})'$, the eigenspaces of X are invariant under $\sigma(\mathcal{A})$. Let μ be the smallest eigenvalue of X and let $\mathcal{M} = \ker(X - I)$ and $\mathcal{N} = \ker(X - \mu I)$. Take any non-zero $x \in \mathcal{M}$.

$$\begin{aligned} \|x\|^2 &= \langle \Phi_A(X)x, x\rangle = \langle (I \otimes X)\tilde{A}^*x, \tilde{A}^*x\rangle \\ &\leq \langle \tilde{A}^*x, \tilde{A}^*x\rangle = \|x\|^2. \end{aligned}$$

From this we must have $(I \otimes X)\tilde{A}^*x = \tilde{A}^*x$ and hence $\tilde{A}^*x \in E \otimes \mathcal{M}$.

Note that if x, y are eigenvectors for X for different eigenvalues then

$$\langle \xi \otimes x, \eta \otimes y\rangle = \langle x, \sigma(\langle \xi, \eta\rangle)y\rangle = 0,$$

for any $\xi, \eta \in E$. Hence if we take any non-zero $x \in \mathcal{M}$ and let y be any eigenvector of X orthogonal to $\mathcal{M}$ we get

$$\langle A(\xi)^*x, y\rangle = \langle \tilde{A}^*x, \xi \otimes y\rangle = 0$$

for any $\xi \in E$. Hence $\mathcal{M}$ is $\mathfrak{A}^*$-invariant. The same argument works for $\mathcal{N}$, as both $\mathcal{M}$ and $\mathcal{N}$ are eigenspaces for extremal values in the spectrum of X. As $\mathcal{M}$ and $\mathcal{N}$ are distinct eigenspaces for a self-adjoint operator, they are orthogonal. Since $\mathcal{V}$ is a finite dimensional, there exists a space $\{0\} \neq \mathcal{M}' \subseteq \mathcal{M}$ of minimal dimension which is $\mathfrak{A}^*$-invariant and a space $\{0\} \neq \mathcal{N}' \subseteq \mathcal{N}$ of minimal dimension which is $\mathfrak{A}^*$-invariant. □

Lemma 2.2.21. *Let (A, σ) be a fully coisometric representation of a C^*-correspondence E over a C^*-algebra $\mathcal{A}$ on a finite dimensional Hilbert space $\mathcal{V}$. Let $\mathfrak{A}$ be the unital algebra generated by the representation (A, σ) and let Φ_A be the map from $\sigma(\mathcal{A})'$ to $\sigma(\mathcal{A})'$ defined in Definition 2.2.17.*

Suppose $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ where both $\mathcal{V}_1$ and $\mathcal{V}_2$ are minimal $\mathfrak{A}^*$-invariant subspaces. Further suppose the representation (A, σ) decomposes into $(B, \sigma_1) \oplus (C, \sigma_2)$ with respect to $\mathcal{V}_1 \oplus \mathcal{V}_2$ with $\mathcal{B}(\mathcal{V}_1) = \mathrm{Alg}\{B(\xi), \sigma_1(a) : \xi \in E, a \in \mathcal{A}\}$ and $\mathcal{B}(\mathcal{V}_2) = \mathrm{Alg}\{C(\xi), \sigma_2(a) : \xi \in E, a \in \mathcal{A}\}$.

If there exists $X \in \sigma(\mathcal{A})'$ such that

1. *$\Phi_A(X) = X$ and*
2. *$X_{21} := P_{\mathcal{V}_2} X P_{\mathcal{V}_1} \neq 0$*

then there is a unitary W such that

$$C(\xi) = W^*B(\xi)W$$

and

$$\sigma_2(a) = W^*\sigma_1(a)W$$

for all $\xi \in E$ and $a \in \mathcal{A}$. Moreover the fixed point set of Φ_A consists of all matrices of the form $\begin{bmatrix} a_{11}I_{\mathcal{V}_1} & a_{12}W^* \\ a_{21}W & a_{22}I_{\mathcal{V}_2}\end{bmatrix}$.

Proof. We can assume that $X = X^*$ and $\|X_{21}\| = 1$ as Φ_A is self-adjoint. We denote by $\tilde{B}$ and $\tilde{C}$ the usual maps from $E \otimes \mathcal{V}_1$ and $E \otimes \mathcal{V}_2$ respectively. Let $\mathcal{M} = \{x \in \mathcal{V}_1 : \|X_{21}v\| = \|v\|\}$. As $\mathcal{V}$ is finite dimensional, $\mathcal{M}$ is non-empty. Note that for any $v \in \mathcal{M}$ we have $X_{21}^* X_{21} v = v$. It follows that $\mathcal{M}$ is a subspace of $\mathcal{V}_1$. Thus if $v \in \mathcal{M}$ and $a \in \mathcal{A}$ we have

$$\|X_{21}\sigma(a)v\|^2 = \langle X_{21}\sigma(a)v, X_{21}\sigma(a)v\rangle = \langle X_{21}^* X_{21} v, \sigma(a^*a)v\rangle = \|\sigma(a)v\|^2.$$

So $\mathcal{M}$ reduces $\sigma(\mathcal{A})$. This tells us that $E \otimes \mathcal{M}$ and $E \otimes (\mathcal{V}_1 \ominus \mathcal{M})$ are orthogonal spaces.

Now take any v in $\mathcal{M}$. Since $\Phi_A(X) = X$, we have that

$$X_{21}v = \tilde{C}(I \otimes X_{21})\tilde{B}^* v.$$

This implies that $\|(I \otimes X_{21})\tilde{B}^* v\| = \|\tilde{B}^* v\| = \|v\|$. Thus $\tilde{B}^* v \in E \otimes \mathcal{M}$ for all $v \in \mathcal{M}$. Take any $\xi \in E$, $v \in \mathcal{M}$ and $w \in \mathcal{V}_1 \ominus \mathcal{M}$.

$$\langle B(\xi)^* v, w\rangle = \langle \tilde{B}^* v, \xi \otimes w\rangle = 0.$$

Thus $B(\xi)^* v \in \mathcal{M}$. We conclude that $\mathcal{M}$ is $\mathfrak{A}^*$-invariant. Hence, by the minimality of $\mathcal{V}_1$, $\mathcal{M}$ is all of $\mathcal{V}_1$. Therefore X_{21} is an isometry from $\mathcal{V}_1$ to $\mathcal{V}_2$. Let $W = X_{21} \in \mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$. For $v \in \mathcal{V}_1$

$$\|v\| = \|Wv\| = \|\tilde{C}(I \otimes W)\tilde{B}^* v\| \le \|(I \otimes W)\tilde{B}^* v\| \le \|v\|.$$

Hence $\tilde{C}$ is an isometry from $\mathrm{Ran}(I \otimes W)\tilde{B}^*$ to the $\mathrm{Ran} W = \mathcal{V}_2$. $\tilde{C}$ is a contraction and so must be zero on the orthogonal complement of $\mathrm{Ran}(I \otimes W)\tilde{B}^*$. It follows that $\tilde{C}^*$ is an isometry from $\mathcal{V}_2$ to $\mathrm{Ran}(I \otimes W)\tilde{B}^*$. Hence $\tilde{C}^* W = (I \otimes W)\tilde{B}^*$. From this it follows that $C(\xi)^* = WB(\xi)^* W^*$ for all $\xi \in E$. Since W is also in the commutant of $\sigma(\mathcal{A})$ it is the desired unitary.

Suppose $Y \in \mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$ and $\left[\begin{smallmatrix} 0 & 0 \\ Y & 0 \end{smallmatrix}\right]$ is fixed by Φ_A, then

$$Y = \tilde{C}(I \otimes Y)\tilde{B}^* = W\tilde{B}(I \otimes W^*)(I \otimes Y)\tilde{B}^* = W\tilde{B}(I \otimes W^* Y)\tilde{B}^*.$$

It follows from Lemma 2.2.20 that W^*Y is a scalar and so Y is a scalar multiple of W. A similar argument works for the other coordinates. $\square$

By Proposition 2.2.13, if $\mathcal{V}'$ is an $\mathfrak{A}^*$-invariant subspace of $\mathcal{V}$ such that $\mathfrak{A}[\mathcal{V}'] = \mathcal{V}$ (i.e. $\mathcal{V}'$ is cyclic for $\mathfrak{A}$) then $\mathfrak{S}[\mathcal{V}'] = \mathcal{H}$. Hence the minimal isometric dilation of the completely contractive representation $(P_{\mathcal{V}'}A(\cdot)|_{\mathcal{V}'}, \sigma(\cdot)|_{\mathcal{V}'})$ is (S, ρ).

Definition 2.2.22. Suppose $\mathfrak{A}$ is an algebra acting on a Hilbert space $\mathcal{V}$, and that $\mathcal{V}'$ is an $\mathfrak{A}^*$-invariant subspace of $\mathcal{V}$ which is cyclic for $\mathfrak{A}$. If $\mathcal{V}'$ has no proper $\mathfrak{A}^*$-invariant subspaces which are cyclic for $\mathfrak{A}$ then we say that $\mathcal{V}'$ is a *minimal cyclic coinvariant subspace (for* $\mathfrak{A}$*) of* $\mathcal{V}$.

When (A, σ) is representation of a C^*-correspondence on a Hilbert space $\mathcal{V}$ and $\mathfrak{A}$ is the unital WOT-closed algebra generated by (A, σ), we call a minimal cyclic coinvariant subspace for $\mathfrak{A}$ a *minimal cyclic coinvariant subspace for* (A, σ).

The following proof is due to Ken Davidson.

Lemma 2.2.23. *Let $\mathcal{V}$ be a finite dimensional Hilbert space. Suppose $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{V})$ is an algebra and that $\mathcal{V}$ is a minimal cyclic coinvariant space for $\mathfrak{A}$. Then $\mathfrak{A}$ is a C^*-algebra.*

Proof. Suppose $\mathcal{L}$ is an $\mathfrak{A}^*$-invariant subspace such that $\mathcal{V} \ominus \mathcal{L}$ is not $\mathfrak{A}^*$-invariant. Let $\mathcal{M} = \mathfrak{A}[\mathcal{L}]$. Then $\mathcal{L} \subsetneq \mathcal{M}$ and $\mathcal{M} \subsetneq \mathcal{V}$. So $\mathcal{V} \ominus \mathcal{M}$ is a non-zero $\mathfrak{A}^*$-invariant subspace such that $\mathcal{V} \ominus \mathcal{M} \subsetneq \mathcal{V} \ominus \mathcal{L}$. We have that $\mathcal{L} \oplus (\mathcal{V} \ominus \mathcal{M})$ is an $\mathfrak{A}^*$-invariant subspace and $\mathfrak{A}[\mathcal{L} \oplus (\mathcal{V} \ominus \mathcal{M})] = \mathcal{V}$. Hence, by our assumption that $\mathcal{V}$ is a minimal cyclic coinvariant space, $\mathcal{V} = \mathcal{L} \oplus (\mathcal{V} \ominus \mathcal{M})$. This is a contradiction. Hence if $\mathcal{L}$ is an $\mathfrak{A}^*$-invariant subspace then $\mathcal{V} \ominus \mathcal{L}$ must also be $\mathfrak{A}^*$-invariant. Since $\mathcal{V}$ is finite dimensional, it follows that $\mathfrak{A}$ is a C^*-algebra. $\square$

Lemma 2.2.24. *Let (A, σ) be a fully coisometric representation of a C^*-correspondence E on a finite dimensional Hilbert space $\mathcal{V}$, and let (S, ρ) be the unique minimal isometric dilation of (A, σ) on a Hilbert space $\mathcal{H}$. Let $\mathfrak{A}$ be the unital algebra generated by the representation (A, σ) and let $\mathfrak{S}$ be the unital* WOT*-closed algebra generated by (S, ρ).*

If $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k$ is a maximal set of pairwise orthogonal minimal $\mathfrak{A}^*$-invariant spaces of $\mathcal{V}$ then $\hat{\mathcal{V}} = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k$ is the unique minimal cyclic coinvariant subspace of $\mathcal{V}$.

Proof. Firstly, $\mathfrak{S}[\hat{\mathcal{V}}]$ is a $\mathfrak{S}$-reducing subspace by Lemma 2.2.6. If $\mathfrak{S}[\hat{\mathcal{V}}]$ is not all of $\mathcal{H}$ then its orthogonal complement in $\mathcal{H}$, $\mathcal{M}$, is also a $\mathfrak{S}$-reducing space. By Lemma 2.2.12 $\mathcal{M} \cap \mathcal{V}$ is a non-zero $\mathfrak{A}^*$-invariant space orthogonal to each $\mathcal{V}_j$ for $1 \leq j \leq k$. This contradicts the maximality of our choice of $\mathcal{V}_1, \ldots, \mathcal{V}_k$. Hence, by Proposition 2.2.13, $\hat{\mathcal{V}}$ is $\mathfrak{A}$-cyclic. Since each $\mathcal{V}_j$ is a minimal $\mathfrak{A}^*$-invariant space and since the $\mathfrak{S}$-reducing spaces $\mathfrak{S}[\mathcal{V}_j]$ are orthogonal by Lemma 2.2.6 it follows that $\hat{\mathcal{V}}$ is indeed a minimal cyclic coinvariant subspace of $\mathcal{V}$.

Now suppose that $\mathcal{W}$ is an $\mathfrak{A}^*$-invariant subspace of $\mathcal{V}$ such that $\mathfrak{S}[\mathcal{W}] = \mathcal{H}$, i.e. $\mathfrak{A}[\mathcal{W}] = \mathcal{V}$. Let $\mathcal{H}_j = \mathfrak{S}[\mathcal{V}_j]$ for each j. We have that $\mathcal{H}_j \subseteq \mathfrak{S}[\mathcal{W}]$ for each j and hence $\mathcal{H}_j \cap \mathcal{W}$ is non-zero. But each $\mathcal{H}_j$ is irreducible by Corollary 2.2.15 and hence $\mathcal{V}_j$ is the unique minimal $\mathfrak{S}^*$-invariant subspace of $\mathcal{H}_j$ by Corollary 2.2.16. It follows that $\mathcal{V}_j$ is contained in $\mathcal{H}_j \cap \mathcal{W}$ for each j. Therefore $\hat{\mathcal{V}} \subseteq \mathcal{W}$. $\square$

Remark 2.2.25. In [17], Lemma 2.2.23 is proved for the case when $\mathfrak{A}$ is the unital algebra generated by a finite dimensional, fully coisometric representation (A, σ) of the C^*-correspondence $\mathbb{C}^n$ over $\mathbb{C}$ ([17, Part of Theorem 5.13]). The proof uses analysis of Φ_A and the fact that $\hat{\mathcal{V}}$ is a direct sum of minimal $\mathfrak{A}^*$-invariant subspaces. We note that the proof presented here shows that the result is in fact just a general result about cyclic, coinvariant subspaces in finite-dimensions, independent of any deeper analysis.

However, that the minimal cyclic coinvariant space is unique is not a general result in finite dimensional linear algebra. For example, the algebra

$$\mathfrak{C} = \left\{ \begin{bmatrix} \lambda & 0 \\ \gamma - \lambda & \gamma \end{bmatrix} : \lambda, \gamma \in \mathbb{C} \right\}$$

in $\mathcal{B}(\mathbb{C}^2)$ has both $\{(x,0) : x \in \mathbb{C}\}$ and $\{(x,x) : x \in \mathbb{C}\}$ as minimal cyclic coinvariant spaces.

While it is shown that the minimal cyclic coinvariant space $\hat{\mathcal{V}}$ in Lemma 2.2.24 is unique, the decomposition of $\hat{\mathcal{V}}$ into a direct sum of minimal coinvariant subspaces is not necessarily unique. For example, suppose $\mathfrak{A}^*$ has two 1-dimensional invariant, orthogonal subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ and that the representation $(P_{\mathcal{V}_1} A(\cdot)|_{\mathcal{V}}, \sigma(\cdot)|_{\mathcal{V}_1})$ is unitarily equivalent to $(P_{\mathcal{V}_2} A(\cdot)|_{\mathcal{V}}, \sigma(\cdot)|_{\mathcal{V}_2})$. Let U be the unitary defining the equivalence. Take a unit vector $v_1 \in \mathcal{V}_1$ and let $v_2 = Uv_1$. Then $\mathcal{V}_1' = \operatorname{span}\{v_1 + v_2\}$ and $\mathcal{V}_2' = \operatorname{span}\{v_1 - v_2\}$ are orthogonal, $\mathfrak{A}^*$-invariant subspaces and

$$\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{V}_1' \oplus \mathcal{V}_2'.$$

We follow the argument given in [17, Theorem 5.13] for the following result. This serves as a converse to Lemma 2.2.19.

Lemma 2.2.26. *Let (A,σ) be a fully coisometric representation of a C^*-correspondence E over a C^*-algebra $\mathcal{A}$. Let $\mathfrak{A}$ be the unital algebra generated by the representation (A,σ) and let Φ_A be the map from $\sigma(\mathcal{A})'$ to $\sigma(\mathcal{A})'$ defined in Definition 2.2.17.*

Suppose $\mathcal{V} = \hat{\mathcal{V}}$, where $\hat{\mathcal{V}} = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k$ is as in Lemma 2.2.24*. Then the fixed point set of Φ_A is equal to the commutant of $\mathfrak{A}$.*

Proof. We have already shown in Lemma 2.2.19 that if $X \in \mathfrak{A}'$ then $\Phi_A(X) = X$. Take $X \in \sigma(\mathcal{A})'$ such that $\Phi_A(X) = X$. Suppose that X is non-scalar. If there is no unitary between $\mathcal{V}_k$ and $\mathcal{V}_l$ intertwining $\mathfrak{A}$ then, by Lemma 2.2.21, $P_{\mathcal{V}_k} X P_{\mathcal{V}_l} = 0$. On the other hand, if $W_{k,l}$ is an intertwining unitary from $\mathcal{V}_k$ to $\mathcal{V}_l$ then Lemma 2.2.21 tells us that $P_{\mathcal{V}_k} X P_{\mathcal{V}_l} = x_{kl} W_{k,l}$ for some scalar x_{kl}, and hence $P_{\mathcal{V}_k} X P_{\mathcal{V}_l}$ is in $\mathfrak{A}'$. It follows that $X \in \mathfrak{A}'$. □

The following theorem summarises our main results.

Theorem 2.2.27. *Suppose E is a C^*-correspondence over a C^*-algebra $\mathcal{A}$. Let (A,σ) be a fully coisometric, finite dimensional representation of E on a Hilbert space $\mathcal{V}$, and let (S,ρ) be the minimal isometric dilation of (A,σ) on $\mathcal{H}$. Let $\mathfrak{A}$ be the unital algebra generated by (A,σ) and $\mathfrak{S}$ be the unital* WOT*-closed algebra generated by (S,ρ).*

If

$$\hat{\mathcal{V}} = \sum_{j=1}^{n}{}^{\oplus} \mathcal{V}_j$$

is a maximal direct sum of minimal, orthogonal $\mathfrak{A}^*$-invariant subspaces of $\mathcal{V}$, then $\hat{\mathcal{V}}$ is the unique *minimal $\mathfrak{A}^*$-invariant subspace such that $\mathfrak{S}[\hat{\mathcal{V}}] = \mathcal{H}$. Further*

$$\mathcal{H} = \sum_{j=1}^{n}{}^{\oplus} \mathcal{H}_j$$

where $\mathcal{H}_j = \mathfrak{S}[\mathcal{V}_j]$.

The representation $(P_{\hat{\mathcal{V}}^\perp} S(\cdot)|_{\hat{\mathcal{V}}^\perp}, \rho(\cdot)|_{\hat{\mathcal{V}}^\perp})$ is an induced representation and $\mathfrak{S}^*|_{\hat{\mathcal{V}}}$ is a C^*-algebra.

We now show that the compression to the minimal cyclic coinvariant space for a finitely correlated, fully coisometric representation is a complete unitary invariant.

Corollary 2.2.28. *Suppose (S, σ) and (T, τ) are finitely correlated, isometric, fully coisometric representations of a C^*-correspondence E on $\mathcal{H}_S$ and $\mathcal{H}_T$ respectively. Let $\mathcal{V}_S$ be the unique minimal cyclic coinvariant subspace for (S, σ) and let $\mathcal{V}_T$ be the unique minimal cyclic subspace for (T, τ).*

Then (S, σ) and (T, τ) are unitarily equivalent if and only if the finite dimensional representations $(P_{\mathcal{V}_S} S(\cdot)|_{\mathcal{V}_S}, \sigma(\cdot)|_{\mathcal{V}_S})$ and $(P_{\mathcal{V}_T} T(\cdot)|_{\mathcal{V}_T}, \tau(\cdot)|_{\mathcal{V}_T})$ are unitarily equivalent.

Proof. Suppose (S, σ) and (T, τ) are unitarily equivalent. Let U be the unitary from $\mathcal{H}_S$ to $\mathcal{H}_T$ intertwining (S, σ) and (T, τ). It follows that $U\mathcal{V}_S$ is invariant under $T(\cdot)^*$ and is cyclic, hence $\mathcal{V}_T \subseteq U\mathcal{V}_S$. Similarly $\mathcal{V}_S \subseteq U^*\mathcal{V}_T$. It follows that $U\mathcal{V}_S = \mathcal{V}_T$ and $(P_{\mathcal{V}_S} S(\cdot)|_{\mathcal{V}_S}, \sigma(\cdot)|_{\mathcal{V}_S})$ and $(P_{\mathcal{V}_T} T(\cdot)|_{\mathcal{V}_T}, \tau(\cdot)|_{\mathcal{V}_T})$ are unitarily equivalent.

Conversely, suppose that $(P_{\mathcal{V}_S} S(\cdot)|_{\mathcal{V}_S}, \sigma(\cdot)|_{\mathcal{V}_S})$ and $(P_{\mathcal{V}_T} T(\cdot)|_{\mathcal{V}_T}, \tau(\cdot)|_{\mathcal{V}_T})$ are unitarily equivalent. Then, by the uniqueness of the minimal isometric dilation, (S, σ) and (T, τ) are unitarily equivalent. □

2.3 Product Systems of C^*-correspondences over $\mathbb{N}^k$

We will now extend our results to product systems of C^*-correspondences. This is the analogue of multivariate operator theory, and so relies on a more sophisticated dilation theory. The key to our analysis will be a trick to reduce to the consideration of a certain C^*-correspondence contained inside our product system (Theorem 2.3.12).

2.3.1 Preliminaries and Notation

The following description of product systems of C^*-correspondences over $\mathbb{N}^k$ follows that of [26] and [69]. Let $\mathcal{A}$ be a unital C^*-algebra. A semigroup E is a *product system of C^*-correspondences* *over* $\mathbb{N}^k$ if there is a semigroup homomorphism $p : E \to \mathbb{N}^k$ such that $E(\mathbf{n}) := p^{-1}(\mathbf{n})$ is a C^*-correspondence over $\mathcal{A}$ and the map $(\xi, \eta) \in E(\mathbf{n}) \times E(\mathbf{m}) \to \xi\eta \in E(\mathbf{n}+\mathbf{m})$ extends to an isomorphism $t_{\mathbf{n},\mathbf{m}}$ from $E(\mathbf{n}) \otimes E(\mathbf{m})$ onto $E(\mathbf{n}+\mathbf{m})$. By $E(\mathbf{0})$ we mean the C^*-algebra $\mathcal{A}$. Letting $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k$ be the standard generating set of $\mathbb{N}^k$, we write E_i for the C^*-correspondence $p^{-1}(\mathbf{e}_i)$. We identify $E(\mathbf{n})$ with $E_1^{n_1} \otimes \ldots \otimes E_k^{n_k}$ when $\mathbf{n} = (n_1, \ldots, n_k)$. It follows that $t_{i,j} := t_{\mathbf{e}_i,\mathbf{e}_j}$ is an isomorphism from $E_i \otimes E_j$ to $E_j \otimes E_i$, for $i \le j$ and $t_{j,i} = t_{i,j}^{-1}$ for $i \le j$. We write $t_{i,i}$ for the identity on E_i^2. We will often suppress the isomorphism and write $E(\mathbf{n}) \otimes E(\mathbf{m}) = E(\mathbf{n}+\mathbf{m})$.

If, for each i, $(A^{(i)}, \sigma)$ is a representation of E_i on a Hilbert space $\mathcal{H}$ and we have the following commutation relation

$$\tilde{A}^{(i)}(I_{E_i} \otimes \tilde{A}^{(j)}) = \tilde{A}^{(j)}(I_{E_j} \otimes \tilde{A}^{(i)})(t_{i,j} \otimes I_{\mathcal{H}})$$

then $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is a (*completely contractive covariant*) *representation of* E *on* $\mathcal{H}$. A representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is said to be *isometric* (resp. *fully coisometric*) if each representation $(A^{(i)}, \sigma)$ is isometric (resp. fully coisometric).

For $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ we define a map $\tilde{A}_{\mathbf{n}}$ from $E(\mathbf{n}) \otimes \mathcal{H}$ to $\mathcal{H}$ by

$$\tilde{A}_{\mathbf{n}} = \tilde{A}^{(1)}_{n_1}(I_{E_1^{n_1}} \otimes \tilde{A}^{(2)}_{n_2}) \ldots (I_{E_1^{n_1}} \otimes \ldots \otimes I_{E_{k-1}^{n_{k-1}}} \otimes \tilde{A}^{(k)}_{n_k})$$

We define a representation $(A_{\mathbf{n}}, \sigma)$ of the C^*-correspondence $E(\mathbf{n})$ by letting

$$A_{\mathbf{n}}(\xi)h = \tilde{A}_{\mathbf{n}}(\xi \otimes h)$$

for each $\xi \in E(\mathbf{n})$ and $h \in \mathcal{H}$.

A representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ of E is said to be *doubly commuting* if it satisfies

$$\tilde{A}^{(j)*}\tilde{A}^{(i)} = (I_{E_j} \otimes \tilde{A}^{(i)})(t_{i,j} \otimes I_{\mathcal{H}})(I_{E_i} \otimes \tilde{A}^{(j)*}).$$

It has been shown in [26] and [69] that the doubly commuting condition is equivalent to what is known as *Nica covariance* [51]. It is easy to check that an isometric, fully coisometric representation is doubly commuting.

Note that if $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is an isometric representation, then for $\mathbf{n} = (n_1, \ldots, n_k)$, $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k$ we have

$$\tilde{A}^*_{\mathbf{m}}\tilde{A}_{\mathbf{n}} = I_{E(\mathbf{n}-(\mathbf{n}-\mathbf{m})_+)} \otimes \tilde{A}^*_{(\mathbf{n}-\mathbf{m})_-}\tilde{A}_{(\mathbf{n}-\mathbf{m})_+}$$

where $(\mathbf{n}-\mathbf{m})_+$ is equal to $n_i - m_i$ in the i^{th} coordinate if $n_i \ge m_i$ and zero in the i^{th} coordinate otherwise, and $(\mathbf{n}-\mathbf{m})_- \in \mathbb{N}^k$ satisfies $\mathbf{n}-\mathbf{m} = (\mathbf{n}-\mathbf{m})_+ - (\mathbf{n}-\mathbf{m})_-$.

We define the Fock space $\mathcal{F}(E)$ of a product space of C^*-correspondences by

$$\mathcal{F}(E) = \sum_{\mathbf{n}\in\mathbb{N}^k}^{\oplus} E(\mathbf{n}).$$

For more details on the construction see [26]. For each $\mathbf{n}$ and $\xi \in E(\mathbf{n})$ define the *creation operator* $T_\xi : \mathcal{F}(E) \to \mathcal{F}(E)$ by

$$T_\xi(\eta) = \xi \otimes \eta$$

for each $\eta \in \mathcal{F}(E)$. The C^*-algebra in $\mathcal{L}(\mathcal{F}(E))$ generated by the creation operators is called the *Toeplitz algebra* associated to E and denoted $\mathcal{T}(E)$. A product system $(E, \mathcal{A})$ is said to have the *normal ordering property* if

$$\mathcal{T}(E) = \overline{\operatorname{span}}\{L(\xi)L(\eta)^* : \xi, \eta \in \cup_{\mathbf{n}\in\mathbb{N}^k} E(\mathbf{n})\}.$$

Let $(S^{(1)}, \ldots, S^{(k)}, \rho)$ be a representation of a product system $(E, \mathcal{A})$ on a Hilbert space $\mathcal{H}$. We denote by I be the identity in $\mathcal{B}(H)$. We call the weak-operator topology closed algebra

$$\mathfrak{S} = \operatorname{Alg}\{I,\ S^{(i)}(\xi_i),\ \rho(a) : a \in \mathcal{A},\ \xi_i \in E_i \text{ for } 1 \leq \overline{i \leq k\}}^{\textsc{wot}}$$

the *unital* WOT*-closed algebra generated by the representation* $(S^{(1)}, \ldots, S^{(k)}, \rho)$.

2.3.2 Minimal Isometric Dilations

Definition 2.3.1. Let $(E, \mathcal{A})$ be a product system over $\mathbb{N}^k$ and let $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ be a representation of E on $\mathcal{V}$. A representation $(S^{(1)}, \ldots, S^{(k)}, \rho)$ on a Hilbert space $\mathcal{H}$ is a *dilation* of $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ if $\mathcal{H}$ contains $\mathcal{V}$ and, for each i, $(S^{(i)}, \rho)$ dilates $(A^{(i)}, \sigma)$. A dilation $(S^{(1)}, \ldots, S^{(k)}, \rho)$ of $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is an *isometric* dilation if $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is an isometric representation. A dilation $(S^{(1)}, \ldots, S^{(k)}, \rho)$ of $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ on $\mathcal{H}$ is *minimal* if $\mathcal{H}$ is the smallest reducing subspace for $\operatorname{Alg}\{\ S^{(i)}(\xi_i),\ \rho(a) : a \in \mathcal{A},\ \xi_i \in E_i$ for $1 \leq i \leq k\}$ which contains $\mathcal{V}$.

Given an arbitrary representation of a product system $(E, \mathcal{A})$ over $\mathbb{N}^k$ it is not always possible to find an isometric dilation. Indeed, if $k \geq 3$ and $\mathcal{A} = E = \mathbb{C}$, then a representation of E is simply k commuting contractions $A_1, \ldots, A_k$. It is known that there are examples of commuting contractions which can not be dilated to commuting isometries, see e.g. [52]. With that said, there are a number of dilation theorems for product systems of C^*-correspondences. We will now review a number of these dilations results that will be useful. The subsequent remarks may help clarify some of the distinctions.

Theorem 2.3.2 (Solel [68])**.** *Let $(E, \mathcal{A})$ be a product system of C^*-correspondences over $\mathbb{N}^2$. Then any representation of E has an isometric dilation.*

Definition 2.3.3. Let $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ be a representation of a product system $(E, \mathcal{A})$ on $\mathcal{H}$. For each $\mathbf{n} \in \mathbb{Z}^k$ we define $A(\mathbf{n})$ to be

$$A(\mathbf{n}) = \tilde{A}^*_{\mathbf{n}_-} \tilde{A}_{\mathbf{n}_+}.$$

Let $(S^{(1)}, \ldots, S^{(k)}, \rho)$ be an isometric dilation of $(A^{(1)}, \ldots, A^{(k)}, \sigma)$. If for each $\mathbf{n} \in \mathbb{Z}^k$, the representation $(S^{(1)}, \ldots, S^{(k)}, \rho)$ satisfies

$$(I_{E(\mathbf{n}_+)} \otimes P_{\mathcal{H}}) S(\mathbf{n})|_{E(\mathbf{n}_+) \otimes \mathcal{H}} = A(\mathbf{n})$$

then $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is a *regular isometric dilation* of $(A^{(1)}, \ldots, A^{(k)}, \sigma)$.

Theorem 2.3.4 (Solel [69]). *Let $(E, \mathcal{A})$ be a product system of C^*-correspondences over $\mathbb{N}^k$ and let $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ be a representation of E. If $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ satisfies the additional condition that, for every $v \subseteq \{1, \ldots, k\}$*

$$\sum_{u \subseteq v} (-1)^{|u|} (I_{\mathbf{e}(v)-\mathbf{e}(u)} \otimes \tilde{A}^*_{\mathbf{e}(u)} \tilde{A}_{\mathbf{e}(u)}) \geq 0, \tag{2.1}$$

where $\mathbf{e}(u) \in \mathbb{N}^k$ is 1 in the i^{th} coordinate if $i \in u$ and zero in the i^{th} coordinate otherwise, then it has a unique minimal regular isometric dilation.

Theorem 2.3.5 (Solel [69]). *Let $(E, \mathcal{A})$ be a product system of C^*-correspondences over $\mathbb{N}^k$ and let $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ be a doubly commuting representation of E. Then the representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ will satisfy (2.1). Further, the minimal regular isometric dilation of $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ will be doubly commuting.*

Theorem 2.3.6 (Shalit [61]). *Let $(E, \mathcal{A})$ be a product system of C^*-correspondences over $\mathbb{N}^k$ and let $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ be a fully coisometric representation of E. Then the representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ has a minimal isometric dilation which is fully coisometric.*

Definition 2.3.7. Let $(E, \mathcal{A})$ be product system of C^*-correspondences over $\mathbb{N}^k$. For a representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ of E on a Hilbert space $\mathcal{H}$ define the *defect operator* for $s \in (0, 1)$

$$\Delta_s = \sum_{\substack{\mathbf{n} \in \mathbb{N}^k \\ \mathbf{n} \leq (1,1,\ldots,1)}} (-s^2)^{(|\mathbf{n}|)} \tilde{A}(\mathbf{n}) \tilde{A}(\mathbf{n})^*,$$

where $|\mathbf{n}| = n_1 + \ldots + n_k$ when $\mathbf{n} = (n_1, n_2, \ldots, n_k)$.

The representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is said to satisfy *the Popescu condition* if there is a $t \in (0, 1)$ such that Δ_s is positive for all $s \in (t, 1)$.

Theorem 2.3.8 (Skalski [64]). *Let $(E, \mathcal{A})$ be a product system of C^*-correspondences over $\mathbb{N}^k$ having the normal ordering property. Let $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ be a representation of E. If $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ satisfies the Popescu condition then it has an isometric dilation.*

Remark 2.3.9 (Remarks on Theorems 2.3.2, 2.3.4, 2.3.6 and 2.3.8). The dilation given in Theorem 2.3.2 is not necessarily unique. Examples of representations which do not dilate uniquely are given by Davidson, Power and Yang in [20]. They also provide an alternative proof of Theorem 2.3.2 for the case that $\mathcal{A} = \mathbb{C}$ and $E_i = \mathbb{C}^{n_i}$ for $i = 1, 2$. Further it is proved that in this setting a minimal isometric dilation of a fully coisometric representation is fully coisometric and unique.

A fully coisometric representation does not necessarily satisfy (2.1). For example if $T_1 = T_2 = S^*$, where S is a unilateral shift on a separable Hilbert space, then the commuting coisometries T_1 and T_2 do not satisfy (2.1). The atomic representations of single vertex k-graphs studied in [19, 22] do satisfy (2.1) since they are doubly commuting. For another example of a non-doubly commuting, fully coisometric representation see Example 2.4.18.

An alternative proof of Theorem 2.3.4 was given by Shalit in [62]. The method of proof in [62] and [61] is to construct a semigroup of commuting contractions from a contractive representation. The result is then deduced from dilation results for semigroups of commuting contractions.

Skalski and Zacharias [65] show that if $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is a doubly commuting representation of E then its minimal isometric dilation is fully coisometric if and only if $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is fully coisometric. We will show in Lemma 2.3.10 that a minimal, isometric dilation of a representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is fully coisometric if and only if $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is fully coisometric, without the assumption that $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is doubly commuting.

It is noted in [64] that if a representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is doubly commuting or coisometric then it will satisfy the Popescu condition. Theorem 2.3.8 is a more general version of a dilation theorem for k-graphs proved by Skalski and Zacharias in [66]. We will look more closely at k-graphs in section 2.4.

The following result is just a higher-rank version of Lemma 2.2.3 and follows much the same argument.

Lemma 2.3.10. *Let $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ be a representation of a product system E on a Hilbert space $\mathcal{V}$ with a minimal isometric dilation $(S^{(1)}, \ldots, S^{(k)}, \rho)$ on a Hilbert space $\mathcal{H}$. Then dilation $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is fully coisometric if and only if $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is fully coisometric.*

Proof. That $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is fully coisometric when $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is follows the same argument as in Lemma 2.2.3.

Conversely, assume that $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is fully coisometric. We will show that $\tilde{S} := \tilde{S}_1$ is a coisometry. That $\tilde{S}_i$ is a coisometry, for $2 \le i \le k$, follows similarly. Note that $\tilde{S}$ is an isometry and so $\tilde{S}\tilde{S}^*$ is a projection on $\mathcal{H}$. Let $\mathcal{M} = (I - \tilde{S}\tilde{S}^*)\mathcal{H}$. Take any $x \in \mathcal{M}$ and $y \in \mathcal{H}$. We have

$$\langle S(\xi_1)^* x, S(\xi_2) y\rangle = \langle x, \tilde{S}(\xi_1 \otimes S(\xi_2) y)\rangle = 0$$

for all $\xi_1, \xi_2 \in E_1$. For $2 \leq i \leq k$ we have

$$\begin{aligned}
\langle S^{(i)}(\eta)^* x, S(\xi)y\rangle &= \langle x, S^{(i)}(\eta)S(\xi)y\rangle \\
&= \langle x, \tilde{S}^{(i)}(I_{E_i} \otimes \tilde{S})(\eta \otimes \xi \otimes y)\rangle \\
&= \langle x, \tilde{S}(I_E \otimes \tilde{S}^{(i)}) \circ (t \otimes I_{\mathcal{H}})(\eta \otimes \xi \otimes y)\rangle \\
&= 0
\end{aligned}$$

for all $\xi \in E$ and $\eta \in E_i$ (where $t = t_{1,i}$). It follows that $\mathcal{M}$ is $\mathfrak{S}^*$-invariant, where $\mathfrak{S}$ is the unital WOT-closed algebra generated by $(S^{(1)}, \ldots, S^{(k)}, \rho)$. The rest of the proof follows the same argument as Lemma 2.2.3. □

Lemma 2.3.11. *Let $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ be a fully coisometric representation of a product system E. Then the minimal isometric dilation of $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is unique up to unitary equivalence.*

Proof. Since $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is fully coisometric it can be dilated by Theorem 2.3.6. It follows from [64, Theorem 2.7] that all doubly commuting, minimal, isometric dilations of a representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ are unitarily equivalent. By Lemma 2.3.10, if $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is a fully coisometric representation then all minimal, isometric dilations are also fully coisometric, and hence they are doubly commuting. It follows that the minimal isometric dilation is unique up to unitary equivalence. □

We now prove a key technical tool. We show that taking the minimal isometric dilation of a fully coisometric representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ gives rise to the minimal isometric representation of the representation $(A_{\mathbf{n}}, \sigma)$ when $\mathbf{n} \geq (1, \ldots, 1)$. This allows us, in Lemma 2.3.18, to prove the analogous result of Lemma 2.2.12 for product systems. In fact, Theorem 2.3.12 allows us to deduce Lemma 2.3.18 from Lemma 2.2.12. Lemma 2.3.18 will play an important role in our analysis, just as Lemma 2.2.12 did in the study of the C^*-correspondence case.

Theorem 2.3.12. *Let $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ be a fully coisometric representation of a product system of C^*-correspondences E on a Hilbert space $\mathcal{V}$ with minimal isometric dilation $(S^{(1)}, \ldots, S^{(k)}, \rho)$ on a Hilbert space $\mathcal{H}$. If $\mathbf{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$ satisfies $n_i \neq 0$ for $1 \leq i \leq k$ then the C^*-correspondence representation $(S_{\mathbf{n}}, \rho)$ of $E(\mathbf{n})$ is the (unique) minimal isometric dilation of $(A_{\mathbf{n}}, \sigma)$.*

Proof. It is clear that $(S_{\mathbf{n}}, \rho)$ is an isometric dilation of $(A_{\mathbf{n}}, \sigma)$ for any $\mathbf{n} \in \mathbb{N}^k$. It remains to show that the dilation is minimal when $n_i \neq 0$ for each i.

For any $\mathbf{n} \in \mathbb{N}^k$ we define $\mathcal{H}_{\mathbf{n}}$ to be the space mapped out by $(S_{\mathbf{n}}, \sigma)$, i.e.

$$\mathcal{H}_{\mathbf{n}} = \mathcal{V} + \bigvee_{\substack{m \in \mathbb{Z} \\ m \geq 0}} \tilde{S}_{\mathbf{n}m}(E(\mathbf{n})^m \otimes \mathcal{V}).$$

Claim (1)**.** If $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$ and $\mathbf{m} \leq \mathbf{n}$, then $\mathcal{H}_{\mathbf{m}} \subseteq \mathcal{H}_{\mathbf{n}}$.

Let $\mathbf{p} = \mathbf{n} - \mathbf{m}$. Take any $v \in \mathcal{V}$ and $\xi \in E(\mathbf{m})$ then

$$\begin{aligned}\tilde{S}_{\mathbf{m}}(\xi \otimes v) &= \tilde{S}_{\mathbf{m}}(I_{E(\mathbf{m})} \otimes \tilde{S}_{\mathbf{p}})(I_{E(\mathbf{m})} \otimes \tilde{S}^*_{\mathbf{p}})(\xi \otimes v) \\ &\in \tilde{S}_{\mathbf{n}}(E(\mathbf{n}) \otimes \mathcal{V}).\end{aligned}$$

That the range of $\tilde{S}^l_{\mathbf{m}}$ is contained in the range of $\tilde{S}^l_{\mathbf{n}}$ for positive integers l follows by a similar argument.

Claim (2)**.** If $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$ and $\mathbf{n} = l\mathbf{m}$ for some positive integer l, then $\mathcal{H}_{\mathbf{m}} = \mathcal{H}_{\mathbf{n}}$.

We know from the first claim that $\mathcal{H}_{\mathbf{m}} \subseteq \mathcal{H}_{\mathbf{n}}$. The reverse inclusion follows from the fact that $\tilde{S}_{\mathbf{n}}$ is isomorphic to

$$\tilde{S}_{\mathbf{m}}(I_{E(\mathbf{m})} \otimes \tilde{S}_{\mathbf{m}}) \ldots (I_{E(\mathbf{m})^{p-1}} \otimes \tilde{S}_{\mathbf{m}}).$$

Claim (3)**.** If $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$ such that $n_i, m_i \neq 0$ for $1 \leq i \leq k$, then $\mathcal{H}_{\mathbf{m}} = \mathcal{H}_{\mathbf{n}}$.

Choose an integer l such that $l\mathbf{m} \geq \mathbf{n}$. Then, by the previous two claims, $\mathcal{H}_{\mathbf{n}} \subseteq \mathcal{H}_{l\mathbf{m}} = \mathcal{H}_{\mathbf{m}}$. The reverse inclusion follows similarly.

Now, since $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is a minimal dilation, we have that $\mathcal{H} = \bigvee_{\mathbf{n} \in \mathbb{N}^k} \mathcal{H}_{\mathbf{n}}$. However, if we fix $\mathbf{n} \in \mathbb{N}^k$ such that $n_i \neq 0$ for each i, then the previous three claims tell us that $\mathcal{H}_{\mathbf{m}} \subseteq \mathcal{H}_{\mathbf{n}}$ for every $\mathbf{m} \in \mathbb{N}^k$. Hence $\mathcal{H} = \mathcal{H}_{\mathbf{n}}$ and so $(S_{\mathbf{n}}, \rho)$ is the minimal isometric dilation of $(A_{\mathbf{n}}, \sigma)$. $\square$

Remark 2.3.13. The condition in Theorem 2.3.12 that $\mathbf{n} \geq (1, 1, \ldots, 1)$ is necessary to guarantee that $(S_{\mathbf{n}}, \rho)$ is the minimal isometric dilation of $(A_{\mathbf{n}}, \sigma)$. For example, let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n : n \geq 0\}$. Define commuting isometries T_1 and T_2 on $\mathcal{H}$ by

$$\begin{aligned}T_1 e_n &= e_{2n} \quad \text{and} \\ T_2 e_n &= e_{3n}.\end{aligned}$$

Then T_1^* and T_2^* are commuting coisometries. Let U_1 and U_2 be the minimal commuting unitaries dilating T_1^* and T_2^*. Note that commuting unitaries are necessarily doubly commuting. We have that for any $n, k \geq 0$

$$\langle U_1 e_3, U_2^k e_n \rangle = \langle e_3, U_2^k e_{2n} \rangle = \langle e_3, T_2^{*k} e_{2n} \rangle = 0,$$

and so U_2 is not the minimal isometric dilation of T_2^*.

In some cases of fully coisometric, atomic representations of single vertex k-graphs, however, it is not necessary for $\mathbf{n} \geq (1, \ldots, 1)$ for Theorem 2.3.12 to be satisfied. See Example 2.4.15 and Proposition 2.4.16.

We now prove a higher rank version of Lemma 2.2.6.

Lemma 2.3.14. *Let $(A^{(1)}, \dots, A^{(k)}, \sigma)$ be a representation of a product system E on a Hilbert space $\mathcal{V}$ with a minimal isometric dilation $(S^{(1)}, \dots, S^{(k)}, \rho)$ on a Hilbert space $\mathcal{H}$. Let $\mathfrak{A}$ and $\mathfrak{S}$ be the unital* WOT*-closed algebra generated by $(A^{(1)}, \dots, A^{(k)}, \sigma)$ and $(S^{(1)}, \dots, S^{(k)}, \rho)$ respectively. Further, suppose that the representation $(S^{(1)}, \dots, S^{(k)}, \rho)$ is doubly commuting. Then if $\mathcal{V}_1$ is an $\mathfrak{A}^*$-invariant subspace of $\mathcal{V}$, $\mathcal{H}_1 = \mathfrak{S}[\mathcal{V}_1]$ reduces $\mathfrak{S}$.*

If $\mathcal{V}_1$ and $\mathcal{V}_2$ are orthogonal $\mathfrak{A}^*$-invariant subspaces the $\mathcal{H}_j = \mathfrak{S}[\mathcal{V}_j]$ for $j = 1, 2$ are mutually orthogonal.

If $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{H}_j \cap \mathcal{V} = \mathcal{V}_j$ for $j = 1, 2$.

Proof. We will prove the first part of the theorem. The remaining parts follow in a similar manner as in Lemma 2.2.6.

First, $\mathcal{V}_1$ is $\mathfrak{A}^*$-invariant, and so $\mathcal{V}_1$ is $\mathfrak{S}^*$-invariant. Elements of the form $\tilde{S}_{\mathbf{n}}(\eta \otimes v)$, with $\mathbf{n} \in \mathbb{N}^k$, $\eta \in E(\mathbf{n})$ and $v \in \mathcal{V}_1$, span a dense subset of $\mathcal{H}_1$. Take $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ and $i \in \{1, \dots, k\}$. Then for any $\xi \in E_i$, $\eta \in E(\mathbf{n})$, $v \in \mathcal{V}_1$, $w \in \mathfrak{S}[\mathcal{V}_1]^\perp$, if $n_i \neq 0$ then

$$\begin{aligned} \langle S^{(i)}(\xi)^* \tilde{S}_{\mathbf{n}}(\eta \otimes v), w \rangle &= \langle \tilde{S^{(i)}}^* \tilde{S}_{\mathbf{n}}(\eta \otimes v), \xi \otimes w \rangle \\ &= \langle I_{E^i} \otimes \tilde{S}_{\mathbf{n}-\mathbf{e}_i}(\eta \otimes v), \xi \otimes w \rangle \\ &= 0, \end{aligned}$$

and so $S^{(i)}(\xi)^* \tilde{S}_{\mathbf{n}}(\eta \otimes v) \in \mathcal{H}_1$. If $n_i = 0$ then, since our dilation is doubly commuting,

$$\begin{aligned} \langle S^{(i)}(\xi)^* \tilde{S}_{\mathbf{n}}(\eta \otimes v), w \rangle &= \langle \tilde{S^{(i)}}^* \tilde{S}_{\mathbf{n}}(\eta \otimes v), \xi \otimes w \rangle \\ &= \langle (I_{E_i} \otimes \tilde{S}_{\mathbf{n}})(t \otimes I_{\mathcal{H}})(I_{E(\mathbf{n})} \otimes \tilde{S^{(i)}}^*)(\eta \otimes v), \xi \otimes w \rangle \\ &= \langle (I_{E_i} \otimes \tilde{S}_{\mathbf{n}})(t \otimes I_{\mathcal{H}})(I_{E(\mathbf{n})} \otimes \tilde{A^{(i)}}^*)(\eta \otimes v), \xi \otimes w \rangle \\ &= 0 \end{aligned}$$

and so again $S^{(i)}(\xi)^* \tilde{S}_{\mathbf{n}}(\eta \otimes v) \in \mathcal{H}_1$. Thus $\mathcal{H}_1$ is $\mathfrak{S}$-reducing. $\square$

Remark 2.3.15. It is natural to ask if there is a higher rank analogue of Lemma 2.2.4. If $(A^{(1)}, \dots, A^{(k)}, \sigma)$ is a representation of E on $\mathcal{V}$ with a minimal isometric dilation $(S^{(1)}, \dots, S^{(k)}, \rho)$ on $\mathcal{H}$, is the restriction of $(S^{(1)}, \dots, S^{(k)}, \rho)$ to $\mathcal{V}^\perp$ an induced representation? The answer is no. From [26] it is known that induced representations are doubly commuting. Looking at the atomic representations studied in [19] and [22], or looking at Example 2.4.15, we see that the restriction to $\mathcal{V}^\perp$ is not, in general, doubly commuting.

2.3.3 Finitely Correlated Representations

Definition 2.3.16. An isometric representation $(S^{(1)}, \ldots, S^{(k)}, \rho)$ of a product system E on a Hilbert space $\mathcal{H}$ is called *finitely correlated* if $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is the minimal isometric dilation of a representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ on a non-zero finite dimensional Hilbert space $\mathcal{V} \subseteq \mathcal{H}$.

In particular, if $\mathfrak{S}$ is the unital WOT-closed algebra generated by $(S^{(1)}, \ldots, S^{(k)}, \rho)$, then $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is finitely correlated if there is a finite dimensional $\mathfrak{S}^*$-invariant subspace $\mathcal{V}$ of $\mathcal{H}$ such that $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is the minimal isometric dilation of the compressed representation $(P_{\mathcal{V}}S^{(1)}(\cdot)|_{\mathcal{V}}, \ldots, P_{\mathcal{V}}S^{(k)}(\cdot)|_{\mathcal{V}}, \rho(\cdot)|_{\mathcal{V}})$.

Remark 2.3.17. In this section we are concerned with finitely correlated fully coisometric representations of product systems. Let $(E, \mathcal{A})$ be a product system of C^*-correspondences over $\mathbb{N}^k$. As in the C^*-correspondence case, assuming existence of a finitely correlated fully coisometric representation of E puts restrictions on the C^*-algebra $\mathcal{A}$. See Remark 2.2.10. The class of product systems of C^*-algebras which exhibit finitely correlated representations includes the k-graphs studied in §2.4.

A class of finitely correlated representations of k-graphs have been studied in [19] (2-graphs) and [22] (k-graphs). These papers consider finitely correlated *atomic* representations of k-graphs. These representations are both isometric and fully coisometric. Atomic representations are an example of *partially isometric* representations, i.e. they are representations defined by row-contractions of partial isometries. Atomic representations of k-graphs are looked at more closely in section 2.4.2. As in the rank 1 case above, the existence of a unique minimal generating space is shown. We will now prove the existence of such a space for a general finitely correlated, isometric, fully coisometric representation of a product system of C^*-correspondences over $\mathbb{N}^k$. We begin with a higher rank version of Lemma 2.2.12.

Lemma 2.3.18. *Let $(S^{(1)}, \ldots, S^{(k)}, \rho)$ be a finitely correlated, fully coisometric representation of a product system E on a Hilbert space $\mathcal{H}$. Let $\mathfrak{S}$ be the unital* WOT*-closed algebra generated by $(S^{(1)}, \ldots, S^{(k)}, \rho)$. Let $\mathcal{V}$ be a finite dimensional $\mathfrak{S}^*$-invariant subspace of $\mathcal{H}$ such that $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is the minimal isometric dilation of the compressed representation $(P_{\mathcal{V}}S^{(1)}(\cdot)|_{\mathcal{V}}, \ldots, P_{\mathcal{V}}S^{(k)}(\cdot)|_{\mathcal{V}}, \rho(\cdot)|_{\mathcal{V}})$.*

Then if $\mathcal{M}$ is a non-zero, $\mathfrak{S}^*$-invariant subspace of $\mathcal{H}$, the subspace $\mathcal{M} \cap \mathcal{V}$ is non-trivial.

Proof. Take any $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ with $n_i \neq 0$ for $1 \leq i \leq k$. By Theorem 2.3.12, $(S_{\mathbf{n}}, \rho)$ is the unique minimal isometric dilation of $(A_{\mathbf{n}}, \sigma)$. The subspace $\mathcal{M}$ is $\mathfrak{S}^*$-invariant and so for any $\xi \in E(\mathbf{n})$, $S_{\mathbf{n}}(\xi)^*\mathcal{M} \subseteq \mathcal{M}$. Let $\mathfrak{S}_{\mathbf{n}}$ be the unital WOT-closed algebra generated by $S_{\mathbf{n}}(E(\mathbf{n}))$ and $\rho(\mathcal{A})$. It follows that $\mathcal{M}$ is invariant under $\mathfrak{S}_{\mathbf{n}}^*$. Hence, by Lemma 2.2.12, $\mathcal{M} \cap \mathcal{V}$ is non-trivial. $\square$

Theorem 2.3.19. *Let $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ be a fully coisometric representation of a product system E on a finite dimensional Hilbert space $\mathcal{V}$, and let $(S^{(1)}, \ldots, S^{(k)}, \rho)$ be the unique minimal isometric dilation of $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ on a Hilbert space $\mathcal{H}$. Let $\mathfrak{A}$ be the unital algebra generated by the representation $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ and let $\mathfrak{S}$ be the unital* WOT*-closed algebra generated by $(S^{(1)}, \ldots, S^{(k)}, \rho)$.*

If $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k$ is a maximal set of pairwise orthogonal minimal $\mathfrak{A}^*$-invariant spaces of $\mathcal{V}$ then $\hat{\mathcal{V}} = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k$ is the unique minimal cyclic coinvariant subspace of $\mathcal{V}$ and $\mathfrak{S}^*|_{\hat{\mathcal{V}}}$ is a C^*-algebra.

Further, if $\mathcal{W}_{\mathbf{m}}$ is the unique, minimal cyclic space for the C^*-correspondence representation $(S_{\mathbf{m}}, \rho)$, where $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ (with $m_i \neq 0$ for $1 \leq i \leq k$), then $\mathcal{W}_{\mathbf{m}} = \hat{\mathcal{V}}$.

Proof. Using Lemma 2.3.14 and Lemma 2.3.18, that $\mathfrak{S}[\hat{\mathcal{V}}] = \mathcal{H}$ follows the same argument as in the C^*-correspondence case. That $\mathfrak{S}^*|_{\hat{\mathcal{V}}}$ is a C^*-algebra follows by Lemma 2.2.23.

Let $\mathbf{m} = (m_1, m_2, \ldots, m_k) \in \mathbb{N}^k$ where $m_i \neq 0$ for $1 \leq i \leq k$. Let $\mathfrak{A}_{\mathbf{m}}$ be the unital algebra generated by the representation $(A_{\mathbf{m}}, \sigma)$ and let $\mathcal{W}$ be the unique minimal cyclic coinvariant space for $\mathfrak{A}_{\mathbf{m}}$. By Theorem 2.3.12 and since $\mathcal{W}$ is unique, $\mathcal{W}$ is contained in any minimal cyclic coinvariant space for $\mathfrak{A}$. In particular $\mathcal{W} \subseteq \hat{\mathcal{V}}$. Note also that $\mathfrak{A}[\mathcal{W}] = \mathcal{V}$ since $\mathfrak{A}_{\mathbf{m}}[\mathcal{W}] = \mathcal{V}$ and $\mathfrak{A}_{\mathbf{m}} \subseteq \mathfrak{A}$. We will show $\mathcal{W}$ is $\mathfrak{A}^*$-invariant.

Define the subspace $\mathcal{U}' \subseteq \mathcal{V}$ by

$$\mathcal{U}' = \sum_{\xi \in E(\mathbf{m}-\mathbf{e}_k)} S_{\mathbf{m}-\mathbf{e}_k}(\xi)^* \mathcal{W}.$$

Note that, by the commutation relations

$$\tilde{S}_{\mathbf{m}-\mathbf{e}_k}(I_{E(\mathbf{m}-\mathbf{e}_k)} \otimes \tilde{S}_{\mathbf{m}}) = \tilde{S}_{\mathbf{m}}(I_{E(\mathbf{m})} \otimes \tilde{S}_{\mathbf{m}-\mathbf{e}_k})(t \otimes I_{\mathcal{H}})$$

where t is the isomorphism $t : E(\mathbf{m}-\mathbf{e}_k) \otimes E(\mathbf{m}) \to E(\mathbf{m}) \otimes E(\mathbf{m}-\mathbf{e}_k)$.

So, if we take vectors $w \in \mathcal{W}$, $v \in \mathcal{V} \ominus \mathcal{U}'$ and $\eta \in E(\mathbf{m})$ and $\xi \in E(\mathbf{m}-\mathbf{e}_k)$ then

$$\begin{aligned}
\langle S_{\mathbf{m}}(\eta)^* S_{\mathbf{m}-\mathbf{e}_k}(\xi)^* w, v\rangle &= \langle (I_{E(\mathbf{m}-\mathbf{e}_k)} \otimes \tilde{S}^*_{\mathbf{m}})\tilde{S}^*_{\mathbf{m}-\mathbf{e}_k} w, \xi \otimes \eta \otimes v\rangle \\
&= \langle (I_{E(\mathbf{m})} \otimes \tilde{S}^*_{\mathbf{m}-\mathbf{e}_k})\tilde{S}^*_{\mathbf{m}} w, (t \otimes I_{\mathcal{H}})(\xi \otimes \eta \otimes v)\rangle \\
&= \langle \tilde{S}^*_{\mathbf{m}} w, (I_{E(\mathbf{m})} \otimes \tilde{S}_{\mathbf{m}-\mathbf{e}_k})(t \otimes I_{\mathcal{H}})(\xi \otimes \eta \otimes v)\rangle.
\end{aligned}$$

Note that $\tilde{S}^*_{\mathbf{m}} w \in E(\mathbf{m}) \otimes \mathcal{W}$, $(I_{E(\mathbf{m})} \otimes \tilde{S}_{\mathbf{m}-\mathbf{e}_k})(t \otimes I_{\mathcal{H}})(\xi \otimes \eta \otimes v)$ is in the space

$$E(\mathbf{m}) \otimes \tilde{S}_{\mathbf{m}-\mathbf{e}_k}(E(\mathbf{m}-\mathbf{e}_k) \otimes (\mathcal{V} \ominus \mathcal{U}')),$$

and $\mathcal{W}$ and $\mathcal{U}'$ are both σ reducing subspaces. It follows that

$$\langle S_{\mathbf{m}}(\eta)^* S_{\mathbf{m}-\mathbf{e}_k}(\xi)^* w, v\rangle = 0,$$

and so $\mathcal{U}'$ is $\mathfrak{A}_{\mathbf{m}}^*$-invariant. By Lemma 2.2.12, $\mathcal{U}'$ has non-trivial intersection with $\mathcal{W}$. Let $\mathcal{U} = \mathcal{W} \cap \mathcal{U}'$.

Suppose $\mathcal{U} \neq \mathcal{W}$. A similar argument to above will show that

$$\sum_{\xi \in E_k} S^{(k)}(\xi)^*(\mathcal{W} \ominus \mathcal{U})$$

has non-trivial intersection with $\mathcal{W}$. Choose $w_1, \ldots, w_n \in \mathcal{W} \ominus \mathcal{U}$ and $\zeta_1, \ldots, \zeta_n \in E_k$ such that $\sum_{i=1}^n S^{(k)}(\zeta_i)^* w_i$ is a non-zero vector in $\mathcal{W}$. Since $(A^{(1)}, \ldots, A^{(k)}, \sigma)$ is fully coisometric we can choose $\eta \in E(\mathbf{m} - \mathbf{e}_k)$ such that

$$w := S_{\mathbf{m}-\mathbf{e}_k}(\eta)^* \sum_{i=1}^n S^{(k)}(\zeta_i)^* w_i$$

is non-zero. Now w is in $\mathcal{W}$ and hence w is in $\mathcal{U} \cap (\mathcal{W} \ominus \mathcal{U})$. This contradiction shows that we must have $\mathcal{U} = \mathcal{W}$. By construction of $\mathcal{U}'$, we have that for any $u \in \mathcal{U}'$ and $\xi \in E_k$, $S^{(k)}(\xi)^* u$ is in $\mathcal{W}$. Hence $\mathcal{W}$ is invariant under $\mathfrak{A}_k^*$, where $\mathfrak{A}_j$ is the unital algebra generated by $(A^{(j)}, \sigma)$. We can similarly show that $\mathcal{W}$ is $\mathfrak{A}_j^*$-invariant for $1 \leq j \leq k-1$, and so $\mathcal{W}$ is $\mathfrak{A}^*$-invariant. Therefore $\mathcal{W} = \hat{\mathcal{V}}$, and thus $\hat{\mathcal{V}}$ is unique. □

Remark 2.3.20. Take a non-zero $\mathbf{m} \in \mathbb{N}^k$ with $\mathbf{m} \not\geq (1, \ldots, 1)$. Let $\mathcal{U}$ be the minimal cyclic coinvariant subspace for the representation $(A_{\mathbf{m}}, \sigma)$ of the C^*-correspondence $E(\mathbf{m})$ and $\hat{\mathcal{V}}$ be the minimal cyclic coinvariant subspace for the representation of the product system, as in Theorem 2.3.19. We necessarily have that $\mathcal{U} \subseteq \hat{\mathcal{V}}$. However given an arbitrary finitely correlated representation we can not say whether $\mathcal{U} = \hat{\mathcal{V}}$ or $\mathcal{U} \subsetneq \hat{\mathcal{V}}$. For the case when $k = 2$ and $\mathbf{m} = (0, 1)$, Example 2.4.15 satisfies $\mathcal{U} = \hat{\mathcal{V}}$ and Example 2.4.17 satisfies $\mathcal{U} \subsetneq \hat{\mathcal{V}}$.

We again conclude that the compression to the unique minimal cyclic subspace for a finitely correlated, fully coisometric representation is a complete unitary invariant.

Corollary 2.3.21. *Suppose $(S^{(1)}, \ldots, S^{(k)}, \sigma)$ and $(T^{(1)}, \ldots, T^{(k)}, \tau)$ are finitely correlated, fully coisometric representations of a product system $(E, \mathcal{A})$ on $\mathcal{H}_S$ and $\mathcal{H}_T$ respectively. Let $\mathcal{V}_S$ be the unique minimal cyclic coinvariant subspace for the representation $(S^{(1)}, \ldots, S^{(k)}, \sigma)$ and let $\mathcal{V}_T$ be the unique minimal cyclic subspace for the representation $(T^{(1)}, \ldots, T^{(k)}, \tau)$.*

Then $(S^{(1)}, \ldots, S^{(k)}, \sigma)$ and $(T^{(1)}, \ldots, T^{(k)}, \tau)$ are unitarily equivalent if and only if the finite dimensional fully coisometric representations $(P_{\mathcal{V}_S} S^{(1)}(\cdot)|_{\mathcal{V}_S}, \ldots, P_{\mathcal{V}_S} S^{(k)}(\cdot)|_{\mathcal{V}_S}, \sigma(\cdot)|_{\mathcal{V}_S})$ and $(P_{\mathcal{V}_T} T^{(1)}(\cdot)|_{\mathcal{V}_T}, \ldots, P_{\mathcal{V}_T} T^{(k)}(\cdot)|_{\mathcal{V}_T}, \tau(\cdot)|_{\mathcal{V}_T})$ are unitarily equivalent.

2.4 Higher Rank Graph Algebras

2.4.1 Graph Algebras

Let G be a directed graph with a countable number of vertices $\mathcal{V}(G)$ and a countable number of edges $\mathcal{E}(G)$. If $e \in \mathcal{E}(G)$ is an edge from a vertex v to a vertex w then we say that v is the *source* of e, denoted $s(e)$, and that w is the *range* of e, denoted $r(e)$. A vertex x is called a *source* if there is no edge e with $r(e) = x$. A *path* of length k in G is a finite collection of edges $e_k e_{k-1} \dots e_1$ such that $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq k-1$. A *cycle* is a path $e_k e_{k-1} \dots e_1$ with $s(e_1) = r(e_k)$. If $x = s(e_1)$ and $y = r(e_k)$ then we say that $e_k e_{k-1} \dots e_1$ is a path from x to y. A graph G is *transitive* if, for any vertices $x, y \in \mathcal{V}(G)$, there is a path from x to y. A graph is *strongly transitive* if it is transitive and it is neither a single cycle nor a graph with one vertex and no edges.

As described in [60, 64, 49] a graph can be described by a C^*-correspondence. We follow the construction of [60] as presented in [64]. Note that in the case of a finite graph this construction is the same as that given in [49].

Let $\mathcal{A} = C_0(\mathcal{V}(G))$ be the C^*-algebra of all functions on $\mathcal{V}(G)$ vanishing at infinity. Let $E(G)$ be the set of functions $\xi : \mathcal{E}(G) \to \mathbb{C}$ which satisfy for each $v \in \mathcal{V}(G)$

$$\xi_v := \sum_{\substack{e \in \mathcal{E}(G) \\ s(e)=v}} |\xi(e)|^2 < \infty$$

and the function $v \to \xi_v$ vanishes at infinity. Define an $\mathcal{A}$-valued inner product on $E(G)$ by

$$\langle \xi, \eta \rangle(v) = \sum_{\substack{e \in \mathcal{E}(G) \\ s(e)=v}} \overline{\xi(e)} \eta(e),$$

for $\xi, \eta \in E(G)$. Define a left action of $\mathcal{A}$ on $E(G)$ by

$$(a\xi)(e) = a(r(e))\xi(e)$$

and a right action by

$$(\xi a)(e) = \xi(e) a(s(e))$$

for $\xi \in E(G)$, $a \in \mathcal{A}$ and $e \in \mathcal{E}(G)$. These make $E(G)$ into a C^*-correspondence over $\mathcal{A}$. We identify the vertex $v \in \mathcal{V}(G)$ with function $\delta_v \in \mathcal{A}$ which sends v to 1 and all other vertices to 0. Similarly, we identify an edge $e \in \mathcal{E}(G)$ with the function $\delta_e \in E(G)$ which sends e to 1 and all other edges to 0.

For a good introduction to graph algebras see [59]. We remark that representations of $E(G)$ coincide with completely contractive representations of G and that the dilation theorem for contractive representations of graphs in [34] and [10] is implied by Theorem 2.2.2.

Denote by $\mathcal{L}_G$ the WOT-closed algebra generated by

$$\{T_\xi, \varphi_\infty(a) : \xi \in E, a \in \mathcal{A}\}$$

acting on the space $\mathcal{H}_G := \mathcal{F}(E(G))$. The algebra $\mathcal{L}_G$ is known as a *free semigroupoid algebra*, see [37]. When G has a single vertex and n edges then $\mathcal{L}_G$ is a *free semigroup algebra*, more commonly denoted $\mathcal{L}_n$.

Finite dimensional representations of graphs are plentiful. Indeed Davidson and Katsoulis show that the finite dimensional representations of a graph G separate points in $\mathcal{L}_G$, [10]. Thus finitely correlated, isometric representations are also plentiful. Provided in [10] is an algorithm for creating finite dimensional representations. Below is a method for creating finite dimensional, fully coisometric representations. A similar example can be found in [34].

Example 2.4.1. Let G be a finite graph with no sources. Let $\mathcal{V}(G) = \{v_1, \ldots, v_n\}$. Let $\mathcal{E}(G)_i = \{e \in \mathcal{E}(G) : r(e) = v_i\} = \{e_{i1}, e_{i2}, \ldots, e_{iC_i}\}$, where C_i is the number of elements in $\mathcal{E}(G)_i$. Let $\mathcal{A}$ and $E(G)$ be as described above. Let $\mathcal{H}$ be a finite dimensional Hilbert space and let $\mathcal{K} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$, where $\mathcal{H}_i = \mathcal{H}$ for each i. We will define a representation (A, σ) of $E(G)$ on $\mathcal{K}$.

For each vertex v_i let $T_i = [T_{i1}, \ldots, T_{iC_i}]$ be a defect free row-contraction on $\mathcal{H}_i$, i.e.

$$\sum_{j=1}^{C_i} T_{ij} T_{ij}^* = I_{\mathcal{H}_i}.$$

Suppose $e_{ij} \in \mathcal{E}(G)_i$ with $s(e_{ij}) = v_l$. Define $A(e_{ij}) \in \mathcal{B}(\mathcal{K}) = M_n(\mathcal{B}(\mathcal{H}))$ by $(A(e_{ij}))_{i,l} = T_{i,j}$ and $(A(e_{ij}))_{k,m} = 0$ when $(k, m) \neq (i, l)$. We define a representation σ of $\mathcal{A}$ on $\mathcal{K}$ by $\sigma(v_i) = P_{\mathcal{H}_i} =: P_{v_i}$ for $1 \leq i \leq n$. Thus

$$\sum_{e \in \mathcal{E}(G)} A(e) A(e)^* = I_{\mathcal{K}}$$

and

$$P_{r(e)} A(e) P_{s(e)} = A(e).$$

It follows that (A, σ) is a finite dimensional, fully coisometric representation of $E(G)$, see [10], [34]. This method readily extends to any graph containing a finite subgraph with no sources.

Strong Double-Cycle Property

We now strengthen our results from section 2.2 for the special case of C^*-correspondences defined by finite graphs with the strong double-cycle property.

Definition 2.4.2. A vertex in G is said to lie on a *double-cycle* if it lies on two, distinct, minimal cycles. We say that G has the *strong double-cycle property* if for every vertex x in G there is a path from x to a vertex lying on a double-cycle.

Example 2.4.3. When $n \geq 2$, a single vertex graph with n edges has the strong double-cycle property. This is the case studied in [17].

Example 2.4.4. If each connected component of G is strongly transitive, then G has the strong double-cycle property.

The following result is proved in [49] and [37] for finite graphs with the strong double-cycle property, and in [18] for when $\mathcal{L}_G = \mathcal{L}_n$ is a free semigroup algebra.

Theorem 2.4.5. *Suppose G is a finite graph with the strong double-cycle property and φ is a weak-$*$ continuous linear functional on $\mathcal{L}_G$ with $\|f\| < 1$. Then there are vectors ξ and ζ in $\mathcal{H}_G$, with $\|\xi\|, \|\zeta\| < 1$, such that $\varphi(A) = \langle A\xi, \zeta\rangle$ for all A in $\mathcal{L}_G$.*

We fix such a graph G with a finitely correlated fully coisometric representation (S, ρ) of $E(G)$ on a Hilbert space $\mathcal{H}$. Let $\mathcal{U}$ be the unique minimal cyclic coinvariant subspace for (S, ρ) and let (A, σ) be the compression of (S, ρ) to $\mathcal{U}$, so that (S, ρ) is the unique minimal dilation of (A, σ). Let $\mathfrak{A}$ be the unital algebra generated by (A, σ) and let $\mathfrak{S}$ be the unital WOT-closed algebra generated by (S, ρ). By Theorem 2.2.27, $\mathcal{U} = \mathcal{U}_1 \oplus \ldots \oplus \mathcal{U}_n$ is a direct sum of minimal $\mathfrak{A}^*$-invariant subspaces and $\mathfrak{A}$ is a C^*-algebra. For each j let $\mathcal{H}_j = \mathfrak{S}[\mathcal{U}_j]$. Let $d = \dim \mathcal{U}$ and let $\{f_1, \ldots, f_d\}$ form an orthonormal basis of $\mathcal{U}$. We now follow the methods in [17] in order to give a full description of $\mathfrak{S}$. In particular, we will show that $\mathfrak{S}$ contains the projection onto $\mathcal{U}$.

For $1 \leq i \leq n$, let q_i be the compression of $\mathfrak{A}$ to $\mathcal{U}_i$, i.e. $q_i(\mathfrak{A}) = P_{\mathcal{U}_i}\mathfrak{A}P_{\mathcal{U}_i} = \mathcal{B}(\mathcal{U}_i)$. Choose a minimal set $H \subseteq \{1, \ldots, n\}$ such that $\sum_{h\in H}^{\oplus} q_h$ is faithful. The minimal ideal $\ker \sum_{h\in H\setminus\{h_0\}}^{\oplus} q_h$ is isomorphic to $\mathcal{B}(\mathcal{U}_{h_0})$. This kernel can be supported on more than one of the $\mathcal{U}_i$'s. We let $H_h \subseteq H$ be the set of indices i where $\mathcal{U}_i$ is supported on $\ker \sum_{g\in H\setminus\{h\}}^{\oplus} q_g$. For each $h \in H$ let m_h be the number of elements in H_h. If we let $\mathcal{W}_h = \sum_{i\in H_h}^{\oplus} \mathcal{U}_i$, then $\mathcal{U} = \sum_{h\in H}^{\oplus} \mathcal{W}_h$. For each $j \in H_h$ there is a spatial, algebra isomorphism σ_j of $\mathcal{B}(\mathcal{U}_h)$ onto $\mathcal{B}(\mathcal{U}_j)$ such that

$$\mathfrak{A}|_{\mathcal{W}_h} = \left\{ \sum_{j\in H_h}^{\oplus} \sigma_j(X) : X \in \mathcal{B}(\mathcal{U}_h) \right\}.$$

For each $h \in H$ let P_h be the projection onto $\mathcal{W}_h$. For each $h \in H$ the projection P_h lies in the centre of $\mathfrak{A}$.

A closer look at Lemma 2.2.4 tells us that for each $\xi \in E$ and $a \in \mathcal{A}$

$$S(\xi) = \begin{bmatrix} A(\xi) & 0 \\ X_\xi & T_\xi^{(\alpha)} \end{bmatrix}, \quad \rho(a) = \begin{bmatrix} \sigma(a) & 0 \\ 0 & \rho(a)|_{\mathcal{V}^\perp} \end{bmatrix}$$

where $\alpha = \dim \mathcal{W}$, with $\mathcal{W} = (\mathcal{U} + \tilde{S}(E \otimes \mathcal{U})) \ominus \mathcal{U}$ as in Lemma 2.2.4. Hence

$$\mathfrak{S} = \begin{bmatrix} \mathfrak{A} & 0 \\ * & \mathcal{L}_G^{(\alpha)} \end{bmatrix}.$$

We denote by $\mathfrak{B}$ the WOT-closed operator algebra on $\mathcal{H}$ spanned by $\mathcal{B}(\mathcal{H})P_{\mathcal{U}}$ and $0_{\mathcal{U}} \oplus \mathcal{L}_G^{(\alpha)}$. The following three proofs follow the arguments of [17, Lemma 4.4], [17, Lemma 5.14] and [17, Corollary 5.3] respectively.

Lemma 2.4.6. *Every weak-$*$ continuous functional on $\mathfrak{B}$ is given by a trace class operator of rank at most $d+1$, where $d = \dim \mathcal{U}$.*

Hence the WOT *and weak-$*$ topologies coincide on $\mathfrak{B}$ and $\mathfrak{S}$.*

Proof. Let φ be a weak-$*$ continuous functional on $\mathfrak{B}$. If $B \in \mathfrak{B}$ then $\varphi(B)$ is determined by $\varphi(BP_{\mathcal{U}})$ and $\varphi(BP_{\mathcal{U}^\perp})$. By the Riesz Representation Theorem there are vectors $y_1, \ldots, y_d \in \mathcal{U}$ such that

$$\varphi(BP_{\mathcal{U}}) = \sum_{i=1}^{d} \langle Bf_i, y_i \rangle.$$

By Theorem 2.4.5 there are vectors $\xi, \zeta \in \mathcal{U}^\perp$ such that $\varphi(A) = \langle A\xi, \zeta \rangle$ for all $A \in \mathcal{L}_G^{(\alpha)}$. Hence

$$\varphi(B) = \sum_{i=1}^{d} \langle Bf_i, y_i \rangle + \langle B\xi, \zeta \rangle,$$

and φ is trace-class of rank at most $d+1$. □

Lemma 2.4.7. *For $h \in H$, let P_h denote the minimal central projections of $\mathfrak{A}$ as above. Then P_h lies in $\mathfrak{S}$. Hence $P_{\mathcal{U}}$ is in $\mathfrak{S}$.*

Proof. Fix a minimal central projection P of $\mathfrak{A}$. Let φ be a non-zero weak-$*$ continuous functional on $\mathfrak{B}$ which is zero on $\mathfrak{S}$. We will show that $\varphi(P) = 0$. It follows immediately that $P \in \mathfrak{S}$.

By Lemma 2.4.6 there are vectors $x, y \in \mathcal{H}^{(d+1)}$ such that $\varphi(A) = \langle A^{(d+1)}x, y \rangle$ for all $A \in \mathfrak{B}$. Let $\mathcal{M} = \mathfrak{S}^{*(d+1)}[y]$. Since φ is zero on $\mathfrak{S}$ it follows that x is orthogonal to $\mathcal{M}$. Let $\mathcal{M}_0 = \mathcal{M} \cap \mathcal{U}^{(d+1)}$. By Lemma 2.2.12, $\mathcal{M}_0$ is non-zero. The subspace $\mathcal{M}_0$ is invariant under the C^*-algebra $\mathfrak{A}^{(d+1)}$ and hence $\mathcal{M}_0$ is the range of a projection Q in the commutant of $\mathfrak{A}^{(d+1)}$.

We decompose $\mathcal{U}^{(d+1)}$ into the following spaces

$$\begin{aligned} &P^{(d+1)}Q\mathcal{U}^{(d+1)} \oplus P^{\perp(d+1)}Q\mathcal{U}^{(d+1)} \oplus P^{(d+1)}Q^\perp\mathcal{U}^{(d+1)} \oplus P^{\perp(d+1)}Q^\perp\mathcal{U}^{(d+1)} \\ &=: \mathcal{M}_{pq} \oplus \mathcal{M}_{p^\perp q} \oplus \mathcal{M}_{pq^\perp} \oplus \mathcal{M}_{p^\perp q^\perp}. \end{aligned}$$

Note that, as Q and $P^{(d+1)}$ are projections in the commutant of $\mathfrak{A}^{(d+1)}$, the four spaces M_{ij} are $\mathfrak{A}^{(d+1)}$-reducing. Also $\mathcal{M}_0 = \mathcal{M}_{pq} \oplus \mathcal{M}_{p^\perp q}$. Letting $\mathcal{H}_{ij} = \mathfrak{S}[\mathcal{M}_{ij}]$ we see that $\mathcal{H}$ decomposes into

$$\mathcal{H} = \mathcal{H}_{pq} \oplus \mathcal{H}_{p^\perp q} \oplus \mathcal{H}_{pq^\perp} \oplus \mathcal{H}_{p^\perp q^\perp}.$$

It follows that $y \in \mathcal{H}_{pq} \oplus \mathcal{H}_{p^\perp q} = \mathfrak{S}[\mathcal{M}_0]$ and so $P_{\mathcal{U}}^{(d+1)} y \in \mathcal{M}_0$. The projection $P_{\mathcal{U}}$ dominates P and so $P^{(d+1)} y \in \mathcal{M}_0$. Hence

$$\varphi(P) = \langle P^{(d+1)} x, y \rangle = \langle x, P^{(d+1)} y \rangle = 0.$$

$\square$

Lemma 2.4.8. *The algebra* $\mathfrak{S} P_{\mathcal{U}} \simeq \sum_{h \in H}^{\oplus} (\mathcal{B}(\mathcal{H}_h) P_h)^{(m_h)}$*, where* $m_h = |H_h|$.

Proof. First suppose that $\mathfrak{A} = \mathcal{B}(\mathcal{U})$, i.e. $\mathcal{U}$ is a minimal $\mathfrak{A}^*$-invariant subspace. By Lemma 2.4.7, the projection $P_{\mathcal{U}}$ is in $\mathfrak{S}$. Hence $\mathfrak{S} P_{\mathcal{U}} = \mathcal{B}(\mathcal{U})$ is in $\mathfrak{S}$. In particular, for any $v \in \mathcal{U}$ the rank 1 operator vv^* is in $\mathfrak{S}$. Note also that $\mathfrak{S}[v] = \mathcal{H}$ for any non-zero $v \in \mathcal{U}$. Hence for any $x \in \mathcal{H}$ there are operators T_k in $\mathfrak{S}$ such that $T_k v$ converges to x. Hence $T_k v v^*$ is in $\mathfrak{S}$. Hence xv^* is in $\mathfrak{S}$ for all $x \in \mathcal{H}$ and $v \in \mathcal{U}$. Therefore $\mathcal{B}(\mathcal{H}) P_{\mathcal{U}}$ is in $\mathfrak{S}$.

Returning to the general case, note that there is a unitary equivalence between $\sum_{j \in H_h}^{\oplus} \mathcal{H}_j$ and $\mathcal{H}_h \otimes \mathbb{C}^{(m_h)}$. Lemma 2.4.7 tells us that $P_{\mathcal{W}_h} \simeq P_h^{(m_h)}$ lies in $\mathfrak{S}$ for each $h \in H$. From the first paragraph it now follows that $\mathfrak{S} P_{\mathcal{U}}$ decomposes as $\sum_{h \in H}^{\oplus} (\mathcal{B}(\mathcal{H}_h) P_h)^{(m_h)}$. $\square$

Combining Lemma 2.4.7 and Lemma 2.4.8 with Theorem 2.2.27 we get the following theorem. When G is a single vertex graph with 2 or more edges, Theorem 2.4.9 is the same as [17, Theorem 5.15].

Theorem 2.4.9. *Let G be a finite graph with the strong double cycle property. Let (A, σ) be fully coisometric, finite dimensional representation of G on a Hilbert space $\mathcal{U}$. Let (S, ρ) be the unique minimal isometric dilation of (A, σ) to a Hilbert space $\mathcal{K}$. Let* $\mathfrak{A} = \mathrm{Alg}\{A(\xi), \sigma(a) : \xi \in E, a \in \mathcal{A}\}$ *and* $\mathfrak{S} = \overline{\mathrm{Alg}\{S(\xi), \rho(a) : \xi \in E, a \in \mathcal{A}\}}^{\mathrm{WOT}}$

If

$$\hat{\mathcal{U}} = \sum_{j=1}^{n}{}^{\oplus} \mathcal{U}_j$$

is a maximal direct sum of minimal $\mathfrak{A}^*$-invariant subspaces of $\mathcal{U}$ then $\hat{\mathcal{U}}$ is the unique *minimal $\mathfrak{A}^*$-invariant subspace such that $\mathfrak{S}[\hat{\mathcal{U}}] = \mathcal{H}$. The compression $\hat{\mathfrak{A}}$ of $\mathfrak{A}$ to $\hat{\mathcal{U}}$ is a C^*-algebra. Writing $\hat{\mathcal{U}}$ as $\sum_{h \in H}^{\oplus} \mathcal{U}_h^{(m_h)}$, where $\mathcal{U}_h$ has dimension d_h and multiplicity m_h then*

$$\hat{\mathfrak{A}} = \sum_{h \in H}{}^{\oplus} M_{d_h} \otimes \mathbb{C}^{m_h}.$$

Let P_h be the projection onto $\mathcal{U}_h$. Then the dilation acts on the space

$$\mathcal{K} = \sum_{h \in H}{}^{\oplus} \mathcal{K}_h^{(m_h)} = \hat{\mathcal{U}} \oplus \mathcal{H}_G^{(\alpha)}$$

where $\mathcal{K}_h = \mathcal{U}_h \oplus \mathcal{H}_h^{(\alpha_h)}$, $\alpha_h = \dim((\tilde{S}(E(G) \otimes \mathcal{U}_h) \ominus \mathcal{U}_h)$ *and*

$$\alpha = \sum_{h \in H} \alpha_h m_h.$$

The algebra $\mathfrak{S}$ *decomposes as*

$$\mathfrak{S} \simeq \sum_{h \in H}^{\oplus} (\mathcal{B}(\mathcal{H}_h) P_h)^{(m_h)} + (0_{\hat{\mathcal{U}}} \oplus \mathcal{L}_G^{(\alpha)}).$$

2.4.2 Higher Rank Graph Algebras

Definition 2.4.10. A *k-graph* (Λ, d) consists of a countable small category Λ, together with a degree functor d from Λ to $\mathbb{N}^k$, satisfying the factorization property: for every $\lambda \in \Lambda$ and $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$ with $d(\lambda) = \mathbf{m} + \mathbf{n}$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = \mathbf{m}$ and $d(\nu) = \mathbf{n}$. For each $\mathbf{n} \in \mathbb{N}^k$ let $\Lambda^{\mathbf{n}} = d^{-1}(\mathbf{n})$. Each k-graph (Λ, d) has a *source* map $s : \Lambda \to \Lambda^0$ and a *range* map $r : \Lambda \to \Lambda^0$.

A k-graph Λ is said to be *finitely aligned* if for each $\lambda, \mu \in \Lambda$ the set

$$\{\nu \in \Lambda : \text{there exists } \alpha, \beta \in \Lambda \text{ such that } \nu = \lambda\alpha = \mu\beta,\ d(\nu) = d(\lambda) \vee d(\mu)\},$$

is finite.

A 1-graph (Λ, d) is simply a graph with vertices Λ^0 and edges Λ^1. A k-graph can be visualized as a multi-coloured graph with vertices Λ^0 and $\Lambda^{\mathbf{e}_i}$ representing a different coloured set of edges for each i.

As in the 1-graph case, a k-graph can be associated with a product system of C^*-correspondences over $\mathbb{N}^k$. Briefly, define a C^*-algebra $\mathcal{A}$ by Λ^0, in the same manner that we used the vertices of a 1-graph to define a C^*-algebra. For $1 \leq i \leq k$ define a C^*-correspondence E_i over $\mathcal{A}$ by $\Lambda^{\mathbf{e}_i}$ in the same manner that we defined a C^*-correspondence using the edges of a 1-graph. The factorisation rule of (Λ, d) will define the isomorphisms $t_{i,j} : E_i \otimes E_j \to E_j \otimes E_i$, and this in turn will define a product system of C^*-correspondences $(E(\Lambda), \mathcal{A})$ over $\mathbb{N}^k$, see [60] or [64] for the details. In [60] it is shown that Toeplitz Λ-families of contractions coincide with isometric representations of $E(\Lambda)$. In [64] it is shown that Λ-contractions coincide with representations of $E(\Lambda)$. Thus there is a $1-1$ correspondence between representations of the k-graph (Λ, d) and representations of $(E(\Lambda), \mathcal{A})$.

When Λ is finitely aligned then $E(\Lambda)$ will satisfy the normal ordering condition. Hence Theorem 2.3.8 can be applied to finitely aligned k-graphs. This is the dilation theorem originally proved in [66].

Let Λ be a k-graph with no sources and with Λ^0 finite. In [66, Theorem 4.7] it is shown that there is a $1-1$ correspondence between states ω on the Cuntz-Pimsner algebra $\mathcal{O}_\Lambda$ and (the unitary equivalence classes of) triples $(\mathcal{V}, \Omega, (S^{(1)}, \ldots, S^{(k)}, \rho))$ where $\mathcal{V}$ is a Hilbert space, $\Omega \in \mathcal{V}$ is norm 1 vector, $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is an isometric representation of $E(\Lambda)$, and $\mathcal{V} = \overline{\mathfrak{S}^*\Omega}$ (where $\mathfrak{S}$ is the algebra generated by $(S^{(1)}, \ldots, S^{(k)}, \rho)$). It is noted in [66] that $(S^{(1)}, \ldots, S^{(k)}, \rho)$ is the minimal isometric dilation of the compression of $(S^{(1)}, \ldots, S^{(k)}, \rho)$ to $\mathcal{V}$. Given this result, it is natural to define what it means for a state on $\mathcal{O}_\Lambda$ to be finitely correlated as follows:

Definition 2.4.11. A state ω on the C^*-algebra $\mathcal{O}_\Lambda$ is *finitely correlated* if its corresponding triple $(\mathcal{V}_\omega, \Omega_\omega, (S_\omega^{(1)}, \ldots, S_\omega^{(k)}, \rho_\omega))$ has the property that $\mathcal{V}_\omega$ is finite dimensional.

When ω is a finitely correlated state on the Cuntz-Pimsner algebra $\mathcal{O}_\Lambda$ with corresponding triple $(\mathcal{V}_\omega, \Omega_\omega, (S_\omega^{(1)}, \ldots, S_\omega^{(k)}, \rho_\omega))$, the representation $(S_\omega^{(1)}, \ldots, S_\omega^{(k)}, \rho_\omega)$ will be finitely correlated. When Λ is a 1-graph with a single vertex and n edges, $\mathcal{O}_\Lambda$ is the Cuntz algebra $\mathcal{O}_n$ and the above definition coincides with the definition of finitely correlated states in [6].

Theorem 2.3.19 and Theorem 2.4.9 together give us the following result.

Proposition 2.4.12. *Let (Λ, d) be a k-graph. Suppose there is an $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ with $n_i \neq 0$ for $1 \leq i \leq k$, such that the 1-graph with vertices Λ^0 and edges defined by $\Lambda^{\mathbf{n}}$ has the strong double-cycle property. Let $(S^{(1)}, \ldots, S^{(k)}, \rho)$ be a finitely correlated, isometric, fully coisometric representation of $E(\Lambda)$ generating a* WOT*-closed algebra $\mathfrak{S}$. Then $\mathfrak{S}$ contains the projection onto its minimal cyclic coinvariant subspace.*

Graphs With a Single Vertex

Suppose (Λ, d) is a k-graph where Λ^0 is a singleton and $\Lambda^{\mathbf{e}_i}$ is finite for $1 \leq i \leq k$. Let $\Lambda^{\mathbf{e}_i} = \{e_l^{(i)} : 1 \leq l \leq m_i\}$, where m_i is the number of elements in $\Lambda^{\mathbf{e}_i}$. Let $S_{m_i \times m_j}$ be the set of permutations on the set of tuples $\{(a, b) : 1 \leq a \leq m_i,\ 1 \leq b \leq m_j\}$. By the factorisation property, for each pair i, j with $1 \leq i < j \leq k$ there is a permutation $\theta_{ij} \in S_{m_i \times m_j}$ such that

$$e_l^{(i)} e_m^{(j)} = e_{m'}^{(j)} e_{l'}^{(i)}$$

when $\theta_{ij}(l, m) = (l', m')$. Let $\theta = \{\theta_{ij} : 1 \leq i < j \leq k\}$. The k-graph Λ can be described as being a unital semigroup $\mathbb{F}_\theta^+$, where $\mathbb{F}_\theta^+$ is the semigroup

$$\langle e_l^{(i)} : e_l^{(i)} e_m^{(j)} = e_{m'}^{(j)} e_{l'}^{(i)} \text{ when } \theta_{ij}(l, m) = (l', m') \rangle.$$

That is, for each i, $e_i^{(1)}, \ldots, e_i^{(m_i)}$ form a copy of the free semigroup $\mathbb{F}_{m_i}^+$ and, when $i \neq j$ and $i < j$, a commutation relation between the e_i's and the e_j's is defined by the permutation θ_{ij}. Note that if we are given arbitrary permutations $\theta_{ij} \in S_{m_i \times m_j}$ for $1 \leq i < j \leq k$ we cannot necessarily form a cancellative semigroup $\mathbb{F}_\theta^+$. However, if $k = 2$ and $\theta \in S_{m_1 \times m_2}$ is any permutation, $\mathbb{F}_\theta^+$ will form a cancellative semigroup, and hence a 2-graph on a single vertex.

Let $(E(\mathbb{F}_\theta^+), \mathcal{A})$ be the product system of C^*-correspondences defined by a k-graph on a single vertex $\mathbb{F}_\theta^+$. It is not hard to see that $\mathcal{A} = \mathbb{C}$ and that $E_i = \mathbb{C}^{m_i}$ for $1 \leq i \leq m$. Let $(A^{(1)}, \dots, A^{(k)}, \sigma)$ be a representation of $(E(\mathbb{F}_\theta^+), \mathcal{A})$ on a Hilbert space $\mathcal{H}$ and define $A_l^{(i)} = A^{(i)}(e_l^{(i)})$. For each i we have that

$$A^{(i)} = [A_1^{(i)}, \dots, A_{m_i}^{(i)}]$$

is a row-contraction. A representation $(A^{(1)}, \dots, A^{(k)}, \sigma)$ is fully coisometric when

$$\sum_{j=1}^{m_i} A_j^{(i)} A_j^{(i)*} = I_{\mathcal{H}}, \tag{2.2}$$

for $1 \leq i \leq k$ i.e. when each row-contraction is defect free. A representation is isometric when $[A_1^{(i)}, \dots, A_{m_i}^{(i)}]$ is a row-isometry for $1 \leq i \leq k$.

Conversely, if $[A_1^{(i)}, \dots, A_{m_i}^{(i)}]$ are k row-contractions which satisfy for $1 \leq i < j \leq k$

$$A_l^{(i)} A_m^{(j)} = A_{m'}^{(j)} A_{l'}^{(i)}$$

when $\theta_{ij}(l, m) = (l', m')$, then they define a representation of the k-graph $\mathbb{F}_\theta^+$.

The k-graph $\mathbb{F}_\theta^+$ is finite and so it is finitely aligned. Thus, by either Theorem 2.3.6 or Theorem 2.3.8 together with Lemma 2.3.11, all fully coisometric representations of $\mathbb{F}_\theta^+$ have a unique minimal isometric, coisometric dilation.

Let $\mathbb{F}^+$ be the unital free semigroup with $m_1 m_2 \dots m_k$ generators

$$\{e_{l_1}^{(1)} e_{l_2}^{(2)} \dots e_{l_k}^{(k)} : 1 \leq l_j \leq m_j\}.$$

This corresponds to the graph with 1-vertex and C^*-correspondence $E(1, 1, \dots, 1)$. If $m_1 \dots m_k \neq 1$, i.e. if $\mathbb{F}^+ \ncong \mathbb{Z}_{\geq 0}$, then it is clear that $\mathbb{F}^+$ has the strong double-cycle property. Thus by Proposition 2.4.12, if $[S_1^{(i)}, \dots, S_{m_i}^{(i)}]$ are defect free row-isometries defining a finitely correlated representation of $\mathbb{F}_\theta^+$, then the WOT-closed algebra they generate contains the projection onto the minimal cyclic coinvariant subspace.

Definition 2.4.13. Let $[A_1^{(i)}, \dots, A_{m_i}^{(i)}]$, for $1 \leq i \leq k$, define a representation of $\mathbb{F}_\theta^+$ on a Hilbert space $\mathcal{H}$. The representation is *atomic* if each $A_l^{(i)}$ is a partial isometry and there is an orthonormal basis $\{\xi_n : n \geq 1\}$ of $\mathcal{H}$ which is permuted, up to scalars, by each partial isometry, i.e. $A_l^{(i)} \xi_n = \alpha \xi_m$ for some m and some $\alpha \in \mathbb{T} \cup \{0\}$.

Atomic representations of k-graphs on a single vertex were studied by Davidson, Power and Yang for 2-graphs [19] and by Davidson and Yang for k-graphs [22]. There the existence of the minimal cyclic coinvariant subspace is shown. The minimal cyclic coinvariant subspace for a finitely correlated, isometric, fully coisometric atomic representation is exhibited by a group construction. That is, a finitely correlated, isometric, fully coisometric atomic representation is shown to be a dilation of a certain representation on $\mathcal{B}(\ell^2(G))$ where G is a group with k generators. The following theorem shows that finitely correlated atomic representations are plentiful.

Theorem 2.4.14 (Davidson, Power and Yang [19, 22])**.** *There are irreducible finite dimensional defect free atomic representations of $\mathbb{F}_\theta^+$ of arbitrarily large dimension.*

Example 2.4.15. Let $\mathbb{F}_\theta^+$ be the two graph where $\theta \in S_{2\times 2}$ is the permutation defined by the cycle $((1,1),(2,2))$. Let $\mathcal{V}$ be a 4 dimensional vector space with orthonormal basis $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$. We define a fully coisometric, atomic representation of $\mathbb{F}_\theta^+$ on $\mathcal{V}$ by row-contractions $[A_1, A_2]$ and $[B_1, B_2]$, where

$$\begin{array}{lll}
A_1\zeta_1 = \zeta_2 & A_1\zeta_3 = \zeta_4 & A_1\zeta_i = 0 \;\text{ for }\; i = 2,4 \\
A_2\zeta_2 = \zeta_1 & A_2\zeta_4 = \zeta_3 & A_1\zeta_i = 0 \;\text{ for }\; i = 1,3 \\
B_1\zeta_2 = \zeta_3 & B_1\zeta_4 = \zeta_1 & B_1\zeta_i = 0 \;\text{ for }\; i = 1,3 \\
B_2\zeta_1 = \zeta_4 & B_2\zeta_3 = \zeta_2 & B_1\zeta_i = 0 \;\text{ for }\; i = 2,4.
\end{array}$$

Let $[S_1, S_2]$ and $[T_1, T_2]$ define the unique minimal isometric dilation of this representation. The representation defined by $[S_1, S_2]$ and $[T_1, T_2]$ will also be atomic [20]. Clearly $\mathcal{V}$ is the minimal cyclic coinvariant subspace for this representation. For $u, w \in \mathbb{F}_2^+$, where $u = i_1 \dots i_l$ and $w = j_i \dots j_m$, we write $S_u T_w$ for

$$S_{i_1} \dots S_{i_l} T_{j_1} \dots T_{j_m}.$$

The set $\{S_u T_w \zeta_i : u, w \in \mathbb{F}_2^+, i = 1,2,3,4\}$ will form an orthonormal basis of $\mathcal{H}$. Since the representation is atomic and fully coisometric each of these basis vectors will be in the range of exactly one S_i and exactly one T_j. It follows that $[S_1, \dots, S_n]$ is the minimal isometric Frazho-Bunce-Popescu dilation of the row-contraction $[A_1, \dots, A_n]$. That is, in this case, it is not necessary to have $\mathbf{m} \geq (1,1)$ in order for the conclusion of Theorem 2.3.12 to be satisfied. As we will see in Proposition 2.4.16, this is true of all finitely correlated atomic representations of periodic single-vertex 2-graphs. Recall, by Remark 2.3.13, that in general we do need the condition that $\mathbf{m} \geq (1,1)$ for Theorem 2.3.12 to hold.

We also have that the minimal cyclic coinvariant subspace for $[S_1, \dots, S_n]$ is all of $\mathcal{V}$. Thus, again, it is not necessary to have $\mathbf{m} \geq (1,1)$ for the conclusion of Theorem 2.3.19 to be satisfied. This is also a general fact about atomic representations. Again, recall that we do require that $\mathbf{m} \geq (1,1)$ in the general case. See Remark 2.3.20 and the following example.

The following lemma applies specifically to *periodic* 2-graphs. We refer to [21] for further details on periodic 2-graphs. We recall that $\mathbb{F}_\theta^+$ with $\theta \in S_{m\times n}$ is $(a, -b)$ periodic when there is a bijection

$$\gamma : \mathbf{m}^a \longrightarrow \mathbf{n}^b$$

such that $e_u f_v = f_{\gamma(u)} e_{\gamma^{-1}(v)}$, where $\mathbf{m}^a$ is the set of words in $\mathbb{F}_m^+$ of length a and $\mathbf{n}^b$ is the set of words in $\mathbb{F}_n^+$ of length b.

Proposition 2.4.16. *Let $\mathbb{F}_\theta^+$ be a periodic 2-graph where $\theta \in S_{m\times n}$ with $2 \leq m \leq n$. Let $S = [S_1, \ldots, S_m]$ and $T = [T_1, \ldots, T_n]$ be defect-free row-isometries on $\mathcal{H}$ defining a finitely correlated, atomic representation of $\mathbb{F}_\theta^+$. Let $\mathcal{V}$ be the minimal cyclic coinvariant subspace for the representation and let A and B be the compressions of S and T to $\mathcal{V}$ respectively. Then the minimal isometric dilation of A is S and the minimal isometric dilation of B is T.*

Proof. Let $Z = \{\zeta_1, \ldots, \zeta_k\}$ be an orthonormal basis for $\mathcal{V}$ on which A and B act atomically. Take any $\zeta_l \in Z$ and $w \in \mathbb{F}_n^+$. We will show that there is a $v \in \mathbb{F}_m^+$ and a basis vector $\zeta \in Z$ such that $S_v\zeta = \mu T_w \zeta_l$ for some scalar μ. This will show that the minimal isometric dilation of A is S. A similar argument will show that the minimal isometric dilation of B is T.

Since $\mathbb{F}_\theta^+$ is periodic we can find a, b large enough, a bijection $\gamma : \mathbf{m}^a \to \mathbf{n}^b$, $u \in \mathbb{F}_m^+$, $v \in \mathbb{F}_n^+$, $\zeta \in Z$ and $\lambda \in \mathbb{T}$ such that

$$T_v S_u \zeta = \lambda \zeta_l \text{ and } T_w T_v S_u = S_{\gamma^{-1}(wv)} T_{\gamma(u)}.$$

We claim that $T_{\gamma(u)}\zeta$ is in $\mathcal{V}$. This will complete the proof.

In order to show that $T_{\gamma(u)}\zeta \in \mathcal{V}$ note that we can find some $w' \in \mathbb{F}_\theta^+$ so that $T_{w'} S_u \zeta \in \mathcal{V}$ and $|w'| = |w| + |v| = b$, since $S_u\zeta \in \mathcal{V}$ by choice of u. But $T_{w'} S_u \zeta = S_{\gamma^{-1}(w')} T_{\gamma(u)} \zeta$. It follows that $T_{\gamma(u)}\zeta$ lies in $\mathcal{V}$.

□

There are examples of finite dimensional, fully coisometric representations which are not partially isometric.

Example 2.4.17. Let $\theta \in S_{2\times 2}$ be the permutation defined by $\theta(1,1) = (1,2)$, $\theta(1,2) = (1,1)$, $\theta(2,1) = (2,2)$ and $\theta(2,2) = (2,1)$, and let $\mathbb{F}_\theta^+$ be the single vertex 2-graph defined by θ. Let $[a_1, a_2]$ be a defect free row-contraction on a finite dimensional Hilbert space $\mathcal{V}$ and $[b_1, b_2]$ be a defect free row-contraction on a finite dimensional Hilbert space $\mathcal{W}$. We will define a representation of $\mathbb{F}_\theta^+$ on $\mathcal{V} \otimes \mathcal{W}^{(2)}$. Define

$$A_1 = a_1 \otimes \begin{bmatrix} 0 & I_\mathcal{W} \\ I_\mathcal{W} & 0 \end{bmatrix} \qquad A_2 = a_2 \otimes \begin{bmatrix} 0 & I_\mathcal{W} \\ I_\mathcal{W} & 0 \end{bmatrix}$$
$$B_1 = I_\mathcal{V} \otimes \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \qquad B_2 = I_\mathcal{V} \otimes \begin{bmatrix} b_2 & 0 \\ 0 & b_1 \end{bmatrix}.$$

Then $[A_1, A_2]$ and $[B_1, B_2]$ define a finite dimensional, fully coisometric representation of $\mathbb{F}_\theta^+$.

Let $\mathcal{V} = \mathcal{W} = \mathbb{C}^2$ and let

$$a_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad a_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$b_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad b_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Construct $[A_1, A_2]$ and $[B_1, B_2]$ as above. Let $\mathcal{U}$ be the minimal cyclic coinvariant subspace for the row-contraction $[A_1, A_2]$. A calculation shows that

$$\mathcal{U} = \operatorname{span}\{e_1, e_3, e_5, e_7\},$$

where $\{e_1, \ldots, e_8\}$ is the standard orthonormal basis for $\mathbb{C}^8$. However, we have that $B_1^* e_1 = e_2 \notin \mathcal{U}$, and so $\mathcal{U}$ is not the minimal cyclic coinvariant subspace for the representation of $\mathbb{F}_\theta^+$ defined by $[A_1, A_2]$ and $[B_1, B_2]$. In fact, the minimal cyclic coinvariant subspace for this representation is all of $\mathbb{C}^8$. This example shows that atomic representations are special in not needing $\mathbf{m} \geq (1, 1)$ in order to satisfy Theorem 2.3.19. It is not true of all representations single vertex 2-graphs.

The construction above works because the permutation θ is very simple. Precisely, if we fix i, θ satisfies $\theta(i, j) = \theta(i, j')$, i.e. i is not changed. Similar constructions of fully coisometric representations of 2-graphs will work for any 2-graph defined by a permutation satisfying this condition. These representations will be doubly commuting.

A general method of constructing finite dimensional, fully coisometric representations of 2-graphs which are not partially isometric has proved hard to find. We give below an example of finite dimensional, fully coisometric representation of a 2-graph which is not doubly commuting.

Example 2.4.18. Let $[A_1, A_2]$ and $[B_1, B_2, B_3]$ be row-contractions on $\mathbb{C}^3$ with

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

Then $[A_1, A_2]$ and $[B_1, B_2, B_3]$ define a fully coisometric representation of $\mathbb{F}_\theta^+$ on $\mathbb{C}^3$ where $\theta \in S_{2\times 3}$ is the cycle

$$((1, 1), (2, 3), (1, 2), (1, 3)).$$

This fully coisometric representation is not doubly commuting.

It is not hard to see that the minimal cyclic coinvariant space for this representation is $\mathbb{C}^2 = \operatorname{span}\{e_1, e_2\}$, where $\{e_1, e_2, e_3\}$ is the standard orthonormal basis for $\mathbb{C}^3$.

Chapter 3

Nonself-adjoint semicrossed products by abelian semigroups

3.1 Introduction

The study of nonself-adjoint semicrossed products began with Arveson [2]. They were further studied by McAsey, Muhly and Saito [45]. In both cases the algebras were described concretely. Peters [55] described the nonself-adjoint semicrossed products as universal algebras for covariant representations. In recent years, Davidson and Katsoulis have shown nonself-adjoint semicrossed products have proven to be a particularly interesting and tractable class of operator algebras [11, 13, 12, 15, 14]. In particular, nonself-adjoint semicrossed product algebras have been shown to be a class where the C^*-envelope is often calculable.

The C^*-envelope of an operator algebra $\mathcal{A}$ was introduced by Arveson [3, 4] as a noncommutative analogue of Shilov boundaries. The existence of the C^*-envelope was first discovered by Hamana [31]. Dritschel and McCullough [23] have since provided an alternative proof of the existence of the C^*-envelope. The viewpoint of Dritschel and McCullough has allowed for the explicit calculation of the C^*-envelope of many operator algebras. In particular, for nonself-adjoint semicrossed products the C^*-envelopes have been studied in [13, 35, 36, 24, 25].

In this chapter we study the nonself-adjoint semicrossed product algebras by semigroups of the form $\mathcal{S} = \sum_{i=1}^{\oplus k} \mathcal{S}_i$, where for each $i \in I$ we have $\mathcal{S}_i$ is a countable subsemigroup of the additive semigroup $\mathbb{R}_+$ containing 0. Our algebras will be universal for Nica-covariant representations, i.e. those representations $\{T_s\}_{s\in\mathcal{S}}$ satisfying $T_s^*T_t = T_tT_s^*$ when $s \wedge t = 0$. Semicrossed product algebras associated to Nica-covariant representations have been widely studied in the C^*-algebra literature [51, 42, 26].

The chapter is divided into three sections. In section 2 Nica-covariant representations are studied independent from dynamical systems. The results of this section may be of interest, even to those not concerned with nonself-adjoint semicrossed products. We show that contractive Nica-covariant representations can be dilated to isometric Nica-covariant representations. This result is well-known for the case of the semigroup $\mathbb{Z}_+^k$, see e.g. [72]. The proof of the existence of an isometric dilation presented here relies on the use of a generalisation of the Schur Product Theorem, and so provides an alternative proof to what is usually presented for $\mathbb{Z}_+^k$.

In section 3 nonself-adjoint semicrossed product algebras are introduced. In section 3.1 we extend our dilation result from section 2 to representations of semicrossed products of C^*-algebras. This result allows us to conclude strong results comparing the different types of semicrossed product algebras. For example, Corollary 3.3.7, tells us that, in the case of a semicrossed product of a C^*-algebra, the universal algebra for *completely isometric* Nica-covariant representations is the same as the universal algebra for *completely contractive* Nica-covariant representations. If we were to work with completely isometric and completely contractive semicrossed algebras without imposing the condition of Nica-covariance on our semigroup representations, then an example due to Varopoulos [73] would show that the analogy of Corollary 3.3.7 would fail in this setting.

In the section 3.2 we consider the C^*-envelope of the isometric semicrossed product algebras. In Theorem 3.3.15 we calculate the C^*-envelope of the isometric semicrossed product as

$$C^*_{env}(\mathcal{A}\,{}_N\!\times_\alpha^{iso}\mathcal{S}) \cong C^*_{env}(\mathcal{A}) \times_\alpha \mathcal{G},$$

where $\mathcal{G}$ is the group generated by $\mathcal{S}$. This result generalises a recent result of Kakariadis and Katsoulis [36], where they worked with the semigroup $\mathcal{S} = \mathbb{Z}_+$.

When $\mathcal{A}$ is a C^*-algebra the Nica-covariance requirement on our representations allows us to view the semicrossed product algebra $\mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$ as a tensor algebra for a product system of C^*-correspondences over $\mathcal{S}$. Thus, from this viewpoint we unite a recent result of Duncan and Peters [25] on the C^*-envelope of a tensor algebra associated with a dynamical system and the results of Kakariadis and Katsoulis on the C^*-envelope of the isometric semicrossed product for a dynamical system.

3.2 Nica-covariant representations

Let $\mathcal{S}$ be the semigroup $\mathcal{S} = \sum_{i\in I}^{\oplus} \mathcal{S}_i$, where for each $i \in I$ we have $\mathcal{S}_i$ is a subsemigroup in the additive semigroup $\mathbb{R}_+$ containing 0. We further assume throughout that $\mathcal{S}$ is the positive cone of the group $\mathcal{G}$ it generates. Denote by $\wedge$ and $\vee$ the join and meet operations on the lattice group $\mathcal{G}$. In section 3 we will be looking at the case when $\mathcal{S}$ is countable. However, we will not need to assume that $\mathcal{S}$ is countable in this section.

Definition 3.2.1. A representation $T : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{S}$ by contractions $\{T_s\}_{s\in\mathcal{S}}$ on a Hilbert space $\mathcal{H}$ is *Nica-covariant* when we have the following relation: if $s \in \mathcal{S}_i$ and $t \in \mathcal{S}_j$ where $i \neq j$ then $T_s^* T_t = T_t T_s^*$.

3.2.1 Isometric Dilations

We wish to show that every Nica-covariant contractive representation of $\mathcal{S}$ can be dilated to an isometric representation. Further, we will show that there is a unique minimal isometric dilation which is Nica-covariant. This result is well known in the discrete $\mathcal{S} = \mathbb{Z}_+^k$ case. If each $\mathcal{S}_i$ is *commensurable*, i.e. if for all $s_1, \ldots, s_n \in \mathcal{S}_i$ there exists $s_0 \in \mathcal{S}_i$ and $k_1, \ldots, k_n \in \mathbb{N}$ such that $s_i = k_i s_0$, then these results have been described by Shalit [63]. We do not impose the condition of commensurability.

The key method to show the existence of the dilation is to use a generalisation of the Schur Product Theorem. To show that there is a minimal Nica-covariant isometric dilation we follow arguments similar to those of Solel [69].

Definition 3.2.2. Let $A = [A_{i,j}]_{1\le i,j\le m}$ and $B = [B_{i,j}]_{1\le i,j\le m}$ be two matrices of operators where each $A_{i,j}$ and $B_{i,j}$ is a bounded operator on a Hilbert space $\mathcal{H}$. The *operator-valued Schur product* of A and B is defined by $A \Box B := [A_{i,j} B_{i,j}]_{1\le i,j\le m}$.

In the above definition, if $\mathcal{H}$ is 1-dimensional then the operation $\Box$ is simply the classical Schur product (or entry-wise product). In the following theorem we will generalise the Schur Product Theorem, which says that the Schur product of two positive matrices is positive. See e.g. [53, Chapter 3].

Theorem 3.2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras in $\mathcal{B}(\mathcal{H})$ such that $\mathcal{A} \subseteq \mathcal{B}'$. Let $A = [A_{i,j}]_{1\le i,j\le m}$ and $B = [B_{i,j}]_{1\le i,j\le m}$ be operator matrices with all $A_{i,j} \in \mathcal{A}$ and $B_{i,j} \in \mathcal{B}$. If $A \ge 0$ and $B \ge 0$ then $A \Box B \ge 0$.*

Proof. Let $\tilde{A} = A \otimes I_m$ and $\tilde{B} = [B_{i,j} \otimes I_m]_{1\le i,j\le m}$. Hence $\tilde{A}$ and $\tilde{B}$ are of the form

$$\tilde{A} = \begin{bmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & A \end{bmatrix}$$

and

$$\tilde{B} = \begin{bmatrix} B_{1,1} & 0 & \dots & 0 & \dots & B_{1,m} & 0 & \dots & 0 \\ 0 & B_{1,1} & \dots & 0 & \dots & 0 & B_{1,m} & \dots & 0 \\ \vdots & & \ddots & \vdots & & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & B_{1,1} & \dots & 0 & 0 & \dots & B_{1,m} \\ \vdots & & & & \vdots & & & & \vdots \\ B_{m,1} & 0 & \dots & 0 & \dots & B_{m,m} & 0 & \dots & 0 \\ 0 & B_{m,1} & \dots & 0 & \dots & 0 & B_{m,m} & \dots & 0 \\ \vdots & & \ddots & \vdots & & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & B_{m,1} & \dots & 0 & 0 & \dots & B_{m,m} \end{bmatrix}$$

It follows that $\tilde{A}$ and $\tilde{B}$ are positive commuting operators. Hence $\tilde{A}\tilde{B}$ is positive.

For each $1 \leq k \leq m$, let P_k be the projection onto the $(m(k-1)+k)^{th}$ copy of $\mathcal{H}$ in $\mathcal{H}^{(m^2)}$, and let $P = \sum_{k=1}^{m} P_k$. Define $R : \mathcal{H}^{(m)} \to \mathcal{H}^{(m^2)}$ by $R\mathbf{h} = P(\mathbf{h}^{\otimes m})$. Hence R is an isometry and for

$$\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix} \in \mathcal{H}^{(m)}$$

we have

$$R\mathbf{h} = \begin{bmatrix} h_1 \\ 0 \\ \vdots \\ 0 \\ h_2 \\ 0 \\ \vdots \\ 0 \\ h_m \end{bmatrix},$$

with m zeroes between h_i and h_{i+1}, $1 \leq i < m$. It follows that $R^*(\tilde{A}\tilde{B})R = A \Box B$. Thus, $A \Box B$ is positive. $\Box$

Let T be a Nica-covariant contractive representation of $\mathcal{S}$ on $\mathcal{H}$. We extend T to a map on all of $\mathcal{G}$ in the following way. Any element $g \in \mathcal{G}$ can be written uniquely as $g = g_+ - g_-$ where $g_-, g_+ \in \mathcal{S}$ and $g_- \wedge g_+ = 0$. Thus we extend T to $\mathcal{G}$ by setting $T_g = T_{g_-}^* T_{g_+} = T_{g_+} T_{g_-}^*$. A well-known theorem of Sz.-Nagy says that T has an isometric dilation if and only if for $s_1, \dots, s_n \in \mathcal{S}$ the operator matrix $[T_{s_j - s_i}]_{1\le i,j\le n}$ is positive (see e.g. [71, Theorem 7.1]). We will need to look more closely at the proof of this later.

In the case when is $\mathcal{S}$ is a subsemigroup of $\mathbb{R}_+$ it has been proved by Mlak [46] that a contractive representation T has an isometric dilation. In the following theorem we will rely on the fact that the representation T restricted to $\mathcal{S}_i$ has an isometric dilation for each i. Then an invocation of Theorem 3.2.3 will give us our result.

Theorem 3.2.4. *Let T be a Nica-covariant contractive representation of the semigroup $\mathcal{S} = \sum_{i\in I}^{\oplus} \mathcal{S}_i$, where each $\mathcal{S}_i$ is subsemigroup of $\mathbb{R}_+$ containing 0. Then T has an isometric dilation.*

Proof. Take $s_1, \ldots, s_n$ in $\mathcal{S}$. By [71, Theorem 7.1] it suffices to show that the operator matrix $[T_{s_j - s_i}]_{1\le i,j\le n}$ is positive. Each s_j is of the form $s_j = \sum_{i\in I} s_j^{(i)}$, where $s_j^{(i)}$ is in $\mathcal{S}_i$. We can choose a finite subset $F \subseteq I$ such that $s_j = \sum_{i\in F} s_j^{(i)}$ for $j = 1, \ldots, n$. Since F is finite we can and will relabel F by $\{1, \ldots, k\}$ for some k. Denote by $T^{(j)}$ the restriction of T to $\mathcal{S}_j$.

By the Nica-covariance property

$$T_{s_j - s_i} = T^{(1)}_{s_j^{(1)} - s_i^{(1)}} \cdots T^{(k)}_{s_j^{(k)} - s_i^{(k)}}.$$

Thus, we can factor the operator matrix $[T_{s_j - s_i}]$ as

$$\begin{aligned}
[T_{s_j - s_i}]_{i,j} &= \left[T^{(1)}_{s_j^{(1)} - s_i^{(1)}} \cdots T^{(k)}_{s_j^{(k)} - s_i^{(k)}}\right]_{i,j} \\
&= \left[T^{(1)}_{s_j^{(1)} - s_i^{(1)}}\right]_{i,j} \square \ldots \square \left[T^{(k)}_{s_j^{(k)} - s_i^{(k)}}\right]_{i,j}.
\end{aligned}$$

Since $\big[T^{(l)}_{s_j^{(l)} - s_i^{(l)}}\big]_{i,j}$ is a positive matrix for $1 \le l \le k$ [46] and since the representation is Nica-covariant, it follows by Theorem 3.2.3 that $[T_{s_j - s_i}]_{i,j}$ is positive. $\square$

In the above we made use of [71, Theorem 7.1] to guarantee the existence of a dilation. We will now pay closer attention to how the dilation there is constructed. Then, following similar arguments of [69], we will show that there is a unique minimal Nica-covariant isometric dilation.

Theorem 3.2.5. *Let T be a Nica-covariant contractive representation of the semigroup $\mathcal{S} = \sum_{i\in I}^{\oplus} \mathcal{S}_i$, where each $\mathcal{S}_i$ is subsemigroup of $\mathbb{R}_+$ containing 0. Then T has a minimal isometric dilation which is Nica-covariant. Further, this dilation is unique.*

Proof. We first sketch the details of the construction of an isometric dilation. Let $\mathcal{H}$ be the space on which the representation T acts. Let $\mathcal{K}_0$ denote the space of all finitely non-zero functions $f : \mathcal{S} \to \mathcal{H}$. For $f, g \in \mathcal{K}_0$ we define

$$\langle f, g \rangle = \sum_{s,t \in \mathcal{S}} \langle T_{t-s} f(t), g(s) \rangle.$$

By Theorem 3.2.4 this defines a positive semidefinite sesquilinear form on $\mathcal{K}_0$. Let

$$\begin{aligned}\mathcal{N} &= \{f \in \mathcal{K}_0 : \langle f, f \rangle = 0\} \\ &= \{f \in \mathcal{K}_0 : \langle f, g \rangle = 0 \text{ for all } g \in \mathcal{K}_0\},\end{aligned}$$

and set $\mathcal{K} = \overline{\mathcal{K}_0/\mathcal{N}}$, where the closure is taken with respect to the norm induced by $\langle \cdot, \cdot \rangle$. We isometrically embed $\mathcal{H}$ in $\mathcal{K}$ by the map $h \mapsto \hat{h}$, where $\hat{h}(s) = \delta_0(s) h$, where $\delta_0(s) = 1$ if $s = 0$ and 0 otherwise.

Now define maps V_s on $\mathcal{K}_0$ by $(V_s f)(t) = f(t - s)$ if $t - s \in \mathcal{S}$ and $(V_s f)(t) = 0$ otherwise. Note that for $f \in \mathcal{K}_0$ and $u \in \mathcal{S}$ we have

$$\begin{aligned}\langle V_u f, V_u f \rangle &= \sum_{s,t} \langle T_{t-s} f(t-u), f(s-u) \rangle \\ &= \sum_{s,t \geq u} \langle T_{t-s} f(t-u), f(s-u) \rangle \\ &= \sum_{s,t \geq 0} \langle T_{(t+u)-(s+u)} f(t), f(s) \rangle \\ &= \sum_{s,t} \langle T_{t-s} f(t), f(s) \rangle = \langle f, f \rangle.\end{aligned}$$

Hence each V_u is isometric on $\mathcal{K}_0$ and leaves $\mathcal{N}$ invariant. It follows that we can extend V_u to an isometry on $\mathcal{K}$ and we have that $\{V_s\}_{s \in \mathcal{S}}$ is an isometric representation of $\mathcal{S}$.

Further, note that for $g \in \mathcal{G}$ and $h, k \in \mathcal{H}$ we have

$$\begin{aligned}\langle V_g \hat{h}, \hat{k} \rangle = \langle V_{g_-}^* V_{g_+} \hat{h}, \hat{k} \rangle &= \langle V_{g_+} \hat{h}, V_{g_-} \hat{k} \rangle \\ &= \sum_{s,t \in \mathcal{S}} \langle T_{t-s} \hat{h}(t - g_+), \hat{k}(s - g_-) \rangle \\ &= \langle T_{g_+ - g_-} h, k \rangle = \langle T_g h, k \rangle.\end{aligned}$$

Thus we have $P_{\mathcal{H}} V_g|_{\mathcal{H}} = T_g$ for all $g \in \mathcal{G}$. In particular $\{V_s\}_{s \in \mathcal{S}}$ is an isometric dilation of $\{T_s\}_{s \in \mathcal{S}}$. It is easily seen to be a minimal isometric dilation. Dilations with the property that $P_{\mathcal{H}} V_g|_{\mathcal{H}} = T_g$ are called *regular* dilations. We want to show that this dilation is Nica-covariant.

Next we will show that if we have $s \in \mathcal{S}_i$ and $\mu \in \mathcal{S}$ such that $s \wedge \mu = 0$ then $V_s^* V_\mu|_{\mathcal{H}} = V_\mu V_s^*|_{\mathcal{H}}$. Take s, μ as described, $\nu \in \mathcal{S}$ and $h, k \in \mathcal{H}$. By the minimality of the dilation it suffices to show that

$$\langle V_s^* V_\mu \hat{h}, V_\nu \hat{k} \rangle = \langle V_\mu V_s^* \hat{h}, V_\nu \hat{k} \rangle.$$

We calculate

$$\begin{aligned}
\langle V_s^* V_\mu \hat{h}, V_\nu \hat{k} \rangle &= \langle V_\nu^* V_s^* V_\mu \hat{h}, \hat{k} \rangle \\
&= \langle V_{(\mu-\nu-s)_-}^* V_{(\mu-\nu-s)_+} \hat{h}, \hat{k} \rangle \\
&= \langle T_{(\mu-\nu-s)_-}^* T_{(\mu-\nu-s)_+} \hat{h}, \hat{k} \rangle.
\end{aligned}$$

Note that, by our choice of s and μ we have that $(\mu-\nu-s)_+ = (\mu-\nu)_+$ and $(\mu-\nu-s)_- = s+(\mu-\nu)_-$. Also $s \wedge (\mu-\nu)_+ = 0$. Thus

$$\begin{aligned}
\langle V_s^* V_\mu \hat{h}, V_\nu \hat{k} \rangle &= \langle T_{(\mu-\nu-s)_-}^* T_{(\mu-\nu-s)_+} \hat{h}, \hat{k} \rangle \\
&= \langle T_{s+(\mu-\nu)_-}^* T_{(\mu-\nu)_+} \hat{h}, \hat{k} \rangle \\
&= \langle T_{(\mu-\nu)_-}^* T_{(\mu-\nu)_+} T_s^* \hat{h}, \hat{k} \rangle \\
&= \langle P_{\mathcal{H}} V_{(\mu-\nu)_-}^* V_{(\mu-\nu)_+} P_{\mathcal{H}} V_s^* \hat{h}, \hat{k} \rangle \\
&= \langle V_{(\mu-\nu)_-}^* V_{(\mu-\nu)_+} V_s^* \hat{h}, \hat{k} \rangle = \langle V_\mu V_s^* \hat{h}, V_\nu \hat{k} \rangle.
\end{aligned}$$

This tells us that the representation V has the Nica-covariant property when restricted to $\mathcal{H}$. We will now extend this to all of $\mathcal{K}$.

By the minimality of the representation V it suffices to show that for $s \in S_i$, $t \in S_j$ where $i \neq j$, $\mu, \nu \in \mathcal{S}$ and $h, k \in \mathcal{H}$ that

$$\langle V_s^* V_t V_\mu \hat{h}, V_\nu \hat{k} \rangle = \langle V_t V_s^* V_\mu \hat{h}, V_\nu \hat{k} \rangle.$$

The right-hand side of the above is

$$\begin{aligned}
\langle V_t V_s^* V_\mu \hat{h}, V_\nu \hat{k} \rangle &= \langle V_\nu^* V_t V_s^* V_\mu \hat{h}, \hat{k} \rangle \\
&= \langle V_\nu^* V_t V_{(\mu-s)_-}^* V_{(\mu-s)_+} \hat{h}, \hat{k} \rangle \\
&= \langle V_\nu^* V_t V_{(\mu-s)_+} V_{(\mu-s)_-}^* \hat{h}, \hat{k} \rangle.
\end{aligned}$$

Note that $t + (\mu - s)_+ = (t + \mu - s)_+$ and $(\mu - s)_- = (t + \mu - s)_-$, hence we have

$$\begin{aligned}
\langle V_t V_s^* V_\mu \hat{h}, V_\nu \hat{k} \rangle &= \langle V_\nu^* V_{(t+\mu-s)_-}^* V_{(t+\mu-s)_+} \hat{h}, \hat{k} \rangle \\
&= \langle V_{\nu+(t+\mu-s)_-}^* V_{(t+\mu-s)_+} \hat{h}, \hat{k} \rangle \\
&= \langle V_{(t+\mu-\nu-s)_-}^* V_{(t+\mu-\nu-s)_+} \hat{h}, \hat{k} \rangle,
\end{aligned}$$

with the last equality coming from the fact that

$$((t+\mu-s)_+ - (t+\mu-s)_- - \nu)_- = (t+\mu-s-\nu)_-$$

and

$$((t+\mu-s)_+ - (t+\mu-s)_- - \nu)_+ = (t+\mu-s-\nu)_+.$$

Hence

$$\begin{aligned}\langle V_t V_s^* V_\mu \hat{h}, V_\nu \hat{k}\rangle &= \langle V^*_{(t+\mu-\nu-s)_-} V_{(t+\mu-\nu-s)_+} \hat{h}, \hat{k}\rangle \\ &= \langle V_\nu^* V_s^* V_t V_\mu \hat{h}, \hat{k}\rangle \\ &= \langle V_s^* V_t V_\mu \hat{h}, V_\nu \hat{k}\rangle.\end{aligned}$$

It follows that V is Nica-covariant.

To show that the dilation is unique we follow a standard argument. Suppose V and W are two minimal isometric Nica-covariant dilations of T on $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Take $h_1, h_2 \in \mathcal{H}$ and $\nu, \mu \in \mathcal{S}$. Then

$$\begin{aligned}\langle V_\mu h_1, V_\nu h_2\rangle &= \langle V_\nu^* V_\mu h_1, h_2\rangle \\ &= \langle V^*_{(\mu-\nu)_-} V_{(\mu-\nu)_+} h_1, h_2\rangle \\ &= \langle T_{\mu-\nu} h_1, h_2\rangle.\end{aligned}$$

Similarly $\langle W_\mu h_1, W_\nu h_2\rangle = \langle T_{\mu-\nu} h_1, h_2\rangle$. Thus the map $U : V_\nu h \mapsto W_\nu h$ extends to a unitary from $\mathcal{K}_1$ to $\mathcal{K}_2$ which fixes $\mathcal{H}$, and the two dilations V and W are unitarily equivalent. □

3.3 Semicrossed product algebras

Throughout let $\mathcal{S}$ be the semigroup $\mathcal{S} = \sum_{i=1}^{\oplus k} \mathcal{S}_i$ where each $\mathcal{S}_i$ is a countable subsemigroup of $\mathbb{R}_+$ containing 0. Further we suppose that $\mathcal{S}$ is the positive cone of the group $\mathcal{G}$ generated by $\mathcal{S}$.

Definition 3.3.1. Let $\mathcal{A}$ be a unital operator algebra. If $\alpha = \{\alpha_s : s \in \mathcal{S}\}$ is a family of completely isometric unital endomorphisms of $\mathcal{A}$ forming an action of $\mathcal{S}$ on $\mathcal{A}$ then we call the triple $(\mathcal{A}, \mathcal{S}, \alpha)$ a *semigroup dynamical system.*

Definition 3.3.2. Let $(\mathcal{A}, \mathcal{S}, \alpha)$ be a semigroup dynamical system. An *isometric (contractive) Nica-covariant representation* of $(\mathcal{A}, \mathcal{S}, \alpha)$ on a Hilbert space $\mathcal{H}$ consists of a pair (σ, V) where σ is a completely contractive representation $\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $V = \{V_s\}_{s\in\mathcal{S}}$ is an isometric (contractive) Nica-covariant representation of $\mathcal{S}$ on $\mathcal{H}$ such that

$$\sigma(A) V_s = V_s \sigma(\alpha_s(A))$$

for all $A \in \mathcal{A}$ and $s \in \mathcal{S}$.

We will be interested in two nonself-adjoint semicrossed product algebras associated to a semigroup dynamical system $(\mathcal{A}, \mathcal{S}, \alpha)$. We define $\mathcal{A}\,{}_N{\times}_\alpha\,\mathcal{S}$ to be the universal algebra for all contractive Nica-covariant representations of $(\mathcal{A}, \mathcal{S}, \alpha)$ and $\mathcal{A}\,{}_N{\times}_\alpha^{iso}\,\mathcal{S}$ to be the universal algebra for all isometric Nica-covariant representations of $(\mathcal{A}, \mathcal{S}, \alpha)$.

The algebras $\mathcal{A} \times_\alpha^{iso} \mathbb{Z}_+$ were introduced by Kakariadis and Katsoulis [36] and have proven to be a more tractable class of algebras than $\mathcal{A} \times_\alpha \mathbb{Z}$. While in general one expects $\mathcal{A}\,{}_N{\times}_\alpha\,\mathcal{S}$ and $\mathcal{A}\,{}_N{\times}_\alpha^{iso}\,\mathcal{S}$ to be different there are times when the two algebras coincide. For example, when $\mathcal{A} = \mathfrak{A}_n$ is the noncommutative disc algebra and $\mathcal{S} = \mathbb{Z}_+$ it follows from [13] that

$$\mathfrak{A}_n \times_\alpha^{iso} \mathbb{Z}_+ \cong \mathfrak{A}_n \times_\alpha \mathbb{Z}_+.$$

Further examples of when the semicrossed product and the isometric semicrossed product are the same for the case $\mathcal{S} = \mathbb{Z}_+$ can be found in [14, Section 12]. When $\mathcal{A}$ is a unital C^*-algebra we will see (Corollary 3.3.7) that

$$\mathcal{A}\,{}_N{\times}_\alpha^{iso}\,\mathcal{S} \cong \mathcal{A}\,{}_N{\times}_\alpha\,\mathcal{S}.$$

Let $\mathcal{P}(\mathcal{A}, \mathcal{S})$ be the algebra of all formal polynomials p of the form

$$p = \sum_{i=1}^{n} \mathcal{V}_{s_i} A_{s_i}$$

where $s_1, \ldots, s_n$ are in $\mathcal{S}$, with multiplication defined by $A\mathcal{V}_s = \mathcal{V}_s \alpha(A)$. If (σ, T) is a contractive Nica-covariant representation of $(\mathcal{A}, \mathcal{S}, \alpha)$ then we can define a representation $\sigma \times T$ of $\mathcal{P}(\mathcal{A}, \mathcal{S})$ by

$$(\sigma \times T)\left(\sum_{i=1}^{n} \mathcal{V}_{s_i} A_{s_i}\right) = \sum_{i=1}^{n} T_{s_i} \sigma(A_{s_i}).$$

We define two norms on $\mathcal{P}(\mathcal{A}, \mathcal{S})$ as follows. For $p \in \mathcal{P}(\mathcal{A}, \mathcal{S})$ let

$$\|p\| = \sup_{\substack{(\sigma,T) \text{ contractive} \\ \text{Nica-covariant}}} \left\{ \|(\sigma \times T)(p)\| \right\}$$

and

$$\|p\|_{iso} = \sup_{\substack{(\sigma,V) \text{ isometric} \\ \text{Nica-covariant}}} \left\{ \|(\sigma \times V)(p)\| \right\}.$$

We can realise our semicrossed product algebras as

$$\mathcal{A}\,{}_N{\times}_\alpha\,\mathcal{S} = \overline{\mathcal{P}(\mathcal{A}, \mathcal{S})}^{\|\cdot\|}$$

and

$$\mathcal{A}\,{}_N{\times}_\alpha^{iso}\,\mathcal{S} = \overline{\mathcal{P}(\mathcal{A}, \mathcal{S})}^{\|\cdot\|_{iso}}.$$

If $(\mathcal{B}, \mathcal{G}, \beta)$ is a dynamical system where β is an action of the group $\mathcal{G}$ on the C^*-algebra $\mathcal{B}$ by automorphisms there is an adjoint operation on $\mathcal{P}(\mathcal{B}, \mathcal{G})$ given by $(\mathcal{V}_g B)^* := \mathcal{V}_{-g}\beta_g^{-1}(B^*)$. If (π, U) is covariant representation of $(\mathcal{B}, \mathcal{G}, \beta)$, then $\{U_s\}_{s\in\mathcal{S}}$ is necessarily a family of commuting unitaries, and hence $\{U_s\}_{s\in\mathcal{S}}$ is automatically Nica-covariant.

Example 3.3.3. Let $(\mathcal{A}, \mathcal{S}, \alpha)$ be a semigroup dynamical system. Let σ be a completely contractive representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Define a completely contractive representation $\tilde{\sigma}$ of $\mathcal{A}$ on $\mathcal{H} \otimes \ell^2(\mathcal{S})$ by

$$\tilde{\sigma}(A)(h_s)_{s\in\mathcal{S}} = (\sigma(\alpha_s(A))h_s)_{s\in\mathcal{S}}$$

for all $A \in \mathcal{A}$ and $(h_s)_{s\in\mathcal{S}} \in \mathcal{H} \otimes \ell^2(\mathcal{S})$.

For each $s \in \mathcal{S}$ define an operator W_s on $\mathcal{H} \otimes \ell^2(\mathcal{S})$ by

$$W_s(h)_t = (h)_{s+t},$$

where $h \in \mathcal{H}$ and $(h)_s \in \mathcal{H} \otimes \ell^2(\mathcal{S})$ is the vector with h in the s^{th} position and 0 everywhere else. Then $(\tilde{\sigma}, W)$ is an isometric Nica-covariant representation of $(\mathcal{A}, \mathcal{S}, \alpha)$.

Note that in the case where each α_s is an automorphism on $\mathcal{A}$ then we can extend this idea to give a Nica-covariant representation $(\hat{\sigma}, U)$ on $\mathcal{H} \otimes \ell^2(\mathcal{G})$ where each U_s is unitary.

Definition 3.3.4. The isometric Nica-covariant representation $(\tilde{\sigma}, W)$ constructed above is called an *induced representation* of $(\mathcal{A}, \mathcal{S}, \alpha)$.

3.3.1 Dilations of Nica-covariant representations

We now consider some dilation results for Nica-covariant representations of a semigroup dynamical system $(\mathcal{A}, \mathcal{S}, \alpha)$ in the case when $\mathcal{A}$ is a C^*-algebra.

In the case that $\mathcal{S} = \mathbb{Z}_+^k$ the following theorem is a special case of a theorem of Solel's [69, Theorem 3.1] which deals with representations of product systems of C^*-correspondences. The result has also been shown by Ling and Muhly [44] for the case $\mathcal{S} = \mathbb{Z}_+^k$ and α is an action on $\mathcal{A}$ by automorphisms.

Theorem 3.3.5. *Let $\mathcal{S} = \sum_{i\in I}^{\oplus} \mathcal{S}_i$ where each $\mathcal{S}_i$ is a countable subsemigroup of $\mathbb{R}_+$ containing 0 and let $(\mathcal{A}, \mathcal{S}, \alpha)$ be a semigroup dynamical system where $\mathcal{A}$ is a unital C^*-algebra. Let (σ, T) be a contractive Nica-covariant representation of $(\mathcal{A}, \mathcal{S}, \alpha)$ on $\mathcal{H}$. Then there is an isometric Nica-covariant representation (π, V) of $(\mathcal{A}, \mathcal{S}, \alpha)$ on $\mathcal{K} \supseteq \mathcal{H}$ such that*

1. *$\pi(A)|_{\mathcal{H}} = \sigma(A)$ for all $A \in \mathcal{A}$*
2. *$P_{\mathcal{H}} V_s|_{\mathcal{H}} = T_s$ for all $s \in \mathcal{S}$.*

Further $\mathcal{K}$ is minimal in the sense that $\mathcal{K} = \bigvee_{s\in\mathcal{S}} V_s\mathcal{H}$.

Proof. Let $\mathcal{K}_0$, $\mathcal{K}$ and $\mathcal{N}$ be as in the proof of Theorem 3.2.5. For each $A \in \mathcal{A}$ we define $\pi_0(A)$ on $\mathcal{K}_0$ by

$$(\pi_0(A)f)(s) = \sigma(\alpha_s(A))f(s),$$

for each $f \in \mathcal{K}_0$ and $s \in \mathcal{S}$. Note that, for $A \in \mathcal{A}$ and $t, s \in \mathcal{S}$ we have

$$\begin{aligned} T_{t-s}\sigma(\alpha_t(A)) &= T_{(t-s)_+}T^*_{(t-s)_-}\sigma(\alpha_t(A)) \\ &= T_{(t-s)_+}\sigma(\alpha_{t+(t-s)_-}(A))T^*_{(t-s)_-} \\ &= \sigma(\alpha_{t+(t-s)_- -(t-s)_+}(A))T^*_{(t-s)_-}T_{(t-s)_+} \\ &= \sigma(\alpha_s(A))T_{t-s}. \end{aligned}$$

It follows that, if $f \in \mathcal{N}$ and $g \in \mathcal{K}_0$ then for each $A \in \mathcal{A}$,

$$\begin{aligned} \langle \pi_0(A)f, g\rangle &= \sum_{s,t}\langle T_{t-s}\sigma(\alpha_t(A))f(t), g(s)\rangle \\ &= \sum_{s,t}\langle T_{t-s}f(t), \sigma(\alpha_s(A^*))g(s)\rangle = 0, \end{aligned}$$

we thus can extend π_0 to a representation π

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K}).$$

It is easy to check that (π, V) form a Nica-covariant representation with the desired properties. □

Remark 3.3.6. In the case when $\mathcal{S} = \sum^{\oplus}_{i\in I}\mathcal{S}_i$ where each $\mathcal{S}_i$ is a subsemigroup of $\mathbb{R}_+$ containing 0 and each $\mathcal{S}_i$ has the extra condition of being commensurable then the statement of Theorem 3.3.5 is a special case of [62, Theorem 4.2]. However, in the proof there, the only place where the commensurable condition is used is in ensuring that contractive Nica-covariant representation of $\mathcal{S}$ has minimal Nica-covariant isometric dilation. As Theorem 3.2.4 and Theorem 3.2.5 provide the existence of minimal Nica-covariant isometric dilations in the case when each $\mathcal{S}_i$ is not necessarily commensurable the proof given in [62] provides an alternate proof of Theorem 3.3.5.

Corollary 3.3.7. *Let $\mathcal{S} = \sum^{\oplus k}_{i=1}\mathcal{S}_i$ where each $\mathcal{S}_i$ is a countable subsemigroup of $\mathbb{R}_+$ containing 0 and let $(\mathcal{A}, \mathcal{S}, \alpha)$ be a semigroup dynamical system where $\mathcal{A}$ is a unital C^*-algebra. Then the norms $\|\cdot\|$ and $\|\cdot\|_{iso}$ on $\mathcal{P}(\mathcal{A}, \mathcal{S})$ are the same. Hence*

$$\mathcal{A}\,{}_N\!\times^{iso}_\alpha \mathcal{S} = \mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}.$$

Proof. Take any $p \in \mathcal{P}(\mathcal{A}, \mathcal{S})$. Since an isometric Nica-covariant representation is itself contractive it follows that $\|p\|_{iso} \leq \|p\|$. Now take a contractive Nica-covariant representation (σ, T) on a Hilbert space $\mathcal{H}$. Let (π, V) be the minimal isometric Nica-covariant dilation of (σ, T). Then

$$\|(\sigma \times T)(p)\| = \|P_{\mathcal{H}}(\pi \times V)(p)P_{\mathcal{H}}\| \leq \|(\pi \times V)(p)\|.$$

Hence $\|p\| \leq \|p\|_{iso}$. □

Remark 3.3.8. Let $(\mathcal{A}, \mathcal{S}, \alpha)$ be a semigroup dynamical system. If $\mathcal{A}$ is a C^*-algebra then $(\mathcal{A}, \mathcal{S}, \alpha)$ can be used to describe a product system of C^*-correspondences over $\mathcal{S}$. Fowler constructs a concrete C^*-algebra which is universal for Nica-covariant completely contractive representations of this product system [26]. It was observed by Solel [69] that the nonself-adjoint Banach algebra formed by the left regular representation of the product system is universal for Nica-covariant completely contractive representations (while Solel was working in $\mathbb{Z}_+^k$ the same reasoning works for countable $\mathcal{S}$). Thus $\mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$ can also be realised as the concrete tensor algebra in the sense of Solel, see [69, Corollary 3.17].

Further, if σ is a faithful representation of $\mathcal{A}$ it follows that the induced representation $(\tilde{\sigma}, W)$ is a completely isometric representation of $\mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$.

The following theorem can be proved by a standard argument in dynamical systems using direct limits of C^*-algebras. As stated below, the result is a special case of [41, Theorem 2.1] and [50, Section 2].

Theorem 3.3.9. *Let $(\mathcal{A}, \mathcal{S}, \alpha)$ be a semigroup dynamical system where $\mathcal{A}$ is a C^*-algebra and each α_s is injective. Then there exists a C^*-dynamical system $(\mathcal{B}, \mathcal{G}, \beta)$ where each β_s is an automorphism, unique up to isomorphism, together with an embedding $i : \mathcal{A} \to \mathcal{B}$ such that*

1. *$\beta_s \circ i = i \circ \alpha_s$, i.e. β dilates α*
2. *$\bigcup_{s \in \mathcal{S}} \beta_s^{-1}(i(\mathcal{A}))$ is dense in $\mathcal{B}$, i.e. $\mathcal{B}$ is minimal.*

Definition 3.3.10. Let $(\mathcal{A}, \mathcal{S}, \alpha)$ and $(\mathcal{B}, \mathcal{G}, \beta)$ be as in Theorem 3.3.9, then we call $(\mathcal{B}, \mathcal{G}, \beta)$ the *minimal automorphic dilation* of $(\mathcal{A}, \mathcal{S}, \alpha)$.

The minimal automorphic dilation of a dynamical system is frequently utilised in the literature. Group crossed product C^*-algebras have a long history and are well understood objects. Thus it is beneficial if one can relate a semicrossed algebra to a crossed product algebra, often the crossed product algebra of the minimal automorphic dilation. We will see in Theorem 3.3.15 that the minimal automorphic dilation plays an important role when calculating the C^*-envelope of crossed product algebras. First we will show now that $\mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$ sits nicely inside $\mathcal{B} \times_\beta \mathcal{G}$. In the case where $\mathcal{S} = \mathbb{Z}_+$ the following has been shown by Kakariadis and Katsoulis [36] and Peters [55].

Theorem 3.3.11. *Let $(\mathcal{A}, \mathcal{S}, \alpha)$ be a semigroup dynamical system where $\mathcal{A}$ is a C^*-algebra and each α_s is injective. Let $(\mathcal{B}, \mathcal{G}, \beta)$ be the minimal automorphic dilation of $(\mathcal{A}, \mathcal{S}, \alpha)$. The $\mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$ is completely isometrically isomorphic to a subalgebra of $\mathcal{B} \times_\beta \mathcal{G}$.*

Further, $\mathcal{A}\,{}_N\!\times_\alpha^{iso} \mathcal{S}$ generates $\mathcal{B} \times_\beta \mathcal{G}$ as a C^*-algebra.

Proof. Let σ be a faithful representation of $\mathcal{A}$ on $\mathcal{H}$. Then the induced representation $\tilde{\sigma} \times W$ is a completely isometric representation of $\mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$, by Remark 3.3.8. We will embed this completely isometric copy of $\mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$ into a completely isometric representation of $\mathcal{B} \times_\beta \mathcal{G}$ by suitably dilating the representation $(\tilde{\sigma}, W)$.

Let i be the embedding of $\mathcal{A}$ into $\mathcal{B}$ as in Theorem 3.3.9. The representation σ also defines a faithful representation of $i(\mathcal{A})$, which we will also denote by σ. We can thus find a representation π of $\mathcal{B}$ on $\mathcal{K} \supseteq \mathcal{H}$ such that $\pi(A)|_{\mathcal{H}} = \sigma(i(A))$ for all $A \in \mathcal{A}$, see e.g. [54, Proposition 4.1.8]. We thus have an induced representation $\hat{\pi} \times U$ of $\mathcal{B} \times_\beta \mathcal{G}$. Restricting π to $\mathcal{A}$ we see that $(\hat{\pi} \circ i) \times U$ is a completely isometric representation of $\mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$, since $\tilde{\sigma} \times W$ is. Further note that $\hat{\pi}$ is faithful on $\bigcup_{s\in\mathcal{S}} \beta_s^{-1}(\mathcal{A})$. By the construction of $\mathcal{B}$, $\hat{\pi}$ is also faithful representation of $\mathcal{B}$. Now, by [54, Theorem 7.7.5], $\tilde{\sigma} \times W$ is a faithful representation of $\mathcal{B} \times_\beta \mathcal{G}$. Hence $\mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$ sits completely isometrically inside $\mathcal{B} \times_\beta \mathcal{G}$.

That $\mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$ generates $\mathcal{B} \times_\beta \mathcal{G}$ as a C^*-algebra follows immediately after considering the algebra $\mathrm{Alg}\{\mathcal{P}(\mathcal{A}, \mathcal{S}), (\mathcal{P}(\mathcal{A}, \mathcal{S}))^*\}$ inside $\mathcal{P}(\mathcal{B}, \mathcal{G})$. □

3.3.2 C^*-Envelopes

Our goal in this subsection is to calculate the C^*-envelope of $\mathcal{A}\,{}_N\!\times_\alpha^{iso} \mathcal{S}$ in the case when α is a family of completely isometric automorphisms on a unital operator algebra $\mathcal{A}$.

If $\mathcal{C}$ is a C^*-algebra which completely isometrically contains $\mathcal{A}$ such that $\mathcal{C} = C^*(\mathcal{A})$ then we call $\mathcal{C}$ a *C^*-cover of $\mathcal{A}$*. If $\mathcal{A}$ is a C^*-algebra, Theorem 3.3.11 says that $\mathcal{B} \times_\beta \mathcal{G}$ is a C^*-cover of $\mathcal{A}\,{}_N\!\times_\alpha^{iso} \mathcal{S}$ when $(\mathcal{B}, \mathcal{G}, \beta)$ is the minimal automorphic dilation of $(\mathcal{A}, \mathcal{S}, \alpha)$.

Definition 3.3.12. Let $\mathcal{A}$ be an operator algebra and let $\mathcal{C}$ be a C^*-cover of $\mathcal{A}$. Let α define an action of $\mathcal{S}$ on $\mathcal{C}$ by faithful *-endomorphisms which leave $\mathcal{A}$ invariant. We define the *relative semicrossed product* $\mathcal{A}\,{}_N\!\times_{\mathcal{C},\alpha} \mathcal{S}$ to be the subalgebra of $\mathcal{C}\,{}_N\!\times_\alpha \mathcal{S}$ generated by the natural copy of $\mathcal{A}$ inside $\mathcal{C}\,{}_N\!\times_\alpha \mathcal{S}$ and the universal isometries $\{\mathcal{V}_s\}_{s\in\mathcal{S}}$.

The idea of a relative semicrossed product was introduced by Kakariadis and Katsoulis [36] when studying semicrossed products by the semigroup $\mathbb{Z}_+$. The key idea is to realise the universal algebra $\mathcal{A}\,{}_N\!\times_\alpha^{iso} \mathcal{S}$ as a relative semicrossed algebra. This allows a concrete place in which to try and discover the C^*-envelope.

The proof of the following proposition follows the same reasoning as the proof of [36, Proposition 2.3]. It is an application of Dritschel and McCullough's [23] result that any representation can be dilated to a maximal representation and Muhly and Solel's [47] result that any maximal representation extends to a *-representation of any C^*-cover.

It is also important to note that if α is an action of $\mathcal{S}$ on an operator algebra $\mathcal{A}$ by completely isometric automorphisms which extend to completely isometric automorphisms of a C^*-cover $\mathcal{C}$ of $\mathcal{A}$, then each α_s necessarily leaves the Shilov boundary ideal $\mathcal{J}$ of $\mathcal{A}$ in $\mathcal{C}$ invariant, see e.g. [14, Proposition 10.6]. We will write $\{\dot{\alpha_s}\}_{s\in\mathcal{S}}$ for the automorphisms on $\mathcal{A}/\mathcal{J}$ induced by the automorphisms $\{\alpha_s\}_{s\in\mathcal{S}}$ on $\mathcal{A}$.

Proposition 3.3.13. *Let $\mathcal{A}$ be an operator algebra and let $\mathcal{C}$ be a C^*-cover of $\mathcal{A}$. Let α be an action of $\mathcal{S}$ on $\mathcal{C}$ by automorphisms that restrict to automorphisms of $\mathcal{A}$. Let $\mathcal{J}$ be the Shilov boundary ideal of $\mathcal{A}$ in $\mathcal{C}$. Then the relative semicrossed products $\mathcal{A}\,{}_N{\times}_{\mathcal{C},\alpha}\,\mathcal{S}$ and $\mathcal{A}/\mathcal{J}\,{}_N{\times}_{\mathcal{C}/\mathcal{J},\dot{\alpha}}\,\mathcal{S}$ are completely isometrically isomorphic.*

Let $(\mathcal{C},\mathcal{S},\alpha)$ be a semigroup dynamical system where $\mathcal{C}$ is a C^*-algebra and each α_s is an automorphism on $\mathcal{C}$. Then it is immediate that the minimal automorphic dilation of $(\mathcal{C},\mathcal{S},\alpha)$ is simply $(\mathcal{C},\mathcal{G},\alpha)$. If we view $\mathcal{G}$ as being a discrete group then $\mathcal{G}$ has a compact dual $\hat{\mathcal{G}}$. Recall that for every character γ in $\hat{\mathcal{G}}$ we can define an automorphism τ_γ on $\mathcal{P}(\mathcal{C},\mathcal{G})$ by

$$\tau_\gamma\left(\sum_{i=1}^n \mathcal{V}_{s_i}A_{s_i}\right) = \sum_{i=1}^n \gamma(s_i)\mathcal{V}_{s_i}A_{s_i}.$$

The automorphism τ_γ extends to an automorphism of $\mathcal{C}\times_\alpha\mathcal{G}$ with $\mathcal{C}$ as its fixed-point set [54, Proposition 7.8.3.]. We call τ_γ a *gauge automorphism*. The gauge automorphisms restrict to automorphisms of $\mathcal{C}\,{}_N{\times}_\alpha\,\mathcal{S}$.

Lemma 3.3.14. *Let $\mathcal{A}$ be a unital operator algebra. Let $\mathcal{C}$ be a C^*-cover of $\mathcal{A}$ and let $\mathcal{J}$ be the Shilov boundary ideal of $\mathcal{A}$ in $\mathcal{C}$. Let α be an action of $\mathcal{S}$ on $\mathcal{C}$ by automorphisms which restrict to completely isometric automorphisms of $\mathcal{A}$. Then*

$$C^*_{env}(\mathcal{A}\,{}_N{\times}_{\mathcal{C},\alpha}\,\mathcal{S}) \cong C^*_{env}(\mathcal{A})\times_{\dot{\alpha}}\mathcal{G}.$$

Proof. By the preceding proposition it suffices to show that

$$C^*_{env}(\mathcal{A}/\mathcal{J}\,{}_N{\times}_{\mathcal{C}/\mathcal{J},\dot{\alpha}}\,\mathcal{S}) \cong \mathcal{C}/\mathcal{J}\times_{\dot{\alpha}}\mathcal{G}.$$

The algebra $\mathcal{A}/\mathcal{J}\,{}_N{\times}_{\mathcal{C}/\mathcal{J},\dot{\alpha}}\,\mathcal{S}$ embeds completely isometrically into $\mathcal{C}/\mathcal{J}\times_{\dot{\alpha}}\mathcal{G}$ and generates it as a C^*-algebra. Let $\mathcal{I}$ be the Shilov boundary ideal of $\mathcal{A}/\mathcal{J}\,{}_N{\times}_{\mathcal{C}/\mathcal{J},\dot{\alpha}}\,\mathcal{S}$ in $\mathcal{C}/\mathcal{J}\times_{\dot{\alpha}}\mathcal{G}$. Suppose that $\mathcal{I}\neq\{0\}$.

The ideal $\mathcal{I}$ is invariant under automorphisms of $\mathcal{C}/\mathcal{J} \times_{\dot{\alpha}} \mathcal{G}$ and hence by the gauge automorphisms of $\mathcal{C}/\mathcal{J} \times_{\dot{\alpha}} \mathcal{G}$. Therefore $\mathcal{I}$ has non-trivial intersection with the fixed points of the gauge automorphisms, i.e. $\mathcal{I} \cap \mathcal{C}/\mathcal{J} \neq \{0\}$. But $\mathcal{I} \cap \mathcal{C}/\mathcal{J}$ is a boundary ideal for $\mathcal{A}$ in $\mathcal{C}/\mathcal{J}$. Hence $\mathcal{I} = \{0\}$. This proves the result. $\square$

We can now prove the main result of this section. This theorem generalises the result of Kakariadis and Katsoulis [36] from the semigroup $\mathbb{Z}_+$ to our more general semigroups $\mathcal{S} = \sum_{i=1}^{\oplus k} \mathcal{S}_i$. From another viewpoint, in the case when $\mathcal{A}$ is a C^*-algebra and $\mathcal{A}\,{}_N\!\times_\alpha^{iso} \mathcal{S} \cong \mathcal{A}\,{}_N\!\times_\alpha \mathcal{S}$ we have that the C^*-envelope of an associated tensor algebra is a crossed product algebra, by Remark 3.3.8 and Corollary 3.3.7. This was shown for abelian C^*-algebras by Duncan and Peters [25].

By [14, Proposition 10.1] the group $\mathrm{Aut}(\mathcal{A})$ of completely isometric automorphisms on the unital operator algebra $\mathcal{A}$ is isomorphic to the group of completely isometric automorphisms on $C^*_{env}(\mathcal{A})$ which leave $\mathcal{A}$ invariant. Thus, if $\{\alpha_s\}_{s\in\mathcal{S}}$ a family of completely isometric automorphisms defining an action of $\mathcal{S}$ on $\mathcal{A}$, then they can be extended to a family completely isometric automorphisms defining an action of $\mathcal{S}$ on $C^*_{env}(\mathcal{A})$.

Theorem 3.3.15. *Let $\mathcal{A}$ be a unital operator algebra. Let α be an action of $\mathcal{S}$ on $\mathcal{A}$ by completely isometric automorphisms. Denote also by α the extension of this action to $C^*_{env}(\mathcal{A})$. Then*

$$C^*_{env}(\mathcal{A}\,{}_N\!\times_\alpha^{iso} \mathcal{S}) \cong C^*_{env}(\mathcal{A}) \times_\alpha \mathcal{G}.$$

Proof. We will show that $\mathcal{A}\,{}_N\!\times_\alpha^{iso} \mathcal{S}$ is isomorphic to a relative semicrossed product. The result will then follow by Lemma 3.3.14.

Let $\{\mathcal{V}_s\}_{s\in\mathcal{S}}$ be the universal isometries in $\mathcal{A}\,{}_N\!\times_\alpha^{iso} \mathcal{S}$ acting on a Hilbert space $\mathcal{H}$. For each $s \in \mathcal{S}$ let $\mathcal{H}_s = \mathcal{H}$ and define maps $\mathcal{V}^{s,t}$ when $s \leq t$

$$\mathcal{V}^{s,t} : \mathcal{H}_s \to \mathcal{H}_t$$

by $\mathcal{V}^{s,t} = \mathcal{V}_{t-s}$. Let $\mathcal{K}$ be the Hilbert space inductive limit of the directed system $(\mathcal{H}_s)_{s\in\mathcal{S}}$.

For each $A \in \mathcal{A}$ the commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\mathcal{V}_s} & \mathcal{H} \\ {\scriptstyle A}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_s^{-1}(A)} \\ \mathcal{H} & \xrightarrow{\mathcal{V}_s} & \mathcal{H} \end{array}$$

defines an operator $\pi(\mathcal{A})$ on $\mathcal{K}$. Thus we have a completely isometric representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$.

Now for each $s, t \in \mathcal{S}$ define operator $U^s_t : \mathcal{H}_s \to \mathcal{H}_s$ by $U^s_t = \mathcal{V}_t$. Passing to the direct limit we get a family of commuting unitaries $\{U_s\}_{s\in\mathcal{S}}$ on $\mathcal{K}$ satisfying

$$\pi(A)U_s = U_s\pi(\alpha_s(A)).$$

The unitaries $\{U_s\}_{s\in\mathcal{S}}$ thus define *-automorphisms of $\mathcal{C} := C^*(\pi(A))$ extending α. Thus

$$\mathcal{A}\,{}_N\!\times_\alpha^{iso}\mathcal{S} \cong \mathcal{A}\,{}_N\!\times_{\mathcal{C},\alpha}\mathcal{S}.$$

The result now follows by Lemma 3.3.14. $\square$

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