Flocking for Multi-Agent Dynamical Systems

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we discuss models for multi-agent dynamical systems. We study the tracking/migration problem for flocks and a theoretical framework for design and analysis of flocking algorithm is presented. The interactions between agents in the systems are denoted by potential functions that act as distance functions, hence, the design of proper potential functions are crucial in modelling and analyzing the flocking problem for multi-agent dynamical systems. Constructions for both non-smooth potential functions and smooth potential functions with finite cut-off are investigated in detail.

The main contributions of this thesis are to extend the literature of continuous flocking models with impulsive control and delay. Lyapunov function techniques and techniques for stability of continuous and impulsive switching system are used, we study the asymptotic stability of the equilibrium of our models with impulsive control and discovery that by applying impulsive control to Olfati-Saber's continuous model, we can remove the damping term and improve the performance by avoiding the deficiency caused by time delay in velocity sensing.

Additionally, we discuss both free-flocking and constrained-flocking algorithm for multi-agent dynamical system, we extend literature results by applying velocity feedbacks which are given by the dynamical obstacles in the environment to our impulsive control and successfully lead to flocking with obstacle avoidance capability in a more energy-efficient way.

Simulations are given to support our results, some conclusions are made and future directions are given.

Acknowledgements

First and foremost, I would like to thank my supervisor Xinzhi Liu and Wei-Chao Xie, whose guidance has been invaluable in my time as a graduate student. I would like to extend a great thanks to my examining committee, Matthew Scott and Sue Ann Campbell, both of whom gave very helpful feedback, in the form of both corrections and suggestions. I would like to extend a thank you to the members of my research group: Mohamad Alwan, Jun Liu, Kexue Zhang, Hongtao Zhang, Shukai Li, Peter Stechlinski and Taghreed Subich, each of whom helped me either directly or indirectly with my research.

Dedication

To my dear Parents and Brothers.

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Chapter 1

Introduction

The beautiful collective behaviour of swarming species such as some bacteria, ant colonies, bee colonies, flocks of birds, schools of fish and other have attracted and fascinated the interest of researchers [19, 42, 41, 40, 37, 47, 59, 56, 52, 35, 27, 57, 39, 45] for many years. Collaborative flocking behaviour that we observe in these groups provides several advantages. The behaviour results in what is sometimes called "collective intelligence" or "swarm intelligence", where groups of relatively simple and "unintelligent" individuals can accomplish very complex tasks using only limited local information and simple rules of behaviour [23]. With the development of technology, including the technology on sensing, computation, information processing, power storage and others, it has become feasible to develop engineered autonomous multi-agent dynamical systems such as systems composed of multi robots, satellites, or ground, air, surface, underwater or deep space vehicles.

The terminology of "swarms" has come to mean a set of agents possessing independent individual dynamics but exhibiting intimately coupled behaviours and collectively performing some tasks [23]. Another terminology to describe such system is the term "multi-agent dynamical systems". Multi-agent dynamical systems have many potential commercial applications such as pollution clear up, search and rescue operations, fire-fighter assistance, surveillance, demining operations and others. These applications range in many different areas such as agriculture (for cultivation or applying pesticides for protection), forestry (for surveillance and early detection of forest fires), in disaster areas (such as for search in areas with radioactive release after a nuclear disaster), border patrol and homeland security, search and coverage in warehouses under fires, fire distinction, health care, etc [45].

In multi-agent dynamical systems, flocking is a form of collective behaviour of a large number of interacting agents with a common group objective [45]. In biology it is reserved for certain species when they are in certain behavioural modes (e.g., honey bees after hive fission occurs and the swarm of bees is searching for, or flying to a new home) [42]. From engineering perspective it is sometimes useful to use biological swarms as examples of behaviour that are achievable in multi-agent dynamical systems technologies. Moreover, operational principles from such biological systems can be used as guidelines in engineering for developing distributed cooperative control, coordination, and learning strategies for autonomous multi-agent dynamical systems. In other words, development of such highly automated systems is likely to benefit from biological principles including modelling of biological swarms, coordination strategy specification, and analysis to show that group dynamics achieve group goals [37].

Multi-agent dynamical systems possess various potential advantages over single-agent systems. First of all, multi-agent systems are more flexible and they can readjust and reorganiz based on the needs of the task under consideration, whereas single-agent systems do not have this capability [23]. Multi-agent dynamical systems can operate in parallel (different agents can concurrently perform different tasks) and therefor in a more efficient manner (provided that appropriate cooperation algorithms are developed), whereas in single-agent systems the agent usually has to finish its current task before starting another task. Multi-agent dynamical systems possess improved robustness properties since if one agent fails the other agents can continue (after reorganization and re-planning if needed) and complete the task, whereas for a single-agent system if the agent fails the task will fail as well [48]. Moreover, multi-agent systems can have improved task capabilities compared to single-agent systems and perform tasks which are not achievable by a single agent. In other words, the set of tasks a multi-agent dynamical system can perform is much larger than those of a single agent, and the range of possible applications and areas of use of a multi-agent dynamical system can be wider compared to those for a single-agent system.

There are several multi-agent dynamical system behaviours and task achievement goals that have been studied in the literature. Early works on understanding and modelling coordinated animal behaviour as well as empirically verifying the developed/proposed models has been performed by biologists. The work in [22] classified the work of biologists into the individual-based (Lagrangian) and continuum (Eulerean) frameworks. Another work which presented a useful background and a review of the swarm modelling concepts and literature such as spatial and non-spatial models, individual-based versus continuum models can be found in [37]. One of the early works within the individual-based framework is done by Breder in [6], where it suggested a simple model composed of a constant attraction term and a repulsion term which is inversely proportional to the square of the distance between two individuals. Similar work was performed by Warburton and Lazarus in [60] where the authors studied also the effect of a family of attraction/repulsion functions on swarm cohesion. An example work within the continuum framework in [35] where it presented a swarm model which is based on non-local interactions of the individuals in the flocks. In [26], a general continuous model for animal group size distribution, which is a non-spatial patch model and constitutes an example work on non-spatial approaches was presented. Other works on model development for biological swarms by mathematical biologists include [21, 34]. The work by Grindrod in [21] was an effort to generate a model for aggregation and clustering of species and considered its stability. While [21] considering a continuum model of a flock, the article in [34] describes a spatially discrete model, showing that the model can describe the flocking behaviour.

There are related studies performed by physicists investigating flocking behaviour. The general approach they take is to model each individual as a particle, which they usually call a self-driven or self-propelled particle, and study the collective behaviour due to their interaction. In particular, they analyze either the dynamic model of the density function or perform simulations based on a model for each individual particle. In [46] Rauch explored a simplified set of swarm models, which were driven by the collective motion of social insects such as ants. In this model the swarm members move in an energy field that models the nutrient or chemotactic profile in biology. In [55] Toner and Tu proposed a non equilibrium continuum model for collective motion of large groups of biological organisms and later in [56] they developed a quantitative continuum theory of flocking. They showed that their model can predict the existence of an ordered phase of flocks, in which all individuals in even arbitrarily large flocks move together [56].

In [9] a simple self-driven lattice-gas model for collective biological motion was introduced, it showed the existence of a transition from individual random walks to collective migration. Similarly, Vicsek in [59], which is a work that has caught attention of the engineering community in the recent years, introduced a simple simulation model for system of self-driven particles. They assumed that particles are moving with constant absolute velocity and at each time step assumed the average direction of motion of the particles in its neighbourhood with some random perturbation. They showed that high noise and low particle density leads to a no transport phase, where the average velocity is zero, whereas in low noise and high particle density the swarm is moving in a particular direction. They called this transition from a stationary state to a mobile state kinetic phase transition. Similarly in [11], they presented experimental results and mathematical model for forming bacterial colonies and collective motion of bacteria. Other results in the same spirit include [10, 12, 13, 58], in [12] a nonequilibrium model was compared to some equilibrium model in ferromagents, in [10] the authors demonstrated similar results in one dimension, in [58] the effect of fluctuations on the collective motion of self-propelled particles was investigated, and in [13] the effect of noise and dimensionality on the scaling behaviour of flocks of self-propelled particles was studied.

The field of coordinated multi-agent dynamical systems has become popular in the past decade in the engineering community as well. One of the earliest works in this field is the work by Reynolds [47] on simulation of a flock of birds in flight using a behavioural model based on few simple rules and only local interactions. Reynolds introduced three heuristic rules that led to flocking. here are three quotes from [47] that describe these rules:

1. Flock Centering: attempt to stay close to nearby flock mates.
2. Obstacle Avoidance: avoid collisions with nearby flock mates.
3. Velocity Matching: attempt to match velocity with nearby flock mates.

These three rules are also known as cohesion, separation and alignment rules in the literature. The main problem with implementation or analysis of the above rules is that they have broad interpretations. The issue of how to interpret Reynolds rules was resolved after publication of more recent papers by Reynolds [48, 49].

Early work on swarm stability is given by Beni and coworkers in [31] and [4]. In [31] they considered a synchronous distributed control method for discrete one and two dimensional swarm structures and prove stability in the presence of disturbances using Lyapunov methods. In [4] they considered a linear model and provided sufficient conditions for asynchronous convergence (without time delays) of the flocks to a synchronously achievable configuration.

Coordinated motion and distributed formation control of agents are important problems in the multi-agent formation control literature. In systems under minimalistic assumptions it might be difficult to achieve even simple formations. In [53] the authors considered asynchronous distributed control and geometric pattern formation of multiple anonymous agents. Other important studies on cooperative control and coordination of swarms of agents and in particular formation control of autonomous air or land vehicles using various different approaches can be found in [61, 43, 24, 3]. In [61] the authors considered cooperative control and coordination of a group of holonomic mobile robots to capture/enclose a target by making group formations. Results of a similar nature using behaviour based strategy can be found also in [3], where they considered a strategy in which the formation behaviour is integrated with other navigational behaviour and present both simulation and implementation results for various types of formations and formation strategies. In [24], the authors described formation control strategies for autonomous air vehicles. They used optimization and graph theory approach to find the best set of communication channels that will keep the aircraft in the desired formation.

Other work on formation control and coordination of multi-agent systems can be found in [2, 14, 15, 17, 33, 36]. In [14, 15], a feedback linearization technique using only local information for controller design to exponentially stabilize the relative distances of the robots in the formation was proposed. Similarly, in [17, 36], the concept of control Lyapunov functions together with formation constraints was used to develop a formation control strategy and prove stability of the formation (formation maintenance). The results in [33], on the other hand, were based on using virtual leaders and artificial potentials for agent interactions in a group of agents for maintenance of a predefined group geometry. By using the system kinetic energy energy and the artificial potential energy as a Lyapunov function closed loop stability was shown. Moreover, a dissipative term was employed in order to achieve asymptotic stability of the formation. In [2], the results in [33] were extended to the case in which the group is moving in a sampled gradient fields.

In comparison with continuous control which had been well studied before, there are not many reports on the design of impulsive control for multi-agent systems with switching topologies. It has been proved that impulsive control approach is effective and robust in synchronization of chaotic systems and complex networks [63], and the advantage of applying impulsive control in a self-driven, communicating multi-agent systems is to reduce energy and communication cost. We can imagine that a bird in a flock will not flap its wings all the time. Motivated by the above discussions, we consider the flocking/formation control problem of multi-agent dynamical systems with switching topologies by hybrid control method, numerical examples and simulations are provided to illustrate the results.

The thesis is organized as follows:

Chapter 2 In this chapter, we give a general background to multi-agent system modelling, some basic concepts of graph theory are introduced and a particle-based framework is presented to describe our problem. Construction for potential functions is investigated in detail.

Chapter 3 In this chapter, we discuss the continuous models for flocking for multi-agent dynamical systems. The concepts of fixed and switching topologies are introduced, a virtual leader is applied into our flocking algorithm and successfully leads to flocking. Stability problem for free-flocking is discussed in detail, furthermore, models for constrained-flocking are also provided.

Chapter 4 In this chapter, we extend the existing continuous flocking models with impulsive control and delay, techniques for stability of impulsive systems are used to analyze the asymptotic stability of the equilibrium of our hybrid flocking models, additionally algorithms for flocking with dynamical obstacle avoidance capability are proposed.

Chapter 5 In this chapter, we give our conclusions and directions for future work.

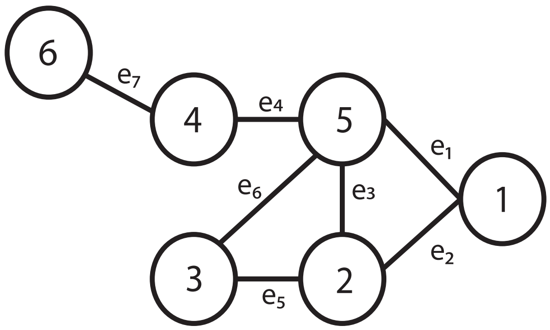
Chapter 2

Muti-Agent Dynamical System Modelling

In order to model multi-agent dynamical systems and study the flocking behaviour and interconnection between agents in the model, one can begin by defining agents to have sensory capabilities (i.e., to sense position or velocity of other agents or sense environmental characteristics), processing ability (a brain or an on-board computer), the ability to communicate or exchange information and the ability to take actions via actuators [23]. Physical agent characteristics along with agent motion dynamics and its sensory and processing capabilities constrain how the agent can move in its environment and the rates at which it can sense and act in spatially distributed areas and interact with the other agents.

Regardless, it is useful to think of the agents as nodes, and arcs between nodes as representing abilities to sense or communicate with other agents. The existence of an arc may depend on sensing range of agents, the properties of the environment, communication network and link imperfections, along with local agent abilities and goals. One can view a multi-agent dynamical system as a set of such communicating agents that work collectively to solve a task.

In this chapter we will focus on how to establish this theoretical framework. We begin by introducing some basic concepts in graph theory [5, 16, 29] to describe agents as nodes and the interconnection between them as edges of a graph. The foundation of graph theory was influenced by the Konigsberg bridge puzzle introduced by Euler in 1736. Later, investigations of social problems by Erdos were a benchmark for the start of formal graph theory [18]. Here we first look into some basic notions of the subject.



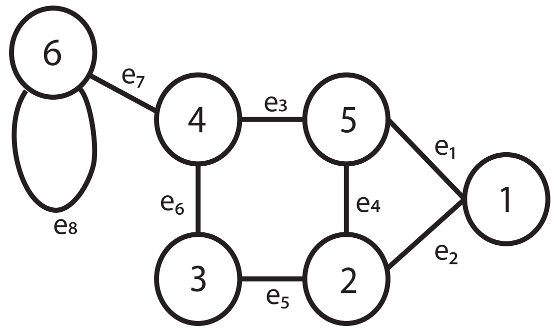


Figure 2.1: (a) A simple graph. (b) A graph with a loop

2.1 Basics of Graph Theory

Definition 2.1. *A graph G (typically written as $G = (V, E)$) is an ordered pair of two sets, a non-empty set $V = V(G)$, called vertex set, consisting of objects $\{i, j, \dots\}$ that are called vertices (sometime also called nodes). Another set $E = E(G)$, called edge set which consists of edges.*

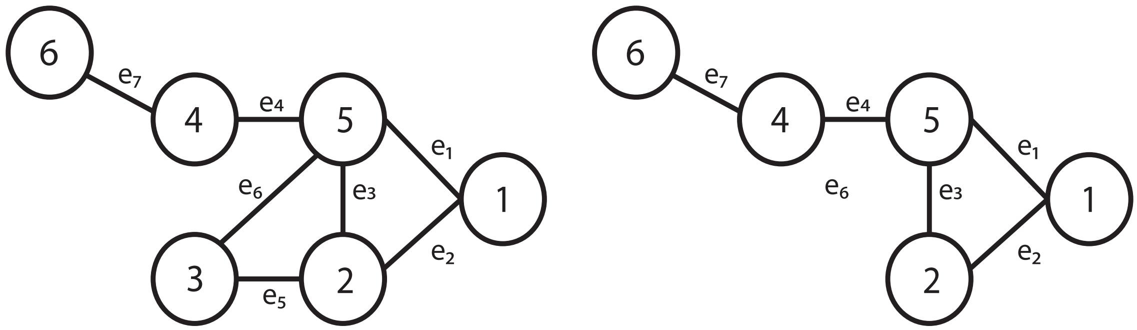
One edge connects two vertices, two vertices can be connected by multiple edges and one edge can connect one vertex to itself. The cardinality of the set of vertices V is called the order of the graph, and the cardinality of the set of edges E is called the size of the graph.

Definition 2.2. *If $e = \{(i, j) : i \text{ and } j \in V(G)\} \in E(G)$, we call the vertices i and j adjacent to each other or connected to each other or neighbours of one another. The edge e can be represented as a pair of vertices (i, j) or denoted by ij.*

Definition 2.3. *An edge (i, i) is called a self-loop or simply a loop. There could be more than one edge with the same vertices, these edges are called parallel edges or multi-edges.*

Fig.2.1(b) shows a graph in which edges e_8 is a self-loop. A graph with no self-loops or multi-edges is called a simple graph. Fig.2.1(a) is an example of a simple graph.

Definition 2.4. *For a simple graph, the degree n_i of a vertex i is the number of vertices which are adjacent to i. The monotonic sequence of degrees of $V = V(G)$ is called degree*



(a) (b)

Figure 2.2: Graph (b) is a subgraph of graph (a)

sequence of the graph G. A vertex with degree zero is called an isolated vertex. A vertex with degree 1 is called a pendant vertex.

In Fig.2.1(a), the vertices v_1, v_2, v_3, v_4 are adjacent to vertex v_5. So the degree of the vertex v_5 is 4.

Definition 2.5. *A graph $G^* = (V^*, E^*)$ is called a subgraph of a graph $G = (V, E)$, if $V^* \subseteq V$ and $E^* \subseteq E$.*

In Fig.2.2 graph (b) is a subgraph of graph (a).

Definition 2.6. *A path in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence.*

Definition 2.7. *A graph is called connected if for every pair of vertices i and j, there exists a path where i and j are end vertices. Otherwise, the graph is called disconnected.*

Definition 2.8. *A directed graph or digraph consists of a set V of vertices $\{i, j, \dots\}$ and a set of edges E which are ordered pairs (i, j) of vertices. We write the edge with ordered pair (i, j) as $i \to j$. i is called the head or initial vertex and j is called the tail or terminal vertex of the edge. The number of edges with i as the initial vertex(resp. terminal vertex) is call the outdegree(resp. indegree) of the vertex i.*

Definition 2.9. *A spanning tree T of a connected, undirected graph G is a tree composed of all the vertices and some (or perhaps all) of the edges of G. A spanning tree of G is a selection of edges of G that form a tree spanning every vertex. That is, every vertex lies in the tree, but no cycles (or loops) are formed.*

2.1.1 Adjacency Matrix

The adjacency matrix of a graph G of n vertices is a $n \times n$ matrix where the non-diagonal entry a_{ij} is the number of edges connecting from vertex i to vertex j, and the diagonal entry a_{ii}, depending on the convention, is either once or twice the number of edges(loops) from vertex i to itself. Undirected graphs often use the former convention of counting loops twice, whereas directed graphs typically use the latter convention. If a graph is undirected, the adjacency matrix is symmetric, and therefore has a complete set of real eigenvalues and an orthogonal eigenvector basis. The set of eigenvalues of the adjacency matrix of a graph is the spectrum of the graph.

Adjacency Matrix: The matrix $A = [a_{ij}]$ with the form

$$a_{ij} = \begin{cases} 1, & if\ ij\ is\ an\ edge \\ 0, & otherwise \end{cases}$$

is the adjacency matrix of the most common simple undirected graph.

Following is the adjacency matrix of the graph in Fig.2.1(b)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

2.1.2 Graph Laplacian

The graph Laplacian matrix is important for analyzing the graph's structure, (which later will be useful in analysis of velocity matching in multi-agent dynamical systems). It is defined in the following way:

Laplacian Matrix: The matrix $L = [a_{ij}]$ with the form

$$a_{ij} = \begin{cases} n_i, & if\ i = j \\ -1, & if\ ij\ is\ an\ edge \\ 0, & otherwise \end{cases}$$

is called the Laplacian matrix of a graph, where n_i denotes the degrees of the vertices i. Following is an example of the corresponding Laplacian of the graph in Fig.2.1(b)

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

Relationship between adjacency matrix and Laplacian martix

For a graph G, let D be the diagonal matrix with entries which are the degrees of vertices, i.e.,

$$D(i,j) = \begin{cases} n_i, & if\ i = j \\ 0, & otherwise \end{cases}$$

where n_i denotes the degrees of the vertices i. The relation between the adjacency matrix A and the graph Laplacian L is

$$L = D - A$$

Laplacian matrix L always has a right eigenvector of $\mathbf{1}_n = (1, \ldots, 1)^T$ associated with eigenvalue $\lambda_1 = 0$. The following lemma from [38] summarizes the basic properties of graph Laplacians:

Lemma 2.1. *Let $G(V, E)$ be an undirected graph of order n with a non-negative adjacency matrix $A = A^T$. Then, the following statements hold:*

1. *L is a positive semidefinite matrix that satisfies the follow sum-of-squares (SOS) property:*

$$z^T L z = \frac{1}{2} \sum_{i,j \in E} a_{ij}(z_j - z_i)^2,\ \ z \in \mathbb{R}^n;$$

2. *The graph G has $c \geq 1$ connected components if and only if $rank(L) = n - c$. Particularly, G is connected if and only if $rank(L) = n - 1$;*

3. *Let G be a connected graph, then*

$$\lambda_2(L) = min_{z \perp \mathbf{1}_n} \frac{z^T L z}{\|z\|^2} > 0,$$

Proof. All three results are well-known in the field of algebraic graph theory and their proofs can be found in Godsil and Royle [25]. □

The quantity $\lambda_2(L)$ is known as algebraic connectivity of a graph [20]. Particularly, we use m-dimensional graph Laplacians defined by

$$\hat{L} = L \otimes \mathbf{1}_m$$

where $\otimes$ denotes the Kronecker product. This multi-dimensional Laplacian satisfies the following SOS property:

$$z^T \hat{L} z = \frac{1}{2} \sum_{(i,j)\in E} a_{ij} \|z_j - z_i\|^2, \; z \in \mathbb{R}^{mn}$$

where $z = (z_1, z_2, \ldots, z_n)^T$ and $z_i \in \mathbb{R}$ for all i. Matrix $\hat{L}$ can be viewed as the Laplacian of a graph with adjacency matrix $\hat{A} = A \otimes \mathbf{1}_m$.

We can use the graph Laplacian to evaluate the graph/network connectivity maintenance. We know that the link/connection between vertex i and vertex j is maintained if $ij \in e$, otherwise this link is considered to be broken. For graph connectivity, a dynamic graph $G(V, E)$ is said to be connected at time t only if there exists a path between any two vertices at time t. To analyze the connectivity of the graph/network, we define $c(t) = \frac{Rank(L(t))}{n-1}$; if $0 \leq c(t) < 1$, the network is broken; if $c(t) = 1$, the network is connected. This property of the Laplacian is very useful to analyze the flocking behaviour of multi-agent dynamical systems.

2.1.3 Particle-Based System

After introducing the basic knowledge of graph theory, now we need to set up a theoretical model to analyze the flocking behaviour of multi-agent dynamical systems. We propose a particle-based model assuming agents in the system are self-driven or self-propelled particles, and possess sensing ability to perceive the position and velocity information of other agents which are within their interaction/communication range. The concept of spatial neighbour of an agent is also introduced.

Let $G = (V, E)$ be a graph and $r_i \in \mathbb{R}^m$ denote the position of agent i for all $i \in V$. The vector $r = (r_1, \ldots, r_n) \in \mathbb{R}^{mn}$ is called the configuration of all nodes of the graph. A framework (or structure) is a pair (G,r) that consists of a graph and the configuration of its nodes.

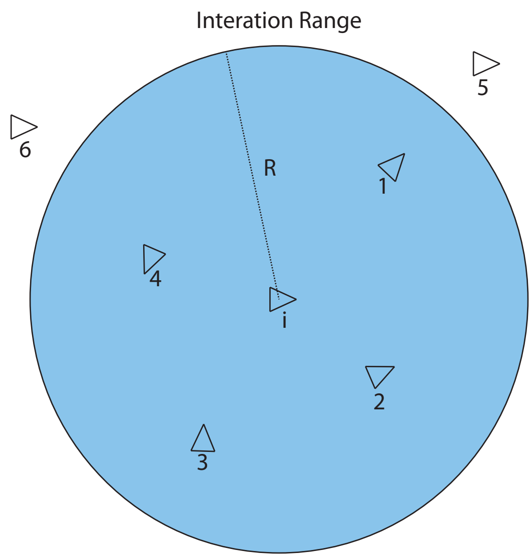


Figure 2.3: An agent and its neighbours in a spherical neighbourhood

Let $R > 0$ denote the interaction range between two agents. An open ball with radius R (indicate in Fig.2.3) determining the set of spatial neighbours of agent i is denoted by

$$N_i = \{j \in V \ : \|r_j - r_i\| < R\}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^m$. Given an interaction range $R > 0$, a spatially induced graph $G(r) = (V, E(r))$ can be specified by V and the set of edges

$$E(r) = \{(i, j) \in V \times V \ : \|r_j - r_i\| < R, i \neq j\}$$

that clearly depends on R. We refer to the graph $G(r)$ as a net and to the structure $(G(r), r)$ as a frame net. The topology of a wireless sensor network with a radio range r is an example of a net [1]. If the interaction range of all agents is the same, the net $G(r)$ becomes an undirected graph. All nets are undirected graphs in this thesis. A net $G(r)$ is generically a digraph under either of the following assumptions [38, 47]:

1. The spherical neighbourhoods of agents do not have the same radius.

2. Every agent uses a conic neighbourhood to determine its neighbours.

2.2 Vicsek's Discrete Model

Among the first groups of physicists who studied flocking from a theoretical perspective were Vicsek. In [57] Vicsek introduced a simple discrete time model with a novel type of dynamics in order to investigate the emergence of self-ordered motion in systems of particles with biologically motivated interaction. In Vicsek's model, particles were driven with a constant absolute velocity and at each time step assumed the average direction of motion of the particles in their neighbourhood with some random perturbation in analogy with the temperature. Numerical simulations were provided to indicate that this model results in rich, realistic dynamics, including a kinetic phase transition. Vicsek's work has an important influence on the later researches of flocking for multi-agent systems since flocking is the kind of coordinated behaviour which combines both position (phase) transition and velocity alignment.

The model is carried out in a square shaped cell of linear size L with periodic boundary conditions. The particles are represented by points moving continuously on this plane. We use the interaction range R as the unit to measure distances ($R = 1$), while the time unit $\Delta t = 1$ is the time interval between two updates of the velocity directions and positions. In most of the simulations we use the simplest initial conditions:

1. At time $t = 0$, all particles' position are randomly distributed in the plane.
2. At time $t = 0$, all particles have the same absolute velocity v.
3. At time $t = 0$, the directions θ of all particles' velocity are randomly distributed.

The velocities v of the particles are determined simultaneously at each time step, and the position of the ith particle updates according to a simple rule which can be described as follow:

$$x_i(t+1) = x_i(t) + v_i(t)\Delta t$$

where the velocity of a particle $v_i(t+1)$ is constructed to have an absolute value v and a direction given by the angle $\theta(t+1)$. This angle is given by the following expression:

$$\theta(t+1) =< \theta(t) >_R + \Delta\theta$$

where $< \theta(t)_R >$ denotes the average direction of the velocities of particles (including particle i) which are in a circle of radius R centering at the given particle. The average direction is given by the angle $arctan[\frac{<sin(\theta(t))>_R}{<cos(\theta(t))>_R}]$ and $\Delta\theta$ is a random number chosen with a uniform probability from the interval $[-\eta/2, \eta/2]$. Thus the term $\Delta\theta$ represents noise which we shall use as a temperaturelike variable. Correspondingly, there are three free parameters for a given system size: η, ρ, and v where v is the distance a particle travels between two updates, and $\rho = v/L^2$ is the density.

We need to investigate the nontrivial behaviour of the transport properties as the two basic parameters of the model, the noise η and the density ρ are varied. We use $v = 0.2$ in the simulations, as $v \to 0$ the particles do not move. For $v \to \infty$ the particles become completely mixed between two updates, and this limit corresponds to the so-called mean-field behaviour of a ferromagnet. We use $v = 0.2$ for which the particles always interact with their actual neighbours and move fast enough to change the configuration after a few updates of the directions. According to the simulations, in a range of the interval $(0.03 < v < 0.3)$, the actual value of v does not affect the results. Figures2.4(a)-(d) demonstrate the velocity fields during runs with various selections for the value of the parameter ρ and η. The actual velocity of a particle is indicated by a small arrow. (a) At t=0 the positions and the directions of velocities are distributed randomly. (b) For small noise the particles tend to form groups moving coherently in random directions. (c) At higher noise the directions of velocities are distributed randomly at t=0 and (d) After 100 time steps the particles still move randomly in random direction.

The emergence of cooperative motion in this model has analogies with the appearance of spatial order in equilibrium systems. This fact and the simplicity of this model suggests that with appropriate modifications, the theoretical methods for describing critical phenomena may be applicable to other kind of equilibrium phase transition such as flocking behaviour. A rigorous proof of convergence by Jadbabaie [30] for Vicseks model was given in Appendix A.

2.3 Double Integrator Agents

In this section we consider a double integrator model for agents. As in the last section, we have introduced some basic background of modelling and Vicsek's discrete velocity consensus model, here we will propose the model of double integrator agents and a systematic method is provided for construction of inter-agent potential function to investigate the flocking behaviour for multi-agent dynamical systems.

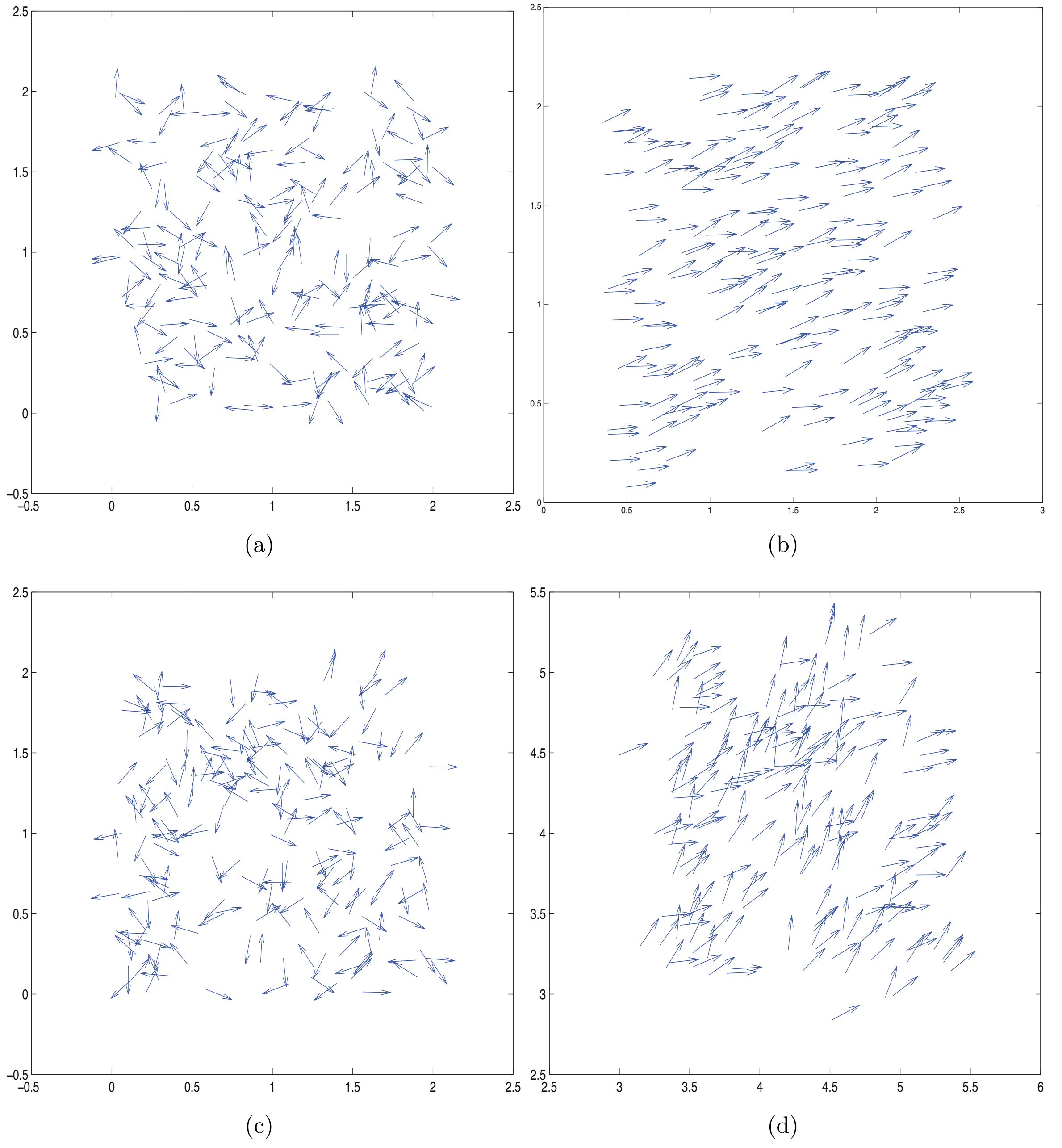


Figure 2.4: (a) low noise at t=0 (b) low noise after 30 time steps (c) high noise at t=0 (d) high noise after 100 time steps

2.3.1 Equation of Motion

We consider a system consists of N interconnecting agents, each with point mass dynamics given by

$$\begin{cases} \dot{r}_i = v_i \\ \dot{v}_i = \frac{1}{M_i} u_i \end{cases} \tag{2.1}$$

where $r_i \in \mathbb{R}^n$ is the position, $v_i \in \mathbb{R}^n$ is the velocity, M_i is the mass and $u_i \in \mathbb{R}^n$ is the (force) control input for the ith agent. The above equations imply that $u_i = M_i \ddot{r}_i$ (force is mass times acceleration). Integrating acceleration once we get velocity, twice and we get position; hence, we use the term "double integrator model." It is assumed that all agents know their own dynamics. For some organisms like bacteria that move in highly viscous environments it can be assumed that $M_i = 0$. If a velocity damping term is used in u_i, we obtain the model proposed by Olfati-Saber [45] which will be studied in Chapter 3 (assuming that $M_i = 1$).

2.3.2 Potential Function Design

Given the agent dynamics in (2.3.1), in this section we will discuss developing control algorithms for obtaining coordinated behaviour for multi-agent dynamical systems. We will solve this problem by using a potential function based approach. We assume all individuals in the system move simultaneously and know the exact relative position of the other individuals. Let $r^T = [r_1^T, r_2^T \dots r_n^T] \in \mathbb{R}^{nn}$ denote the vector of concatenated states of all the agents. In this section the control input u_i of individual i will have the form

$$u_i = -\nabla_{r_i} J(r)$$

where $J : \mathbb{R}^{nn} \to \mathbb{R}$ is a potential function which represents the interaction (i.e., the attraction and repulsion relationship) between the individual agents and needs to be chosen by the designer based on the flocking application under consideration and the desired behaviour from the system. We will discuss which properties the potential functions should satisfy for different problems and present results for some potential functions.

Aggregation is one of the most basic behaviour seen in flocks in nature (such as insect colonies) and is sometimes the initial phase in collective tasks performed by a flock. Below, we discuss how to achieve aggregation for the single integrator model in (2.3.1). If only simple aggregation is desired from the multi-agent dynamical system, then the potential function J can be selected as $J(r) = J_{aggregation}(r)$ where

$$J_{aggregation} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [J_a(\|r_i - r_j\|) - J_r(\|r_i - r_j\|)].$$

Here, $J_a : \mathbb{R}^+ \to \mathbb{R}$ represents the attraction component, whereas $J_r : \mathbb{R}^+ \to \mathbb{R}$ represents the repulsion component of the potential function. Although not the only choice, the above potential function is very intuitive since it represents an interplay between attraction and repulsion components. Note also that it is based only on the relative distances between the agents and not the absolute agent positions.

Given the above type of potential function, the control input of individual $i, j = 1, \ldots, N$ can be calculated as

$$u_i = - \sum_{j=1, j \neq i}^{n} [\nabla_{r_i} J_a(\|r_i - r_j\|) - \nabla_{r_i} J_r(\|r_i - r_j\|)]$$

Note that since $\dot{v}_i = u_i$, the motion of the individual is along the negative gradient and leads to a descent motion towards a minimum of the potential function J. Moreover, since the function $J_a(\|r\|)$ and $J_r(\|r\|)$ create a potential field of attraction and repulsion, respectively, around each individual, the above property restricts the motion of the individuals toward each other along the gradient of these potentials (i.e., along the combined gradient field of $J_a(\|r\|)$ and $J_r(\|r\|)$).

One can show that, because of the chain rule and the definition of the functions J_a and J_r, the equalities

$$\nabla_r J_a(\|r\|) = r g_a(\|r\|)$$
$$\nabla_r J_r(\|r\|) = r g_r(\|r\|)$$

are always satisfied for some some function $g_a : \mathbb{R}^+ \to \mathbb{R}$ and $g_r : \mathbb{R}^+ \to \mathbb{R}$. Here $g_a : \mathbb{R}^+ \to \mathbb{R}^+$ represents the attraction term, whereas $g_r : \mathbb{R}^+ \to \mathbb{R}^+$ represents the repulsion term. Note also that the combined term $-r g_a(\|r\|)$ represents the actual attraction whereas the combined term $r g_r(\|r\|)$ represents the actual repulsion, and they both act on the line connecting the two interaction individuals, but in opposite directions. The vector r determines the alignment, it guarantees that the interaction vector is along the line on which r is located), and it also affects the magnitude of the attraction and repulsion components. The terms $g_a(\|r\|)$ and $g_r(\|r\|)$, on the other hand, affect correspondingly only the magnitude of the attraction and repulsion, whereas their difference determine the direction of the interaction along vector r. Let's define the function $g(\cdot)$ as

$$g(r) = -r[g_a(\|r\|) - g_r(\|r\|)]. \tag{2.2}$$

We call the function $g(\cdot)$ an attraction/repulsion function and assume that on large distances attraction dominates, while on short distances repulsion dominates, and that there is a unique distance at which the attraction and the repulsion balance. In other words, we assume that $g(\cdot)$ satisfies the following assumptions.

Definition 2.10. *The function $g(\cdot)$ in (2.2) and the corresponding $g_a(\cdot)$ and $g_r(\cdot)$ are such that there exist a unique distance d at which we have $g_a(d) = g_r(d)$. Moreover, we have $g_a(\|r\|) > g_r(\|r\|)$ for $\|r\| > d$ and $g_r(\|r\|) > g_a(\|r\|)$ for $\|r\| < d$.*

Moreover for the attraction/repulsion function $g(\cdot)$ defined as above we have $g(r) = -g(-r)$, in other words, the above $g(\cdot)$ functions are odd. This is an important feature of the $g(\cdot)$ functions that lead to reciprocity in the inter-agent relations and interactions.

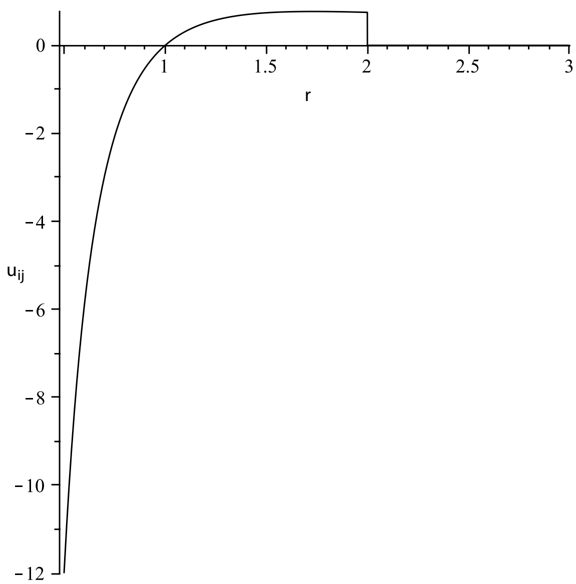
Non-smooth Potential Fucntion

In order to satisfy the above assumptions, the potential function designer should choose the attraction and repulsion potential such that the minimum of $J_a(\|r_i - r_j\|)$ occurs on $\|r_i - r_j\| = 0$, whereas the minimum of $-J_r(\|r_i - r_j\|)$ occurs on $\|r_i - r_j\| \to \infty$, and the minimum of the combined $J_a(\|r_i - r_j\|) - J_r(\|r_i - r_j\|)$ occurs at $\|r_i - r_j\| = d$. In other words, at $\|r_i - r_j\| = d$ the attraction/repulsion potential between two interacting individuals has a global minimum, however, when there are more than two individuals, the minimum of the combined potential does not necessarily occur at $\|r_i - r_j\| = d$ for all $j \neq i$. Moreover, there exist a family of minima. So we can view $J(r)$ as the potential average of the multi-agents system, whose value depends on the inter-individual distances (such that it's high when the agents are either far from each other or too close to each other) and the motion of all the agents is towards a unique global minimum energy configuration.

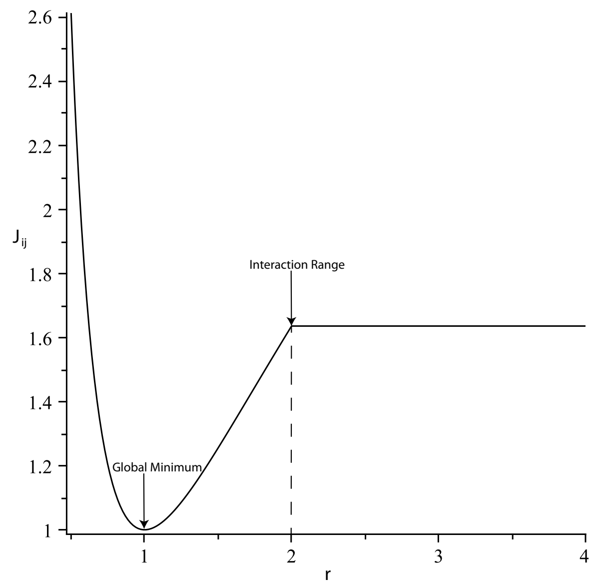
One potential function which satisfies the above conditions, including Assumption 2.10, and has been used in Tanner's model [28] is:

$$J_{ij} = \begin{cases} \frac{1}{\|r_i - r_j\|^2} + log\|r_i - r_j\|^2, & \|r_i - r_j\| < R \\ V_R & \|r_i - r_j\| \geq R \end{cases} \tag{2.3}$$

where R denotes the interaction range between two agents and $V_R = 1/R^2 + logR^2$ is a positive constant. Its corresponding attraction/repulsion function can be calculates as



(a)



(b)

Figure 2.5: (a) A Discontinuous Attraction/Repulsion Function. (b) A Non-Smooth Potential Function with R=2

$$u_{ij} = \frac{-2}{\|r_i - r_j\|^3} + \frac{2}{\|r_i - r_j\|} \tag{2.4}$$

which is indicated in Fig.2.5(a). We can see that the attraction/repulsion function is zero at $r = 1$ which is the global minimum of the corresponding potential function, indicating that the attraction and repulsion are balance at this equilibrium position. Also notice that due to the property that the potential function is not smooth at $R = 2$, so in the Fig.2.5(a) of the attraction/repulsion function we can see there is a jump at $R = 2$, which implies that the interaction between two agents is suddenly disappear when they are out of their interaction range. Moreover, from Fig.2.5(b) we notice that this non-smooth potential function is not differentiable and goes to infinity at 0, which means we need to apply an unbounded repulsive force to avoid collision between two agents.

Smooth Potential Function

The non-smooth potential function introduced in Tanner's model is not differentiable at 0, this is due to the fact that the map $\|r\|$ is not differentiable at $r = 0$. Moreover, we want to construct a potential function with a finite cut-off at R, which mean the potential function is zero for all $r \geq R$. Hence, in order to construct a new smooth potential function with a finite cut-off, we need to introduce σ-Norms and Bump functions.

Definition 2.11. *The σ-norm of a vector is a map $\mathbb{R}^m \to \mathbb{R}^+$ defined as*

$$\|r\|_\sigma = \frac{1}{\epsilon}[\sqrt{1 + \epsilon\|r\|^2} - 1]$$

with a parameter $\epsilon > 0$ and a gradient $\sigma_\epsilon(r) = \nabla\|r\|_\sigma$ given by

$$\sigma_\epsilon(r) = \frac{r}{\sqrt{1 + \epsilon\|r\|^2}} = \frac{r}{1 + \epsilon\|r\|_\sigma}$$

Notice the map $\|r\|_\sigma$ is differentiable everywhere whereas $\|r\|$ is not differentiable at $r = 0$. Later this property of σ-norms is used for construction of smooth collective potential functions for multi-agent dynamical system.

Definition 2.12. *A bump function is a scalar function $ph(r)$ that smoothly varies between 0 and 1.*

Here we use bump functions for construction of smooth potential functions with finite cut-offs, one possible choice is the following bump function introduced in [50]

$$ph(r) = \begin{cases} 1, & r \in [0, h) \\ \frac{1}{2}[1 + cos(\pi\frac{r-h}{1-h})], & r \in [h, 1] \\ 0 & otherwise \end{cases} \tag{2.5}$$

where $h \in (0, 1)$. One can show that $ph(r)$ is a C^1-smooth function with the property that $ph^{'}(r) = 0$ over the interval $[1, \infty)$ and $|ph^{'}(r)|$ is uniformly bounded in r. A example of (2.5) with $h = 0.2$ is given is Fig.2.6

A collective potential function $\psi(r)$ is a smooth version of a deviation energy function[45] with a scalar pairwise potential that has a finite cut-off. This means that there exists a finite interaction range $R > 0$ such that $\psi\prime(r) = 0, \forall r > R$. This feature turns out to be the fundamental source of scalability of our flocking algorithms. A common approach to

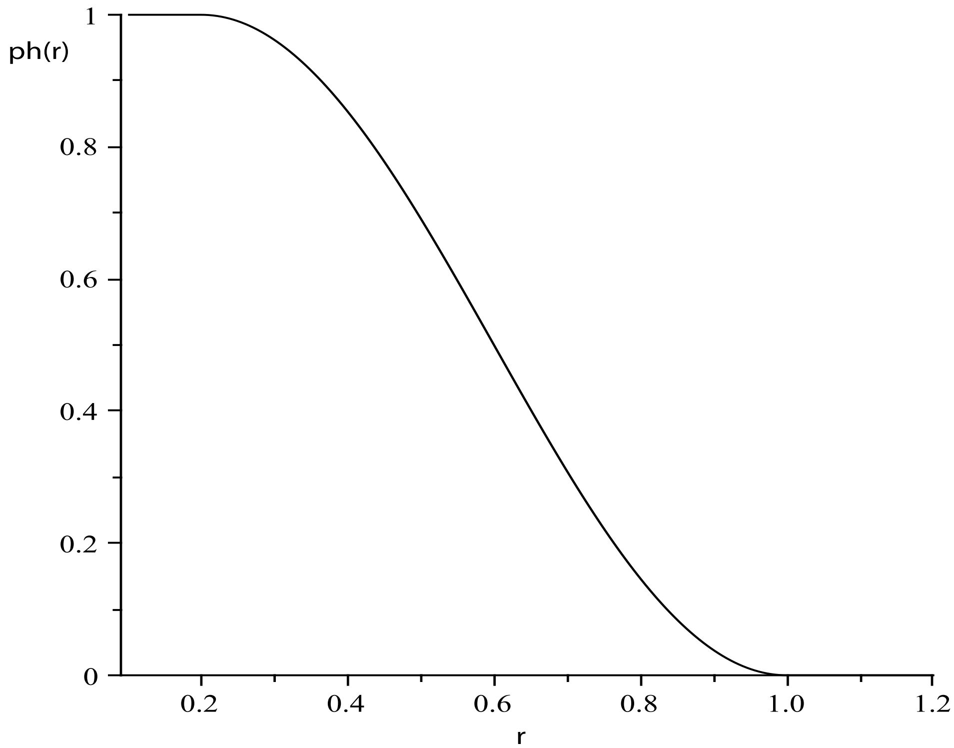


Figure 2.6: A Bump Function with h=0.2

create a pairwise potential with a finite cut-off is "soft cutting" in which a pairwise potential is multiplied by a bump function [38]. Here we use an alternative approach by softly cutting attraction/repulsion functions and then using their integrals as pairwise potentials. This way the derivative of the bump function never appears in the attraction/repulsion function, and thereby a negative bump in the action function near $r = R$ is avoided.

Let $\psi(r) : \mathbb{R}^+ \to \mathbb{R}^+$ be an attraction/repulsion pairwise potential function with a global minimum at $r = d$ and a finite cut-off at R. Then the following function

$$\varphi(r) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \psi(\|r_i - r_j\| - d)$$

is a collective potential function that is not differentiable at singular configurations in which two distinct nodes coincide, or $r_i = r_j$. To resolve this problem, we use the set of algebraic constraints that are written in terms of σ-norms as

$$\|r_i - r_j\|_\sigma = d_\alpha, \ \forall j \in N_i(r)$$

where $N_i(r)$ denotes all the neighbours of agent i, $d_\alpha = \|d\|_\sigma$. These constraints induce a smooth collective potential function of the form:

$$V(r) = \frac{1}{2}\sum_i \sum_{j \neq i} \psi_\alpha(\|r_i - r_j\|_\sigma) \tag{2.6}$$

where $\psi_\alpha(r)$ is a smooth pairwise attraction/repulsion potential with a finite cut-off at $r_\alpha = \|r\|_\sigma$ and a global minimum at $r = d_\alpha$.

To construct a smooth pairwise potential with finite cut-off, we integrate an attraction/repulsion function $\phi_\alpha(r)$ that vanishes for all $r \geq R_\alpha$. Define this attraction/repulsion function as

$$\phi_\alpha(r) = ph(\frac{r}{R_\alpha})\phi(r - d_\alpha) \tag{2.7}$$

$$\phi(r) = \frac{1}{2}[(a+b)\sigma_1(r+c) + (a-b)] \tag{2.8}$$

where $\sigma_1(r) = r/\sqrt{1+r^2}$ and $\phi(r)$ is an uneven sigmoidal function with parameters that satisfy $0 < a \leq b$, $c = |a-b|/\sqrt{4ab}$ to guarantee $\phi(0) = 0$. The pairwise attraction/repulsion potential $\psi_\alpha(r)$ is defined as

$$\psi_\alpha(r) = \int_{d_\alpha}^{r} \phi_\alpha(s)ds \tag{2.9}$$

Functions ϕ_α and $\psi_\alpha(r)$ are indicated in Fig.2.7

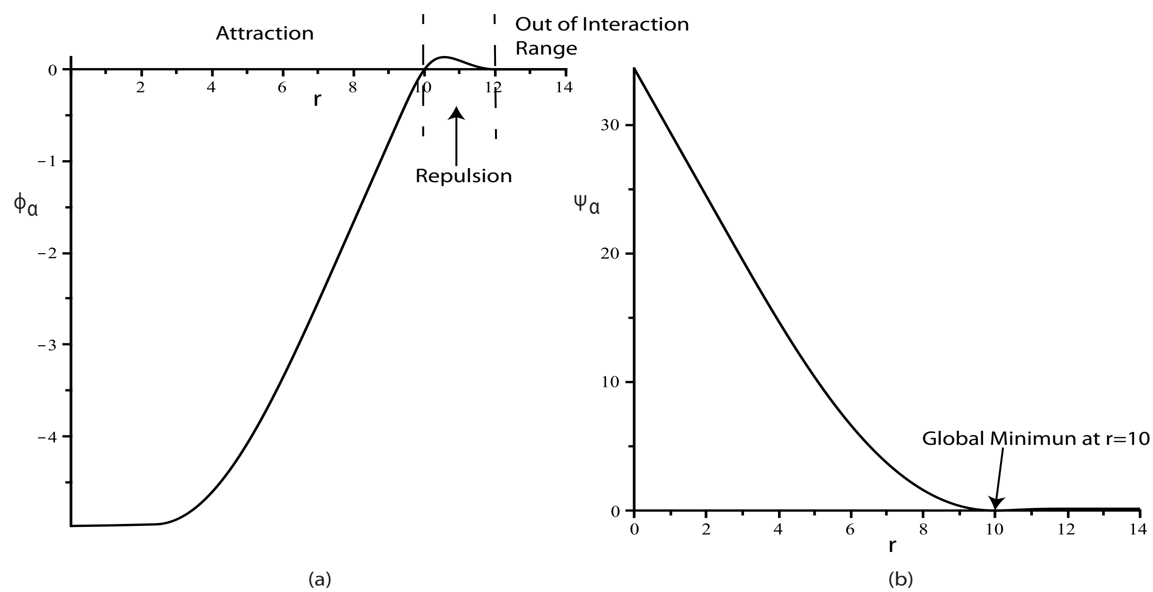


Figure 2.7: The Attraction/Repulsion an Potential Function with finite cut-offs : (a) $\phi_{\alpha(r)}$, (b) $\psi_{\alpha}(r)$

Chapter 3

Flocking via Continuous Control

In last chapter we propose a double integrator model, we consider a system consists of N interacting agents with point mass dynamics given by

$$\begin{cases} \dot{r}_i = v_i \\ \dot{v}_i = \frac{1}{M_i} u_i \end{cases} \tag{3.1}$$

where $r_i \in \mathbb{R}^n$ is the position, $v_i \in \mathbb{R}^n$ is the velocity, here we assume the mass M_i of every agent in the model to be 1. In this chapter we will focus on using the potential function and navigational feedback(a global objective of all the agents) to design the term u_i to control the agents in order to achieve the flocking phenomenon. One essential rule which leads to flocking is that our control algorithms have to maintain the network connectivity of the system, which means all agents in the system can interact with at least one of its neighbours. Hence, in this chapter we will first give the definition of α-Lattices which is convenient for us to analyze the formation of the flocks, then we will provide a control rule u_i without any navigational feedback for both fixed topologies and switching topologies. We will show that navigational feedback is necessary to avoid fragmentation of the flock. Then we will investigate the model proposed by Olfati-Saber [45] which includes a navigational feedback in the control input term that successfully lead to flocking. Stability analysis for Olfati-Saber's model is provided and time delay and obstacle avoidance ability of agents are also considered.

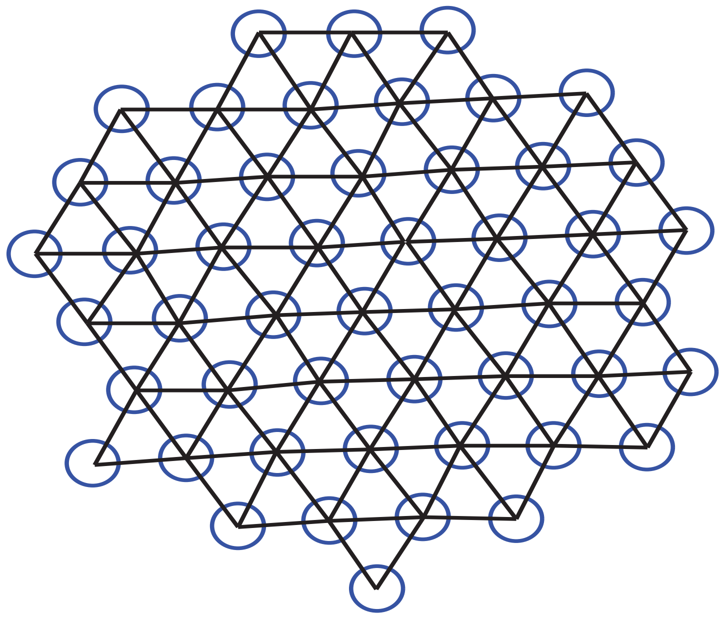


Figure 3.1: Example of 2D α-lattices

3.1 α-Lattices and Quasi α-Lattices

One of our objectives is to design a flocking algorithm with abilities that allow the group of dynamical agents to maintain rigid inter-agents distances over a net $G(r)$. Hence we should consider the following set of inter-agent algebraic constraints:

$$\|r_j - r_i\| = d, \ \forall j \in N_i(r) \tag{3.2}$$

Definition 3.1. *(α-lattice) An α-lattice is a configuration r satisfying the set of constraints in 3.2. We refer to d and $k = R/d$ as the scale and ratio of an α-lattice respectively.*

All edges of a structure $(G(r), r)$ that are induced by an α-lattice have the same length. One common examples of 2D α-lattice is illustrated in Fig.3.1

We also need to use a slightly deformed version of α-lattice that is defined as follows:

Definition 3.2. *(quasi α-lattice) A quasi α-lattice is a configuration r satisfying the following set of inequality constraints:*

$$-\delta \leq \|r_j - r_i\| - d \leq \delta, \ \forall (i,j) \in E(r)$$

where $\delta << d$ is the edge-length uncertainty, d is the scale and $k = r/d$ is the ratio of the quasi α-lattice.

3.2 Flocking Without Navigational Feedback

In this section we will study a well-known model without navigational feedback which is proposed by Tanner in [54] in 2003. In this model Tanner used a damping term and an artificial potential function which describes the attractive and repulsive behaviour between agents in the system. The motion of each agent is determined by two factors:

1. Attraction to the other agents over long distances.
2. Repulsion from the other agents over short distances.

Tanner's model can be described as follows:

$$\begin{cases} \dot{r}_i = v_i \\ \dot{v}_i = u_i \quad i = 1, \ldots, N \end{cases}$$

The control input can be divided into two components:

$$u_i = a_i + \alpha_i.$$

The first component a_i is attributed to an artificial potential function V_i, which depends on the relative position information between agent i and its neighbours. The second component α_i regulates the velocity vectors of agent i to the average of that of its neighbours.

And the potential function is defined as follows:

$$V_{ij} = \begin{cases} \frac{1}{\|r_{ij}\|^2} + \log \|r_{ij}\|^2, & \|r_{ij}\| < R \\ V_R, & \|r_{ij}\| \geq R \end{cases}$$

where $r_{ij} = r_i - r_j$. For agent i the (total) potential V_i is formed by summing the potentials due to each of its neighbours:

$$V_i \triangleq (N - |N_i|)V_R + \sum_{j \in N_i} V_{ij}(\|r_{ij}\|) \tag{3.3}$$

where $|N_i|$ is number of neighbours of agent i. The control law u_i is defined as:

$$u_i = -\sum_{j \in N_i}(v_i - v_j) - \sum_{j \in N_i} \nabla_{r_i} V_{ij} \tag{3.4}$$

Tanner considered two different situations. One is that the topology of the group of agents is fixed, which means the interacting neighbours of each agent is fixed. The second situation is that the topologies of the group of agents are changing with time (dynamical). The analysis of those two situations are discussed in following sections.

3.2.1 Stable Flocking With Fixed Topology

In this section we will give the stability analysis for Tanner's model in [54] with fixed topologies. Let us consider the following positive semi-definite function

$$W = \frac{1}{2}\sum_{i=1}^{N}(V_i + v_i^T v_i)$$

The following sets

$$\Omega = \{(v_i, r_{ij}) | W \leq c\} \tag{3.5}$$

are compact sets in the space of agent velocities and relative distances. This is because the set $\{r_{ij}, v_i\}$ such that $W \leq c$, for $c > 0$, is closed by continuity. Boundedness, on the other hand, follows from connectivity: from $W \leq c$ we have that $V_{ij} \leq c$. Connectivity ensures that a path connecting nodes i and j has length at most $N-1$. Thus $\|r_{ij}\| \leq V_{ij}^{-1}(c(N-1))$. Similarly, $v_i^T v_i \leq c$ yielding $\|v\|_i \leq \sqrt{c}$. Due to V_i being symmetric with respect to r_{ij} and the fact that $r_{ij} = -r_{ji}$,

$$\frac{\partial V_{ij}}{\partial r_{ij}} = \frac{\partial V_{ij}}{\partial r_i} = -\frac{\partial V_{ij}}{\partial r_j} \tag{3.6}$$

and therefore it follows:

$$\frac{d}{dt}\sum_{i=1}^{N}\frac{1}{2}V_i = \sum_{i=1}^{N}\nabla_{r_i} V_i v_i \tag{3.7}$$

Theorem 3.1. *(Tanner et al.(2003) [54]) Consider a system of N mobile agents with dynamics (3.2), each steered by control law (3.4) and assume that the graph is connected. Then all agent velocity vectors become asymptotically the same, collisions between inter-connected agents are avoided and the system approaches a configuration that minimizes all agent potentials.*

Proof. Taking the time derivative of W, we have:

$$\dot{W} = \frac{1}{2}\sum_{i=1}^{N} \dot{V}_i - \sum_{i=1}^{N} v_i^T \big(\sum_{j \in N_i} (v_i - v_j) + \nabla_{r_i} V_i\big) \tag{3.8}$$

due to the symmetric nature of V_{ij}, this can be simplified to

$$\begin{aligned} \dot{W} &= \sum_{i=1}^{N} v_i^T \nabla_{r_i} V_i - \sum_{i=1}^{N} v_i^T \big(\sum_{j \in N_i} (v_i - v_j) + \nabla_{r_i} V_i\big) \\ &= -\sum_{i=1}^{N} v_i^T \sum_{j \in N_i} (v_i - v_j) \\ &= -v^T (L \otimes I_2) v \end{aligned}$$

where v is the state vector of all agent velocity vectors, L is the Laplacian of the neighbouring graph and $\otimes$ denotes the Kronecker matrix product. Writing the quadratic form explicitly,

$$\dot{W} = -v_x^T L v_x - v_y^T L v_y \tag{3.9}$$

where v_x and v_y are the state vectors of the components of the agent velocities along x and y directions respectively. For a connected graph G, L is positive semidefinite and the eigenvector associated with the single zero eigenvalue is $\mathbf{1}$. Thus $\dot{W} = 0$ implies that both v_x and v_y belong to span$\{\mathbf{1}\}$. This means that all agent velocities have the same components and are therefore equal. It follows immediately that $\dot{r}_{ij} = 0, \forall (i,j) \in N \times N$. Application of Lasalle's invariance principle establishes convergence of system trajectories to $S = \{v | \dot{W} = 0\}$. In S, the agent velocity dynamics become:

$$\dot{v} = -\begin{bmatrix} \nabla_{r_1} V_1 \\ \vdots \\ \nabla_{r_N} V_N \end{bmatrix} = -(A \otimes I_2) \begin{bmatrix} \vdots \\ \nabla_{r_{ij}} V_{ij} \\ \vdots \end{bmatrix} \tag{3.10}$$

where A is the adjacency matrix of the fixed graph. $\dot{v}$ can be expanded to

$$\dot{v}_x = -A[\nabla_{r_{ij}} V_{ij}]x$$
$$\dot{v}_y = -A[\nabla_{r_{ij}} V_{ij}]y$$

Thus, $\dot{v}_x$ and $\dot{v}_y$ belong in the range of the adjacency matrix A. For a connected graph, range(A) =span$\{\mathbf{1}\}^\perp$ and therefore

$$\dot{v}_x, \dot{v}_y \in span\{\mathbf{1}\}^\perp \tag{3.11}$$

In an invariant set within S,

$$v_x, v_y \in span\{\mathbf{1}\} \Rightarrow \dot{v}_x, \dot{v}_y \in span\{\mathbf{1}\}. \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$\dot{v}_x, \dot{v}_y \in span\{\mathbf{1}\} \cap span\{\mathbf{1}\}^\perp = \{0\}. \tag{3.13}$$

Thus, in steady state agent velocities must not change. Furthermore, from (3.10) it follows that in steady state the potential V_i of each agent i is minimized. Interconnected agents cannot collide since this will result in $V_i \to \infty$ and the system departing Ω, which is a contradiction since Ω is positively invariant. □

3.2.2 Stable Flocking With Switching Topologies

In this section, the topologies of the group of agents in the system are no longer fixed. We begin by proposing a similar theorem for flocking with switching topologies as in last section,

Theorem 3.2. *(Tanner et al.(2003) [54]) Consider a system of N mobile agents with dynamics (3.2), each steered by control law (3.4) and assume that the neighbouring graph is connected. Then all pairwise velocity differences converge asymptotically to zero, collisions between the agents are avoided, and the system approaches a configuration that minimizes all agent potentials.*

The proof for flocking with switching topologies is similar to the case of fixed topology and is omitted here, a detailed proof given by Tanner can be found in [54].

One thing that needs to be mentioned is that in both cases of Tanner's model, an important assumption which must be satisfied for successful flocking is that the neighbouring

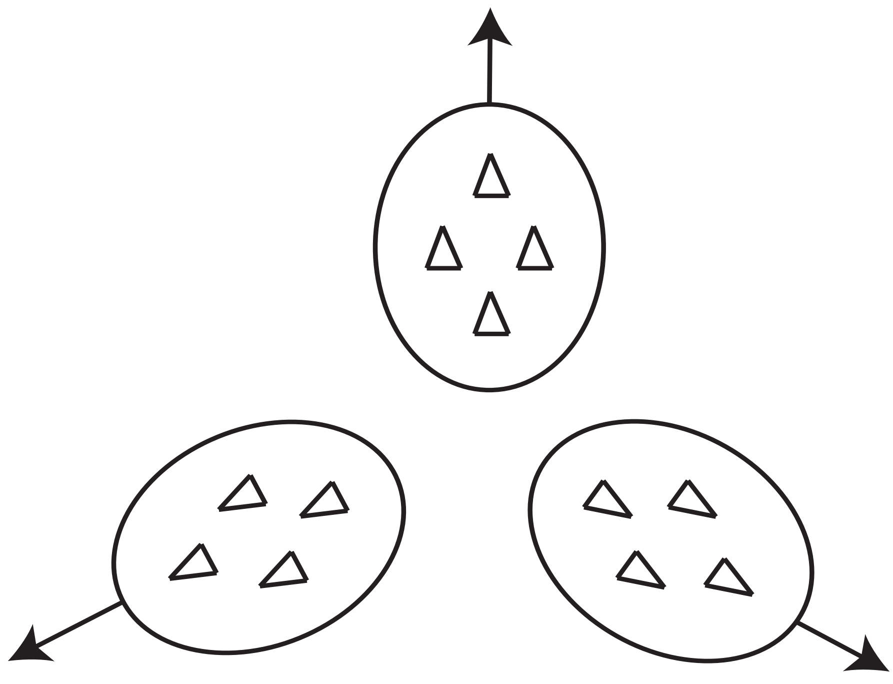


Figure 3.2: Fragmentation Phenomenon

graph G must remain connected. This guarantees that the network is always connected in both cases. If this crucial assumption is not satisfied, one possible situation is that the initial positions of some agents are too far away from the rest, then the neighbour set of those agents are empty, which means they cannot interact with any other agents and leads to fragmentation of the flock. An example of fragmentation phenomenon is given in Fig.3.2.

3.3 Flocking With Navigational Feeback

The model without navigational feedback can only lead to flocking for a very restricted set of initial states, if the network connectivity assumption is not satisfied it may lead to fragmentation. In [45], Olfati-Saber introduced three kinds of agents based on Tanner's model: $\alpha - agents$, $\beta - agents$ and $\gamma - agents$. $\alpha - agent$ refer to a physical agent which has the same meaning as previous model, while $\beta - agent$ is introduced as a representation of all nearby obstacles whenever the $\alpha - agent$ is in close proximity of an obstacle , and $\gamma - agent$ denotes a common group objective or virtual leader of the flock, which eliminates the fragmentation phenomenon. By applying these three new terms in the control inputs, Olfati-Saber's algorithm has successfully led to flocking with obstacles avoidance ability.

One significant capability of Olfati-Saber's algorithm is to allow a group of dynamic agents to maintain identical/quasi-identical inter-agent distances between interacting agents, i.e. group dynamics that satisfy the algebraic constrains in (3.2). We will present a distributed algorithm for flocking in free-space, or free-flocking. (The flocking algorithm with obstacle avoidance capabilities is presented in section 3.5.) We refer to a physical agent with equation of motion $\ddot{r}_i = u_i$ as an α-agent. α-agents correspond to birds, or member of a flock. An α-agent has a tendency to stay at a distance $d > 0$ from all of its neighbouring α-agents, this is the reason behind the name α-lattice. In free-flocking, each α-agent has a control input that consists of three components:

$$\begin{cases} \dot{r}_i = v_i, \\ \dot{v}_i = u_i, \quad i = 1, \dots, N, \end{cases} \tag{3.14}$$

$$u_i = f_i^g + f_i^d + f_i^\gamma \tag{3.15}$$

where $f_i^g = -\nabla_{r_i} V(r)$ is a gradient-based term, f_i^d is a velocity consensus/alignment term that acts as damping force, and f_i^γ is a navigational feedback due to a group objective. Example of a group objective is a destination where a flock moves towards during migration. Olfati-Saber proposed an algorithm that can be used for creation of flocking motion in $\mathbb{R}^m$ as follows:

$u_i = u_i^\alpha + u_i^\gamma$, or

$$u_i = \sum_{j \in N_i} \phi_\alpha(\|r_j - r_i\|_\sigma) n_{ij} + \sum_{j \in N_i} a_{ij}(r)(v_j - v_i) + f_i^\gamma(r_i, v_i) \tag{3.16}$$

where n_{ij} is a vector along the line connecting r_i to r_j and is given by

$$n_{ij} = \frac{r_j - r_i}{\sqrt{1 + \epsilon \|r_j - r_i\|^2}} \tag{3.17}$$

and ϕ_α is defined as in (2.8), $[a_{ij}]$ is the adjacency matrix of the net $G(r)$ and f_i^γ is the navigational feedback is given by:

$$f_i^\gamma(r_i, v_i, r_\gamma, v_\gamma) = -c_1(r_i - r_\gamma) - c_2(v_i - v_\gamma),\ c_1, c_2 > 0. \tag{3.18}$$

The pair $(r_\gamma, v_\gamma) \in \mathbb{R}^m \times \mathbb{R}^m$ is the state of a γ-agent. A γ-agent is a dynamic/static agent that represents a group objective. Let (r_d, v_d) be a fixed pair of m-vectors that denote the initial position and velocity of a γ-agent. A dynamic γ-agent has the following model

$$\begin{cases} \dot{r}_\gamma = v_\gamma \\ \dot{v}_\gamma = f_\gamma(r_\gamma, v_\gamma) \end{cases} \tag{3.19}$$

with $(r_\gamma(0), v_\gamma(0)) = (r_d, v_d)$. A static γ-agent has a fixed state that is equal to (r_d, v_d) for all time. The design of $f_\gamma(r_\gamma, v_\gamma)$ for a dynamic γ-agent is part of tracking control design for a group of agents. For example, the choice of $f_\gamma = 0$ leads to a γ-agent that moves along a straight line with a desired velocity v_d.

3.3.1 Collective Dynamics

The collective dynamics of a group of α-agents applying protocol (3.16) is in the form:

$$\textit{collective dynamics} : \begin{cases} \dot{r} &= v \\ \dot{v} &= -\nabla V(r) - \hat{L}(r)v + f_\gamma(r, v, r_\gamma, v_\gamma) \end{cases} \tag{3.20}$$

where $V(r)$ is a smooth collective potential function given in (2.6) and $\hat{L}(r)$ is the m-dimensional Laplacian of the net $G(r)$ with a position-dependent adjacency matrix $A(r) = [a_{ij}(r)]$, r, v are the state vector of r_i, v_i respectively.

The first expected result is that with $f_\gamma = 0$, system (3.20) is a dissipative particle system with Hamiltonian:

$$H(r, v) = V(r) + \sum_{i=1}^{N} \|v_i\|^2 \tag{3.21}$$

This is due to $\dot{H} = -v^T \hat{L}(r) v \leq 0$ and the fact that the multi-dimensional graph Laplacian $\hat{L}(r)$ is a positive semidefinite matrix for all r. The key in stability analysis of collective dynamics is employing a correct coordinate system that allows the use of LaSalle's invariance principle. One naive approach is to use $H(r, v)$ in the (r, v)-coordinates. The reason such an approach does not work is that one cannot establish the boundedness of solutions. During fragmentation, the solution cannot remain bounded. Therefore, Olfati-Saber proposed the use of a moving frame to analyze the stability of flocking motion.

3.3.2 Decomposition Dynamics

Consider a moving frame that is centred at r_c, the centre of mass of all agents. Let $Ave(z) = \frac{1}{n}\sum_{i=1}^{n} z_i$ denote the average of the z_i's with $z = col(z_1, \ldots, z_n)$. Let $r_c = Ave(r)$ and $v_c = Ave(v)$ denote the position and velocity of the origin of the moving frame. Then $\dot{r_c}(t) = v_c(t)$ and $\dot{v}_c(t) = Ave(u(t))$. Our objective is to separate the analysis of the motion of the centre of the group with respect to the reference frame from the collective motion of the agents in the moving frame. The position and velocity of agent i in the moving frame are given by

$$\begin{cases} \hat{r_i} & = r_i - r_c \\ \hat{v_i} & = v_i - v_c \end{cases} \tag{3.22}$$

The relative position and velocities remain the same in the moving frame, i.e. $\hat{r_j} - \hat{r_i} = r_j - r_i$ and $\hat{v_j} - \hat{v_i} = v_j - v_i$. Thus, $V(r) = V(\hat{r})$ and $\nabla V(r) = \nabla V(\hat{r})$. The control input in the moving frame can be expressed as

$$u_i^\alpha = \sum_{j \in N_i} \phi_\alpha(\|\hat{r_j} - \hat{r_i}\|_\sigma) n_{ij} + \sum_{j \in N_i} a_{ij}(\hat{r})(\hat{v_j} - \hat{v_i}) \tag{3.23}$$

with $a_{ij}(\hat{r}) = ph(\|\hat{r_j} - \hat{r_i}\|_\sigma)/r_\alpha$. Now we will present a decomposition lemma that is the basis for posing a structural stability problem for “dynamic flocks” (a dynamic network with a topology that is a connected net and nodes that are particles).

Lemma 3.1. *(Decomposition) (Olfati-Saber et al.(2004) [45]) Suppose that the navigational feedback $f_\gamma(r, v)$ is linear, i.e. there exists a decomposition of $f_\gamma(r, v)$ of the following form:*

$$f_\gamma(r, v, r_\gamma, v_\gamma) = g(\hat{r}, \hat{v}) + h(r_c, v_c, r_\gamma, v_\gamma). \tag{3.24}$$

Then, the collective dynamics of a group of agents can be decomposed as n second-order systems in the moving frame:

$$\textit{structural dynamics} : \begin{cases} \dot{\hat{r}} & = \hat{v} \\ \dot{\hat{v}} & = -\nabla V(\hat{r}) - L(\hat{v}) + g(\hat{r}, \hat{v}) \end{cases} \tag{3.25}$$

and one second-order system in the reference frame:

$$\textit{translational dynamics} : \begin{cases} \dot{r_c} & = v_c \\ \dot{v_c} & = h(r_c, v_c, r_\gamma, v_\gamma) \end{cases} \tag{3.26}$$

where

$$g(\hat{r}, \hat{v}) = -c_1\hat{r} - c_2\hat{v} \tag{3.27}$$

$$h(r_c, v_c, r_\gamma, v_\gamma) = -c_1(r_c - r_\gamma) - c_2(v_c - v_\gamma) \tag{3.28}$$

and (r_γ, v_γ) *is the state of* γ*-agent.*

A detailed proof of this lemma is given by Olfati-Saber in [45].

3.3.3 Stability Analysis

According to the decomposition lemma, we are now at the position to define stable flocking motion as the combination of the following forms of stability properties:

1. Stability of certain equilibria of the structural dynamics in the moving frame.
2. Stability of a desired equilibrium of the translational dynamics in the reference frame.

The challenging part of stability analysis for a flocking algorithm is to establish part 1. Analysis of part 2 is far more simple than part 1. As far as animal behaviour is concerned, the translational dynamics of a flock does not necessarily have to possess an equilibrium point that is stable or asymptotically stable. A flock of birds could circle an area over and over or move in a erratic and unpredictable manner. However, from an engineering perspective, an overall control over the collective behaviour of a flock is highly desired. In fact, the reason to perform flocking for UAVs is to steer a group of vehicles from point A to B as a whole. Thus, in robotics or engineering applications, performing the second task becomes very crucial. As a consequence, flocking protocols such as (3.16) that account for the group objective are beneficial for engineering applications.

The significant differences between Tanner's model and Olfati-Saber's model are due to the differences in their perspective structural dynamics. Given Tanner's model which has no navigational feedback, one obtains the following structural dynamics:

$$\Sigma_1 : \begin{cases} \dot{\hat{r}} & = \hat{v} \\ \dot{\hat{v}} & = -\nabla V(\hat{r}) - L(\hat{r})\hat{v} \end{cases} \tag{3.29}$$

with a positive semidefinite Laplacian matrix $\hat{L}(r)$. In comparison, the structural dynamics of a group of agents with a γ-agent (navigational feedback) in Olfati-Saber's model is in the form:

$$\Sigma_2 : \begin{cases} \dot{\hat{r}} & = \hat{v} \\ \dot{\hat{v}} & = -\nabla U_\lambda(\hat{r}) - D(\hat{r})\hat{v} \end{cases} \tag{3.30}$$

where U_λ is called the aggregate potential function and is defined by

$$U_\lambda(\hat{r}) = V(\hat{r}) + \lambda J(\hat{r}). \tag{3.31}$$

The map $J(\hat{r}) = \frac{1}{2}\sum_{i=1}^{n} \|\hat{r}_i\|^2$ is the moment of inertia of all agents and $\lambda = c_1 > 0$ is a parameter of the navigational feedback, and the matrix $D(\hat{r}) = c_2 I_m + L(\hat{r})$ is a positive definite matrix with $c_2 > 0$.

Define the structural dynamics of system (3.29) and (3.30) as follows:

$$H(\hat{r}, \hat{v}) = V(\hat{r}) + K(\hat{v}) \tag{3.32}$$

$$H_\lambda(\hat{r}, \hat{v}) = U_\lambda(\hat{r}) + K(\hat{v}) \tag{3.33}$$

where $K(\hat{v}) = \frac{1}{2}\sum_{i}^{n} \|\hat{v}_i\|^2$ is the kinetic energy of the agents in the moving frame. We now need to define "cohesion of a group" and "flocks".

Definition 3.3. *(a cohesive group) Let $(\hat{r}(t), \hat{v}(t))$ be the state trajectory of a group of dynamic agents over the time interval $[t_0, t_f]$. We say the group is cohesive for all $t \in [t_0, t_f]$ if there exists a ball of radius $R > 0$ centred at $r_c(t) = Ave(r(t))$ that contains all the agents for all time $t \in [t_0, t_f]$, i.e. $\exists R > 0 : \|\hat{r}\| \leq R, \forall t \in [t_0, t_f]$.*

Definition 3.4. *(flocks, quasi-flocks, dynamic flocks) The configuration r of a set of points ν is called a flock with interaction range R if the net $G(r)$ is connected. r is called a quasi-flock if $G(r)$ has a giant component (i.e. a connected subgraph with relatively large number of nodes). A group of α-agents are called a dynamic flock over the time interval $[t_0, t_f)$ if at every moment $t \in [t_0, t_f)$, they are a flock.*

Theorem 3.3. *(Olfati-Saber et al.(2004) [45]) Consider a group of α-agents with structural dynamics Σ_1 (3.29). Let $\omega_c = \{(\hat{r}, \hat{v}) | H(\hat{r}, \hat{v}) \leq c\}$ be a set of the Hamiltonian $H(\hat{r}, \hat{v})$ of (3.29) such that for any solution starting in ω_c, the group of agents is cohesive for all $t \geq 0$. Then, the follow statements hold:*

1. *Solution of the structural dynamics converges to an equilibrium $(\hat{r}^*, 0)$ with a configuration $\hat{r}^*$ that is an α-lattice.*

2. *The velocity of all agents asymptotically match in the reference frame.*

3. *Given $c < c^* = \Psi_\alpha(0)$, no inter-agent collisions occur for all $t \leq 0$.*

Proof. Any solution $(r(t), v(t))$ of the collective dynamics of α-agents with structural dynamics Σ_1 is uniquely mapped to a solution $(\hat{r}(t), \hat{v}(t))$ of the structural dynamics. We have

$$\dot{H}(\hat{r}, \hat{v}) = -\hat{v}^T \hat{L}(\hat{r})\hat{v} = -\frac{1}{2} \sum_{(i,j)\in\epsilon(\hat{r})} a_{ij}(\hat{r})\|\hat{v}_j - \hat{v}_i\|^2 \leq 0 \tag{3.34}$$

which means the structural energy $H(\hat{r}, \hat{v})$ is non-increasing for all $t \geq 0$. In addition, $H(\hat{r}(t), \hat{v}(t)) \leq c$ for all $t \geq 0$ implies Ω_c is an invariant set. This guarantees that the velocity mismatch is upper bounded by c because of

$$K(\hat{v}(t)) \leq H(\hat{r}(t), \hat{v}(t)) \leq c, \ \forall t \geq 0.$$

By assumption, for any solution starting in Ω_c, the group is cohesive in all time $t \geq 0$. Hence, there exists an $R > 0$ such that $\|\hat{r}(t)\| \leq R, \forall t \geq 0$. The combination of boundedness of velocity mismatch and group cohesion guarantees boundedness of solution of Σ_1 starting in Ω_c. This fact is the result of the following inequality:

$$\|(\hat{r}(t), \hat{v}(t))\|^2 = \|\hat{r}(t)\|^2 + \|\hat{v}(t)\|^2 \leq R^2 + 2c = C \tag{3.35}$$

where $C > 0$ is a constant.

From LaSalle's invariance principle, all the solutions of Σ_1 starting in Ω_1 converge to the largest invariant set in $E = \{(\hat{r}, \hat{v}) \in \Omega_c : \dot{H} = 0\}$. However since the group of α-agents constitutes a dynamic flock for all $t \geq 0$, $G(r(t))$ is a connected graph for all $t \geq 0$. Thus, based on equation (3.34), we know that the velocities of all agents match in the moving frame, i.e. $\hat{v}_1 = \cdots = \hat{v}_n$. But $\sum_i \hat{v}_i = 0$, therefor, $\hat{v}_i = 0$ for all i. This means that the velocity of all agents asymptotically match in the reference frame, or $v_1 = \cdots = v_n$, which proves part 2. Moreover, the configuration $\hat{r}$ asymptotically converges to a fixed configuration $\hat{r}^*$ that is an extrema of $V(x)$, which means $\nabla V(\hat{r}^*) = 0$.

Since any solution of the system starting at certain equilibria such as local maxima or saddle points remain in those equilibria for all time, not all solutions of the system converge to a local minima. However, anything but a local minima is an unstable equilibria. Thus, almost every solution of the system converges to an equilibrium $(\hat{r}^*, 0)$ where $\hat{r}^*$ is a local minima of $V(\hat{r})$. According to Theorem 3.4, every local minima of $V(\hat{r})$ is an α-lattice. Therefore, $\hat{r}^*$ is an α-lattice and asymptotically all inter-agent distances between neighbouring α-agents become equal to d. This finishes the proof of parts 1 and 2. We prove part 3 by contradiction. Assume there exists a time $t = t_1 > 0$ so that two distinct agents k, l collide, or $r_k(t_1) = r_l(t_1)$. for all $t \geq 0$, we have

$$\begin{aligned} V(r(t)) &= \frac{1}{2} \sum_i \sum_{j \neq i} \psi_\alpha(\|r_j - r_i\|_\sigma) \\ &= \psi_\alpha(\|r_k(t) - r_l(t)\|_\sigma) + \frac{1}{2} \sum_{i \in \nu \setminus \{k,l\}} \sum_{j \in \nu \setminus \{i,k,l\}} \psi_\alpha(\|r_j - r_i\|_\sigma) \\ &\geq \psi_\alpha(\|r_k(t) - r_l(t)\|_\sigma). \end{aligned}$$

Hence, $V(r(t_1)) \geq \psi_\alpha(0) = c^*$. But the velocity mismatch is a non-negative quantity and Ω_c is an invariant set of H. This yields:

$$V(r(t)) = H(\hat{r}(t), \hat{v}(t)) - K(\hat{v}(t)) \leq H(\hat{r}(t), \hat{v}(t)) \leq c \leq c^*, \ \forall t \geq 0$$

which is in contradiction with an earlier inequality $V(r(t_1)) \geq c^*$. Therefore, no two agents collide at any time $t \geq 0$. □

Theorem 3.4. *Every local minima of $V(r)$ is an α-lattice and vice versa.*

A detailed proof given by Olfati-Saber can be found in [45].

The following result provides a global stability analysis with structural dynamics Σ_2 (3.30) that is useful for creation of flocking motion for generic sets of initial conditions. In comparison to Theorem 3.3, no assumptions regarding group cohesion or connectivity of the net are made in the following theorem:

Theorem 3.5. *(Olfati-Saber et al.(2004) [45]) Consider a group of α-agents with structural dynamics Σ_2 (3.30) and $c_1, c_2 > 0$. Assume that the initial kinetic function $K(\hat{v}(0))$ and inertia $J(\hat{r}(0))$ are finite. Then, the following statements hold:*

1. *The group of agents remain cohesive for all $t \geq 0$.*
2. *Solution of Σ_2 (3.30) asymptotically converges to an equilibrium $(\hat{r}^*_\lambda, 0)$ where $\hat{r}^*_\lambda$ is a local minima of $U_\lambda(\hat{r})$.*
3. *The velocity of all agents asymptotically match in the reference frame.*

4. *Assume the initial structural energy of the agents is less than* $(k+1)c^*$ *with* $c^* = \psi_\alpha(0)$ *and* $k > 0$. *Then, at most* k *distinct pairs of* α*-agents could possibly collide* ($k = 0$ *guarantees a collision-free motion).*

Proof. First, note that the multi-agent system with structural dynamics Σ_2 and Hamiltonian $H_\lambda(\hat{r}, \hat{v}) = U_\lambda(\hat{r}) + K(\hat{v})$ is a strictly dissipative particle system in the moving frame because it satisfies

$$\dot{H}_\lambda(\hat{r}, \hat{v}) = -\hat{v}^T(c_2 I_m + \hat{L}(\hat{r}))\hat{v} = -c_2(\hat{v}^T\hat{v}) - \hat{v}^T\hat{L}(\hat{r})\hat{v} < 0, \ \forall \hat{v} \neq 0. \tag{3.36}$$

Hence, the structural energy $H(\hat{r}, \hat{v})$ is monotonically decreasing for all $(\hat{r}, \hat{v})$ and

$$H_\lambda(\hat{r}(t), \hat{v}(t)) \leq H_0 = H_\lambda(\hat{r}(0), \hat{v}(0)) < \infty.$$

The finiteness of $H_0 = V(\hat{r}(0)) + \lambda J(\hat{r}(0)) + K(\hat{v}(0))$ follows from the assumption that the collective potential, the inertia and the velocity mismatch are all initially finite. Thus for all $t \geq 0$, we have

$$\begin{aligned} U_\lambda(\hat{r}(t)) &\leq H_0 \\ K(\hat{v}(t)) &\leq H_0 \end{aligned}$$

But $U_\lambda = V(\hat{r}) + \frac{\lambda}{2}\hat{r}^T\hat{r}$ with $\lambda > 0$ and $V(\hat{r}) \geq 0$ for all $\hat{r}$, therefore

$$\hat{r}^T(t)\hat{r}(t) \leq \frac{2H_0}{\lambda}, \ \forall t \geq 0.$$

This guarantees the cohesion of the group of α-agents for all $t \geq 0$ because the position of all agents remains in a ball of radius $R = \sqrt{2H_0/\lambda}$ centred at r_c. This cohesion property together with boundedness of velocity mismatch, or $K(\hat{v}(t)) \leq H_0$, guarantees boundedness of solutions of the structural dynamics Σ_2. To see this, let $z = col(\hat{r}, \hat{v})$, then

$$\|z(t)\|^2 = \hat{r}^T(t)\hat{r}(t) + \hat{v}^T(t)\hat{v}(t) \leq 2(\frac{1}{\lambda} + 1)H_0 = C(\lambda) < \infty.$$

Part 2 follows from LaSalle's invariance principle. Notice that $\dot{H}_\lambda(\hat{r}, \hat{v}) = 0$ implies $\hat{v} = 0$. Thus similar to the argument in the proof of Theorem 3.3, almost every solution of the multi-agent system asymptotically converges to an equilibrium point $z^*_\lambda = (\hat{r}^*_\lambda, 0)$ where $\hat{r}^*_\lambda$ is a local minima of the aggregate potential function $U_\lambda(\hat{r})$.

Part 3 follows from the fact that $\hat{v}$ asymptotically vanishes. Thus, the velocities of all agents asymptotically match in the reference frame.

To prove part 4, suppose $H_0 < (k+1)c^*$ and there are more than k distinct pairs of agents that collide at a given time $t_1 \geq 0$. Hence, there must be at least $k+1$ distinct pairs of agents that collide at time t_1. This implies the collective potential of the particle system at time $t = t_1$ is at least $(k+1)\psi_\alpha(0)$. However, we have

$$H_0 = V(\hat{r}(0)) + \lambda J(\hat{r}(0)) + K(\hat{v}(0)) \geq V(\hat{r}(0)) \geq (k+1)\psi_\alpha(0).$$

This contradicts the assumption that $H_0 < (k+1)c^*$. Hence, no more than k distinct pairs of agents can possibly collide at any time $t \geq 0$. Finally, with $k = 0$, no two agents can collide. □

Theorem 3.5 establishes some critical properties of collective behaviour of a group of agents with structural dynamic Σ_2 (3.30) including cohesion, convergence, asymptotic velocity matching and collision avoidance without the network connectivity assumption.

3.3.4 Simulation

In this section we will present a simulation result for Olfati-Saber's model with structural dynamic Σ_2 (3.30), a computer animation is available and the following parameters remain fixed throughout the simulation: $d = 12, R = 1.2d, \epsilon = 0.1$ (for σ-norm), $a = b = 5$ for $\phi(z)$, $h = 0.2$ for the bump function of $\phi_\alpha(z)$, and the step-size in the simulation is $0.01s$. In addition, the position of a static γ-agent is marked with a $*$ sign. The initial positions and velocity of all 50 α-agents are uniformly chosen at random form the box $[-10, 10]^2$. A flock is formed in Fig. 3.3 (d) and maintained thereafter. The number of edges of the dynamic graph $G(r(t))$ increases in time and has a tendency to render the net connected. Numerical measurements indicate that the final conformation is a low-energy quasi α-lattice that induces a flock. These observations are in close agreement with our analysis in last section.

Moreover, we define a velocity mismatch function with respect to all α-agents and the γ-agent as :$M(t) = \sum_{i=1}^{N}(\|v_i - v_\gamma\|)^2$ and plot it in Fig.3.4, we can see that the difference between α-agents and the γ-agent asymptotically converges to zero, which also denotes that the velocity mismatch between all α-agents asymptotically converge to zero. We can see that our simulation is consistent with our study in last section.

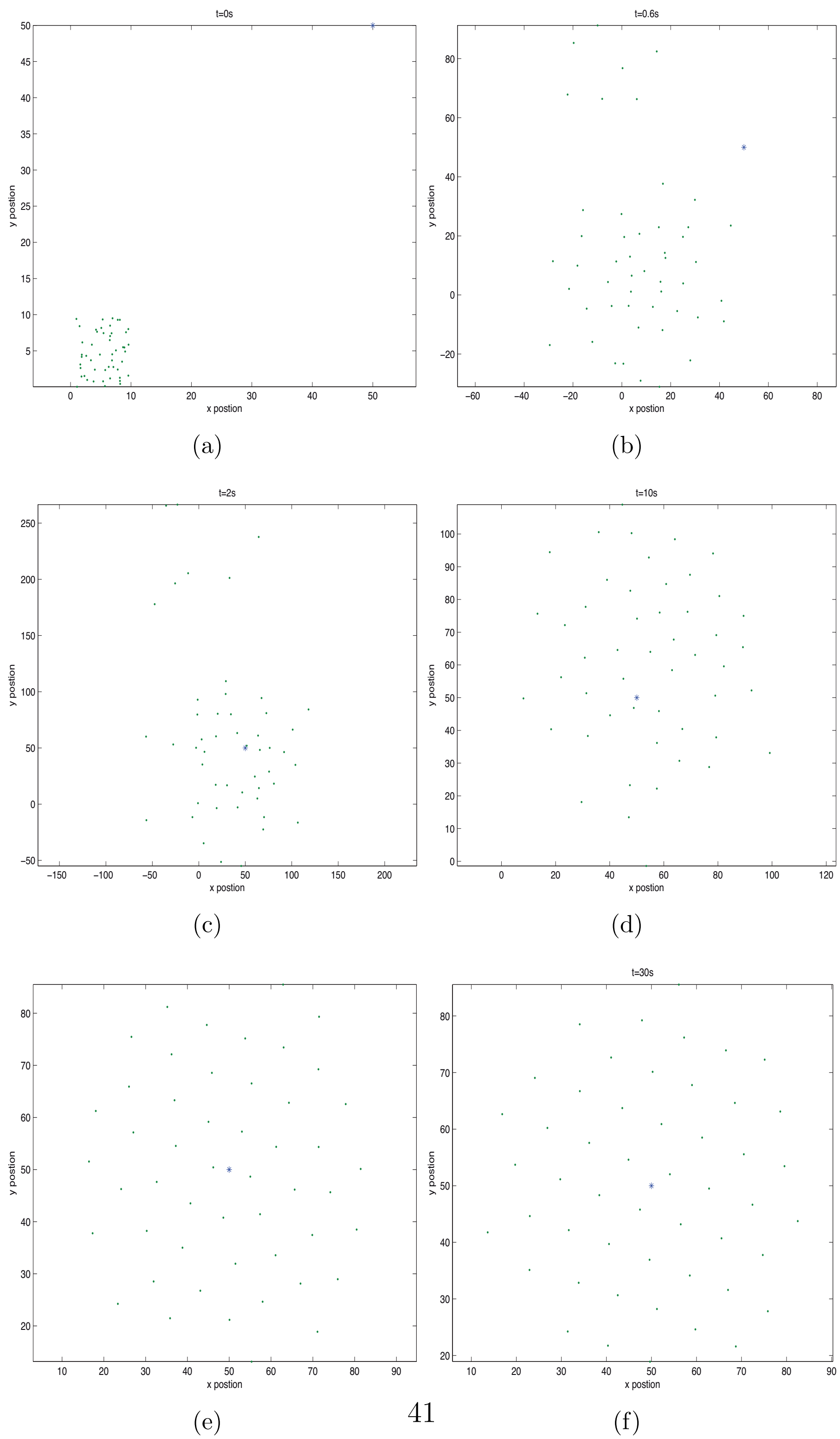


Figure 3.3: Flocking in Free-Space for n=50 agents.

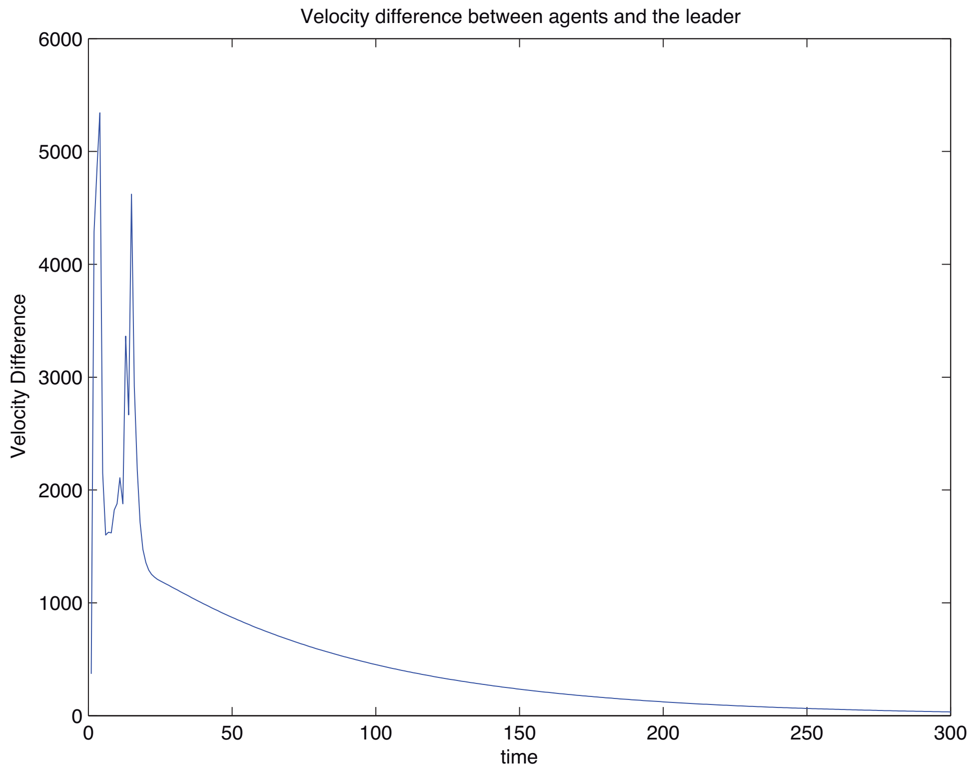


Figure 3.4: Velocity Mismatch

3.4 Communication Time Delay

In this section, we will study flocking for multi-agent dynamical system with time delay. Due to the finite speed of transmission and spreading as well as traffic congestions, there are usually time delay in spreading and communication in reality. Therefore, time delay should be considered in designing controls for multi-agent systems. For consensus problems, the influence of communication delay has been studied (Moreau 2004; Reza Olfati-Saber and Murray 2004; Xiao and Wang 2006; Hu and Hong 2007). For formation problems, the effect of communication delays has also been studied in Liu and Tian (2008), Ristian and Cesare (2008). It's remarkable that there are few results for flocking of multi-agent system with time delay in communication (Yang and Zhang 2010). So in this section we will first study a set of control laws with time delay for multi-agent system without leader, and the control laws applied to each agent relies on the state information. Then we will show that all agent velocities become asymptotically the same and avoidance of collision between agents is ensured. Next, we will study a set of control laws with time delay for multi-agent dynamical systems with a virtual leader. In this case the control law applied to each agent relies on the state information and the navigational feedback which is similar with our discussion in previous two sections. With this control law all agents can follow the virtual leader and collision-free can also be ensured.

3.4.1 Without Navigational Feedback

Given any dynamic graph $G(r(t)) = (\nu, E(t))$, define the set of control laws as following:

$$u_i = -\sum_{i=1}^{N} a_{ij}(v_i - v_j(t-\tau)) - \sum_{i=1}^{N} a_{ij}\nabla_{r_i} V_{ij} \tag{3.37}$$

where τ is the coupling time delay and $[a_{ij}]$ is the adjacency matrix. $\nabla_{r_i} V_{ij}$ corresponds to a vector in the direction of the negative gradient of an artificial potential function defined as follows:

$$V_{ij}(r_{ij}) = \frac{1}{R^2 - \|r_{ij}\|^2}, \ \|r_{ij}\|^2 \in (0, R) \tag{3.38}$$

with $r_{ij} = r_i - r_j$. This allows both collision free and maintenance of the connectivity of the network. Notice that V_{ij} grows unbounded when $\|r_{ij}\| \to R^-$, which means we apply an unbounded force to ensure that if an agent a is a neighbour of the other agent b, then a can never move out of the neighbourhood of agent b.

For the multi-agent dynamical system under consideration here, the relationship between neighbouring agents (the interconnection topology) changes over time. Hence, the system under control inputs (3.37) result in a dynamical switching system. Let t_p for $p = 1, 2, \ldots$ denote the switching times when the topology of $G(r(t))$ changes, and define a switching signal $\sigma(t) : [t_0 : \infty) \to C_N$ associated with connected graphs, we have the following theorem.

Theorem 3.6. *(Zhengquan et al.(2011) [62]) Assume a multi-agent dynamical system with control law (3.37), then for any pair of switching times $t_p < t_q$ the switching signal $\sigma(t)$ satisfies $E(t_p) \subseteq E(t_q)$ and the collision avoidance can always be guaranteed.*

Proof. Consider the following positive semidefinite Lyapunov function:

$$V_g = \sum_{i=1}^{N} v_i^T v_i + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} V_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} \int_{-\tau}^{0} v_i^T(t+\eta) v_i(t+\eta) d\eta$$

For any $c > 0$, let $\omega_c = \{(r_{ij}, v_i) | V_g \le c\}$ denote the level sets of V_g and observe that

$$
\begin{aligned}
\dot{V}_g &= 2\sum_{i=1}^{N} v_i^T(-\sum_{i=1}^{N} a_{ij}(v_i - v_j(t-\tau)) - \sum_{j=1}^{N} a_{ij}\nabla_{r_i} V_{ij}) \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\dot{V}_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[v_i^T v_i - v_i^T(t-\tau)v_i(t-\tau)] \\
&= -2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}v_i^T(v_i - v_j(t-\tau)) - 2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}v_i^T\nabla_{r_i} V_{ij} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\dot{V}_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[v_i^T v_i - v_i^T(t-\tau)v_i(t-\tau)].
\end{aligned}
$$

Note however that due to the symmetric nature of V_{ij},

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\dot{V}_{ij} &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\dot{r}_{ij}^T\nabla_{r_{ij}} V_{ij} \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\dot{r}_i^T\nabla_{r_{ij}} V_{ij} - \dot{r}_j^T\nabla_{r_{ij}} V_{ij}) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\dot{r}_i^T\nabla_{r_{ij}} V_{ij} + \dot{r}_j^T\nabla_{r_{ij}} V_{ji}) \\
&= 2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\dot{r}_i^T\nabla_{r_{ij}} V_{ij} \\
&= 2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}v_i^T\nabla_{r_{ij}} V_{ij}.
\end{aligned}
$$

Thus, $\dot{V}_g$ can be simplified to

$$\begin{aligned}
\dot{V}_g &= -2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}v_i^T(v_i - v_j(t-\tau)) + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[v_i^T v_i - v_i^T(t-\tau)v_i(t-\tau)] \\
&= -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}v_i^T v_i + 2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}v_i^T v_j(t-\tau) - \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}v_i^T(t-\tau)v_i(t-\tau) \\
&= -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[v_i^T v_i - 2v_i^T v_j(t-\tau) + v_j^T(t-\tau)v_j(t-\tau)] \\
&= -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[v_i - v_j(t-\tau)]^T[v_i - v_j(t-\tau)] \\
&\leq 0.
\end{aligned}$$

Hence, for any switching signal $\sigma(t)$, $\dot{V}_g$ is negative semidefinite. The negative semidefiniteness of $\dot{V}_g$ ensures V_g is non-increasing for all time. In addition, the boundedness of V_g implies that for any $(i, j) \in E$, V_{ij} is bounded by c and

$$V_{ij} \leq V_g \leq c << \infty$$

However, if for some $(i, j) \in E$, $\|r_{ij}\| \to R$ or $\|r_{ij}\| \to R$ imply $V_{ij}(r_{ij}) \to \infty$. Thus it follows that $\|r_{ij}\| < R$ for all $(i, j) \in E$ and $t \in [t_p, t_{p+1})$. In other words, all links in net $G(r(t))$ maintained between any switching times, this implies that $E(t_p) \subseteq E(t_{p+1})$. At the same time, the collision between any two interacting agents can be avoided. □

Theorem 3.7. *(Zhengquan et al.(2011) [62]) By taking the control law in (3.37) and assume that the initial net $G(r(t_0))$ is connected, velocities of all agents in the multi-agent dynamical system become asymptotically the same and avoidance of collisions between agents is ensured.*

Proof. Notice that in a net $G(r(t))$ the total number of vertices is finite so the total number of switching times of the multi-agent system is also finite. Hence the switching signal $\sigma(t)$ eventually becomes constant, i.e. $\sigma(t) \to \sigma$. It follows from Theorem 3.6 that if $G(r(t_0))$ is connected, $G(r(t))$ is connected for all time $t \geq t_0$ and eventually $\sigma(t) \to \sigma \in C_N$. So we can essentially study the convergence of the system once the switching signal has converged and the network topology is fixed. As in theorem 3.6, for any switching signal σ the potential V_g is positive definite and

$$\dot{V}_g = -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[v_i - v_j(t-\tau)]^T[v_i - v_j(t-\tau)] \leq 0.$$

The level set $\omega_c = \{(r_{ij}, v_i)|V_g \leq c, c > 0\}$ is closed by continuity and is bounded by the connectivity of the position neighbouring graph. By LaSalles' invariance principle, every solution starting in ω_c asymptotically converges to the largest invariant set in $\{(r_{ij}, v_i)|\dot{V}_g = 0\}$. $\dot{V}_g = 0$ implies that $v_1 = \cdots = v_N$. This means that the velocities of all agents asymptotically become the same. Therefor we have

$$\frac{d}{dt}\|r_i - r_j\|^2 = 2(r_i - r_j)^T(v_i - v_j) = 0,$$

and the distances between agents remain the same. □

3.4.2 With Navigational Feedback

Similar to our discussion in last two sections, now we will apply the communication time delay to the control with navigational feedback. Our objective is to enable the entire group move at a desired velocity v_0 and maintain constant distances between agents. Remember that the desired velocity v_0 is supposed to be a constant. We take the control law u_i for agent i to be the following:

$$u_i = -\sum_{i=1}^{N} a_{ij}(v_i - v_j(t-\tau)) - \sum_{i=1}^{N} a_{ij}\nabla_{r_i}V_{ij} - b_i(v_i - v_0) \tag{3.39}$$

where $b_i > 0$ denotes the weight of influence of the reference signal on the motion of agent i.

Define the error vectors as follows:

$$\begin{cases} e_i^p = r_i - v_0 t \\ e_i^v = v_i - v_0 \end{cases}$$

where t is the time variable, e_i^v represents the velocity difference vector between the actual velocity and the desired velocity (velocity of the virtual leader) of agent i. It is easy to see that $\dot{e}_i^p = e_i^v$ and $\dot{e}_i^v = \dot{v}_i$. Hence, the error dynamics is given by

$$\begin{cases} \dot{e}_i^p = e_i^v \\ \dot{e}_i^v = u_i, \ \ i = 1, 2, \ldots, N. \end{cases} \tag{3.40}$$

By the definition of V_{ij}, it follows that $V_{ij}(\|r_{ij}\|) = V_{ij}(\|e_{ij}^p\|) \doteq \hat{V}_{ij}$, where $e_{ij}^p = e_i^p e_j^p$. Thus, the control input for agent i in the error system has the following form:

$$u_i = -\sum_{i=1}^{N} a_{ij}([e_i^v - e_j^v(t-\tau)] + \nabla_{e_i^p}\hat{V}_{ij}) - b_i e_i^v. \quad (3.41)$$

Similar to the discussion for no leader's case, we have the following theorem:

Theorem 3.8. *(Zhengquan et al.(2011) [62]) Assume the multi-agent system with control rule (3.41), for any pair of switching times $t_p < t_q$, the switching signal $\sigma(t)$ satisfies $\epsilon(t_p) \subseteq \epsilon(t_q)$ and the collision avoidance can be always guaranteed.*

Proof. Consider the following positive semidefinite function:

$$\begin{aligned} W_g = &\sum_{i=1}^{N} e_i^{vT} e_i^v + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\hat{V}_{ij} \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\int_{-\tau}^{0} e_i^{vT}(t+\eta)e_i^v(t+\eta)d\eta. \end{aligned}$$

For any $c > 0$, let $\omega_c = \{(e_{ij}^p, e_i^v) | W_G \le c\}$ denote the level sets of V_g and observe that

$$\begin{aligned} \dot{W}_g = &-2\sum_{i=1}^{N} e_i^{vT}\{\sum_{j=1}^{N} a_{ij}([e_i^v - e_j^v(t-\tau)] + \nabla_{e_i^p}\hat{V}_{ij}) + b_i e_i^v\} \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\dot{\hat{V}}_{ij} + \sum_{i=1}^{N}\sum_{j=1} a_{ij}[e_i^{vT}e_i^v - e_i^{vT}(t-\tau)e_i(t-\tau)] \\ = &-2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}e_i^{vT}[e_i^v - e_j^v(t-\tau)] - 2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}e_i^{vT}\nabla_{e_i^p}\hat{V}_{ij} \\ &- 2\sum_{i=1}^{N} b_i e_i^{vT}e_i^v + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\dot{\hat{V}}_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[e_i^{vT}e_i^v - e_i^{vT}(t-\tau)e_i(t-\tau)]. \end{aligned}$$

Notice that due to the symmetric nature of V_{ij}

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\dot{\hat{V}}_{ij} = 2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}e_i^{vT}\nabla_{e_i^p}\hat{V}_{ij}$$

Thus $\dot{W}_g$ can be simplified to

$$\begin{aligned}
\dot{W}_g &= -2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} e_i^{vT}[e_i^v - e_j^v(t-\tau)] \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[e_i^{vT}e_i^v - e_i^{vT}(t-\tau)e_i(t-\tau)] - 2\sum_{i=1}^{N} b_i e_i^{vT} e_i^v \\
&= -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} e_i^{vT} e_i^v + 2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} e_i^{vT} e_j^v(t-\tau) \\
&- \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} e_i^{vT}(t-\tau)e_i(t-\tau) - 2\sum_{i=1}^{N} b_i e_i^{vT} e_i^v \\
&= -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[e_i^{vT}e_i^v - 2e_i^{vT}e_j^v(t-\tau) + e_j^{vT}(t-\tau)e_j(t-\tau)] - 2\sum_{i=1}^{N} b_i e_i^{vT} e_i^v \\
&= -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[e_i^v - e_j^v(t-\tau)]^T[e_i^v - e_j^v(t-\tau)] - 2\sum_{i=1}^{N} b_i e_i^{vT} e_i^v \\
&\leq 0
\end{aligned}$$

Hence, for any switching signal $\sigma(t)$, $\dot{W}_g$ is negative semidefinite. The negative semidefiniteness of $\dot{W}_g$ ensures W_g is non-increasing for all time. Moreover, the boundedness of W_g implies that for any $(i,j) \in E$, $\hat{V}_{ij}$ is bounded by c

$$\hat{V}_{ij} \leq W_g \leq c << \infty.$$

However, if for some $(i,j) \in E$, $\|r_{ij}\| \to R$ or $\|e_{ij}^p\| \to 0$, then $\hat{V}_{ij}(e_{ij}^p) \to \infty$. Thus it follows that $\|e_{ij}^p\| < R$ for all $(i,j) \in E$ and $t \in [t_p, t_{p+1})$. In other words, all links in $G(r(t))$ are maintained between switching times, which implies that $E(t_p) \subseteq E(t_{p+1})$. At the same time, the collision between any two interacting agents can be avoided. □

Theorem 3.9. *(Zhengquan et al.(2011) [62]) By taking the control law (3.41), assume that the initial net $G(r(t_0))$ is connected, velocities of all agents in the multi-agent system approach asymptotically the leader's velocity and avoidance of collisions between agents is ensured.*

Proof. Notice that in a net $G(r(t))$ the total number of vertices is finite so the total number of switching times of the multi-agent system is also finite. Hence the switching signal $\sigma(t)$ eventually becomes constant, i.e. $\sigma(t) \to \sigma$. It follows from Theorem 3.8 that if $G(r(t_0))$ is connected, $G(r(t))$ is connected for all time $t \geq t_0$ and eventually $\sigma(t) \to \sigma \in C_N$. So we can essentially study the convergence of the system once the switching signal has converged and the network topology is fixed. As in theorem 3.8, for any switching signal $\sigma(t)$ the potential W_g is positive definite and

$$\begin{aligned}\dot{W}_g &= -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[e_i^v - e_j^v(t-\tau)]^T[e_i^v - e_j^v(t-\tau)] - 2\sum_{i=1}^{N} b_i e_i^{vT} e_i^v \\ &\leq 0.\end{aligned}$$

The level set $\omega_c = \{(e_{ij}^p, e_i^p) | W_g \leq c, c > 0\}$ is closed by continuity and is bounded by the connectivity of the position neighbouring graph. By LaSalle's invariance principle, every solution starting in ω_c asymptotically converges to the largest invariant set in $\{(e_{ij}^p, e_i^v) | \dot{W}_g = 0\}$. $\dot{W}_g = 0$ implies that $e_1^v = \cdots = e_N^v = 0$. This occurs only when $v_1 = \cdots = v_N = v_0$ for $i = 1, 2, \ldots, N$. Thus all agents' velocities in the error system remain the same and all equal to zero in the steady-state. Moreover, we have

$$\frac{d}{dt}\|e_i^p - e_j^p\|^2 = 2(e_i^p - e_j^p)^T(e_i^v - e_j^v) = 0,$$

which implies the distances between agents are invariant.

□

3.5 Obstacle Avoidance Ability

In this section, we will present a distributed flocking algorithm with multiple obstacle-avoidance capability. The main idea is to use agent-based representation of all nearby obstacles by creating a new species of agents called β-agents [38]. A β-agent is a kinematic agent that is induced by an α-agent whenever the α-agent is in close proximity of an obstacle. In the following, we will define the notion of a β-agent and specify the interaction protocol between an α-agent and a β-agent. Moreover, we restrict our study to obstacles that lie in a connected convex region of $\mathbb{R}^m$ with boundaries that are smooth manifolds. More explicitly we mainly focus on obstacles that are either spheres or infinite walls as shown in Fig.3.5. We can approach to obstacle avoidance by summarizing in the following steps:

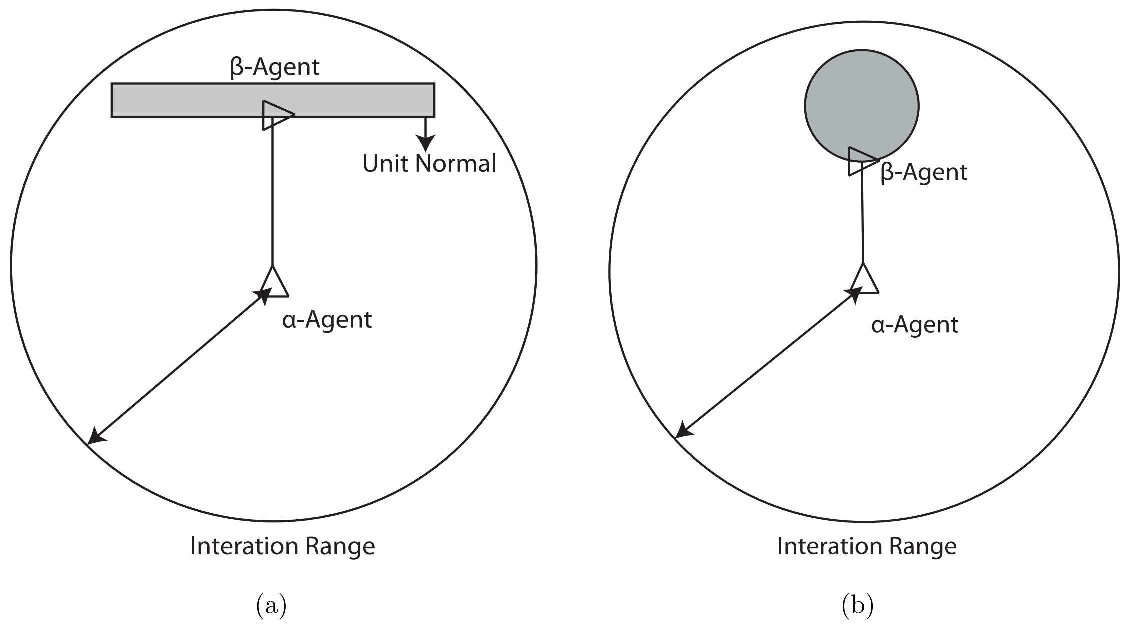


Figure 3.5: β-agent representation of obstacles: (a) a wall and (b) a spherical obstacle.

1. Determine the indices N_i^β of the set of obstacles O_k that are neighbours of α-agent i.
2. Create a (virtual) β-agent at $r_{i,k}^\beta$ on the boundary of a neighbouring obstacle O_k by projection. $r_{i,k}^\beta$ satisfies
$$r_{i,k}^\beta = argmin_{x \in O_k} \|x - r_i\| \tag{3.42}$$
and O_k is either a closed ball or a closed half space on one side of a hyperplane.
3. Add a term $\psi_\beta(\|r_{i,k}^\beta - r_i\|_\sigma)$ to the potential function of a group of α-agents corresponding to each β-agent at $r_{i,k}^\beta$ ($\psi_\beta(z)$ to be defined).

Let $\nu_\alpha = \{1, 2, \ldots, n\}$ and $\nu_\beta = \{1', 2', \ldots, l'\}$ denote the set of indices of α-agents and obstacles (β-agents), respectively. Notice that the prime in elements of ν_β is used to guarantee that $\nu_\alpha \cap \nu_\beta = \emptyset$. An α-agent is called a neighbour of an obstacle $O_k (k \in \nu_\beta)$ if and only if the ball $B_{r'}(r_i)$ and O_k overlap as shown in Fig.3.5. This form of neighbourhood between an α-agent and an obstacle is a mutual property. Moreover, an α-agent could have multiple neighbouring obstacles. Particularly, this occurs when a group of agents intend to pass through a narrow pathway so that an agent might come within close proximity of multiple obstacles.

We define the sets of α-neighbours and β-neighbours of an α-agent $i \in \nu_\alpha$ as follows:

$$N_i^\alpha = \{j \in \nu_\alpha : \|r_j - r_i\| < R\}$$
$$N_i^\beta = \{k \in \nu_\beta : \|r_{i,k}^\beta - r_i\| < R^\beta\}$$

where $R, R^\beta > 0$ are interaction ranges of an α-agent with neighbouring α-agents and β-agents, respectively. Here we choose $R^\beta < R$.

The sets of α-neighbours and β-neighbours of an α-agent $i \in \nu_\alpha$ naturally define an (α, β)-net that is a spatially induced graph in the form

$$G_{\alpha,\beta}(r) = G_\alpha(r) + G_\beta(r)$$

where $G_\alpha(r) = (\nu_\alpha, \epsilon_\alpha(r))$ is a net induced by configuration of all α-agents and $G_\beta(r) = (\nu_\beta, \epsilon_\beta(r))$ is a directed bipartite graph induced by r and the set of obstacles $O = \{O_k : k \in \nu_\beta\}$ where $\epsilon_\beta(r) \subseteq \nu_\alpha \times \nu_\beta$. The condition $\nu_\alpha \cap \nu_\beta = \emptyset$ guarantees well-posedness of the definition of the bipartite graph $G_\beta(r)$. More explicitly we have:

$$\epsilon_\alpha(r) = \{(i, j) : i \in \nu_\alpha, j \in N_i^\alpha\}$$
$$\epsilon_\beta(r) = \{(i, k) : i \in \nu_\alpha, k \in N_i^\beta\}$$

and $G_{\alpha,\beta}(r) = (\nu_\alpha \cup \nu_\beta, \epsilon_\alpha(r) \cup \epsilon_\beta(r))$. Similarly, an (α, β)-frame net is a structure $(G_{\alpha,\beta}(r), r, \hat{r})$ where $\hat{r}$ denotes the configuration of all β-agents.

The new set of inter-agent and agent-to-obstacle algebraic constraints for an α-agent is specifies as follows:

$$\begin{cases} \|r_j - r_i\| = d, & \forall j \in N_i^\alpha \\ \|r_{i,k}^\beta - r_i\| = d^\beta, & \forall k \in N_i^\beta \end{cases} \tag{3.43}$$

A constrained α-lattice denoted by (r, O) consists of an α-lattice r and a set of obstacles O that satisfy the set of constraints in (3.43). The relevant ratios of a constrained α-lattice are $k = R/d$ and $k^\beta = d^\beta/d = R^\beta/R$.

To achieve flocking in presence of obstacles, we use the following multi-species collective potential function for the multi-agent system:

$$V(r) = c_1^\alpha V_\alpha(r) + c_1^\beta V_\beta(r) + c_1^\gamma V_\gamma(r) \tag{3.44}$$

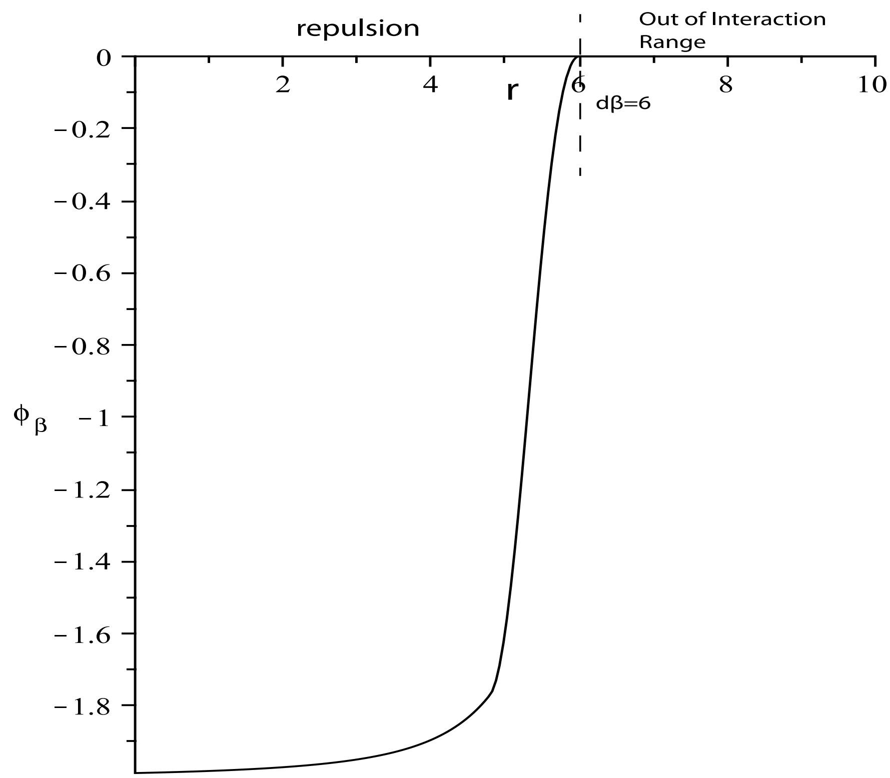


Figure 3.6: A Repulsive Action Function with $d_\beta = 6$

where the $c_1^\alpha, c_1^\beta, c_1^\gamma$ are positive constants and (α,α), (α,β), (α,γ) interaction potentials are defined as follows:

$$V_\alpha(r) = \sum_{i\in\nu_\alpha} \sum_{j\in\nu_\alpha\setminus\{i\}} \psi_\alpha(\|r_j - r_i\|_\sigma) \tag{3.45}$$

$$V_\beta(r) = \sum_{i\in\nu_\alpha} \sum_{k\in N_i^\beta} \psi_\beta(\|r_{i,k}^\beta - r_i\|_\sigma) \tag{3.46}$$

$$V_\gamma(r) = \sum_{i\in\nu_\alpha} (\sqrt{1 + \|r_i - r_\gamma\|^2} - 1). \tag{3.47}$$

The function $V_\gamma(r)$ relates to the navigational feedback of a group of α-agents. The heterogeneous adjacency matrix between an α-agent at r_i and its neighbouring β-agent at $r_{i,k}^\beta$ is defined as

$$b_{i,k}(r) = ph(\|r^{\beta}_{i,k} - r_i\|_{\sigma}/d_{\beta})$$

where $d_{\beta} < R_{\beta}$ with $d_{\beta} = \|d^{\beta}\|_{\sigma}, R_{\beta} = \|R^{\beta}\|_{\sigma}$. Define the *repulsive action function* as follows:

$$\phi_{\beta}(z) = \rho_h(z/d_{\beta})(\sigma_1(z - d_{\beta}) - 1) \tag{3.48}$$

with $\sigma_1(z) = \frac{z}{\sqrt{1+z^2}}$. Notice that $\phi_{\beta}(z)$ vanishes smoothly to zero at $z = d_{\beta}$ and remains zero for all $z \geq d_{\beta}$ as shown in Fig 3.6. Similarly define a *repulsive pairwise potential* $\psi_{\beta}(z)$ in the form

$$\psi_{\beta}(z) = \int_{d_{\beta}}^{z} \phi_{\beta}(s)ds \geq 0. \tag{3.49}$$

From the point-of view of application, it is unreasonable to apply unbounded forces to α-agents (vehicles/robots/animals). We avoid the use of repulsive potential functions with unbounded derivatives such as $1/z$ or $log(z)$ that are well-known examples of barrier functions. Clearly, $-2 < \psi'_{\beta}(z) \leq 0$ for all $z \in \mathbb{R}$ and thereby the derivative of $\psi_{\beta}(z)$ is uniformly bounded. Then we are ready to present the flocking algorithm with obstacle-avoidance capability:

We can divide the control input into three terms:

$$u_i = u_i^{\alpha} + u_i^{\beta} + u_i^{\gamma} \tag{3.50}$$

where u_i^{α} denotes the (α, α) interaction terms, u_i^{β} denotes the (α, β) interaction terms and u_i^{γ} is a distributed navigational feedback. We explicitly specify each term as follows:

$$u_i^{\alpha} = - c_1^{\alpha} \nabla_{r_i} V_{\alpha}(r) + c_2^{\alpha} \sum_{j \in N_i^{\alpha}} a_{ij}(r)(v_j - v_i) \tag{3.51}$$

$$= c_1^{\alpha} \sum_{j \in N_i^{\alpha}} \phi_{\alpha}(\|r_j - r_i\|_{\sigma}) n_{ij} + c_2^{\alpha} \sum_{j \in N_i^{\alpha}} a_{ij}(v_j - v_i) \tag{3.52}$$

$$u_i^{\beta} = - c_1^{\beta} \nabla_{r_i} V_{\beta}(r) + c_2^{\beta} \sum_{k \in N_i^{\beta}} b_{i,k}(r)(v^{\beta}_{i,k} - v_i) \tag{3.53}$$

$$= c_1^{\beta} \sum_{k \in \hat{N}_i} \phi_{\beta}(\|r^{\beta}_{i,k} - r_i\|_{\sigma}) n^{\beta}_{i,k} + c_2^{\beta} \sum_{k \in N_i^{\beta}} b_{i,k}(r)(v^{\beta}_{i,k} - v_i) \tag{3.54}$$

$$u_i^{\gamma} = - c_1^{\gamma}(r_i - r_{\gamma}) - c_2^{\gamma}(v_i - v_{\gamma}) \tag{3.55}$$

where $\sigma_1(z) = z/\sqrt{1 + \|z\|^2}$ and c_η^ν are positive constants for all $\eta = 1, 2$ and $\nu = \alpha, \beta, \gamma$. The pair (r_γ, v_γ) is the state of a static/dynamic γ-agent. The vectors n_{ij} and $n_{i,k}^\beta$ are given by

$$n_{ij} = \frac{r_j - r_i}{\sqrt{1 + \epsilon\|r_j - r_i\|^2}}$$
$$n_{i,k}^\beta = \frac{r_{i,k}^\beta - r_i}{\sqrt{1 + \epsilon\|r_{i,k}^\beta - r_i\|^2}}.$$

In terms of sensing requirements, we assume that every α-agent is equipped with range sensors that allow the agent to measure the relative position between the closest point on an obstacle and itself. Both radars and laser radars can be used as range sensors, therefore this assumption is feasible in practice. For the purpose of simulation of flocking motion, we use the projection of the position of an α-agent on the boundary of an obstacle.

Given an obstacle O_k and its neighbouring α-agent with state (r_i, v_i), the position and velocity of a β-agent on a wall or a sphere is given by the following lemma:

Lemma 3.2. *(Olfati-Saber et al.(2004) [45]) Let $r_{i,k}^\beta, v_{i,k}^\beta$ with $(i, k) \in \nu_\alpha \times \nu_\beta$ denote the position and velocity of a β-agent generated by an α-agent with state (r_i, v_i) on an obstacle O_k. Then*

1. *For an obstacle with a hyperplane boundary that has a unit normal a_k and passes through the point y_k, the position and velocity of the β-agent are determined by*

$$r_{i,k}^\beta = Pr_i + (I - P)y_k, \ v_{i,k}^\beta = Pv_i$$

2. *For a spherical obstacle with radius R_k centred at y_k, the position and velocity of the β-agent are given by*

$$r_{i,k}^\beta = \mu r_i + (1 - \mu)y_k, \ r_{i,k}^\beta = \mu P v_i$$

where $\mu = R_k/\|r_i - y_k\|, a_k = (r_i - y_k)/\|r_i - y_k\|$, and $P = I - a_k a_k^T$

The following lemma demonstrates that the second term in u_i^β is in fact a valid damping force. This fact indicates that the overall particle system is dissipative.

Lemma 3.3. *(Olfati-Saber et al.(2004) [45]) The force f^β between α-agents and β-agents with elements $f_i^\beta = \sum_{k\in N_i^\beta} b_{i,k}(r_i - r_{i,k}^\beta)$ is a valid damping force, i.e. Let $K_r = \frac{1}{2}\sum_i \|v_i\|^2$ and suppose $\dot{r}_i = f_i^\beta$, the $\dot{K}_r \leq 0$.*

Another question is whether the particle system obtained by applying control input (3.50) is dissipative. The answer in this case is not as predictable as the case of interactions among α-agents. The reason is that in free-flocking , every α-agent reciprocates the action of its neighbouring α-agents, but in constrained flocking the (α, β)-net is a unidirectional graph.

Theorem 3.10. *(Olfati-Saber et al.(2004) [45]) Consider a multi-agent system applying control input (3.50). Assume that the γ-agent is a static agent with a fixed state $(r_\gamma, v_\gamma) = (r_d, v_d)$. Define the energy function $H(r, v) = V(r) + T(r, v)$ with kinetic energy $T(r, v) = \frac{1}{2}\sum_{i=1}^N \|v_i\|$. Suppose there exists a finite time $t_0 \geq 0$ such that the average velocity of all agents satisfies the following condition*

$$\frac{n}{2} < v_c(t), v_d > \leq T(r(t), v(t)), \;\; \forall t \geq t_0. \tag{3.56}$$

Then, the energy of the system is non-increasing (i.e. $\dot{H}(r(t), v(t)) \leq 0$) along the trajectory of the collective dynamics of the multi-agent system for all $t \geq t_0$

Proof. By direct differentiation, we have

$$\begin{aligned}
\dot{H}(r,v) =& < \nabla V_\alpha(r), v > + < \nabla_r V_\beta(r), v > + < \nabla_{r^\beta}, v^\beta > + \sum_{i\in\nu_\alpha} < v_i, u_i^\alpha + u_i^\beta + u_i^\gamma > \\
& c_1^\beta < \nabla_{r^\beta} V_\beta(r), v^\beta > + c_2^\alpha \sum_{(i,j)\in\epsilon_\alpha(r)} a_{ij}(r) < v_i, v_j - v_i > \\
& + c_2^\beta \sum_{i\in\nu_\alpha}\sum_{k\in N_i^\beta} b_{i,k} < v_i, v_{i,k}^\beta - v_i > - c_2^\gamma \sum_{i\in\nu_\alpha} < v_i, v_i - v_d >
\end{aligned}$$

But

$$\begin{aligned}
< \nabla_{r^\beta} V_\beta(r), v^\beta > &= \sum_{i\in\nu_\alpha}\sum_{k\in N_i^\beta} < \nabla_{r_{i,k}^\beta} V_\beta(r), v_{i,k}^\beta > \\
&= \sum_{i\in\nu_\alpha}\sum_{k\in N_i^\beta} \phi_\beta(\|r_j - r_i\|_\sigma) < n_{i,k}^\beta, v_{i,k}^\beta > \\
&= 0
\end{aligned}$$

because $v_{i,k}^{\beta}$ is tangent to the surface of the obstacle, whereas $n_{i,k}^{\beta}$ is orthogonal to the surface of the obstacle. Thus, $\dot{H}(r,v)$ satisfies

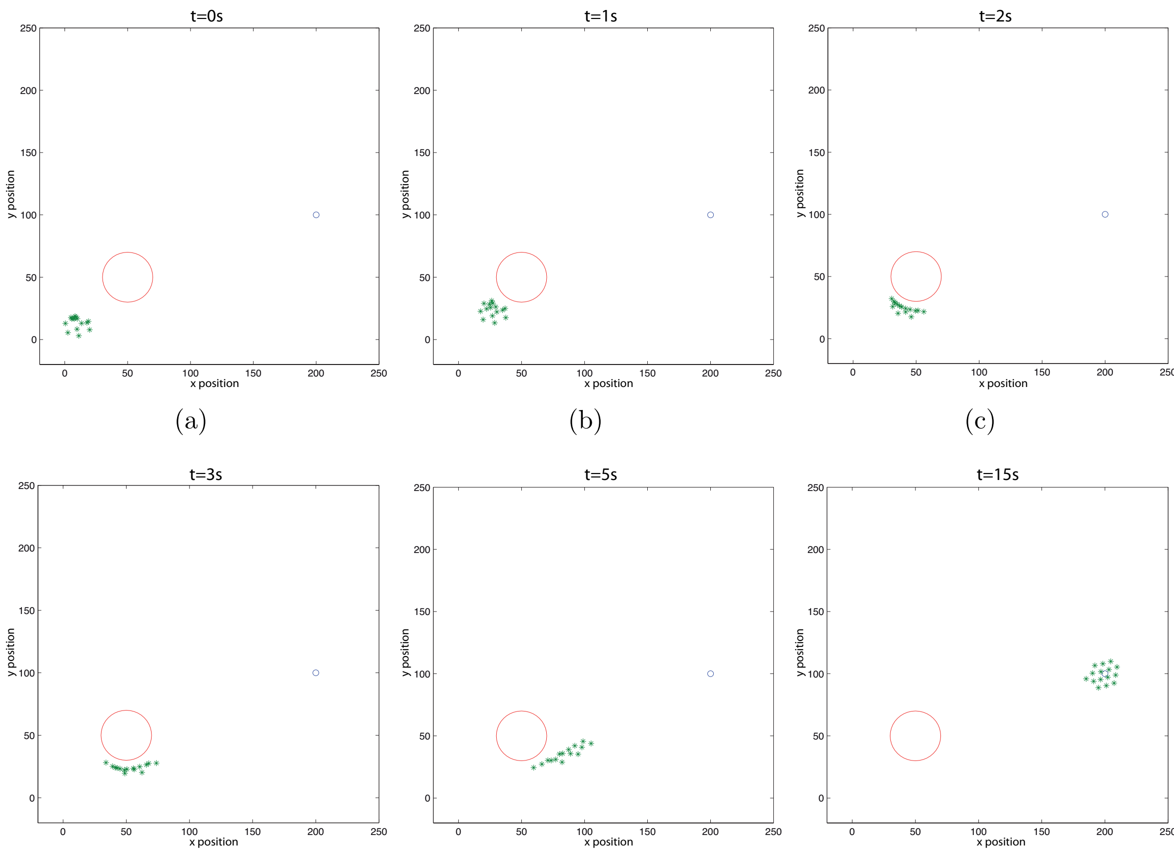
$$\dot{H}(r,v) = -c_2^{\alpha}(v^T\hat{L}(r)v) + c_2^{\beta}\sum_{i\in\nu_\alpha}\sum_{k\in N_i^{\beta}} b_{i,k} < v_i, v_{i,k}^{\beta} - v_i > -2c_2^{\gamma}(T(r,v) - \frac{n}{2}(v_d^T\dot{v}_c)) \leq 0,\ \forall t \geq t_0$$

where the second term in the last inequality is negative semidefinite based on Lemma. 3.3 and the term $T(r,v) - \frac{n}{2}(v_d^T\dot{v}_c)$ is by assumption non-negative for all $t \geq t_0$. □

The interpretation of condition (3.56) for a group of particles with equal velocities is interesting. In this case, $v_c = v_i$ for all i, and therefore (3.56) reduces to the inequality $v_c^T v_d \leq \|v_c\|^2$. Let $\theta_{c,d}$ denote the misalignment angle between vectors v_c and v_d in R^m, i.e. $cos(\theta_{c,d}) =< v_c, \dot{v}_d > /(\|v_c\|\|\dot{v}_d\|)$. Suppose $v_c, v_d \neq 0$, then the group has to by sufficiently agile, or $\|v_c\| \geq v_0 = \|v_d\|cos(\theta_{c,d})$. Intuitively, this can be interpreted as a collective effort by the group to keep up with the desired velocity v_d. For a γ-agent with $v_d = 0$, condition (3.56) trivially holds.

3.5.1 Simulation

In Fig.3.7 we present a simulation result for a multi-agent system with input control (3.50), a computer animation is available and the following parameters remain fixed throughout the simulation: $d = 12, R = 1.2d, \epsilon = 0.1$ (for σ-norm), $a = b = 5$ for $\phi(z)$, $h = 0.2$ for the bump function of $\phi_\alpha(z)$ and $h = 0.8$ for $\phi_\beta(z)$, and the step-size in the simulation is $0.01s$. In addition, the position of a static γ-agent is marked with a o sign, a spherical obstacle which centred at $(50, 50)$ with radius $R = 20$ is denoted by a red circle. The initial positions and velocity of all 15 α-agents are uniformly chosen at random form the box $[-20, 20]^2$. One can observe that the agents avoid collision with both other agents and the spherical obstacle as moving forward, finally form an α-lattice shape around the γ-agent (navigational feedback). This has been numerically verified for the entire trajectory of the agents and it is consistent with our analysis.



(a) (b) (c)

(d) (e) (f)

Figure 3.7: Flocking in Presence of Obstacles for n=15 Agents and A Spherical Obstacle.

Chapter 4

Flocking via Impulsive Control

In comparison with continuous control which had been studied in last two chapters, there has been little work done on the design of impulsive control for multi-agent systems with switching topologies. Impulsive control has some advantages compared with continuous control, e.g., it is more flexible and energy-efficient than continuous control. Moreover, it has been proved that impulsive control approach is effective and robust in synchronization of chaotic systems and complex networks [63], and the controllers used in impulsive method usually have relatively simple structure. The advantage of applying impulsive control in a self-driven, communicating multi-agent systems is to reduce energy and communication cost. We can imagine that a bird in a flock is not flapping its wings all the time. Inspired by this idea, we will propose a model which consists of both continuous and impulsive controls in this chapter. For the continuous part, we will apply a smooth potential function which guarantees cohesive and collision-free capabilities of agents in the system. Additionally, we apply impulsive controls to the system for the velocity alignment and consensus with the virtual leader (navigational feedback).

4.1 Impulsive Consensus Problem

In this section we will start with studying the consensus problem for the multi-agent dynamical system. It can be described as multi-agent systems to develop distributed control rules based on local information that enable all agents to reach a global agreement on certain interest [44], which can be understood as a special case of flocking problem for multi-agent dynamical systems.

We will consider the system with switching interconnection topologies in the problem. Suppose that there is an infinite sequence of bounded, non-overlapping, contiguous time-intervals $[t_{i-1}, t_i), i = 1, 2, \ldots,$ starting at $t_0 = 0$. As we introduced in last chapter, $G = \{G_1, G_2, \ldots, G_n\}$ is a set of the graphs with all possible topologies, which includes all possible interconnection graphs including n agents and a leader. Denote $P = \{1, 2, \ldots, n\}$ as its index set. To describe the variable interconnection topology, we define a switching signal $\sigma : [0, +\infty) \to P$, which is piecewise-constant. Therefore, $N_i(t), a_{ij}(t)$ are all time varying, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$. Moreover, in each time interval, $L_p, p \in P$ associated with the switching interconnection topology is also switching. However, in each time interval, L_p are time-invariant for some $p \in P$.

In this section, all the considered agents share a common state space $\mathbb{R}^m$, which is $r_i = (r_i^1, r_i^2, \ldots, r_i^m)^T$ and $v_i = (v_i^1, v_i^2, \ldots, v_i^m)^T$. The leader of the multi-agent dynamical system is assumed to be dynamic. Its underlying dynamics can be expressed as follows:

$$\begin{cases} \dot{r}_0 = v_0 \\ \dot{v}_0 = a(t) = a_0(t) + \delta(t) \end{cases} \tag{4.1}$$

where $a(t)$ is the input (accelaration). We assume that $a_0(t)$ is known and $\delta(t)$ is unknown but bounded by a given upper bound L, i.e. $\|\delta(t)\| \leq L$.

The dynamics of each agent are described as follows:

$$\begin{cases} \dot{r}_i = v_i + u_i^r \\ \dot{v}_i = a_0(t) + u_i^v \end{cases} \tag{4.2}$$

where $u_i^r, u_i^v \in \mathbb{R}^m, i = 1, 2, \ldots, n$ are control inputs.

Our object is to design appropriate control inputs of follower agents to track the leader. Applying impulsive control to (4.1), we have the following consensus algorithm for the multi-agent system,

$$
\begin{cases}
\dot{r}_i & = v_i,\ (t \neq t_k) \\
\Delta r_i(t_k) & = r_i(t_k^+) - r_i(t_k^-) \\
& = C_k(\sum_{j \in N_i(t_k^-)} a_{ij}(t_k^-)(r_i(t_k^-) - r_j(t_k^-))),\ (t = t_k) \\
r_i(t_0^+) & = r_i(t_0),\ (t_0 \geq 0) \\
\dot{v}_i & = a_0(t),\ (t \neq t_k) \\
\Delta v_i(t_k) & = v_i(t_k^+) - v_i(t_k^-) \\
& = C_k(\sum_{j \in N_i(t_k^-)} a_{ij}(t_k^-)(v_i(t_k^-) - v_j(t_k^-))),\ (t = t_k) \\
v_i(t_0^+) & = v_i(t_0),\ (t_0 \geq 0)
\end{cases} \tag{4.3}
$$

where $\Delta r_i(t_k), \Delta v_i(t_k)$ are the jump of the state and velocity of following agent i at the time instant t_k, $v_i(t_k^+) = \lim_{h \to 0^+} v_i(t_k + h)$ and $v_i(t_k^-) = \lim_{h \to 0^-} v_i(t_k + h)$, we let $v_i(t_k^-) = v_i(t_k)$ for simplicity. $C_k \in \mathbb{R}^{m \times m}$ is the impulsive controller gain at the moment t_k. The moments of impulse satisfy

$$
0 \leq t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots,\ \lim_{k \to \infty} t_k = \infty,
$$

where Δt_k is the impulsive interval and satisfy

$$
\Delta t_k = t_k - t_{k-1} \leq \tau < \infty,\ (k = 1, 2, \dots)
$$

Denote

$$
\begin{cases}
r_i - r_0 = \xi_i \\
v_i - v_0 = \eta_i
\end{cases}
$$

where $\xi_i, \eta_i \in \mathbb{R}^m$ and denote

$$
e = (\xi_1^T, \xi_2^T, \cdots, \xi_n^T, \eta_1^T, \eta_2^T, \cdots, \eta_n^T) \in \mathbb{R}^{2mn}.
$$

Then the error system with (4.1) and (4.1) can be written as:

$$
\begin{cases}
\dot{e}(t) = Fe(t) + \hat{\delta},\ (t \neq t_k) \\
\Delta e(t_k) = \hat{C}_k e(t_k),\ (t = t_k) \\
e(t_0^+) = e(t_0)
\end{cases} \tag{4.4}
$$

where

$$
F = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \otimes I_m, \hat{C}_k = \begin{pmatrix} (L_{\tau_k} + B_{\tau_k}) \otimes C_k & 0 \\ 0 & (L_{\tau_k} + B_{\tau_k}) \otimes C_k \end{pmatrix}, \hat{\delta} = \begin{pmatrix} 0 \\ \mathbf{1}_n \times \delta(t) \end{pmatrix}
$$

and $\Delta e = (\Delta\xi_1^T, \dots, \Delta\xi_n^T, \Delta\eta_1^T, \dots, \Delta\eta_n^T)^T$, $\mathbf{1}_n = (1, 1, \dots, 1) \in \mathbb{R}^m$ and $\otimes$ denotes the Kronecker product. L_{τ_k} and B_{τ_k} are associated with the switching interconnection graph at time t_k.

Lemma 4.1. *Suppose that the directed graph G has a directed spanning tree and the root of such tree has access to the leader. Then, all the eigenvalues of the matrix $L + B$ have positive real parts.*

Now we are ready to provide the analysis of the impulsive consensus problem for multi-agent system with fixed and switching topologies. Firstly, we consider the graph with fixed topology. Let β_k be the maximum eigenvalue of matrix $(I + \hat{C}_k)^T(I + \hat{C}_k)$, where $\hat{C}_k$ is associated with the impulsive moment t_k.

Theorem 4.1. *(Qing et al.(2012) [44]) Assume that the interconnection graph among the agents is fixed and the graph G has a directed spanning tree. If β_k and Δt_k satisfy the condition that there exists $\xi > 1$ such that*

$$\eta(t_k - t_{k-1}) + ln(\xi \cdot \beta_k) < 0$$

then the impulsive controlled multi-agent system (4.4) is asymptotically stable at origin.

Proof. Choose the candidate Lyapunov functional as

$$V(e(t)) = (e(t))^T e(t)$$

For any $t \in (t_{k-1}, t_k]$, the right and upper Dini's derivative of $V(e(t))$ along the trajectory of (4.4) is

$$\begin{aligned} D^+V(e(t)) &= (\dot{e}(t))^T e(t) + (e(t))^T \dot{e}(t) \\ &= (e(t))^T (F + F^T) e(t) + 2\hat{\delta} e(t) \\ &= (e(t))^T (F + F^T) e(t) + 2L\|e(t)\| \\ &\leq (1 + 2L) V(e(t)) = \eta V(e(t)) \end{aligned}$$

For $L > 0$, we have $\eta > 1$. Moreover, for any $t \in (t_{k-1}, t_k]$,

$$V(e(t)) \leq V(e(t_{k-1}^+)) exp\{\eta(t - t_{k-1})\} \tag{4.5}$$

On the other hand, it follows from the second equation of system 4.4 that

$$\begin{aligned} V(e(t_k^+)) &= e(t_k)^T e(t_k) \\ &= (e(t_k))^T (I + \hat{C}_{\tau_k})^T (I + \hat{C}_{\tau_k}) e(t_k) \\ &\leq \beta_k V(e(t_k)), \ (k = 1, 2, \dots) \end{aligned} \tag{4.6}$$

Let $k = 1$ in inequality (4.5), we have

$$V(e(t)) \leq V(e(t_0))exp\{\eta(t - t_0)\}, t \in (t_0, t_1],$$

which lead to

$$V(e(t_1)) \leq V(e(t_0))exp\{\eta(t_1 - t_0)\}$$

Let $k = 1$ in inequality (4.6), we have

$$V(e(t_1^+)) \leq \beta_1 V(e(t_1)) \leq \beta_1 V(e(t_0))exp\{\eta(t_1 - t_0)\}$$

Therefore, for $t \in (t_1, t_2]$,

$$\begin{aligned} V(e(t)) &\leq V(e(t_1^+))exp\{\eta(t - t_1)\} \leq \beta_1 V(e(t_0))exp\{\eta(t - t_0)\}, \\ V(e(t_2)) &\leq \beta_1 V(e(t_0))exp\{\eta(t_2 - t_0)\} \\ V(e(t_2^+)) &\leq \beta_2 V(e(t_2)) \leq \beta_1 \beta_2 V(e(t_0))exp\{\eta(t_2 - t_0)\} \end{aligned}$$

Generally for $t \in (t_k, t_{k+1}]$,

$$V(e(t)) \leq \beta_1 \cdots \beta_k V(e(t_0))exp(\eta(t - t_0))$$

From the condition $\eta(t_k - t_{k-1}) + ln(\xi \cdot \beta_k) < 0$, we have

$$\beta_k \cdot exp(\eta(t_k - t_{k-1})) < \frac{1}{\xi}, \ (k = 1, 2, \dots).$$

So we have

$$\begin{aligned} V(e(t)) &\leq \beta_1 \cdots \beta_k V(e(t_0)) \cdot exp(\eta(t - t_k)) \cdot exp(\eta(t_k - t_0)) \\ &\leq \frac{1}{\xi^k} V(e(t_0)) \cdot exp(\eta(t - t_k)), \ t \in (t_k, t_{k+1}]. \end{aligned} \tag{4.7}$$

Notice that $\eta > 0$, $\|t - t_k\|$ is a finite constant, $\frac{1}{\xi^k} \to 0$ when $k \to \infty$, hence, the origin is globally asymptotically stable. □

When the interconnection graph is switching, let β_{τ_k} be the maximum eigenvalue of matrix $(I + \hat{C}_{\tau_k})^T(I + \hat{C}_{\tau_k})$, where $\hat{C}_{\tau_k}$ is associated with the impulsive gain C_k and matrix $L_{\tau_k} + B_{\tau_k}$. Because there are finite possible interconnection graphs, we can take β_k =max$\{\beta_{\tau_k} : \tau_k \in P\}$. Following the similar proof of theorem 4.1, we can get the theorem below.

Theorem 4.2. *(Qing et al.(2012) [44]) Assume that the interconnection graph among the agents is switching and the graphs $G_1, G_2, \ldots, G_n$ all have directed spanning tree. If β_k and Δt_k satisfy the same condition as in theorem 4.1, then the impulsive controlled multi-agent system (4.4) is asymptotically stable at origin.*

Remark 1. If $\xi = 1$, 4.7 can be written as

$$\begin{aligned} V(e(t)) &\leq V(e(t_0)) \cdot exp(\eta(t - t_k)) \\ &\leq V(e(t_0))e^{\tau \cdot \eta} \end{aligned}$$

where τ is the maximum impulsive interval. So $V(e(t)) < A$ as $k \to \infty$, where A is a constant. That is to say, the origin is stable but not asymptotically stable.

Remark 2. When the graphs $G_1, G_2, \ldots, G_n$ have spanning trees, from Lemma.4.1, all eigenvalues of matrix $L_{\tau_k} + B_{\tau_k}$ have positive real part. Choose $C_k = diag\{c_k^1, c_k^2, \ldots, c_k^m\}, c_k^i \in (-1, 0)$, then the eigenvalues of matrix $\hat{C}_k$ have negative real part. By this means, the condition in theorem 4.1 will be satisfied.

4.2 Impulsive Flocking Problem

In last section we have studied the impulsive consensus problem for multi-agent system and proven that with the impulsive controller under certain conditions, the error between the following agents and the leader will asymptotically converged to zero. Notice that this kind of impulsive control will result in a jump of both the state and velocity of all follower agents. For the flocking problem for multi-agent dynamical systems, it is not reasonable or applicable to apply a jump of the state of the agents from both engineering and biological perspectives. Hence, what we can do is to apply such impulsive control to only the velocities of agents in the systems as velocity pulses, which may be caused by a suddenly acceleration by a high-pressure jet engine of an unmanned vehicle from an engineering perspective, or a bird flapping its wings during a long-distant migration from an biological perspective.

Inspired by these ideas, in this section we will propose a hybrid system model, in which we apply continuous controls to the states of agents in the multi-agent dynamical system in order to regulate the formation of the flock. Moreover, we apply impulsive controls to the velocities of agents to achieve velocity alignment and consensus of all agents with the virtual leader (navigational feedback). For the continuous control part, we will generate a smooth potential function and use the negative gradient of this potential function as the acceleration inputs, which is similar with our discussion in Chapter 3, while at every moment t_k we apply a impulsive control to the velocity which is similar with our discussion in the impulsive consensus problem.

4.2.1 Without Damping

In this section, we will propose a hybrid system model without damping, the dynamics of all agents in the multi-agent dynamical system can be described as follows,

$$\begin{cases} \dot{r}_i = v_i & \\ \dot{v}_i = -\nabla_{r_i} V(r) - c_1(r_i - r_0) & t \neq t_k \\ \Delta v_i(t_k) = c_2(v_i(t_k^-) - v_0(t_k^-)) & t = t_k \end{cases} \tag{4.8}$$

where $\Delta v_i(t_k)$ is the jump of the velocity of agent i at the time instant t_k, for simplicity we let $v_i(t_k^+) = v_i(t_k)$, $c_2 \in \mathbb{R}^{m\times m}$ is the impulsive controller gain at the moment t_k. The moments of impulse satisfy

$$0 \leq t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots, \lim_{k\to\infty} t_k = \infty,$$

where Δt_k is the impulsive interval and satisfy

$$\Delta t_k = t_k - t_{k-1} \leq \tau < \infty, \ (k = 1, 2, \dots)$$

For the potential function $V(r)$ we will use the same definition as in (2.6).

As our discussion for continuous control in Chapter 3, we can analyze stability by studying the collective dynamics of the model, firstly we can write the collective dynamics of system (4.8) in the following form:

$$\begin{cases} \dot{r} = v & \\ \dot{v} = -\nabla V(r) + f_1(r, r_0) & t \neq t_k \\ \Delta v = f_2(v(t_k^-), v_0(t_k^-)) & t = t_k \end{cases} \tag{4.9}$$

where $f_1(r, r_0)$ and $f_2(v, v_0)$ are position and velocity feedback given by the virtual leader respectively.

Our objective is to separate the analysis of the motion of the centre of the group with respect to the reference frame from the collective motion of the particles in the moving frame. The position and velocity of agent i in the moving frame is given by:

$$\begin{cases} \hat{r}_i & = r_i - r_c \\ \hat{v}_i & = v_i - v_c \end{cases}$$

The relative position and velocities remain the same in the moving frame, i.e. $\hat{r}_j - \hat{r}_i = r_j - r_i$ and $\hat{v}_j - \hat{v}_i = v_j - v_i$. Thus, $V(r) = V(\hat{r})$ and $\nabla V(r) = \nabla V(\hat{r})$.

Lemma 4.2. *Suppose that the navigational feedback $f_1(r, r_0)$ and $f_2(v, v_0)$ are linear, i.e., there exists decompositions of $f_1(r, r_0)$ and $f_2(v, v_0)$ in the following form:*

$$\begin{cases} f_1(r, r_0) = g_1(\hat{r}) + \mathbf{1}_n \otimes h_1(r_c, r_0) \\ f_2(v, v_0) = g_2(\hat{v}) + \mathbf{1}_n \otimes h_2(v_c, v_0) \end{cases}$$

Then, the collective dynamics of a group of agents can be decomposed as n second-order systems in the moving frame:

$$\textit{structural dynamics}: \begin{cases} \dot{\hat{r}} = \hat{v} \\ \dot{\hat{v}} = -\nabla V(\hat{r}) + g_1(\hat{r}) & t \neq t_k \\ \Delta\hat{v} = g_2(\hat{v}(t_k^-)) & t = t_k \end{cases} \tag{4.10}$$

and one second-order system in the reference frame:

$$\textit{translational dynamics} \begin{cases} \dot{r}_c = v_c \\ \dot{v}_c = h_1(r_c, r_0) & t \neq t_k \\ \Delta\hat{v}_c = h_2(r_c(t_k^-), v_0(t_k^-)) & t = t_k \end{cases}$$

where

$$\begin{cases} g_1(\hat{r}) = -c_1\hat{r} \\ g_2(\hat{v}) = -c_2\hat{v} \\ h_1(r_c, r_0) = -c_1(r_c - r_0) \\ h_2(v_c, v_0) = -c_2(v_c - v_0) \end{cases} \tag{4.11}$$

and (r_0, v_0) is the state of the virtual leader (navigational feedback).

Proof. By definition, $\dot{x}_c = v_c$ and $\dot{\hat{x}} = \hat{v}$. Given system 4.9 we have

$$\begin{cases} \ddot{r} = -\nabla V(r) + f_1(r, r_0) & t \neq t_k \\ \Delta v = f_2(v(t_k^-), v_0(t_k^-)) & t = t_k \end{cases}$$

Due to the fact that G(r) is an undirected graph,we can get

$$
\begin{aligned}
Ave(\nabla V(r)) &= \frac{1}{n}\sum_{i=1}^{n}\frac{\partial V(r)}{\partial r_i} \\
&= \frac{1}{n}\sum_{i=1}^{n}[\sum_{j\in N_i}\phi_\alpha(\|r_j - r_i\|)(r_j - r_i)] = 0
\end{aligned}
$$

for all r, where ϕ_α is defined in (2.8). Assume $f_1(r, r_0)$ and $f_2(v, v_0)$ are linear controllers in the form:

$$
\begin{cases}
f_1(r, r_0) = -c_1(r - \mathbf{1}_n \otimes r_0) \\
f_2(v, v_0) = -c_2(v - \mathbf{1}_n \otimes v_0)
\end{cases}
$$

where $\mathbf{1}_n = (1, \ldots, 1)^T \in \mathbb{R}^n$. Then $f_1(r, r_0)$ can be decomposed into two terms:

$$
\begin{aligned}
f_1(r, r_0) &= -c_1(r - \mathbf{1}_n \otimes r_0) \\
&= -c_1(r - \mathbf{1}_n \otimes r_c + \mathbf{1}_n \otimes r_c - \mathbf{1}_n \otimes r_0) \\
&= -c_1(\hat{r} + \mathbf{1}_n \otimes r_c - \mathbf{1}_n \otimes r_0) \\
&= -c_1(\hat{r} + \mathbf{1}_n \otimes (r_c - r_0)) \\
&= g_1(\hat{r}) + \mathbf{1}_n \otimes h_1(r_c, r_0)
\end{aligned}
$$

similarly,

$$
f_2(v, v_0) = g_2(\hat{v}) + \mathbf{1}_n \otimes h_2(v_c, v_0) \tag{4.12}
$$

where $g_1(\hat{r}), g_2(\hat{v}), h_1(r_c, r_0), h_2(v_c, v_0)$ are defined in (4.11). Notice that $Ave(\hat{r}) = Ave(\hat{v}) = 0$, hence, $Ave(g_1(\hat{r})) = Ave(g_2(\hat{v})) = 0$, moreover

$$
\begin{aligned}
\ddot{r}_c &= \ddot{Ave}(r) = Ave(\ddot{r}) \\
&= Ave(-\nabla V(r) + f_1(r, r_0)) \\
&= Ave(-\nabla V(r)) + Ave(f_1(r, r_0)) \\
&= Ave(g_1(\hat{r})) + Ave(\mathbf{1}_n \otimes h_1(r_c, r_0)) \\
&= h_1(r_c, r_0)
\end{aligned}
$$

and

$$\begin{aligned}\Delta v_c &= \Delta(Ave(v)) = Ave(\Delta(v)) \\ &= Ave(f_2(v, v_0)) \\ &= Ave(g_2(\hat{v}) + \mathbf{1}_n \otimes h_2(v_c, v_0)) \\ &= h_2(v_c, v_0).\end{aligned}$$

The translational dynamics can be written as follows:

$$\begin{cases} \ddot{r_c} = h_1(r_c, r_0) & t \neq t_k \\ \Delta\hat{v_c} = h_2(r_c(t_k^-), v_0(t_k^-)) & t = t_k \end{cases}. \tag{4.13}$$

On the other hand, $\hat{r} = r - \mathbf{1}_n \otimes r_c$ which gives

$$\begin{aligned}\ddot{\hat{r}} &= -\nabla V(r) + f_1(r, r_0) - \mathbf{1}_n \otimes h_1(r_c, r_0) \\ &= -\nabla V(\hat{r}) + g_1(\hat{r}),\ t \neq t_k\end{aligned}$$

Similarly,

$$\Delta\hat{v} = f_2(v, v_0) - \mathbf{1}_n \otimes h_2(v_c, v_0) = g_2(\hat{v}),\ t = t_k \tag{4.14}$$

□

Then we can analyze the stability of the structural dynamics. Define the Hamiltonian $H(\hat{r}(t), \hat{v}(t)) = U_\lambda(r) + K(\hat{v})$ where

$$U_\lambda(\hat{r}) = V(\hat{r}) + \lambda J(\hat{r})$$

where $J(\hat{r}) = \frac{1}{2}\hat{r}(t)^T\hat{r}(t)$ is the moment of inertia of all agents and $\lambda = c_1 > 0$ is the parameter of the navigational feedback, $K(\hat{v}) = \frac{1}{2}\hat{v}(t)^T\hat{v}(t)$ is the velocity mismatch function, or the kinetic energy of the agents in the moving frame.

Theorem 4.3. *Consider a group of agents applying protocol (4.8) with $c_1, c_2 > 0$ and structural dynamics (4.10). Assume that the initial velocity mismatch $K(\hat{v}(0))$ and inertia $J(\hat{r}(0))$ are finite. Then, the following statements hold:*

1. *The group of agents remain cohesive for all $t > 0$.*

2. *Almost every solution of (4.10) asymptotically converges to an equilibrium point $(\hat{r}^*_\lambda, 0)$ where $\hat{r}^*_\lambda$ is a local minima of $U_\lambda(\hat{r})$.*

3. *The velocity of all agents asymptotically match in the reference frame.*

4. *Assume the initial structural energy of the particle system is less than c^* with $c^* = \psi_\alpha(0)$ and $k \in \mathbb{Z}^+$. Then, no two agents ever collide.*

Proof. First, note that the multi-agent system with structural dynamics (4.10) and Hamiltonian $H(\hat{r}(t), \hat{v}(t)) = U_\lambda(r) + K(\hat{v})$ is a strictly dissipative system in the moving frame because for any $t \in [t_{k-1}, t_k)$, the right and upper Dini's derivative of $H(\hat{r}(t), \hat{v}(t))$ along the trajectory of (4.10) is:

$$\begin{aligned} D^+ H(\hat{r}(t), \hat{v}(t)) &= \nabla V(\hat{r}(t))^T \dot{\hat{r}}(t) + \frac{1}{2} c_1 (\dot{\hat{r}}(t)^T \hat{r}(t) + \hat{r}(t)^T \dot{\hat{r}}(t)) + \frac{1}{2} (\dot{\hat{v}}(t)^T \hat{v}(t) + \hat{v}(t)^T \dot{\hat{v}}(t)) \\ &= \nabla V(\hat{r}(t))^T \hat{v}(t) + \frac{1}{2} c_1 (\hat{v}(t)^T \hat{r}(t) + \hat{r}(t)^T \hat{v}(t)) \\ &+ \frac{1}{2} ((-\nabla V(\hat{r}(t)) - c_1 \hat{r}(t))^T \hat{v}(t) + \hat{v}(t)^T (-\nabla V(\hat{r}(t)) - c_1 \hat{r}(t))) \\ &= 0. \end{aligned}$$

Hence, for any $t \in [t_{k-1}, t_k)$,

$$H(\hat{r}(t), \hat{v}(t)) = L_k, \tag{4.15}$$

on the other hand, it follows from the third equation of (4.10) that:

$$\begin{aligned} H(\hat{r}(t_k), \hat{v}(t_k)) &= V(\hat{r})(t_k^-) + \frac{1}{2} c_1 \hat{r}(t_k^-)^T \hat{r}(t_k^-) + \frac{1}{2} (\hat{v}(t_k^-) + \Delta \hat{v})^T (\hat{v}(t_k^-) + \Delta \hat{v}) \\ &= V(\hat{r})(t_k^-) + \frac{1}{2} c_1 \hat{r}(t_k^-)^T \hat{r}(t_k^-) + \frac{1}{2} (1 - c_2)^2 \hat{v}(t_k^-)^T \hat{v}(t_k^-). \end{aligned}$$

Choose $(1-c_2)^2 < 1$, we have $H(\hat{r}(t_k), \hat{v}(t_k)) < H(\hat{r}(t_k^-), \hat{v}(t_k^-)),\ \forall \hat{v} \neq 0$, so there exists a $\beta_k \in [0, 1]$ that:

$$H(\hat{r}(t_k), \hat{v}(t_k)) < \beta_k H(\hat{r}(t_k^-), \hat{v}(t_k^-)),\ \forall \hat{v} \neq 0,\ k = 1, 2, \cdots \tag{4.16}$$

Let k=1 in (4.15), we have

$$H(\hat{r}(t), \hat{v}(t)) = L_1,\ t \in [t_0, t_1).$$

Let k=1 in inequality (4.16), we have

$$H(\hat{r}(t_1), \hat{v}(t_1)) < \beta_1 H(\hat{r}(t_1^-), \hat{v}(t_1^-)) = \beta_1 L_1, \ \forall \hat{v} \neq 0.$$

Let k=2 in (4.15) and (4.16), which gives

$$H(\hat{r}(t_2), \hat{v}(t_2)) < \beta_2 H(\hat{r}(t_2^-), \hat{v}(t_2^-)) = \beta_2 H(\hat{r}(t_1), \hat{v}(t_1)) < \beta_2 \beta_1 L_1, \ \forall \hat{v} \neq 0.$$

Generally, for $t \in [t_k, t_{k+1})$,

$$H(\hat{r}(t), \hat{v}(t)) < \beta_1 \beta_2 \cdots \beta_k L_1, \ \forall \hat{v} \neq 0,$$

since $\beta_k \in [0, 1]$, choose $\eta = max(\beta_k) \leq 1$, so that

$$H(\hat{r}(t), \hat{v}(t)) < \eta^k L_1 < L_1 \quad t \in [t_k, t_{k+1}), \ \forall \hat{v} \neq 0 \ \forall k.$$

Hence, the structural energy $H(\hat{r}, \hat{v})$ is non-increasing for all $(\hat{r}, \hat{v})$ and

$$H(\hat{r}(t), \hat{v}(t)) \leq H_0 = H(\hat{r}(0), \hat{v}(0)) < \infty.$$

The finiteness of $H_0 = V(\hat{r}(0)) + \lambda J(\hat{r}(0)) + K(\hat{v}(0))$ follows from the assumption that the collective potential, the inertia and the velocity mismatch are all initially finite. Thus, for all $t \geq 0$, we have

$$U_\lambda(\hat{r}(t)) \leq H_0, \ K(\hat{v}(0)) \leq H_0.$$

But $U_\lambda(\hat{r}) = V(\hat{r}) + \frac{\lambda}{2}\hat{r}^T\hat{r}$ with $\lambda > 0$ and $V(\hat{r}) \geq 0$ for all $\hat{r}$, therefore

$$\hat{r}^T(t)\hat{r}(t) \leq \frac{2H_0}{\lambda}, \forall t \geq 0.$$

This guarantees the cohesion of the group of all agents for all $t \geq 0$ because the position of all agents remains in a ball of radius $R = \sqrt{2H_0/\lambda}$ centred at $\hat{r}_c$. This cohesion property together with boundedness of velocity mismatch, or $K(\hat{v}(t)) \leq H_0$, guarantees boundedness of solution of the structural dynamics (4.10). To see this, let $z = (\hat{r}, \hat{v})^T$, then

$$\|z(t)\|^2 = \hat{r}^T(t)\hat{r}(t) + \hat{v}^T(t)\hat{v}(t) \leq 2(\frac{1}{\lambda} + 1)H_0 < \infty.$$

Part 2. follows from that $H(\hat{r}, \hat{v})$ is always decreasing for all $\hat{v} \neq 0$ due to the velocity impulsive control, hence, $H(\hat{r}, \hat{v})$ is invariant only if $\hat{v} = 0$, i.e., $H(\hat{r}_\lambda^*, 0) = U_\lambda(\hat{r}_\lambda^*)$, where

$\hat{r}^*_\lambda$ is a local minima of the aggregate potential function $U_\lambda(\hat{r})$, thus, almost every solution of the multi-agent system asymptotically converges to an equilibrium point $(\hat{r}^*_\lambda, 0)$.

Part 3. follows from the fact that $\hat{v}$ asymptotically vanishes. Thus, the velocities of all agents asymptotically match in the reference frame.

To prove part 4., suppose $H_0 < c^*$ and there are two agents collide at a given time $t^* \geq 0$. This implies the collective potential of the multi-agent system at time $t = t^*$ is at least $\psi_\alpha(0)$. However, we have

$$H_0 = V(\hat{r}(0)) + \lambda J(\hat{r}(0)) + K(\hat{v}(0)) \geq V(\hat{r}(0)) \geq \psi_\alpha(0).$$

This contradicts the assumption that $H_0 < c^*$. Hence, no two agents can possibly collide at any time $t \geq 0$.

□

4.2.2 Simulation

Here we present a simulation result for the hybrid system model without damping, the structural dynamics is defined in (4.10), a computer animation is available and the following parameters remain fixed throughout the simulation: $d = 12, R = 1.2d, \epsilon = 0.1$ (for σ-norm), $a = b = 5$ for $\phi(z)$, $h = 0.2$ for the bump function of $\phi_\alpha(z)$, and the step-size in the simulation is $0.01s$ and the length of the impulsive interval $\tau = 1s$. In addition, the position of a static virtual leader is marked with a o sign. The initial positions and velocity of all 50 agents are uniformly chosen at random form the box $[-10, 10]^2$. A flock is formed in Fig.4.1(e) and maintained thereafter. The number of edges of the dynamic graph $G(r(t))$ increases by time and has a tendency to render the net connected. Numerical measurements indicate that the final conformation is a low-energy quasi α-lattice that induces a flock. These observations are in close agreement with our analysis in last section.

Moreover, we define a velocity mismatch function with respect to all agents and the virtual agent as :$M(t) = \sum_{i=1}^{N}(\|v_i - v_0\|)^2$ and plot it in Fig.4.2, we can see clearly the velocity pulse effect at every impulsive moment $t_k = 100$ time-steps, and the velocity difference between all agents and the virtual leader asymptotically converges to zero, which also denotes that the velocity mismatch between all agents asymptotically converge to zero, we can see that our simulation is consistent with our study.

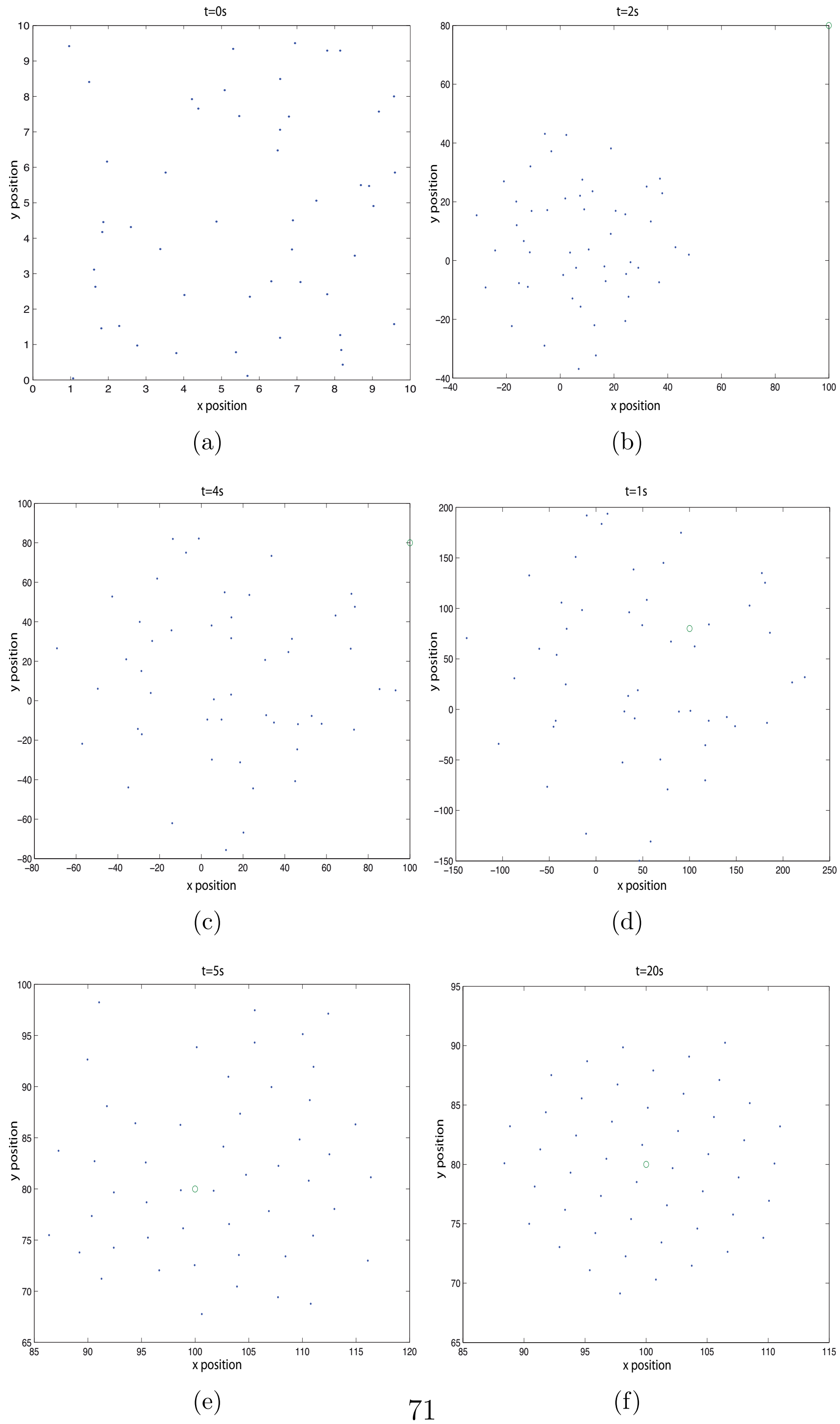


Figure 4.1: Flocking via Impulsive Control without Damping for n=50 agents.

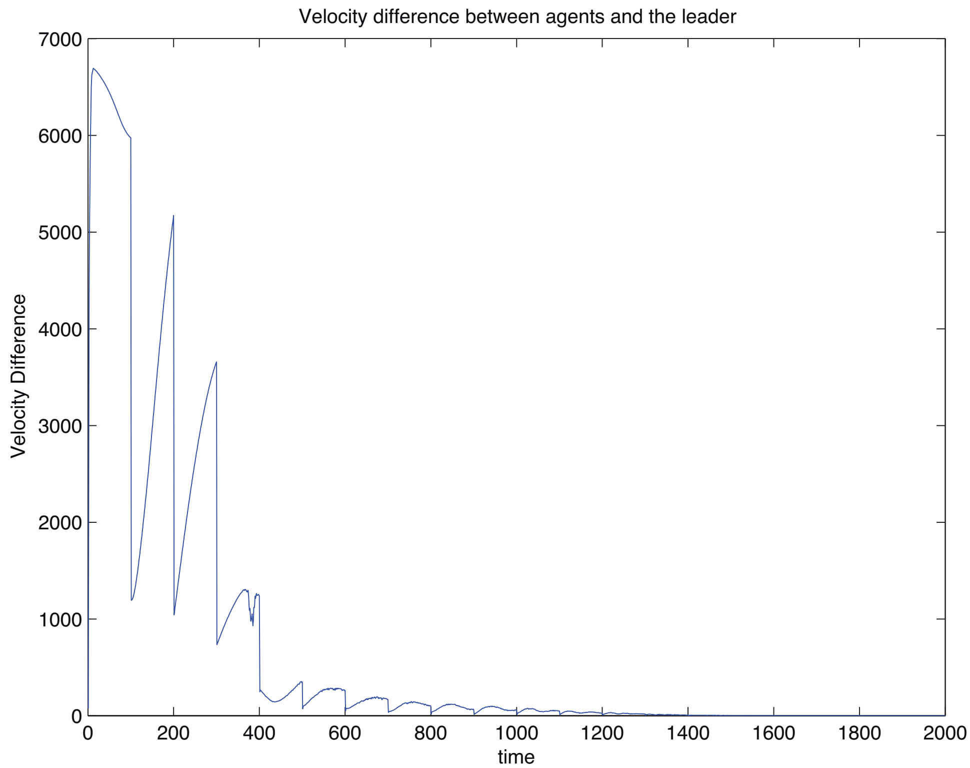


Figure 4.2: Velocity Mismatch

4.3 Coupling Time Delay

In last section we have discussed a hybrid system model without damping, which successfully leads to flocking. In this section, we introduce a damping term into our control rule. Due to the limitation of velocity sensors on agents, (we assume that the position sensors are more effective and accurate than velocity sensors so there's no delay in position sensors), we need to consider the coupling time delay in our control law while introducing the damping term. The dynamics of a flock of agents can be described in following form:

$$\begin{cases} \dot{r}_i = v_i & \\ \dot{v}_i = -\nabla_{r_i} V(r) - c_1(r_i - r_0) - \sum_{j \in N_i} a_{ij}(r)(v_i - v_j(t-\tau)) & t \neq t_k \\ \Delta v_i(t_k) = c_2(v_i(t_k^-) - v_0(t_k^-)) & t = t_k \end{cases} \tag{4.17}$$

where $V(r) = \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij}(\|r_i - r_j\|)$ is defined in (2.6), $\nabla_{r_i} V(r)$ corresponds to a vector in the direction of the gradient of an artificial potential function, and τ is the coupling communication time delay due to the velocity sensors. $\Delta v_i(t_k)$ is the jump of the velocity of agent i at the time instant t_k, for simplicity we let $v_i(t_k^+) = v_i(t_k)$, $c_2 \in \mathbb{R}^{m \times m}$ is the impulsive controller gain at the moment t_k. The moments of impulsive satisfy

$$0 \le t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots, \ \lim_{k \to \infty} t_k = \infty,$$

where Δt_k is the impulsive interval and satisfy

$$\Delta t_k = t_k - t_{k-1} \le \hat{\tau} < \infty, \ (k = 1, 2, \dots)$$

r_0, v_0, a_0 are the state of the virtual leader of all agents which satisfy the following:

$$\begin{cases} \dot{r_0} = v_0 \\ \dot{v_0} = 0 \end{cases}$$

Our objective is to control the entire group move at a desired velocity v_0 and maintain constant distance between agents. Notice that the desired velocity is that $v_0(t) = v_0(t - \tau)$. Define the error vectors $e_i^r = r_i - r_0$, $e_i^v = v_i - v_0$ where e_i^r, e_i^v represent the position and velocity difference between agent i and virtual leader, notice that

$$\begin{aligned} \dot{e}_i^v &= -\nabla_{r_i} V(r) - c_1(r_i - r_0) - \sum_{j \in N_i} a_{ij}(r)(v_i - v_j(t - \tau)) \\ &= -\nabla_{e_i^r} V(e^r) - c_1(e_i^r) - \sum_{j \in N_i} a_{ij}(r)((v_i - v_0) - (v_j(t - \tau) - v_0(t - \tau))) \\ &= -\nabla_{e_i^r} V(e^r) - c_1(e_i^r) - \sum_{j \in N_i} a_{ij}(r)(e_i^v - e_j^v(t - \tau)), \end{aligned}$$

hence, the error dynamics is given by:

$$\begin{cases} \dot{e}_i^r = e_i^v \\ \dot{e}_i^v = -\nabla_{e_i^r} V(e^r) - c_1(e_i^r) - \sum_{j \in N_i} a_{ij}(r)(e_i^v - e_j^v(t - \tau)) & t \neq t_k \\ \Delta e_i^v(t_k) = -c_2 e_i^v(t_k^-) & t = t_k \end{cases} . \tag{4.18}$$

Theorem 4.4. *Consider a group of agents applying control rule (4.17) with error dynamics (4.18). Then the solution of (4.17) asymptotically converges to an equilibrium point* (r^*, v_0) *where* r^* *is a local minima of* $U_\lambda(r)$ *and* v_0 *is the velocity of the virtual leader.*

Proof. For $t \in [t_{k-1}, t_k)$, consider the following positive semi-definite Lyapunov function:

$$W = \frac{1}{2}\sum_{i=1}^{N}(e_i^v)^T e_i^v + V(e^r) + \frac{c_1}{2}\sum_{i=1}^{N}(e_i^r)^T(e_i^r) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(r)\int_{t-\tau}^{t}(e_i^v(s))^T(e_i^v(s))ds,$$

for any $t \in [t_{k-1}, t_k)$, the right and upper Dini's derivative of W along the trajectory of (4.18) is:

$$\begin{aligned}
D^+W =& \sum_{i=1}^{N}(e_i^v)^T[-\nabla_{e_i^r}V(e^r) - c_1 e_i^r - \sum_{j\in N_i} a_{ij}(r)(e_i^v - e_j^v(t-\tau))] + \sum_{i=1}^{N}(\nabla_{e_i^r}V(e^r))^T e_i^v \\
&+ c_1\sum_{i=1}^{N}(e_i^r)^T e_i^v + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(r)[(e_i^v(t))^T e_i^v(t) - (e_i^v(t-\tau))^T e_i^v(t-\tau)] \\
=& -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(r)[(e_i^v(t))^T(e_i^v(t) - e_j^v(t-\tau))] \\
&+ \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(r)[(e_i^v(t)^T)(e_i^v(t)) - (e_i^v(t-\tau))^T(e_i^v(t-\tau))] \\
=& \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(r)[-\frac{1}{2}(e_i^v(t))^T(e_i^v(t)) + (e_i^v(t))^T(e_j^v(t-\tau)) - \frac{1}{2}(e_i^v(t-\tau))^T(e_i^v(t-\tau))] \\
=& -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(r)[e_i^v(t) - e_j^v(t-\tau)]^T[e_i^v(t) - e_j^v(t-\tau)] \leq 0.
\end{aligned}$$

Hence, for all $t \in [t_{k-1}, t_k)$, $W(t) < W(t_{k-1})$ for all $e_i^v(t) \neq e_j^v(t-\tau)$, moreover, it follows from the third equation of (4.18) that:

$$
\begin{aligned}
W(t_k) &= \frac{1}{2}\sum_{i=1}^{N}(e_i^v(t_k))^T e_i^v(t_k) + V(e^r(t_k)) + \frac{c_1}{2}\sum_{i=1}^{N}(e_i^r(t_k))^T(e_i^r(t_k)) \\
&+ \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(r)\int_{t_k-\tau}^{t_k}(e_i^v(s))^T(e_i^v(s))ds \\
&= \frac{1}{2}(1-c_2)^2\sum_{i=1}^{N}(e_i^v(t_k^-))^T e_i^v(t_k^-) + V(e^r(t_k^-)) + \frac{c_1}{2}\sum_{i=1}^{N}(e_i^r(t_k^-))^T(e_i^r(t_k^-)) \\
&+ \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(r)\int_{t_k^- -\tau}^{t_k^-}(e_i^v(s))^T(e_i^v(s))ds.
\end{aligned}
$$

Choose $(1-c_2)^2 < 1$, i.e. $0 < c_2 < 2$, we have $W(t_k) < W(t_k^-)$ for all $e_i^v(t) \neq 0$. Therefor, for $t \in [t_{k-1}, t_k], \forall k$

$$W(t_k) < W(t_k^-) < W(t_{k-1}),\ \forall k$$

which indicates that the Lyapunov function is decreasing along the solution trajectory when $e_i^v(t) \neq e_j^v(t-\tau), e_i^v(t) \neq 0$. Hence, the solution must converge to the set $\{(e^{r^*}, e^v)\ |e_i^v(t) = e_j^v(t-\tau) = 0, \nabla_{e^{r^*}} U_\lambda(e^{r^*}(t)) = 0\}$, where $U_\lambda(e^r) = V(e^r) + \frac{c_1}{2}(e^r)^T(e^r)$ denotes the aggregate potential function, so the entire group of agents will asymptotically move at desired velocity v_0 and the solution of (4.17) will asymptotically converges to an equilibrium point (r^*, v_0). □

Notice that to achieve our goal, we have the assumption that $v_0(t) = v_0(t-\tau)$ which requires the velocity function of the virtual leader to be a constant or periodic with period τ, when this assumption is not satisfied, an alternative way to achieve the same goal is to apply the same coupling communication time delay to the agent itself, i.e.

$$
\begin{cases}
\dot{r}_i = v_i & \\
\dot{v}_i = -\nabla_{r_i} V(r) - c_1(r_i - r_0) - \sum_{j \in N_i} a_{ij}(r)(v_i(t-\tau) - v_j(t-\tau)) + a_0 & t \neq t_k \\
\Delta v_i(t_k) = c_2(v_i(t_k^-) - v_0(t_k^-))) & t = t_k
\end{cases}
$$

the proof is similar to previous one and is omitted here.

4.3.1 Simulation

Here we present a simulation result for the hybrid system model with communication time delay in velocity, the dynamics is defined in 4.17 , a serial of snapshots is provided and the following parameters remain fixed throughout the simulation: $d = 12, R = 1.2d, \epsilon = 0.1$ (for σ-norm), $a = b = 5$ for $\phi(z)$, $h = 0.2$ for the bump function of $\phi_\alpha(z)$, and the step-size in the simulation is $0.01s$ and the length of the impulsive interval $0.3s$ and the delay $\tau = 0.1s$. In addition, the position of a static virtual leader is marked with a o sign. The initial positions and velocity of all 50 agents are uniformly chosen at random form the box $[-10, 10]^2$. A flock is formed in Fig.4.3(d) and maintained thereafter. The number of edges of the dynamic graph $G(r(t))$ increases by time and has a tendency to render the net connected. Numerical measurements indicate that the final conformation is a low-energy quay α-lattice that induces a flock. These observations are in close agreement with our analysis in last section.

Moreover, we define a velocity mismatch function with respect to all agents and the virtual agent as :$M(t) = \sum_{i=1}^{N}(\|v_i - v_0\|)^2$ and plot it in Fig.4.4, we can see clearly the velocity pulse effect at every impulsive moment $t_k = 30$ time-steps, and the velocity difference between all agents and the virtual leader asymptotically converges to zero, which also denotes that the velocity mismatch between all agents asymptotically converge to zero, we can see that our simulation is consistent with our study.

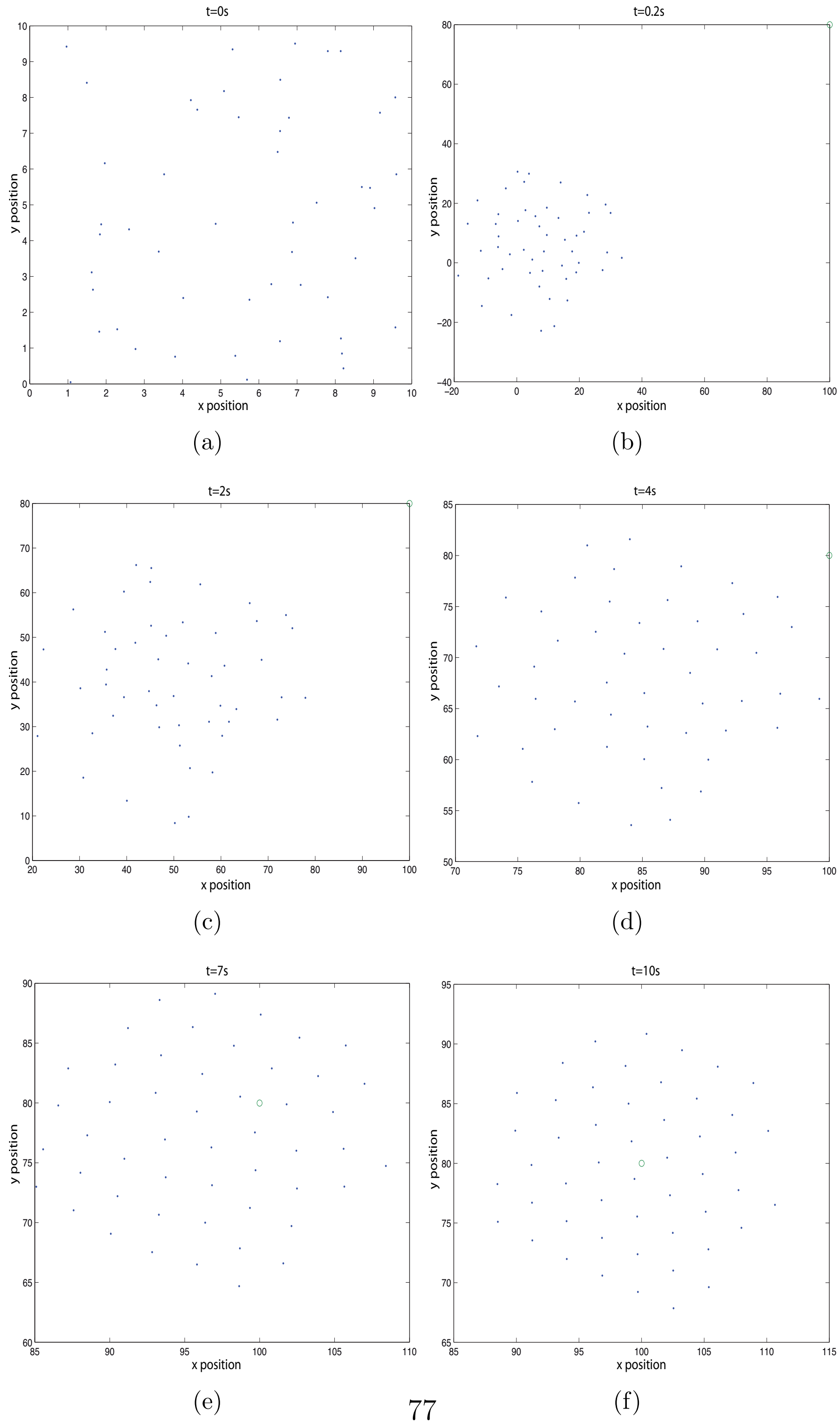


Figure 4.3: Flocking via Impulsive Control with Time Delay for n=50 agents.

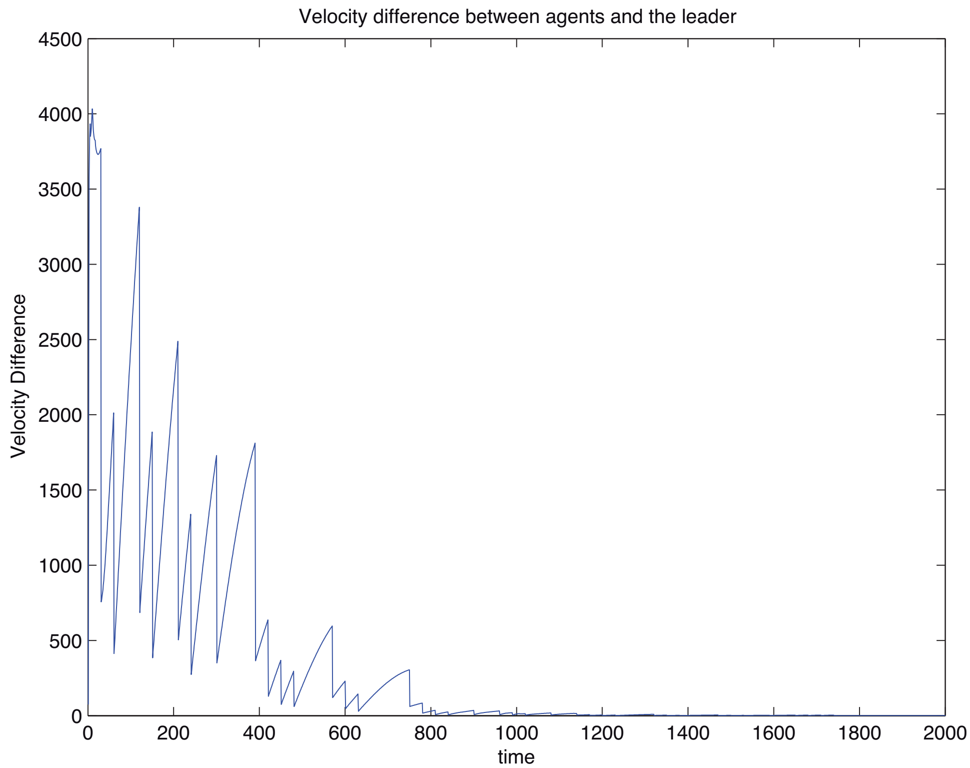


Figure 4.4: Velocity Mismatch

4.4 Dynamic Obstacle Avoidance

In this section, we will present our hybrid control flocking algorithm with dynamic obstacle avoidance capability. Recall in Chapter 3 we have defined agent-based representation of all nearby obstacle by creating a new species of agents called β-agents. Here we will divide the feedback control which is based on the information given by all nearby obstacles to a continuous control and an impulsive control. For the continuous control we create a position-based potential function between obstacles and agents as we did in Chapter 3, while for the impulsive control we add the velocity difference between dynamic obstacles and agents as a damping term. Our hybrid control algorithm can be described as following:

$$\begin{cases} \dot{r}_i = v_i \\ \dot{v}_i = -\nabla_{r_i} V_\alpha(r) - \nabla_{r_i} V_\beta(r) - c_1^\gamma (r_i - r_0) & t \neq t_k \\ \Delta v_i(t_k) = c_2^\gamma (v_i(t_k^-) - v_0(t_k^-)) + c_2^\beta \sum_{k \in N_i^\beta} b_{i,k}(r)(v_{i,k}^\beta(t_k^-) - v_i(t_k^-)) & t = t_k \end{cases} \tag{4.19}$$

where V_α is the potential function between inter-agents and V_β is the potential function between agents and obstacles respectively, which can be define as following:

$$V_\alpha(r) = \sum_{i \in \nu_\alpha} \sum_{j \in \nu_\alpha \setminus \{i\}} \psi_\alpha(\|r_j - r_i\|_\sigma)$$

$$V_\beta(r) = \sum_{i \in \nu_\alpha} \sum_{k \in N_i^\beta} \psi_\beta(\|r_{i,k}^\beta - r_i\|_\sigma)$$

The adjacency matrix between an α-agent at r_i and its neighbouring obstacle (β-agent) at $r_{i,k}^\beta$ is defined as

$$b_{i,k}(r) = ph(\|r_{i,k}^\beta - r_i\|_\sigma / d_\beta)$$

where $d_\beta < r_\beta$ with $d_\beta = \|d^\beta\|_\sigma, r_\beta = \|r^\beta\|_\sigma$. $c_1^\gamma, c_2^\gamma, c_2^\beta$ are positive constants and (r_0, v_0) is the state of the virtual leader (navigational feedback).

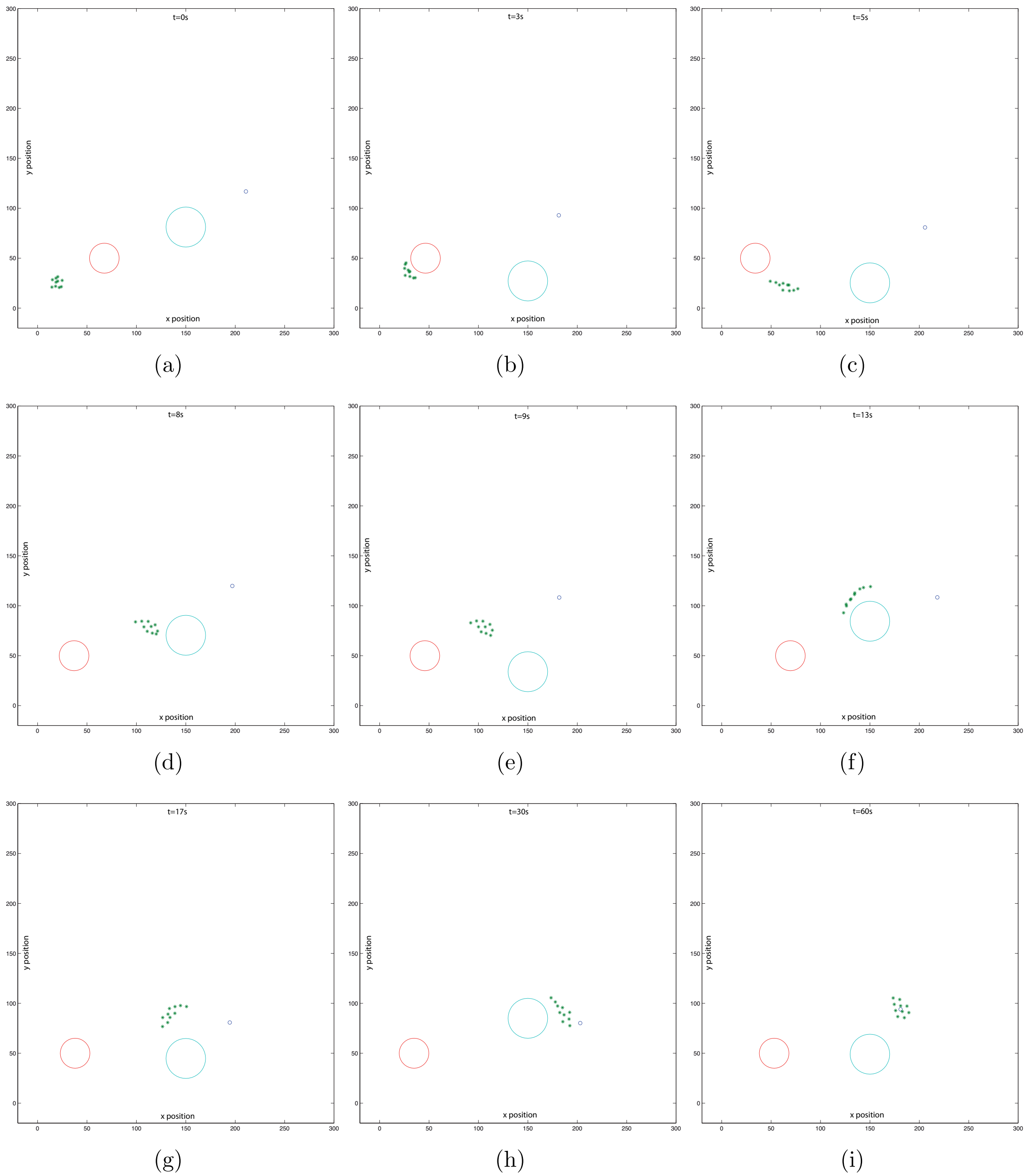
We assume that every agent is equipped with a radar sensor that allows the agent to measure the relative position and velocity between the closest point on an obstacle and itself, given a spherical obstacle with radius R_k centred at o_k, the position and velocity of the obstacle (β-agent) are given by

$$r_{i,k}^\beta = \mu r_i + (i - \mu) y_k, \; r_{i,k}^\beta = \mu P r_i$$

where $\mu = R_k / \|r_i - y_k\|, a_k = (r_i - y_k)/\|r_i - y_k\|$, and $P = I - a_k a_k^T$.

Analysis of an equilibrium state of a group of dynamic agents that perform flocking in presence of dynamic obstacle makes less sense when the flock does not pass around all the obstacles. To be more precise, it is less interesting to analyze the stability of the equilibrium of collective dynamics of a flock while some obstacles are permanently present. This assumes that after some finite time $t_1 > 0$, no α-agent ever comes near an obstacle, the case reduces to analysis of free-flocking that has already been presented. Hence, we only present several simulation results here for our hybrid control algorithm with dynamic obstacle avoidance capabilities.

In the simulation, we introduce two spherical dynamic obstacles which are indicated in Fig.4.5, one centre at $(50, 50)$ with radius $r = 15$ at $t = 0s$ and it's moving in horizontal position reciprocally with velocity $v = 0.5$ between $(30, 50)$ and $(70, 50)$, while the other one centre at $(150, 50)$ with radius $r = 20$ at $t = 0s$ and it's moving in vertical position reciprocally with velocity $v = 1.1$ between $(150, 20)$ and $(150, 80)$. Also the virtual leader which is indicated by a small dot is moving at velocity $v = 1$ along a circle centre at $(200, 100)$ with radius $r = 20$. Simulations show that our hybrid control algorithm has successfully led to flocking and the group of agents can avoid collision with nearby obstacles and finally converge to the same velocity of the virtual leader and maintain the network connectivity.



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 4.5: Flocking with dynamic obstacle avoidance

Chapter 5

Conclusion and Future Direction

Flocking is a form of collective behaviour of large number of interacting agents with a common group objective. For many decades, scientists from many diverse disciplines including animal behaviour, physics and biophysics, social sciences, and computer science have been fascinated by the emergence of flocking in groups of agents via local interactions [19, 42, 41, 40, 37, 47, 59, 56, 52, 35, 27, 57, 39, 45]. Coordinated motion has received significant recent attention in cooperative robotics (i.e., to make the agents to aggregate and stay in a tight group, achieve a spatial pattern or a geometric formation, or tack a moving object [45]) and biology (i.e., formation of fruiting bodies by bacteria, foraging behaviour of ants, or cohesive flight of warms of bees [41, 37]). Such problems are also closely related to a group reaching a consensus or agreement (since often preference can be linked to position[44]). The problem of distributed agreement is a problem which has been popular in the multi-agent dynamical systems community.

Dynamics of multi-agent systems can exhibit rich and complex characteristics. Combining agent, sensing, and communication characteristics results in a multi-scale system with local spatio-temporal agent action dynamically combining into an "emergent" global pattern of group behaviour [23]. There are other problems such as task allocation or scheduling, executing spatially distributed tasks, or task execution under time or energy or other constraints. For example, task allocation often raised in multi-agent dynamical system problems where the "tasks" arise via interactions with the environment, and the tasks must be allocated across the agents and properly scheduled in time for efficient execution, methods from distributed scheduling, load balancing, and assignment are integrated into coordinated motion methods to achieve a multi-objective method that mush balance tight group cohesion with the pressing need to complete tasks. This thesis is concerned with the coordinated motion of the group of agents to migrate to certain destination, or track a group leader,while network connectivity is maintained and both inter-agent and obstacles collision avoidance capabilities are ensured, both continuous and impulsive control have been considered to achieve these goals.

In chapter 2, we firstly introduce some basic concepts in graph theory in order to establish a theoretical framework for modelling the flocking problem for multi-agent dynamical system, then we study the discrete model given by Vicsek [59], in which the agents of the system applying a very simple rule by averaging the velocity of their neighbouring agents plus some random noise, simulation results show that under low noise and high density situation agents can achieve a global consensus of velocity by using only local information of neighbours, this discrete model gives us a first impression of what consists flocking for multi-agent dynamical systems. Moreover, we present a continuous model of double integrator agents using the Newton's law, and how to design the potential function between the agents, which is main method we use to denote the inter-agent interactions in this thesis. In chapter 3, we study the continuous model for flocking for multi-agent dynamical systems, by introducing a virtual leader into our model, we successfully eliminate the fragmentation phenomenon and ensure network connectivity of the system at the same time. Lyapunov techniques for continuous system has been used to prove the asymptotic stability of the equilibrium state of the system, communication time delay is also considered in our model. At the end of chapter 2 we give a detailed investigation on how to model the obstacles in the environment and present an algorithm to allow the agents to possess obstacle avoidance capabilities. In chapter 4, we further extend the existing continuous model to a hybrid control model, which is a more general model for flocking problem for multi-agent dynamical systems, and this chapter is the main contribution of this thesis. We use the velocity feedbacks given by the virtual leader as impulsive controls while the potential function between agents as the continuous controls, and our model successfully lead to flocking, by applying this hybrid control method we find out that we can remove the damping terms which appear in the continuous control model in chapter 3, hence, we can effectively reduce the influence subjected to sensing time delay in velocity and reduce the cost as well. Lyapunov techniques for continuous and impulsive systems is used and stability analysis is also provided. At the end of this chapter we present a hybrid control algorithm to allow agents to possess dynamic obstacle avoidance capabilities, simulation results illustrate our algorithm is in agreement with our analysis.

Further, we can study the stability criteria for our hybrid control model for flocking with dynamic obstacle avoidance capabilities, and obstacles other than a stripe or spherical can be constructed in the future. It is not difficult to imagine that given a non-convex obstacle there might be multiple β-agents which are equally close to an agent. Assuming that all of these β-agents repel an agent in the system, one might expect that in certain situations there would be a conflict of tasks between obstacle avoidance by a group of agents and achieving a group objective by a virtual leader. An example of this might be the agent, obstacle (β-agent) and the virtual are located on a same line, or the agents are trapped by a convex obstacle. In these cases, the agents can not pass through the "wall" and reach their destination. Hence, further study of using a more sophisticated obstacle avoidance approach eliminates getting trapped behind an obstacle can be done in the future. A successful example of such approach is the use of gyroscopic forces for obstacle avoidance in [7, 8]. The influence of random perturbation in the environment can also be considered in the future.

Study on flocking with predictive mechanisms can also be investigated in the future. In [32] the authors present a discrete-time MPC flocking protocol to estimated future position of neighbours. This model successfully improve the convergent speed compared with Olfati-Saber's model and achieve a better quasi α-lattice structure. In [51] the authors use a game theory approach to multi-agent team cooperation to ensure team cooperation by considering a combination of individual costs as a team cost function. Optimization algorithms and formation controls other than α-lattice can also be considered.

APPENDICES

Appendix A

Convergent Analysis for Vicsek's Model

Before studying the stability of Vicsek's model, we need to introduce some concepts in matrix analysis which will be very useful in the later on proof.

Definition A.1. ***Nonegative Matrix*** *Let $A = [a_{ij}]_{m\times n}$, A is said to be a nonnegative matrix ($A \geq 0$) if all $a_{ij} \geq 0$, positive matrix ($A > 0$) if all $a_{ij} > 0$.*

Definition A.2. ***Primitive Matrix*** *A nonnegative matrix $A \in \mathbb{R}^{n\times n}$ is said to be primitive if and only if $A^m > 0$ for some $m \geq 1$.*

Definition A.3. ***Stochastic Matrix*** *A nonnegative matrix $A \in \mathbb{R}^{n\times n}$ with the property that all its row sums are 1 is said to be a (row) stochastic matrix. (since each row may be thought of as a discrete probability distribution on a sample space with n points).*

Definition A.4. ***Ergodic*** *A stochastic matrix $A \in \mathbb{R}^{n\times n}$ with the property that $\lim_{i\to\infty} M^i$ is a matrix of rank 1 is called ergodic.*

Theorem A.1. *Primitive stochastic matrices are ergodic.*

Theorem A.2. *Let $M_1, M_2, \ldots, M_m$ be a finite set of ergodic matrices with the property that for each sequence $M_{i_1}, M_{i_2}, \ldots, M_{i_j}$ of positive length, the matrix product $M_{i_j} M_{i_{j-1}}, \ldots, M_{i_1}$ is ergodic. Then for each infinite sequence, $M_{i_1}, M_{i_2}, \ldots$ there exist a row vector c such that*

$$\lim_{j\to\infty} M_{i_j} M_{i_{j-1}} \cdots M_{i_1} = \mathbf{1}c$$

Recall Vicsek's model which is a simple discete-time model of n autonomous agents all moving in the plane with the same velocity but different direction. Each agent's direction is updated based on the rule that taking the average of its own direction plus its neighbours. The mathematical model without noise can be described as followed:

$$\theta_i(t+1) = < \theta_i(t) >_r \tag{A.1}$$

where t is a discrete-time index taking values in the nonnegative integers, and $< \theta_i(t) >_r$ is the average of the directions of agent i and its neighbours neighbours at time t, we can it explicitly as

$$< \theta_i(t) >_r = \frac{1}{1+n_i(t)}(\theta_i(t) + \sum_{j \in N_i(t)} \theta_j(t)) \tag{A.2}$$

where $n_i(t)$ denotes the number of neighbours of agent i at time t. The discrete update equations determined by $(A.1)$ and $(A.2)$ depends on the relationships between neighbours at every time t. These relationship can be described as a simple, undirected graph with vertex set $\{1, 2, \ldots, n\}$ which is determined so that the edge (i, j) describe the relationship between agent i and j. Since the relationship between neighbours is time variant, so the graph with describes them, or more explicitly, the adjacency matrix $[a_{ij}]_{n\times n}$ is also time change over time. To deal with this we will need to consider all the possibilities of such graphs or adjacency matrices $[a_{ij}]_{n\times n}$. Here we will use the symbol P to denote a suitably defined set, indexing the class of all simple graph G_P define on n vertices. The update rules determined by (1) and (2) can be written in vector form. For each $p \in P$, define

$$F_p = (I + D_p)^{-1}(A_p + I) \tag{A.3}$$

where A_p is the adjacency matrix of graph G_p and D_p is the diagonal matrix whose ith diagonal element is the *degree* of vertex i within the graph, which denotes the number of edges incident to vertex i.

Notice that $f_{ij} = (a_{ij} + \delta_{ij})/(1 + d_i)$, where f_{ij} denotes the elements of F_p, $\delta_{ij} = 0$ for all $i \neq j$, $\delta_{ij} = 1$ only if $i = j$, d_i is the degree of vertex i. So we have $\sum_{j=1}^{n} f_{ij} = \sum_{j=1}^{n}(a_{ij} + \delta_{ij})/(1 + d_i) = (d_i + 1)/(d_i + 1) = 1$, which indicates that F_p is a stochastic matrix. Moreover, if A_p is connected, it is known that $(I + A_p)^m$ becomes a positive matrix for m sufficiently large

it's easy to show if $(I + A_p)^m$ has all positive entries, then so does F_p^m. Hence F_p is also a primitive matrix if A_p is connected. By theorem 1, F_p is ergodic.

Then the system can be written as

$$\theta(t+1) = F_{\sigma(t)}\theta(t), \qquad t \in \{0, 1, 2, \ldots\} \tag{A.4}$$

where θ is the direction vector $\theta = [\theta_1 \quad \theta_2 \cdots \theta_n]'$and $\sigma : \{0, 1, \ldots\} \to P$ is a switching signal whose value at time t denotes the index of the graph representing the agents' relationship at time t.

In our stability analysis we want to show that for a large class of switching signals, or any initial condition, the directions of all the agents in the systems will finally converge to a same steady state direction θ_{ss}, which is equivalent to write

$$\lim_{t \to \infty} \theta(t) = \theta_{ss}\mathbf{1} \tag{A.5}$$

where $\mathbf{1} = [1 \quad 1 \quad \cdots 1]'_{n\times 1}$. Intuitively this can not always be true unless we have some additional conditions on the connectivity of the Graph $G_{\sigma(t)}$, to show this let's consider two extreme situations:

1. the interaction range r is very small so that there is at least one agent, say i, which never acquires any neighbours. Mathematically this means the graph $G_{\sigma(t)}$ is never connected and vertex i is isolated for all t.
2. the interaction range r is large enough that it ensures every agent in the system can interact with all other agents for all t. Mathematically this means the graph $G_{\sigma(t)}$ is connected and $\sigma(t)$ is fixed for all t.

In situation 1, since there's at least one agent can never interact with any other agents, such that the systems state θ can not converge to the steady state θ_{ss} for certain initial conditions, in situation 2 convergence of of θ to $\theta_{ss}\mathbf{1}$ can easily be established because with σ fixed, (A.5) is a linear, time-invariant, discrete-time system. So the most interesting situation is between these two extremes when σ changes with time, $G_{\sigma(t)}$ is not necessarily connected for all t, but no strictly proper subsets of $G_{\sigma(t)}$'s vertices is isolated from the rest for all t. Motivated by this idea, we will introduce a new concept called *jointly connected*, which is a weaker condition than connected of a graph.

Definition A.5. *Jointly Connected Let G be the union of a collection of simply graphs $\{G_{p_1}, G_{p_2}, \ldots, G_{p_m}\}$, which means G with vertex set $\mathbb{V}$ and edge set $\mathbb{E}$ equaling the union of the edge sets of all the graphs in the collection. We say that such a collection $\{G_{p_1}, G_{p_2}, \ldots, G_{p_m}\}$ is jointly connected if the union G of its members is a connected graph.*

Definition A.6. *Linked Together N agents are called linked together across a time interval $[t, \tau]$ if the collection of graph $\{G_{\sigma(t)}, G_{\sigma_(t+1)}, \ldots, G_{\sigma_(\tau)}\}$ encountered along the interval is jointly connected.*

Notice that if n agents is jointly connected, then situation 1 is included while situation 2 is eliminated, then we can establish the sufficient condition for the convergence of the system to a stead state.

Theorem A.3. *Let $\theta(0)$ be fixed and let $\sigma : \{0, 1, 2, \ldots\} \to P$ be a switching signal for which there exists an infinite sequence of contiguous, nonempty, bounded, time-intervals $[t_i, t_{i+1})$, $i \geq 0$, starting at $t_0 = 0$, with the property that across each such interval, the n agents are linked together. Then*

$$\lim_{t \to \infty} \theta(t) = \theta_{ss} \mathbf{1} \tag{A.6}$$

where θ_{ss} is a number depending only on $\theta(0)$ and σ.

Before proving theorem 3, we need two more lemmas.

Lemma A.1. *Let $m \geq 2$ be a positive integer and let $A_1, A_2, \ldots, A_m$ be nonnegative $n \times n$ matrices. Suppose that the diagonal elements of all of the A_i are positive and let u and ρ denote the smallest and largest of these, respectively. Then*

$$A_1 A_2 \cdots A_m \geq (\frac{\mu^2}{2\rho})^{m-1}(A_1 + A_2 + \cdots + A_m). \tag{A.7}$$

Proof. Let $\delta = \mu^2/2\rho$. It will be shown by induction that

$$A_1 A_2 \cdots A_m \geq \delta^{i-1}(A_1 + A_2 + \cdots + A_i) \tag{A.8}$$

holds for $i \in \{2, 3, \ldots, m\}$. Toward this end, note that it's possible to write each A_i as $A_i = \mu I + B_i$ where B_i is nonnegative. Then, for any $j, k \in \{1, 2, \ldots, m\}$

$$A_j A_k = (\mu I + B_j)(\mu I + B_k) = \mu^2 I + \mu(B_j + B_k) + B_j B_k.$$

Hence

$$A_j A_k \geq \mu^2 I + \mu(B_j + B_k) \geq \mu^2 I + \tfrac{\mu^2}{2\rho}(B_j B_k) = \delta((\rho I + B_j) + (\rho I + B_k))$$

Since $(\rho I + B_j) \geq A_j$ and $(\rho I + B_k) \geq A_k$ it follows that

$$A_j A_k \geq \delta(A_j + A_k) \quad \forall j, k \in \{1, 2, \ldots, m\}. \tag{A.9}$$

Setting $j = 1$ and $k = 2$ proves that (A.8) holds for $i = 2$. Now suppose that $m > 2$ and that (8) hold for $i \in \{2, 3, \ldots, l\}$ where l is some integer in $\{2, 3, \ldots, m-1\}$. Then $A_1 A_2 \cdots A_{l+1} = (A_1, \ldots, A_l)A_{l+1}$ so by the inductive hypothesis

$$A_1 A_2 \cdots A_{l+1} \geq \delta^{l-1}(A_1 + A_2 + \cdots + A_l)A_{l+1} \tag{A.10}$$

However, using (A.9) l times, we can write

$$(A_1 + A_2 + \cdots + A_l)A_{l+1} \geq \delta((A_1 + A_{l+1}) + (A_2 + A_{l+1}) + \cdots + (A_l + A_{l+1})).$$

Thus

$$(A_1 + A_2 + \cdots + A_l)A_{l+1} \geq \delta(A_1 + A_2 + \cdots + A_{l+1})$$

This and (A.10) imply that (A.8) holds for $i = l+1$. Therefore, by induction the statement is true for all $i \in \{2, 3, \ldots, m\}$ □

Lemma A.2. *Let $\{p_1, p_2, \ldots, p_m\}$ be a set of indices in P for which $\{G_{p_1}, G_{p_2}, \ldots, G_{p_m}\}$ is a jointly connected collection of graphs. Then the matrix product $F_{p_1} F_{p_2}, \cdots, F_{p_m}$ is ergodic.*

Proof. Let $F = (I + D)^{-1}(I + A)$ where A and D are respectively the adjacency matrix and diagonal degree matrix of the union of the collection of graphs P for which $\{G_{p_1}, G_{p_2}, \ldots, G_{p_m}\}$. Since the collection is jointly connected, its union is connected which means F is connected, so F is primitive. By Lemma 1,

$$F_{p_1} F_{p_2} \cdots F_{p_m} \geq \gamma(F_{p_1} + F_{p_2} + \cdots + F_{p_m}) \tag{A.11}$$

where γ is a positive constant depending on the matrices in the product. Since for $i \in \{1, 2, \ldots, m\}$, $F_{p_i} = (I + D_{p_i})^{-1}(I + A_{p_i})$ and $D > D_{p_i}$, it must have that $F_{p_i} \geq (I + D)^{-1}(I + A_{p_i})$, $i \in \{1, 2, \ldots, m\}$. From this and (A.11) it follows that

$$F_{p_1} F_{p_1} \cdots F_{p_m} \geq \gamma(I + D)^{-1}(mI + A_{p_1} + A_{p_2} + \cdots + A_{p_m}) \tag{A.12}$$

However, $A_{p_1} + A_{p_2} + \cdots + A_{p_m} \geq A$ and $mI \geq I$ so

$$F_{p_1} F_{p_1} \cdots F_{p_m} \geq \gamma F$$

Since the product $F_{p_1} F_{p_1} \cdots F_{p_m}$ is bounded below by a primitive matrix γF, the product must be primitive as well, moreover, $F_{p_1} F_{p_1} \cdots F_{p_m}$ is also a stochastic matrix, so it must be ergodic. □

Then we can are ready to prove theorem 3 now,

Proof. Let T denote the least upper bound on the lengths of the intervals $[t_i, t_{i+1})$, $i \geq 0$. By assumption $T < \infty$. Let $\Phi(t,t) = I, t \geq 0$, and $\Phi(t,\tau) \triangleq F_{\sigma_{t-1}} \cdots F_{\sigma_{\tau+1}} F_{\sigma_\tau}, t > \tau \geq 0$. Clearly $\theta(t) = \Phi(t,0)\theta(0)$. To complete the theorem's proof, it's enough to show that

$$\lim_{t\to\infty} \Phi(t,0) = \mathbf{1}c \tag{A.13}$$

for some row vector c since this would imply (A.6) with $\theta_{ss} \triangleq c\theta(0)$. In view of Lemma 2, the constraints on σ imply that each such matrix product $\Phi(t_{j+1}, t_j), j \geq 0$, is ergodic. Moreover the set of possible $\Phi(t_{j+1}, t_j), j \geq 0$, must be finite because each $\Phi(t_{j+1}, t_j)$ is a product of at most T matrices from $\{F_p : p \in P\}$ which is a finite set. But $\Phi(t_j, 0) = \Phi(t_j, t_{j-1})\Phi(t_{j-1}, t_{j-2}), \ldots, \Phi(t_1, t_0)$. Therefor by theorem 2

$$\lim_{j\to\infty} \Phi(t_j,0) = \mathbf{1}c \tag{A.14}$$

which is equivalent to (A.13). $\square$

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