# Algebraic Methods and Monotone Hurwitz Numbers

by

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## AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Mathieu Guay-Paquet

# Abstract

We develop algebraic methods to solve *join-cut equations*, which are partial differential equations that arise in the study of permutation factorizations. Using these techniques, we give a detailed study of the recently introduced *monotone Hurwitz numbers*, which count factorizations of a given permutation into a fixed number of transpositions, subject to some technical conditions known as *transitivity* and *monotonicity*.

Part of the interest in monotone Hurwitz numbers comes from the fact that they have been identified as the coefficients in a certain asymptotic expansion related to the *Harish-Chandra-Itzykson-Zuber integral*, which comes from the theory of random matrices and has applications in mathematical physics. The connection between random matrices and permutation factorizations goes through representation theory, with symmetric functions in the *Jucys-Murphy elements* playing a key role.

As the name implies, monotone Hurwitz numbers are related to the more classical *Hurwitz numbers*, which count permutation factorizations regardless of monotonicity, and for which there is a significant body of work. Our results for monotone Hurwitz numbers are inspired by similar results for Hurwitz numbers; we obtain a *genus expansion* for the related generating functions, which yields explicit formulas and a polynomiality result for monotone Hurwitz numbers. A significant difference between the two cases is that our methods are purely algebraic, whereas the theory of Hurwitz numbers relies on some fairly deep results in algebraic geometry.

Despite our methods being algebraic, it seems that there should be a connection between monotone Hurwitz numbers and geometry, although this is currently missing. We give some evidence for this connection by identifying some of the coefficients in the monotone Hurwitz genus expansion with coefficients in the classical Hurwitz genus expansion known to be *Hodge integrals* over the moduli space of curves.

# Acknowledgements

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On the technical side, I would like to thank Matthew Skala for creating a high-quality thesis template and making it available to the public, which no doubt saved me from many headaches.

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*To my friends,*
*for making this a wonderful ride*
*so far.*

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# List of Symbols

| | |
|---|---|
| $\mathsf{A}, \mathsf{B}$ | Generically, elements of $\mathbb{Q}\mathcal{S}$, page 7. |
| $\alpha, \beta$ | Generically, integer partitions, page 3. |
| $\rho, \sigma, \tau$ | Generically, permutations in $\mathcal{S}_n$, $n \geq 0$, page 7. |
| $[A]B$ | The coefficient of $A$ in the expansion of $B$ in a suitable basis, page 7. |
| $\mathsf{A} \otimes \mathsf{B}$ | The concatenation product of $\mathsf{A}$ and $\mathsf{B}$ in $\mathbb{Q}\mathcal{S}$, page 29. |
| $\mathsf{A}^{\otimes k}$ | The $k$th concatenation power of $\mathsf{A}$ in $\mathbb{Q}\mathcal{S}$, page 29. |
| $\lvert\alpha\rvert$ | The size of the partition $\alpha$, page 3. |
| $\lvert\mathrm{aut}(\alpha)\rvert$ | The quantity $\prod_{i\geq 1} m_i(\alpha)!$, page 3. |
| $\alpha \vdash n$ | A partition of the integer $n$, page 3. |
| $\mathsf{C}_\alpha$ | A conjugacy class sum in $\mathbb{Q}\mathcal{S}_n$, $n = \lvert\alpha\rvert$, page 8. |
| ch | The characteristic map $\mathbb{Q}\mathcal{S} \to \overline{\Lambda}_{\mathbb{Q}}$, page 28. |
| $\chi^\alpha_\beta$ | The character of the irreducible representation $V^\alpha$, evaluated at a permutation of cycle type $\beta$, page 9. |
| $\mathrm{cyc}(\sigma)$ | The cycle type of the permutation $\sigma$, page 24. |
| $c(k, T)$ | Content of the box labelled $k$ in the standard Young tableau $T$, page 10. |
| $c_{g,\alpha}$ | A coefficient in the rational form of $\vec{\mathbf{H}}_g$, page 50. |
| $hc_{g,\alpha}$ | A coefficient in the rational form of $\mathbf{H}_g$, page 78. |
| $\mathrm{D}_k$ | The differential operator $p_k \frac{\partial}{\partial p_k}$, page 52. |
| $\mathcal{D}$ | The differential operator $\sum_{k\geq 1} k p_k \frac{\partial}{\partial p_k}$, page 52. |
| $\Delta_i$ | The $i$th lifting operator on $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$, page 40. |

$\mathrm{E}_k$ The differential operator $q_k \frac{\partial}{\partial q_k}$, page 52.

$\mathcal{E}$ The differential operator $\sum_{k\geq 1} k q_k \frac{\partial}{\partial q_k}$, page 52.

$\eta$ The power series $\sum_{k\geq 1}(2k+1)\binom{2k}{k} q_k \in \mathbb{Q}[[\mathbf{q}]]$, page 49.

$\eta(y_1)$ An auxiliary power series in $\mathbb{Q}[[\mathbf{q},\mathbf{y}]]$, page 51.

$\eta_j$ The power series $\sum_{k\geq 1}(2k+1)k^j\binom{2k}{k} q_k \in \mathbb{Q}[[\mathbf{q}]]$, page 49.

$\eta_j(y_1)$ An auxiliary power series in $\mathbb{Q}[[\mathbf{q},\mathbf{y}]]$, page 51.

ev The Jucys-Murphy evaluation map $\Lambda_{\mathbb{Q}[t]} \to \mathrm{FH}$, page 25.

$\overline{\mathrm{E}}_k$ The differential operator $\overline{q}_k \frac{\partial}{\partial \overline{q}_k}$, page 82.

$\overline{\mathcal{E}}$ The differential operator $\sum_{k\geq 1} k \overline{q}_k \frac{\partial}{\partial \overline{q}_k}$, page 82.

$\overline{\eta}$ The power series $\sum_{k\geq 1} k^{k+1} q_k/k! \in \mathbb{Q}[[\mathbf{q}]]$, page 78.

$\overline{\eta}_j$ The power series $\sum_{k\geq 1} k^{k+j+1} q_k/k! \in \mathbb{Q}[[\mathbf{q}]]$, page 78.

$\mathsf{F}^\alpha$ An orthogonal idempotent of $\mathbb{Q}\mathcal{S}_n$, $n = |\alpha|$, page 9.

FH The Farahat-Higman algebra in $\mathbb{Q}\mathcal{S}$, page 24.

$\mathbf{F}$ A guess for the generating function $\vec{\mathbf{H}}_0 \in \mathbb{Q}[[\mathbf{p}]]$, page 54.

$f^\alpha$ The dimension of the irreducible representation $V^\alpha$, page 9.

$f_\alpha$ A forgotten symmetric function, page 6.

$\gamma$ The power series $\sum_{k\geq 1}\binom{2k}{k} q_k \in \mathbb{Q}[[\mathbf{q}]]$, page 49.

$\gamma(y_1)$ An auxiliary power series in $\mathbb{Q}[[\mathbf{q},\mathbf{y}]]$, page 51.

$\mathrm{GZ}_n$ The Gelfand-Zetlin algebra in $\mathbb{Q}\mathcal{S}_n$, page 10.

$\overline{\gamma}$ The power series $\sum_{k\geq 1} k^k q_k/k! \in \mathbb{Q}[[\overline{\mathbf{q}}]]$, page 78.

$g$ A nonnegative integer, the genus, page 22.

$\mathbf{H}$ The generating function in $\mathbb{Q}[[\mathbf{p},t]]$ for transitive transposition factorizations, page 37.

$\mathsf{H}$ The generating function in $\mathbb{Q}\mathcal{S}[[t]]$ for transitive transposition factorizations, page 32.

$\mathbf{H}_g$ The generating function in $\mathbb{Q}[[\mathbf{p}]]$ for transitive transposition factorizations of genus $g$, page 38.

$\mathbf{H}^*$ The generating function in $\mathbb{Q}[[\mathbf{p}, t]]$ for all transposition factorizations, page 36.

$\mathsf{H}^*$ The generating function in $\mathbb{Q}\mathcal{S}[[t]]$ for all transposition factorizations, page 32.

$H^r(\alpha)$ A Hurwitz number, counting certain factorizations of length $r$, page 34.

$H_g(\alpha)$ A Hurwitz number, counting certain factorizations of genus $g$, page 33.

$\vec{\mathbf{H}}$ The generating function in $\mathbb{Q}[[\mathbf{p}, t]]$ for transitive monotone factorizations, page 47.

$\vec{\mathsf{H}}$ The generating function in $\mathbb{Q}\mathcal{S}[[t]]$ for transitive monotone factorizations, page 31.

$\vec{\mathbf{H}}_g$ The generating function in $\mathbb{Q}[[\mathbf{p}, t]]$ for transitive monotone factorizations of genus $g$, page 47.

$\vec{\mathbf{H}}^*$ The generating function in $\mathbb{Q}[[\mathbf{p}, t]]$ for all monotone factorizations, page 42.

$\vec{\mathsf{H}}^*$ The generating function in $\mathbb{Q}\mathcal{S}[[t]]$ for all monotone factorizations, page 31.

$\vec{H}^r(\alpha)$ A monotone Hurwitz number, counting certain factorizations of length $r$, page 34.

$\vec{H}_g(\alpha)$ A monotone Hurwitz number, counting certain factorizations of genus $g$, page 34.

$h_\alpha$ The symmetric function $\prod_{i=1}^{\ell(\alpha)} h_{\alpha_i}$, page 5.

$h_k$ A complete symmetric function, page 5.

$\mathrm{id}_n$ The identity permutation in $\mathcal{S}_n$, page 12.

$\mathsf{J}$ The list $\mathsf{J}_1, \mathsf{J}_2, \ldots$, page 25.

$\mathsf{J}_k$ The $k$th Jucys-Murphy element in $\mathbb{Q}\mathcal{S}$, page 25.

$\mathsf{J}_{k,n}$ The $k$th Jucys-Murphy element in $\mathbb{Q}\mathcal{S}_n$, page 9.

$\mathsf{K}_\alpha$ A reduced class sum in $\mathcal{Z}$, page 24.

$\ell(\alpha)$ The length of the partition $\alpha$, page 3.

$\Lambda_R$ The ring of symmetric functions with coefficients in $R$ (typically $R = \mathbb{Q}$), page 5.

$\overline{\Lambda}_{\mathbb{Q}}$ The graded completion of $\Lambda_{\mathbb{Q}}$, page 6.

$m_\alpha$ A monomial symmetric function, page 4.

$m_i(\alpha)$ The multiplicity of $i$ in the partition $\alpha$, page 3.

$[n]$ The ground set $\{1, 2, 3, \dots, n\}$, page 6.

$\mathsf{n}$ The element $\sum_{n\geq 0} n\,\mathrm{id}_n$ in $\mathcal{Z}$, page 24.

$n$ A nonnegative integer, the size of the ground set, page 6.

$n^{\overline{k}}$ The rising product $n(n+1)(n+2)\cdots(n+k-1)$, page 49.

$\Pi_i$ The $i$th projection operator on $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$, page 40.

$e_\alpha$ The symmetric function $\prod_{i=1}^{\ell(\alpha)} e_{\alpha_i}$, page 5.

$e_k$ An elementary symmetric function, page 5.

$p_\alpha$ The symmetric function $\prod_{i=1}^{\ell(\alpha)} p_{\alpha_i}$, page 5.

$p_k$ A power sum symmetric function, page 5.

$\mathbb{Q}\mathcal{S}$ The symmetric group algebra $\mathbb{Q}\mathcal{S} = \prod_{n\geq 0} \mathbb{Q}\mathcal{S}_n$, page 6.

$\mathbb{Q}\mathcal{S}_n$ The group algebra of $\mathcal{S}_n$ over $\mathbb{Q}$, page 6.

$\overline{\mathbf{q}}$ The countable set of indeterminates $\overline{q}_1, \overline{q}_2, \dots$, related to $\mathbf{p}$ by $\overline{q}_j = p_j \exp(j\overline{\gamma})$, page 80.

$\mathbf{q}$ The countable set of indeterminates $q_1, q_2, \dots$, related to $\mathbf{p}$ by $q_j = p_j(1-\gamma)^{-2j}$, page 50.

$\mathrm{rank}(\sigma)$ The rank of the permutation $\sigma$, page 12.

$\mathrm{redcyc}(\sigma)$ The reduced cycle type of the permutation $\sigma$, page 24.

$\sigma_k$ The $k$th partial product of a transposition factorization of $\sigma$, page 12.

$\mathrm{Split}_{i\to j}$ The $i$-to-$j$ splitting operator on $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$, page 40.

$\mathcal{S}_n$ The symmetric group on $[n]$, page 6.

$s_\alpha$ A Schur symmetric function, page 5.

$V^\alpha$ The irreducible representation of $\mathcal{S}_n$ indexed by $\alpha$, $n = |\alpha|$, page 9.

$v_T$ Element of Young's orthogonal basis indexed by the standard Young tableau $T$, page 10.

$\mathbf{w}$ The countable set of indeterminates $w_1, w_2, \dots$, related to $\overline{\mathbf{y}}$ by $w_i = \overline{y}_i \exp(w_i)$, page 81.

$\overline{\xi}_k$ The power series $\Pi_1 \, w_1^k/(1-w_1)^k \in \mathbb{Q}[[\overline{\mathbf{q}}]]$, page 80.

$\xi_k$ The power series $\Pi_1(1-4y_1)^{-\frac{k}{2}} - 1 \in \mathbb{Q}[[\mathbf{q}]]$, page 72.

$\mathbf{x}$ The countable set of indeterminates $x_1, x_2, \ldots$, page 39.

$\overline{\mathbf{y}}$ The countable set of indeterminates $\overline{y}_1, \overline{y}_2, \ldots$, related to $\mathbf{x}$ by $\overline{y}_i = x_i \exp(\overline{\gamma})$, page 81.

$\mathbf{y}$ The countable set of indeterminates $y_1, y_2, \ldots$, related to $\mathbf{x}$ by $y_i = x_i(1-\gamma)^{-2}$, page 51.

$\mathcal{Z}$ The centre of $\mathbb{Q}\mathcal{S}$, page 8.

$\mathcal{Z}_n$ The centre of $\mathbb{Q}\mathcal{S}_n$, page 9.

# Chapter 1

# Preliminaries

## 1.1 Outline

In this thesis, we develop algebraic methods to solve join-cut equations, which are partial differential equations that arise in the study of permutation factorizations, and apply these techniques to the study of monotone Hurwitz numbers.

Much of this work is part of a larger project by Goulden, Novak and the author [16, 17, 18], which introduced monotone Hurwitz numbers as a combinatorial device to resolve long-standing questions about a certain asymptotic expansion of the Harish-Chandra-Itzykson-Zuber integral from mathematical physics and the theory of random matrices. In the course of this work, several striking similarities emerged between the coefficients of this asymptotic expansion and the well-known Hurwitz numbers, and this is the source of the name "monotone Hurwitz numbers".

Note that this thesis focuses on the case of so-called *single* monotone Hurwitz numbers, by analogy with the single Hurwitz numbers. There is also a theory of *double* monotone Hurwitz numbers which follows the theory of double Hurwitz numbers. We will not be discussing double monotone Hurwitz theory further here, but the interested reader is referred to [16, 17] and to Carrell [3], where it is shown that double monotone Hurwitz numbers are piecewise polynomial and satisfy the 2-Toda hierarchy.

Our results for monotone Hurwitz numbers are inspired by similar results for Hurwitz numbers; we obtain a genus expansion for the related generating functions, which yields explicit formulas and a polynomiality result for monotone Hurwitz numbers, and identify the extreme coefficients in the genus expansion, both at the low end and at the high end. A significant difference in our approach is that our methods are purely algebraic, whereas the theory of Hurwitz numbers relies on some fairly deep results in algebraic geometry. However, the results themselves strongly suggest that monotone Hurwitz numbers should have some kind of geometric content.

The thesis is organized as follows.

Chapter 1 discusses the larger setting for the monotone Hurwitz problem. After a brief review of some standard notation and some representation theoretic aspects of the symmetric group algebra, we turn to the problem of enumerating transposition factorizations. Section 1.6 defines many terms and contains a few results for transposition factorizations, illustrated through a topological construction. Sections 1.7 and 1.8 describe two important connections between the centre of the symmetric group algebra and the ring of symmetric functions. The first one is via the Jucys-Murphy evaluation map and the Farahat-Higman algebra, and respects the composition of permutations; the second one is via the characteristic map, and respects the disjoint union of permutations. The chapter concludes with the definition of Hurwitz and monotone Hurwitz numbers, our main objects of study. No substantial new results are given, but some aspects of the presentation are novel.

Chapter 2 is concerned with setting up the join-cut equations for the Hurwitz and monotone Hurwitz generating functions, which are the main tool used in our investigations, through a combinatorial analysis. Along the way, some convenient operators are introduced, as a bit of notation which reflects the underlying combinatorics while being easy to manipulate algebraically. The join-cut equations for the Hurwitz case are not new, but for the monotone Hurwitz case they first appeared in recent work by Goulden, Novak and the author [16, 17, 18].

Chapter 3 deals with the technical aspects of solving the join-cut equations for monotone Hurwitz generating functions. This relies crucially on a Lagrangian change of variables, introduced in Section 3.2. Then, Section 3.3 contains the details of the verification for genus zero, and Section 3.4 contains the details of the solution for higher genera, leading to a very specific rational form for the monotone Hurwitz generating functions in higher genera. As with the join-cut equations, this work also appears in [16, 17, 18].

Chapter 4 explores some of the consequences of the solution given in Chapter 3. In particular, this includes a general polynomiality result for monotone Hurwitz numbers, an explicit formula for genus one, and a discussion of the extreme coefficients which appear in the rational form for the higher genera monotone Hurwitz generating functions, both at the low end (the Bernoulli terms) and at the high end (the Witten terms). The results concerning the Witten terms are entirely new, and are a key piece of evidence pointing to a geometric interpretation for monotone Hurwitz numbers.

Chapter 5 gives a short account of some previous results in Hurwitz theory which are remarkably similar to the new results in monotone Hurwitz theory, and a proof of a recurrence for the Witten terms which is structurally identical to the recurrence of Chapter 4. This recurrence is the only new result in this chapter.

Chapter 6 concludes with a discussion of the many similarities and a few of the differences between Hurwitz theory and monotone Hurwitz theory. In particular, it contains the formal statement and proof of the correspondence between the Witten terms for the Hurwitz and monotone Hurwitz generating functions implied by the recurrences of Chapter 4 and Chapter 5, which strongly suggests that a (currently missing) geometric interpretation for monotone Hurwitz numbers should exist.

Appendix A and Appendix B contain the numerical coefficients for the rational forms of the monotone and classical Hurwitz generating functions, respectively, for genus 2, 3, 4 and 5.

## 1.2 Standard Notation

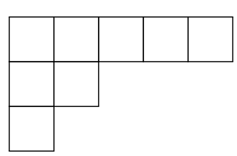
We begin by reviewing some notation used in this thesis. For partitions and symmetric functions, we use mainly standard notation, as can be found in Macdonald [34].

Given an integer $n \geq 0$, a **partition** $\alpha$ of $n$ is a weakly decreasing list $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers whose sum is $n$, called the **parts** of $\alpha$. We write $\alpha \vdash n$ to indicate that $\alpha$ is a partition of $n$. The **size** of $\alpha$ is $|\alpha| = n$, and its **length** is $\ell(\alpha) = k$. For partitions whose parts are at most 9, we may dispense with the parentheses and commas to avoid visual clutter, and we may use exponents to denote repeated parts of the same size; for example, we may write 442111 or $4^2 21^3$ for the partition $(4, 4, 2, 1, 1, 1)$. The number of parts equal to a given integer $i$ in a partition $\alpha$ is called the **multiplicity** $m_i(\alpha)$ of $i$ in $\alpha$, sometimes written simply as $m_i$ if the partition $\alpha$ is clear from the context. The quantity

$$|\mathrm{aut}(\alpha)| = \prod_{i \geq 1} m_i!,$$

which is the number of permutations of the parts of $\alpha$ which leave it globally unchanged, is also frequently used.

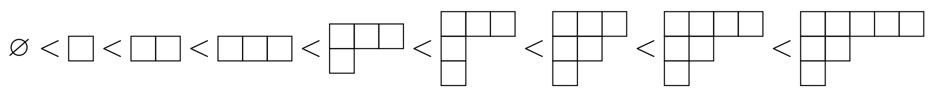
A partition $\alpha \vdash n$ can also be identified with its **Ferrers diagram**, which is a graphical representation of the set $\{(i, j) \mid 1 \leq i \leq \ell(\alpha),\ 1 \leq j \leq \alpha_i\}$. The elements of this set are typically called **boxes** or **cells**, and arranged so that the box $(i, j)$ is in row $i$ and column $j$. The English convention for the coordinates of boxes is the same as the usual convention for the coordinates of matrix entries, so that, *e.g.*, the Ferrers diagram for the partition $\alpha = (5, 2, 1)$ is



This representation of partitions naturally suggests the **containment order** on partitions: if $\alpha \vdash n$ and $\beta \vdash m$ are partitions such that the Ferrers diagram for $\alpha$ is contained in the Ferrers diagram for $\beta$, then we write $\alpha \leq \beta$. Given a saturated chain

$$(0) = \alpha^0 < \alpha^1 < \cdots < \alpha^n = \alpha$$

from the empty partition $(0)$ to a partition $\alpha \vdash n$ with respect to the containment order, we can represent it as a **standard Young tableau**, that is, a labelling of the boxes of the Ferrers diagram of $\alpha$ with the numbers $1, 2, \ldots, n$ such that the boxes labelled $1, 2, \ldots, m$ form the Ferrers diagram of $\alpha^m$. For example, the chain



is represented by the standard Young tableau

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 7 | 8 |
| 4 | 6 | | | |
| 5 | | | | |

Note that standard Young tableaux are characterized by the fact that the labels $\{1, 2, \ldots, n\}$ are each used exactly once, and the box labels increase along each row and along each column. Similarly, a **semistandard Young tableau** of shape $\alpha \vdash n$ is a labelling of the boxes of the Ferrers diagram of $\alpha$ by positive integers, possibly repeated, subject to the restriction that the box labels increase *weakly* along each row and *strictly* along each column. If a semistandard Young tableau contains $t_i$ copies of the label $i$ for $i = 1, 2, 3, \ldots$, then its **type** is the vector $(t_1, t_2, t_3, \ldots)$. For example, the labelling

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 |
| 2 | 3 | | | |
| 5 | | | | |

is a semistandard Young tableau of shape 521 and of type $(3, 2, 2, 0, 1, 0, 0, \ldots)$.

Given a countable set of indeterminates $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ and a partition $\alpha \vdash n$ with $k$ parts, the **monomial symmetric function** $m_\alpha(\mathbf{x})$ is the sum[1]

$$m_\alpha(\mathbf{x}) = \sum x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

[1]Here, the usage of $m_\alpha$ for monomial symmetric functions is a standard but unfortunate clash of notation with the usage of $m_i$ for the multiplicity of a part in a partition. However, the meaning should generally be clear from the context. The two notations can also be distinguished by whether the subscript is a partition or a single integer.

of all distinct monomials that can be obtained by choosing distinct indices $i_1, i_2, \dots, i_k$ for the indeterminates, so-named because they are invariant under any permutation of the indeterminates $\mathbf{x}$. Then, the **ring of symmetric functions** with rational coefficients in the indeterminates $\mathbf{x}$, $\Lambda_{\mathbb{Q}}(\mathbf{x})$, is the set of all finite $\mathbb{Q}$-linear combinations of monomial symmetric functions in the indeterminates $\mathbf{x}$, with addition, subtraction and multiplication defined in the obvious way. This ring is naturally graded by total degree, so that $m_\alpha(\mathbf{x})$ has degree $|\alpha|$.

Some other standard symmetric functions are the **power sum** symmetric functions

$$p_k(\mathbf{x}) = \sum_{i \geq 1} x_i^k, \qquad k \geq 1,$$

the **elementary** symmetric functions

$$e_k(\mathbf{x}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad k \geq 1,$$

and the **complete** symmetric functions

$$h_k(\mathbf{x}) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad k \geq 1.$$

For a partition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, we write $p_\alpha(\mathbf{x})$ for the product of power sums $p_{\alpha_1}(\mathbf{x}) p_{\alpha_2}(\mathbf{x}) \cdots p_{\alpha_k}(\mathbf{x})$, and similarly for $e_\alpha(\mathbf{x})$ and $h_\alpha(\mathbf{x})$. In addition to the set $\{m_\alpha(\mathbf{x}) \mid \alpha \vdash n,\, n \geq 0\}$, the sets $\{p_\alpha(\mathbf{x}) \mid \alpha \vdash n,\, n \geq 0\}$, $\{e_\alpha(\mathbf{x}) \mid \alpha \vdash n,\, n \geq 0\}$ and $\{h_\alpha(\mathbf{x}) \mid \alpha \vdash n,\, n \geq 0\}$ are bases of $\Lambda_{\mathbb{Q}}(\mathbf{x})$. A fifth significant basis of the symmetric functions is given by the **Schur** symmetric functions $s_\alpha(\mathbf{x})$, also indexed by the set of all partitions $\alpha$, can be defined as

$$s_\alpha(\mathbf{x}) = \sum_T x_1^{t_1(T)} x_2^{t_2(T)} x_3^{t_3(T)} \cdots ,$$

where the sum is over all semistandard Young tableaux $T$ of shape $\lambda$, and $(t_1(T), t_2(T), t_3(T), \dots)$ is the type of $T$.

It is much more common to work with symmetric functions through one of these five bases, rather than explicitly through the indeterminates $\mathbf{x}$, so we will generally erase these indeterminates from the notation, and assume that there is an unspecified and anonymous set of indeterminates in the background.

The ring $\Lambda_{\mathbb{Q}}$ also comes equipped with a standard symmetric and positive definite inner product, called the **Hall inner product**. It can be characterized by the fact that the pairs

$$m_\alpha \longleftrightarrow h_\alpha, \qquad p_\alpha \longleftrightarrow \frac{p_\alpha}{\prod_{i \geq 1} i^{m_i(\alpha)} \cdot m_i(\alpha)!}, \qquad s_\alpha \longleftrightarrow s_\alpha$$

are dual bases, in the sense that

$$\langle m_\alpha, h_\beta \rangle = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for the other two pairs. This leads to the sixth and final classical basis of the symmetric functions, the **forgotten** symmetric functions $f_\alpha$, defined as the dual basis of the elementary symmetric functions $e_\alpha$.

We will sometimes need to deal with symmetric functions of infinite degree, that is, infinite $\mathbb{Q}$-linear combinations of monomial (or power sum, or…) symmetric functions. Since the ring $\Lambda_\mathbb{Q}$ is graded, we can do this by passing to the natural completion $\overline{\Lambda}_\mathbb{Q}$, which is similar to a ring of formal power series. In fact, $\overline{\Lambda}_\mathbb{Q}$ can be described as the ring $\mathbb{Q}[[\mathbf{p}]]$ of formal power series in the power sums $\mathbf{p} = (p_1, p_2, \dots)$, and we will use this notation almost exclusively starting in Chapter 2.

## 1.3 The Symmetric Group

For $n \geq 0$,

- let $[n]$ be the **ground set** $\{1, 2, \dots, n\}$;
- let $\mathcal{S}_n$ be the **symmetric group** on $[n]$, that is, the set of permutations of $[n]$ under composition;
- let $\mathbb{Q}\mathcal{S}_n$ be the **group algebra** of $\mathcal{S}_n$, that is, the set of formal $\mathbb{Q}$-linear combinations of permutations in $\mathcal{S}_n$, with multiplication defined by extending the group operation $\mathbb{Q}$-linearly; and
- let $\mathbb{Q}\mathcal{S} = \prod_{n \geq 0} \mathbb{Q}\mathcal{S}_n$ be the **symmetric group algebra**, that is, the Cartesian product of the group algebras of the symmetric groups $\mathcal{S}_n$ for all $n \geq 0$, with multiplication defined component-wise.

*Remark* 1.3.1. For our purposes, the relevant properties of the set $[n]$ are that it is an $n$-element set equipped with a total ordering and that we have a natural chain of inclusions

$$[0] \subseteq [1] \subseteq [2] \subseteq \cdots \subseteq [n] \subseteq \cdots .$$

The labels $1, 2, \dots, n$ for the elements of $[n]$ are otherwise immaterial.

In some cases, we can attach a combinatorial meaning to the coefficients of an element of the symmetric group algebra. These elements then behave like generating functions, in the sense that some algebraic operations on the elements correspond to combinatorial operations on their coefficients.

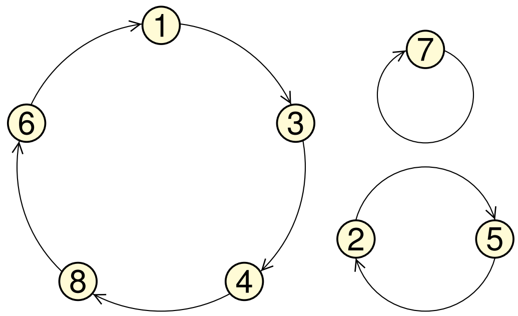



Figure 1.1: The permutation diagram for the permutation $\sigma \in \mathcal{S}_8$ with $\sigma(1) = 3$, $\sigma(2) = 5$, $\sigma(3) = 4$, $\sigma(4) = 8$, $\sigma(5) = 2$, $\sigma(6) = 1$, $\sigma(7) = 7$, $\sigma(8) = 6$.

*Example* 1.3.2. For example, if we define $\mathsf{A}$ by

$$\mathsf{A} = \sum_{\substack{\sigma \in \mathcal{S}_n,\, n \geq 0 \\ \sigma \text{ an involution}}} \sigma,$$

then $[\sigma]\mathsf{A}^2$ (that is, the coefficient of $\sigma$ in $\mathsf{A}^2$) is the number of ways of writing $\sigma$ as a product of two involutions, $[\sigma]\mathsf{A}^3$ is the number of ways of writing $\sigma$ as a product of three involutions, and so on.

In general, if $[\sigma]\mathsf{A}$ is the number of ways of putting an $\mathcal{A}$-structure on a permutation $\sigma$ (for some notion of $\mathcal{A}$-structure) and $[\rho]\mathsf{B}$ is the number of ways of putting a $\mathcal{B}$-structure on a permutation $\rho$, then $[\tau]\mathsf{AB}$ is the number of ways of factoring the permutation $\tau$ as $\tau = \sigma\rho$ and putting an $\mathcal{A}$-structure on $\sigma$ and a $\mathcal{B}$-structure on $\rho$. This point of view will be particularly useful to us for the rest of this chapter and for Chapter 2, as we deal with the combinatorics of some permutation factorization problems. Note that this can also be extended to the case where the coefficients $[\sigma]\mathsf{A}$ and $[\rho]\mathsf{B}$ are ordinary or exponential generating functions instead of integers.

A permutation $\sigma \in \mathcal{S}_n$ can be represented as a directed graph with vertex set $[n]$ and arcs $i \to \sigma(i)$ for each $i \in [n]$, as in Figure 1.1. From this representation, it is clear that any permutation can be decomposed uniquely into a set of disjoint cycles, and this leads to the disjoint cycle notation: we write

$$\sigma = (a_1\, a_2\, \cdots\, a_{\ell_1})(b_1\, b_2\, \cdots\, b_{\ell_2}) \cdots (z_1\, z_2\, \cdots\, z_{\ell_k}), \tag{1.1}$$

where $a_i, b_i, \ldots, z_i \in [n]$, for the permutation with the $k$ disjoint cycles

$$a_1 \to a_2 \to \cdots \to a_{\ell_1} \to a_1,$$
$$b_1 \to b_2 \to \cdots \to b_{\ell_2} \to b_1,$$

$$\vdots$$
$$z_1 \to z_2 \to \cdots \to z_{\ell_k} \to z_1.$$

The cycle lengths $\ell_1, \ell_2, \ldots, \ell_k$ determine a partition of $n$, called the **cycle type** of $\sigma$. Note that cycles of length one are usually suppressed from the notation, but they are still counted in the cycle type of $\sigma$.

*Remark* 1.3.3. Our convention in this thesis is that permutations are multiplied left-to-right. That is, if $\sigma, \rho$ are permutations in $\mathcal{S}_n$, then $\sigma\rho$ is the permutation whose action on $[n]$ is obtained by applying $\sigma$ first, then $\rho$. In terms of the usual function composition notation, this corresponds to $\rho \circ \sigma$. For example, if $\sigma = (1\,2\,3)(4\,5)$ and $\rho = (1\,2\,3\,4\,5)$, then we have

$$\sigma\rho = (1\,2\,3)(4\,5) \cdot (1\,2\,3\,4\,5) = (1\,3\,2\,4)(5).$$

If $\sigma$ has the disjoint cycle structure given in (1.1), and $\rho \in \mathcal{S}_n$ is any permutation, then the conjugate $\sigma$ by $\rho$ is

$$\rho^{-1}\sigma\rho = (\rho(a_1)\,\rho(a_2)\,\cdots\,\rho(a_{\ell_1}))\cdots(\rho(z_1)\,\rho(z_2)\,\cdots\,\rho(z_{\ell_k})),$$

so that the cycle type of a permutation is invariant under conjugation. In fact, the cycle type is a complete invariant, hence the conjugacy classes of $\mathcal{S}_n$ can be labelled by the partitions $\alpha \vdash n$. It follows that an element $\mathsf{A}$ of the symmetric group algebra $\mathbb{Q}\mathcal{S}$ is in the centre $\mathcal{Z}$ of the algebra exactly when the coefficient $[\sigma]\mathsf{A}$ only depends on the cycle type of $\sigma$. If the coefficients $[\sigma]\mathsf{A}$ have a combinatorial meaning and give the answer to some enumeration problem on permutations, then this problem is said to be **central** when $\mathsf{A}$ is in $\mathcal{Z}$.

*Example* 1.3.4. The set of all permutations which are involutions is closed under conjugation, so if $\mathsf{A}$ is the sum of all involutions as above, then $\mathsf{A}$ belongs to the centre $\mathcal{Z}$. As a consequence, every power $\mathsf{A}^k$, $k \geq 0$ of this element is also in the centre. The coefficient $[\sigma]\mathsf{A}^k$ is the number ways to write the permutation $\sigma$ as a product of $k$ involutions, so this is an example of a central problem.

For an example of a non-central problem, consider the problem of counting the inversions of a permutation $\sigma$, that is, the number of pairs $(i, j)$ with $i < j$ and $\sigma(i) > \sigma(j)$. This number does not depend only on the cycle type of $\sigma$; for example, the permutation $(1\,2)(3)$ has one inversion, but the permutation $(1\,3)(2)$ has three. Correspondingly, the element $\mathsf{B} = \sum_{\sigma \in \mathcal{S}_n,\, n \geq 0} (\#\text{ inversions of } \sigma) \cdot \sigma$ does not belong to the centre $\mathcal{Z}$.

For $\alpha \vdash n$, $n \geq 0$, let $\mathsf{C}_\alpha \in \mathbb{Q}\mathcal{S}$ be the **conjugacy class sum** for cycle type $\alpha$, that is, the sum of all permutations of cycle type $\alpha$. Then, we have the following equivalent characterization of the centre of the symmetric group algebra: $\mathsf{A}$ is in $\mathcal{Z}$ if and only if it can be written as a $\mathbb{Q}$-linear combination of the conjugacy class sums. Since the centre is closed under multiplication, this leads to one of the most basic problems about the symmetric group algebra, namely, the determination of the connection coefficients of the conjugacy class sums. Given partitions $\alpha, \beta, \gamma \vdash n$, what is $[\mathsf{C}_\alpha]\mathsf{C}_\beta\mathsf{C}_\gamma$? Combinatorially, the answer is simple; it is the number of ways to factor an arbitrary fixed permutation of cycle type $\alpha$ as a product of a permutation of cycle type $\beta$ and a permutation of cycle type $\gamma$. Numerically, however, this appears to be very hard to compute in general for large cycle types. Since many important enumeration problems can be phrased in terms of this basic problem, anything nontrivial that can be said about the answer is worthwhile.

## 1.4 Representation Theory

Let $\mathcal{Z}_n$ be the centre of $\mathbb{Q}\mathcal{S}_n$. The conjugacy class sums $\mathsf{C}_\alpha$ for $\alpha \vdash n$ form a basis of this algebra over $\mathbb{Q}$, but from representation theory, we know that there is another basis consisting of the **orthogonal idempotents** $\mathsf{F}^\alpha$, also indexed by partitions $\alpha \vdash n$. As orthogonal idempotents, these elements satisfy $\mathsf{F}^\alpha\mathsf{F}^\beta = 0$ for $\alpha \neq \beta$ and $\mathsf{F}^\alpha\mathsf{F}^\alpha = \mathsf{F}^\alpha$, so unlike the conjugacy class basis, it is very easy to multiply elements of $\mathcal{Z}_n$ expressed in the orthogonal idempotent basis. The change of basis between these two bases is given by

$$\mathsf{C}_\alpha = |\mathsf{C}_\alpha| \sum_{\beta \vdash n} \frac{\chi_\alpha^\beta}{f^\beta} \mathsf{F}^\beta, \qquad \mathsf{F}^\alpha = \frac{f^\alpha}{n!} \sum_{\beta \vdash n} \chi_\beta^\alpha \mathsf{C}_\beta, \qquad \alpha \vdash n,$$

where $f^\alpha$ is the **dimension** of the irreducible representation of $\mathcal{S}_n$ indexed by the partition $\alpha$ and $\chi_\beta^\alpha$ is the associated **irreducible character** of this representation evaluated at a permutation of cycle type $\beta$. From this, it follows that we can express the (generalized) connection coefficients of the conjugacy class sums as

$$[\mathsf{C}_\alpha]\mathsf{C}_{\beta^1}\mathsf{C}_{\beta^2}\cdots\mathsf{C}_{\beta^k} = \frac{\left|\mathsf{C}_{\beta^1}\right|\left|\mathsf{C}_{\beta^2}\right|\cdots\left|\mathsf{C}_{\beta^k}\right|}{n!} \sum_{\gamma \vdash n} \frac{\chi_\alpha^\gamma \chi_{\beta^1}^\gamma \chi_{\beta^2}^\gamma \cdots \chi_{\beta^k}^\gamma}{(f^\gamma)^{k-1}}.$$

While this is a very useful expression for theoretical purposes, in practice computing the sum over all $\approx \exp(\pi\sqrt{2n/3})/(4n\sqrt{3})$ partitions of $n$ is very expensive.

Given a partition $\alpha \vdash n$, let $V^\alpha$ be the irreducible $\mathbb{Q}\mathcal{S}_n$-module corresponding to the character $\chi^\alpha$. This module has a $\mathbb{Q}$-basis indexed by the set of standard Young tableaux of shape $\alpha$, called **Young's orthogonal basis**. There are many different constructions of this basis (see, *e.g.*, Sagan [43]), but for our purposes, it will be convenient to give a characterization in terms of the Jucys-Murphy elements.

The **Jucys-Murphy elements**, introduced independently by Jucys [30] and Murphy [38], are elements of the group algebra $\mathbb{Q}\mathcal{S}_n$ defined by

$$\mathsf{J}_{k,n} = \sum_{j<k} (j\,k) = (1\,k) + (2\,k) + \cdots + (k-1\,k), \qquad 1 \leq k \leq n. \tag{1.2}$$

For our purposes, the first important fact about these elements is that they commute with each other. Indeed, $\mathsf{J}_{k,n}$ is invariant under conjugation by any permutation in $\mathcal{S}_n$ which fixes the sets $\{1,\ldots,k\}$, $\{k\}$, and $\{k+1,\ldots,n\}$, so $\mathsf{J}_{k,n}$ commutes with any linear combination of these permutations. The second important fact about Jucys-Murphy elements is that, as $\mathbb{Q}$-linear operators acting on $V^\alpha$, they have the elements of Young's orthogonal basis as eigenvectors; if $v_T$ is the basis element indexed by the standard Young tableau $T$, then

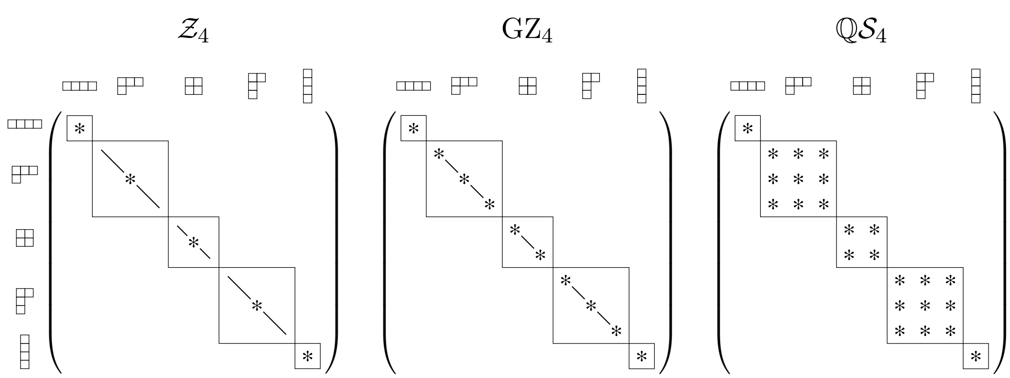
$$\mathsf{J}_{k,n} v_T = c(k,T) v_T, \tag{1.3}$$

where $c(k,T) = j - i$ is called the **content** of the box $(i,j)$ labelled $k$ in $T$. Note that this uniquely determines the vectors $v_T$ up to scaling. Using this fact, it is possible to show that there exist polynomials $p_T(\mathsf{J}_{1,n},\ldots,\mathsf{J}_{n,n})$ such that

$$p_T(\mathsf{J}_{1,n},\ldots,\mathsf{J}_{n,n}) v_{T'} = \begin{cases} v_{T'} & \text{if } T' = T, \\ 0 & \text{otherwise.} \end{cases}$$

Since the elements of $\mathbb{Q}\mathcal{S}_n$ are uniquely determined by their action on the modules $V^\alpha$ for $\alpha \vdash n$, it follows that the subalgebra of $\mathbb{Q}\mathcal{S}_n$ generated by the Jucys-Murphy elements is exactly the set of elements which act diagonally on Young's orthogonal basis. In particular, this is a maximal commutative subalgebra of $\mathbb{Q}\mathcal{S}_n$, called the **Gelfand-Zetlin algebra** $\mathrm{GZ}_n$, and we have the inclusions $\mathcal{Z}_n \subseteq \mathrm{GZ}_n \subseteq \mathbb{Q}\mathcal{S}_n$, as illustrated in Figure 1.2.

By (1.3), the Jucys-Murphy elements can be related to the irreducible characters $\chi^\alpha$ in a simple way; and by their definition (1.2) as a sum of transpositions,




$$\dim \mathcal{Z}_4 = \sum_{\alpha\vdash 4} 1 \qquad \dim \mathrm{GZ}_4 = \sum_{\alpha\vdash 4} f^\alpha \qquad \dim \mathbb{Q}\mathcal{S}_4 = \sum_{\alpha\vdash 4} \left(f^\alpha\right)^2$$

Figure 1.2: The action of the centre $\mathcal{Z}_n$, the Gelfand-Zetlin algebra $\mathrm{GZ}_n$, and the full symmetric group algebra $\mathbb{Q}\mathcal{S}_n$ on Young's orthogonal basis, for $n = 4$. The basis elements are grouped together according to the shape of the associated standard Young tableaux.

their action on the conjugacy classes $\mathsf{C}_\alpha$ can be investigated combinatorially. Thus, the Jucys-Murphy elements have a dual nature, in some sense, and provide a bridge between the algebraic world of representation theory and the combinatorial world of transposition factorizations, which we turn to in Section 1.5.

*Remark* 1.4.1. Although historically Young's orthogonal basis was defined before the Jucys-Murphy elements, there are now constructions of Young's orthogonal basis which start from the Jucys-Murphy elements. Together with a focus on the chain of inclusions $\mathbb{Q}\mathcal{S}_1 \subseteq \mathbb{Q}\mathcal{S}_2 \subseteq \cdots \subseteq \mathbb{Q}\mathcal{S}_n$, this lies at the heart of the modern approach to the representation theory of the symmetric groups developed by Okounkov and Vershik [42] (see also the recent book of Ceccherini-Silberstein, Scarabotti and Tolli [4] for a detailed account).

## 1.5 Transposition Factorizations

A permutation of the form $(a\,b)$ is called a **transposition**. The action of multiplication by a transposition on the cycles of a permutation is particularly simple, as illustrated in Figure 1.3: if the elements $a, b$ are in different cycles of $\sigma$, then $\sigma \cdot (a\,b)$ is obtained by joining these two cycles together; if $a, b$ are in the same cycle of $\sigma$, then $\sigma \cdot (a\,b)$ is obtained by cutting this cycle in two. In the first case, the transposition $(a\,b)$ is called a **join** for $\sigma$, and in the second case it is called a **cut** for $\sigma$.

Products of transpositions will be of particular interest to us. We will refer to a sequence

$$(a_1\,b_1),\ (a_2\,b_2),\ \ldots,\ (a_r\,b_r)$$

such that

$$(a_1\,b_1)(a_2\,b_2)\cdots(a_r\,b_r) = \sigma$$

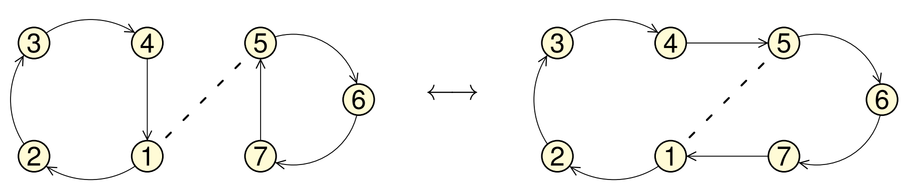



Figure 1.3: Multiplying the permutation $(1\,2\,3\,4)(5\,6\,7)$ on the right by the transposition $(1\,5)$ joins the two cycles together. Conversely, multiplying the permutation $(1\,2\,3\,4\,5\,6\,7)$ on the right by this transposition would split it back into two cycles.

as a **transposition factorization** of $\sigma$ of **length** $r$. By convention, we always assume that $a_i < b_i$, and refer to the sequence $(b_1, b_2, \dots, b_r)$ as the **signature** of the factorization. The factorization is said to be **monotone** if the signature is a weakly increasing sequence, and **strictly monotone** if the signature is a strictly increasing sequence.

*Remark* 1.5.1. As can be guessed from Lemma 1.5.1, the "theory" of strictly monotone factorizations is fairly boring. We introduce this concept mainly as an extended example, with some similarities to the theories of non-monotone transposition factorization and of (weakly) monotone factorizations, but where more explicit computations can be given.

The **partial products** of this factorization are

$$\sigma_k = \prod_{i=1}^{k} (a_i\, b_i), \qquad 0 \le k \le r,$$

and we can think of a transposition factorization of $\sigma$ as building up to $\sigma$ from the identity permutation, one transposition at a time. The $i$th factor $(a_i\, b_i)$ is called a **join** or a **cut** of the factorization according to whether it is a join or a cut of the partial product $\sigma_{i-1}$. The initial permutation $\sigma_0 = \mathrm{id}_n$ has $n$ cycles and each factor either decreases or increases the number of cycles by one, depending on whether it is a join or a cut. It follows that the number $r$ of factors is at least

$$n - (\#\text{ cycles of } \sigma),$$

the **rank** of $\sigma$, and $r$ has the same parity as this number. Permutation factorizations which satisfy the bound $r \ge \mathrm{rank}(\sigma)$ with equality are said to be **minimal**, and as the following lemma shows, they exist for every permutation $\sigma$.

**Lemma 1.5.1.** *Every permutation $\sigma$ has a unique strictly monotone factorization, and its length is the rank of $\sigma$.*

*Proof.* Consider a strictly monotone factorization

$$(a_1\, b_1)(a_2\, b_2) \cdots (a_r\, b_r) = \sigma.$$

If $r = 0$, then $\sigma = \mathrm{id}$. Otherwise, the last transposition is the only one which moves the element $b_r$, and no transposition moves any element greater than $b_r$, so $b_r$ must be the greatest non-fixed point of $\sigma$, and we must have $a_r = \sigma(b_r)$. Thus, the last transposition is uniquely determined by $\sigma$. The same argument shows that the transposition $(a_i\, b_i)$ is uniquely determined by the partial product $\sigma_i$. Since $\sigma_{i-1} = \sigma_i \cdot (a_i\, b_i)$ these partial products are uniquely determined by $\sigma$, which shows that $\sigma$ has at most one strictly monotone factorization.

Now, suppose we are given a permutation $\sigma$ of rank $r$. We can construct partial products $\sigma_r, \sigma_{r-1}, \dots, \sigma_0$ by starting with $\sigma_r = \sigma$ and taking $\sigma_{i-1} = \sigma_i \cdot (a_i\, b_i)$

as above for $i = r, r-1, \dots, 1$. Note that $a_i$ and $b_i$ are in the same cycle of $\sigma_i$, so $(a_i\, b_i)$ is a cut for $\sigma_i$, and the rank of $\sigma_{i-1}$ is one less than the rank of $\sigma_i$. It follows that $\sigma_0 = \mathrm{id}$, so this process does yield a transposition factorization. Also, by construction, $a_i < b_i$, and $\sigma_{i-1}$ fixes every element greater than or equal to $b_i$, so the signature of this factorization is strictly increasing. □

Note that the Jucys-Murphy element

$$\mathsf{J}_{k,n} = (1\,k) + (2\,k) + \cdots + (k-1\,k) \in \mathbb{Q}\mathcal{S}_n$$

is the sum of all transpositions in $\mathcal{S}_n$ with signature $k$, so that for a given permutation $\sigma \in \mathcal{S}_n$, the coefficient

$$[\sigma]\mathsf{J}_{b_1,n}\mathsf{J}_{b_2,n}\cdots\mathsf{J}_{b_r,n}$$

is the number of transposition factorizations of $\sigma$ with signature $(b_1, b_2, \dots, b_r)$. In particular, we can rephrase Lemma 1.5.1 as the following generating function equation in $\mathbb{Q}\mathcal{S}_n[t]$, where $t$ is an indeterminate:

$$(1 + t\mathsf{J}_{1,n})(1 + t\mathsf{J}_{2,n})\cdots(1 + t\mathsf{J}_{n,n}) = \sum_{\sigma\in\mathcal{S}_n} t^{\mathrm{rank}(\sigma)}\sigma. \tag{1.4}$$

Similarly, the generating function for monotone factorizations in $\mathbb{Q}\mathcal{S}_n[[t]]$, where $t$ is an ordinary marker for the length of the factorizations, is:

$$(1 - t\mathsf{J}_{1,n})^{-1}(1 - t\mathsf{J}_{2,n})^{-1}\cdots(1 - t\mathsf{J}_{n,n})^{-1}. \tag{1.5}$$

Part of the interest in counting monotone factorizations comes from the fact that this generating function is equal to

$$\left(\sum_{\sigma\in\mathcal{S}_n} (-t)^{\mathrm{rank}(\sigma)}\sigma\right)^{-1},$$

and its coefficients have been identified as giving an asymptotic expansion of the unitary Weingarten function in the large $N$ limit, from the theory of matrix integration over the classical groups (see Collins [5] for the unitary Weingarten function, and Novak [40] for the connection to this generating function).

*Remark* 1.5.2. Since the Jucys-Murphy elements commute, for any reordering $(c_1, c_2, \dots, c_r)$ of a given signature $(b_1, b_2, \dots, b_r)$, we have

$$\mathsf{J}_{b_1,n}\mathsf{J}_{b_2,n}\cdots\mathsf{J}_{b_r,n} = \mathsf{J}_{c_1,n}\mathsf{J}_{c_2,n}\cdots\mathsf{J}_{c_r,n},$$

so that any given permutation $\sigma \in \mathcal{S}_n$ has as many factorizations with one signature as with the other. Thus, although there is an apparent asymmetry in the definition of monotone factorizations between *increasing* and *decreasing* signatures, this difference is immaterial for the purposes of enumerating them.

## 1.6 Graphs and Surfaces

The purpose of this section is to describe a way of visualizing permutation factorizations as topological surfaces. As such, we take a very naive view of topology and refrain from proving any of the topological facts. We also define a few related concepts pertaining to transposition factorizations and use the surfaces to illustrate some bijections. However, the actual definitions and bijections are purely combinatorial.

Given a transposition factorization

$$(a_1\, b_1)(a_2\, b_2)\cdots(a_r\, b_r) = \sigma \in \mathcal{S}_n,$$

we can construct a graph embedded on a surface to visualize the factorization. (Technically speaking, the result is an embedding of a vertex-labelled, edge-labelled loopless multigraph on a not necessarily connected, oriented surface with boundary, where the vertices lie on the boundary, the edges lie in the interior, and the faces are homeomorphic to discs, with each face containing exactly one arc of the surface's boundary.) This embedding can be described by decomposing it into vertices, boundary arcs, internal edges, and faces:

- The **vertices** and **boundary arcs** are given by the permutation diagram of the product $\sigma$; that is, the vertex set is the ground set $[n]$, and there is a boundary arc $i \to \sigma(i)$ for each vertex $i$.

- The internal **edges** are given by the factors of the permutation factorization; that is, for each transposition $(a_k\, b_k)$, there is an internal edge labelled $k$ joining the vertices $a_k$ and $b_k$. Note that the labelling of edges is important, as a given transposition may appear more than once in the factorization, leading to multiple edges between two vertices.

- The **faces** are oriented discs whose boundaries are given by following elements of the ground set $[n]$ through the partial products of the permutation factorization; that is, for each element $i \in [n]$, consider the sequence

  $$\sigma_r(i), \quad \sigma_{r-1}(i), \quad \sigma_{r-2}(i), \quad \ldots, \quad \sigma_2(i), \quad \sigma_1(i), \quad \sigma_0(i).$$

  This determines a walk from the vertex $\sigma(i)$ (written as $\sigma_r(i)$) to the vertex $i$ (written as $\sigma_0(i)$) if we traverse the edge labelled $k$, corresponding to the factor $(a_k b_k)$, whenever $\sigma_k(i) = a_k$ and $\sigma_{k-1}(i) = b_k$ or vice versa. Taking this walk together with the boundary arc $i \to \sigma(i)$ gives an oriented closed walk, to which we can glue an oriented disc, yielding a face of the embedding. Note that each internal edge is traversed exactly once in each direction when tracing the face boundaries, so that gluing the faces together along the internal edges does produce an oriented surface with the correct boundary.

*Example* 1.6.1. This construction is illustrated for the permutation factorization

$$(1\,4)(6\,8)(3\,4)(5\,7)(2\,7)(1\,8)(5\,7)(1\,3) = (1\,3\,4\,8\,6)(2\,5)(7)$$

in Figures 1.4 and 1.5.

The top of Figure 1.4 shows the graph formed from the vertices and the internal edges. The middle shows the resulting oriented surface, with the vertices and boundary arcs present, but with the internal edges erased; the left component is a disc together with a handle glued on, making it homeomorphic to a punctured torus, and the right component is formed from two discs joined by a handle, making it homeomorphic to a cylinder. The bottom shows the whole graph embedding, with the surface cut up so that it can be flattened; to recover the embedding, the dashed edges should be identified according to the number and direction of chevrons they are marked with, following the usual topological convention.

The top two rows of Figure 1.5 show the neighbourhoods local to each vertex, with the neighbouring vertices repeated if necessary, and with the faces labelled $\{A, B, C, D, E, F, G, H\}$. The bottom two rows show the faces of the embedding, with the same labelling, corresponding to the progression of the elements $\{1, 2, 3, 4, 5, 6, 7, 8\}$ through the factorization, respectively.

In both figures, every flat part of the surface is shown so that the clockwise direction matches the orientation of the surface.

This shows how to construct a graph embedding from a permutation factorization. Conversely, given a graph embedding, the only conditions for it to come from a transposition factorization are that:

1. the vertices lie on the boundary, the edges lie in the interior, and the faces be homeomorphic to discs, each face containing exactly one arc of the surface's boundary; and

2a. the labels of the internal edges be decreasing along each face's boundary in clockwise order, starting and stopping with the (unlabelled) arc of the surface's boundary; or, equivalently

2b. the labels of the internal edges be increasing around each vertex in clockwise order, starting with the outgoing surface boundary arc and stopping with the incoming surface boundary arc.

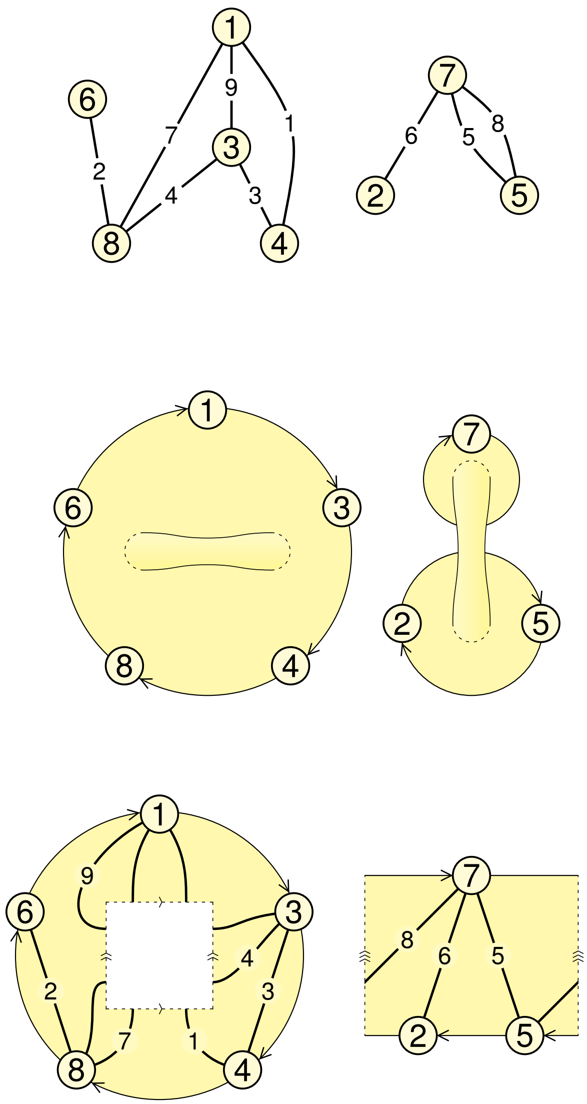



Figure 1.4: The global data for the topological construction associated to the permutation factorization $(1\,4)(6\,8)(3\,4)(5\,7)(2\,7)(1\,8)(5\,7)(1\,3) = (1\,3\,4\,8\,6)(2\,5)(7)$. From top to bottom: the graph of the factorization; the unadorned surface; and the embedding of the graph on the surface.

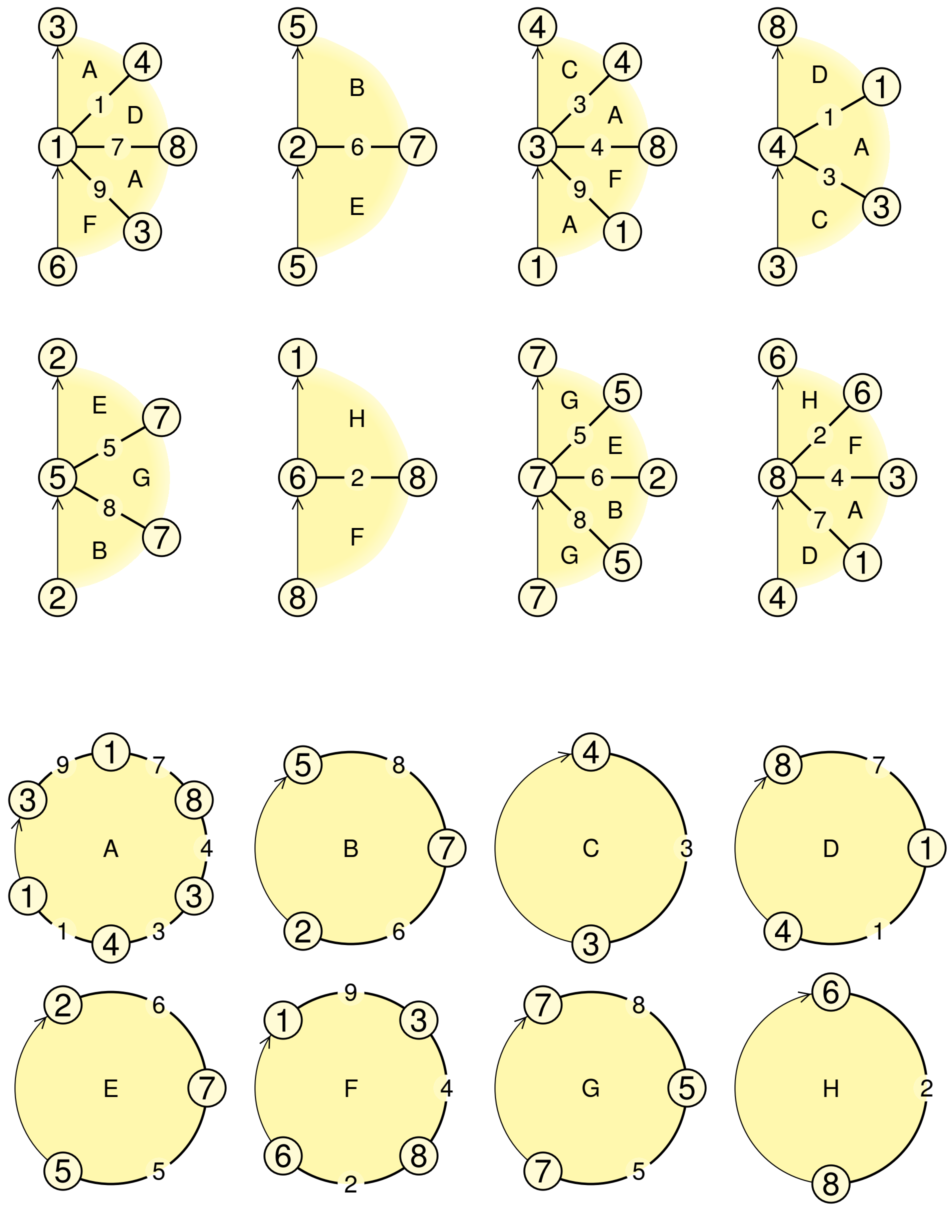



Figure 1.5: The local data for the topological construction associated to the permutation factorization $(1\,4)(6\,8)(3\,4)(5\,7)(2\,7)(1\,8)(5\,7)(1\,3) = (1\,3\,4\,8\,6)(2\,5)(7)$. From top to bottom: the neighbourhood of each vertex in the graph embedding; and the faces of the graph embedding.

From this point of view, it is fairly easy to enumerate transposition factorizations in certain special cases, as in Theorem 1.6.1 and Theorem 1.6.2 below.

**Theorem 1.6.1.** *(Dénes, [8]). The number of minimal transposition factorizations of the permutation* $(1\,2\,\cdots\,n)$ *is* $n^{n-2}$.

*Proof.* A transposition factorization is uniquely determined by the corresponding graph of internal edges, together with the its vertex and edge labels, and every (loopless multi-)graph with vertices labelled by $[n]$ and edges labelled by $[r]$ determines a factorization of some permutation in $\mathcal{S}_n$ into $r$ transpositions. Thus, it is enough to show that there are $n^{n-2}$ graphs corresponding to minimal factorization of $(1\,2\,\cdots\,n)$.

Given the graph of a factorization of $(1\,2\,\cdots\,n)$, the vertex labels can be recovered from the knowledge of

- the permutation $\sigma = (1\,2\,\cdots\,n)$,
- the edge labels, and
- the label of a single vertex, say $n$,

since one can use the edge labels to walk along the graph from any vertex $i$ to the vertex of its image $\sigma(i)$. In particular, the graph must be connected. For a minimal factorization of $(1\,2\,\cdots\,n)$, this connected graph has $r = n - 1$ edges, so it must be a tree.

In fact, given any tree with edges labelled $1, 2, \ldots, n-1$ and a vertex labelled $n$, this procedure of walking along the graph to recover the missing vertex labels always works; regardless of the vertex labels, the $n-1$ transpositions of the corresponding factorization must all be joins, so the product can only be a full

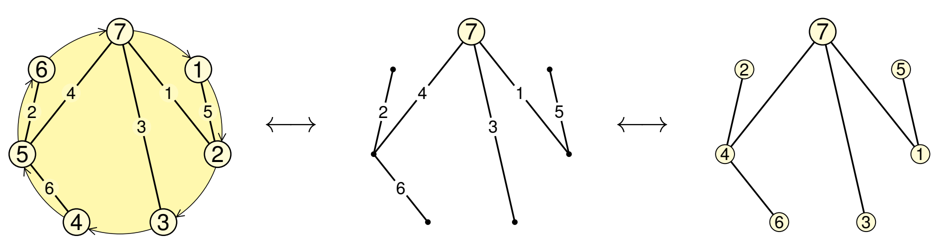



Figure 1.6: The graph embedding corresponding to the minimal transposition factorization $(2\,7)(5\,6)(3\,7)(5\,7)(1\,2)(4\,5) = (1\,2\,3\,4\,5\,6\,7)$.

cycle. Thus, this data is equivalent to the data of a minimal factorization of $(1\,2\,\cdots\,n)$.

Now, note that the set of trees with a "non-redundant" labelling as above is in bijection with the set of trees with vertices labelled by $[n]$; the correspondence is given by "pushing" the labels $1, 2, \ldots, n-1$ from the edges to the vertices away from $n$, or in reverse by "pulling" these labels from the vertices to the edges towards $n$. (See Figure 1.6 for an illustration of this.) Threading all these correspondences together, we obtain a bijection between minimal transposition factorizations of $(1\,2\,\cdots\,n)$ and trees on the vertex set $[n]$, of which there are $n^{n-2}$. □

For Theorem 1.6.2, it is convenient to have a name to refer to permutations such as

$$(9\,8\,7)(6\,5\,4)(3)(2\,1),$$

whose disjoint cycle representation can be obtained by writing down by the decreasing list $n, n-1, \ldots, 1$ and adding parentheses, so we call them **decreasing permutations**. Formally speaking, these permutations are characterized by the property that for each $i \in [n]$, $i$ is either the smallest element in its cycle, or $i \to i-1$.

We will also need some terms to describe the graphs associated to minimal monotone factorizations of decreasing permutations. A **forest** is a graph which is a disjoint union of trees, or in other words, a graph which contains no cycles. A **rooted** forest is a forest together with a choice of **root** vertex for each component. For each non-root vertex, its **parent** is the unique adjacent vertex closer to a root vertex, and it is said to be a **child** of its parent. For each vertex, its **up-degree** is its number of children. An **ordered** forest is a rooted forest together with a total ordering on the set of its roots and, for each vertex, a total ordering on the set of its children.

*Example* 1.6.2. The right-hand side of Figure 1.7 gives an example of an ordered forest. The root vertices are labelled $(9, 3, 2)$ in order from left to right, and the children of each vertex are located directly above it, also in order from left to right. For instance, the vertex 7 has up-degree 3, the list of its children is $(6, 5, 4)$, and its parent is the vertex 9.

Given a vertex in a rooted forest, its **ancestry** is the list of vertices on the unique path from a root to it. In an ordered forest, we can compare ancestries lexicographically, that is, in dictionary order, and this gives a total ordering on the vertices of the forest, called the **depth-first search** ordering. Note that the structure of an ordered forest can be recovered from its underlying forest and its depth-first search ordering, but in general not every total ordering on the vertices of a forest comes from a depth-first search ordering.

*Example* 1.6.3. For the ordered forest of Figure 1.7, the list of ancestries is

$$\begin{array}{llllll} 1: & (2,1) & 4: & (9,7,4) & 7: & (9,7) \\ 2: & (2) & 5: & (9,7,5) & 8: & (9,8) \\ 3: & (3) & 6: & (9,7,6) & 9: & (9). \end{array}$$

Since the children of each vertex happen to be ordered according to their vertex labels in decreasing order, the sorted list of ancestries is

$$(9), \quad (9,8), \quad (9,7), \quad (9,7,6), \quad (9,7,5), \quad (9,7,4), \quad (3), \quad (2), \quad (2,1),$$

and the depth-first search ordering of the vertices is $(9,8,7,6,5,4,3,2,1)$.

**Theorem 1.6.2.** *Given a minimal monotone factorization of a decreasing permutation in $\mathcal{S}_n$, the associated graph is an ordered forest, where the depth-first search ordering of the vertices is $n, n-1, \ldots, 1$.*

*Conversely, given an ordered forest on $n$ unlabelled vertices, its vertices can be labelled $n, n-1, \ldots, 1$ in depth-first search order. There is then a unique way to label it edges to obtain the associated graph of a minimal monotone factorization of a decreasing permutation.*

*Remark* 1.6.4. Matsumoto and Novak [35] give an equivalent bijection between minimal monotone factorizations and certain parking functions. We restate it here in terms of ordered forests since they appear naturally as part of the topological construction described in this section.

*Remark* 1.6.5. Note that by the nature of the depth-first search ordering, for every edge in the ordered forests described in Theorem 1.6.2, the parent vertex label is larger than the child vertex label. Since the edges correspond directly to the transpositions of the associated transposition factorization, it follows that the up-degree of the vertex labelled $i$, $i \in [n]$, is the number of times $i$ appears in the signature of the factorization. In particular, for *strictly* monotone factorizations, the up-degree is always zero or one, so the component trees of the corresponding ordered forests are simply paths.

*Proof.* We proceed by induction on $n$, with a trivial base case of $n = 0$. For $n > 0$, let $\sigma \in \mathcal{S}_n$ be a decreasing factorization, let

$$F: \qquad (a_1\, b_1)(a_2\, b_2)\cdots(a_r\, b_r) = \sigma$$

be a minimal monotone factorization, and consider the partial factorizations

$$F_k: \qquad (a_1\, b_1)(a_2\, b_2)\cdots(a_k\, b_k) = \sigma_k, \qquad 0 \le k \le r.$$

A factorization is minimal if and only if each of its transpositions is a join, so it follows that each $F_k$ is a minimal monotone factorization of the corresponding

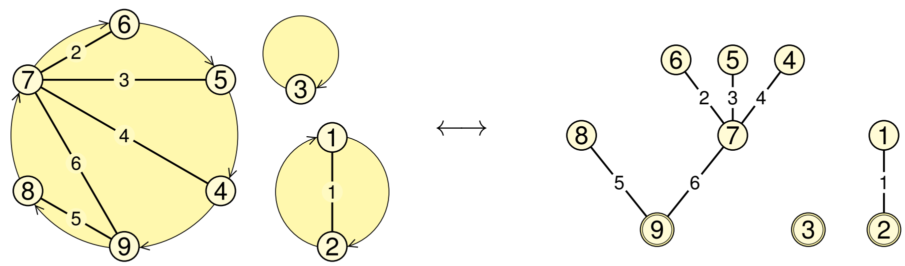



Figure 1.7: The graph embedding corresponding to the minimal monotone factorization $(1\,2)(6\,7)(5\,7)(4\,7)(8\,9)(7\,9) = (9\,8\,7\,6\,5\,4)(3)(2\,1)$.

partial product $\sigma_k$. For $k > 0$, as in the proof of Lemma 1.5.1, this implies that $b_k$ is the largest non-fixed point of $\sigma_k$, and $a_k$ and $b_k$ are in the same cycle of $\sigma_k$. If we let $b'_k = \sigma_k^{-1}(a_k)$ and $a'_k = \sigma_k^{-1}(b_k)$, then the action of $(a_k\, b_k)$ on the cycle of $\sigma_k$ containing $a_k$ and $b_k$ is given by

$$(b_k \,\cdots\, b'_k\, a_k \,\cdots\, a'_k) \cdot (a_k\, b_k) = (b_k \,\cdots\, b'_k)(a_k \,\cdots\, a'_k).$$

In particular, if $\sigma_k$ is a decreasing permutation, then so is $\sigma_{k-1} = \sigma_k \cdot (a_k\, b_k)$, and it follows that each $\sigma_k$ is a decreasing permutation.

Now, let $j$ be maximal such that $n$ is a fixed point of $\sigma_j$. Then, by construction, for $k > j$, we have $b_k = n$ and the cycle of $\sigma_{k-1}$ containing $n$ consists of the elements $i$ with $n \geq i > a_k$. Thus, we have $n > a_{j+1} > a_{j+2} > \cdots > a_r$, and these are the $r - j + 1$ largest elements at the top of a cycle of $\sigma_j$. By the induction hypothesis, the graph of the factorization $F_j$ restricted to the ground set $[n-1]$ is an ordered forest where the depth-first search ordering of the vertices is $n-1, n-2, \ldots, 1$. The first $r - j$ component trees of this ordered forest must be rooted at $a_{j+1}, a_{j+2}, \ldots, a_r$. Adding the vertex $n$ and edges for the transpositions $(a_{j+1}\, n), (a_{j+2}\, n), \ldots, (a_r\, n)$ to this ordered forest gives the graph of the full factorization $F$. Furthermore, if the vertex $n$ is taken as a new first root, with children $a_{j+1}, a_{j+2}, \ldots, a_r$ in that order, then the depth-first search ordering of the resulting ordered forest is $n, n-1, \ldots, 1$; this is because the vertices $n, n-1, \ldots, i$, where $i = \sigma^{-1}(n)$, gain $n$ at the start of their ancestry, while the vertices $i-1, i-2, \ldots, 1$ have their ancestry unaffected.

For the converse, note that the edge labels of the graph of a minimal monotone factorization of a decreasing permutation are uniquely determined by the vertex labels; by monotonicity, the transpositions $(a_1\, b_1), (a_2\, b_2), \ldots, (a_r\, b_r)$ occur in weakly increasing order of $b_k$, and by the argument above, for a given value of $b_k$, they occur in strictly decreasing order of $a_k$. Thus, given an ordered forest on $n$ unlabelled vertices, the vertex labels can be recovered by depth-first search, and the edge labels can be recovered uniquely, and the construction above can be reversed to construct a minimal monotone factorization of a decreasing permutation. □

**Corollary 1.6.3.** *(Matsumoto and Novak [35]). If a decreasing permutation has cycle type* $\alpha \vdash n$*, then it has* $\prod_{i=1}^{\ell(\alpha)} \frac{1}{\alpha_i}\binom{2\alpha_i-2}{\alpha_i-1}$ *minimal monotone factorizations.*

*Proof.* In this case, the minimal monotone factorizations correspond to ordered forests on $n$ unlabelled vertices, with the additional restriction that the component ordered trees have numbers of vertices given by the parts of $\alpha$ in some fixed order. Each of these component ordered trees can be chosen independently, and the number of ordered trees on $\alpha_i$ unlabelled vertices is the Catalan number $\frac{1}{\alpha_i}\binom{2\alpha_i-2}{\alpha_i-1}$. □

Given a transposition factorization, the corresponding surface may not be connected, but it can be split up into **connected components**. By construction, the connected components of the surface correspond exactly to the connected components of the graph of internal edges. Combinatorially, the vertex sets of the components can be described as the orbits of the subgroup

$$\langle (a_1\, b_1), (a_2\, b_2), \ldots, (a_r\, b_r) \rangle \subseteq \mathcal{S}_n$$

generated by the factors, acting on the ground set $[n]$; by abuse of language, we may refer to these orbits as connected components of the factorization as well. The surface is connected exactly when the subgroup acts transitively on the ground set (meaning that it has a single orbit), in which case the factorization is said to be **transitive**.

This notion of connectivity also lets us refine our previous classification of factors of a transposition factorization as either joins or cuts; given the components of the partial factorization

$$(a_1\, b_1)(a_2\, b_2) \cdots (a_{i-1}\, b_{i-1}) = \sigma_{i-1},$$

the next factor $(a_i\, b_i)$ can be either a cut, in which case $a_i$ and $b_i$ are necessarily in the same component, or it can be a join with $a_i$ and $b_i$ in different components, in which case it is called an **essential join** for the factorization, or it can be a join with $a_i$ and $b_i$ in the same component already, in which case it is called a **redundant join** for the factorization.

Topologically, this notion of redundant joins corresponds to the notion of **genus** of the surface. Indeed, if the surface is connected and has genus $g$, its Euler characteristic can be computed as

$$\chi = 2 - 2g - (\#\text{ boundary components}),$$

or alternatively, using the decomposition into vertices, boundary arcs, internal edges and faces, as

$$\chi = (\#\text{ vertices}) - (\#\text{ boundary arcs}) - (\#\text{ internal edges}) + (\#\text{ faces}),$$

from which it follows that

$$r = n + \ell(\alpha) + 2g - 2,$$

where $\alpha$ is the cycle type of $\sigma$. Then, using the fact that

$$\ell(\alpha) = n - (\#\text{ joins}) + (\#\text{ cuts})$$

and that $n-1$ of the joins must be essential joins to get a connected surface, we get

$$g = (\#\text{ redundant joins}).$$

Thus, we say that a transitive transposition factorization has **genus** $g$ if it has $g$ redundant joins. If $g = 0$, we say that it is **minimal transitive**.

*Remark* 1.6.6. For permutations with a single cycle, all transposition factorizations are automatically transitive, so the notions of minimal factorization and minimal transitive factorization coincide in this case. The two notions diverge for permutations with more than one cycle, and this can be seen topologically: the surface for a minimal factorization consists a separate closed disc for each cycle, while the surface for a minimal transitive factorization consists of a sphere with an open disc removed for each cycle.

*Remark* 1.6.7. Given the notion of representing transposition factorizations as graphs embedded on surfaces, one approach to enumerating transposition factorizations is to fix a surface and then to count embeddings on it. Here, fixing a surface means specifying the number of connected components, together with the genus and number of boundary components of each one. One benefit of this is that the number of vertices is not fixed, so that this approach is amenable to the use of generating functions. In the physics literature, this leads to the notion of topological recursion, as described by Eynard and Orantin [12]; see also Bouchard and Mariño [2] and Eynard, Mulase and Safnuk [11]. Our approach in this thesis is similar in some sense, but the technical details are quite different.

## 1.7 The Farahat-Higman Algebra

We have a natural chain of set inclusions

$$[0] \hookrightarrow [1] \hookrightarrow [2] \hookrightarrow \cdots \hookrightarrow [n] \hookrightarrow [n+1] \hookrightarrow \cdots,$$

and this corresponds to a natural chain of group inclusions

$$\mathcal{S}_0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \mathcal{S}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{S}_n \hookrightarrow \mathcal{S}_{n+1} \hookrightarrow \cdots,$$

where permutations of the set $[n]$ can be extended to permutations of the set $[n+1]$ by adding the element $n+1$ as a fixed point. Since these inclusions respect the group structure of the symmetric groups, they can be extended $\mathbb{Q}$-linearly to give ring inclusions

$$\begin{aligned}\mathbb{Q}\mathcal{S}_0 \hookrightarrow \mathbb{Q}\mathcal{S}_1 \hookrightarrow \mathbb{Q}\mathcal{S}_2 \hookrightarrow \cdots \hookrightarrow \mathbb{Q}\mathcal{S}_n \hookrightarrow \mathbb{Q}\mathcal{S}_{n+1} \hookrightarrow \cdots,\\ \mathrm{GZ}_0 \hookrightarrow \mathrm{GZ}_1 \hookrightarrow \mathrm{GZ}_2 \hookrightarrow \cdots \hookrightarrow \mathrm{GZ}_n \hookrightarrow \mathrm{GZ}_{n+1} \hookrightarrow \cdots.\end{aligned}$$

We might expect to get a similar chain of ring inclusions for the centres of the group algebras, but since the inclusion $\mathbb{Q}\mathcal{S}_n \hookrightarrow \mathbb{Q}\mathcal{S}_{n+1}$ treats the element $n+1$ differently from the other elements of $[n+1]$, the centre $\mathcal{Z}_n$ does not map to the centre $\mathcal{Z}_{n+1}$. Still, there is some common structure between the centres, as evidenced by equations involving the conjugacy class sums such as

$$\mathsf{C}_{21^{n-2}}\mathsf{C}_{21^{n-2}} = \binom{n}{2}\mathsf{C}_{1^n} + 3\mathsf{C}_{31^{n-3}} + 2\mathsf{C}_{2^21^{n-4}} \in \mathcal{Z}_n, \tag{1.6}$$

which holds uniformly for all values of $n \geq 4$. As we will see, this common structure is captured by the Farahat-Higman algebra, a subring of $\mathbb{Q}\mathcal{S} = \prod_{n\geq 0}\mathbb{Q}\mathcal{S}_n$.

To introduce the Farahat-Higman algebra, we need the concept of reduced cycle type. For a permutation $\sigma \in \mathcal{S}_n$, let $\mathrm{cyc}(\sigma) \vdash n$ be its cycle type. Then, the **reduced cycle type** $\mathrm{redcyc}(\sigma)$ of $\sigma$ is obtained from the partition $\mathrm{cyc}(\sigma)$ by subtracting one from each of its parts, and discarding any resulting parts of size zero. Thus, $\mathrm{redcyc}(\sigma)$ is a partition of the integer $|\mathrm{cyc}(\sigma)| - \ell(\mathrm{cyc}(\sigma)) = \mathrm{rank}(\sigma)$. Given $n$, the cycle type of $\sigma$ can be recovered from its reduced cycle type, but unlike $\mathrm{cyc}(\sigma)$, $\mathrm{redcyc}(\sigma)$ is preserved by the natural inclusion $\mathcal{S}_n \hookrightarrow \mathcal{S}_{n+1}$.

Now, for every partition $\alpha$, let $\mathsf{K}_\alpha \in \mathcal{Z} \subseteq \mathbb{Q}\mathcal{S}$ be the **reduced class sum** for the reduced cycle type $\alpha$, defined by

$$\mathsf{K}_\alpha = \sum_{\substack{\sigma\in\mathcal{S}_n, n\geq 0\\ \mathrm{redcyc}(\sigma)=\alpha}} \sigma,$$

and let $\mathsf{n} \in \mathbb{Q}\mathcal{S}$ be the element defined by

$$\mathsf{n} = \sum_{n\geq 0} n\,\mathrm{id}_n,$$

where $\mathrm{id}_n$ is the identity permutation in $\mathcal{S}_n$. Then, the **Farahat-Higman algebra** FH is the $\mathbb{Q}$-subalgebra of $\mathcal{Z}$ generated by $\mathsf{n}$ and the reduced class sums $\mathsf{K}_\alpha$. With these definitions, (1.6) can be rewritten as

$$\mathsf{K}_1\mathsf{K}_1 = \binom{\mathsf{n}}{2}\mathsf{K}_0 + 3\mathsf{K}_2 + 2\mathsf{K}_{11} \in \mathrm{FH}.$$

Farahat and Higman [13] showed that for partitions $\alpha \vdash n$ and $\beta \vdash m$, the product $\mathsf{K}_\alpha \mathsf{K}_\beta$ has an expansion of the form

$$\mathsf{K}_\alpha \mathsf{K}_\beta = \sum_{k=0}^{\lfloor \frac{n+m}{2} \rfloor} \sum_{\gamma \vdash n+m-2k} c_{\alpha,\beta}^{\gamma}(\mathsf{n}) \mathsf{K}_\gamma,$$

where each $c_{\alpha,\beta}^{\gamma}$ is a polynomial of degree at most $2k = |\alpha| + |\beta| - |\gamma|$. Thus, the reduced class sums $\mathsf{K}_\alpha$ form a basis of FH over the polynomial ring $\mathbb{Q}[\mathsf{n}]$. Furthermore, we have a natural filtration on FH if we take $\mathsf{n}$ to have degree 1 and $\mathsf{K}_\alpha$ to have degree $|\alpha|$.

We can give another description of the Farahat-Higman algebra in terms of the **global Jucys-Murphy elements**

$$\mathsf{J}_k = \sum_{n \geq k} \mathsf{J}_{k,n} \in \mathbb{Q}\mathcal{S}.$$

Since only finitely many of these global Jucys-Murphy elements are nonzero under each of the projections $\mathbb{Q}\mathcal{S} \to \mathbb{Q}\mathcal{S}_n$, $n \geq 0$, symmetric functions of $\mathsf{J}_1, \mathsf{J}_2, \dots$ are well-defined elements of $\mathbb{Q}\mathcal{S}$. By (1.4), the $r$th elementary symmetric function is given by

$$e_r(\mathsf{J}_1, \mathsf{J}_2, \dots) = \sum_{\substack{\sigma \in \mathcal{S}_n, n \geq 0 \\ \operatorname{rank}(\sigma) = r}} \sigma = \sum_{\alpha \vdash r} \mathsf{K}_\alpha,$$

so it is an element of FH, and it follows that every symmetric function of $\mathsf{J}_1, \mathsf{J}_2, \dots$ is an element of FH. We can also allow polynomials in $\mathsf{n}$ as coefficients for these symmetric functions and stay within the Farahat-Higman algebra. Thus, we have an evaluation map

$$\begin{aligned} \operatorname{ev}\colon \Lambda_{\mathbb{Q}[t]}(\mathbf{x}) &\to \Lambda_{\mathbb{Q}[\mathsf{n}]}(\mathsf{J}) \subseteq \mathrm{FH} \\ t &\mapsto \mathsf{n} \\ x_i &\mapsto \mathsf{J}_i, \qquad i \geq 1, \end{aligned}$$

where we write $\mathsf{J}$ for the list $\mathsf{J}_1, \mathsf{J}_2, \dots$. Since $e_r(\mathsf{J}_1, \mathsf{J}_2, \dots) \in \mathrm{FH}$ has degree $r$, it follows that this map is degree-preserving, in the sense that symmetric functions of total degree at most $d$ in the indeterminates $t, x_1, x_2, \dots$ map to elements of degree at most $d$ in the Farahat-Higman algebra.

One consequence of the evaluation map is that various transposition factorization enumeration problems are automatically central, namely, the ones whose generating functions can be expressed as symmetric functions in the Jucys-Murphy elements. In particular, by (1.5), the generating function for monotone factorizations of length $r$ is the $r$th complete symmetric function in the Jucys-Murphy elements, so this is a central problem. Another important example is that of monomial symmetric functions, where we have the following result.

**Lemma 1.7.1.** *(Matsumoto and Novak [35]). If $\alpha, \beta \vdash r$ are partitions of the same size, then $[\mathsf{K}_\alpha] m_\beta(\mathsf{J})$ is the number of ordered forests where the $i$th tree has $\alpha_i$ edges, and the up-degrees of the non-leaf vertices are given by the parts of $\beta$ in some order.*

*In particular, $[\mathsf{K}_\alpha] m_\beta(\mathsf{J}) = 0$ if $\beta$ is not a refinement of $\alpha$, and $[\mathsf{K}_\alpha] m_\beta(\mathsf{J}) = 1$ if $\beta = \alpha$.*

*Proof.* For a partition $\beta \vdash r$, the monomial symmetric function $m_\beta(\mathsf{J})$ is the generating function for the following problem, as can be seen by expanding it as a sum of products of Jucys-Murphy elements: how many monotone factorizations of length $r$ of a permutation $\sigma \in \mathcal{S}_n$ have a signature with some element repeated $\beta_1$ times, another repeated $\beta_2$ times, another repeated $\beta_3$ times, etc.?

By centrality, we can restrict our attention to decreasing permutations, and then according to Theorem 1.6.2, the answer is simple in the case of minimal monotone factorizations, that is, when $r = \operatorname{rank}(\sigma)$; then, the monotone factorizations are in bijection with ordered forests with tree sizes given by $\operatorname{cyc}(\sigma)$ and up-degrees of non-leaf vertices given by $\beta$. Since a tree of size one consists only of a leaf vertex, the answer depends on the reduced cycle type $\operatorname{redcyc}(\sigma)$ rather than on the cycle type $\operatorname{cyc}(\sigma)$, and $\operatorname{redcyc}(\sigma)$ gives the numbers of edges in each tree, rather than the number of vertices. □

**Corollary 1.7.2.** *(Diaconis and Greene, [9]; Corteel, Goupil and Schaeffer, [6]). The Jucys-Murphy evaluation map $\operatorname{ev}\colon \Lambda_{\mathbb{Q}[t]}(\mathbf{x}) \to \mathrm{FH}$ is a ring isomorphism.*

*Proof.* First, note that ev identifies the subring $\mathbb{Q}[t]$ of $\Lambda_{\mathbb{Q}[t]}(\mathbf{x})$ with the subring $\mathbb{Q}[\mathsf{n}]$ of FH, so that we can view ev as a linear map with respect to this common subring. Now, order the set of all partitions in increasing order of size and refinement, and take the monomial symmetric functions as an ordered basis for $\Lambda_{\mathbb{Q}[t]}(\mathbf{x})$ over $\mathbb{Q}[t]$ and the reduced class sums as an ordered basis for FH over $\mathbb{Q}[\mathsf{n}]$. Then, by Lemma 1.7.1 and degree considerations in the Farahat-Higman algebra, the matrix for ev is upper triangular, with ones on the diagonal. In particular, it is invertible. □

*Remark* 1.7.1. Early on, Jucys [30] showed that every projection of the map ev to $\mathbb{Q}\mathcal{S}_n$, $n \geq 0$, gives a surjection onto the centre $\mathcal{Z}_n$, but did not consider whether the combined map to $\mathbb{Q}\mathcal{S} = \bigoplus_{n\geq 0} \mathbb{Q}\mathcal{S}_n$ is bijective. This was also shown independently by Murphy [38, 39]. Later, Diaconis and Greene [9] essentially gave a proof that ev is invertible by considering power sum symmetric functions. This approach was also taken independently by Corteel, Goupil and Schaeffer [6].

Since the Jucys-Murphy evaluation map is an isomorphism, it follows that $\mathrm{FH} = \Lambda_{\mathbb{Q}[\mathsf{n}]}(\mathsf{J})$ has bases over $\mathbb{Q}[\mathsf{n}]$ given by the classical bases for the ring of symmetric functions, in addition to the basis of reduced class sums. This leads to the problem of expressing these bases in terms of each other.

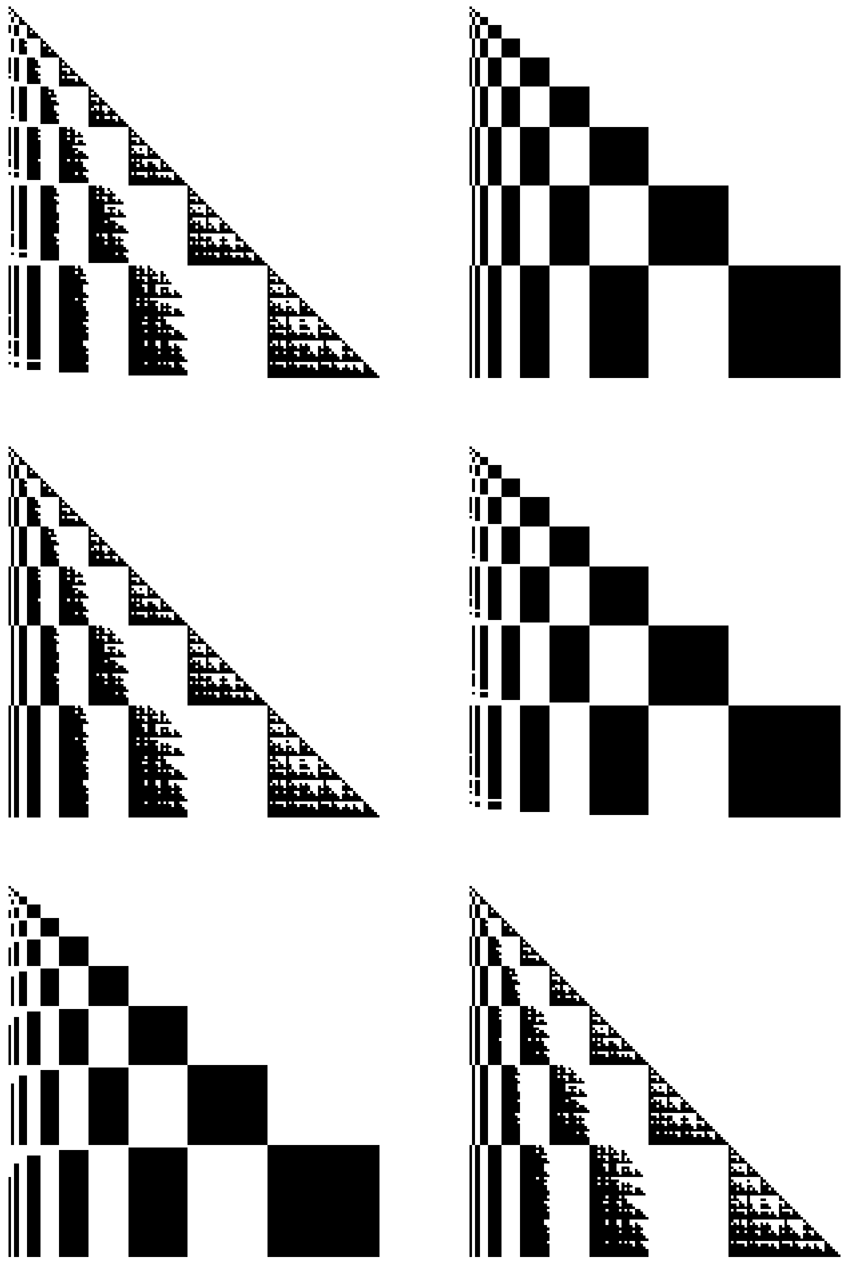


Figure 1.8: The pattern of nonzero coefficients up to degree 10 for various change of basis matrices, from classical bases of the symmetric function ring $\Lambda_{\mathbb{Q}[\mathsf{n}]}(\mathsf{J})$, to the basis of reduced class sums $\mathsf{K}_\alpha$ of the Farahat-Higman algebra FH. From left to right, and top to bottom: monomial symmetric functions, complete symmetric functions; power sum symmetric functions, Schur symmetric functions; elementary symmetric functions, and forgotten symmetric functions.

When expressing other bases in terms of the reduced class sums $\mathsf{K}_\alpha$, this is called the class expansion problem. Figure 1.8 gives a visual representation of the change of basis matrices from the classical bases of symmetric functions to the reduced class sum basis, where the pattern of nonzero entries is drawn in black. In each case, there is a clear block structure, corresponding to the different degrees of symmetric functions and to the different ranks of the reduced class sums. The lower triangular nature of the block structure comes from the fact that the grading on symmetric functions is compatible with the filtration on the Farahat-Higman algebra, and the checkerboard pattern comes from the fact that even-degree symmetric functions of the Jucys-Murphy elements only have even permutations in their expansions, and similarly for odd-degree symmetric functions.

From this diagram, it is also apparent that the bases of monomial symmetric functions, power sum symmetric function, and forgotten symmetric functions are all triangular (and not just block triangular) with respect to the reduced class sums. What is not apparent is that the monomial symmetric functions are actually *unitriangular* with respect to the reduced class sums, meaning that the diagonal entries are all ones, unlike the power sum and forgotten symmetric functions. Thus, it may be that the monomial symmetric functions are the more natural basis when studying the Jucys-Murphy evaluation map.

Another possible natural basis is given by some homogeneous symmetric function introduced by Macdonald [34, I.7, Example 25] (see also Goulden and Jackson [19]) to study the coefficients $[\mathsf{K}_\alpha]\mathsf{K}_\beta\mathsf{K}_\gamma$ when $|\alpha| = |\beta| + |\gamma|$, called the top connection coefficients. As noted in [6], up to a predictable sign change, the change of basis matrix from these symmetric functions to the reduced class sums is *block* unitriangular, meaning that the diagonal blocks are identity matrices. In fact, it is easy to see that there is a unique basis of the symmetric functions which is both homogeneous and block unitriangular with the reduced class sums; studying this basis may yield some insight on the exact correspondence between the natural grading for symmetric functions on one hand, and the natural filtration for the Farahat-Higman algebra on the other hand.

## 1.8 The Characteristic Map

The Jucys-Murphy evaluation map $\mathrm{ev}\colon \Lambda_{\mathbb{Q}[t]} \to \mathrm{FH}$ gives one link between symmetric functions and the symmetric group algebra $\mathbb{Q}\mathcal{S}$, but another link is given by the characteristic map $\mathrm{ch}\colon \mathbb{Q}\mathcal{S} \to \overline{\Lambda}_{\mathbb{Q}}$. There is already an algebra structure on $\mathbb{Q}\mathcal{S}$ given by the component-wise product, but through the characteristic map, we will define another algebra structure on $\mathbb{Q}\mathcal{S}$ which is relevant to the combinatorics of permutation factorizations.

As a $\mathbb{Q}$-linear map, the characteristic map is defined by $\mathrm{ch}(\sigma) = p_\alpha/n!$, where $\alpha \vdash n$ is the cycle type of $\sigma$, and $p_\alpha$ is a power sum symmetric function. It is well-known from the representation theory of symmetric groups that, when restricted to the centre $\mathcal{Z} = \prod_{n\geq 0} \mathcal{Z}_n$, the characteristic map satisfies

$$\mathrm{ch}(\mathsf{C}_\alpha) = \frac{p_\alpha}{\prod_{i\geq 1} i^{m_i(\alpha)} \cdot m_i(\alpha)!}, \qquad \mathrm{ch}(\mathsf{F}^\alpha) = \frac{f^\alpha s_\alpha}{n!}, \qquad \mathrm{ch}(\chi^\alpha) = s_\alpha$$

for $\alpha \vdash n$, $n \geq 0$, where $\chi^\alpha = \sum_{\beta\vdash n} \chi^\alpha_\beta \mathsf{C}_\beta$ is an element of $\mathcal{Z}_n$ representing the irreducible character indexed by $\alpha$. Also, using the standard inner products on $\mathcal{Z}$ and $\Lambda_\mathbb{Q}$, we have

$$\langle \chi^\alpha, \chi^\beta \rangle = \langle s_\alpha, s_\beta \rangle = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise,} \end{cases}$$

so the characteristic map is an isometry.

As a linear map, the characteristic map is already useful as a way to relate change of basis matrices in $\Lambda_\mathbb{Q}$ and in $\mathcal{Z}$, but the multiplication in $\Lambda_\mathbb{Q}$ also corresponds to a meaningful representation-theoretic operation in $\mathcal{Z}$. If $U$ is a group representation of $\mathcal{S}_k$ for some $k \leq n$ and $V$ is a group representation of $\mathcal{S}_{n-k}$, then $U \otimes V$ is a group representation of the group $\mathcal{S}_k \times \mathcal{S}_{n-k}$, which we can view as a subgroup of $\mathcal{S}_n$ by letting $\mathcal{S}_k$ act on $\{1, 2, \ldots, k\}$ and $\mathcal{S}_{n-k}$ act on $\{k+1, \ldots n\}$. Then, there is a corresponding induced representation $W = U \otimes V \uparrow^{\mathcal{S}_n}_{\mathcal{S}_k \times \mathcal{S}_{n-k}}$ of $\mathcal{S}_n$. If we let $\chi^U = \sum_{\sigma\in\mathcal{S}_k} \mathrm{trace}_U(\sigma)$ be the element representing $U$ in $\mathcal{Z}_k$, and similarly for $\chi^V$ and $\chi^W$, then we have

$$\mathrm{ch}(\chi^W) = \mathrm{ch}(\chi^U)\,\mathrm{ch}(\chi^V).$$

Thus, the characteristic map relates the multiplication of symmetric functions to induced representations of the symmetric group.

While this is a notable description of the relation between multiplication and the characteristic map, there is another, more combinatorial description which will be more useful to us. There are $\binom{n}{k}$ ways to embed the sets $[k]$ and $[n-k]$ as complementary subsets of the set $[n]$ in an order-preserving way, and for each of these, there is an embedding of the group $\mathcal{S}_k \times \mathcal{S}_{n-k}$ into $\mathcal{S}_n$. Given permutations $\sigma \in \mathcal{S}_k$ and $\rho \in \mathcal{S}_{n-k}$, their **concatenation product** $\sigma \otimes \rho \in \mathbb{Q}\mathcal{S}_n$ is the sum of the $\binom{n}{k}$ order-preserving embeddings of the pair $(\sigma, \rho)$ into $\mathcal{S}_n$. If we extend this operation bilinearly to the symmetric group algebra $\mathbb{Q}\mathcal{S}$, then we immediately have

$$\mathrm{ch}(\mathsf{A} \otimes \mathsf{B}) = \mathrm{ch}(\mathsf{A})\,\mathrm{ch}(\mathsf{B}), \qquad \mathsf{A}, \mathsf{B} \in \mathbb{Q}\mathcal{S},$$

and we will write $\mathsf{A}^{\otimes k}$ for the concatenation product

$$\underbrace{\mathsf{A} \otimes \mathsf{A} \otimes \cdots \otimes \mathsf{A}}_{k \text{ times}}.$$

Note that for conjugacy class sums, we have

$$\mathsf{C}_\alpha \otimes \mathsf{C}_\beta = \frac{|\mathrm{aut}(\alpha \cup \beta)|}{|\mathrm{aut}(\alpha)|\,|\mathrm{aut}(\beta)|}\mathsf{C}_{\alpha\cup\beta},$$

where $\alpha \cup \beta$ is the partition obtained by taking all the parts of $\alpha$ and all the parts of $\beta$ and sorting them in decreasing order, so the concatenation product preserves the centre $\mathcal{Z}$ of $\mathbb{Q}\mathcal{S}$.

For elements of $\mathbb{Q}\mathcal{S}$ whose coefficients have a combinatorial meaning attached, the concatenation product is compatible with a cycle-by-cycle decomposition of permutations. Roughly speaking, suppose we have an element $\mathsf{A} \in \mathbb{Q}\mathcal{S}$ such that $[\sigma]\mathsf{A}$ is the number of ways of putting an $\mathcal{A}$-structure on a permutation $\sigma$, for some notion of $\mathcal{A}$-structure which is independent of order-preserving relabellings of the ground set. Similarly, suppose $[\rho]\mathsf{B}$ is the number of ways of putting a $\mathcal{B}$-structure on a permutation $\rho$. Then, $[\tau]\mathsf{A}\otimes\mathsf{B}$ is the number of ways of choosing some cycles of $\tau$, putting an $\mathcal{A}$-structure on them, and putting a $\mathcal{B}$-structure on the remaining cycles. For some permutation problems, this can be used to talk about decompositions into connected components, as shown by the following propositions and their proofs.

*Remark* 1.8.1. Note that Proposition 1.8.1 and its proof are included here mainly as a template for Propositions 1.8.2 and 1.8.3, the benefit of the strictly monotone case being that all the generating functions involved have simple, explicit formulas.

**Proposition 1.8.1.** *Let $\mathsf{R} \in \mathbb{Q}\mathcal{S}[[t]]$ be the generating function for transitive strictly monotone factorizations, where $t$ is an ordinary marker for the length of factorizations, so that $[t^r\sigma]\mathsf{R}$ is the number of transitive strictly monotone factorizations of length $r$ of the permutation $\sigma$. Similarly, let $\mathsf{R}^*$ be the generating function for all strictly monotone factorizations. Then, we have*

$$\sum_{k\geq 0} \frac{\mathsf{R}^{\otimes k}}{k!} = \mathsf{R}^*,$$

*so that* $\exp(\mathrm{ch}(\mathsf{R})) = \mathrm{ch}(\mathsf{R}^*)$*, or equivalently,* $\mathrm{ch}(\mathsf{R}) = \log(\mathrm{ch}(\mathsf{R}^*))$*.*

*Proof.* We know from Lemma 1.5.1 that every permutation $\sigma$ has a unique strictly monotone factorization, and that its length is $\mathrm{rank}(\sigma)$. Also, none of the transpositions can be a cut, so each cycle of $\sigma$ is a connected component of the factorization. Thus, only the permutations with a single cycle have a transitive strictly monotone factorization, and we have

$$\mathsf{R} = \sum_{k\geq 1} t^{k-1}\mathsf{C}_k.$$

Now, consider the generating function $\mathsf{R}\otimes\mathsf{R}$. The coefficient $[\tau]\mathsf{R}\otimes\mathsf{R}$ is zero unless $\tau$ has exactly two cycles, in which case it counts the number of pairs $(F, G)$

where $F$ is a transitive strictly monotone factorization of one of the cycles of $\tau$, and $G$ is a transitive strictly monotone factorization of the other cycle of $\tau$. If $F$ is the factorization

$$(a_1\, b_1)(a_2\, b_2)\cdots(a_r\, b_r) = \sigma$$

and $G$ is the factorization

$$(c_1\, d_1)(c_2\, d_2)\cdots(c_s\, d_s) = \rho,$$

then $F$ and $G$ operate on disjoint subsets of the ground set of $\tau$, so the factors of $F$ commute with the factors of $G$, and there is a unique rearrangement of the product

$$(a_1\, b_1)\cdots(a_r\, b_r)(c_1\, d_1)\cdots(c_s\, d_s) = \sigma\rho = \tau$$

which interleaves the factors of $F$ and $G$ to give a strictly monotone factorization of $\tau$. Conversely, given a permutation $\tau$ with two cycles, its unique strictly monotone factorization can be restricted to a transitive strictly monotone factorization on each of the two cycles, and this corresponds to two ordered pairs $(F, G)$ as above. Thus,

$$\frac{\mathsf{R}\otimes\mathsf{R}}{2} = \sum_{\substack{\alpha\vdash n, n\geq 0\\ \ell(\alpha)=2}} t^{|\alpha|-2}\mathsf{C}_\alpha$$

is the generating function for strictly monotone factorizations with two connected components. Similarly,

$$\frac{\mathsf{R}^{\otimes k}}{k!} = \sum_{\substack{\alpha\vdash n, n\geq 0\\ \ell(\alpha)=k}} t^{|\alpha|-k}\mathsf{C}_\alpha$$

is the generating function for strictly monotone factorizations with $k$ connected components, and the sum

$$\sum_{k\geq 0}\frac{\mathsf{R}^{\otimes k}}{k!} = \sum_{\alpha\vdash n, n\geq 0} t^{|\alpha|-\ell(\alpha)}\mathsf{C}_\alpha = \mathsf{R}^*$$

is the generating function for all strictly monotone factorizations. □

**Proposition 1.8.2.** *Let $\vec{\mathsf{H}} \in \mathbb{Q}\mathcal{S}[[t]]$ be the generating function for transitive monotone factorizations, where $t$ is an ordinary marker for the length of factorizations, so that $[t^r\sigma]\vec{\mathsf{H}}$ is the number of transitive monotone factorizations of length $r$ of the permutation $\sigma$. Similarly, let $\vec{\mathsf{H}}^*$ be the generating function for all monotone factorizations. Then, we have*

$$\sum_{k\geq 0}\frac{\vec{\mathsf{H}}^{\otimes k}}{k!} = \vec{\mathsf{H}}^*,$$

*so that* $\exp(\mathrm{ch}(\vec{\mathsf{H}})) = \mathrm{ch}(\vec{\mathsf{H}}^*)$*, or equivalently,* $\mathrm{ch}(\vec{\mathsf{H}}) = \log(\mathrm{ch}(\vec{\mathsf{H}}^*))$.

*Proof.* The proof is essentially the same as the proof of Proposition 1.8.1, even though the generating function $\vec{\mathsf{H}}$ doesn't have a simple expression in this case. If $\sigma$ and $\rho$ are permutations obtained by restricting $\tau$ to two complementary subsets of its ground set, and $F, G$ are the transitive monotone factorizations

$$\begin{aligned} F: &\qquad (a_1\, b_1)(a_2\, b_2)\cdots(a_r\, b_r) = \sigma, \\ G: &\qquad (c_1\, d_1)(c_2\, d_2)\cdots(c_s\, d_s) = \rho, \end{aligned}$$

then $F$ and $G$ operate on disjoint subsets of the ground set of $\tau$, so the factors of $F$ commute with the factors of $G$. There is a unique rearrangement of the product

$$(a_1\, b_1)\cdots(a_r\, b_r)(c_1\, d_1)\cdots(c_s\, d_s) = \sigma\rho = \tau$$

which interleaves the factors of $F$ and $G$ while keeping the relative order of the factors of $F$ and keeping the relative order of the factors of $G$ to give a monotone factorization of $\tau$. Conversely, given a monotone factorization of $\tau$ with two connected components, it can be restricted to an ordered pair of transitive monotone factorizations in two ways. In general, a monotone factorization of $\tau$ with $k$ connected components can be restricted to an ordered $k$-tuple of transitive monotone factorizations in $k!$ ways by choosing an ordering on the connected components, so the generating function for monotone factorizations with exactly $k$ components is $\vec{\mathsf{H}}^{\otimes k}/k!$. □

**Proposition 1.8.3.** *Let $\mathsf{H} \in \mathbb{Q}\mathcal{S}[[t]]$ be the generating function for transitive transposition factorizations, where $t$ is an exponential marker for the length of factorizations, so that $[t^r\sigma/r!]\mathsf{H}$ is the number of transitive transposition factorizations of length $r$ of the permutation $\sigma$. Similarly, let $\mathsf{H}^*$ be the generating function for all transposition factorizations. Then, we have*

$$\sum_{k\geq 0} \frac{\mathsf{H}^{\otimes k}}{k!} = \mathsf{H}^*,$$

*so that* $\exp(\mathrm{ch}(\mathsf{H})) = \mathrm{ch}(\mathsf{H}^*)$, *or equivalently,* $\mathrm{ch}(\mathsf{H}) = \log(\mathrm{ch}(\mathsf{H}^*))$.

*Proof.* As with Proposition 1.8.2, the proof has the same structure as the proof of Proposition 1.8.1, but the fact that $t$ is an exponential marker rather than an ordinary marker for the length of factorizations is a significant difference. If $\sigma$ and $\rho$ are permutations obtained by restricting $\tau$ to two complementary subsets of its ground set, and $F, G$ are the transitive transposition factorizations

$$\begin{aligned} F: &\qquad (a_1\, b_1)(a_2\, b_2)\cdots(a_r\, b_r) = \sigma, \\ G: &\qquad (c_1\, d_1)(c_2\, d_2)\cdots(c_s\, d_s) = \rho, \end{aligned}$$

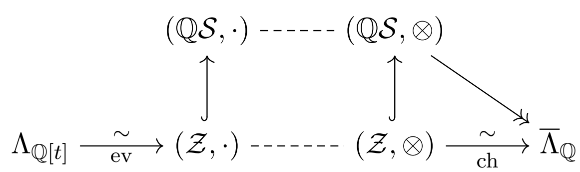
then there is generally more than one way to rearrange the product

$$(a_1\, b_1)\cdots(a_r\, b_r)(c_1\, d_1)\cdots(c_s\, d_s) = \sigma\rho = \tau$$

to interleave the factors of $F$ and $G$ while keeping the relative order of factors within each factorization to obtain a transposition factorization of $\tau$. In fact, any of the $\binom{r+s}{r}$ ways of interleaving the factors works. In general, for a transposition factorization of $\tau$ with $k$ connected components, the number of ways of interleaving the factors of the restrictions of this factorization to its connected components is a multinomial coefficient, and this is exactly compensated by the exponential nature of the marker $t$ for the number of transpositions in a factorization. □

Thus, the use of the characteristic map in these cases allows an algebraic treatment of the combinatorial decomposition into connected components.

*Remark* 1.8.2. The relationship between the (centre of the) symmetric group algebra and the ring of symmetric functions can be summarized by the diagram



Note that the symmetric functions on the left and the symmetric functions on the right have different base rings, which is an important distinction in this case. Also, although each one is isomorphic to the symmetric group algebra in some sense, this is for different product structures; the Jucys-Murphy evaluation map preserves the composition product structure, whereas the characteristic map preserves the concatenation product.

In particular, the logarithm and the exponential in an expression such as $\log(\mathrm{ch}(\mathrm{ev}(\exp(p_1))))$ have very different natures.

## 1.9 Classical and Monotone Hurwitz Numbers

Having defined all the required preliminary notions, we close this chapter with a definition of Hurwitz numbers and monotone Hurwitz numbers, which are the subject of the remainder of the thesis.

The **Hurwitz number** $H_g(\alpha)$ (see [27, 28]) can be defined as the number of branched coverings (up to isomorphism) of the Riemann sphere by a surface of genus $g$ with ramification data given the partition $\alpha$, as follows. After fixing the ramification locus and a set of branch cuts on the sphere, the branched covering can be thought of as a set of disjoint sheets, labelled $1, 2, \ldots, n$, and glued along the branch cuts in some way. This gluing can be described by considering the preimage of a small clockwise (say) circle around each ramification point, which can be encoded as a permutation in $\mathcal{S}_n$. Then, a branched cover is counted by

$H_g(\alpha)$ if the permutation for the first ramification point has cycle type $\alpha$, and the permutations for the other ramification points are all transpositions.

Since the choice of ramification locus and branch cuts is arbitrary, we may assume that the first ramification point is at the origin, with the other ramification points arranged clockwise on the unit circle, and the branch cuts are line segments from the origin to the unit circle, forming a star graph. Then, the permutations $\sigma, (a_1\, b_1), (a_2\, b_2), \ldots, (a_r\, b_r)$ which encode the gluing are characterized by the equation

$$(a_1\, b_1)(a_2\, b_2) \cdots (a_r\, b_r) = \sigma,$$

and the requirement that this transposition factorization be transitive. Furthermore, a branched covering is uniquely determined up to isomorphism by this monodromy data, and the length $r$ of the factorization is determined by the Riemann-Hurwitz formula,

$$r = n + \ell(\alpha) + 2g - 2. \tag{1.7}$$

Thus, the Hurwitz number $H_g(\alpha)$ could also be defined as the number of transitive transposition factorizations of length $r$ of all permutations of cycle type $\alpha \vdash n$. Per Proposition 1.8.3, this is also the coefficient of $t^r p_\alpha / r!n!$ in $\mathrm{ch}(\mathsf{H})$.

*Remark* 1.9.1. Given a cycle type $\alpha$, the parameters $r$ and $g$ are equivalent, in that either one is uniquely determined by the other, but either one may be more natural in a given situation. We will write $H^r(\alpha)$ and $H_g(\alpha)$ interchangeably to refer to Hurwitz numbers, with the understanding that (1.7) always holds.

*Remark* 1.9.2. Note that the branched coverings counted by $H_g(\alpha)$ can be related to the topological construction of Section 1.6 as follows. Given a branched covering with the branch cuts in the form of a star, consider the planar dual graph $G$ of this star, which consists of a single vertex and $r$ loops. Then, the $n$ preimages of the vertex of $G$ are the vertices of the construction, and the preimages of the loops of $G$ give the internal edges of the construction. If a suitable directed loop is added to $G$, then its preimage gives the boundary arcs of the construction, and removing the interiors of these boundary arcs completes the picture.

By analogy, the **monotone Hurwitz number** $\vec{H}^r(\alpha)$ is defined as the number of transitive *monotone* factorizations of length $r$ of all permutations of cycle type $\alpha \vdash n$; per Proposition 1.8.2, $\vec{H}^r(\alpha)$ is the coefficient of $t^r p_\alpha / n!$ in $\mathrm{ch}(\vec{\mathsf{H}})$. Since monotone factorizations are transposition factorizations, the relation (1.7) also holds for them, and we will write $\vec{H}_g(\alpha)$ and $\vec{H}^r(\alpha)$ interchangeably, as with Hurwitz numbers.

The monotone Hurwitz numbers were introduced by Goulden, Novak and the author [16, 17] as the coefficients in an asymptotic expansion of the Harish-Chandra-Itzykson-Zuber integral [26, 29].

For clarity, we will often refer to the Hurwitz numbers as *classical* Hurwitz numbers, to distinguish them from their monotone counterparts.

# Chapter 2

# Join-Cut Equations

The algebraic approach to monotone Hurwitz numbers that we develop in this thesis is based on a join-cut equation for the associated generating function, very similar to the join-cut equation of Goulden and Jackson [20] for the generating function for classical Hurwitz numbers. In this chapter, we give a quick review of the join-cut equation for classical Hurwitz numbers and the combinatorial join-cut analysis that leads up to it, then use the same techniques to establish a join-cut equation for monotone Hurwitz numbers.

While these join-cut equations are combinatorially straightforward, they *are* second-order nonlinear partial differential equations in infinitely many variables, so they can be difficult to manipulate. To address this, we introduce three families of convenient algebraic operators.

## 2.1 Hurwitz Numbers

### 2.1.1 All Transposition Factorizations

First, we review the combinatorial join-cut analysis which leads to the join-cut differential operator.

**Lemma 2.1.1.** *(Goulden, [14]). Let* $\mathsf{A}$ *be an element of* $\mathbb{Q}\mathcal{S}$*, and let* $\mathsf{K}_1 \in \mathbb{Q}\mathcal{S}$ *be the sum of all transpositions. Then, we have*

$$\mathrm{ch}(\mathsf{K}_1\mathsf{A}) = \frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2}{\partial p_i \partial p_j} + (i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}\right)\mathrm{ch}(\mathsf{A}).$$

*Thus, when translated through the characteristic map* ch*, the action of* $\mathsf{K}_1$ *on* $\mathsf{A}$ *can be expressed as a differential operator.*

*Proof.* This is essentially a restatement in the context of the characteristic map of the permutation equations

$$(a_1\, a_2\, \cdots\, a_i)(b_1\, b_2\, \cdots\, b_j)\cdot(a_1\, b_1) = (a_1\, a_2\, \cdots\, a_i\, b_1\, b_2\, \cdots\, b_j),$$

$$(a_1\, a_2\, \cdots\, a_i\, b_1\, b_2\, \cdots\, b_j) \cdot (a_1\, b_1) = (a_1\, a_2\, \cdots\, a_i)(b_1\, b_2\, \cdots\, b_j),$$

which express the fact that multiplying a permutation by a transposition either joins two cycles into one, or breaks one cycle into two. By $\mathbb{Q}$-linearity, it is enough to check the statement when $\mathsf{A}$ is a single permutation $\sigma \in \mathcal{S}_n$, $n \geq 0$, in which case $\mathsf{K}_1\mathsf{A}$ is the sum of all permutations that can be obtained by multiplying $\sigma$ by a transposition $(a\, b) \in \mathcal{S}_n$. The transposition $(a\, b)$ will either be a join or a cut for $\sigma$, depending on whether $a$ and $b$ are in the same cycle of $\sigma$. Let us first consider joins. Let $\sigma$ have $m_i$ cycles of length $i$, so that $\mathrm{ch}(\sigma) = p_1^{m_1} p_2^{m_2} \cdots / n!$. Then, for $i, j \geq 1$, there are $im_i$ ways to choose an element $a$ in a cycle of length $i$, and either $jm_j$ or $j(m_j - 1)$ ways to choose an element $b$ in a (different) cycle of length $j$, depending on whether $j$ is distinct from $i$. For each of these choices, multiplying $\sigma$ by $(a\, b)$ joins a cycle of length $i$ and a cycle of length $j$ into a cycle of length $i + j$. Summing over these choices and accounting for the symmetry between $(a\, b)$ and $(b\, a)$, we get

$$\sum_{(a\, b) \text{ a join}} \mathrm{ch}(\sigma \cdot (a\, b)) = \frac{1}{2} \sum_{i,j \geq 1} ij p_{i+j} \frac{\partial^2 \mathrm{ch}(\sigma)}{\partial p_i \partial p_j}.$$

Now, let us consider cuts. For $i, j \geq 1$, there are $m_{i+j}$ ways to choose a cycle of length $i + j$, and then there are $i + j$ ways to choose $a$ and $b$ in that cycle so that multiplying $\sigma$ by $(a\, b)$ puts $a$ in a cycle of length $i$ and $b$ in a cycle of length $j$. Summing over these choices and accounting for the symmetry between $(a\, b)$ and $(b\, a)$ again, we get

$$\sum_{(a\, b) \text{ a cut}} \mathrm{ch}(\sigma \cdot (a\, b)) = \frac{1}{2} \sum_{i,j \geq 1} (i + j) p_i p_j \frac{\partial \mathrm{ch}(\sigma)}{\partial p_{i+j}}. \qquad \square$$

Using this, we can easily find a differential equation which characterizes the generating function for all transposition factorizations, with no transitivity or monotonicity requirement.

**Theorem 2.1.2.** *(Goulden, [15]). Let* $\mathsf{H}^* = \exp(t\mathsf{K}_1) \in \mathbb{Q}\mathcal{S}[[t]]$ *be the generating function for all transposition factorizations, where* $t$ *is an exponential marker for the factorization lengths, so that* $[t^r \sigma / r!]\mathsf{H}^*$ *is the number of factorizations of the permutation* $\sigma$ *into* $r$ *transpositions. Let* $\mathbf{H}^* = \mathrm{ch}(\mathsf{H}^*) \in \mathbb{Q}[[\mathbf{p}, t]]$ *be its image under the characteristic map. Then,* $\mathbf{H}^*$ *is uniquely determined by the partial differential equation*

$$\frac{\partial}{\partial t} \mathbf{H}^* = \frac{1}{2} \sum_{i,j \geq 1} \left( ij p_{i+j} \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} + (i + j) p_i p_j \frac{\partial}{\partial p_{i+j}} \right) \mathbf{H}^*$$

*with initial condition* $[t^0]\mathbf{H}^* = \exp(p_1)$*.*

*Proof.* For a solution which is a formal power series in $t$, the coefficient of $t^{r+1}$ can be computed directly from the coefficient of $t^r$, so existence and uniqueness follow directly. To show that $\mathbf{H}^* = \mathrm{ch}(\mathsf{H}^*)$ is a solution, note that the operators $\frac{\partial}{\partial t}$ and ch commute, so we have

$$\frac{\partial}{\partial t}\mathbf{H}^* = \mathrm{ch}\left(\frac{\partial}{\partial t}\mathsf{H}^*\right) = \mathrm{ch}\left(\frac{\partial}{\partial t}\exp(t\mathsf{K}_1)\right) = \mathrm{ch}\left(\mathsf{K}_1\exp(t\mathsf{K}_1)\right) = \mathrm{ch}(\mathsf{K}_1\mathsf{H}^*).$$

Then, by Lemma 2.1.1, we have

$$\mathrm{ch}(\mathsf{K}_1\mathsf{H}^*) = \frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2}{\partial p_i\partial p_j} + (i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}\right)\mathbf{H}^*.$$

For the initial condition, note that the identity element $\mathrm{id} \in \mathbb{Q}\mathcal{S}$ is the sum of the identity elements $\mathrm{id}_n \in \mathcal{S}_n$ for all $n \geq 0$, so we have

$$[t^0]\mathbf{H}^* = \sum_{n\geq 0}\mathrm{ch}(\mathrm{id}_n) = \sum_{n\geq 0}\frac{p_1^n}{n!} = \exp(p_1). \qquad \square$$

*Remark* 2.1.1. Although we gave an algebraic proof of the join-cut equation for $\mathbf{H}^*$, the terms in it have a very straightforward combinatorial interpretation, and the proof could be made combinatorial. Specifically, the equation can be read as saying, "to get a transposition factorization of length $r+1$, take a transposition factorization of length $r$ and either add a join or add a cut", and the initial condition can be read as saying, "for each $n \geq 0$, the only transposition factorization of length zero in $\mathcal{S}_n$ is the empty factorization of the identity permutation $\mathrm{id}_n$". However, a substantial amount of bookkeeping is involved in counting the contributions of all cases exactly and making sure that the correct scaling is used.

### 2.1.2 Transitive Transposition Factorizations

From the join-cut equation for the generating function for all transposition factorizations of Theorem 2.1.2, we can deduce a join-cut equation which characterizes the generating function for *transitive* transposition factorizations.

**Corollary 2.1.3.** *(Goulden, Jackson and Vainshtein, [23]). Let $\mathsf{H} \in \mathbb{Q}\mathcal{S}[[t]]$ be the generating function for transitive transposition factorizations, where $t$ is an exponential marker for the factorization lengths, so that $[t^r\sigma/r!]\mathsf{H}$ is the number of transitive factorizations of the permutation $\sigma$ into $r$ transpositions. Let $\mathbf{H} = \mathrm{ch}(\mathsf{H}) \in \mathbb{Q}[[\mathbf{p}, t]]$ be its image under the characteristic map. Then, $\mathbf{H}$ is uniquely determined by the partial differential equation*

$$\frac{\partial\mathbf{H}}{\partial t} = \frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial\mathbf{H}}{\partial p_i}\frac{\partial\mathbf{H}}{\partial p_j} + ijp_{i+j}\frac{\partial^2\mathbf{H}}{\partial p_i\partial p_j} + (i+j)p_ip_j\frac{\partial\mathbf{H}}{\partial p_{i+j}}\right)$$

*with initial condition $[t^0]\mathbf{H} = p_1$.*

*Proof.* This follows directly from Theorem 2.1.2 by substituting $\exp(\mathsf{H})$ for $\mathsf{H}^*$, as per Proposition 1.8.3, and using the chain rule. □

*Remark* 2.1.2. As above, we gave an algebraic proof of the join-cut equation for the generating function for transitive transposition factorizations, but its terms have a simple combinatorial interpretation, and the proof could be made combinatorial. Specifically, the partial differential equation can be read as saying, "to get a transitive transposition factorization of length $r+1$, either add an essential join to a transposition factorization of length $r$ with two components, or add a redundant join to a transitive transposition factorization of length $r$, or add a cut to a transitive transposition factorization of length $r$", and the initial condition can be read as saying, "the only transitive transposition factorization of length zero is the empty factorization of the identity permutation in $\mathcal{S}_1$". Again, however, the algebraic proof avoids having to deal with a substantial amount of bookkeeping.

### 2.1.3 Breakdown by Transitive Genus

As noted in Sections 1.6 and 1.9, a transitive factorization of a permutation of cycle type $\alpha \vdash n$ into $r$ transpositions has a genus $g$, defined by

$$r = n + \ell(\alpha) + 2g - 2,$$

and this $g$ is always a non-negative integer. For our purposes, it will be useful to group the coefficients of the Hurwitz generating function according to their genus.

**Definition 2.1.4.** The **genus** $g$ **generating function** for Hurwitz numbers is

$$\mathbf{H}_g = \sum_{n\geq 1}\sum_{\alpha\vdash n} H_g(\alpha)\frac{1}{(n+\ell(\alpha)+2g-2)!}\frac{p_\alpha}{n!}.$$

We can unpack the differential equation from Corollary 2.1.3 for $\mathbf{H}$ into a separate differential equation for each $\mathbf{H}_g$, as follows.

**Theorem 2.1.5.** *(Goulden, Jackson and Vainshtein, [23]). The genus zero generating function for Hurwitz numbers $\mathbf{H}_0 \in \mathbb{Q}[[\mathbf{p}]]$ is uniquely determined by the partial differential equation*

$$\sum_{k\geq 1}\left((k+1)p_k\frac{\partial \mathbf{H}_0}{\partial p_k}\right) - 2\mathbf{H}_0 - \frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial \mathbf{H}_0}{\partial p_i}\frac{\partial \mathbf{H}_0}{\partial p_j} + (i+j)p_ip_j\frac{\partial \mathbf{H}_0}{\partial p_{i+j}}\right) = 0$$

*with initial condition $[p_1]\mathbf{H}_0 = 1$. For $g \geq 1$, the genus $g$ generating function for Hurwitz numbers $\mathbf{H}_g \in \mathbb{Q}[[\mathbf{p}]]$ is uniquely determined by the partial differential equation*

$$\sum_{k\geq 1}\left((k+1)p_k\frac{\partial \mathbf{H}_g}{\partial p_k}\right)+(2g-2)\mathbf{H}_g$$
$$-\frac{1}{2}\sum_{i,j\geq 1}\left(2ijp_{i+j}\frac{\partial \mathbf{H}_0}{\partial p_i}\frac{\partial \mathbf{H}_g}{\partial p_j}+(i+j)p_ip_j\frac{\partial \mathbf{H}_g}{\partial p_{i+j}}\right)$$
$$=\frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2 \mathbf{H}_{g-1}}{\partial p_i\partial p_j}+\sum_{g'=1}^{g-1}ijp_{i+j}\frac{\partial \mathbf{H}_{g'}}{\partial p_i}\frac{\partial \mathbf{H}_{g-g'}}{\partial p_j}\right)$$

*and the generating functions* $\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_{g-1}$.

*Proof.* Let $\mathbf{A}=\sum_{g\geq 0}\mathbf{H}_g u^g$, so that as a generating function, $\mathbf{A}$ has the same coefficients as $\mathbf{H}$, but has an ordinary marker $u$ for the genus $g$ instead of an exponential marker $t$ for the number of transpositions $r$. Since $r=n+\ell(\alpha)+2g-2$, the generating functions

$$t\frac{\partial \mathbf{H}}{\partial t} \qquad \text{and} \qquad \sum_{k\geq 1}\left((k+1)p_k\frac{\partial \mathbf{A}}{\partial p_k}\right)+2u\frac{\partial \mathbf{A}}{\partial u}-2\mathbf{A}$$

also have the same coefficients. Then, by comparison with Corollary 2.1.3, it follows that $\mathbf{A}$ is uniquely determined by the partial differential equation

$$\sum_{k\geq 1}\left((k+1)p_k\frac{\partial \mathbf{A}}{\partial p_k}\right)+2u\frac{\partial \mathbf{A}}{\partial u}-2\mathbf{A}$$
$$=\frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial \mathbf{A}}{\partial p_i}\frac{\partial \mathbf{A}}{\partial p_j}+uijp_{i+j}\frac{\partial^2 \mathbf{A}}{\partial p_i\partial p_j}+(i+j)p_ip_j\frac{\partial \mathbf{A}}{\partial p_{i+j}}\right)$$

with initial condition $[u^1p_1]\mathbf{A}=1$. Extracting the coefficient of $u^g$ from this equation for $g=0$ and $g\geq 1$ and reorganizing terms gives the stated equations for $\mathbf{H}_0$ and $\mathbf{H}_g$, $g\geq 1$. ☐

*Remark* 2.1.3. Although they may seem unwieldy, the differential equations of Theorem 2.1.5 are actually quite useful. They were used by Goulden and Jackson [20, 22, 21] to prove the correctness of explicit formulas for $g=0,1,2$, and in Chapter 5, we use them to relate the asymptotics of Hurwitz numbers and monotone Hurwitz numbers.

## 2.2 Convenient Operators

To perform manipulations on generating functions in $\mathbb{Q}[[\mathbf{p}]]$ such as $\mathbf{H}_g$, it will be convenient to have a ready supply of auxiliary indeterminates, so we introduce a countable set $\mathbf{x}=(x_1,x_2,\ldots)$ of new indeterminates and form the ring $\mathbb{Q}[[\mathbf{p},\mathbf{x}]]$.

Unlike power sum symmetric functions, these new indeterminates should be thought of as completely interchangeable.

Then, the following $\mathbb{Q}$-linear operators can be used to relate the new indeterminates $\mathbf{x}$ to the power sums $\mathbf{p}$ and manipulate them.

**Definition 2.2.1.** The $i$th **lifting operator** $\Delta_i$ is the $\mathbb{Q}[[\mathbf{x}]]$-linear differential operator on the ring $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$ defined by

$$\Delta_i = \sum_{k \geq 1} k x_i^k \frac{\partial}{\partial p_k}, \qquad i \geq 1.$$

**Definition 2.2.2.** The $i$th **projection operator** $\Pi_i$ is the $\mathbb{Q}[[\mathbf{p}]]$-linear idempotent operator on the ring $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$ defined by

$$\Pi_i = [x_i^0] + \sum_{k \geq 1} p_k [x_i^k], \qquad i \geq 1.$$

Note that $\Pi_i$ is also $\mathbb{Q}[[x_j]]$-linear for $j \neq i$.

**Definition 2.2.3.** Let $F(x_i)$ be an element of $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$, considered as a power series in $x_i$, and let $j \geq 1$ be an index other than $i \geq 1$. Then the $i$-to-$j$ **splitting** operator is defined by

$$\operatorname*{Split}_{i \to j} F(x_i) = \frac{x_j F(x_i) - x_i F(x_j)}{x_i - x_j} + F(0),$$

so that $\operatorname{Split}_{i \to j} 1 = \operatorname{Split}_{i \to j} x_i = 0$ and

$$\operatorname*{Split}_{i \to j} x_i^k = x_i^{k-1} x_j + x_i^{k-2} x_j^2 + \cdots + x_i x_j^{k-1}, \qquad k \geq 2.$$

Note that $\operatorname{Split}_{i \to j}$ is $\mathbb{Q}[[\mathbf{p}]]$-linear, and $\mathbb{Q}[[x_k]]$-linear for $k \neq i$.

*Remark* 2.2.1. Note that the indeterminates $\mathbf{x} = (x_1, x_2, \dots)$ are interchangeable in the definition of the lifting, projection and splitting operators, in the sense that the definition of $\Delta_j$ is obtained by replacing every occurrence of $x_i$ by $x_j$ in the definition of $\Delta_i$, and similarly for $\Pi_i$ and $\operatorname{Split}_{i \to j}$. To cut down on the proliferation of indices, we will generally state results involving these operators using explicit numeric indices and write, *e.g.*, $\operatorname{Split}_{1 \to 2}$ instead of $\operatorname{Split}_{i \to j}$.

Notationally, the lifting and projection operators essentially translate between subscripts on the power sums $\mathbf{p}$ and exponents on the indeterminates $\mathbf{x}$. This makes it easier to express some natural combinatorial operations on permutations as algebraic operations on generating functions in $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$.

For example, if a cycle of length $k$ is represented by a power sum $p_k$, then as noted in Section 1.8, the concatenation product on permutations corresponds to multiplication of generating functions. However, if a cycle of length $k$ is given a label $i$ and represented by as $x_i^k$, then multiplication of generating functions corresponds to the operation of joining cycles with the same label instead. Conversely, if $x_1^k$ represents a cycle of length $k$ and $x_2$ is an unused indeterminate, then applying the $\mathrm{Split}_{1\to 2}$ operator to a generating function corresponds to the operation of cutting the cycle labelled 1 in two cycles labelled 1 and 2 in all possible ways.

More concretely, the lifting, projection and splitting operators can be used to rewrite the differential operators of Lemma 2.1.1 (and Theorem 2.1.2) in a way that mirrors the combinatorial considerations which appear in its proof. The operator which accounts for multiplying a permutation by a transposition $(a\,b)$ which is a join is

$$\frac{1}{2}\sum_{i,j\geq 1} ijp_{i+j}\frac{\partial^2}{\partial p_i \partial p_j} = \frac{1}{2}\,\Pi_1\,\Delta_1^2,$$

where the first application of $\Delta_1$ accounts for the number of ways of picking $a$ while recording the length of the cycle containing it as the exponent of $x_1$; the second application of $\Delta_1$ accounts for the number of ways of picking $b$ in a different cycle while recording the length of this cycle as the exponent $x_1$; the implicit multiplication of these two powers of $x_1$ accounts for joining these two cycles; the application of $\Pi_1$ converts the exponent of the auxiliary indeterminate $x_1$ to a power sum subscript; and the division by two accounts for the symmetry between $a$ and $b$. The operator which accounts for multiplying a permutation by a cut is

$$\frac{1}{2}\sum_{i,j\geq 1}(i+j)p_i p_j\frac{\partial}{\partial p_{i+j}} = \frac{1}{2}\,\Pi_1\,\Pi_2\,\mathop{\mathrm{Split}}_{1\to 2}\Delta_1,$$

where again the application of $\Delta_1$ accounts for the number of ways of picking $a$ while recording the length of the cycle containing it as the exponent of $x_1$; the application of $\mathrm{Split}_{1\to 2}$ accounts for the ways in which the cycle containing $a$ could be split in two, while recording the lengths of these cycles as the exponents of $x_1$ and $x_2$; the application of $\Pi_1\,\Pi_2$ converts these exponents to power sum subscripts; and the division by two is again for symmetry.

Similarly, the differential equation of Corollary 2.1.3 can be rewritten as

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2}\,\Pi_1\,\Pi_2\Big(\big(\,\Delta_1\,\mathbf{H}\big)^2 + \Delta_1^2\,\mathbf{H} + \mathop{\mathrm{Split}}_{1\to 2}\Delta_1\,\mathbf{H}\Big),$$

which can also be given a fairly direct combinatorial interpretation.

These examples illustrate the fact that the lifting, projection and splitting operators can be used to describe simple combinatorial operations on permutations, but as we will see in Chapter 3, their power comes from the fact that they are also compatible with algebraic changes of variables.

As a final note, we have the following easy lemma, which can be used to move projection operators to the outside of an expression; this typically makes computations easier.

**Lemma 2.2.4.** *For $F \in \mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$, we have the identity*

$$\Delta_1 \, \Pi_2 \, F = \Pi_2 \, \Delta_1 \, F + x_1 \left. \frac{\partial F}{\partial x_2} \right|_{x_2 = x_1}.$$

*Proof.* The identity can be verified directly on monomials in $\mathbf{p}$ and $\mathbf{x}$, and by $\mathbb{Q}$-linearity, it extends to all of $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$. □

## 2.3 Monotone Hurwitz Numbers

### 2.3.1 All Monotone Factorizations

Now, we turn to the combinatorial analysis leading to a join-cut equation for the generating function for monotone factorizations. To simplify the bookkeeping involved, it will be useful to give a combinatorial interpretation to the coefficients of a few auxiliary generating functions.

*Example* 2.3.1. Let $\vec{\mathbf{H}}^* = \mathrm{ch}(\vec{\mathsf{H}}^*) \in \mathbb{Q}[[\mathbf{p}, t]]$, so that for $r \geq 0$ and $\alpha \vdash n$, the coefficient $[t^r p_\alpha / n!]\vec{\mathbf{H}}^*$ is the number of monotone factorizations of length $r$ of all permutations in $\mathcal{S}_n$ of cycle type $\alpha$. In particular, there are 20 permutations in $\mathcal{S}_5$ of cycle type $\alpha = (3, 1, 1)$:

| | | | | |
|---|---|---|---|---|
| (1)(2)(3 4 5) | (1)(3)(2 4 5) | (1)(4)(2 3 5) | (1)(5)(2 3 4) | (2)(3)(1 4 5) |
| (1)(2)(4 3 5) | (1)(3)(4 2 5) | (1)(4)(3 2 5) | (1)(5)(3 2 4) | (2)(3)(4 1 5) |
| (2)(4)(1 3 5) | (2)(5)(1 3 4) | (3)(4)(1 2 5) | (3)(5)(1 2 4) | (4)(5)(1 2 3) |
| (2)(4)(3 1 5) | (2)(5)(3 1 4) | (3)(4)(2 1 5) | (3)(5)(2 1 4) | (4)(5)(2 1 3) |

Each of these permutations has exactly 3 monotone factorizations of length 2, so we have $[t^2 p_{311}/5!]\vec{\mathbf{H}}^* = 60$. The corresponding contribution to $\Delta_1 \, \vec{\mathbf{H}}^*$ is

$$180 \frac{t^2 p_{11} x_1^3}{5!} + 120 \frac{t^2 p_{31} x_1}{5!},$$

and we can interpret this as follows: in the list of 60 monotone factorizations, there are 180 instances of an element of the ground set in a cycle of length 3 (marked by $x_1^3$), and 120 instances of an element in a cycle of length 1 (marked by $x_1$). Note that, in each case, the selected element of the ground set is equally likely to be 1, 2, 3, 4 or 5, so we can rewrite this contribution as

$$36 \frac{t^2 p_{11} x_1^3}{4!} + 24 \frac{t^2 p_{31} x_1}{4!},$$

with the interpretation that out of the 60 monotone factorizations, there are 36 where the element 5 (say) is in a cycle of length 3, and 24 where the element 5 is in a cycle of length 1.

Now, what happens if we apply a second lifting operator, $\Delta_2$, to this rewritten contribution? The effect of $\Delta_2$ corresponds to selecting an element of the ground set in all possible ways, and marking its cycle by $x_2^k$ instead of $p_k$, where $k$ is the cycle length; here, we exclude any element whose cycle is already marked by some power of $x_1$ instead of a power sum. Thus, the contribution of the 60 monotone factorizations to $\Delta_2 \, \Delta_1 \, \vec{\mathbf{H}}^*$ can be written as

$$72\frac{t^2 p_1 x_1^3 x_2}{4!} + 72\frac{t^2 p_1 x_1 x_2^3}{4!} + 24\frac{t^2 p_3 x_1 x_2}{4!},$$

indicating that

- in the 36 monotone factorizations where the element 5 is in a cycle of length 3 (marked by $x_1^3$), there are 72 instances of another element of the ground set being in a cycle of length 1 (marked by $x_2$);
- in the 24 monotone factorizations where the element 5 is in a cycle of length 1 (marked by $x_1$), there are 72 instances of another element of the ground set being in a cycle of length 3 (marked by $x_2^3$);
- in the same 24 monotone factorizations where the element 5 is in a cycle of length 1, there are 24 instances of another element of the ground set being in a different cycle of length 1 (marked by $x_2$).

These observations are formalized in the following lemma.

**Lemma 2.3.1.** *Let $\vec{\mathsf{H}}^* \in \mathbb{Q}\mathcal{S}[[t]]$ be the generating function for all monotone factorizations, where $t$ is an ordinary marker for the factorization lengths, so that $[t^r \sigma]\vec{\mathsf{H}}^*$ is the number of monotone factorizations of the permutation $\sigma$ into $r$ transpositions. Let $\vec{\mathbf{H}}^* = \mathrm{ch}(\vec{\mathsf{H}}^*) \in \mathbb{Q}[[\mathbf{p}, t]]$ be its image under the characteristic map. Let $\alpha \vdash n$ be a partition with at least one part of size $k$, and let $\alpha - k$ be the partition obtained by removing one of these parts. Then,*

1. *The coefficient $[t^r p_{\alpha-k} x_1^k / n!] \, \Delta_1 \, \vec{\mathbf{H}}^*$ is the number of triples $(\sigma, F, a)$ where $\sigma \in \mathcal{S}_n$ is a permutation of cycle type $\alpha$, $F$ is a monotone factorization of $\sigma$ into $r$ transpositions, and $a \in [n]$ is an element of the ground set in a cycle of $\sigma$ of length $k$;*
2. *The coefficient $[t^r p_{\alpha-k} x_1^k / (n-1)!] \, \Delta_1 \, \vec{\mathbf{H}}^*$ is the number of pairs $(\sigma, F)$ where $\sigma \in \mathcal{S}_n$ is a permutation of cycle type $\alpha$, $F$ is a monotone factorization of $\sigma$ into $r$ transpositions, and the element $n$ of the ground set is in a cycle of $\sigma$ of length $k$; and*

3. *if $\alpha - k$ has at least one part of size $j$, then the coefficient $[t^r p_{\alpha-j-k} x_1^k x_2^j/(n-1)!]\, \Delta_2\, \Delta_1\, \vec{\mathbf{H}}^*$ is the number of triples $(\sigma, F, a)$ where $\sigma \in \mathcal{S}_n$ is a permutation of cycle type $\alpha$, $F$ is a monotone factorization of $\sigma$ into $r$ transpositions, the element $n$ of the ground set is in a cycle of $\sigma$ of length $k$, and $a \in [n]$ is an element of the ground set in a different cycle, of length $j$.*

*Proof.* 1. If $\alpha$ has $m_k$ parts of size $k$, then we have

$$[t^r p_{\alpha-k} x_1^k/n!]\, \Delta_1\, \vec{\mathbf{H}}^* = k m_k [t^r p_\alpha/n!]\vec{\mathbf{H}}^*.$$

Since $\vec{\mathbf{H}}^* = \mathrm{ch}(\vec{\mathsf{H}}^*)$, the coefficient $[t^r p_\alpha/n!]\vec{\mathbf{H}}^*$ is the number of pairs $(\sigma, F)$ where $\sigma \in \mathcal{S}_n$ is a permutation of cycle type $\alpha$ and $F$ is a monotone factorization of $\sigma$ into $r$ transpositions. For each of these, there are $k m_k$ elements $a \in [n]$ which are in a cycle of $\sigma$ of length $k$.

2. Since the problem of counting monotone factorizations is central, as noted in Section 1.7, for any permutation $\sigma \in \mathcal{S}_n$ and any element $a \in [n]$, the number of pairs $(\sigma, F)$ where $F$ is a monotone factorization of $\sigma$ is equal to the number of pairs $((a\,n)\sigma(a\,n), F')$ where $F'$ is a monotone factorization of the conjugate permutation $(a\,n)\sigma(a\,n)$. Furthermore, the element $n$ is in a cycle of $\sigma$ of length $k$ exactly when $a$ is in a cycle of $(a\,n)\sigma(a\,n)$ of length $k$.

   Thus, for each pair $(\sigma, F)$ where $n$ is in a cycle of $\sigma$ of length $k$, there are $n$ triples $((a\,n)\sigma(a\,n), F', a)$ where $a$ is in a cycle of $(a\,n)\sigma(a\,n)$ of length $k$. Since $\sigma$ can be recovered from the knowledge of $(a\,n)\sigma(a\,n)$ and $a$, this gives a 1-to-$n$ correspondence between the pairs $(\sigma, F)$ counted here and the triples $(\sigma, F, a)$ counted in part 1. Thus, the number of these pairs is

$$\frac{1}{n}[t^r p_{\alpha-k} x_1^k/n!]\, \Delta_1\, \vec{\mathbf{H}}^* = [t^r p_{\alpha-k} x_1^k/(n-1)!]\, \Delta_1\, \vec{\mathbf{H}}^*$$

   as claimed.

3. This part follows from part 2 by essentially the same argument as in part 1. ∎

Using these combinatorial interpretations for the coefficients of $\Delta_1\, \vec{\mathbf{H}}^*$ and $\Delta_2\, \Delta_1\, \vec{\mathbf{H}}^*$, we can give a differential equation which characterizes $\vec{\mathbf{H}}^*$.

**Theorem 2.3.2.** *Let $\vec{\mathsf{H}}^* \in \mathbb{Q}\mathcal{S}[[t]]$ be the generating function for all monotone factorizations, where $t$ is an ordinary marker for the factorization lengths, so that $[t^r \sigma]\vec{\mathsf{H}}^*$ is the number of monotone factorizations of the permutation $\sigma$ into $r$ transpositions. Let $\vec{\mathbf{H}}^* = \mathrm{ch}(\vec{\mathsf{H}}^*) \in \mathbb{Q}[[\mathbf{p}, t]]$ be its image under the characteristic map. Then, $\vec{\mathbf{H}}^*$ is uniquely determined by the partial differential equation*

$$\Delta_1\, \vec{\mathbf{H}}^* = t\, \Delta_1^2\, \vec{\mathbf{H}}^* + t\, \Pi_2 \operatorname*{Split}_{1\to 2} \Delta_1\, \vec{\mathbf{H}}^* + x_1 \vec{\mathbf{H}}^* \tag{2.1}$$

*with initial condition $[p_0]\vec{\mathbf{H}}^* = 1$.*

*Proof.* To prove uniqueness, let $\alpha \vdash n$ be a nonempty partition, let $k \geq 1$ be the size of one of its parts, let $r \geq 0$, and consider the coefficient of $t^r p_{\alpha-k} x_1^k$ of both sides of (2.1). For the left-hand side, this is $km_k[t^r p_\alpha]\vec{\mathbf{H}}^*$, where $m_k \geq 1$ is the number of parts of $\alpha$ of size $k$. For the right-hand side, this is an expression involving only coefficients $[t^{r'} p_{\alpha'}]\vec{\mathbf{H}}^*$ where either $r' < r$ and $|\alpha'| = |\alpha|$, or where $r' = r$ and $|\alpha'| < |\alpha|$. Thus, by induction on $r + |\alpha|$, with the base case of $|\alpha| = 0$ handled by the initial condition, every coefficient of $\vec{\mathbf{H}}^*$ is uniquely determined.

To prove that $\vec{\mathbf{H}}^*$ is indeed a solution, consider the coefficient of $t^r p_{\alpha-k} x_1^k/(n-1)!$ of both sides of (2.1) and their combinatorial interpretation, where again $\alpha$ is an arbitrary nonempty partition, $k \geq 1$ is the size of one of its parts, and $r \geq 0$. For the left-hand side, as noted in Lemma 2.3.1, this is the number of pairs $(\sigma, F)$ where $\sigma \in \mathcal{S}_n$ is a permutation of cycle type $\alpha$, $F$ is a monotone factorization of $\sigma$ into $r$ transpositions, and the element $n$ of the ground set is in a cycle of $\sigma$ of length $k$. Let $F$ be the factorization

$$(a_1\, b_1)(a_2\, b_2) \cdots (a_r\, b_r) = \sigma.$$

The pairs $(\sigma, F)$ can be split into three disjoint sets, corresponding to the three terms of the right-hand side of (2.1), based on whether

1. the transposition $(a_r\, b_r)$ is a join and $b_r = n$, or
2. the transposition $(a_r\, b_r)$ is a cut and $b_r = n$, or
3. no transposition involves the element $n$.

**Case 1:** $(a_r\, b_r)$ is a join with $b_r = n$ precisely when the shorter monotone factorization

$$F' : \qquad (a_1\, b_1)(a_2\, b_2) \cdots (a_{r-1}\, b_{r-1}) = \sigma_{r-1}$$

is such that $a_r$ is in a cycle of $\sigma_{r-1}$ of some length $j$, $b_r = n$ is in a different cycle of $\sigma_{r-1}$ of length $k - j$. By Lemma 2.3.1, the number of triples $(\sigma_{r-1}, F', a_r)$ satisfying these conditions for a given value of $j$ is simply $[t^{r-1} p_{\alpha-k} x_1^{k-j} x_2^j/(n-1)!]\, \Delta_2\, \Delta_1\, \vec{\mathbf{H}}^*$. Then, the total number of pairs $(\sigma, F)$ counted in this case is

$$[t^r p_{\alpha-k} x_1^k/(n-1)!] t\, \Delta_1^2\, \vec{\mathbf{H}}^*,$$

which is the first term on the right-hand side of (2.1).

**Case 2:** Similarly, $(a_r\, b_r)$ is a cut with $b_r = n$ precisely when $F'$ is such that $a_r$ is in the same cycle of $\sigma_{r-1}$ as $n$, which must have length $j + k$ for some part size $j$ of $\alpha - k$. For a given value of $j$, the number of triples $(\sigma_{r-1}, F', a_r)$ satisfying these conditions is $[t^{r-1} p_{\alpha-j-k} x_1^{j+k}/(n-1)!]\, \Delta_1\, \vec{\mathbf{H}}^*$.

Then, the total number of pairs $(\sigma, F)$ counted in this case is

$$[t^r p_{\alpha-k} x_1^k/(n-1)!]\, t\, \Pi_2 \operatorname*{Split}_{1\to 2} \Delta_1 \vec{\mathbf{H}}^*,$$

which is the second term on the right-hand side of (2.1).

**Case 3:** By monotonicity, if neither of the previous cases applies, then there is no transposition in the factorization $F$ involving $n$. In this case, $n$ must be a fixed point of $\sigma$, so $k = 1$, and $F$ can be any monotone factorization of length $r$ of the restriction $\sigma'$ of $\sigma$ to the ground set $[n-1]$. By the definition of $\vec{\mathbf{H}}^*$, the number of pairs $(\sigma', F)$ in this case is $[t^r p_{\alpha-k}/(n-1)!]\vec{\mathbf{H}}^*$. This can be rewritten as

$$[t^r p_{\alpha-k} x_1^k/(n-1)!] x_1 \vec{\mathbf{H}}^*,$$

which is the third and final term on the right-hand side of (2.1).

The only remaining coefficients of (2.1) that are left to verify are those with $\alpha$ being the empty partition, but for these coefficients the equation is $0 = 0$, which is trivially satisfied. As for the initial condition, note that the only monotone factorization of the empty permutation in $\mathcal{S}_0$ is the empty factorization. □

*Remark* 2.3.2. Note that unlike Theorem 2.1.2, the existence of a solution to (2.1) is not entirely trivial from an algebraic point of view, since in general the partial differential equation imposes more than one constraint on each coefficient $[t^r p_\alpha]\vec{\mathbf{H}}^*$; for example, extracting the coefficient of $t^1 x_1^2 p_3$ and the coefficient of $t^1 x_1^3 p_2$ from (2.1) gives two different ways of computing the coefficient $[t^1 p_{32}]\vec{\mathbf{H}}^*$ in terms of coefficients of the form $[t^0 p_\alpha]\vec{\mathbf{H}}^*$, and these two ways must be consistent for existence to hold. Instead, the proof relies on the combinatorial interpretation of the coefficients of $\vec{\mathbf{H}}^*$.

In contrast, applying the projection operator $\Pi_1$ to (2.1) gives the slightly weaker differential equation

$$\sum_{i\geq 1} i p_i \frac{\partial}{\partial p_i} \vec{\mathbf{H}}^* = t \sum_{i,j\geq 1} \left( ij p_{i+j} \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} + (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} \right) \vec{\mathbf{H}}^* + p_1 \vec{\mathbf{H}}^*, \quad (2.2)$$

for which existence and uniqueness of a solution are both trivial to show algebraically; that is, each coefficient $[t^r p_\alpha]\vec{\mathbf{H}}^*$ is uniquely determined by (2.2) only once, so to speak.

While these two differential equations each uniquely determine $\vec{\mathbf{H}}^*$, it is unclear how to derive (2.1) from (2.2) without using the combinatorial interpretation of $\vec{\mathbf{H}}^*$.

### 2.3.2 Transitive Monotone Factorizations

From the differential equation of Theorem 2.3.2 for the generating function for all monotone factorizations, we can deduce a differential equation which characterizes the generating function for *transitive* monotone factorizations.

**Corollary 2.3.3.** *Let* $\vec{\mathsf{H}} \in \mathbb{Q}\mathcal{S}[[t]]$ *be the generating function for transitive monotone factorizations, where* $t$ *is an ordinary marker for the factorization lengths, so that* $[t^r\sigma]\vec{\mathsf{H}}$ *is the number of transitive monotone factorizations of the permutation* $\sigma$ *into* $r$ *transpositions. Let* $\vec{\mathbf{H}} = \mathrm{ch}(\vec{\mathsf{H}}) \in \mathbb{Q}[[\mathbf{p}, t]]$ *be its image under the characteristic map. Then,* $\vec{\mathbf{H}}$ *is uniquely determined by the partial differential equation*

$$\Delta_1 \vec{\mathbf{H}} = t(\Delta_1 \vec{\mathbf{H}})^2 + t\,\Delta_1^2 \vec{\mathbf{H}} + t\,\Pi_2 \operatorname*{Split}_{1\to 2} \Delta_1 \vec{\mathbf{H}} + x_1$$

*with initial condition* $[p_0]\vec{\mathbf{H}} = 0$.

*Proof.* This follows directly from Theorem 2.3.2 by substituting $\exp(\vec{\mathsf{H}})$ for $\vec{\mathsf{H}}^*$, as per Proposition 1.8.3, and using the chain rule. □

*Remark* 2.3.3. Note that this differential equation for $\vec{\mathbf{H}}$ can also be proved combinatorially. In this case, the first three terms on the right-hand side correspond, respectively, to the last transposition in a transitive monotone factorization being an essential join, a redundant join, and a cut; and the fourth term on the right-hand side corresponds to the only empty transitive monotone factorization, which is the empty factorization for the identity permutation in $\mathcal{S}_1$.

*Remark* 2.3.4. In fact, the statement of Corollary 2.3.3 can be subtly strengthened to the statement that $\mathbf{A} = \Delta_1 \vec{\mathbf{H}} \in \mathbb{Q}[[\mathbf{p}, x_1, t]]$ is the unique solution of the equation

$$\mathbf{A} = t\mathbf{A}^2 + t\,\Delta_1 \mathbf{A} + t\,\Pi_2 \operatorname*{Split}_{1\to 2} \mathbf{A} + x_1$$

with initial condition $[p_0 x^0]\mathbf{A} = 0$. The fact that $\mathbf{A}$ is a solution follows directly from Corollary 2.3.3, but showing uniqueness requires a bit more work, since the map $\Delta_1$ is not surjective.

### 2.3.3 Breakdown by Transitive Genus

As with Hurwitz numbers (see Section 2.1.3), it will be useful to group the coefficients of the monotone Hurwitz generating function according to their genus.

**Definition 2.3.4.** The **genus $g$ generating function** for monotone Hurwitz numbers is

$$\vec{\mathbf{H}}_g = \sum_{n\geq 1} \sum_{\alpha \vdash n} \vec{H}_g(\alpha) \frac{p_\alpha}{n!}.$$

Again, we can unpack the differential equation from Corollary 2.3.3 for $\vec{\mathbf{H}}$ into a separate differential equation for each $\vec{\mathbf{H}}_g$.

**Theorem 2.3.5.**

1. *The generating function* $\Delta_1 \vec{\mathbf{H}}_0 \in \mathbb{Q}[[\mathbf{p}, x_1]]$ *is uniquely determined by the partial differential equation*

$$\Delta_1 \vec{\mathbf{H}}_0 - \Pi_2 \operatorname*{Split}_{1\to 2} \Delta_1 \vec{\mathbf{H}}_0 - (\Delta_1 \vec{\mathbf{H}}_0)^2 - x_1 = 0 \tag{2.3}$$

*with initial condition* $[p_0 x_1^0]\, \Delta_1 \vec{\mathbf{H}}_0 = 0$.

2. *For* $g \geq 1$, *the generating function* $\Delta_1 \vec{\mathbf{H}}_g \in \mathbb{Q}[[\mathbf{p}, x_1]]$ *is uniquely determined by* $\Delta_1 \vec{\mathbf{H}}_0, \Delta_1 \vec{\mathbf{H}}_1, \ldots, \Delta_1 \vec{\mathbf{H}}_{g-1}$ *and by the partial differential equation*

$$\left(1 - 2\,\Delta_1 \vec{\mathbf{H}}_0 - \Pi_2 \operatorname*{Split}_{1\to 2}\right) \Delta_1 \vec{\mathbf{H}}_g = \Delta_1^2 \vec{\mathbf{H}}_{g-1} + \sum_{g'=1}^{g-1} \Delta_1 \vec{\mathbf{H}}_{g'} \cdot \Delta_1 \vec{\mathbf{H}}_{g-g'} \tag{2.4}$$

*with initial condition* $[p_0 x_1^0]\, \Delta_1 \vec{\mathbf{H}}_g = 0$.

3. *For* $g \geq 0$, *the generating function* $\vec{\mathbf{H}}_g$ *is uniquely determined by the generating function* $\Delta_1 \vec{\mathbf{H}}_g$ *and the fact that* $[p_0]\vec{\mathbf{H}}_g = 0$.

*Proof.* Let $\mathbf{A} = \sum_{g\geq 0} \vec{\mathbf{H}}_g u^g$, so that as a generating function, $\mathbf{A}$ has the same coefficients as $\vec{\mathbf{H}}$, but has an ordinary marker $u$ for the genus $g$ instead of an ordinary marker $t$ for the number of transpositions $r$. Since $r = n + \ell(\alpha) + 2g - 2$, by comparison with Corollary 2.3.3, it follows that $\mathbf{A} \in \mathbb{Q}[[\mathbf{p}, u]]$ is uniquely determined by the partial differential equation

$$\Delta_1 \mathbf{A} = (\Delta_1 \mathbf{A})^2 + u\,\Delta_1^2 \mathbf{A} + \Pi_2 \operatorname*{Split}_{1\to 2} \Delta_1 \mathbf{A} + x_1$$

with initial condition $[p_0]\mathbf{A} = 0$. As noted in Remark 2.3.4, this differential equation actually uniquely determines $\Delta_1 \mathbf{A} \in \mathbb{Q}[[\mathbf{p}, x_1, u]]$. Extracting the coefficient of $u^g$ from this equation for $g = 0$ and $g \geq 1$ and reorganizing terms gives the stated equations for $\Delta_1 \vec{\mathbf{H}}_0$ and $\Delta_1 \vec{\mathbf{H}}_g$, $g \geq 1$. Finally, note that given $\Delta_1 \vec{\mathbf{H}}_g$, we can compute

$$\Pi_1 \Delta_1 \vec{\mathbf{H}}_g = \sum_{n\geq 1} \sum_{\alpha \vdash n} \vec{H}_g(\alpha) \frac{p_\alpha}{(n-1)!},$$

which uniquely determines every coefficient of $\vec{\mathbf{H}}_g$ except for the constant term. $\square$

# Chapter 3

# Monotone Hurwitz Generating Functions

## 3.1 Results

In this chapter, our goal is to state and prove explicit formulas for the genus zero and genus one generating functions for monotone Hurwitz numbers of Section 2.3.3, and a general form for the higher genus generating functions. These formulas, given in terms of some power series $\gamma, \eta, \eta_1, \eta_2, \ldots \in \mathbb{Q}[[\mathbf{p}]]$ defined in Section 3.2.2 except for genus zero, are as follows.

**Theorem 3.1.1.** *The genus zero generating function for monotone Hurwitz numbers is given by*

$$\vec{\mathbf{H}}_0 = \sum_{n \geq 1} \sum_{\alpha \vdash n} \frac{p_\alpha}{|\operatorname{aut} \alpha|} (2n+1)^{\overline{\ell(\alpha)-3}} \prod_{j=1}^{\ell(\alpha)} \binom{2\alpha_j}{\alpha_j},$$

*where*

$$(2n+1)^{\overline{k}} = (2n+1)(2n+2)\cdots(2n+k)$$

*denotes a rising product with $k$ factors, and by convention*

$$(2n+1)^{\overline{k}} = \frac{1}{(2n+k+1)^{\overline{-k}}}$$

*for $k < 0$.*

*Remark* 3.1.1. We initially conjectured this formula for genus zero monotone Hurwitz numbers after generating extensive numerical data, using the group algebra approach described in Chapter 1, together with the character theory and generating function capabilities of Sage [44]. In particular, the case where $\alpha$ has $\ell(\alpha) = 3$ parts was very suggestive, since the formula then breaks down into a product of three terms. This was also a first indication of the striking similarities between monotone Hurwitz numbers and classical Hurwitz numbers.

**Theorem 3.1.2.** *Let $\gamma, \eta, \eta_1, \eta_2, \dots \in \mathbb{Q}[[\mathbf{p}]]$ be as defined in Section 3.2.2. Then, the genus one generating function for monotone Hurwitz numbers is given by*

$$\vec{\mathbf{H}}_1 = \tfrac{1}{24} \log \frac{1}{1-\eta} - \tfrac{1}{8} \log \frac{1}{1-\gamma}.$$

**Theorem 3.1.3.** *Let $\gamma, \eta, \eta_1, \eta_2, \dots \in \mathbb{Q}[[\mathbf{p}]]$ be as defined in Section 3.2.2. Then, for $g \geq 2$, the genus $g$ generating function for monotone Hurwitz numbers has the form*

$$\vec{\mathbf{H}}_g = -c_{g,(0)} + \sum_{d=0}^{3g-3} \sum_{\alpha \vdash d} \frac{c_{g,\alpha}\, \eta_\alpha}{(1-\eta)^{2g-2+\ell(\alpha)}},$$

*where the constants $c_{g,\alpha}$ are rational numbers.*

These formulas are all obtained starting from the genus-wise monotone join-cut equations of Theorem 2.3.5. In the case of genus zero, the overall approach is one of guessing (the explicit formula) and checking (that it satisfies the join-cut equation (2.3)), and forms Section 3.3. For $g \geq 1$, the overall approach is to invert the differential operator on the left-hand side of the join-cut equation (2.4) as a linear operator and to show that its inverse preserves the appropriate subspaces, which is done in Section 3.4. In both cases, the technical details rely crucially on an implicit change of variables introduced in Section 3.2.

## 3.2 Algebraic Framework

In this section, we introduce the algebraic framework surrounding our solution of the monotone join-cut equations. This consists of a change of variables, some auxiliary series, and descriptions of the operators from Section 2.2 in terms of these new variables and series.

### 3.2.1 Change of Variables

In working with the monotone Hurwitz generating functions, it is convenient to change variables from $\mathbf{p} = (p_1, p_2, \dots)$ to $\mathbf{q} = (q_1, q_2, \dots)$, where

$$q_j = p_j \left( 1 - \sum_{k \geq 1} \binom{2k}{k} q_k \right)^{-2j}, \qquad j \geq 1. \tag{3.1}$$

This change of variables is invertible, and can be carried out using the Lagrange Implicit Function Theorem in many variables (see [25]). In order to work consistently in the transformed variables $\mathbf{q}$, it will be useful to have descriptions of the lifting, projection and splitting operators in terms of $\mathbf{q}$, and when considering these operators, the change of variables from $\mathbf{p}$ to $\mathbf{q}$ also corresponds to a change of variables from $\mathbf{x} = (x_1, x_2, \dots)$ to $\mathbf{y} = (y_1, y_2, \dots)$, defined by

$$y_i = x_i \left(1 - \sum_{k\geq 1} \binom{2k}{k} q_k\right)^{-2}, \qquad i \geq 1, \tag{3.2}$$

and this is also invertible. Note that like the indeterminates $\mathbf{x} = (x_1, x_2, \dots)$, the indeterminates $\mathbf{y} = (y_1, y_2, \dots)$ are usually interchangeable, so we will generally state results involving them using explicit numeric indices.

We can express the indeterminates $\mathbf{p}$ and $\mathbf{q}$ as formal power series in terms of each other using (3.1), so we can identify the rings $\mathbb{Q}[[\mathbf{p}]]$ and $\mathbb{Q}[[\mathbf{q}]]$. Using (3.2), we can further identify the rings $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$ and $\mathbb{Q}[[\mathbf{q}, \mathbf{y}]]$.

### 3.2.2 Auxiliary Series

The power series $\gamma$, $\eta$ and $\eta_j$, $j \geq 1$ which appear in the statement of Theorem 3.1.2 and Theorem 3.1.3, are defined by

$$\begin{aligned}
\gamma &= \sum_{k\geq 1} \binom{2k}{k} q_k, \\
\eta &= \sum_{k\geq 1} (2k+1) \binom{2k}{k} q_k, \\
\eta_j &= \sum_{k\geq 1} (2k+1) k^j \binom{2k}{k} q_k, \qquad j \geq 1.
\end{aligned}$$

In particular, $\gamma$ is defined so that the change of variables between $\mathbf{p}$ and $\mathbf{q}$ becomes $q_j = p_j(1-\gamma)^{-2j}$ and the change of variables between $\mathbf{x}$ and $\mathbf{y}$ becomes $y_i = x_i(1-\gamma)^{-2}$.

It will occasionally be useful to have parallel versions of these power series where $q_k$ is replaced by $y_1^k$, so we also define the power series

$$\begin{aligned}
\gamma(y_1) &= \sum_{k\geq 1} \binom{2k}{k} y_1^k = (1-4y_1)^{-\frac{1}{2}} - 1, \\
\eta(y_1) &= \sum_{k\geq 1} (2k+1) \binom{2k}{k} y_1^k = (1-4y_1)^{-\frac{3}{2}} - 1, \\
\eta_j(y_1) &= \sum_{k\geq 1} (2k+1) k^j \binom{2k}{k} y_1^k = \left(y_1 \frac{\partial}{\partial y_1}\right)^j (1-4y_1)^{-\frac{3}{2}}, \qquad j \geq 1.
\end{aligned}$$

Note that these series can all be expressed as polynomials in the quantity $(1-4y_1)^{-\frac{1}{2}}$.

### 3.2.3 Differential Operators and Lifting Operators

For $k \geq 1$, consider the differential operators

$$\mathrm{D}_k = p_k \frac{\partial}{\partial p_k}, \qquad \mathcal{D} = \sum_{k \geq 1} k p_k \frac{\partial}{\partial p_k}, \qquad \mathrm{E}_k = q_k \frac{\partial}{\partial q_k}, \qquad \mathcal{E} = \sum_{k \geq 1} k q_k \frac{\partial}{\partial q_k},$$

defined on $\mathbb{Q}[[\mathbf{p}]] = \mathbb{Q}[[\mathbf{q}]]$, then extended to $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]] = \mathbb{Q}[[\mathbf{q}, \mathbf{y}]]$ by $\mathbb{Q}[[\mathbf{x}]]$-linearity (and *not* by $\mathbb{Q}[[\mathbf{y}]]$-linearity).

As $\mathbb{Q}$-linear operators, the operators $\mathrm{D}_1, \mathrm{D}_2, \ldots$ and $\mathcal{D}$ have the set $\{p_\alpha \colon \alpha \vdash n,\, n \geq 0\}$ as an eigenbasis, and consequently they commute with each other. Similarly, the operators $\mathrm{E}_1, \mathrm{E}_2, \ldots$ and $\mathcal{E}$ have the set $\{q_\alpha \colon \alpha \vdash n,\, n \geq 0\}$ as an eigenbasis and commute with each other. However, these two families of operators don't commute with each other. By using the defining relations (3.1) to compute the action of $\mathrm{E}_k$ on $p_j$, the operator identity

$$\mathrm{E}_k = \mathrm{D}_k - \frac{2q_k}{1-\gamma}\binom{2k}{k}\mathcal{D}, \qquad k \geq 1 \tag{3.3}$$

can be verified. It follows that

$$\mathcal{E} = \frac{1-\eta}{1-\gamma}\mathcal{D},$$

and we can deduce that

$$\mathrm{D}_k = \mathrm{E}_k + \frac{2q_k}{1-\eta}\binom{2k}{k}\mathcal{E}, \qquad k \geq 1. \tag{3.4}$$

Thus, we can write each family of differential operators in terms of the other. Using this fact, together with (3.1) and the $\mathbb{Q}[[\mathbf{x}]]$-linearity of these differential operators, we can compute the action of the lifting operator $\Delta_1$ on $\mathbf{q}$ and $\mathbf{y}$, obtaining

$$\Delta_1(q_k) = k y_1^k + k q_k \frac{4y_1}{(1-4y_1)^{\frac{3}{2}}(1-\eta)}, \quad \Delta_1(y_k) = y_k \frac{4y_1}{(1-4y_1)^{\frac{3}{2}}(1-\eta)}, \quad k \geq 1.$$

It follows that we can express the lifting operators as

$$\Delta_1 = \sum_{k \geq 1}\left(k y_1^k \frac{\partial}{\partial q_k}\right) + \frac{4y_1}{(1-4y_1)^{\frac{3}{2}}(1-\eta)} \sum_{k \geq 1}\left(k q_k \frac{\partial}{\partial q_k} + y_k \frac{\partial}{\partial y_k}\right). \tag{3.5}$$

### 3.2.4 Projection, Splitting, and Coefficient Extraction Operators

Unlike the lifting operators, which are $\mathbb{Q}[[\mathbf{x}]]$-linear, the projection and splitting operators are $\mathbb{Q}[[\mathbf{p}]]$-linear (or equivalently, $\mathbb{Q}[[\mathbf{q}]]$-linear), which makes it easier to express them in terms of $\mathbf{q}$ and $\mathbf{y}$. Using (3.2), we immediately have

$$[y_1^k]F(x_1) = (1-\gamma)^{2k}[x_1^k]F(x_1), \qquad k \geq 1,$$

and it follows that[1]

$$\Pi_1 = [x_1^0] + \sum_{k\geq 1} p_k[x_1^k] = [y_1^0] + \sum_{k\geq 1} q_k[y_1^k],$$

$$\operatorname*{Split}_{1\to 2} F(x_1) = \frac{x_2F(x_1) - x_1F(x_2)}{x_1 - x_2} + F(0) = \frac{y_2F(x_1) - y_1F(x_2)}{y_1 - y_2} + F(0),$$

where $F(x_1) \in \mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$ is an arbitrary element, considered as a power series in $x_1$.

Like $[x_1]$ and $[y_1]$, the coefficient extraction operators $[p_\alpha]$ and $[q_\alpha]$ defined on $\mathbb{Q}[[\mathbf{p}]] = \mathbb{Q}[[\mathbf{q}]]$ can be expressed in terms of each other, but this is not quite as straightforward. Using a multivariate version of the Lagrange Implicit Function Theorem, we obtain the following.

**Lemma 3.2.1.** *If $\alpha \vdash n$ is a partition and $F$ is an element of $\mathbb{Q}[[\mathbf{p}]]$, or equivalently of $\mathbb{Q}[[\mathbf{q}]]$, then*

$$[p_\alpha]F = [q_\alpha]\frac{(1-\eta)F}{(1-\gamma)^{2n+1}}.$$

*Proof.* Let $\phi_j = (1-\gamma)^{-2j}$, so that (3.1) becomes $q_j = p_j\phi_j$, $j \geq 1$. Then, from the multivariate Lagrange Implicit Function Theorem [25, Theorem 1.2.9], we have

$$\begin{aligned} [p_\alpha]F &= [q_\alpha]F\,\phi_\alpha \det\left(\delta_{ij} - q_j\frac{\partial}{\partial q_j}\log\phi_i\right)_{i,j\geq 1} \\ &= [q_\alpha]F\,\phi_\alpha \det\left(\delta_{ij} - \frac{2iq_j}{1-\gamma}\binom{2j}{j}\right)_{i,j\geq 1}, \end{aligned}$$

where $\phi_\alpha = \prod_j \phi_{\alpha_j}$. We have $\phi_\alpha = (1-\gamma)^{-2n}$, and using the fact that $\det(I + M) = 1 + \operatorname{trace}(M)$ for any matrix $M$ of rank zero or one, we can evaluate the determinant as

$$\det\left(\delta_{ij} - q_j\frac{\partial}{\partial q_j}\log\phi_i\right)_{i,j\geq 1} = 1 - \sum_{k\geq 1}\frac{2kq_k}{1-\gamma}\binom{2k}{k} = \frac{1-\eta}{1-\gamma}.$$

Substituting, we obtain

$$[p_\alpha]F = [q_\alpha]\frac{(1-\eta)F}{(1-\gamma)^{2n+1}}.$$ □

[1] Note that $y_2F(x_1)$ and $y_1F(x_2)$ are not typos; although we have the identification $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]] = \mathbb{Q}[[\mathbf{q}, \mathbf{y}]]$, the expansion of $F(x_1)$ as a power series in $y_1$ is not equal to $F(y_1)$.

## 3.3 Genus Zero

In this section, we prove Theorem 3.1.1, which gives an explicit formula for the genus zero generating function $\vec{\mathbf{H}}_0$ for monotone Hurwitz numbers. Our strategy is to define the series $\mathbf{F}$ by

$$\mathbf{F} = \sum_{n\geq 1}\sum_{\alpha\vdash n} \frac{p_\alpha}{|\mathrm{aut}\,\alpha|}(2n+1)^{\overline{\ell(\alpha)-3}} \prod_{j=1}^{\ell(\alpha)} \binom{2\alpha_j}{\alpha_j},$$

and then to show that it satisfies the genus zero monotone join-cut equation (2.3). The main difficulty lies in finding a closed form for $\mathbf{F}$, or rather for $\Delta_1\,\mathbf{F}$. The first step is to find a closed form for a related series, as follows.

**Theorem 3.3.1.** *We have*

$$(2\mathcal{D}-2)(2\mathcal{D}-1)(2\mathcal{D})\mathbf{F} = \frac{(1-\gamma)^3}{1-\eta} - 1.$$

*Proof.* From the definition of $\mathbf{F}$, for any $\alpha \vdash n$ with $n \geq 1$, we have

$$\begin{aligned}[p_\alpha](2\mathcal{D}-2)(2\mathcal{D}-1)(2\mathcal{D})\mathbf{F} &= \frac{1}{|\mathrm{aut}(\alpha)|}(2n-2)^{\overline{\ell(\alpha)}} \prod_{j=1}^{\ell(\alpha)} \binom{2\alpha_j}{\alpha_j} \\ &= \frac{(-1)^{\ell}\ell(\alpha)!}{|\mathrm{aut}(\alpha)|}\binom{2-2n}{\ell(\alpha)} \prod_{j=1}^{\ell(\alpha)} \binom{2\alpha_j}{\alpha_j},\end{aligned}$$

and we conclude that

$$[p_\alpha](2\mathcal{D}-2)(2\mathcal{D}-1)(2\mathcal{D})\mathbf{F} = [q_\alpha](1-\gamma)^{2-2n}.$$

Applying Lemma 3.2.1 to translate between between the coefficient extraction operators $[q_\alpha]$ and $[p_\alpha]$, it we obtain

$$[p_\alpha](2\mathcal{D}-2)(2\mathcal{D}-1)(2\mathcal{D})\mathbf{F} = [p_\alpha]\frac{(1-\gamma)^3}{1-\eta}.$$

This holds for $\alpha \vdash n$ and $n \geq 1$, and the result follows after computing the constant term separately. □

The second step is to peel off the operators $(2\mathcal{D}-2)$, $(2\mathcal{D}-1)$ and $(2\mathcal{D})$ to get closer to a closed form for $\Delta_1\,\mathbf{F}$, as follows.

**Theorem 3.3.2.** *For $k \geq 1$, we have*

$$\mathrm{D}_k\mathbf{F} = \frac{1}{2k(2k-1)}\binom{2k}{k}q_k - \sum_{j\geq 1}\frac{2j+1}{2(j+k)(2k-1)}\binom{2j}{j}\binom{2k}{k}q_j q_k.$$

*Proof.* As notation local to this proof, let

$$\mathbf{F}''' = (2\mathcal{D}-2)(2\mathcal{D}-1)(2\mathcal{D})\mathbf{F}, \quad \mathbf{F}'' = (2\mathcal{D}-1)(2\mathcal{D})\mathbf{F}, \quad \mathbf{F}' = (2\mathcal{D})\mathbf{F}.$$

To prove the result, we use the operator identity

$$(1-\gamma)^i(2\mathcal{E}-i)\Big((1-\gamma)^{-i}G\Big) = \frac{1-\eta}{1-\gamma}(2\mathcal{D}-i)(G), \tag{3.6}$$

which holds for any integer $i$ and any formal power series $G$, as can be checked by using the product rule and the fact that $\mathcal{E}(\gamma) = (\eta-1)/2$. This allows us to express the differential operators $(2\mathcal{D}-2)$, $(2\mathcal{D}-1)$ and $(2\mathcal{D})$ in terms of the operators $(2\mathcal{E}-2)$, $(2\mathcal{E}-1)$ and $(2\mathcal{E})$, which we can invert by recalling that they have $\{q_\alpha \colon \alpha \vdash n,\, n \geq 0\}$ as an eigenbasis.

We proceed in stages. First we invert $(2\mathcal{D}-2)$ by applying (3.6) with $i=2$ to Theorem 3.3.1, obtaining

$$\begin{aligned}
\mathbf{F}'' &= (2\mathcal{D}-2)^{-1}(\mathbf{F}''') \\
&= \frac{1}{2} + (2\mathcal{D}-2)^{-1}\left(\frac{(1-\gamma)^3}{1-\eta}\right) \\
&= \tfrac{1}{2} + (1-\gamma)^2(2\mathcal{E}-2)^{-1}(1) \\
&= \tfrac{1}{2} - \tfrac{1}{2}(1-\gamma)^2,
\end{aligned}$$

after checking separately that $[p_1]\mathbf{F}'' = 2$. (This needs to be checked because the kernel of $(2\mathcal{D}-2)$ is spanned by $p_1$.)

Next we apply $\mathrm{D}_k$ to $\mathbf{F}''$ via (3.4). This is straightforward, and gives

$$\mathrm{D}_k\mathbf{F}'' = \frac{(1-\gamma)^2}{1-\eta}\binom{2k}{k}q_k.$$

Now we invert $(2\mathcal{D}-1)$ by applying (3.6) with $i=1$, which gives

$$\begin{aligned}
\mathrm{D}_k\mathbf{F}' &= (2\mathcal{D}-1)^{-1}(\mathrm{D}_k\mathbf{F}'') \\
&= (1-\gamma)(2\mathcal{E}-1)^{-1}\left(\binom{2k}{k}q_k\right) \\
&= (1-\gamma)\frac{1}{2k-1}\binom{2k}{k}q_k.
\end{aligned}$$

Finally, we invert $(2\mathcal{D})$ by applying (3.6) with $i=0$, giving

$$\begin{aligned}
\mathrm{D}_k\mathbf{F} &= (2\mathcal{D})^{-1}(\mathrm{D}_k\mathbf{F}') \\
&= (2\mathcal{E})^{-1}\left((1-\eta)\frac{1}{2k-1}\binom{2k}{k}q_k\right)
\end{aligned}$$

$$= (2\mathcal{E})^{-1}\left(\frac{1}{2k-1}\binom{2k}{k}q_k - \sum_{j\geq 1}\frac{2j+1}{2k-1}\binom{2j}{j}\binom{2k}{k}q_jq_k\right)$$
$$= \frac{1}{2k(2k-1)}\binom{2k}{k}q_k - \sum_{j\geq 1}\frac{2j+1}{2(j+k)(2k-1)}\binom{2j}{j}\binom{2k}{k}q_jq_k.$$

Again, the constant term needs to be checked separately, since the kernel of $(2\mathcal{D})$ consists of the constants, but clearly $\mathrm{D}_k\mathbf{F}$ has no constant term. □

We are now able to evaluate $\Delta_1\,\mathbf{F}$ as a messy but closed form in terms of $\mathbf{q}$ and $\mathbf{y}$.

**Corollary 3.3.3.** *We have*

$$\Delta_1\,\mathbf{F} = \Pi_2\left(1-\sqrt{1-4y_1} - \frac{y_1}{2(y_1-y_2)}\left(1-\sqrt{\frac{1-4y_1}{1-4y_2}}\right)\right).$$

*Proof.* From Theorem 3.3.2, we have

$$\begin{aligned}\Delta_1\,\mathbf{F} &= \sum_{k\geq 1}\frac{ky_1^k}{q_k}\mathrm{D}_k\mathbf{F}\\ &= \sum_{k\geq 1}\frac{1}{2(2k-1)}\binom{2k}{k}y_1^k - \sum_{j,k\geq 1}\frac{(2j+1)k}{2(j+k)(2k-1)}\binom{2j}{j}\binom{2k}{k}y_1^kq_j\\ &= \Pi_2\left(2G(y_1,0)-G(y_1,y_2)\right),\end{aligned}$$

where the power series $G(y_1,y_2)$ is defined by

$$G(y_1,y_2) = \sum_{j\geq 0}\sum_{k\geq 1}\frac{(2j+1)k}{2(j+k)(2k-1)}\binom{2j}{j}\binom{2k}{k}y_1^ky_2^j.$$

Then, the computation

$$\begin{aligned}G(y_1,y_2) &= \int_0^1 y_1t(1-4y_1t)^{-\frac{1}{2}}(1-4y_2t)^{-\frac{3}{2}}\frac{\mathrm{d}t}{t}\\ &= \left[\frac{-y_1}{2(y_1-y_2)}(1-4y_1t)^{\frac{1}{2}}(1-4y_2t)^{-\frac{1}{2}}\right]_{t=0}^{1}\\ &= \frac{y_1}{2(y_1-y_2)}\left(1-\sqrt{\frac{1-4y_1}{1-4y_2}}\right)\end{aligned}$$

completes the proof. □

Finally, using the above closed form for $\Delta_1 \mathbf{F}$, we can verify that $\mathbf{F}$ satisfies the genus zero join-cut equation for monotone Hurwitz numbers and conclude that $\vec{\mathbf{H}}_0 = \mathbf{F}$, which completes the proof of Theorem 3.1.1.

**Theorem 3.3.4.** *The series* $\mathbf{F}$ *satisfies the genus zero monotone join-cut equation* (2.3) *of Theorem 2.3.5,*

$$\Delta_1 \mathbf{F} - \Pi_2 \operatorname*{Split}_{1\to 2} \Delta_1 \mathbf{F} - (\Delta_1 \mathbf{F})^2 - x_1 = 0.$$

*Thus,* $\mathbf{F}$ *is the genus zero generating function* $\vec{\mathbf{H}}_0$ *for monotone single Hurwitz numbers.*

*Proof.* Corollary 3.3.3 gives $\Delta_1 \mathbf{F} = \Pi_2 \, A(y_1, y_2)$, where

$$A(y_1, y_2) = 1 - \sqrt{1 - 4y_1} - \frac{y_1}{2(y_1 - y_2)} \left( 1 - \sqrt{\frac{1 - 4y_1}{1 - 4y_2}} \right).$$

We can rewrite each of the terms in the join-cut equation as

$$\begin{aligned} \Delta_1 \mathbf{F} &= \Pi_2 \, \Pi_3 \left( A(y_1, y_2) \right), \\ \Pi_2 \operatorname*{Split}_{1\to 2} \Delta_1 \mathbf{F} &= \Pi_2 \, \Pi_3 \left( \frac{y_2 A(y_1, y_3) - y_1 A(y_2, y_3)}{y_1 - y_2} \right), \\ (\Delta_1 \mathbf{F})^2 &= \Pi_2 \, \Pi_3 \left( A(y_1, y_2) A(y_1, y_3) \right), \\ x_1 = y_1 (1 - \gamma)^2 &= \Pi_2 \, \Pi_3 \left( y_1 \left( 2 - \frac{1}{\sqrt{1 - 4y_2}} \right) \left( 2 - \frac{1}{\sqrt{1 - 4y_3}} \right) \right) \end{aligned}$$

to get an expression of the form

$$\Pi_2 \, \Pi_3 \; B(y_1, y_2, y_3).$$

This series $B(y_1, y_2, y_3)$ itself is not zero, but a straightforward computation shows that the series

$$\tfrac{1}{2} B(y_1, y_2, y_3) + \tfrac{1}{2} B(y_1, y_3, y_2),$$

obtained by symmetrizing with respect to $y_2$ and $y_3$, is zero. Thus we have

$$\Pi_2 \, \Pi_3 \; B(y_1, y_2, y_3) = \Pi_2 \, \Pi_3 \left( \tfrac{1}{2} B(y_1, y_2, y_3) + \tfrac{1}{2} B(y_1, y_3, y_2) \right) = 0,$$

which completes the verification. ∎

## 3.4 Higher Genera

In this section, we prove Theorems 3.1.2 and 3.1.3, which give an explicit formula for the genus one generating function $\vec{\mathbf{H}}_1$ for monotone Hurwitz numbers, and a general form for the higher genus generating functions $\vec{\mathbf{H}}_g$, $g \geq 2$. Our strategy is to use the expression

$$\Delta_1 \vec{\mathbf{H}}_0 = \Pi_2 \left(1 - \sqrt{1-4y_1} - \frac{y_1}{2(y_1 - y_2)} \left(1 - \sqrt{\frac{1-4y_1}{1-4y_2}}\right)\right)$$

from Corollary 3.3.3 in the join-cut equation (2.4) for higher genera from Theorem 2.3.5, that is,

$$\left(1 - 2\,\Delta_1 \vec{\mathbf{H}}_0 - \Pi_2 \operatorname*{Split}_{1\to 2}\right) \Delta_1 \vec{\mathbf{H}}_g = \Delta_1^2 \vec{\mathbf{H}}_{g-1} + \sum_{g'=1}^{g-1} \Delta_1 \vec{\mathbf{H}}_{g'}\, \Delta_1 \vec{\mathbf{H}}_{g-g'},$$

to compute an inverse for the $\mathbb{Q}$-linear operator

$$\left(1 - 2\,\Delta_1 \vec{\mathbf{H}}_0 - \Pi_2 \operatorname*{Split}_{1\to 2}\right)$$

which appears on the left-hand side. Then, $\Delta_1 \vec{\mathbf{H}}_g$ can be computed for $g \geq 1$, and $\vec{\mathbf{H}}_g$ can be recovered.

*Remark* 3.4.1. While Theorem 3.1.3 is stated as an existential result, the proof essentially gives an algorithm to compute the rational constants $c_{g,\alpha}$ which appear as the coefficients of $\vec{\mathbf{H}}_g$ for $g \geq 2$. Provided that the rational constants for lower genera are known, the computation is quite feasible in practice. Appendix A lists the values for $g = 2, 3, 4, 5$.

### 3.4.1 Rings and Generators

The proof of Theorem 3.1.2 that we give in this section mostly involves direct computation of various algebraic expressions while keeping track of their general form. To do this, we focus on a few subrings of $\mathbb{Q}[[\mathbf{q}, \mathbf{y}]]$, illustrated in Figure 3.1.[2] In addition, we also keep track of a certain notion of degree, defined as follows.

**Definition 3.4.1.** Let $F$ be an element of $\mathbb{Q}[(1-4y_1)^{-1}; \eta_k(1-\eta)^{-1}]_{k\geq 1}$, that is, a polynomial in the quantities $(1-4y_1)^{-1}$ and $\eta_1(1-\eta)^{-1}, \eta_2(1-\eta)^{-1}, \ldots$. Then, its **weighted degree** $\deg F$ is its degree as a polynomial in these quantities, where $(1-4y_1)^{-1}$ has degree 1, and $\eta_k(1-\eta)^{-1}$ has degree $k$.

[2] Note that we use the subscript "$k \geq 1$" on some of these rings to indicate families of generators, so that, *e.g.*, the ring $\mathbb{Q}[(1-\eta)^{-1}; \eta_k]_{k\geq 1}$ is the set of polynomials in the quantities $(1-\eta)^{-1}$ and $\eta_1, \eta_2, \ldots$.

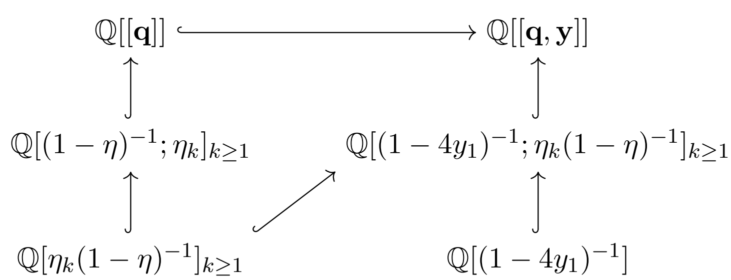



Figure 3.1: The relationship between some rings appearing in this section.

With this notion of degree, Theorem 3.1.2 can be restated as follows: if $g \geq 2$, then the quantity

$$\frac{\vec{\mathbf{H}}_g + c_{g,(0)}}{(1-\eta)^{2-2g}}$$

is an element of $\mathbb{Q}[\eta_k(1-\eta)^{-1}]_{k\geq 1}$ of weighted degree at most $3g-3$.

Also, as can be seen from the following computational lemma, the lifting operator $\Delta_1$ plays well with the quantities $(1-4y_1)^{-1}$, $(1-\eta)^{-1}$, $\eta_1$, $\eta_2$, …, and with the notion of weighted degree.

**Lemma 3.4.2.** *We have*

$$\begin{aligned}
\Delta_1 y_k &= y_k(\eta(y_1)-\gamma(y_1))(1-\eta)^{-1}, \qquad k \geq 1,\\
\Delta_1 \gamma &= \tfrac{1}{2}(\eta(y_1)-\gamma(y_1))(1-\gamma)(1-\eta)^{-1},\\
\Delta_1 \eta &= \eta_1(y_1) + (\eta(y_1)-\gamma(y_1))\eta_1(1-\eta)^{-1},\\
\Delta_1 \eta_k &= \eta_{k+1}(y_1) + (\eta(y_1)-\gamma(y_1))\eta_{k+1}(1-\eta)^{-1}, \qquad k \geq 1.
\end{aligned}$$

*Proof.* The first equation is a special case of (3.5), which we record here again for convenience. The other three equations can be obtained by applying Lemma 2.2.4 to the equations $\gamma = \Pi_2\,\gamma(y_2)$, $\eta = \Pi_2\,\eta(y_2)$, and $\eta_k = \Pi_2\,\eta_k(y_2)$, after noting that the operators $x_2\frac{\partial}{\partial x_2}$ and $y_2\frac{\partial}{\partial y_2}$ are equal. □

### 3.4.2 Inverting the Left-Hand Side Operator

**Theorem 3.4.3.** *If $g \geq 1$ and $\vec{\mathbf{H}}_g$ is the genus $g$ generating function for monotone Hurwitz numbers, then the quantity*

$$\frac{\Delta_1\,\vec{\mathbf{H}}_g}{(1-\eta)^{1-2g}(1-4y_1)^{-\frac{1}{2}}}$$

*is an element of $\mathbb{Q}[(1-4y_1)^{-1};\eta_k(1-\eta)^{-1}]_{k\geq 1}$ of weighted degree at most $3g-1$.*

*Proof.* We proceed by induction on $g$, using the higher genus monotone join-cut equation (2.4) to solve for $\Delta_1 \vec{\mathbf{H}}_g$ in terms of $\Delta_1 \vec{\mathbf{H}}_0, \Delta_1 \vec{\mathbf{H}}_1, \ldots, \Delta_1 \vec{\mathbf{H}}_{g-1}$. The proof involves several subclaims with independent proofs; they are organized below as separate propositions.

**Proposition 3.4.4.** *For $g \geq 1$, the quantity*

$$\frac{\Delta_1^2 \vec{\mathbf{H}}_{g-1} + \sum_{g'=1}^{g-1} \Delta_1 \vec{\mathbf{H}}_{g'}\ \Delta_1 \vec{\mathbf{H}}_{g-g'}}{(1-\eta)^{2-2g}}$$

*is an element of $\mathbb{Q}[(1-4y_1)^{-1}; \eta_k(1-\eta)^{-1}]_{k\geq 1}$ of weighted degree at most $3g-1$.*

*Proof.* This is where the induction hypothesis is used. For the base case of $g = 1$, we need to compute $\Delta_1^2 \vec{\mathbf{H}}_0$. Recall from Corollary 3.3.3 that

$$\Delta_1 \vec{\mathbf{H}}_0 = \Pi_2 \left(1 - \sqrt{1-4y_1} - \frac{y_1}{2(y_1 - y_2)} \left(1 - \sqrt{\frac{1-4y_1}{1-4y_2}}\right)\right)$$

Using Lemma 2.2.4 and Lemma 3.4.2, together with the fact that $\Pi_2(1-4y_2)^{-\frac{3}{2}} = 1 + \eta$, we obtain

$$\Delta_1^2 \vec{\mathbf{H}}_0 = y_1^2(1-4y_1)^{-2} \tag{3.7}$$

after a tedious but straightforward algebraic computation. For the case of $g \geq 2$, it follows from the induction hypothesis and Lemma 3.4.2 that

$$\frac{\Delta_1^2 \vec{\mathbf{H}}_{g-1}}{(1-\eta)^{2-2g}} + \frac{\sum_{g'=1}^{g-1} \Delta_1 \vec{\mathbf{H}}_{g'}\ \Delta_1 \vec{\mathbf{H}}_{g-g'}}{(1-\eta)^{2-2g}}$$

is an element of $\mathbb{Q}[(1-4y_1)^{-1}; \eta_k(1-\eta)^{-1}]_{k\geq 1}$ of weighted degree at most $3g-1$. □

**Proposition 3.4.5.** *For $g \geq 1$, we have*

$$\left(1 - 2\,\Delta_1 \vec{\mathbf{H}}_0 - \Pi_2 \operatorname*{Split}_{1\to 2}\right) \Delta_1 \vec{\mathbf{H}}_g = (1 - \mathrm{T}) \left((1-\eta)(1-4y_1)^{\frac{1}{2}}\, \Delta_1 \vec{\mathbf{H}}_g\right), \tag{3.8}$$

*where* T *is the* $\mathbb{Q}[[\mathbf{q}]]$*-linear operator defined by*

$$\mathrm{T}(F) = (1-\eta)^{-1}\, \Pi_2 \left((1-4y_2)^{-\frac{3}{2}} \operatorname*{Split}_{1\to 2} \left((1-4y_1)F\right)\right).$$

*Remark* 3.4.2. As we will see in Proposition 3.4.6, the operator T is locally nilpotent, so as a linear transformation, it can be thought of as a strictly upper triangular matrix. Then, (3.8) essentially says that, to a first approximation, we have

$$\left(1 - 2\,\Delta_1 \vec{\mathbf{H}}_0 - \Pi_2 \operatorname*{Split}_{1\to 2}\right) \Delta_1 \vec{\mathbf{H}}_g \approx (1-\eta)(1-4y_1)^{\frac{1}{2}}\, \Delta_1 \vec{\mathbf{H}}_g.$$

In other words, the left-hand side operator is very close to being a simple multiplication by a fixed element of $\mathbb{Q}[[\mathbf{q}, \mathbf{y}]]$.

*Proof.* Since $\Delta_1 \vec{\mathbf{H}}_g$ has no constant term as a power series in $y_1$, we have

$$\operatorname*{Split}_{1\to 2} \Delta_1 \vec{\mathbf{H}}_g = \frac{y_2\, \Delta_1 \vec{\mathbf{H}}_g - y_1\, \Delta_2 \vec{\mathbf{H}}_g}{y_1 - y_2}$$

and

$$(1-4y_2)^{-\frac{3}{2}} \operatorname*{Split}_{1\to 2} \left((1-4y_1)^{\frac{3}{2}}\, \Delta_1 \vec{\mathbf{H}}_g\right) = \frac{y_2(1-4y_2)^{-\frac{3}{2}}(1-4y_1)^{\frac{3}{2}}\, \Delta_1 \vec{\mathbf{H}}_g - y_1\, \Delta_2 \vec{\mathbf{H}}_g}{y_1 - y_2}.$$

Then, using the fact that

$$\Delta_1 \vec{\mathbf{H}}_0 = \Pi_2 \left(1 - \sqrt{1-4y_1} - \frac{y_1}{2(y_1-y_2)}\left(1 - \sqrt{\frac{1-4y_1}{1-4y_2}}\right)\right)$$

and

$$(1-\eta)(1-4y_1)^{\frac{1}{2}} = \Pi_2 \left(2 - (1-4y_2)^{-\frac{3}{2}}\right)(1-4y_1)^{\frac{1}{2}},$$

we can rewrite (3.8) in the form

$$\Pi_2 \left(\frac{F(y_1,y_2)\, \Delta_1 \vec{\mathbf{H}}_g - y_1\, \Delta_2 \vec{\mathbf{H}}_g}{y_1 - y_2}\right) = \Pi_2 \left(\frac{G(y_1,y_2)\, \Delta_1 \vec{\mathbf{H}}_g - y_1\, \Delta_2 \vec{\mathbf{H}}_g}{y_1 - y_2}\right)$$

where $F(y_1,y_2), G(y_1,y_2)$ are algebraic expressions in $y_1$ and $y_2$. In fact, a direct computation shows that $F(y_1,y_2) = G(y_1,y_2)$, so (3.8) holds. □

**Proposition 3.4.6.** *The operator* T *defined in Proposition 3.4.5 sends the ring sends the ring* $\mathbb{Q}[(1-4y_1)^{-1}; \eta_k(1-\eta)^{-1}]_{k\geq 1}$ *to itself. As an operator from* $\mathbb{Q}[(1-4y_1)^{-1}; \eta_k(1-\eta)^{-1}]_{k\geq 1}$ *to itself,* T *is locally nilpotent and preserves weighted degrees.*

*Proof.* Since T is $\mathbb{Q}[[\mathbf{q}]]$-linear, in particular, it is $\mathbb{Q}[\eta_k(1-\eta)^{-1}]_{k\geq 1}$-linear, and it is enough to show that the statement holds for the basis $1, 4y_1(1-4y_1)^{-1}, 4y_1(1-4y_1)^{-2}, \ldots$. We have $\mathrm{T}(1) = 0$, and for $k \geq 1$, we have

$$\mathrm{T}\left(4y_1(1-4y_1)^{-k}\right) = (1-\eta)^{-1}\, \Pi_2 (1-4y_2)^{-\frac{3}{2}} F_k(y_1,y_2),$$

where

$$F_k(y_1,y_2) = \operatorname*{Split}_{1\to 2} \left(4y_1(1-4y_1)^{1-k}\right) = \left((1-4y_1)^{-1} - 1\right)\left((1-4y_2)^{-1} - 1\right) \sum_{i=0}^{k-1} (1-4y_1)^{-i}(1-4y_2)^{1-k+i}. \tag{3.9}$$

Now, note that the elements

$$1, \quad \frac{\eta_1(y_2)}{(1-4y_2)^{-\frac{3}{2}}}, \quad \frac{\eta_2(y_2)}{(1-4y_2)^{-\frac{3}{2}}}, \quad \ldots, \quad \frac{\eta_k(y_2)}{(1-4y_2)^{-\frac{3}{2}}}$$

are polynomials in $(1-4y_2)^{-1}$ of degree $0, 1, 2, \ldots, k$ respectively, so we can write

$$F_k(y_1, y_2) = G_{k,0}(y_1) + \sum_{j=1}^{k} G_{k,j}(y_1) \frac{\eta_j(y_2)}{(1-4y_2)^{-\frac{3}{2}}}, \tag{3.10}$$

where, for $j = 0, 1, \ldots, k$, $G_{k,j}(y_1)$ is a polynomial in $(1-4y_1)^{-1}$ of degree $k-j$. In fact, setting $y_2 = 0$ in (3.10) and comparing with (3.9) shows that $G_{k,0}(y_1) = 0$. It follows that $\mathrm{T}\left(4y_1(1-4y_1)^{-k}\right)$ is an element of $\mathbb{Q}[(1-4y_1)^{-1}; \eta_k(1-\eta)^{-1}]_{k\geq 1}$ of weighted degree at most $k$. Furthermore, its degree in $(1-4y_1)^{-1}$ alone is strictly less than $k$, and it follows that repeated application of T to any element of $\mathbb{Q}[(1-4y_1)^{-1}; \eta_k(1-\eta)^{-1}]_{k\geq 1}$ eventually yields zero. □

By Proposition 3.4.4, the right-hand side of the higher genus monotone join-cut equation (2.4) is an element of $\mathbb{Q}[(1-4y_1)^{-1}; \eta_k(1-\eta)^{-1}]_{k\geq 1}$, up to a power of $(1-\eta)^{-1}$. Since the operator T is $\mathbb{Q}[(1-\eta)^{-1}]$-linear and locally nilpotent on $\mathbb{Q}[(1-4y_1)^{-1}; \eta_k(1-\eta)^{-1}]_{k\geq 1}$, it follows that the left-hand side operator

$$\left(1 - 2\,\Delta_1 \vec{\mathbf{H}}_0 - \Pi_2 \operatorname*{Split}_{1\to 2}\right) = (1-\mathrm{T})(1-\eta)(1-4y_1)^{\frac{1}{2}}$$

is invertible, with inverse given by

$$\left(1 - 2\,\Delta_1 \vec{\mathbf{H}}_0 - \Pi_2 \operatorname*{Split}_{1\to 2}\right)^{-1} = (1-\eta)^{-1}(1-4y_1)^{-\frac{1}{2}}(1 + \mathrm{T} + \mathrm{T}^2 + \cdots). \tag{3.11}$$

Together with the degree bound of Proposition 3.4.4, this completes the proof of Theorem 3.4.3. □

### 3.4.3 Inverting the Lifting Operator

Having solved for $\Delta_1 \vec{\mathbf{H}}_g$ for $g \geq 1$, we can now recover $\vec{\mathbf{H}}_g$. This is essentially an integration problem for rational functions, so it shouldn't be surprising that logarithms appear in $\vec{\mathbf{H}}_1$. For higher genera, the denominators all have higher degree, so no logarithms appear; for $g \geq 2$, the main difficulty lies in showing that there exists an antiderivative of the right form. Once this is done, establishing degree bounds is fairly easy.

Given the previous results of this section, Theorem 3.1.2 follows easily. For convenience, we restate it here as a corollary.

**Corollary 3.4.7.** *The genus one generating function* $\vec{\mathbf{H}}_1$ *for monotone Hurwitz numbers is given by*

$$\vec{\mathbf{H}}_1 = \tfrac{1}{24}\log\frac{1}{1-\eta} - \tfrac{1}{8}\log\frac{1}{1-\gamma}.$$

*Proof.* Using (3.7) and (3.11) to solve the genus one monotone join-cut equation (2.4), we get

$$\Delta_1 \vec{\mathbf{H}}_1 = \frac{2\eta_1(y_1) - 3\eta(y_1) + 3\gamma(y_1)}{48(1-\eta)} + \frac{(\eta(y_1)-\gamma(y_1))\eta_1}{24(1-\eta)^2}.$$

Since the kernel of $\Delta_1$ on $\mathbb{Q}[[\mathbf{p}]]$ is simply $\mathbb{Q}$, this uniquely determines $\vec{\mathbf{H}}_1$ up to a constant. Thus, the statement of the theorem can be checked by using Lemma 3.4.2, together with the fact that $\vec{\mathbf{H}}_1$ has no constant term. □

For higher genera, the following theorem establishes the existence of an appropriate "antiderivative" for $\Delta_1 \vec{\mathbf{H}}_g$.

**Theorem 3.4.8.** *For* $g \geq 2$*, the genus* $g$ *generating function* $\vec{\mathbf{H}}_g$ *for monotone Hurwitz numbers is a polynomial in the quantities* $(1-\eta)^{-1}$ *and* $\eta_1, \eta_2, \ldots$.

*Proof.* We proceed by first using our knowledge of the form of $\Delta_1 \vec{\mathbf{H}}_g$ for $g \geq 2$ to show that

$$\frac{\sum_{k\geq 1} \mathrm{E}_k \vec{\mathbf{H}}_g}{(1-\eta)^{-2}} \in \mathbb{Q}[(1-\eta)^{-1}; \eta_k]_{k\geq 1}.$$

We are then able to invert the operator $\sum_{k\geq 1} \mathrm{E}_k$ to obtain the result.

Note that the elements

$$1, \quad \frac{\eta(y_1)-\gamma(y_1)}{(1-4y_1)^{-\frac{1}{2}}}, \quad \frac{\eta_1(y_1)}{(1-4y_1)^{-\frac{1}{2}}}, \quad \frac{\eta_2(y_1)}{(1-4y_1)^{-\frac{1}{2}}}, \quad \ldots$$

are polynomials in $(1-4y_1)^{-1}$ of degree $0, 1, 2, \ldots$ respectively, so by Theorem 3.4.3, we know that we can write

$$\frac{\Delta_1 \vec{\mathbf{H}}_g}{(1-\eta)^{1-2g}} = F_{g,0}(1-4y_1)^{-\frac{1}{2}} + F_{g,1}\big(\eta(y_1)-\gamma(y_1)\big) + \sum_{j=2}^{3g-1} F_{g,j}\eta_{j-1}(y_1), \tag{3.12}$$

where, for $j = 0, 1, \ldots, 3g-1$, $F_{g,j}$ is an element of $\mathbb{Q}[\eta_k(1-\eta)^{-1}]_{k\geq 1}$. If we set $y_1 = 0$ in (3.12), we get $F_{g,0} = 0$, since $\Delta_1 \vec{\mathbf{H}}_g$ has no constant term as a power series in $y_1$. Next, note that when we are dealing with polynomials in $(1-4y_1)^{-1}$, we can evaluate them at $y_1 = \infty$, or equivalently, at $(1-4y_1)^{-1} = 0$. By Proposition 3.4.5, if we apply the operator

$$(1-\mathrm{T})(1-\eta)(1-4y_1)^{\frac{1}{2}}$$

to (3.12) and evaluate at $y_1 = \infty$, we get

$$\left. \frac{\Delta_1^2 \vec{\mathbf{H}}_{g-1}}{(1-\eta)^{1-2g}} \right|_{y_1=\infty} = F_{g,1}(\eta - 1) + \sum_{j=2}^{3g-1} F_{g,j}\eta_{j-1}.$$

For $g = 1$, the left-hand side is nonzero, but for $g \geq 2$, it follows from Theorem 3.4.3 and Lemma 3.4.2 that the left-hand side is zero. Thus, for $g \geq 2$, we have

$$F_{g,1}(\eta - 1) + \sum_{j=2}^{3g-1} F_{g,j}\eta_{j-1} = 0. \tag{3.13}$$

Now, we turn to the computation of $\sum_{k\geq 1} \mathrm{E}_k \vec{\mathbf{H}}_g$. Using (3.3), we have

$$\sum_{k\geq 1} \mathrm{E}_k \vec{\mathbf{H}}_g = \left( \sum_{k\geq 1} \mathrm{D}_k - \frac{2\gamma}{1-\gamma}\mathcal{D} \right) \vec{\mathbf{H}}_g = \Pi_1 \left( \left( y_1 \frac{\partial}{\partial y_1} \right)^{-1} - \frac{2\gamma}{1-\gamma} \right) \Delta_1 \vec{\mathbf{H}}_g.$$

By direct computation, we also have

$$\begin{aligned} \Pi_1 \left( \left( y_1 \frac{\partial}{\partial y_1} \right)^{-1} - \frac{2\gamma}{1-\gamma} \right) (\eta_1(y_1) - \gamma_1(y_1)) &= \frac{-2\gamma}{1-\gamma}(\eta - 1), \\ \Pi_1 \left( \left( y_1 \frac{\partial}{\partial y_1} \right)^{-1} - \frac{2\gamma}{1-\gamma} \right) \eta_1(y_1) &= \eta - \frac{2\gamma}{1-\gamma}\eta_1, \\ \Pi_1 \left( \left( y_1 \frac{\partial}{\partial y_1} \right)^{-1} - \frac{2\gamma}{1-\gamma} \right) \eta_j(y_1) &= \eta_{j-1} - \frac{2\gamma}{1-\gamma}\eta_j, \qquad j \geq 2. \end{aligned}$$

Thus, applying the operator

$$\Pi_1 \left( \left( y_1 \frac{\partial}{\partial y_1} \right)^{-1} - \frac{2\gamma}{1-\gamma} \right)$$

to (3.12) and using relation (3.13), we have

$$\frac{\sum_{k\geq 1} \mathrm{E}_k \vec{\mathbf{H}}_g}{(1-\eta)^{1-2g}} = F_{g,2}\eta + \sum_{j=3}^{3g-1} F_{g,j}\eta_{j-2}(y_1)$$

for $g \geq 2$. In particular, we have

$$\frac{\sum_{k\geq 1} \mathrm{E}_k \vec{\mathbf{H}}_g}{(1-\eta)^{-2}} \in \mathbb{Q}[(1-\eta)^{-1}; \eta_k(1-\eta)^{-1}]_{k\geq 1}. \tag{3.14}$$

Finally, note that $\eta, \eta_1, \eta_2, \ldots$ are all eigenvectors of the differential operator $\sum_{k\geq 1} \mathrm{E}_k$ with eigenvalue 1, since they are purely linear in $\mathbf{q} = (q_1, q_2, \ldots)$. Thus, up to an additive rational constant, we have

$$\left(\sum_{k\geq 1} \mathrm{E}_k\right)^{-1} \frac{\eta_\alpha}{(1-\eta)^{j+\ell(\alpha)}} = \frac{\eta_\alpha}{\eta^{\ell(\alpha)}} \int_0^1 \frac{(\eta t)^{\ell(\alpha)}}{(1-\eta t)^{j+\ell(\alpha)}} \frac{\mathrm{d}t}{t}. \tag{3.15}$$

For $j \geq 2$, the integral is $\eta^{\ell(\alpha)}$ times a polynomial in $(1-\eta)^{-1}$. In conjunction with (3.14), this shows that $\vec{\mathbf{H}}_g$ is a polynomial in $(1-\eta)^{-1}$ and $\eta_1, \eta_2, \ldots$, as required. □

We can now deduce Theorem 3.1.3, which we restate here as a corollary for convenience.

**Corollary 3.4.9.** *For $g \geq 2$, the genus $g$ generating function for monotone Hurwitz numbers has the form*

$$\vec{\mathbf{H}}_g = -c_{g,(0)} + \sum_{d=0}^{3g-3} \sum_{\alpha \vdash d} \frac{c_{g,\alpha}\, \eta_\alpha}{(1-\eta)^{2g-2+\ell(\alpha)}},$$

*where the constants $c_{g,\alpha}$ are rational numbers.*

*Proof.* From Lemma 3.4.2, we have that if

$$\frac{F}{(1-\eta)^i} \in \mathbb{Q}[\eta_k (1-\eta)^{-1}]_{k\geq 1}$$

and it has weighted degree $j$, then

$$\frac{\Delta_1 F}{(1-\eta)^{i-1}(1-4y_1)^{-\frac{1}{2}}} \in \mathbb{Q}[(1-4y_1)^{-1}; \eta_k (1-\eta)^{-1}]_{k\geq 1}$$

and it has weighted degree $j+2$. Since the kernel of $\Delta_1$ on $\mathbb{Q}[[\mathbf{p}]]$ is $\mathbb{Q}$, it follows from Theorems 3.4.3 and 3.4.8 that for some $c \in \mathbb{Q}$,

$$\frac{\vec{\mathbf{H}}_g - c}{(1-\eta)^{2-2g}}$$

is an element of $\mathbb{Q}[\eta_k(1-\eta)^{-1}]_{k\geq 1}$ of weighted degree at most $3g-3$. This shows that $\vec{\mathbf{H}}_g$ is of the form

$$\vec{\mathbf{H}}_g = c + \sum_{d=0}^{3g-3} \sum_{\alpha \vdash d} \frac{c_{g,\alpha}\, \eta_\alpha}{(1-\eta)^{2g-2+\ell(\alpha)}},$$

and the fact that $c = -c_{g,(0)}$ follows from the fact that $\vec{\mathbf{H}}_g$ has no constant term as a power series. □

# Chapter 4

# Monotone Hurwitz Numbers

Having established some explicit formulas and a general form for the genus-by-genus generating functions for monotone Hurwitz numbers in Chapter 3, we now turn to some consequences of this.

First, we give a method for generating explicit formulas for monotone Hurwitz numbers from the generating functions $\vec{\mathbf{H}}_g$ in Section 4.1, and observe that we get a polynomiality result out of it. In Section 4.2, we use this method for the genus one case, and the resulting formula is strikingly similar to a known formula for classical Hurwitz numbers.

Then, we discuss the rational constants appearing at the extremes of the general form

$$\vec{\mathbf{H}}_g = -c_{g,(0)} + \sum_{d=0}^{3g-3} \sum_{\alpha \vdash d} \frac{c_{g,\alpha}\, \eta_\alpha}{(1-\eta)^{2g-2+\ell(\alpha)}}$$

from Theorem 3.1.3 for $g \geq 2$. Section 4.3 deals with the low end, that is, the coefficients $c_{g,\alpha}$ with $|\alpha| = d = 0$, which can be computed in terms of Bernoulli numbers; and Section 4.4 deals with the high end coefficients, with $d = 3g-3$, where we establish a recurrence relation which doesn't depend on the coefficients with $d < 3g-3$.

The significance of the results of this chapter is mainly apparent when comparing them to similar results in classical Hurwitz theory. In particular, the reason for referring to the high end coefficients $c_{g,\alpha}$ of the monotone Hurwitz generating functions as **Witten terms** comes from the fact that they are, up to a known power of 2, equal to the high end coefficients $\overline{c}_{g,\alpha}$ of the classical Hurwitz generating functions, which are closely related to Witten's conjecture [47]. However, we hold off a discussion of this until Chapter 6, after the relevant parts of Hurwitz theory are discussed in Chapter 5.

## 4.1 Polynomiality

A key consequence of Theorems 3.1.1, 3.1.2 and 3.1.3 is that they imply a polynomiality result for the monotone single Hurwitz numbers themselves, that is, the coefficients of the genus-by-genus generating functions studied in Chapter 3. Note that Theorem 4.1.1 is the exact analogue of Theorem 5.1.5 for Hurwitz numbers.

**Theorem 4.1.1.** *For each pair $(g,\ell)$ with $(g,\ell) \notin \{(0,1),(0,2)\}$, there is a polynomial $\vec{P}_{g,\ell}$ in $\ell$ variables such that, for all partitions $\alpha \vdash n$ with $\ell$ parts,*

$$\vec{H}_g(\alpha) = \frac{n!}{|\mathrm{aut}(\alpha)|} \vec{P}_{g,\ell}(\alpha_1, \ldots, \alpha_\ell) \prod_{i=1}^{\ell} \binom{2\alpha_i}{\alpha_i},$$

*where $\vec{H}_g(\alpha) = [p_\alpha/n!]\vec{\mathbf{H}}_g$ is a genus $g$ monotone Hurwitz number.*

*Proof.* For $g = 0$, this follows from the explicit formula for the genus zero generating function $\vec{\mathbf{H}}_0$ for monotone Hurwitz numbers given in Theorem 3.1.1. For $g \geq 1$, by using the Lagrange Implicit Function Theorem (see Lemma 3.2.1), we have

$$\vec{H}_g(\alpha) = n![p_\alpha]\vec{\mathbf{H}}_g = n![q_\alpha]\frac{(1-\eta)\vec{\mathbf{H}}_g}{(1-\gamma)^{2n+1}}.$$

Given the formula for $\vec{\mathbf{H}}_1$ from Theorem 3.1.2 and the general form from Theorem 3.1.3, the power series on the right-hand side can be expanded as an infinite sum of (rational number multiples of) terms of the form

$$\binom{-2n-1}{u} \gamma^u \eta^{v_0} \eta_1^{v_1} \eta_2^{v_2} \cdots \eta_k^{v_k},$$

where $k$ and $u, v_0, v_1, \ldots, v_k \geq 0$ are non-negative integers. However, since the series $\gamma, \eta, \eta_1, \eta_2, \ldots$ are all purely linear in the indeterminates $\mathbf{q}$, only the finitely many terms with $u + v_0 + v_1 + \cdots + v_k = \ell$ contribute to the coefficient of $q_\alpha$. For fixed $u$, the binomial coefficient $\binom{-2n-1}{u}$ is a polynomial in the parts of $\alpha$, and given the definition of the series $\gamma, \eta, \eta_1, \eta_2, \ldots$, the contribution to the coefficient of $q_\alpha$ is a polynomial in the parts of $\alpha$ multiplied by the factor

$$\frac{1}{|\mathrm{aut}(\alpha)|} \prod_{i=1}^{\ell} \binom{2\alpha_i}{\alpha_i}.$$

It follows that $\vec{H}_g(\alpha)$ has the stated form. ∎

## 4.2 Genus One Formula

As an illustration of Theorem 4.1.1, we can compute an explicit formula for monotone Hurwitz numbers in the genus one case. This is very similar in flavour to the explicit genus one formula of Theorem 5.1.2 for classical Hurwitz numbers.

**Theorem 4.2.1.** *For $\alpha \vdash n$, the genus one monotone single Hurwitz number $\vec{H}_1(\alpha)$ is given by*

$$\vec{H}_1(\alpha) = \frac{1}{24}\frac{n!}{|\mathrm{aut}(\alpha)|}\prod_{i=1}^{\ell(\alpha)}\binom{2\alpha_i}{\alpha_i}$$
$$\times\left((2n+1)^{\overline{\ell(\alpha)}} - 3(2n+1)^{\overline{\ell(\alpha)-1}} - \sum_{k=2}^{\ell(\alpha)}(k-2)!(2n+1)^{\overline{\ell(\alpha)-k}}e_k(2\alpha+1)\right), \tag{4.1}$$

*where*

$$(2n+1)^{\overline{k}} = (2n+1)(2n+2)\cdots(2n+k)$$

*denotes a rising product with $k$ factors, and $e_k(2\alpha+1)$ is the $k$th elementary symmetric polynomial of the quantities $\{2\alpha_i+1 \mid i = 1,2,\dots,\ell(\alpha)\}$.*

*Proof.* Recall from Theorem 3.1.2 that the genus one generating function for monotone Hurwitz numbers is

$$\vec{\mathbf{H}}_1 = \tfrac{1}{24}\log\frac{1}{1-\eta} - \tfrac{1}{8}\log\frac{1}{1-\gamma}. \tag{4.2}$$

For the first term in (4.2), using Lemma 3.2.1, we have

$$\begin{aligned}
[p_\alpha]\log\frac{1}{1-\eta}
&= [q_\alpha]\frac{1-\eta}{(1-\gamma)^{2n+1}}\log\frac{1}{1-\eta}\\
&= [q_\alpha]\left(\sum_{j\geq 0}(2n+1)^{\overline{j}}\frac{\gamma^j}{j!}\right)\left(\eta - \sum_{k\geq 2}(k-2)!\frac{\eta^k}{k!}\right)\\
&= [q_\alpha]\left((2n+1)^{\overline{\ell-1}}\frac{\gamma^{\ell-1}\eta}{(\ell-1)!} - \sum_{k=2}^{\ell}(k-2)!(2n+1)^{\overline{\ell-k}}\frac{\gamma^{\ell-k}}{(\ell-k)!}\frac{\eta^k}{k!}\right).
\end{aligned}$$

By iterating the product rule, we get

$$\begin{aligned}
|\mathrm{aut}(\alpha)|\,[q_\alpha]\frac{\gamma^{\ell(\alpha)-k}}{(\ell(\alpha)-k)!}\frac{\eta^k}{k!}
&= \frac{\partial^{\ell(\alpha)}}{\partial q_\alpha}\left(\frac{\eta^k}{k!}\frac{\gamma^{\ell(\alpha)-k}}{(\ell(\alpha)-k)!}\right)\\
&= \prod_{i=1}^{\ell(\alpha)}\binom{2\alpha_i}{\alpha_i}\sum_{1\le i_1<\cdots<i_k\le\ell(\alpha)}(2\alpha_{i_1}+1)(2\alpha_{i_2}+1)\cdots(2\alpha_{i_k}+1)\\
&= \prod_{i=1}^{\ell(\alpha)}\binom{2\alpha_i}{\alpha_i}e_k(2\alpha+1).
\end{aligned}$$

Thus,

$$\begin{aligned}
[p_\alpha]\log\frac{1}{1-\eta} &= \frac{1}{|\mathrm{aut}(\alpha)|}\prod_{i=1}^{\ell(\alpha)}\binom{2\alpha_i}{\alpha_i}\\
&\quad\times\left((2n+1)^{\overline{\ell(\alpha)}}-\sum_{k=2}^{\ell(\alpha)}(k-2)!(2n+1)^{\overline{\ell(\alpha)-k}}e_k(2\alpha+1)\right). \qquad (4.3)
\end{aligned}$$

For the second term in (4.2), using Lemma 3.2.1 again and the fact that $p_\alpha$ and $q_\alpha$ are eigenvectors for the differential operators $\mathcal{D}$ and $\mathcal{E}$ respectively, we get

$$\begin{aligned}
[p_\alpha]\log\frac{1}{1-\gamma} &= \tfrac{1}{n}[p_\alpha]\mathcal{D}\log\frac{1}{1-\gamma}\\
&= \tfrac{1}{n}[q_\alpha]\mathcal{E}\left(\frac{1}{2n(1-\gamma)^{2n}}\right)\\
&= [q_\alpha]\left(\frac{1}{2n(1-\gamma)^{2n}}\right)\\
&= [q_\alpha](2n+1)^{\overline{\ell(\alpha)-1}}\frac{\gamma^{\ell(\alpha)}}{\ell(\alpha)!}.
\end{aligned}$$

It follows that

$$[p_\alpha]\log(1-\gamma)^{-1} = \frac{1}{|\mathrm{aut}(\alpha)|}\prod_{i=1}^{\ell(\alpha)}\binom{2\alpha_i}{\alpha_i}\times(2n+1)^{\overline{\ell(\alpha)-1}}. \qquad (4.4)$$

Combining (4.3) and (4.4) with (4.2) completes the proof. □

## 4.3 Bernoulli Terms

There is a known formula for monotone Hurwitz numbers $\vec{H}_g(\alpha)$ for arbitrary genus $g$ in the case where $\alpha = (n)$ is a partition with a single part, due to Matsumoto and Novak (see [36]). By comparing this with the general form of Theorem 3.1.3, we can compute the rational constants $c_{g,(0)}$ for $g \geq 2$.

**Theorem 4.3.1.** *For $g \geq 2$, the rational constant $c_{g,(0)}$ appearing in Theorem 3.1.3 is given by*

$$c_{g,(0)} = \frac{-B_{2g}}{2g(2g-2)},$$

*where $B_{2g}$ is a Bernoulli number.*

*Proof.* To compute the monotone single Hurwitz number $\vec{H}_g(\alpha)$ for $\alpha = (n)$, we can expand the expression for $\vec{\mathbf{H}}_g$ given in Theorem 3.1.3 as a power series in $\eta, \eta_1, \eta_2, \ldots$, and then further expand this as a power series in $\mathbf{p} = (p_1, p_2, \ldots)$, throwing away any terms of degree higher than one at each step. (Note that the difference $p_k - q_k$ only has terms of degree at least 2 in either $\mathbf{p}$ or $\mathbf{q}$, so doing this with $\mathbf{q}$ instead of $\mathbf{p}$ would give the same coefficients.) This yields the expression

$$\begin{aligned}
\vec{H}_g((n)) &= [p_n/n!]\vec{\mathbf{H}}_g \\
&= [p_n/n!]\left((2g-2)c_{g,(0)}\eta + \sum_{k=1}^{3g-3} c_{g,(k)}\eta_k\right) \\
&= \frac{(2n)!}{n!}\left((2g-2)c_{g,(0)}\,(2n+1) + \sum_{k=1}^{3g-3} c_{g,(k)}(2n+1)d^k\right).
\end{aligned}$$

For fixed $g$, this expression is $(2n)!/n!$ times a polynomial in $n$, and evaluating this polynomial at $n = 0$ gives $(2g-2)c_{g,(0)}$. In contrast, according to Matsumoto and Novak's formula [36, Equation (48)], we have

$$\vec{H}_g((n)) = \frac{(2n)!}{n!}\binom{2g-2+2n}{2g-2}\frac{1}{2g(2g-1)}\left[\frac{z^{2g}}{(2g)!}\right]\left(\frac{\sinh(z/2)}{z/2}\right)^{2n-2}.$$

Again, for fixed $g$, this expression is $(2n)!/n!$ times a polynomial in $n$. Evaluating this polynomial at $n = 0$ gives

$$\begin{aligned}
(2g-2)c_{g,(0)} &= \frac{1}{2g(2g-1)}\left[\frac{z^{2g}}{(2g)!}\right]\left(\frac{\sinh(z/2)}{z/2}\right)^{-2} \\
&= \frac{1}{2g(2g-1)}\left[\frac{z^{2g}}{(2g)!}\right]\left(z\frac{\partial}{\partial z} - 1\right)\frac{-z}{e^z - 1},
\end{aligned}$$

and $z/(e^z - 1)$ is the exponential generating function for the Bernoulli numbers.

□

## 4.4 Witten Terms

To obtain information on the high end coefficients of the genus $g$ generating function $\vec{\mathbf{H}}_g$ for monotone Hurwitz numbers, that is, the rational numbers $c_{g,\alpha}$ of Theorem 3.1.3 such that $|\alpha| = 3g - 3$, we can take the higher genus monotone join-cut equation (2.4) and restrict our attention to the top degree terms. This yields the recurrence of Theorem 4.4.6 below, which is structurally identical to the recurrence of Theorem 5.1.7 in the classical Hurwitz case.

While the auxiliary power series $\eta, \eta_1, \eta_2, \dots$ defined in Section 3.2.2 are sufficient to express the generating functions $\vec{\mathbf{H}}_g$ for $g \geq 1$, they are not sufficient to express some of the intermediate expressions which appear in (2.4). For this, it will be convenient to introduce a different set of auxiliary power series $\xi_1, \xi_2, \dots,$ as follows.

**Definition 4.4.1.** For $k \geq 1$, let $\xi_k = \Pi_1 (1 - 4y_1)^{-\frac{k}{2}} - 1$.

Note that $\gamma = \xi_1$ and $\eta = \xi_3$. For $k \geq 1$, we have $\eta_k = \Pi_1 \left( y_1 \frac{\partial}{\partial y_1} \right)^k (1 - 4y_1)^{-\frac{3}{2}}$, and we have $y_1 \frac{\partial}{\partial y_1} (1 - 4y_1)^{-\frac{1}{2}} = \frac{1}{2}\left( (1 - 4y_1)^{-\frac{3}{2}} - (1 - 4y_1)^{-\frac{1}{2}} \right)$, so it follows that $\eta_k$ is a $\mathbb{Q}$-linear combination of the power series $\xi_3, \xi_5, \xi_7, \dots, \xi_{2k+3}$. Thus, expressions given in terms of $\gamma, \eta, \eta_1, \eta_2, \dots$ can be rewritten in terms of the power series $\xi_k$ with odd indices; the even indices appear when dealing with expressions like $\Pi_1 \Delta_1^2 \vec{\mathbf{H}}_{g-1}$ or $\Pi_1 \left( \Delta_1 \vec{\mathbf{H}}_{g'} \cdot \Delta_1 \vec{\mathbf{H}}_{g-g'} \right)$.

**Definition 4.4.2.** The notion of **weighted degree** from Definition 3.4.1 can be extended to elements of $\mathbb{Q}[[\mathbf{q}, \mathbf{y}]]$ which are polynomials (or sometimes power series) in the quantities $(1 - 4y_i)^{-\frac{1}{2}}$ and $\xi_1, \xi_2, \dots,$ by giving $(1 - 4y_i)^{-\frac{1}{2}}$ degree $1/2$ and giving $\xi_k$ degree $(k - 3)/2$.

With this extended notion of weighted degree, some elements have half-integer, or even negative degree, but this does not cause any particular problems. For example, if we adopt the convention that non-zero constant polynomials have degree 0 while the zero polynomial has degree $-\infty$, then the degree of a product is the sum of the degrees of the factors, as usual.

However, implicit in this definition of weighted degree is the assumption that whatever elements of $\mathbb{Q}[[\mathbf{q}, \mathbf{y}]]$ we are working with cannot be written in terms of $(1 - 4y_i)^{-\frac{1}{2}}$ and $\xi_1, \xi_2, \dots$ in more than one way. The following proposition justifies this assumption.

**Proposition 4.4.3.** *The elements $\xi_1, \xi_2, \dots \in \mathbb{Q}[[\mathbf{q}]]$ are algebraically independent over $\mathbb{Q}$, and they are purely linear in the indeterminates $\mathbf{q}$. Thus, two power series in a finite set of these elements are equal if and only if they have the same coefficients.*

*Proof.* It is clear from their definition that $\xi_1, \xi_2, \dots$ are linear in the indeterminates $\mathbf{q}$ with no constant term, so it is enough to consider only homogeneous polynomials in these elements to show that they are algebraically independent. (This is also why the last part of the proposition holds for power series instead of just for polynomials.)

Consider a polynomial dependency of homogeneous degree $d$, that is, an equation of the form

$$\sum_{k_1 \le k_2 \le \cdots \le k_d} a_{k_1,k_2,\dots,k_d} \xi_{k_1} \xi_{k_2} \cdots \xi_{k_d} = 0,$$

where the (finitely many) coefficients $a_{k_1,k_2,\dots,k_d}$ are rational numbers. By the definition of $\xi_1, \xi_2, \dots,$ we can rewrite this as

$$\Pi_1 \, \Pi_2 \dots \Pi_d \sum_{k_1 \le k_2 \le \cdots \le k_d} a_{k_1,k_2,\dots,k_d} \prod_{i=1}^{d} \big( (1-4y_i)^{-\frac{k_i}{2}} - 1 \big) = 0.$$

This is equivalent to the symmetrized equation

$$\Pi_1 \, \Pi_2 \dots \Pi_d \sum_{k_1 \le k_2 \le \cdots \le k_d} \frac{a_{k_1,k_2,\dots,k_d}}{d!} \sum_{\sigma \in \mathcal{S}_d} \prod_{i=1}^{d} \big( (1-4y_{\sigma(i)})^{-\frac{k_i}{2}} - 1 \big) = 0,$$

averaged over all permutations of the indeterminates $y_1, y_2, \dots, y_d$. In turn, by virtue of being symmetrized, this equation is equivalent to the equation

$$\sum_{k_1 \le k_2 \le \cdots \le k_d} \frac{a_{k_1,k_2,\dots,k_d}}{d!} \sum_{\sigma \in \mathcal{S}_d} \prod_{i=1}^{d} \big( (1-4y_{\sigma(i)})^{-\frac{k_i}{2}} - 1 \big) = 0,$$

obtained by removing the projection operators. It can be seen that the power series $(1-4y_1)^{-\frac{k}{2}} - 1$ for $k \ge 1$ are linearly independent over $\mathbb{Q}$, and by definition the indeterminates $y_1, y_2, \dots, y_d$ are algebraically independent over $\mathbb{Q}$, so it follows that the coefficients $a_{k_1,k_2,\dots,k_d}$ are all zero. $\square$

Given that expansions in terms of the power series $\xi_1, \xi_2, \dots$ are unique when they exist, the notion of weighted degree from Definition 4.4.2 is well-defined, and the notion of top degree terms is also well-defined.

**Definition 4.4.4.** Let $R$ be the subring of $\mathbb{Q}[[\mathbf{q}, \mathbf{y}]]$ consisting of the elements which have a well-defined, finite weighted degree. Given an element $F \in R$, it can be expressed as a (possibly infinite) sum of monomials in the quantities $(1-4y_i)^{-\frac{1}{2}}$, $i \ge 1$, and $\xi_1, \xi_2, \dots$. Let $\operatorname{Top} F$ be the sum of those monomials of degree equal to $\deg F$. This defines the **top degree extraction operator** $\operatorname{Top}\colon R \to R$.

The following computational lemma records how the lifting, projection and splitting operators interact with top degree terms of elements of $\mathbb{Q}[[\mathbf{q},\mathbf{y}]]$ of finite weighted degree.

**Lemma 4.4.5.**

1. *We have*

$$\begin{aligned}
\Delta_1(1-4y_1)^{-\frac{1}{2}} &= \tfrac{1}{2}\big((1-4y_1)^{-\frac{3}{2}}-(1-4y_1)^{-\frac{1}{2}}\big)^2(1-\xi_3)^{-1},\\
\Delta_1(1-4y_2)^{-\frac{1}{2}} &= \tfrac{1}{2}\big((1-4y_2)^{-\frac{3}{2}}-(1-4y_2)^{-\frac{1}{2}}\big)\\
&\qquad\times\big((1-4y_1)^{-\frac{3}{2}}-(1-4y_1)^{-\frac{1}{2}}\big)(1-\xi_3)^{-1},\\
\Delta_1\,\xi_k &= \tfrac{k}{2}\big((1-4y_1)^{-\frac{3}{2}}-(1-4y_1)^{-\frac{1}{2}}\big)\\
&\qquad\times\big((\xi_{k+2}-\xi_k)(1-\xi_3)^{-1}+(1-4y_1)^{-\frac{k-1}{2}}\big),\qquad k\geq 1.
\end{aligned}$$

*Thus, if* $F\in\mathbb{Q}[[\mathbf{q},\mathbf{y}]]$ *with* $\deg F=d$ *and* $d\neq 0$*, then* $\deg\Delta_1 F=d+\frac{5}{2}$*, and* $\operatorname{Top}\Delta_1 F=\operatorname{Top}\Delta_1\operatorname{Top}F$*.*

2. *If* $F\in\mathbb{Q}[[\mathbf{q},\mathbf{y}]]$ *with* $[y_1^0]F=0$ *and* $\deg F=d$*, then* $\deg\Pi_1 F=d-\frac{3}{2}$*, and* $\operatorname{Top}\Pi_1 F=\operatorname{Top}\Pi_1\operatorname{Top}F$*.*

3. *For* $k\geq 2$*, we have*

$$\operatorname{Top}\operatorname*{Split}_{1\to 2}(1-4y_1)^{-k}=\sum_{i=1}^{k}(1-4y_1)^{-i}(1-4y_2)^{-(k+1-i)}.$$

*Proof.*

1. The values of $\Delta_1(1-4y_1)^{-\frac{1}{2}}$ and $\Delta_1(1-4y_2)^{-\frac{1}{2}}$ can be computed directly from the definition of $\Delta_1$ given in (3.5). Then, using the expression for $\Delta_1\,\Pi_2$ given in Lemma 2.2.4, the value of $\Delta_1\,\xi_k=\Delta_1\,\Pi_2\big((1-4y_2)^{-\frac{k}{2}}-1\big)$ can be computed, recalling that we have the operator identity $x_2\frac{\partial}{\partial x_2}=y_2\frac{\partial}{\partial y_2}$.

2. The statement about $\Pi_1$ follows directly from the definition of $\xi_k$ and the definition of weighted degree.

3. If we let $z_i=(1-4y_i)^{-1}$, so that $y_i=\frac{1}{4}(1-z_i^{-1})$, then we have

$$\begin{aligned}
\operatorname{Top}\operatorname*{Split}_{1\to 2}(1-4y_1)^{-k} &= \frac{y_2z_1^k-y_1z_2^k}{y_1-y_2}+1\\
&= \frac{(z_2-1)z_1^{k+1}-(z_1-1)z_2^{k+1}}{z_1-z_2}+1,
\end{aligned}$$

and the result follows directly. ∎

With these tools in place, we can give the promised description of the high end coefficients of the genus $g$ generating function $\vec{\mathbf{H}}_g$ for monotone Hurwitz numbers, as follows.

**Theorem 4.4.6.** *Let $\xi_1, \xi_2, \ldots \in \mathbb{Q}[[\mathbf{q}]]$ be the auxiliary power series defined in Definition 4.4.1, and let $\vec{\mathbf{H}}_g$ be the genus $g$ generating function for monotone Hurwitz numbers. Then, for $g \geq 2$, we have*

$$\operatorname{Top} \vec{\mathbf{H}}_g = \sum_{\alpha \vdash (3g-3)} \frac{c_{g,\alpha}}{(1-\xi_3)^{2g-2+\ell(\alpha)}} \prod_{i=1}^{\ell(\alpha)} \frac{(2\alpha_i+1)!}{4^{\alpha_i} \cdot \alpha_i!} \xi_{2\alpha_i+3}, \tag{4.5}$$

*where the $c_{g,\alpha}$ are the rational numbers appearing in Theorem 3.1.3 for which $|\alpha| = 3g-3$; these rational numbers can be computed recursively from the equation*

$$\Theta \operatorname{Top} \vec{\mathbf{H}}_g = \operatorname{Top} \Pi_1 \left( \Delta_1^2 \operatorname{Top} \vec{\mathbf{H}}_{g-1} + \sum_{g'=1}^{g-1} \Delta_1 \operatorname{Top} \vec{\mathbf{H}}_{g'} \cdot \Delta_1 \operatorname{Top} \vec{\mathbf{H}}_{g-g'} \right), \tag{4.6}$$

*where $\Theta$ is the differential operator defined by*

$$\Theta\, \xi_k = \frac{k}{2} \xi_{k+1} - \frac{k}{4} \sum_{i=3}^{k+1} \xi_i \xi_{k+4-i}, \tag{4.7}$$

*together with the equation*

$$\operatorname{Top} \vec{\mathbf{H}}_1 = \tfrac{1}{24} \log(1-\xi_3)^{-1}. \tag{4.8}$$

*Proof.* First, note that the expression (4.5) comes directly from Theorem 3.1.3, after noting that

$$\operatorname{Top} \eta_k = \tfrac{3}{2} \cdot \tfrac{5}{2} \cdot \tfrac{7}{2} \cdot \cdots \cdot \tfrac{2k+1}{2} \xi_{2k+3}.$$

Next, note that (4.6) comes from applying the operator $\operatorname{Top} \Pi_1$ to the higher genus monotone join-cut equation (2.4). The right-hand sides of (4.6) and (2.4) clearly match. For the left-hand sides, we are dealing with a $\mathbb{Q}$-linear differential operator in each case, so it suffices to compare their actions on a suitable basis; since $\operatorname{Top} \vec{\mathbf{H}}_g$ is a sum of monomials in the power series $\xi_3, \xi_5, \xi_7, \ldots$ with odd indices, Lemma 4.4.5 can be used with the expression from Proposition 3.4.5 for the left-hand side operator to obtain (4.7), after a straightforward computation.

As for (4.8), it comes from Theorem 3.1.2 by applying the operator $\operatorname{Top} \Pi_1$.

Thus, all the equations given in the theorem statement hold, but it remains to show that (4.6) uniquely determines the rational numbers $c_{g,\alpha}$ for $\alpha \vdash 3g-3$ for a given genus $g \geq 2$, if $\operatorname{Top} \vec{\mathbf{H}}_1, \operatorname{Top} \vec{\mathbf{H}}_2, \ldots, \operatorname{Top} \vec{\mathbf{H}}_{g-1}$ are known. In this case, the right-hand side of (4.6) is uniquely determined.

As for the left-hand side, every monomial in $\operatorname{Top} \vec{\mathbf{H}}_g$ is of the form

$$a_{k_1,k_2,\ldots,k_d} \xi_{2k_1+1} \xi_{2k_2+1} \cdots \xi_{2k_d+1},$$

where $k_1, k_2, \ldots, k_d \geq 1$ are integers and $a_{k_1,k_2,\ldots,k_d} \in \mathbb{Q}$. The image of this monomial under the operator $\Theta$ is a sum of monomials with $d$ and $d+1$ factors each, exactly one of which has an even index. Thus, the coefficients $a_{k_1,k_2,\ldots,k_d}$ can be recovered sequentially, in increasing order of $d$, and this uniquely specifies the numbers $c_{g,\alpha}$, $\alpha \vdash 3g-3$. $\square$

# Chapter 5

# Classical Hurwitz Generating Functions

The goal of this chapter is to give a short account of some of the main results concerning classical Hurwitz numbers which parallels the theory of monotone Hurwitz numbers as outlined in the previous chapters. We also give a new result (Theorem 5.1.7) which establishes an essentially identical recurrence for the top degree terms in the classical Hurwitz case to the monotone Hurwitz case (Theorem 4.4.6). For a discussion of the similarities and differences between the classical and monotone versions of Hurwitz theory, see Chapter 6.

## 5.1 Overview

As noted in Section 1.9, Hurwitz [27, 28] first identified the numbers bearing his name in the context of counting certain branched covers of the Riemann sphere with specified ramification data, and showed that these branched covers could be identified with transitive transposition factorizations (which is the definition that we have been using). He also stated a closed formula for the genus zero case, equivalent to the following theorem, and gave a sketch of a proof.

**Theorem 5.1.1.** *(Hurwitz, [27, 28], Strehl, [45]; Goulden and Jackson, [20]). The genus zero generating function for Hurwitz numbers is given by*

$$\mathbf{H}_0 = \sum_{n \geq 1} \sum_{\alpha \vdash n} \frac{p_\alpha}{|\mathrm{aut}\,\alpha|} n^{\ell(\alpha)-3} \prod_{j=1}^{\ell(\alpha)} \frac{\alpha_j^{\alpha_j}}{\alpha_j!}.$$

This was forgotten for a long time, during which certain special cases of this formula were rediscovered. In particular, Dénes [8] recovered the $\ell(\alpha) = 1$ case, Arnol′d [1] the $\ell(\alpha) = 2$ case, while Crescimanno and Taylor [7] dealt with the other extreme of $\alpha = 1^n$. Following this, Goulden and Jackson [20] used their join-cut equation to state and prove the complete formula for genus zero. This was also proved independently by Strehl [45], who filled in the details of Hurwitz's original argument.

Goulden, Jackson and Vainshtein [23] extended this to a conjectural complete formula for genus one Hurwitz numbers, as follows.

**Theorem 5.1.2.** *(Vakil, [46]; Goulden and Jackson, [22]). The genus one Hurwitz numbers are given by*

$$H_1(\alpha) = \frac{1}{24}\frac{n!}{|\mathrm{aut}(\alpha)|}(n+\ell(\alpha))!\prod_{i=1}^{\ell(\alpha)}\frac{\alpha_i^{\alpha_i}}{\alpha_i!} \times \left(n^{\ell(\alpha)} - n^{\ell(\alpha)-1} - \sum_{k=2}^{\ell(\alpha)}(k-2)!n^{\ell(\alpha)-k}e_k(\alpha)\right). \quad (5.1)$$

This was proved by Vakil [46] using techniques from algebraic geometry, and also by Goulden and Jackson [22] using the join-cut equation again, in the following equivalent form.

**Theorem 5.1.3.** *(Goulden and Jackson, [22]). Let $\overline{\gamma}, \overline{\eta}, \overline{\eta}_1, \overline{\eta}_2, \ldots \in \mathbb{Q}[[\mathbf{p}]]$ be as defined in Section 5.2.1. Then, the genus one generating function for Hurwitz numbers is given by*

$$\mathbf{H}_1 = \tfrac{1}{24}\log\frac{1}{1-\overline{\eta}} - \tfrac{1}{24}\overline{\gamma}.$$

Using the same techniques, Goulden and Jackson [21] were also able to give explicit formulas for the genus 2 and 3 generating functions $\mathbf{H}_2, \mathbf{H}_3$ for Hurwitz numbers, and conjectured the following general form.

**Theorem 5.1.4.** *(Goulden, Jackson and Vakil, [24]). Let $\overline{\gamma}, \overline{\eta}, \overline{\eta}_1, \overline{\eta}_2, \ldots \in \mathbb{Q}[[\mathbf{p}]]$ be as defined in Section 5.2.1. Then, for $g \geq 2$, the genus $g$ generating function for Hurwitz numbers has the form*

$$\mathbf{H}_g = \sum_{d=2g-3}^{3g-3}\sum_{\alpha\vdash d}\frac{\overline{c}_{g,\alpha}\,\overline{\eta}_\alpha}{(1-\overline{\eta})^{2g-2+\ell(\alpha)}},$$

*where the constants $\overline{c}_{g,\alpha}$ are rational numbers.*

This is closely related to the following polynomiality result.

**Theorem 5.1.5.** *For each pair $(g,\ell)$ with $(g,\ell) \notin \{(0,1),(0,2)\}$, there is a polynomial $P_{g,\ell}$ in $\ell$ variables such that, for all partitions $\alpha \vdash n$ with $\ell$ parts,*

$$H_g(\alpha) = \frac{n!}{|\mathrm{aut}(\alpha)|}(n+\ell+2g-2)!P_{g,\ell}(\alpha_1,\ldots,\alpha_\ell)\prod_{i=1}^{\ell}\frac{\alpha_i^{\alpha_i}}{\alpha_i!},$$

*where $H_g(\alpha) = [p_\alpha/n!]\mathbf{H}_g$ is a genus $g$ Hurwitz number.*

Similarly to the situation for monotone Hurwitz numbers, where the corresponding polynomiality result follows directly from the general form of the genus expansion of the generating functions, Theorem 5.1.5 can be seen as a consequence of Theorem 5.1.4. However, the connection actually goes the other way, in the sense that Goulden, Jackson and Vakil [24] deduced Theorem 5.1.4 from the remarkable ELSV formula, which gives an expression for the polynomials appearing in Theorem 5.1.5 in terms of the cohomology of the moduli space of curves.

**Theorem 5.1.6.** *(Ekedahl, Lando, Shapiro and Vainshtein, [10]). The polynomials $P_{g,\ell}$ of Theorem 5.1.5 are given by*

$$P_{g,\ell}(\alpha_1,\ldots,\alpha_\ell) = \int_{\overline{\mathcal{M}}_{g,\ell}} \frac{1-\lambda_1+\cdots+(-1)^g\lambda_g}{(1-\alpha_1\psi_1)\cdots(1-\alpha_\ell\psi_\ell)}. \tag{5.2}$$

Here, $\overline{\mathcal{M}}_{g,\ell}$ is the (compact) moduli space of stable $\ell$-pointed genus $g$ curves, $\psi_1,\ldots,\psi_m$ are (complex codimension 1) classes corresponding to these $\ell$ marked points, and $\lambda_k$ is the (complex codimension $k$) $k$th Chern class of the Hodge bundle. The denominators in Equation (5.2) should be interpreted formally, as geometric series. Then, the integral over $\overline{\mathcal{M}}_{g,\ell}$ kills all but the terms of codimension $\dim\overline{\mathcal{M}}_{g,\ell} = 3g-3+\ell$, and "intersects" these terms on $\overline{\mathcal{M}}_{g,\ell}$. The result is a polynomial in the quantities $\alpha_1,\ldots\alpha_\ell$ whose coefficients are intersections of $\psi$ classes and up to one $\lambda$ class, and these intersections are known as Hodge integrals.

From this, the coefficients $\overline{c}_{g,\alpha}$ of $\mathbf{H}_g$ from Theorem 5.1.4 can also be identified up to sign as Hodge integrals. In fact, the high end coefficients, that is, the rational numbers $\overline{c}_{g,\alpha}$ with $|\alpha| = 3g-3$, are precisely the Hodge integrals which only involve $\psi$ classes, and no $\lambda$ class.

These Hodge integrals are also the coefficients of the free energy from Witten's conjecture [47] (which now has several proofs; see, for instance, [33, 31, 32, 37, 41]), also known as the Gromov-Witten potential of a point. For this reason, we refer to them as the **Witten terms** of the expansion for $\mathbf{H}_g$ from Theorem 5.1.4.

The rest of this chapter is devoted to the proof of the following theorem, giving a recursion which uniquely determines the Witten term of $\mathbf{H}_g$. The actual proof can be found at the end of Section 5.3, but first we must develop an algebraic framework for classical Hurwitz generating functions which parallels the framework of Section 3.2 for monotone Hurwitz generating functions. Many of the same concepts are used, but they must be adapted to a different Lagrangian change of variables, and there are enough differences that the translation from one framework to the other is not entirely mechanical.

**Theorem 5.1.7.** *Let $\overline{\xi}_1, \overline{\xi}_2, \ldots \in \mathbb{Q}[[\mathbf{p}]]$ be the auxiliary power series defined in Section 5.2.1, and let $\mathbf{H}_g$ be the genus $g$ generating function for Hurwitz numbers. Then, for $g \geq 2$, we have*

$$\operatorname{Top} \mathbf{H}_g = \sum_{\alpha \vdash (3g-3)} \frac{2^{3g-3} \overline{c}_{g,\alpha}}{(1-\overline{\xi}_3)^{2g-2+\ell(\alpha)}} \prod_{i=1}^{\ell(\alpha)} \frac{(2\alpha_i+1)!}{4^{\alpha_i} \cdot \alpha_i!} \overline{\xi}_{2\alpha_i+3}, \tag{5.3}$$

*where the $\overline{c}_{g,\alpha}$ are the rational numbers appearing in Theorem 5.1.4 for which $|\alpha| = 3g-3$; these rational numbers can be computed recursively from the equation*

$$\overline{\Theta} \operatorname{Top} \mathbf{H}_g = \frac{1}{4} \operatorname{Top} \Pi_1 \left( \Delta_1^2 \operatorname{Top} \mathbf{H}_{g-1} + \sum_{g'=1}^{g-1} \Delta_1 \operatorname{Top} \mathbf{H}_{g'} \cdot \Delta_1 \operatorname{Top} \mathbf{H}_{g-g'} \right), \tag{5.4}$$

*where $\overline{\Theta}$ is the differential operator defined by*

$$\overline{\Theta}\, \overline{\xi}_k = \frac{k}{2} \overline{\xi}_{k+1} - \frac{k}{4} \sum_{i=3}^{k+1} \overline{\xi}_i \overline{\xi}_{k+4-i}, \tag{5.5}$$

*together with the equation*

$$\operatorname{Top} \mathbf{H}_1 = \tfrac{1}{24} \log(1-\overline{\xi}_3)^{-1}. \tag{5.6}$$

## 5.2 Algebraic Framework

As with the monotone Hurwitz generating functions and join-cut equations, it is convenient to work in some transformed variables for the classical Hurwitz generating functions and join-cut equations. The following changes of variables are due to Goulden and Jackson (see [20, 22, 21]), but we adapt them to our notation to highlight the similarities between the classical case and the monotone case.

### 5.2.1 Changes of Variables

In the case of Hurwitz generating functions, the natural change of variables from $\mathbf{p} = (p_1, p_2, \ldots)$ to $\overline{\mathbf{q}} = (\overline{q}_1, \overline{q}_2, \ldots)$, a new set of indeterminates, is defined by

$$\overline{q}_j = p_j \exp\left( j \sum_{k \geq 1} \frac{k^k}{k!} \overline{q}_k \right), \qquad j \geq 1. \tag{5.7}$$

This is an invertible change of variables, and it can be computed explicitly using the Lagrange Implicit Function Theorem in many variables (see [25]). The change of variables from $\mathbf{p}$ to $\overline{\mathbf{q}}$ also corresponds to a change of variables from $\mathbf{x} = (x_1, x_2, \dots)$ to another new set of indeterminates $\overline{\mathbf{y}} = (\overline{y}_1, \overline{y}_2, \dots)$ defined by

$$\overline{y}_i = x_i \exp\left(\sum_{k\geq 1} \frac{k^k}{k!}\overline{q}_k\right), \qquad i \geq 1, \tag{5.8}$$

which is also invertible.

In the monotone Hurwitz case, the power series

$$(1-4y_1)^{-\frac{1}{2}} = \sum_{k\geq 0} \binom{2k}{k} y_1^k$$

frequently appeared. The analogous power series in the classical Hurwitz case is not algebraic in $\overline{y}_1$, so it is convenient to introduce a further change of variables, from $\overline{\mathbf{y}} = (\overline{y}_1, \overline{y}_2, \dots)$ to a final new set of indeterminates $\mathbf{w} = (w_1, w_2, \dots)$, defined by

$$w_i = \overline{y}_i \exp(w_i), \qquad i \geq 1,$$

so that

$$w_i = \sum_{k\geq 1} \frac{k^{k-1}}{k!} \overline{y}_i^k.$$

Then, the analogue of $(1-4y_1)^{-\frac{1}{2}}$ is the series

$$\frac{w_1}{1-w_1} = \sum_{k\geq 1} \frac{k^k}{k!} \overline{y}_1^k.$$

The power series $\overline{\gamma}$, $\overline{\eta}$ and $\overline{\eta}_j$, $j \geq 1$ which appear in the statement of Theorem 5.1.3 and Theorem 5.1.4, are defined by

$$\overline{\gamma} = \sum_{k\geq 1} \frac{k^k}{k!}\overline{q}_k,$$

$$\overline{\eta} = \sum_{k\geq 1} k \cdot \frac{k^k}{k!}\overline{q}_k,$$

$$\overline{\eta}_j = \sum_{k\geq 1} k^{j+1} \cdot \frac{k^k}{k!}\overline{q}_k, \qquad j \geq 1.$$

In particular, $\overline{\gamma}$ is defined so that the change of variables between $\mathbf{p}$ and $\overline{\mathbf{q}}$ becomes $\overline{q}_j = p_j \exp(j\overline{\gamma})$ and the change of variables between $\mathbf{x}$ and $\overline{\mathbf{y}}$ becomes $\overline{y}_i = x_i \exp(\overline{\gamma})$.

As in the monotone Hurwitz case, we have the operator identity

$$\Pi_1 = [x_1^0] + \sum_{k\geq 1} p_k [x_1^k] = [\overline{y}_1^0] + \sum_{k\geq 1} \overline{q}_k [\overline{y}_1^k],$$

and we also have

$$x_1 \frac{\partial}{\partial x_1} = \overline{y}_1 \frac{\partial}{\partial \overline{y}_1} = \frac{w_1}{1-w_1} \frac{\partial}{\partial w_1},$$

so it follows that

$$\begin{aligned}
\overline{\gamma} &= \Pi_1 \frac{w_1}{1-w_1}, \\
\overline{\eta} &= \Pi_1 \frac{w_1}{(1-w_1)^3}, \\
\overline{\eta}_j &= \Pi_1 \left( \frac{w_1}{1-w_1} \frac{\partial}{\partial w_1} \right)^j \frac{w_1}{(1-w_1)^3}, \qquad j \geq 1.
\end{aligned}$$

Finally, the power series $\overline{\xi}_1, \overline{\xi}_2, \ldots$ which appear in the statement of Theorem 5.1.7 are defined by

$$\overline{\xi}_k = \Pi_1 \left( \frac{w_1}{1-w_1} \right)^k, \qquad k \geq 1.$$

Note that each $\overline{\xi}_k$ is purely linear in the indeterminates $\overline{\mathbf{q}}$, and in fact the matrix of coefficients of $\overline{\xi}_1, \overline{\xi}_2, \ldots$ in terms of $\overline{q}_1, \overline{q}_2, \ldots$ is a triangular matrix with ones on the diagonal. Thus, this matrix is invertible, and its inverse gives the coefficients of $\overline{q}_1, \overline{q}_2, \ldots$ in terms of $\overline{\xi}_1, \overline{\xi}_2, \ldots$. From this, it follows that we can identify the rings $\mathbb{Q}[[\mathbf{p}]]$, $\mathbb{Q}[[\overline{\mathbf{q}}]]$ and $\mathbb{Q}[[\overline{\xi}_k]]_{k \geq 1}$; furthermore, any two of $x_i$, $\overline{y}_i$ and $w_i$ can be expressed in terms of each other as power series over this identified ring, so we can also identify the rings $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]]$, $\mathbb{Q}[[\overline{\mathbf{q}}, \overline{\mathbf{y}}]]$ and $\mathbb{Q}[[\overline{\xi}_j; w_k/(1-w_k)]]_{j,k \geq 1}$. This will be convenient in Section 5.3, since weighted degrees in the classical Hurwitz case are defined in terms of the generators $\overline{\xi}_j$ and $w_k/(1-w_k)$.

### 5.2.2 Differential Operators

When manipulating the classical Hurwitz join-cut equations, the basic differential operators

$$\mathrm{D}_k = p_k \frac{\partial}{\partial p_k}, \qquad \mathcal{D} = \sum_{k \geq 1} k p_k \frac{\partial}{\partial p_k}, \qquad \overline{\mathrm{E}}_k = \overline{q}_k \frac{\partial}{\partial \overline{q}_k}, \qquad \overline{\mathcal{E}} = \sum_{k \geq 1} k \overline{q}_k \frac{\partial}{\partial \overline{q}_k},$$

for $k \geq 1$, will again be useful, so we discuss them here. They are defined on $\mathbb{Q}[[\mathbf{p}]] = \mathbb{Q}[[\overline{\mathbf{q}}]]$, then extended to $\mathbb{Q}[[\mathbf{p}, \mathbf{x}]] = \mathbb{Q}[[\overline{\mathbf{q}}, \overline{\mathbf{y}}]]$ by $\mathbb{Q}[[\mathbf{x}]]$-linearity.

As in the monotone Hurwitz case, the $\mathbb{Q}$-linear operators $\mathrm{D}_1, \mathrm{D}_2, \ldots$ and $\mathcal{D}$ have the set $\{p_\alpha \colon \alpha \vdash n,\, n \geq 0\}$ as an eigenbasis, and the operators $\overline{\mathrm{E}}_1, \overline{\mathrm{E}}_2, \ldots$ and $\overline{\mathcal{E}}$ have the set $\{\overline{q}_\alpha \colon \alpha \vdash n,\, n \geq 0\}$ as an eigenbasis. By using the defining relations (5.7) to compute the action of $\overline{\mathrm{E}}_k$ on $p_j$, the operator identity

$$\overline{\mathrm{E}}_k = \mathrm{D}_k - \frac{k^k \overline{q}_k}{k!} \mathcal{D}, \qquad k \geq 1 \tag{5.9}$$

can be verified. It follows that

$$\overline{\mathcal{E}} = (1-\overline{\eta})\mathcal{D},$$

and we can deduce that

$$\mathrm{D}_k = \overline{\mathrm{E}}_k + \frac{k^k \overline{q}_k}{k!(1-\overline{\eta})}\overline{\mathcal{E}}, \qquad k \geq 1. \tag{5.10}$$

Using this fact, together with (5.7) and the $\mathbb{Q}[[\mathbf{x}]]$-linearity of these differential operators, we can compute the action of the lifting operator $\Delta_1$ on $\overline{\mathbf{q}}$ and $\overline{\mathbf{y}}$, obtaining

$$\Delta_1(\overline{q}_k) = k\overline{y}_1^k + k\overline{q}_k \frac{w_1}{(1-w_1)^3(1-\overline{\eta})}, \quad \Delta_1(\overline{y}_k) = \overline{y}_k \frac{w_1}{(1-w_1)^3(1-\overline{\eta})}, \quad k \geq 1.$$

Thus, we can express the lifting operators as

$$\Delta_1 = \sum_{k\geq 1}\left(k\overline{y}_1^k \frac{\partial}{\partial \overline{q}_k}\right) + \frac{w_1}{(1-w_1)^3(1-\overline{\eta})}\sum_{k\geq 1}\left(k\overline{q}_k\frac{\partial}{\partial\overline{q}_k} + \overline{y}_k \frac{\partial}{\partial \overline{y}_k}\right). \tag{5.11}$$

## 5.3 Witten Terms

To obtain information on the high end coefficients of the genus $g$ generating function $\mathbf{H}_g$ for Hurwitz numbers, that is, the rational numbers $\overline{c}_{g,\alpha}$ of Theorem 5.1.4 such that $|\alpha| = 3g-3$, we can take the higher genus join-cut equation from Theorem 2.1.5 and restrict our attention to the top degree terms. This requires a suitable notion of weighted degree for the auxiliary power series introduced in Section 5.2.1, and some work in rewriting the join-cut equation in terms of these power series.

**Definition 5.3.1.** Let $F$ be an element of $\mathbb{Q}[[\overline{\mathbf{q}},\overline{\mathbf{y}}]]$. Then, $F$ can be expanded as a power series in the quantities $\overline{\xi}_1, \overline{\xi}_2, \ldots$ and $\frac{w_1}{1-w_1}, \frac{w_2}{1-w_2}, \ldots$ in a unique way. The **weighted degree** $\deg F$ of $F$ is the maximum degree of a monomial in this expansion, where we set $\deg \overline{\xi}_k = (k-3)/2$ and $\deg w_k/(1-w_k) = 1/2$. By convention, $\deg F = \infty$ if there are monomials of arbitrarily high degree in the expansion, and $\deg F = -\infty$ if $F = 0$.

**Definition 5.3.2.** Let $R$ be the subring of $\mathbb{Q}[[\overline{\mathbf{q}},\overline{\mathbf{y}}]]$ consisting of those elements $F$ such that $\deg F < \infty$. For such an element $F \in \mathbb{Q}[[\overline{\mathbf{q}},\overline{\mathbf{y}}]]$, let $\operatorname{Top} F$ be the sum of those monomials in its expansion in terms of $\overline{\xi}_1, \overline{\xi}_2, \ldots$ and $\frac{w_1}{1-w_1}, \frac{w_2}{1-w_2}, \ldots$ of degree equal to $\deg F$. This defines the **top degree extraction operator** $\operatorname{Top}\colon R \to R$.

The following computational lemma records how the top degree extraction operator interacts with the lifting and projection operators, as well as the series $\overline{\gamma}, \overline{\eta}, \overline{\eta}_1, \overline{\eta}_2, \ldots$.

**Lemma 5.3.3.**

1. *We have*

$$\operatorname{Top}\overline{\gamma} = \overline{\xi}_1, \qquad \operatorname{Top}\overline{\eta} = \overline{\xi}_3, \qquad \operatorname{Top}\overline{\eta}_k = \frac{(2k+1)!}{2^k \cdot k!}\overline{\xi}_{2k+3}, \quad k \geq 1.$$

2. *We have*

$$\begin{aligned} \Delta_1 \frac{w_1}{1-w_1} &= \frac{w_1^2}{(1-w_1)^6(1-\overline{\eta})}, \\ \Delta_1 \frac{w_2}{1-w_2} &= \frac{w_1 w_2}{(1-w_1)^3(1-w_2)^3(1-\overline{\eta})}, \\ \Delta_1 \overline{\xi}_k &= \frac{k w_1}{(1-w_1)^3}\left(\frac{\overline{\xi}_{k+2} + 2\overline{\xi}_{k+1} + \overline{\xi}_k}{1-\overline{\eta}} + \Big(\frac{w_1}{1-w_1}\Big)^{k-1}\right), \qquad k \geq 1, \end{aligned}$$

*so that*

$$\begin{aligned} \operatorname{Top}\Delta_1 \frac{w_1}{1-w_1} &= \Big(\frac{w_1}{1-w_1}\Big)^6 (1-\overline{\xi}_3)^{-1}, \\ \operatorname{Top}\Delta_1 \frac{w_2}{1-w_2} &= \Big(\frac{w_1}{1-w_1}\Big)^3 \Big(\frac{w_2}{1-w_2}\Big)^3 (1-\overline{\xi}_3)^{-1}, \\ \operatorname{Top}\Delta_1 \overline{\xi}_k &= k\Big(\frac{w_1}{1-w_1}\Big)^3 \left(\overline{\xi}_{k+2}(1-\overline{\xi}_3)^{-1} + \Big(\frac{w_1}{1-w_1}\Big)^{k-1}\right), \qquad k \geq 1. \end{aligned}$$

*Thus, if* $F \in \mathbb{Q}[[\overline{\mathbf{q}}, \overline{\mathbf{y}}]]$ *with* $\deg F = d$ *and* $d \neq 0$*, then* $\deg \Delta_1 F = d + \frac{5}{2}$*, and* $\operatorname{Top}\Delta_1 F = \operatorname{Top}\Delta_1 \operatorname{Top} F$*.*

3. *If* $F \in \mathbb{Q}[[\overline{\mathbf{q}}, \overline{\mathbf{y}}]]$ *with* $[\overline{y}_1^0]F = 0$ *and* $\deg F = d$*, then* $\deg \Pi_1 F = d - \frac{3}{2}$*, and* $\operatorname{Top}\Pi_1 F = \operatorname{Top}\Pi_1 \operatorname{Top} F$*.*

*Proof.*

1. Note that $\overline{\gamma} = \overline{\xi}_1$ and $\overline{\eta} = \overline{\xi}_3 + 2\overline{\xi}_2 + \overline{\xi}_1$. For $k \geq 1$, we have

$$\overline{\eta}_k = \Pi_1 \left(\overline{y}_1 \frac{\partial}{\partial \overline{y}_1}\right)^{k+1} \frac{w_1}{1-w_1}$$

and

$$\overline{y}_1 \frac{\partial}{\partial \overline{y}_1} = \left(\Big(\frac{w_1}{1-w_1}\Big)^3 + 2\Big(\frac{w_1}{1-w_1}\Big)^2 + \Big(\frac{w_1}{1-w_1}\Big)\right)\frac{\partial}{\partial w_1},$$

so it follows by induction on $k$ that the leading term of $\overline{\eta}_k$ is

$$\Pi_1\, 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2k+1) \cdot \Big(\frac{w_1}{1-w_1}\Big)^{2k+3}.$$

2. The values of $\Delta_1 \frac{w_1}{1-w_1}$ and $\Delta_1 \frac{w_2}{1-w_2}$ can be computed directly from the expression (5.11) for $\Delta_1$, recalling that $\overline{y}_1 \frac{\partial}{\partial \overline{y}_1} = \frac{w_1}{1-w_1} \frac{\partial}{\partial w_1}$. Then, the value of $\Delta_1 \overline{\xi}_k = \Delta_1 \Pi_2 \left(\frac{w_2}{1-w_2}\right)^k$ can be computed using the expression for $\Delta_1 \Pi_2$ given in Lemma 2.2.4, recalling that $x_2 \frac{\partial}{\partial x_2} = \frac{w_1}{1-w_1} \frac{\partial}{\partial w_1}$ as well. To compute the top degree terms, note that we have $\frac{w_1}{(1-w_1)^3} = \left(\frac{w_1}{1-w_1}\right)^3 + 2\left(\frac{w_1}{1-w_1}\right)^2 + \left(\frac{w_1}{1-w_1}\right)$.

3. The statement about $\Pi_1$ follows directly from the definition of $\overline{\xi}_k$ and the definition of weighted degree. □

Having established a suitable notion of top degree terms, we now turn to the task of rewriting the higher genus join-cut equation for Hurwitz generating functions. The following two propositions give the details.

**Proposition 5.3.4.** *The genus zero generating function* $\mathbf{H}_0$ *for Hurwitz numbers satisfies the equation*

$$\Delta_1 \mathbf{H}_0 = \Pi_2 \left( w_1 - \frac{\overline{y}_2}{\overline{y}_1 - \overline{y}_2} + \frac{w_2}{(w_1 - w_2)(1 - w_2)} + \frac{w_2}{1 - w_2} \right).$$

*Proof.* This is essentially a repackaging of some computational results from [20] in our notation. Proposition 3.1 from [20] states that

$$\frac{\partial}{\partial p_j} \mathbf{H}_0 = \frac{j^{j-2}}{j!} \exp(j\overline{\gamma}) - \frac{j^{j-1}}{j!} \sum_{k \geq 1} \frac{k^{k+1}}{k!} \frac{\exp(j\overline{\gamma})\overline{q}_k}{j+k}, \qquad j \geq 1,$$

from which we deduce that

$$\Delta_1 \mathbf{H}_0 = \sum_{j \geq 1} j x_1^j \frac{\partial}{\partial p_j} \mathbf{H}_0 = \sum_{j \geq 1} \frac{j^{j-1}}{j!} \overline{y}_1^j - \Pi_2 \sum_{j,k \geq 1} \frac{j^j}{j!} \frac{k^{k+1}}{k!} \frac{\overline{y}_1^j \overline{y}_2^k}{j+k}. \tag{5.12}$$

In the course of proving Proposition 3.2 from [20], the identity

$$\sum_{j \geq 0} \sum_{k \geq 1} \frac{j^j}{j!} \frac{k^{k+1}}{k!} \frac{\overline{y}_1^j \overline{y}_2^k}{j+k} = \frac{\overline{y}_2}{\overline{y}_1 - \overline{y}_2} - \frac{w_2}{(w_1 - w_2)(1 - w_2)} \tag{5.13}$$

is obtained. Setting $\overline{y}_1 = 0$ yields the term corresponding to $j = 0$,

$$\sum_{k \geq 1} \frac{k^{k+1}}{k!} \frac{\overline{y}_2^k}{k} = \frac{w_2}{1 - w_2}. \tag{5.14}$$

Subtracting (5.14) from (5.13) gives the second term of (5.12). □

**Proposition 5.3.5.** *For $g \geq 1$, the join-cut equation of Theorem 2.1.5 for the genus $g$ generating function $\mathbf{H}_g$ for Hurwitz numbers can be rewritten using the notation of Section 5.2 as*

$$\Pi_1(1-w_1)\,\Delta_1\,\mathbf{H}_g + \left(\sum_{k\geq 1}\overline{\mathrm{E}}_k\right)\mathbf{H}_g + (2g-2)\mathbf{H}_g$$
$$-\frac{1}{2}\,\Pi_1\,\Pi_2(1-w_1)^{-1}(1-w_2)^{-1}\frac{\frac{w_2}{1-w_2}\,\Delta_1\,\mathbf{H}_g - \frac{w_1}{1-w_1}\,\Delta_2\,\mathbf{H}_g}{\frac{w_1}{1-w_1}-\frac{w_2}{1-w_2}}$$
$$= \frac{1}{2}\,\Pi_1\left(\Delta_1^2\,\mathbf{H}_{g-1} + \sum_{g'=1}^{g-1}\Delta_1\,\mathbf{H}_{g'}\cdot\Delta_1\,\mathbf{H}_{g-g'}\right).$$

*Proof.* Recall that the join-cut equation of Theorem 2.1.5 for higher genera is

$$\sum_{k\geq 1}\left((k+1)p_k\frac{\partial \mathbf{H}_g}{\partial p_k}\right) + (2g-2)\mathbf{H}_g$$
$$-\frac{1}{2}\sum_{i,j\geq 1}\left(2ijp_{i+j}\frac{\partial \mathbf{H}_0}{\partial p_i}\frac{\partial \mathbf{H}_g}{\partial p_j} + (i+j)p_ip_j\frac{\partial \mathbf{H}_g}{\partial p_{i+j}}\right)$$
$$= \frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2 \mathbf{H}_{g-1}}{\partial p_i\partial p_j} + \sum_{g'=1}^{g-1} ijp_{i+j}\frac{\partial \mathbf{H}_{g'}}{\partial p_i}\frac{\partial \mathbf{H}_{g-g'}}{\partial p_j}\right). \tag{5.15}$$

We proceed by rewriting each term of (5.15). Using (5.9) to translate from $\mathrm{D}_k$ to $\overline{\mathrm{E}}_k$, we have

$$\sum_{k\geq 1}\left((k+1)p_k\frac{\partial \mathbf{H}_g}{\partial p_k}\right) = \mathcal{D}\mathbf{H}_g + \left(\sum_{k\geq 1}\mathrm{D}_k\right)\mathbf{H}_g$$
$$= (1+\overline{\gamma})\mathcal{D}\mathbf{H}_g + \left(\sum_{k\geq 1}\overline{\mathrm{E}}_k\right)\mathbf{H}_g.$$

Next, as noted in Section 2.2, we have

$$-\frac{1}{2}\sum_{i,j\geq 1}(i+j)p_ip_j\frac{\partial \mathbf{H}_g}{\partial p_{i+j}} = -\frac{1}{2}\,\Pi_1\,\Pi_2\,\underset{1\to 2}{\mathrm{Split}}\,\Delta_1\,\mathbf{H}_g,$$

and for the right-hand side, we have

$$\frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2 \mathbf{H}_{g-1}}{\partial p_i\partial p_j} + \sum_{g'=1}^{g-1} ijp_{i+j}\frac{\partial \mathbf{H}_{g'}}{\partial p_i}\frac{\partial \mathbf{H}_{g-g'}}{\partial p_j}\right)$$
$$= \frac{1}{2}\,\Pi_1\left(\Delta_1^2\,\mathbf{H}_{g-1} + \sum_{g'=1}^{g-1}\Delta_1\,\mathbf{H}_{g'}\cdot\Delta_1\,\mathbf{H}_{g-g'}\right).$$

The term $(2g-2)\mathbf{H}_g$ needs no rewriting, so the only term remaining of (5.15) is

$$-\frac{1}{2}\sum_{i,j\geq 1} 2ijp_{i+j}\frac{\partial \mathbf{H}_0}{\partial p_i}\frac{\partial \mathbf{H}_g}{\partial p_j} = -\,\Pi_1\left(\Delta_1\,\mathbf{H}_0\cdot\Delta_1\,\mathbf{H}_g\right),$$

which we can split up and rewrite further using Proposition 5.3.4. We have

$$\begin{aligned}-\,\Pi_1\left(\Delta_1\,\mathbf{H}_0\cdot\Delta_1\,\mathbf{H}_g\right) = \Pi_1\,\Pi_2\Bigg((1-w_1)\,\Delta_1\,\mathbf{H}_g - \Big(1+\frac{w_2}{1-w_2}\Big)\,\Delta_1\,\mathbf{H}_g \\ + \frac{\overline{y}_2\,\Delta_1\,\mathbf{H}_g}{\overline{y}_1-\overline{y}_2} - \frac{\frac{w_2}{1-w_2}\,\Delta_1\,\mathbf{H}_g}{w_1-w_2}\Bigg),\end{aligned} \tag{5.16}$$

and the first two terms of (5.16) give

$$\Pi_1(1-w_1)\,\Delta_1\,\mathbf{H}_g, \qquad -(1+\overline{\gamma})\mathcal{D}\mathbf{H}_g.$$

For the last two terms of (5.16), we can symmetrize with respect to $\overline{y}_1$ and $\overline{y}_2$ to obtain

$$\begin{aligned}\Pi_1\,\Pi_2\,\frac{\overline{y}_2\,\Delta_1\,\mathbf{H}_g}{\overline{y}_1-\overline{y}_2} &= \frac{1}{2}\,\Pi_1\,\Pi_2\left(\frac{\overline{y}_2\,\Delta_1\,\mathbf{H}_g}{\overline{y}_1-\overline{y}_2} + \frac{\overline{y}_1\,\Delta_2\,\mathbf{H}_g}{\overline{y}_2-\overline{y}_1}\right) \\ &= \frac{1}{2}\,\Pi_1\,\Pi_2\,\underset{1\to 2}{\mathrm{Split}}\,\Delta_1\,\mathbf{H}_g\end{aligned}$$

and

$$\begin{aligned}-\,\Pi_1\,\Pi_2\,\frac{\frac{w_2}{1-w_2}\,\Delta_1\,\mathbf{H}_g}{w_1-w_2} &= -\frac{1}{2}\,\Pi_1\,\Pi_2\left(\frac{\frac{w_2}{1-w_2}\,\Delta_1\,\mathbf{H}_g}{w_1-w_2} + \frac{\frac{w_1}{1-w_1}\,\Delta_2\,\mathbf{H}_g}{w_2-w_1}\right) \\ &= -\frac{1}{2}\,\Pi_1\,\Pi_2\,\frac{\frac{w_2}{1-w_2}\,\Delta_1\,\mathbf{H}_g - \frac{w_1}{1-w_1}\,\Delta_2\,\mathbf{H}_g}{(1-w_1)(1-w_2)\Big(\frac{w_1}{1-w_1}-\frac{w_2}{1-w_2}\Big)}.\end{aligned}$$

After rewriting each of the terms of (5.15) as above, the terms

$$(1+\overline{\gamma})\mathcal{D}\mathbf{H}_g, \qquad \frac{1}{2}\,\Pi_1\,\Pi_2\,\underset{1\to 2}{\mathrm{Split}}\,\Delta_1\,\mathbf{H}_g$$

cancel out, leaving the claimed equation. □

Finally, we can give the promised proof of the recursion for the top degree coefficients of classical Hurwitz generating functions.

*Proof of Theorem 5.1.7.* This proof closely parallels the proof of Theorem 4.4.6.

The expressions (5.3) and (5.6) come from applying the operator Top to Theorems 5.1.3 and 5.1.4, using Lemma 5.3.3.

The equation (5.4) comes from applying the operator Top to the join-cut equation for Hurwitz numbers as written in Proposition 5.3.5, scaled by a factor of $\frac{1}{2}$. The right-hand sides clearly match, and for the left-hand sides, we are dealing with $\mathbb{Q}$-linear differential operators, so it suffices to compare their actions on the basis $\overline{\xi}_1, \overline{\xi}_2, \ldots$. As noted in Lemma 5.3.3, for $k \geq 1$, we have

$$\Delta_1 \overline{\xi}_k = \frac{kw_1}{(1-w_1)^3} \left( \frac{\overline{\xi}_{k+2} + 2\overline{\xi}_{k+1} + \overline{\xi}_k}{1-\overline{\eta}} + \left(\frac{w_1}{1-w_1}\right)^{k-1} \right),$$

and since $\frac{w_1}{(1-w_1)^2} = \left(\frac{w_1}{1-w_1}\right)^2 + \frac{w_1}{1-w_1}$, it follows that

$$\frac{1}{2} \operatorname{Top} \Pi_1 (1-w_1)\, \Delta_1 \overline{\xi}_k = \frac{k}{2}\overline{\xi}_{k+1} + \frac{k}{2}\overline{\xi}_{k+2}\overline{\xi}_2(1-\overline{\xi}_3)^{-1}. \tag{5.17}$$

The terms $\left(\sum_{j\geq 1} \overline{\mathrm{E}}_j\right)\overline{\xi}_k$ and $(2g-2)\overline{\xi}_k$ have strictly smaller weighted degree, so they simply disappear. For $j \geq 2$, we have

$$\frac{\frac{w_2}{1-w_2}\left(\frac{w_1}{1-w_1}\right)^j - \frac{w_1}{1-w_1}\left(\frac{w_2}{1-w_2}\right)^j}{\frac{w_1}{1-w_1} - \frac{w_2}{1-w_2}} = \sum_{i=1}^{j-1} \left(\frac{w_1}{1-w_1}\right)^i \left(\frac{w_2}{1-w_2}\right)^{j-i},$$

so it follows that

$$\begin{aligned} -\frac{1}{4} \operatorname{Top} \Pi_1\, \Pi_2 (1-w_1)^{-1}(1-w_2)^{-1} \frac{\frac{w_2}{1-w_2}\, \Delta_1 \overline{\xi}_k - \frac{w_1}{1-w_1}\, \Delta_2 \overline{\xi}_k}{\frac{w_1}{1-w_1} - \frac{w_2}{1-w_2}} \\ = -\frac{k}{2}\overline{\xi}_{k+2}\overline{\xi}_2\overline{\xi}_3(1-\overline{\xi}_3)^{-1} - \frac{k}{4}\sum_{i=2}^{k+2}\overline{\xi}_i\overline{\xi}_{k+4-i}. \end{aligned} \tag{5.18}$$

The combined expression from (5.17) and (5.18) matches the definition of $\overline{\Theta}\, \overline{\xi}_k$ through (5.5).

This shows that all the equations given in the theorem statement hold; given this, the proof that the rational numbers $\overline{c}_{g,\alpha}$ for $\alpha \vdash 3g-3$ are uniquely determined is identical to the argument given at the end of the proof of Theorem 4.4.6. ☐

# Chapter 6

# Conclusion, and a Hint of Geometry

Having developed a theory of monotone Hurwitz numbers in Chapters 3 and 4 and reviewed the corresponding parts classical Hurwitz theory in Chapter 5, we conclude this thesis with a brief discussion of the many similarities, both qualitative and quantitative, and some of the differences between the monotone and the classical cases. In particular, we highlight the fact that the top degree, or Witten, terms for the generating functions in both cases are essentially equal, which strongly suggests a (currently missing) geometric interpretation in the monotone case.

## 6.1 A Tempting Analogy

As noted in the course of Chapter 4, each of our theorems about monotone Hurwitz numbers has an analogue in the classical Hurwitz world, and it is tempting to posit a kind of dictionary for translating between the two worlds. For example, in genus one, we have the formula

$$\vec{H}_1(\alpha) = \frac{1}{24}\frac{n!}{|\mathrm{aut}(\alpha)|}\prod_{i=1}^{\ell(\alpha)}\binom{2\alpha_i}{\alpha_i} \times \left((2n+1)^{\overline{\ell(\alpha)}} - 3(2n+1)^{\overline{\ell(\alpha)-1}} - \sum_{k=2}^{\ell(\alpha)}(k-2)!(2n+1)^{\overline{\ell(\alpha)-k}}e_k(2\alpha+1)\right) \tag{4.1}$$

from Theorem 4.2.1, and from Theorem 5.1.2 we have

$$H_1(\alpha) = \frac{1}{24}\frac{n!}{|\mathrm{aut}(\alpha)|}(n+\ell(\alpha))!\prod_{i=1}^{\ell(\alpha)}\frac{\alpha_i^{\alpha_i}}{\alpha_i!}$$

$$\times \left( n^{\ell(\alpha)} - n^{\ell(\alpha)-1} - \sum_{k=2}^{\ell(\alpha)} (k-2)! n^{\ell(\alpha)-k} e_k(\alpha) \right). \quad (5.1)$$

The extra factor of $(n + \ell(\alpha))!$ in (5.1) can be attributed to the fact that the generating function $\mathbf{H} \in \mathbb{Q}[[\mathbf{p}, t]]$ has $t$ as an exponential marker for the number of transpositions, whereas the generating function $\vec{\mathbf{H}} \in \mathbb{Q}[[\mathbf{p}, t]]$ has $t$ as an ordinary marker; the numbers $\alpha_i^{\alpha_i}$ are related to the count of labelled trees, whereas the numbers $\binom{2\alpha_i}{\alpha_i}$ are related to unlabelled trees; and in going from (5.1) to (4.1), powers of $n$ are replaced by rising powers of $(2n + 1)$, and each $\alpha_i$ is replaced by $(2\alpha_i + 1)$. However, there does not seem to be a simple way to make this correspondence completely precise, let alone rigorous.

It is also tempting to attribute many of the similarities between monotone and classical Hurwitz numbers to the fact that they are both uniquely determined by very similar join-cut equations. Indeed, recall that the (global, transitive) monotone Hurwitz join-cut equation of Corollary 2.3.3 can be written as

$$\frac{\Delta_1 \vec{\mathbf{H}} - x_1}{t} = (\Delta_1 \vec{\mathbf{H}})^2 + \Delta_1^2 \vec{\mathbf{H}} + \Pi_2 \operatorname*{Split}_{1\to 2} \Delta_1 \vec{\mathbf{H}}, \quad (6.1)$$

and the (global, transitive) classical Hurwitz join-cut equation of Corollary 2.1.3 can be written as

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2} \Pi_1 \left( (\Delta_1 \mathbf{H})^2 + \Delta_1^2 \mathbf{H} + \Pi_2 \operatorname*{Split}_{1\to 2} \Delta_1 \mathbf{H} \right), \quad (6.2)$$

the main difference being that the left-hand side of (6.1) is a divided difference, whereas for (6.2) it is a derivative. Also, our methods for verifying the genus zero and one solutions are similar to those of [20, 22] from a high level perspective. However, the technical details are sufficiently different that we kept expecting the analogy to break down. Indeed, the analogy does seem to break down somewhat beyond genus one: as discussed in Chapter 5, the only known proof of the genus expansion for $\mathbf{H}_g$ for $g \geq 2$ given in Theorem 5.1.4 uses the ELSV formula, a high-powered result from algebraic geometry; in contrast, our proof of the genus expansion for $\vec{\mathbf{H}}_g$ for $g \geq 2$, given in Theorem 3.1.3, is purely algebraic.

It may be possible to salvage the analogy between monotone and classical Hurwitz generating functions for $g \geq 2$, but this does not seem to be a straightforward, mechanical process. Having an algebraic proof of Theorem 5.1.4 would be interesting, however.

## 6.2 Witten Terms

On a different note, there is an important aspect of the theory where the similarities are not only qualitative, but numerical, as the following theorem shows.

**Theorem 6.2.1.** *For $g \geq 2$ and $\alpha \vdash 3g-3$, let $c_{g,\alpha}$ be the top degree coefficient of the genus $g$ monotone Hurwitz generating function indexed by $\alpha$ in the expansion from Theorem 3.1.3, and let $\overline{c}_{g,\alpha}$ be the corresponding top degree coefficient of the genus $g$ classical Hurwitz generating function in the expansion from Theorem 5.1.4. Then, we have*

$$c_{g,\alpha} = 2^{3g-3}\overline{c}_{g,\alpha}.$$

*Proof.* This follows by comparing the recurrences of Theorems 4.4.6 and 5.1.7 for the Witten terms of monotone and classical Hurwitz generating functions $\vec{\mathbf{H}}_g$ and $\mathbf{H}_g$, respectively, via the correspondence

$$\begin{aligned} c_{g,\alpha} &\leftrightarrow 2^{3g-3}\overline{c}_{g,\alpha} & \operatorname{Top}\vec{\mathbf{H}}_g &\leftrightarrow \operatorname{Top}\mathbf{H}_g \\ \xi &\leftrightarrow \overline{\xi} & \operatorname{Top}\Pi_1\,\Delta_1^2\,\xi_k &\leftrightarrow \frac{1}{4}\operatorname{Top}\Pi_1\,\Delta_1^2\,\overline{\xi}_k \\ \Theta &\leftrightarrow \overline{\Theta} & \operatorname{Top}\Pi_1\left(\Delta_1\,\xi_j\cdot\Delta_1\,\xi_k\right) &\leftrightarrow \frac{1}{4}\operatorname{Top}\Pi_1\left(\Delta_1\,\overline{\xi}_j\cdot\Delta_1\,\overline{\xi}_k\right). \end{aligned}$$

The two recurrences are structurally identical, and they have the same initial conditions, so it follows that $c_{g,\alpha} = 2^{3g-3}\overline{c}_{g,\alpha}$. □

Recall from Chapter 5 that these top degree coefficients $\overline{c}_{g,\alpha}$ for classical Hurwitz numbers are actually Hodge integrals (specifically, the ones which only involve $\psi$ classes and no $\lambda$ class), and as such they are intimately related to the cohomology of the moduli space of curves $\overline{\mathcal{M}}_{g,\ell}$. The other coefficients $\overline{c}_{g,\alpha}$ are also Hodge integrals, namely the ones which do involve a $\lambda$ class. Thus, given that the top degree coefficients $c_{g,\alpha}$ for monotone Hurwitz numbers are also Hodge integrals (up to a predictable power of two), we conjecture that the other coefficients $c_{g,\alpha}$ have a geometric significance, perhaps related to the cohomology of some other natural space, in which case there may be an ELSV-type formula for the polynomials $\vec{P}_{g,\ell}$ of Theorem 4.1.1, similar to Theorem 5.1.6. It is unclear to us what exact form such a formula would take, but there may be a clue in the fact that the summation sets for the generating functions $\vec{\mathbf{H}}_g$ and $\mathbf{H}_g$, $g \geq 2$ given in Theorems 3.1.3 and 5.1.4 differ slightly: in the classical Hurwitz case, only terms of degree between $2g-3$ and $3g-3$ appear, whereas in the monotone Hurwitz case, terms of degree lower than $2g-3$ also appear.

# Appendix A

# Coefficients for Monotone Hurwitz Generating Functions

The following equations give the genus two and three generating functions for the monotone Hurwitz numbers, as described in Theorem 3.1.3. Tables A.1, A.2, A.3 and A.4 give the coefficients for genus 2, 3, 4 and 5, respectively.

$$720\vec{\mathbf{H}}_2 = -3 + \frac{3}{(1-\eta)^2} + \frac{5\eta_3 - 6\eta_2 - 5\eta_1}{(1-\eta)^3} + \frac{29\eta_{21} - 10\eta_{11}}{(1-\eta)^4} + \frac{28\eta_{111}}{(1-\eta)^5}.$$

$$\begin{aligned}
90720\vec{\mathbf{H}}_3 = 90 &+ \frac{-90}{(1-\eta)^4} + \frac{70\eta_6 + 63\eta_5 - 377\eta_4 - 189\eta_3 + 667\eta_2 + 126\eta_1}{(1-\eta)^5} \\
&+ \frac{1078\eta_{51} + 2012\eta_{42} + 1214\eta_3^2 + 1209\eta_{41}}{(1-\eta)^6} \\
&+ \frac{1998\eta_{32} - 3914\eta_{31} - 2627\eta_{22} - 2577\eta_{21} + 1967\eta_{11}}{(1-\eta)^6} \\
&+ \frac{8568\eta_{411} + 26904\eta_{321} + 5830\eta_{222} + 10092\eta_{311}}{(1-\eta)^7} \\
&+ \frac{13440\eta_{221} - 20322\eta_{211} - 4352\eta_{111}}{(1-\eta)^7} \\
&+ \frac{44520\eta_{3111} + 86100\eta_{2211} + 49980\eta_{2111} - 15750\eta_{1111}}{(1-\eta)^8} \\
&+ \frac{162120\eta_{21111} + 31080\eta_{11111}}{(1-\eta)^9} + \frac{68600\eta_{111111}}{(1-\eta)^{10}}.
\end{aligned}$$

Table A.1: Coefficients for the genus 2 monotone Hurwitz generating function in the form $\vec{\mathbf{H}}_2 = \frac{1}{240}\big((1-\eta)^{-2}-1\big) + \sum_{d=1}^{3}\sum_{\alpha\vdash d} c_{2,\alpha}\eta_\alpha(1-\eta)^{-2-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^4 3^2 5^1 = 720 = 6!$.

| $\alpha$ | $N \cdot c_{2,\alpha}$ | $\alpha$ | $N \cdot c_{2,\alpha}$ | $\alpha$ | $N \cdot c_{2,\alpha}$ |
|---|---|---|---|---|---|
| **3** | **5** | **21** | **29** | **111** | **28** |
| 2 | −6 | 11 | −10 | | |
| 1 | −5 | | | | |

Table A.2: Coefficients for the genus 3 monotone Hurwitz generating function in the form $\vec{\mathbf{H}}_3 = \frac{-1}{1008}\big((1-\eta)^{-4}-1\big) + \sum_{d=1}^{6}\sum_{\alpha\vdash d} c_{3,\alpha}\eta_\alpha(1-\eta)^{-4-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^5 3^4 5^1 7^1 = 90720 = 9!/4$.

| $\alpha$ | $N \cdot c_{3,\alpha}$ | $\alpha$ | $N \cdot c_{3,\alpha}$ | $\alpha$ | $N \cdot c_{3,\alpha}$ |
|---|---|---|---|---|---|
| **6** | **70** | **51** | **1078** | **411** | **8568** |
| 5 | 63 | **42** | **2012** | **321** | **26904** |
| 4 | −377 | **33** | **1214** | **222** | **5830** |
| 3 | −189 | 41 | 1209 | 311 | 10092 |
| 2 | 667 | 32 | 1998 | 221 | 13440 |
| 1 | 126 | 31 | −3914 | 211 | −20322 |
| | | 22 | −2627 | 111 | −4352 |
| | | 21 | −2577 | | |
| **3111** | **44520** | 11 | 1967 | | |
| **2211** | **86100** | | | | |
| 2111 | 49980 | **21111** | **162120** | | |
| 1111 | −15750 | 11111 | 31080 | $[\mathbf{1^6}]$ | **68600** |

Table A.3: Coefficients for the genus 4 monotone Hurwitz generating function in the form $\vec{\mathbf{H}}_4 = \frac{1}{1440}\big((1-\eta)^{-6} - 1\big) + \sum_{d=1}^{9} \sum_{\alpha \vdash d} c_{4,\alpha} \eta_\alpha (1-\eta)^{-6-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^8 3^5 5^2 7^1 = 10886400 = 3 \cdot 10!$.

| $\alpha$ | $N \cdot c_{4,\alpha}$ | $\alpha$ | $N \cdot c_{4,\alpha}$ | $\alpha$ | $N \cdot c_{4,\alpha}$ |
|---|---|---|---|---|---|
| **9** | **700** | **81** | **20860** | **711** | **320544** |
| 8 | 3360 | **72** | **56820** | **621** | **1559328** |
| 7 | −2151 | **63** | **106860** | **531** | **2594400** |
| 6 | −25260 | **54** | **146100** | **522** | **1677480** |
| 5 | −2154 | 71 | 100048 | **441** | **1535040** |
| 4 | 74640 | 62 | 252288 | **432** | **4831680** |
| 3 | 11561 | 53 | 426240 | **333** | **980000** |
| 2 | −98100 | 44 | 254190 | 611 | 1495680 |
| 1 | −7956 | 61 | −16679 | 521 | 6620304 |
| | | 52 | −16677 | 431 | 9676368 |
| | | 43 | −4131 | 422 | 6339228 |
| | | 51 | −557480 | 332 | 7771680 |
| | | 42 | −1075560 | 511 | 205344 |
| | | 33 | −650592 | 421 | 1242384 |
| | | 41 | −201854 | 331 | 866256 |
| **6111** | **3297504** | 32 | −365544 | 322 | 1259496 |
| **5211** | **21227136** | 31 | 1116016 | 411 | −6017460 |
| **4311** | **30631104** | 22 | 769422 | 321 | −19277040 |
| **4221** | **39502848** | 21 | 349845 | 222 | −4256028 |
| **3321** | **48172320** | 11 | −400212 | 311 | −3015120 |
| **3222** | **20748112** | | | 221 | −4192608 |
| 5111 | 14602560 | **51111** | **24907680** | 211 | 7975908 |
| 4211 | 83635104 | **42111** | **185243520** | 111 | 887776 |
| 3311 | 51414048 | **33111** | **113077440** | | |
| 3221 | 134339520 | **32211** | **437085600** | | |
| 2222 | 14666190 | **22221** | **93963100** | | |
| 4111 | 4786824 | 41111 | 101876880 | **411111** | **143330880** |
| 3211 | 26980800 | 32111 | 654024000 | **321111** | **1126957440** |
| 2221 | 12703040 | 22211 | 426890800 | **222111** | **967702400** |
| 3111 | −40578272 | 31111 | 43375920 | 311111 | 523729920 |
| 2211 | −79416204 | 22111 | 120797880 | 221111 | 1707291600 |
| 2111 | −22258488 | 21111 | −184899680 | 211111 | 233397360 |
| 1111 | 8080814 | 11111 | −18654020 | 111111 | −94694320 |
| $\mathbf{[3, 1^6]}$ | **633785600** | $\mathbf{[2, 1^7]}$ | **2128810880** | $\mathbf{[1^9]}$ | **581398720** |
| $\mathbf{[2^2, 1^5]}$ | **2447642400** | $[1^8]$ | 666360800 | | |
| $[2, 1^6]$ | 1982016960 | | | | |
| $[1^7]$ | 112171360 | | | | |

Table A.4: Coefficients for the genus 5 monotone Hurwitz generating function in the form $\vec{\mathbf{H}}_5 = \frac{-1}{1056}\big((1-\eta)^{-8}-1\big) + \sum_{d=1}^{12}\sum_{\alpha\vdash d} c_{5,\alpha}\eta_\alpha(1-\eta)^{-8-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^9 3^6 5^2 7^1 11^1 = 718502400 = 18 \cdot 11!$.

| $\alpha$ | $N \cdot c_{5,\alpha}$ | $\alpha$ | $N \cdot c_{5,\alpha}$ |
|---|---|---|---|
| **[12]** | **3080** | | |
| [11] | 32340 | 82 | 12766216 |
| [10] | 76978 | 73 | 28096776 |
| 9 | −226677 | 64 | 44440448 |
| 8 | −979089 | 55 | 25768902 |
| 7 | 516582 | 81 | −7431999 |
| 6 | 4592728 | 72 | −19253708 |
| 5 | −185493 | 63 | −34574748 |
| 4 | −10858021 | 54 | −46386130 |
| 3 | −683232 | 71 | −37982436 |
| 2 | 12607524 | 62 | −97588484 |
| 1 | 546480 | 53 | −165452460 |
| | | 44 | −98920039 |
| **[11, 1]** | **150920** | 61 | 3943998 |
| **[10, 2]** | **540320** | 52 | 1320790 |
| **93** | **1378080** | 43 | −5239846 |
| **84** | **2633120** | 51 | 131627986 |
| **75** | **3848880** | 42 | 258485354 |
| **66** | **2179800** | 33 | 156327858 |
| [10, 1] | 1553596 | 41 | 30510125 |
| 92 | 5244624 | 32 | 59066122 |
| 83 | 12387216 | 31 | −209783532 |
| 74 | 21589064 | 22 | −146989935 |
| 65 | 28319256 | 21 | −46058760 |
| 91 | 3939870 | 11 | 65027556 |

Table A.4: Coefficients for the genus 5 monotone Hurwitz generating function in the form $\vec{\mathbf{H}}_5 = \frac{-1}{1056}\big((1-\eta)^{-8} - 1\big) + \sum_{d=1}^{12} \sum_{\alpha \vdash d} c_{5,\alpha} \eta_\alpha (1-\eta)^{-8-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^9 3^6 5^2 7^1 11^1 = 718502400 = 18 \cdot 11!$. (cont.)

| $\alpha$ | $N \cdot c_{5,\alpha}$ | $\alpha$ | $N \cdot c_{5,\alpha}$ | $\alpha$ | $N \cdot c_{5,\alpha}$ |
|---|---|---|---|---|---|
| $[\mathbf{10}, \mathbf{1^2}]$ | **3799488** | 333 | −268996608 | | |
| **921** | **25006784** | 611 | −715895628 | 6221 | 12739765632 |
| **831** | **58283456** | 521 | −3207547232 | 5411 | 13432174800 |
| **822** | **37599056** | 431 | −4698449776 | 5321 | 42840873504 |
| **741** | **100727424** | 422 | −3107511892 | 5222 | 9289463040 |
| **732** | **158528832** | 332 | −3809388000 | 4421 | 25400829696 |
| **651** | **131700576** | 511 | −121563384 | 4331 | 31156686000 |
| **642** | **245068224** | 421 | −683668664 | 4322 | 40516064424 |
| **633** | **149469408** | 331 | −466606288 | 3332 | 16557865632 |
| **552** | **141442680** | 322 | −680575360 | 7111 | 1603025072 |
| **543** | **407987520** | 411 | 1794375864 | 6211 | 12403476800 |
| **444** | **80396160** | 321 | 5811747320 | 5311 | 21260651808 |
| 911 | 37912160 | 222 | 1297709514 | 5221 | 28063554816 |
| 821 | 233149312 | 311 | 674424616 | 4411 | 12677991612 |
| 731 | 497978432 | 221 | 962519712 | 4321 | 82861040064 |
| 722 | 324186368 | 211 | −1894749012 | 4222 | 18234425924 |
| 641 | 774992016 | 111 | −156973608 | 3331 | 17095134144 |
| 632 | 1236792672 | | | 3322 | 33814979088 |
| 551 | 447940560 | **9111** | **64463168** | 6111 | −1092238136 |
| 542 | 1696308360 | **8211** | **580868288** | 5211 | −6103240264 |
| 533 | 1039594320 | **7311** | **1225401408** | 4311 | −8122710288 |
| 443 | 1233434400 | **7221** | **1579161408** | 4221 | −9735976536 |
| 811 | 98750784 | **6411** | **1894318272** | 3321 | −11151467712 |
| 721 | 573934384 | **6321** | **5959816896** | 3222 | −4420223544 |
| 631 | 1122565040 | **6222** | **1280911280** | 5111 | −8570428240 |
| 622 | 742330412 | **5511** | **1092881328** | 4211 | −49461789124 |
| 541 | 1555143672 | **5421** | **8130486144** | 3311 | −30419666640 |
| 532 | 2545967280 | **5331** | **4962347808** | 3221 | −79961047848 |
| 442 | 1520737326 | **5322** | **6399177840** | 2222 | −8790023242 |
| 433 | 1882393536 | **4431** | **5867072640** | 4111 | −2954008640 |
| 711 | −116306424 | **4422** | **3781428288** | 3211 | −16614083992 |
| 621 | −518517104 | **4332** | **9231765312** | 2221 | −7861090512 |
| 531 | −812310976 | **3333** | **938698992** | 3111 | 14896133208 |
| 522 | −509211400 | 8111 | 616312928 | 2211 | 29366294892 |
| 441 | −471615312 | 7211 | 5133749984 | 2111 | 6503917904 |
| 432 | −1382412600 | 6311 | 9798998880 | 1111 | −2355420606 |

Table A.4: Coefficients for the genus 5 monotone Hurwitz generating function in the form $\vec{\mathbf{H}}_5 = \frac{-1}{1056}\big((1-\eta)^{-8}-1\big) + \sum_{d=1}^{12}\sum_{\alpha\vdash d} c_{5,\alpha}\eta_\alpha(1-\eta)^{-8-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^9 3^6 5^2 7^1 11^1 = 718502400 = 18 \cdot 11!$. (cont.)

| $\alpha$ | $N \cdot c_{5,\alpha}$ | $\alpha$ | $N \cdot c_{5,\alpha}$ |
|---|---|---|---|
| **81111** | **816347840** | 33221 | 556311704256 |
| **72111** | **8872174080** | 32222 | 120527969100 |
| **63111** | **16749384960** | 61111 | 18700344080 |
| **62211** | **32357793600** | 52111 | 168890451312 |
| **54111** | **22845389952** | 43111 | 249143583744 |
| **53211** | **107796401856** | 42211 | 492561868656 |
| **52221** | **46297494320** | 33211 | 609394623552 |
| **44211** | **63712382976** | 32221 | 535293918544 |
| **43311** | **77807068416** | 22222 | 35295888320 |
| **43221** | **200466870912** | 51111 | −6222432128 |
| **42222** | **21529880680** | 42111 | −35816319704 |
| **33321** | **81589525248** | 33111 | −19708525056 |
| **33222** | **52586584512** | 32211 | −63659195160 |
| 71111 | 7384805120 | 22221 | −10969834260 |
| 62111 | 73238584320 | 41111 | −71621395912 |
| 53111 | 123161943168 | 32111 | −461641675440 |
| 52211 | 239991560160 | 22211 | −302490323520 |
| 44111 | 73004276928 | 31111 | −31345362708 |
| 43211 | 698112996768 | 22111 | −87671917168 |
| 42221 | 302362543560 | 21111 | 81769223712 |
| 33311 | 142749070464 | 11111 | 6733061940 |

Table A.4: Coefficients for the genus 5 monotone Hurwitz generating function in the form $\vec{\mathbf{H}}_5 = \frac{-1}{1056}\big((1-\eta)^{-8} - 1\big) + \sum_{d=1}^{12}\sum_{\alpha \vdash d} c_{5,\alpha}\eta_\alpha(1-\eta)^{-8-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^9 3^6 5^2 7^1 11^1 = 718502400 = 18 \cdot 11!$. (cont.)

| $\alpha$ | $N \cdot c_{5,\alpha}$ | $\alpha$ | $N \cdot c_{5,\alpha}$ |
|---|---|---|---|
| **711111** | **8101040640** | **6111111** | **64516144000** |
| **621111** | **98452939200** | **5211111** | **830121821760** |
| **531111** | **164025597120** | **4311111** | **1198705052160** |
| **522111** | **422364096000** | **4221111** | **3857350728000** |
| **441111** | **96939929856** | **3321111** | **4712368066560** |
| **432111** | **1219613826816** | **3222111** | **8088674242880** |
| **422211** | **785274409920** | **2222211** | **1562274051800** |
| **333111** | **248287522560** | 5111111 | 499085146560 |
| **332211** | **1438698026304** | 4211111 | 5651771257440 |
| **322221** | **617752911160** | 3311111 | 3467697045120 |
| **222222** | **26528209400** | 3221111 | 22498949124960 |
| 611111 | 68349240960 | 2222111 | 9733013752000 |
| 521111 | 746061624000 | 4111111 | 1106867534080 |
| 431111 | 1085173682208 | 3211111 | 10800863673600 |
| 422111 | 2817663932160 | 2221111 | 11835459617520 |
| 332111 | 3457210990464 | 3111111 | 61131778400 |
| 322211 | 4488775799200 | 2211111 | 363422074400 |
| 222221 | 583001703900 | 2111111 | −1890764250760 |
| 511111 | 164269607040 | 1111111 | −110968630080 |
| 421111 | 1594210009392 | | |
| 331111 | 986016130176 | **51111111** | **416170437760** |
| 322111 | 5191236614256 | **42111111** | **5413828259840** |
| 222211 | 1708221819920 | **33111111** | **3307359538560** |
| 411111 | −15555347808 | **32211111** | **25538516203200** |
| 321111 | −51154596416 | **22221111** | **13696254017600** |
| 222111 | −5309017560 | 41111111 | 2889658242240 |
| 311111 | −432354874952 | 32111111 | 32198143084800 |
| 221111 | −1412409052824 | 22211111 | 41767355057120 |
| 211111 | −198453609508 | 31111111 | 5714203288640 |
| 111111 | 49368073216 | 22111111 | 26277999193600 |
| | | 21111111 | 753318362720 |
| $[\mathbf{1^{12}}]$ | **5931880416000** | 11111111 | −724713053680 |
| **411111111** | **2170719788160** | $[\mathbf{3}, \mathbf{1^9}]$ | **9044935715200** |
| **321111111** | **27303458927360** | $[\mathbf{2^2}, \mathbf{1^8}]$ | **52368137852800** |
| **222111111** | **40990175638720** | $[2, 1^9]$ | 46039949278400 |
| 311111111 | 13155841012160 | $[1^{10}]$ | 6167827881600 |
| 221111111 | 68240395386240 | | |
| 211111111 | 22168311605440 | $[\mathbf{2}, \mathbf{1^{10}}]$ | **29382811427200** |
| 111111111 | 360221597120 | $[1^{11}]$ | 10757398220800 |

# Appendix B

# Coefficients for Classical Hurwitz Generating Functions

The following equations give the genus two and three generating functions for the classical Hurwitz numbers, as described in Theorem 5.1.4. Tables B.1, B.2, B.3 and B.4 give the coefficients for genus 2, 3, 4 and 5, respectively.

$$5760\mathbf{H}_2 = \frac{5\overline{\eta}_3 - 12\overline{\eta}_2 + 7\overline{\eta}_1}{(1-\overline{\eta})^3} + \frac{29\overline{\eta}_{21} - 25\overline{\eta}_{11}}{(1-\overline{\eta})^4} + \frac{28\overline{\eta}_{111}}{(1-\overline{\eta})^5}.$$

$$\begin{aligned}5806080\mathbf{H}_3 = {} & \frac{70\overline{\eta}_6 - 294\overline{\eta}_5 + 410\overline{\eta}_4 - 186\overline{\eta}_3}{(1-\overline{\eta})^5} \\ & + \frac{1078\overline{\eta}_{51} + 2012\overline{\eta}_{42} + 1214\overline{\eta}_{33} - 3876\overline{\eta}_{41}}{(1-\overline{\eta})^6} \\ & + \frac{-6156\overline{\eta}_{32} + 4658\overline{\eta}_{31} + 3002\overline{\eta}_{22} - 1860\overline{\eta}_{21}}{(1-\overline{\eta})^6} \\ & + \frac{8568\overline{\eta}_{411} + 26904\overline{\eta}_{321} + 5830\overline{\eta}_{222}}{(1-\overline{\eta})^7} \\ & + \frac{-25968\overline{\eta}_{311} - 33642\overline{\eta}_{221} + 25770\overline{\eta}_{211} - 2790\overline{\eta}_{111}}{(1-\overline{\eta})^7} \\ & + \frac{44520\overline{\eta}_{3111} + 86100\overline{\eta}_{2211} - 110600\overline{\eta}_{2111} + 21420\overline{\eta}_{1111}}{(1-\overline{\eta})^8} \\ & + \frac{162120\overline{\eta}_{21111} - 62440\overline{\eta}_{11111}}{(1-\overline{\eta})^9} + \frac{68600\overline{\eta}_{111111}}{(1-\overline{\eta})^{10}}.\end{aligned}$$

Table B.1: Coefficients for the genus 2 classical Hurwitz generating function in the form $\mathbf{H}_2 = \sum_{d=1}^{3} \sum_{\alpha\vdash d} \bar{c}_{2,\alpha}\overline{\eta}_\alpha(1-\overline{\eta})^{-2-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^7 3^2 5^1 = 5760 = 8 \cdot 6!$.

| $\alpha$ | $N \cdot \bar{c}_{2,\alpha}$ | $\alpha$ | $N \cdot \bar{c}_{2,\alpha}$ | $\alpha$ | $N \cdot \bar{c}_{2,\alpha}$ |
|---|---|---|---|---|---|
| **3** | **5** | **21** | **29** | **111** | **28** |
| 2 | −12 | 11 | −25 | | |
| 1 | 7 | | | | |

Table B.2: Coefficients for the genus 3 classical Hurwitz generating function in the form $\mathbf{H}_3 = \sum_{d=3}^{6} \sum_{\alpha\vdash d} \bar{c}_{3,\alpha}\overline{\eta}_\alpha(1-\overline{\eta})^{-4-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^{11}3^4 5^1 7^1 = 5806080 = 16 \cdot 9!$.

| $\alpha$ | $N \cdot \bar{c}_{3,\alpha}$ | $\alpha$ | $N \cdot \bar{c}_{3,\alpha}$ | $\alpha$ | $N \cdot \bar{c}_{3,\alpha}$ |
|---|---|---|---|---|---|
| **6** | **70** | **51** | **1078** | **411** | **8568** |
| 5 | −294 | **42** | **2012** | **321** | **26904** |
| 4 | 410 | **33** | **1214** | **222** | **5830** |
| 3 | −186 | 41 | −3876 | 311 | −25968 |
| | | 32 | −6156 | 221 | −33642 |
| | | 31 | 4658 | 211 | 25770 |
| | | 22 | 3002 | 111 | −2790 |
| **3111** | **44520** | 21 | −1860 | | |
| **2211** | **86100** | | | | |
| 2111 | −110600 | **21111** | **162120** | | |
| 1111 | 21420 | 11111 | −62440 | $[\mathbf{1^6}]$ | **68600** |

Table B.3: Coefficients for the genus 4 classical Hurwitz generating function in the form $\mathbf{H}_4 = \sum_{d=5}^{9} \sum_{\alpha \vdash d} \overline{c}_{4,\alpha} \overline{\eta}_\alpha (1-\overline{\eta})^{-6-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^{17}3^5 5^2 7^1 = 5573836800 = 2^9 \cdot 3 \cdot 10!$.

| $\alpha$ | $N \cdot \overline{c}_{4,\alpha}$ | $\alpha$ | $N \cdot \overline{c}_{4,\alpha}$ | $\alpha$ | $N \cdot \overline{c}_{4,\alpha}$ |
|---|---|---|---|---|---|
| **9** | **700** | **81** | **20860** | **711** | **320544** |
| 8 | −4480 | **72** | **56820** | **621** | **1559328** |
| 7 | 10856 | **63** | **106860** | **531** | **2594400** |
| 6 | −11648 | **54** | **146100** | **522** | **1677480** |
| 5 | 4572 | 71 | −121104 | **441** | **1535040** |
| | | 62 | −297024 | **432** | **4831680** |
| | | 53 | −496320 | **333** | **980000** |
| | | 44 | −293520 | 611 | −1673760 |
| | | 61 | 267128 | 521 | −7234992 |
| | | 52 | 581112 | 431 | −10471104 |
| | | 43 | 845856 | 422 | −6771744 |
| | | 51 | −262896 | 332 | −8285760 |
| **6111** | **3297504** | 42 | −500928 | 511 | 3308232 |
| **5211** | **21227136** | 33 | −308208 | 421 | 12463200 |
| **4311** | **30631104** | 41 | 96012 | 331 | 7652160 |
| **4221** | **39502848** | 32 | 160020 | 322 | 9903768 |
| **3321** | **48172320** | | | 411 | −2915136 |
| **3222** | **20748112** | **51111** | **24907680** | 321 | −9347856 |
| 5111 | −15300320 | **42111** | **185243520** | 222 | −2029888 |
| 4211 | −85865472 | **33111** | **113077440** | 311 | 960120 |
| 3311 | −52533600 | **32211** | **437085600** | 221 | 1280160 |
| 3221 | −135800112 | **22221** | **93963100** | | |
| 2222 | −14627536 | 41111 | −100916480 | **411111** | **143330880** |
| 4111 | 26587232 | 32111 | −638070720 | **321111** | **1126957440** |
| 3211 | 126832944 | 22211 | −412129200 | **222111** | **967702400** |
| 2221 | 54750864 | 31111 | 150292800 | 311111 | −494988480 |
| 3111 | −20345136 | 22111 | 389394600 | 221111 | −1597854720 |
| 2211 | −39738384 | 21111 | −95886000 | 211111 | 609107520 |
| 2111 | 5760720 | 11111 | 4320540 | 111111 | −50823360 |
| $[\mathbf{3}, \mathbf{1^6}]$ | **633785600** | $[\mathbf{2}, \mathbf{1^7}]$ | **2128810880** | $[\mathbf{1^9}]$ | **581398720** |
| $[\mathbf{2^2}, \mathbf{1^5}]$ | **2447642400** | $[1^8]$ | −590352000 | | |
| $[2, 1^6]$ | −1802857280 | | | | |
| $[1^7]$ | 247474080 | | | | |

Table B.4: Coefficients for the genus 5 classical Hurwitz generating function in the form $\mathbf{H}_5 = \sum_{d=7}^{12} \sum_{\alpha \vdash d} \bar{c}_{5,\alpha} \overline{\eta}_\alpha (1 - \overline{\eta})^{-8-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^{21}3^6 5^2 7^1 11^1 = 2942985830400 = 2^{11} \cdot 3 \cdot 12!$.

| $\alpha$ | $N \cdot \bar{c}_{5,\alpha}$ | $\alpha$ | $N \cdot \bar{c}_{5,\alpha}$ |
|---|---|---|---|
| $[\mathbf{12}]$ | **3080** | 65 | −22284528 |
| [11] | −27720 | 91 | 4336816 |
| [10] | 101552 | 82 | 13220672 |
| 9 | −187440 | 73 | 28156272 |
| 8 | 171848 | 64 | 43703728 |
| 7 | −61320 | 55 | 25243728 |
| | | 81 | −7506496 |
| $[\mathbf{11},\mathbf{1}]$ | **150920** | 72 | −20923968 |
| $[\mathbf{10},\mathbf{2}]$ | **540320** | 63 | −40084848 |
| **93** | **1378080** | 54 | −55010352 |
| **84** | **2633120** | 71 | 6493128 |
| **75** | **3848880** | 62 | 16513568 |
| **66** | **2179800** | 53 | 28109424 |
| [10, 1] | −1266848 | 44 | 16711128 |
| 92 | −4199712 | 61 | −2207520 |
| 83 | −9832608 | 52 | −5150880 |
| 74 | −17022432 | 43 | −7726320 |

Table B.4: Coefficients for the genus 5 classical Hurwitz generating function in the form $\mathbf{H}_5 = \sum_{d=7}^{12} \sum_{\alpha \vdash d} \overline{c}_{5,\alpha} \overline{\eta}_\alpha (1-\overline{\eta})^{-8-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^{21}3^6 5^2 7^1 11^1 = 2942985830400 = 2^{11} \cdot 3 \cdot 12!$. (cont.)

| $\alpha$ | $N \cdot \overline{c}_{5,\alpha}$ | $\alpha$ | $N \cdot \overline{c}_{5,\alpha}$ | $\alpha$ | $N \cdot \overline{c}_{5,\alpha}$ |
|---|---|---|---|---|---|
| $[\mathbf{10}, \mathbf{1^2}]$ | **3799488** | 531 | −1265742672 | 6311 | −7190206848 |
| **921** | **25006784** | 522 | −822637008 | 6221 | −9284781264 |
| **831** | **58283456** | 441 | −750789696 | 5411 | −9820417728 |
| **822** | **37599056** | 432 | −2399841216 | 5321 | −31032567456 |
| **741** | **100727424** | 333 | −491782752 | 5222 | −6679493040 |
| **732** | **158528832** | 611 | 121192944 | 4421 | −18354230400 |
| **651** | **131700576** | 521 | 541195872 | 4331 | −22457502144 |
| **642** | **245068224** | 431 | 794495520 | 4322 | −29003059008 |
| **633** | **149469408** | 422 | 521078880 | 3332 | −11829077568 |
| **552** | **141442680** | 332 | 643371960 | 7111 | 1346923072 |
| **543** | **407987520** | 511 | −38631600 | 6211 | 9957224464 |
| **444** | **80396160** | 421 | −154526400 | 5311 | 16708366224 |
| 911 | −29586304 | 331 | −96579000 | 5221 | 21626744832 |
| 821 | −178983200 | 322 | −128772000 | 4411 | 9892861440 |
| 731 | −379102080 | | | 4321 | 62820255168 |
| 722 | −244886928 | **9111** | **64463168** | 4222 | 13551628816 |
| 641 | −587052096 | **8211** | **580868288** | 3331 | 12841233504 |
| 632 | −927898560 | **7311** | **1225401408** | 3322 | 24928395888 |
| 551 | −338912760 | **7221** | **1579161408** | 6111 | −1977893104 |
| 542 | −1267309440 | **6411** | **1894318272** | 5211 | −13010553072 |
| 533 | −775234080 | **6321** | **5959816896** | 4311 | −18976788864 |
| 443 | −917009280 | **6222** | **1280911280** | 4221 | −24663253296 |
| 811 | 93868368 | **5511** | **1092881328** | 3321 | −30324126624 |
| 721 | 517489888 | **5421** | **8130486144** | 3222 | −13137176976 |
| 631 | 985223776 | **5331** | **4962347808** | 5111 | 1453932480 |
| 622 | 637618000 | **5322** | **6399177840** | 4211 | 8392008240 |
| 541 | 1347934944 | **4431** | **5867072640** | 3311 | 5178983040 |
| 532 | 2140011024 | **4422** | **3781428288** | 3221 | 13575112320 |
| 442 | 1267225536 | **4332** | **9231765312** | 2222 | 1482114480 |
| 433 | 1554319968 | **3333** | **938698992** | 4111 | −424947600 |
| 711 | −150642896 | 8111 | −462478016 | 3211 | −2124738000 |
| 621 | −750182944 | 7211 | −3794809920 | 2221 | −944328000 |

Table B.4: Coefficients for the genus 5 classical Hurwitz generating function in the form $\mathbf{H}_5 = \sum_{d=7}^{12} \sum_{\alpha \vdash d} \bar{c}_{5,\alpha} \overline{\eta}_\alpha (1-\overline{\eta})^{-8-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^{21}3^6 5^2 7^1 11^1 = 2942985830400 = 2^{11} \cdot 3 \cdot 12!$. (cont.)

| $\alpha$ | $N \cdot \bar{c}_{5,\alpha}$ | $\alpha$ | $N \cdot \bar{c}_{5,\alpha}$ |
|---|---|---|---|
| **81111** | **816347840** | 33311 | −99989277696 |
| **72111** | **8872174080** | 33221 | −387259147968 |
| **63111** | **16749384960** | 32222 | −83331307752 |
| **62211** | **32357793600** | 61111 | 14134545920 |
| **54111** | **22845389952** | 52111 | 122789940240 |
| **53211** | **107796401856** | 43111 | 178319167488 |
| **52221** | **46297494320** | 42211 | 346197930672 |
| **44211** | **63712382976** | 33211 | 424561720704 |
| **43311** | **77807068416** | 32221 | 366312529968 |
| **43221** | **200466870912** | 22222 | 23704867648 |
| **42222** | **21529880680** | 51111 | −18692200560 |
| **33321** | **81589525248** | 42111 | −141712477280 |
| **33222** | **52586584512** | 33111 | −87119205888 |
| 71111 | −5349023680 | 32211 | −339641236704 |
| 62111 | −52332480640 | 22221 | −73563956928 |
| 53111 | −87462441792 | 41111 | 12277437480 |
| 52211 | −169354163088 | 32111 | 79380747600 |
| 44111 | −51731130880 | 22211 | 51978843840 |
| 43211 | −490282146816 | 31111 | −3187107000 |
| 42221 | −210979948256 | 22111 | −8498952000 |

Table B.4: Coefficients for the genus 5 classical Hurwitz generating function in the form $\mathbf{H}_5 = \sum_{d=7}^{12} \sum_{\alpha \vdash d} \overline{c}_{5,\alpha} \overline{\eta}_\alpha (1 - \overline{\eta})^{-8-\ell(\alpha)}$. Top degree coefficients are set in bold. Note that the coefficients are scaled by $N = 2^{21}3^6 5^2 7^1 11^1 = 2942985830400 = 2^{11} \cdot 3 \cdot 12!$. (cont.)

| $\alpha$ | $N \cdot \overline{c}_{5,\alpha}$ | $\alpha$ | $N \cdot \overline{c}_{5,\alpha}$ |
|---|---|---|---|
| **711111** | **8101040640** | **6111111** | **64516144000** |
| **621111** | **98452939200** | **5211111** | **830121821760** |
| **531111** | **164025597120** | **4311111** | **1198705052160** |
| **522111** | **422364096000** | **4221111** | **3857350728000** |
| **441111** | **96939929856** | **3321111** | **4712368066560** |
| **432111** | **1219613826816** | **3222111** | **8088674242880** |
| **422211** | **785274409920** | **2222211** | **1562274051800** |
| **333111** | **248287522560** | 5111111 | −340128268480 |
| **332211** | **1438698026304** | 4211111 | −3811966238080 |
| **322221** | **617752911160** | 3311111 | −2332563340800 |
| **222222** | **26528209400** | 3221111 | −15048857725440 |
| 611111 | −47948134080 | 2222111 | −6472981354840 |
| 521111 | −517093044160 | 4111111 | 705047949760 |
| 431111 | −748552870912 | 3211111 | 6712913518080 |
| 422111 | −1932091298368 | 2221111 | 7239046306800 |
| 332111 | −2364392612736 | 3111111 | −707258650560 |
| 322211 | −3051596341024 | 2211111 | −2757001354800 |
| 222221 | −393874176080 | 2111111 | 335828437560 |
| 511111 | 113285406080 | 1111111 | −8286478200 |
| 421111 | 1064547594880 | | |
| 331111 | 652723416576 | **51111111** | **416170437760** |
| 322111 | 3378948408528 | **42111111** | **5413828259840** |
| 222211 | 1093196341776 | **33111111** | **3307359538560** |
| 411111 | −132235783680 | **32211111** | **25538516203200** |
| 321111 | −1056336943200 | **22221111** | **13696254017600** |
| 222111 | −915106536960 | 41111111 | −1918814374400 |
| 311111 | 75370430520 | 32111111 | −21207448859520 |
| 221111 | 246647210040 | 22211111 | −27360937148160 |
| 211111 | −16572956400 | 31111111 | 3402113225280 |
| | | 22111111 | 15407697757760 |
| | | 21111111 | −2824078561920 |
| $\mathbf{[1^{12}]}$ | **5931880416000** | 11111111 | 131114712960 |
| **411111111** | **2170719788160** | $\mathbf{[3, 1^9]}$ | **9044935715200** |
| **321111111** | **27303458927360** | $\mathbf{[2^2, 1^8]}$ | **52368137852800** |
| **222111111** | **40990175638720** | $[2, 1^9]$ | −29322334764800 |
| 311111111 | −8539520232320 | $[1^{10}]$ | 3348064473600 |
| 221111111 | −44063442068800 | | |
| 211111111 | 12492588440640 | $\mathbf{[2, 1^{10}]}$ | **29382811427200** |
| 111111111 | −899669453760 | $[1^{11}]$ | −6761652374400 |

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