Throughput and Expected-Rate in Wireless Block Fading Systems

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis deals with wireless channels in uncorrelated block fading environment with Rayleigh distribution. All nodes are assumed to be oblivious to their forward channel gains; however, they have perfect information about their backward channel gains. We also assume a stringent decoding delay constraint of one fading block that makes the definition of ergodic (Shannon) capacity meaningless. In this thesis, we focus on two different systems. In each case, the throughput and expected-rate are analyzed.

First, the point-to-point multiple-antenna channel is investigated in chapter 2. We prove that in multiple-input single-output (MISO) channels, the optimum transmission strategy maximizing the throughput is to use all available antennas and perform equal power allocation with uncorrelated signals. Furthermore, to increase the expected-rate, multi-layer coding (the broadcast approach) is applied. Analogously, we establish that sending uncorrelated signals and performing equal power allocation across all available antennas at each layer is optimum. A closed form expression for the maximum continuous-layer expected-rate of MISO channels is also obtained. Moreover, we investigate multiple-input multiple-output (MIMO) channels, and formulate the maximum throughput in the asymptotically low and high SNR regimes and also asymptotically large number of transmit or receive antennas by obtaining the optimum transmit covariance matrix. Furthermore, a distributed antenna system, wherein two single-antenna transmitters want to transmit a common message to a single-antenna receiver, is considered. It is shown that this system has the same outage probability and hence, throughput and expected-rate, as a point-to-point 2×1 MISO channel.

In chapter 3, the problem of dual-hop transmission from a single-antenna source to a single-antenna destination via two parallel full-duplex single-antenna relays under the above assumptions is investigated. The focus of this chapter is on simple, efficient, and practical relaying schemes to increase the throughput and expected-rate at the destination. For this purpose, various combinations of relaying protocols and multi-layer coding are proposed. For the decode-forward (DF) relaying, the maximum finite-layer expected-rate as well as two upper-bounds on the continuous-layer expected-rate are obtained. The main feature of the proposed DF scheme is that the layers being decoded at both relays are added coherently at the destination although each relay has no information about the number of layers being successfully decoded by the other relay. It is proved that the optimum coding scheme is transmitting uncorrelated signals via the relays. Next, the maximum expected-rate of ON/OFF based amplify-forward (AF) relaying is analytically formulated. For further performance improvement, a hybrid decode-amplify-forward (DAF) relaying strategy, adopting multi-layer coding at the source and relays, is proposed and its maximum throughput and finite-layer expected-rate are presented. Moreover, the maximum throughput and expected-rate in the compress-forward (CF) relaying adopting multi-layer coding, using optimal quantizers and Wyner-Ziv compression at the relays, are fully derived. All theoretical results are illustrated by numerical simulations. As it turns out from the results, when the ratio of the relay power to the source power is high, the CF relaying outperforms DAF (and hence outperforms both DF and AF relaying); otherwise, DAF scheme is superior.

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To my dear parents,

for all I have ever accomplished

Table of Contents

List of Figures

List of Abbreviations

i.i.d.	Independent and Identically Distributed
AWGN	Additive White Gaussian Noise
CSI	Channel State Information
SISO	Single-Input Single-Output
MISO	Multiple-Input Single-Output
SIMO	Single-Input Multiple-Output
MIMO	Multiple-Input Multiple-Output
PDF	Probability Density Function
CDF	Cumulative Density Function
SNR	Signal-to-Noise Ratio
DF	Decode-Forward
AF	Amplify-Forward
DAF	Decode-Amplify-Forward
CF	Compress-Forward
RC	Relay-Cooperation
MSE	Mean-Square Error

Notations

$\Pr\{A\}$	The probability of event A
$\mathbb{E}(\cdot)$	The expected operation
$\mathrm{Var}(\cdot)$	The variance operation
ln	Natural logarithm (rates are expressed in *nats*)
$f_{\mathrm{x}}(\cdot)$	The probability density function (PDF) of random variable x
$F_{\mathrm{x}}(\cdot)$	The cumulative density function (CDF) of random variable x
$\overline{F}(x)$	$\triangleq 1 - F(x)$
$F'(x)$	$\triangleq \frac{\mathrm{d}F(x)}{\mathrm{d}x}$
$\vec{X}$	A vector
$\mathbf{Q}$	A matrix
$\mathrm{tr}(\mathbf{Q})$	The trace of $\mathbf{Q}$
$\mathbf{I}_{n_t}$	The $n_t \times n_t$ identity matrix
s^o	The optimum solution with respect to the variable s
$.^*$	The conjugation operator
$.^{\mathrm{T}}$	The matrix transpose operator
$.^\dagger$	The matrix conjugate transpose operator
$\Re(\cdot)$	The real part of complex variables
$\Im(\cdot)$	The imaginary part of complex variables

$\lvert \cdot \rvert$	The absolute value or modulus operator
det	The determinant operator
$\mathrm{eig}_\ell(\mathbf{Q})$	The ℓ'th ordered eigenvalue of matrix $\mathbf{Q}$
h_ℓ	The ℓ'th component of vector $\vec{h}$
$h_{\ell,k}$	The (ℓ, k)'th entry of matrix $\mathbf{H}$
$\mathcal{CN}(0,1)$	The complex circularly symmetric Gaussian distribution with zero mean and unit variance
$\mathcal{N}(\mu, \sigma^2)$	The Gaussian distribution with mean μ and variance σ^2
$\mathcal{W}(\cdot)$	The Lambert W-function, also called the omega function, which is the inverse function of $f(W) = We^W$ [17, 18]
$\mathrm{E}_1(x)$	$\triangleq \int_x^\infty \frac{e^{-t}}{t} \mathrm{d}t,\ x \geq 0$, the exponential integral function
$\Gamma(n, x)$	$\triangleq \int_x^\infty t^{n-1} e^{-t} \mathrm{d}t$, the upper incomplete gamma function
$\Gamma(n)$	$\triangleq \Gamma(n, 0)$
$\digamma(n)$	$\triangleq \frac{\Gamma'(n)}{\Gamma(n)}$, the Eüler's digamma function [37]
$\mathcal{Q}(x)$	$\triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} \mathrm{d}t$, the $\mathcal{Q}$-function
$\mathbf{R}_{XY}$	The covariance matrix of random variables X and Y
$\mathbf{R}_{XY \mid W,Z,\cdots}$	The conditional covariance matrix of random variables X and Y
$\mathcal{H}(\cdot)$	The differential entropy function
$\mathcal{I}(\cdot;\cdot)$	The mutual information function
$\mathcal{U}(\cdot)$	The unit step function

Chapter 1

Introduction

The information theoretic aspects of wireless channels have received wide attention [24]. The widespread applications of wireless networks, along with many recent results in the network information theory area, have motivated efficient strategies for practical applications [22]. Fading is often used for modeling the wireless channels [13]. Also, the assumption of channel state information (CSI) at the transmitter side is not of practical relevance in some applications (e.g., systems with many receivers). In this thesis, we investigate two important system configurations.

1.1 Multiple-Antenna Systems

It has been shown that multiple-antenna arrays have the ability to reach higher transmission rates [30, 31, 46, 52, 75–77, 84]. With no delay constraint, the ergodic nature of the fading channel can be experienced by sending very large transmission blocks, and the ergodic capacity is well studied [13]. When the channel variation is slow, the channel can be estimated relatively accurately at the receiver. By assuming perfect CSI at the receiver but no CSI at the transmitter, Telatar [77] showed that the ergodic capacity of MIMO channels is achieved by sending an uncorrelated circularly symmetric zero mean equal power complex Gaussian codebook on all transmit antennas.

Due to the stringent delay constraint for the problem in consideration, the transmission block length is forced to be shorter than the dynamics of the slow fading process, though still large enough to yield a reliable communication. The performance of such channels are usually evaluated by outage capacity [78]. The notion of capacity versus outage was introduced in [13, 57]. It has been proved that in uncorrelated MISO channels, the optimum transmit strategy minimizing the outage probability is to use a fraction of all available transmit antennas and perform equal power allocation with uncorrelated signals [2, 3, 43].

The maximum throughput is an important performance measure in block fading channels [5, 6, 44, 92], which is defined as the maximum of the product of the transmission rate and the probability of successful transmission using a single-layer code (see Definition 1.1). As mentioned in [43], the results on the outage probability cannot be directly applied to this metric due to the maximization. In Section 2.3, we prove that to achieve the maximum throughput in an uncorrelated MISO channel, the optimum transmit strategy is to send equal power uncorrelated signals from all available antennas (see Theorem 2.1).

The maximum average achievable rate is another performance measure which is important in some applications [66]. In order to increase the average achievable rate, Shamai and Steiner [67] proposed a broadcast approach (multi-layer coding) for a point-to-point block fading channel with no CSI at the transmitter. Since the average achievable rate increases with the number of code layers, they reached the highest average achievable rate using a continuous-layer (infinite-layer) code. Numerical algorithms have been proposed to find the optimum layers' power distribution in single-user MIMO channels [60]. The broadcast approach was applied to a dual-hop single-relay channel in [59, 70], a channel with two collocated cooperative users in [71], the relay channel in [88], the diamond channel in [89,90], and a packet erasure channel in [26, 29]. Multi-layer coding can also achieve the maximum average achievable rate in a block fading multiple-access channel with no CSI at the transmitters [53] and channels with quantized limited feedback [69]. The optimized trade-off between the QoS and network coverage in a multicast network was derived in [55] using the broadcast approach. Multi-layer coding was later applied to joint source-channel coding scenarios [27, 28, 38, 39] to minimize the received distortion by layered source transmission and successive refinement [25] concepts.

In this thesis, we derive the maximum expected-rate of MISO channels, which is defined as the maximum average decodable rate when a multi-layer code is transmitted (see Definition 1.2). Theorem 2.2 proves that to maximize the expected-rate in MISO channels, it is optimum to transmit equal power independent signals on all available antennas in each layer. Using the continuous-layer coding approach, the maximum expected-rate of MISO channels is then obtained and formulated in closed form in Theorem 2.3.

To evaluate the maximum throughput in uncorrelated MIMO channels, the distribution of the instantaneous mutual information is crucial [93]. In [40, 82], it is shown that the distribution of the instantaneous mutual information in MIMO channels is always very close to the Gaussian distribution. The mean and variance of this equivalent Gaussian distribution were derived in [40] for asymptotic ranges of the number of antennas. As this distribution is not tractable in general MIMO channels, in this thesis, we consider four asymptotic cases: asymptotically low SNR regime, asymptotically high SNR regime, asymptotically large number of transmit antennas, and asymptotically large number of receive antennas. In all four cases, the optimum covariance matrix is obtained and the maximum throughput expression is derived.

Afterwards, the maximum throughput and maximum expected-rate of a distributed antenna system with two single-antenna transmitters and one single-antenna receiver is obtained. It is also proved that any achievable throughput, expected-rate, ergodic capacity, and outage capacity in a MISO channel with two transmit antennas are also achievable in this channel.

1.2 Diamond Channel

The growing demand for quality of service (QoS) and network coverage inspires the use of several intermediate wireless nodes to help the communication among distant nodes, which is referred to as relaying or multi-hopping. Many papers analyze the information theoretic and communication aspects of relay networks. An information theoretic view of the three-node relay channel [80] was proposed by Cover and El Gamal in [20], which was generalized in [48] and [86] for multi-user and multi-relay networks. In [20], two different coding strategies were introduced. In the first strategy, originally named "cooperation" and later known as "decode-forward" (DF), the relay decodes the transmitted message and cooperates with the source to send the message in the next block. In the second strategy, "compress-forward" (CF), the relay compresses the received signal and sends it to the destination. Besides studying the DF and CF strategies, the authors in [23, 42, 50, 64] have studied the "amplify-forward" (AF) strategy for the Gaussian relay network. In AF relaying, the relay amplifies and transmits its received signal to the destination. Despite its simplicity, AF relaying performs well in many scenarios. El-Gamal and Zahedi [23] employed AF relaying in the single relay channel and derived the single letter characterization of the maximum achievable rate using a simple linear scheme (assuming frequency division and AWGN channel).

The problems of transmission between a disconnected source and destination via two parallel intermediate nodes, a.k.a. the diamond channel, were analyzed in [64] for the AWGN channels and in [63] for the case where the relays transmit in orthogonal frequency bands/time slots. There are also some asymptotic analyses on a source to destination communication via parallel relays with fading channels where the forward channels are known at both the transmitter and relays sides, see [34] and references therein. Diversity gains in a parallel relay network using distributed space-time codes, where CSI is only at the receivers, was presented in [4, 41]. Many papers also analyzed the diamond channel in half-duplex mode, for example see [9, 62].

Here, we consider the problem of maximum expected-rate in the diamond channel. A good application for this network is a TV broadcasting system from a satellite to cellphones through base stations. In second generation digital video broadcasting (DVB-S2), satellites multicast high-speed data rates to mobile users [1]. Hence, users with better channels might receive additional services, such as high definition TV signal [54]. The growing adoption of broadcasting mobile TV services suggests that it has the potential to become a mass market application. However, the quality and success of such services are governed by guaranteeing a good coverage, particularly in areas that are densely populated. We suggest the use of relays to provide better coverage in such strategically important areas. The main transmitter which is a central TV broadcasting unit uses two parallel relays in each area with large density to improve coverage (see Fig. 1.1). According to the large number of relay pairs covering their respective areas and also the large number of users in each designated area, neither the main transmitter nor the relays can access the forward CSI.

We investigate various relaying strategies in conjunction with multi-layer coding scheme for the dual-hop channel with parallel relays where neither the source (main transmitter) nor the relays access the forward channels. We also assume that channel gains are fixed during two consecutive blocks of transmission. The main focus of this chapter is on simple and efficient schemes, since the relays can not buffer multiple packets and also handle large delays. Different relaying strategies such as DF, AF, hybrid DF-AF (DAF), and CF are considered.

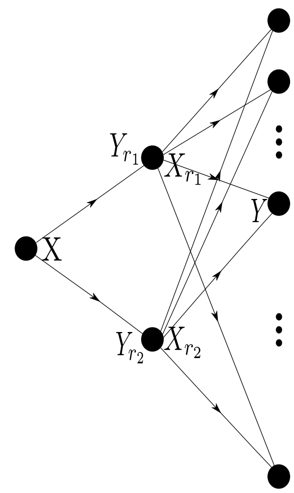

Figure 1.1: Dual-hop multicast transmission via two parallel relays.

In DF relaying, a combination of multi-layering and space-time coding is proposed, such that the common layers, decoded at both relays, are decoded at the destination cooperatively. Note that each relay has no information about the number of layers being decoded by the other relay. The destination decodes from the first layer up to the layer that the channel condition allows. After decoding all common layers, the layers decodable at just one relay are decoded. It is proved that the optimal coding strategy is transmitting uncorrelated signals via the relays. Since the DF relaying in conjunction with continuous-layer coding is a seemingly intractable problem, the maximum finite-layer expected-rate is analyzed. Furthermore, two upper-bounds for the maximum continuous-layer expected-rate in DF are obtained. In the DF relaying, the relays must know the codebook of the source and have enough time to decode the received signal. In the networks without these conditions, AF relaying is considered next. Both the throughput and expected-rate, using a space-time code permutation between the relays, are derived. In the same direction and for further performance improvement, at the cost of increased complexity, a hybrid DF and AF scheme called DAF is proposed. In multi-layer DAF, each relay decode-and-forwards a portion of the layers and amplify-and-forwards the rest. Afterwards, a multi-layer CF relaying is presented. In the CF relaying, the relays do not decode their received signals; instead, compress the signals by performing the optimal quantization in the Wyner-Ziv sense [85], which means each relay quantizes its received signal relying on the side information from the other relay. Besides the proposed achievable expected-rates, some upper bounds based on the channel enhancement idea and the max-flow min-cut theorem [21] are obtained. As it turns out from the numerical results, in all the proposed relaying strategies combined with multi-layer coding, the maximum expected-rate increases with the number of code layers. It is also shown that when the ratio of the relay power to the source power is large, the CF relaying outperforms DAF, and hence outperforms both DF and AF; otherwise, DAF is the superior scheme. Here, ON/OFF based AF is always outperformed by either DF or CF. This is in contrast to the full-duplex AWGN diamond channel in which CF is always outperformed by either DF or AF [65].

1.3 Definitions

In the following, the performance metrics which are widely used are defined.

Definition 1.1. *The throughput $\mathcal{R}_s$ is the average achievable rate when a single-layer code with a fixed rate R is transmitted, i.e., the transmission rate times the probability of successful transmission. The maximum throughput, namely $\mathcal{R}_s^m$, is the maximum of the throughput over all transmit covariance matrices $\mathbf{Q}$, and transmission rates R. Mathematically,*

$$\mathcal{R}_s^m \triangleq \max_{\substack{R,\mathbf{Q} \\ \mathrm{tr}(\mathbf{Q}) \leq P}} \Pr\left\{\mathcal{I} \geq R\right\} R, \tag{1.1}$$

where $\mathcal{I}$ is the instantaneous mutual information function of the channel whose arguments are dropped when they are clear.

Definition 1.2. *The expected-rate $\mathcal{R}_f$ is the average achievable rate when a multi-layer* *code is transmitted, i.e., the statistical expectation of the achievable rate. The maximum expected-rate, namely* $\mathcal{R}_f^m$*, is the maximum of the expected-rate over all transmit covariance matrices and transmission rates in each layer, and all power distributions of the layers. Mathematically,*

$$\mathcal{R}_f^m \triangleq \max_{\substack{R_i, P_i, \mathbf{Q}_i \\ \mathrm{tr}(\mathbf{Q}_i) \leq P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^{K} \Pr\left\{\mathcal{I}_i \geq R_i\right\} R_i, \tag{1.2}$$

where R_i*,* $\mathbf{Q}_i$*, and* $\mathcal{I}_i$ *are the transmission rate, transmit covariance matrix, and instantaneous mutual information in the i'th layer, respectively.*

If a continuum of code layers are transmitted, the maximum continuous-layer (infinite-layer) expected-rate, namely $\mathcal{R}_c^m$*, is given by maximizing the continuous-layer expected-rate over the layers' power distribution.*

Continuous-layer and infinite-layer are used interchangeably throughout this thesis. We also use multi-layer coding and broadcast approach interchangeably.

Definition 1.3. *The ergodic capacity* C_{erg} *is the maximum expected value of the instantaneous mutual information* $\mathcal{I}$ *over all transmit covariance matrices* $\mathbf{Q}$*. Mathematically,*

$$C_{\mathrm{erg}} \triangleq \max_{\substack{\mathbf{Q} \\ \mathrm{tr}(\mathbf{Q}) \leq P}} \mathbb{E}\left(\mathcal{I}\right). \tag{1.3}$$

Note that throughout the thesis, all rates are expressed in *nats*.

Chapter 2

Maximum Throughput and Expected-Rate in Multiple-Antenna Systems

2.1 Problem Setup

A MIMO channel with n_t transmit antennas and n_r receive antennas is defined as a channel with the following input-output relationship:

$$\vec{Y} = \mathbf{H}\vec{X} + \vec{Z}, \tag{2.1}$$

where $\vec{Y}$ is the received signal, $\mathbf{H} \sim [\mathcal{CN}(0,1)]_{n_r \times n_t}$ is the channel matrix, $\vec{Z} \sim [\mathcal{CN}(0,1)]_{n_r \times 1}$ is the independent and identically distributed (i.i.d.) additive white Gaussian noise (AWGN), and $\vec{X}$ is the transmitted signal under the following total power constraint:

$$\mathbb{E}\left(\vec{X}^\dagger \vec{X}\right) = \mathbb{E}\left(\mathrm{tr}\left(\vec{X}\vec{X}^\dagger\right)\right) = \mathrm{tr}\left(\mathbb{E}\left(\vec{X}\vec{X}^\dagger\right)\right) \leq P. \tag{2.2}$$

Defining $\mathbf{Q}$ as the transmit covariance matrix, i.e., $\mathbf{Q} = \mathbb{E}\left(\vec{X}\vec{X}^{\dagger}\right)$, the instantaneous mutual information is

$$\mathcal{I} = \ln\det\left(\mathbf{I}_{n_r} + \mathbf{H}\mathbf{Q}\mathbf{H}^{\dagger}\right) = \ln\det\left(\mathbf{I}_{n_t} + \mathbf{Q}\mathbf{H}^{\dagger}\mathbf{H}\right). \tag{2.3}$$

In a MISO channel, the channel coefficients are represented by a vector $\vec{h}^T \sim [\mathcal{CN}(0,1)]_{n_t \times 1}$, and

$$Y = \vec{h}\vec{X} + Z. \tag{2.4}$$

The main focus of this chapter is to solve the following problems.

Problem 2.1. *To obtain the optimum transmit covariance matrix, denoted by $\mathbf{Q}^o$, which maximizes the throughput $\mathcal{R}_s$ in the MISO channel.*

Theorem 2.1 proves that the optimum transmit strategy is to transmit uncorrelated signals on all antennas with equal powers, i.e., $\mathbf{Q}^o = \frac{P}{n_t}\mathbf{I}_{n_t}$, and provides the maximum throughput expression.

Problem 2.2. *To derive the optimum transmit covariance matrix in each layer, i.e., $\mathbf{Q}_i^o$, for finite-layer coding in the MISO channel, which maximizes the expected-rate $\mathcal{R}_f$.*

As we shall see in Theorem 2.2, the optimum transmit covariance matrix in each layer is in the form of $\mathbf{Q}_i^o = \frac{P_i}{n_t}\mathbf{I}_{n_t}$, and the maximum expected-rate is given by Eq. (2.29).

Problem 2.3. *To derive the maximum continuous-layer expected-rate $\mathcal{R}_c^m$ in the MISO channel.*

The closed form expression of the maximum continuous-layer expected-rate is derived in Theorem 2.3.

In the MIMO channel, the PDF of the instantaneous mutual information $\mathcal{I}$ is not known even for the simplest case of $\mathbf{Q} = \frac{P}{n_t}\mathbf{I}_{n_t}$, although there are some approximations in literature for asymptotic cases. In the next step, the maximum throughputs in four asymptotic cases of the MIMO channel are addressed.

Problem 2.4. *To derive the maximum throughput of the MIMO channel in asymptotically*

- *low SNR regime*
- *high SNR regime*
- *large number of transmit antennas*
- *large number of receive antennas*

Different MIMO approximations are exploited to solve Problem 2.4. For asymptotically low SNR regime, the MISO results are carried over and the maximum throughput and maximum expected-rate are formulated. For asymptotically high SNR regime, Wishart distribution properties [79] are used to obtain the maximum throughput. For asymptotically large number of transmit or receive antennas, Gaussian approximations for the instantaneous mutual information presented in [40] are utilized. As we shall see in Section 2.5, in all aforementioned asymptotic regimes, the optimum transmit covariance matrix which maximizes the throughput is $\mathbf{Q}^o = \frac{P}{n_t}\mathbf{I}_{n_t}$.

In the last problem of this chapter, a distributed antenna system consisting of two single-antenna transmitters with common messages and a single-antenna receiver is considered.

Problem 2.5. *To find the minimum outage probability, the maximum throughput, and the maximum expected-rate in a two-transmitter distributed antenna system.*

Theorem 2.7 establishes that any achievable outage probability in the 2×1 MISO channel is also achievable in the two-transmitter distributed antenna system in Problem 2.5. Hence, both channels experience the same instantaneous mutual information distribution and thereby, all MISO channel results are applied here with $n_t = 2$.

2.2 Upper-Bound

In the following, we present three propositions which show that the maximum throughput and maximum expected-rate are upper-bounded by Eq. (2.10).

Proposition 2.1. *In fading channels, the maximum throughput is less than or equal to the ergodic capacity.*

Proof. The proof is based on the Markov inequality [58], that is if $f_{\mathrm{x}}(x) = 0$ for $x < 0$, then, for $\alpha > 0$, $\Pr\{x \geq \alpha\} \leq \frac{\mathbb{E}(x)}{\alpha}$. Therefore, $\forall R > 0$,

$$\Pr\{\mathcal{I} \geq R\} \leq \frac{\mathbb{E}(\mathcal{I})}{R}, \tag{2.5}$$

so that

$$\mathcal{R}_s^m = \max_{\substack{R,\mathbf{Q}\\ \mathrm{tr}(\mathbf{Q})\leq P}} \Pr\{\mathcal{I} \geq R\} R \leq \max_{\substack{\mathbf{Q}\\ \mathrm{tr}(\mathbf{Q})\leq P}} \mathbb{E}(\mathcal{I}), \tag{2.6}$$

and Eq. (2.6) results because $\max_{\mathbf{Q},\mathrm{tr}(\mathbf{Q})\leq P} \mathbb{E}(\mathcal{I})$ equals the ergodic capacity. □

Proposition 2.2. *In Rayleigh fading channels, the maximum expected-rate is less than or equal to the ergodic capacity.*

Proof. From Eq. (1.2) it follows that

$$
\begin{aligned}
\mathcal{R}_f^m &= \max_{\substack{R_i, P_i, \mathbf{Q}_i \\ \mathrm{tr}(\mathbf{Q}_i) \le P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^{K} \Pr\left\{\mathcal{I}_i \ge R_i\right\} R_i \\
&\overset{(a)}{\le} \max_{\substack{P_i, \mathbf{Q}_i \\ \mathrm{tr}(\mathbf{Q}_i) \le P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^{K} \mathbb{E}\left(\mathcal{I}_i\right) \\
&\overset{(b)}{=} \max_{\substack{P_i, \mathbf{Q}_i \\ \mathrm{tr}(\mathbf{Q}_i) \le P_i \\ \sum_{i=1}^K P_i = P}} \mathbb{E}\left(\sum_{i=1}^{K} \mathcal{I}_i\right) \\
&= \max_{\substack{P_i, \mathbf{Q}_i \\ \mathrm{tr}(\mathbf{Q}_i) \le P_i \\ \sum_{i=1}^K P_i = P}} \mathbb{E}\left(\sum_{i=1}^{K} \ln \frac{\det\left(\mathbf{I}_{n_t} + \sum_{j=i}^{K} \mathbf{Q}_j \mathbf{H}^\dagger \mathbf{H}\right)}{\det\left(\mathbf{I}_{n_t} + \sum_{j=i+1}^{K} \mathbf{Q}_j \mathbf{H}^\dagger \mathbf{H}\right)}\right) \\
&= \max_{\substack{P_i, \mathbf{Q}_i \\ \mathrm{tr}(\mathbf{Q}_i) \le P_i \\ \sum_{i=1}^K P_i = P}} \mathbb{E}\left(\ln \prod_{i=1}^{K} \frac{\det\left(\mathbf{I}_{n_t} + \sum_{j=i}^{K} \mathbf{Q}_j \mathbf{H}^\dagger \mathbf{H}\right)}{\det\left(\mathbf{I}_{n_t} + \sum_{j=i+1}^{K} \mathbf{Q}_j \mathbf{H}^\dagger \mathbf{H}\right)}\right) \\
&= \max_{\substack{P_i, \mathbf{Q}_i \\ \mathrm{tr}(\mathbf{Q}_i) \le P_i \\ \sum_{i=1}^K P_i = P}} \mathbb{E}\left(\ln\det\left(\mathbf{I}_{n_t} + \sum_{i=1}^{K} \mathbf{Q}_i \mathbf{H}^\dagger \mathbf{H}\right)\right),
\end{aligned}
\tag{2.7}
$$

where (a) follows from Proposition 2.1, and (b) follows from the fact that expectation and summation commute. Defining $\mathbf{Q} \triangleq \sum_{i=1}^{K} \mathbf{Q}_i$, we get

$$
\mathrm{tr}\left(\mathbf{Q}\right) = \mathrm{tr}\left(\sum_{i=1}^{K} \mathbf{Q}_i\right) = \sum_{i=1}^{K} \mathrm{tr}\left(\mathbf{Q}_i\right) \le \sum_{i=1}^{K} P_i = P. \tag{2.8}
$$

Inserting Eq. (2.8) into Eq. (2.7), we obtain

$$
\mathcal{R}_f^m \le \max_{\substack{\mathbf{Q} \\ \mathrm{tr}(\mathbf{Q}) \le P}} \mathbb{E}\left(\ln\det\left(\mathbf{I}_{n_t} + \mathbf{Q}\mathbf{H}^\dagger\mathbf{H}\right)\right) = \max_{\substack{\mathbf{Q} \\ \mathrm{tr}(\mathbf{Q}) \le P}} \mathbb{E}\left(\mathcal{I}\right), \tag{2.9}
$$

and Eq. (2.9) results because $\max_{\mathbf{Q},\mathrm{tr}(\mathbf{Q})\leq P} \mathbb{E}\left(\mathcal{I}\right)$ equals the ergodic capacity. □

Propositions 2.1 and 2.2 lead to the fact that the maximum throughput and maximum expected-rate are upper-bounded by the ergodic capacity. Proposition 2.3 presents the ergodic capacity of the MISO channel in closed form.

Proposition 2.3. *The ergodic capacity in an $n_t \times 1$ MISO Rayleigh fading channel with total power constraint P is given by*

$$\begin{aligned} C_{\text{erg}} = e^{\frac{n_t}{P}} \mathrm{E}_1\left(\frac{n_t}{P}\right) \sum_{\ell=0}^{n_t-1} \frac{\left(-n_t\right)^{\ell}}{\ell! P^{\ell}} \\ + \sum_{\ell=1}^{n_t-1} \sum_{k=0}^{\ell-1} \frac{\left(-1\right)^{k}}{\left(\ell-k\right) k!} \sum_{m=0}^{\ell-k-1} \frac{\left(n_t\right)^{k+m}}{m! P^{k+m}}, \end{aligned} \tag{2.10}$$

where $\mathrm{E}_1\left(\cdot\right)$ is the exponential integral function. The ergodic capacity in a $1 \times n_r$ single-input multiple-output (SIMO) channel with total power constraint P equals the ergodic capacity of an $n_r \times 1$ MISO channel with total power constraint $n_r P$.

Proof. We offer the proof in Appendix A. □

2.3 Maximum Throughput in MISO Channels

Let the transmitted signal $\vec{X}$ be a single-layer code with rate $R = \ln\left(1 + Ps\right)$. In the MISO channel, the maximum throughput in Eq. (1.1) can be rewritten as

$$\mathcal{R}_s^m = \max_{\substack{R,\mathbf{Q} \\ \mathrm{tr}(\mathbf{Q})\leq P}} \Pr\left\{\ln\left(1 + \vec{h}\mathbf{Q}\vec{h}^{\dagger}\right) \geq R\right\} R, \tag{2.11}$$

where $\mathbf{Q}$ is the covariance matrix of $\vec{X}$, i.e., $\mathbf{Q} = \mathbb{E}\left(\vec{X}\vec{X}^{\dagger}\right)$.

For transmission rate R, the throughput is $\mathcal{R}_s = \overline{\mathcal{P}}_{\text{out}}(R)R$, where $\mathcal{P}_{\text{out}}(R)$ is the outage probability of a fixed transmission rate R. It is conjectured in [77], and a decade later proved in [2,3] that the optimum transmit strategy minimizing the outage probability is to send uncorrelated circularly symmetric zero mean equal power complex Gaussian signals from a fraction of antennas. Thus, here, one can restrict the transmit covariance matrix $\mathbf{Q}$ to diagonal matrices whose diagonal entries are either zero or a constant subject to the total power constraint P.

In following, Theorem 2.1 proves that the optimum solution with respect to R, denoted by R^o, maximizing $\overline{\mathcal{P}}_{\text{out}}(R)R$ is less than $\ln(1+P)$. In this range of the transmission rate, the optimum transmit strategy which minimizes the outage probability and consequently, maximizes the throughput is to use all available antennas. Equation (2.12) yields the maximum throughput of an $n_t \times 1$ MISO block Rayleigh fading channel.

Theorem 2.1. *In a single-layer $n_t \times 1$ MISO block Rayleigh fading channel, the optimum transmit covariance matrix which maximizes the throughput is $\mathbf{Q^o} = \frac{P}{n_t}\mathbf{I}_{n_t}$. The maximum throughput is given by*

$$\mathcal{R}_s^m = \max_{0<s<1} \frac{\Gamma(n_t, n_t s)}{(n_t-1)!} \ln(1+Ps)\,. \tag{2.12}$$

Proof. As pointed out above, we can restrict our attention to assume that l_t out of n_t transmit antennas are active and perform equal power allocation. Equation (2.11) is simplified to

$$\begin{aligned}\mathcal{R}_s^m &= \max_{R,l_t} \Pr\left\{\ln\left(1+\frac{P}{l_t}\sum_{\ell=1}^{l_t}|h_\ell|^2\right) \geq R\right\} R \\ &= \max_{s,l_t} \Pr\left\{\sum_{\ell=1}^{l_t}|h_\ell|^2 \geq l_t s\right\} \ln(1+Ps)\end{aligned}$$

$$= \max_{s,l_t} \overline{F}_{\mathrm{a}}(l_t s) \ln(1 + Ps), \tag{2.13}$$

where $a \triangleq \sum_{\ell=1}^{l_t} |h_\ell|^2$ is gamma-distributed and thereby, $\overline{F}_{\mathrm{a}}(x) = \frac{\Gamma(l_t,x)}{\Gamma(l_t)}$. The first derivative of $\mathcal{R}_s(s) = \overline{F}_{\mathrm{a}}(l_t s) \ln(1 + Ps)$ with respect to s is

$$\mathcal{R}'_s(s) = \overline{F}_{\mathrm{a}}(l_t s) \frac{P}{1 + Ps} - l_t f_{\mathrm{a}}(l_t s) \ln(1 + Ps). \tag{2.14}$$

Let us define the following functions,

$$r(s) \triangleq \frac{\overline{F}_{\mathrm{a}}(l_t s)}{l_t f_{\mathrm{a}}(l_t s)}, \tag{2.15}$$

$$g(s,P) \triangleq \ln(1 + Ps)^{\frac{1+Ps}{P}}. \tag{2.16}$$

As such, we get

$$\begin{cases} \mathcal{R}'_s(s) > 0 \quad \text{iff} \quad r(s) > g(s,P), \\ \mathcal{R}'_s(s) = 0 \quad \text{iff} \quad r(s) = g(s,P), \\ \mathcal{R}'_s(s) < 0 \quad \text{iff} \quad r(s) < g(s,P). \end{cases} \tag{2.17}$$

Noting $\overline{F}_{\mathrm{a}}(x) = \frac{\Gamma(l_t,x)}{\Gamma(l_t)}$ and $f_{\mathrm{a}}(x) = \frac{x^{l_t-1}e^{-x}}{\Gamma(l_t)}$, we have

$$r(s) = \frac{\Gamma(l_t, l_t s)}{l_t (l_t s)^{l_t-1} e^{-l_t s}} = \frac{\Gamma(l_t, l_t s)}{l_t^{l_t} s^{l_t-1} e^{-l_t s}}. \tag{2.18}$$

For positive integer arguments of m, $\Gamma(m,x) = (m-1)! e^{-x} \sum_{\ell=0}^{m-1} \frac{x^\ell}{\ell!}$. Inserting the above equation into Eq. (2.18) yields

$$r(s) = \frac{(l_t - 1)! e^{-l_t s} \sum_{\ell=0}^{l_t-1} \frac{(l_t s)^\ell}{\ell!}}{l_t (l_t s)^{l_t-1} e^{-l_t s}}$$

$$= \frac{1}{l_t} + \frac{1}{l_t} \sum_{\ell=0}^{l_t-2} \frac{(l_t-1)\dots(\ell+1)}{(l_t s)^{l_t-\ell-1}}$$
$$= \frac{1}{l_t} + \frac{1}{l_t} \sum_{\ell=0}^{l_t-2} \prod_{k=0}^{l_t-\ell-2} \frac{l_t-k-1}{l_t s}. \tag{2.19}$$

As $\frac{l_t-k-1}{l_t s} < 1$ for $s \geq 1$, replacing in Eq. (2.19) gives

$$r(s) \leq \frac{1}{l_t} + \frac{1}{l_t} \sum_{\ell=0}^{l_t-2} \prod_{k=0}^{l_t-\ell-2} 1 = \frac{1}{l_t} + \frac{l_t-1}{l_t} = 1, \quad \forall s \geq 1. \tag{2.20}$$

From Eq. (2.19), $\lim_{s\to 0} r(s) = +\infty$.

On the other hand, the first derivative of $g(s,P)$ with respect to P is

$$\frac{\partial g(s,P)}{\partial P} = \frac{sP - \ln(1+sP)}{P^2}$$
$$= \frac{1}{P^2} \ln \frac{e^{sP}}{1+sP}$$
$$= \frac{1}{P^2} \ln \left(1 + \frac{1}{1+sP} \sum_{k=2}^{\infty} \frac{(sP)^k}{k!}\right) > 0. \tag{2.21}$$

Therefore, $g(s,P)$ is a strictly increasing function with respect to P. As a result,

$$g(s,P) > \lim_{P\to 0} \ln(1+Ps)^{\frac{1+Ps}{P}} = s. \tag{2.22}$$

Comparing Eq. (2.20), Eq. (2.22), $\lim_{s\to 0} r(s) = +\infty$, and $g(0,P) = 0$, we get

$$\begin{cases} r(s) > g(s,P) & s = 0, \\ r(s) < g(s,P) & s \geq 1. \end{cases} \tag{2.23}$$

Inserting Eq. (2.23) into Eq. (2.17) yields

$$\begin{cases} \mathcal{R}_s'(s) > 0 & s = 0, \\ \mathcal{R}_s'(s) < 0 & s \geq 1. \end{cases} \tag{2.24}$$

Since $\mathcal{R}_s(s)$ is a continuous function, according to Eq. (2.24), for all positive integer values of l_t and positive values of P, one can conclude that $\mathcal{R}_s(s)$ takes its maximum at $0 < s^o < 1$.

Jorswieck and Boche [43] proved that when $P > e^R - 1$, or equivalently $s < 1$, the optimum transmission strategy to minimize the outage probability is to use all available antennas with equal power allocation. Since $\forall l_t$, $0 < s^o < 1$, the optimum strategy maximizing the throughput is to use all available antennas and perform equal power allocation. The maximum throughput is given by Eq. (2.12).

□

Remark 2.1. *In point-to-point single-input single-output (SISO) channels, by substituting* $n_t = 1$ *in Eq.* (2.12)*, the optimum solution with respect to* s *is* $s^o = \frac{1}{\mathcal{W}_0(P)} - \frac{1}{P}$*, where* $\mathcal{W}_0(\cdot)$ *is the zero branch of the Lambert W-function [17, 18]. Therefore,*

$$\mathcal{R}_s^m = e^{\frac{1}{P} - \frac{1}{\mathcal{W}_0(P)}} \ln\left(\frac{P}{\mathcal{W}_0(P)}\right). \tag{2.25}$$

From Proposition 2.3, the ergodic capacity in this channel is

$$C_{\mathrm{erg}} = e^{\frac{1}{P}} \mathrm{E}_1\left(\frac{1}{P}\right). \tag{2.26}$$

Remark 2.2. *Note that* $g(s, P)$ *is a strictly increasing function with respect to* s *and* P*, and* $r(s)$ *is a strictly decreasing function with respect to* s *and increases with the number of* *transmit antennas. Therefore, the solution to* $r(s) = g(s, P)$*, i.e.,* s^o*,*

- *decreases with* P*. In asymptotically high SNR regime,* $s^o \to 0$.
- *increases with* n_t*. In asymptotically large number of transmit antennas,* $s^o \to 1$.

As a byproduct result of Theorem 2.1 and remark 2.2, we have the following.

Corollary 2.1. *In the asymptotically large number of transmit antennas MISO channel, the maximum throughput is given by*

$$\mathcal{R}_s^m = \lim_{s \to 1} \frac{\Gamma(n_t, n_t s)}{(n_t - 1)!} \ln(1 + Ps) \overset{n_t \to \infty}{\longrightarrow} \ln(1 + P). \tag{2.27}$$

Remark 2.3. *In a correlated MISO channel wherein the transmitter does neither know the CSI nor the channel correlation, the outage probability is a Schur-convex (resp. Schur-concave) function of the channel covariance matrix for* $P > e^R - 1$ *(resp.* $P < \frac{e^R - 1}{2}$*) [43]. According to Theorem 2.1, in the maximum throughput of the MISO channel, i.e.,* $\overline{\mathcal{P}}_{out}(R^o)R^o$*, we have* $e^{R^o} - 1 < P$*. Hence, in this range of the transmission rate,* $\mathcal{R}_s$ *is a Schur-concave function of the channel covariance matrix, i.e., channel correlation decreases the throughput. In terms of the impact of correlation in the MISO channel with no CSI at the transmitter, the behavior of the maximum throughput is similar to the behavior of the ergodic capacity which is also a Schur-concave function of the channel covariance matrix [14]. For the definition of Schur-convexity and majorization theory see [51].*

2.4 Maximum Expeted-Rate in MISO Channels

A block fading channel can be modeled by an equivalent broadcast channel whose receiver channels represent any fading coefficient realization. The expected-rate of a fading channel is equal to a weighted sum-rate of its equivalent broadcast channel in which the weights distribution is the complementary CDF (tail distribution) of the channel gain [66]. In broadcast channels, any maximum weighted sum-rate with positive value weights is on the capacity region [55]. Since superposition (multi-layer) coding achieves the capacity region of degraded broadcast channels [12, 22, 32], it is the optimum coding strategy to maximize the average achievable rate in any block fading channel whose equivalent broadcast channel is degraded [67]. An example for such channels is the SISO channel [19]. Although multi-layer coding is not the optimum coding strategy in MISO channels, it increases the average achievable rate of the channel. Numerical results for the maximum continuous-layer expected-rate of MISO and SIMO block Rayleigh fading channels were presented by Steiner and Shamai for transmitters with no CSI and partial CSI in [72] and [73], respectively. Here, the optimum transmit covariance matrix at each code layer is obtained, and consequently, the maximum expected-rate of the MISO channel is analytically formulated. Note that the maximum expected-rate of the SIMO channel with n_t receive antennas can be calculated using the same formula by replacing P with n_tP in Eq. (2.39).

In order to enhance the lucidity of this section, we divide it into two subsections. Section 2.4.1 presents the maximum expected-rate of the MISO channel when a finite-layer code is transmitted. The more code layers, the higher expected-rate. Hence, a continuous-layer (infinite-layer) code yields the highest expected-rate of the channel. The maximum continuous-layer expected-rate of the MISO channel is derived in Section 2.4.2 in closed form.

2.4.1 Finite-Layer Code

In finite-layer coding approach, the transmitter sends a K-layer code $\vec{X} = \sum_{i=1}^{K} \vec{X}_i$. Let P_i be the signal power in the i'th layer with rate $R_i = \ln\left(1 + \frac{P_i s_i}{1+I_i s_i}\right)$, where $I_i = \sum_{j=i+1}^{K} P_j$

is the power of the upper layers while decoding the i'th layer. The maximum expected-rate in Eq. (1.2) is simplified to

$$\mathcal{R}_f^m = \max_{\substack{R_i, P_i, \mathbf{Q}_i \\ \operatorname{tr}(\mathbf{Q}_i) \leq P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^{K} \Pr\left\{ \ln\left(1 + \frac{\vec{h}\mathbf{Q}_i\vec{h}^\dagger}{1 + \vec{h}\sum_{j=i+1}^{K}\mathbf{Q}_j\vec{h}^\dagger}\right) \geq R_i \right\} R_i. \tag{2.28}$$

Theorem 2.2 presents the optimum covariance matrix in each layer which maximizes the expected-rate in the MISO channel.

Theorem 2.2. *In a finite-layer $n_t \times 1$ MISO block Rayleigh fading channel, the optimum transmit covariance matrix in each layer which maximizes the expected-rate is $\mathbf{Q}_i^o = \frac{P_i}{n_t}\mathbf{I}_{n_t}$, where P_i is the power allocated to the i'th layer. The maximum K-layer expected-rate is given by*

$$\mathcal{R}_f^m = \max_{\substack{0 < s_i < 1, P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^{K} \frac{\Gamma\left(n_t, n_t s_i\right)}{(n_t - 1)!} \ln\left(1 + \frac{P_i s_i}{1 + \sum_{j=i+1}^{K} P_j s_i}\right). \tag{2.29}$$

Proof. Since in the absence of CSI at the transmitter in uncorrelated MISO channels, the outage probability does not depend on the directions of the transmit covariance matrix $\mathbf{Q}$ [81, 87], the problem is diagonalized. Therefore, the expected-rate received at the destination is simplified to

$$\mathcal{R}_f = \sum_{i=1}^{K} \Pr\left\{ \ln\left(1 + \frac{P_i \sum_{\ell=1}^{n_t} \delta_\ell |h_\ell|^2}{1 + I_i \sum_{\ell=1}^{n_t} \eta_\ell |h_\ell|^2}\right) \geq R_i \right\} R_i, \tag{2.30}$$

where δ_ℓ and η_ℓ are the power fraction and upper-layer interference portion at the ℓ'th antenna in the i'th layer, respectively, subject to $\sum_{\ell=1}^{n_t} \delta_\ell = \sum_{\ell=1}^{n_t} \eta_\ell = 1$. Equation (2.30) can be rewritten as

$$\mathcal{R}_f = \sum_{i=1}^{K} \Pr\left\{ \sum_{\ell=1}^{n_t} \left(\delta_\ell + s_i I_i \delta_\ell - s_i I_i \eta_\ell\right) |h_\ell|^2 \geq s_i \right\} R_i. \tag{2.31}$$

As $\sum_{\ell=1}^{n_t} \left(\delta_\ell + s_i I_i \delta_\ell - s_i I_i \eta_\ell\right) = 1$, to minimize $\Pr\left\{\sum_{\ell=1}^{n_t} \left(\delta_\ell + s_i I_i \delta_\ell - s_i I_i \eta_\ell\right) |h_\ell|^2 < s_i\right\}$, $\forall i$, the optimum value of $\delta_\ell + s_i I_i \delta_\ell - s_i I_i \eta_\ell$ must be either zero or a constant independent of ℓ for any positive value of s_i [74]. Hence, up to now, the optimum solution to Eq. (2.31) is to choose either $\delta_\ell = \eta_\ell = \frac{1}{l_{t_i}}$ or $\delta_\ell = \eta_\ell = 0$, that is to use l_{t_i} out of n_t antennas with power $\frac{P_i}{l_{t_i}}$ in each layer. Therefore, Eq. (2.31) is simplified to

$$\mathcal{R}_f = \sum_{i=1}^{K} \Pr\left\{ \sum_{\ell=1}^{l_{t_i}} |h_\ell|^2 \geq l_{t_i} s_i \right\} R_i = \sum_{i=1}^{K} \overline{F}_{\mathrm{a}_i}\left(l_{t_i} s_i\right) R_i, \tag{2.32}$$

where $a_i = \sum_{\ell=1}^{l_{t_i}} |h_\ell|^2$. In the remainder of the proof, we shall show that the optimum solution with respect to l_{t_i} is $l_{t_i}^o = n_t$, $\forall i$. Analogous to the throughput case in Theorem 2.1, let us define

$$\mathcal{R}_s(s_i) \triangleq \overline{F}_{\mathrm{a}_i}\left(l_{t_i} s_i\right) \ln\left(1 + \frac{P_i s_i}{1 + I_i s_i}\right), \tag{2.33}$$

$$r(s_i) \triangleq \frac{\overline{F}_{\mathrm{a}_i}(l_{t_i} s_i)}{l_{t_i} f_{\mathrm{a}_i}(l_{t_i} s_i)}, \tag{2.34}$$

$$g(s_i, P_i, I_i) \triangleq \frac{\left(1 + I_i s_i\right)\left(1 + \left(I_i + P_i\right) s_i\right)}{P_i} \ln\left(1 + \frac{P_i s_i}{1 + I_i s_i}\right). \tag{2.35}$$

Note that $g(0, P_i, I_i) = 0$, $\lim_{s_i \to 0} r(s_i) = +\infty$, and Eqs. (2.17) and (2.20) still hold by redefining $\mathcal{R}_s(s_i)$, $r(s_i)$, and $g(s_i, P_i, I_i)$ as above, and with s replaced by s_i.

Defining $\hat{P}_i \triangleq \frac{P_i}{1+I_i s_i}$, from Eq. (2.22) and noting $I_i s_i \geq 0$, we have

$$g(s_i, P_i, I_i) = (1 + I_i s_i) \frac{\left(1 + \frac{P_i s_i}{1+I_i s_i}\right)}{\frac{P_i}{1+I_i s_i}} \ln\left(1 + \frac{P_i s_i}{1 + I_i s_i}\right)$$
$$\geq \ln\left(1 + \hat{P}_i s_i\right)^{\frac{(1+\hat{P}_i s_i)}{\hat{P}_i}} > s_i, \quad \forall s_i \geq 1. \tag{2.36}$$

Therefore, Eqs. (2.23) and (2.24) still hold with the above functions, and lead to $0 < s_i^o < 1$. This directly corresponds to the proof of Theorem 2.1 and shows that the optimum power allocation strategy is to use all available antennas with equal power allocation in each layer, i.e., $\mathbf{Q}_i^o = \frac{P_i}{n_t}\mathbf{I}_{n_t}$, and the maximum expected-rate is given by Eq. (2.29).

□

2.4.2 Continuous-Layer Code

In the continuous-layer coding, a continuum of code layers is transmitted. Similar to finite-layer coding in Section 2.4.1, the receiver decodes the signal from the lowest layer up to the layer that the channel condition allows.

Theorem 2.3 yields a closed form expression for the maximum continuous-layer expected-rate in the MISO channel by optimizing the power distribution over the layers.

Theorem 2.3. *In the MISO block Rayleigh fading channel, the maximum continuous-layer expected-rate obtained by optimizing the power distribution over the layers is given by*

$$\mathcal{R}_c^m = \mathcal{R}(s_1) - \mathcal{R}(s_0), \tag{2.37}$$

where,

$$\mathcal{R}(s) = e^{-s}\sum_{\ell=1}^{n_t-1}\frac{1}{\ell!}\left(s^\ell - (n_t+1-\ell)(\ell-1)!\sum_{k=0}^{\ell-1}\frac{s^k}{k!}\right) \\ +e^{-s} - (n_t+1)\mathrm{E}_1(s). \tag{2.38}$$

s_0 and s_1 are the solutions to

$$\begin{cases} \sum_{\ell=0}^{n_t-1}\frac{(n_t-1)!}{\ell! s_0^{n_t-\ell}} = 1 + \frac{P}{n_t}s_0, \\ \sum_{\ell=0}^{n_t-1}\frac{(n_t-1)!}{\ell! s_1^{n_t-\ell}} = 1, \end{cases} \tag{2.39}$$

respectively.

Proof. Based on Theorem 2.2, transmitting each of the code layers on all available antennas and performing equal power allocation is optimum. As showed in [67], the maximum continuous-layer expected-rate of fading channels with general distribution is given by

$$\mathcal{R}_c^m = \max_{I(s)}\int_0^\infty \overline{F}_{\mathrm{a}}(s)\frac{-sI'(s)}{1+sI(s)}\mathrm{d}s. \tag{2.40}$$

Noting $\overline{F}_{\mathrm{a}}(s) = \frac{\Gamma(n_t,s)}{\Gamma(n_t)} = e^{-s}\sum_{\ell=0}^{n_t-1}\frac{s^\ell}{\ell!}$, we have

$$\mathcal{R}_c^m = \max_{I(s)}\int_0^\infty \frac{-se^{-s}I'(s)}{1+sI(s)}\sum_{\ell=0}^{n_t-1}\frac{s^\ell}{\ell!}\mathrm{d}s. \tag{2.41}$$

The optimization solution to Eq. (2.41) with respect to $I(s)$ under the total power constraint $\frac{P}{n_t}$ at each antenna is found using variation methods [33]. By solving the corresponding

Eüler equation [33], we come up with the final solution as follows,

$$\mathcal{R}_c^m = \int_{s_0}^{s_1} e^{-s}\left(\frac{n_t+1}{s}-1\right)\sum_{\ell=0}^{n_t-1}\frac{s^\ell}{\ell!}\mathrm{d}s, \tag{2.42}$$

where boundaries s_0 and s_1 are the solutions to $\sum_{\ell=0}^{n_t-1}\frac{(n_t-1)!}{\ell! s_0^{n_t-\ell}} = 1+\frac{P}{n_t}s_0$ and $\sum_{\ell=0}^{n_t-1}\frac{(n_t-1)!}{\ell! s_1^{n_t-\ell}} = 1$, respectively. The indefinite integral (antiderivative) of Eq. (2.42) is given by Eq. (2.38) (the derivation steps are deferred to Appendix B). Applying the integration limits completes the proof.

□

Remark 2.4. *By substituting $n_t = 1$ in Theorem 2.3, the maximum continuous-layer expected-rate of the SISO channel is*

$$\mathcal{R}_c^m = 2\mathrm{E}_1\left(\frac{2}{1+\sqrt{1+4P}}\right) - 2\mathrm{E}_1(1) - e^{\frac{-2}{1+\sqrt{1+4P}}} + e^{-1}, \tag{2.43}$$

which is consistent with the result of [67]. As pointed out earlier, one can model a point-to-point block Rayleigh fading channel with an equivalent broadcast channel. According to the degradedness of the equivalent SISO broadcast channel, and the optimality of superposition (multi-layer) coding for such channels [12, 22, 32], the maximum continuous-layer expected-rate of the SISO channel, i.e., Eq. (2.43)*, represents its maximum average achievable rate [67].*

Remark 2.5. *Since the equivalent broadcast channel of the MISO channel is not degraded [83], its maximum continuous-layer expected-rate is not the maximum average achievable rate of the channel. For example, in asymptotically low SNR regime, the multiple-access scheme provides a higher average achievable rate in the MISO channel. In the multiple-access scheme, the antennas send independent messages, and the receiver decodes as much as it can.*

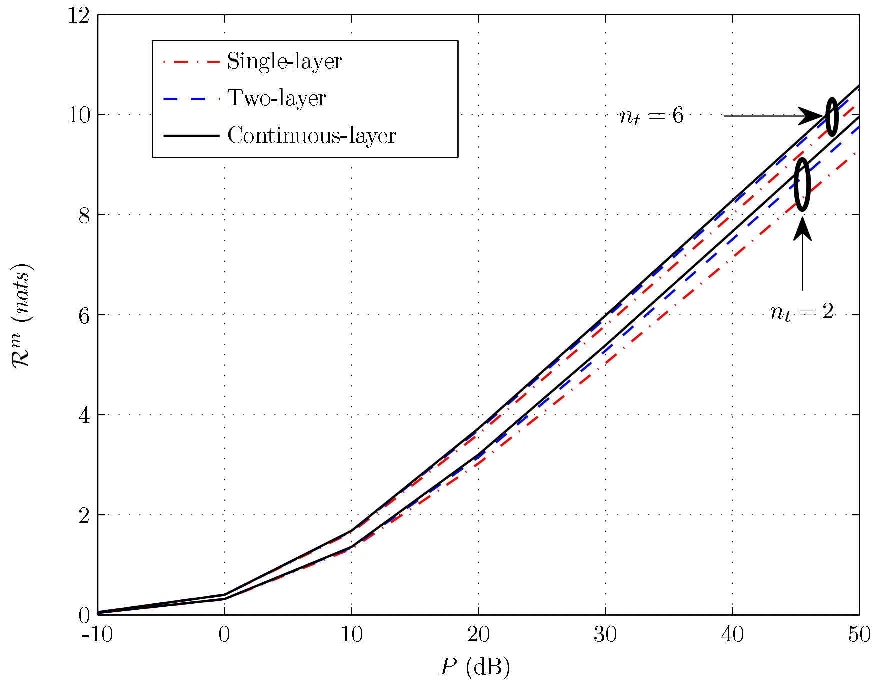

Figure 2.1: Maximum throughput, maximum two-layer expected-rate, and maximum continuous-layer expected-rate (all in *nats*) in the MISO channel with $n_t = 2$ and $n_t = 6$.

Remark 2.6. *Similar to remark 2.3, one can conclude that for* $0 < s_i^o < 1$, $\forall i$, *the maximum expected-rate of the MISO channel with uninformed transmitter is a Schur-concave function of the channel covariance matrix, that is channel correlation reduces the maximum expected-rate.*

Figure 2.1 compares the maximum throughput (red dashed-dotted line), maximum expected-rate with two-layer coding (blue dashed line), and maximum expected-rate with continuous-layer coding (black solid line) of the MISO channel for $n_t = 2$ and $n_t = 6$.

2.5 Maximum Throughput in MIMO Channels

The throughput maximization problem in the MIMO channel is less tractable than that corresponding to the MISO channel. Since in the Gaussian MIMO channel, in the sense of the outage probability, the optimum eigenvectors of the transmit covariance matrix always correspond to the eigenvectors of the channel correlation matrix [81], one can restrict the transmit covariance matrix to be diagonal in the problem of interest.

Recall from Section 2.1, in an $n_t \times n_r$ MIMO channel, the PDF of the instantaneous mutual information in Eq. (2.3) does not lend itself to a closed form expression. In order to analyze the throughput, it is necessary to characterize this PDF. There are some approximations for the PDF of the instantaneous mutual information in literature, e.g., approximations on the distribution of the eigenvalues of $\mathbf{HH}^\dagger$ in MIMO channels with asymptotically large number of antennas at both the transmitter and receiver sides [16, 68].

In a MIMO channel with $\mathbf{Q} = \frac{P}{n_t}\mathbf{I}_{n_t}$, the PDF of the instantaneous mutual information can be well approximated by the Gaussian distribution with the same mean and variance [40, 82], i.e.,

$$\mathcal{I} \sim \mathcal{N}\left(\mu(n_t, n_r), \sigma^2(n_t, n_r)\right), \tag{2.44}$$

where

$$\begin{cases} \mu(n_t, n_r) = \mathbb{E}\left(\mathcal{I}\right), \\ \sigma^2(n_t, n_r) = \mathrm{Var}\left(\mathcal{I}\right). \end{cases} \tag{2.45}$$

Note that for $\mathbf{Q} = \frac{P}{n_t}\mathbf{I}_{n_t}$, $\mu(n_t, n_r)$ equals the ergodic capacity of an $n_t \times n_r$ MIMO channel, which is a strictly increasing function with respect to n_t and n_r [77]. This Gaussian distribution approximation allows the throughput maximization to be expressed as

$$\begin{aligned} \mathcal{R}_s^m &= \max_R \Pr\left\{\mathcal{I} \geq R\right\} R \\ &= \max_R \mathcal{Q}\left(\frac{R - \mu(n_t, n_r)}{\sigma(n_t, n_r)}\right) R. \end{aligned} \tag{2.46}$$

With $z \triangleq \frac{R-\mu(n_t,n_r)}{\sigma(n_t,n_r)}$, Eq. (2.46) leads to

$$\mathcal{R}_s^m = \max_z \mathcal{Q}(z)\left(\sigma(n_t,n_r)z + \mu(n_t,n_r)\right) \tag{2.47}$$

$$= \mathcal{Q}(z^o)\left(\sigma(n_t,n_r)z^o + \mu(n_t,n_r)\right), \tag{2.48}$$

where z^o is the solution to

$$-\frac{1}{\sqrt{2\pi}}e^{-\frac{{z^o}^2}{2}}\left(\sigma(n_t,n_r)z^o + \mu(n_t,n_r)\right) + \sigma(n_t,n_r)\mathcal{Q}(z^o)=0. \tag{2.49}$$

Since the existing approximations for the PDF of the instantaneous mutual information in the MIMO channel are not tractable enough to analyze the maximum throughput in general case, four asymptotic cases are investigated. In all four cases, it is shown that the optimum transmit strategy is to use all available antennas. It seems reasonable to conjecture that the above statement holds with the general MIMO channel. To test the claim, Fig. 2.2 shows the maximum throughput in a MIMO channel with 10 receive antennas. Note that the number of transmit antennas varies from 1 to 20 and the total power P sweeps the range of -10 dB to 50 dB.

2.5.1 Asymptotically Low SNR Regime

For small SNR values, the eigenvalues of $\mathbf{QH}^\dagger\mathbf{H}$ are small enough to approximate the following,

$$\prod_{\ell=1}^{n_t}\left(1+\mathrm{eig}_\ell\left(\mathbf{QH}^\dagger\mathbf{H}\right)\right) \approx 1 + \sum_{\ell=1}^{n_t}\mathrm{eig}_\ell\left(\mathbf{QH}^\dagger\mathbf{H}\right), \tag{2.50}$$

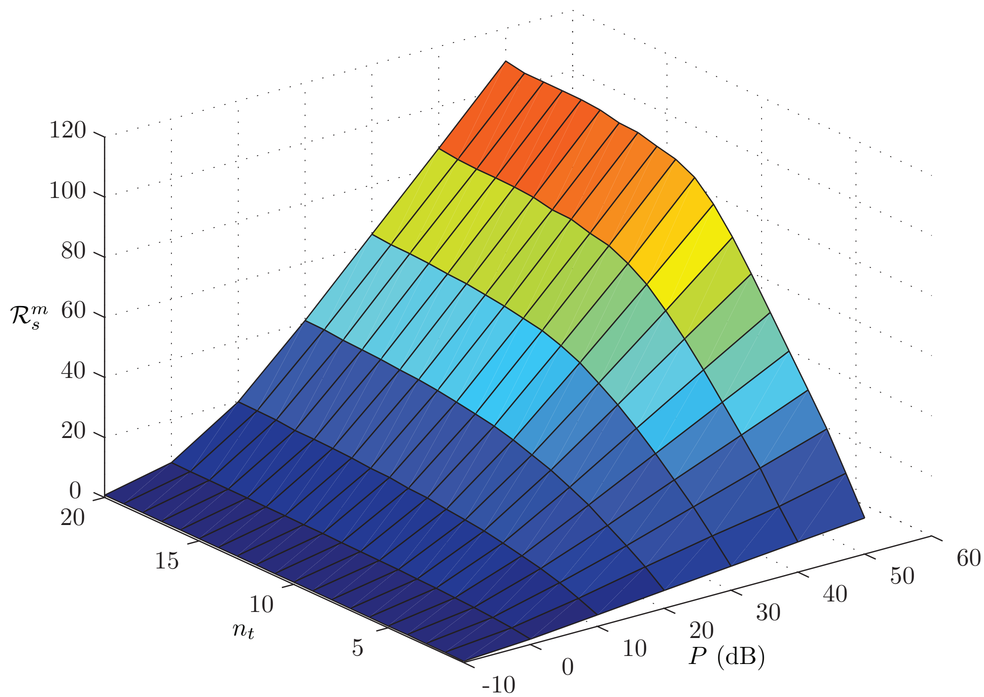

Figure 2.2: Maximum throughput (in *nats*) in a MIMO channel with 10 receive antennas ($n_r = 10$).

where $\mathrm{eig}_\ell(\cdot)$ is the ℓ'th ordered eigenvalue of matrix. Therefore, the instantaneous mutual information of Eq. (2.3) can be approximated by

$$\begin{aligned}\mathcal{I} &= \ln\det\left(\mathbf{I}_{n_t} + \mathbf{Q}\mathbf{H}^\dagger\mathbf{H}\right)\\ &= \ln\prod_{\ell=1}^{n_t}\left(1+\mathrm{eig}_\ell\left(\mathbf{Q}\mathbf{H}^\dagger\mathbf{H}\right)\right)\\ &\approx \ln\left(1+\sum_{\ell=1}^{n_t}\mathrm{eig}_\ell\left(\mathbf{Q}\mathbf{H}^\dagger\mathbf{H}\right)\right). \end{aligned} \tag{2.51}$$

Using Eq. (2.51), we can prove the following proposition on the optimum transmit covariance matrix which maximizes the throughput in the asymptotically low SNR regime MIMO channel.

Proposition 2.4. *The optimum transmit strategy maximizing the throughput in the asymptotically low SNR regime MIMO channel is transmitting independent signals and performing equal power allocation across all available antennas. The maximum throughput is*

$$\mathcal{R}_s^m = \max_{0<s<n_r}\frac{\Gamma\left(n_t n_r, n_t s\right)}{\left(n_t n_r - 1\right)!}\ln\left(1+Ps\right). \tag{2.52}$$

Proof. Let $\delta_\ell P$ denote the allocated power to the ℓ'th antenna subject to $\sum_{\ell=1}^{n_t}\delta_\ell = 1$. From Eq. (2.51), the instantaneous mutual information for low SNR values can be expressed as,

$$\begin{aligned}\mathcal{I} &\approx \ln\left(1+\sum_{\ell=1}^{n_t}\mathrm{eig}_\ell\left(\mathbf{Q}\mathbf{H}^\dagger\mathbf{H}\right)\right)\\ &= \ln\left(1+\mathrm{tr}\left(\mathbf{Q}\mathbf{H}^\dagger\mathbf{H}\right)\right)\\ &= \ln\left(1+P\sum_{\ell=1}^{n_t}\sum_{k=1}^{n_r}\delta_\ell\left|h_{\ell,k}\right|^2\right). \end{aligned} \tag{2.53}$$

Equation (2.53) corresponds to the instantaneous mutual information in the MISO channel. Therefore, the optimum transmit strategy minimizing the outage probability in the asymptotically low SNR regime MIMO channel is to transmit independent signals and perform equal power allocation across a fraction of available antennas.

Assume that the transmitter has allocated equal power to l_t out of n_t transmit antennas. The maximum throughput is given by

$$\mathcal{R}_s^m = \max_s \frac{\Gamma\left(l_t n_r, l_t s\right)}{(l_t n_r - 1)!} \ln\left(1 + Ps\right). \tag{2.54}$$

With $\hat{s} \triangleq \frac{s}{n_r}$, Eq. (2.54) leads to

$$\mathcal{R}_s^m = \max_{\hat{s}} \frac{\Gamma\left(l_t n_r, l_t n_r \hat{s}\right)}{(l_t n_r - 1)!} \ln\left(1 + P n_r \hat{s}\right). \tag{2.55}$$

Equation (2.55) corresponds to the maximum throughput expression of the MISO channel, i.e., Eq. (2.13), with $l_t n_r$ transmit antennas and total power $P n_r$. According to Theorem 2.1, the optimum transmit strategy is to use all available antennas and $0 < \hat{s} < 1$, and equivalently $0 < s < n_r$.

□

In the same direction, the finite-layer expected-rate is given by Proposition 2.5.

Proposition 2.5. *The optimum transmit strategy maximizing the expected-rate of the asymptotically low SNR regime MIMO channel is transmitting independent signals and performing equal power allocation across all available antennas in each code layer. The maximum finite-layer expected-rate is*

$$\mathcal{R}_f^m = \max_{\substack{0<s_i<n_r, P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^{K} \frac{\Gamma\left(n_t n_r, n_t s_i\right)}{(n_t n_r - 1)!} \ln\left(1 + \frac{P_i s_i}{1 + \sum_{j=i+1}^{K} P_j s_i}\right). \tag{2.56}$$

Proof. At the i'th layer, let $\delta_\ell P_i$ and $\eta_\ell I_i$ denote the allocated power and upper-layers power at the ℓ'th antenna subject to $\sum_{\ell=1}^{n_t} \delta_\ell = \sum_{\ell=1}^{n_t} \eta_\ell = 1$, and $I_i = \sum_{j=i+1}^{K} P_j$. Following the same steps in Eq. (2.53), the i'th layer instantaneous mutual information can be approximated by

$$\mathcal{I}_i \approx \ln\left(1 + \frac{P_i \sum_{\ell=1}^{n_t} \sum_{k=1}^{n_r} \delta_\ell \left|h_{\ell,k}\right|^2}{1 + I_i \sum_{\ell=1}^{n_t} \sum_{k=1}^{n_r} \eta_\ell \left|h_{\ell,k}\right|^2}\right). \tag{2.57}$$

Equation (2.57) corresponds to the instantaneous mutual information of the multi-layer MISO channel in Section 2.4.1. The proof is completed by following the steps in the proof of Theorem 2.2 and Proposition 2.4. □

Corresponding to Theorem 2.3, we have the following proposition for continuous-layer coding in the low SNR MIMO channels.

Proposition 2.6. *The maximum continuous-layer expected-rate in the asymptotically low SNR regime MIMO channel is given by*

$$\mathcal{R}_c^m = \mathcal{R}(s_1) - \mathcal{R}(s_0), \tag{2.58}$$

where,

$$\begin{aligned}\mathcal{R}(s) = e^{-s} \sum_{\ell=1}^{n_t n_r - 1} \frac{1}{\ell!} \left(s^\ell - (n_t n_r + 1 - \ell)(\ell - 1)! \sum_{k=0}^{\ell-1} \frac{s^k}{k!}\right) \\ + e^{-s} - (n_t n_r + 1)\mathrm{E}_1(s).\end{aligned} \tag{2.59}$$

s_0 and s_1 are the solutions to

$$\begin{cases} \sum_{\ell=0}^{n_t n_r - 1} \frac{(n_t n_r - 1)!}{\ell! s_0^{n_t n_r - \ell}} = 1 + \frac{P}{n_t} s_0, \\ \sum_{\ell=0}^{n_t n_r - 1} \frac{(n_t n_r - 1)!}{\ell! s_1^{n_t n_r - \ell}} = 1, \end{cases} \tag{2.60}$$

respectively.

Remark 2.7. *Analogous to the MISO channel, in the asymptotically low SNR regime MIMO channel with uninformed transmitter, channel correlation decreases the maximum throughput and maximum expected-rate.*

2.5.2 Asymptotically High SNR Regime

For large SNR values, we take advantages of Wishart distribution properties. In order to enhance the lucidity of this section, let us define $p \triangleq \min\{n_t, n_r\}$, $n \triangleq \max\{n_t, n_r\}$, and

$$\mathbf{W} = \begin{cases} \mathbf{H}^\dagger \mathbf{H} & n_t \leq n_r, \\ \mathbf{H}\mathbf{H}^\dagger & n_t > n_r. \end{cases} \tag{2.61}$$

Matrix $\mathbf{W}$ has a central complex p-variate Wishart distribution with scale matrix $\mathbf{I}_p$ and n degrees of freedom [8, 56, 61].

Theorem 2.4 yields the maximum throughput in the asymptotically high SNR regime MIMO channel by obtaining the optimum transmit covariance matrix $\mathbf{Q}^o$.

Theorem 2.4. *The optimum transmit strategy maximizing the throughput in the asymptotically high SNR regime MIMO channel is sending independent signals and performing equal power allocation across all available antennas. The maximum throughput is*

$$\mathcal{R}_s^m = \max_s \overline{F}_{\mathrm{a}} \left(\frac{n_t^p s}{P^{p-1}} \right) \ln(1 + Ps) \tag{2.62}$$

$$= \max_{z} \mathcal{Q}(z) \left(z \sqrt{\frac{\pi^2}{6} p - \sum_{k=0}^{p-1} \sum_{\ell=1}^{n-k-1} \frac{1}{\ell^2}} + p \left(\digamma(1) + \ln \left(\frac{P}{n_t} \right) \right) + \sum_{k=0}^{p-1} \sum_{\ell=1}^{n-k-1} \frac{1}{\ell} \right), \quad (2.63)$$

where $-\digamma(1) \approx 0.577215$ *is the Eüler-Mascheroni constant,* $a \triangleq \prod_{\ell=1}^{p} a_{\ell,\ell}^2$*, and* $a_{\ell,\ell}^2, \forall \ell$ *are independent gamma-distributed with scale 1 and shape* $n - \ell + 1$*, i.e.,* $f_{\mathrm{a}_{\ell,\ell}^2}(x) = \frac{\Gamma(n-\ell+1,x)}{(n-\ell)!}$.

Proof. Again, we first assume that l_t out of n_t transmit antennas are active. Then, we shall see that the optimum solution is $l_t^o = n_t$. Define the index set $Z(\mathbf{Q}) \triangleq \{\ell : q_{\ell,\ell} = 0\}$. Denote by $\mathbf{Q}_{l_t}$ the matrix obtained from $\mathbf{Q}$ by eliminating of all the ℓ'th rows and columns with $\ell \in Z(\mathbf{Q})$. Clearly, $\mathbf{Q}_{l_t}$ has full rank. We divide the proof into two parts: Part i) $l_t \leq n_r$, Part ii) $l_t \geq n_r$. We wish to show that in both cases, the throughput is a strictly increasing function with respect to l_t.

Part i):

In high SNR regime, the eigenvalues of $\mathbf{Q}_{l_t}\mathbf{H}^\dagger\mathbf{H}$ are large. The instantanous mutual information can be well approximated by

$$\begin{aligned}
\mathcal{I} &= \ln\det\left(\mathbf{I}_{l_t} + \mathbf{Q}\mathbf{H}^\dagger\mathbf{H}\right) \\
&= \ln \prod_{\ell=1}^{l_t} \left(1 + \mathrm{eig}_\ell\left(\mathbf{Q}\mathbf{H}^\dagger\mathbf{H}\right)\right) \\
&\approx \ln \prod_{\ell=1}^{l_t} \left(\mathrm{eig}_\ell\left(\mathbf{Q}_{l_t}\mathbf{H}^\dagger\mathbf{H}\right)\right) \\
&= \ln\det\left(\mathbf{Q}_{l_t}\mathbf{H}^\dagger\mathbf{H}\right) \\
&= \ln\det\mathbf{Q}_{l_t} + \ln\det\left(\mathbf{H}^\dagger\mathbf{H}\right) \\
&= \ln\det\mathbf{Q}_{l_t} + \ln\det\mathbf{W}. \qquad (2.64)
\end{aligned}$$

Clearly, the CDF of $\ln\det\mathbf{W}$ decreases by the use of more antennas. We shall now show that $\ln\det \mathbf{Q}_{l_t}$ and thereby, $\mathcal{I}$ increases with the number of active antennas. It is straightforward to verify that the solution to the maximization problem $\max \det\mathbf{Q}_{l_t}$ subject to $\mathrm{tr}\left(\mathbf{Q}_{l_t}\right) = P$ over diagonal matrices is $\mathbf{Q}_{l_t} = \frac{P}{l_t}\mathbf{I}_{l_t}$. Therfore, Eq. (2.64) is simplified as follows

$$\mathcal{I} \approx l_t \ln\left(\frac{P}{l_t}\right) + \ln\det\mathbf{W}. \tag{2.65}$$

For $P > el_t$,

$$\frac{\partial\left(l_t \ln\left(\frac{P}{l_t}\right)\right)}{\partial l_t} = \ln\left(\frac{P}{l_t}\right) - 1 > 0. \tag{2.66}$$

As a result, in high SNR regime, the instantaneous mutual information $\mathcal{I}$ is strictly increasing with respect to the number of transmit antennas.

Part ii):

In this case, we approximate the instantaneous mutual information as follows.

$$\begin{aligned}
\mathcal{I} &= \ln\det\left(\mathbf{I}_{n_r} + \mathbf{H}\mathbf{Q}\mathbf{H}^\dagger\right) \\
&= \ln\prod_{\ell=1}^{n_r}\left(1 + \mathrm{eig}_\ell\left(\mathbf{H}\mathbf{Q}\mathbf{H}^\dagger\right)\right) \\
&\approx \ln\prod_{\ell=1}^{n_r}\left(\mathrm{eig}_\ell\left(\mathbf{H}\mathbf{Q}_{l_t}\mathbf{H}^\dagger\right)\right) \\
&= \ln\det\left(\mathbf{H}\mathbf{Q}_{l_t}\mathbf{H}^\dagger\right).
\end{aligned} \tag{2.67}$$

Based on Telatar's conjecture [77], let us assume that the transmitter performs equal power allocation on l_t out of n_t transmit antennas. Therefore,

$$\begin{aligned}\mathcal{I} &\approx n_r \ln\left(\frac{P}{l_t}\right) + \ln\det\left(\mathbf{H}\mathbf{H}^\dagger\right) \\ &= n_r \ln\left(\frac{P}{l_t}\right) + \ln\det\mathbf{W}. \end{aligned} \tag{2.68}$$

In the following, we shall establish that the maximum throughput of the channel is strictly increasing with respect to l_t. From the maximization problem of Eq. (2.47), the maximum throughput can be equivalently expressed as

$$\mathcal{R}_s^m = \max_z \mathcal{Q}(z)\left(\sigma(l_t, n_r)z + \mu(l_t, n_r)\right), \tag{2.69}$$

with

$$\mu(l_t, n_r) = \mathbb{E}\left(\ln\det\mathbf{W}\right) + p\ln\left(\frac{P}{l_t}\right), \tag{2.70}$$

$$\sigma^2(l_t, n_r) = \mathrm{Var}\left(\ln\det\mathbf{W}\right). \tag{2.71}$$

A central complex Wishart-distributed matrix $\mathbf{W}$ satisfies [79]

$$\mathbb{E}\left(\ln\det\mathbf{W}\right) = \sum_{k=0}^{p-1} \digamma(n-k), \tag{2.72}$$

$$\mathrm{Var}\left(\ln\det\mathbf{W}\right) = \sum_{k=0}^{p-1} \digamma'(n-k). \tag{2.73}$$

For natural arguments, the Eüler's digamma function and its derivative, i.e., $\digamma(m)$ and $\digamma'(m)$, can be expressed as

$$\digamma(m) = \digamma(1) + \sum_{\ell=1}^{m-1} \frac{1}{\ell}, \tag{2.74}$$

$$F'(m) = \frac{\pi^2}{6} - \sum_{\ell=1}^{m-1} \frac{1}{\ell^2}, \tag{2.75}$$

with $-F(1) = -\Gamma'(1) = \lim_{m\to\infty} \left(\sum_{\ell=1}^{m} \frac{1}{\ell} - \ln(m)\right) \approx 0.577215$, often referred to as the Eüler-Mascheroni constant. Inserting Eq. (2.75) into Eq. (2.73) and then into Eq. (2.71) to obtain

$$\sigma^2(l_t, n_r) = \frac{\pi^2}{6} n_r - \sum_{k=0}^{n_r-1} \sum_{\ell=1}^{l_t-k-1} \frac{1}{\ell^2}, \tag{2.76}$$

we see that $\sigma^2(l_t, n_r)$ is a monotonically decreasing function with respect to l_t. Whereas $\mu(l_t, n_r)$ is a strictly increasing function with respect to both l_t and n_r as it represents the ergodic capacity of the high SNR $l_t \times n_r$ MIMO channel. On the other hand, $\sigma^2(l_t, n_r) = \sum_{k=0}^{p-1} F'(n-k)$ is a monotonically increasing function with respect to n_r, because of the Basel problem, i.e., $\lim_{m\to\infty} \sum_{\ell=1}^{m} \frac{1}{\ell^2} = \frac{\pi^2}{6}$, which verifies that $F'(m) \geq 0$.

As the $\mathcal{Q}$-function is upper-bounded by the Chernoff bound, i.e., $\mathcal{Q}(z) \leq \frac{1}{2} e^{-\frac{z^2}{2}},\ z \geq 0$, we have for $z \geq 0$,

$$\begin{aligned} &-\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \left(\sigma(l_t, n_r) z + \mu(l_t, n_r)\right) + \sigma(l_t, n_r) \mathcal{Q}(z) \\ &\leq -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma(l_t, n_r) \left(z + \frac{\mu(l_t, n_r)}{\sigma(l_t, n_r)} - \sqrt{\frac{\pi}{2}} \right) \overset{(a)}{<} 0, \end{aligned} \tag{2.77}$$

where (a) follows the fact that $z \geq 0$ and $\frac{\mu(l_t,n_r)}{\sigma(l_t,n_r)} - \sqrt{\frac{\pi}{2}} > 0$ as P and thereby $\mu(l_t, n_r)$ is large. From Eqs. (2.49) and (2.77), one immediately finds that $z^o < 0$. Recall from Eq. (2.48), the maximum throughput is a strictly increasing function with respect to l_t because $\mathcal{R}_s^m$ is a strictly increasing function with respect to $\mu(l_t, n_r)$, a monotonically decreasing function with respect to $\sigma(l_t, n_r)$, and $z^o < 0$.

Thus, in both parts, i.e., $l_t \leq n_r$ and $l_t \geq n_r$, $\mathcal{R}_s^m$ is a strictly increasing function with respect to l_t. We conclude that in the asymptotically high SNR regime MIMO channel, the maximum throughput is a strictly increasing function with respect to the number of active transmit antennas, and hence, $l_t^o = n_t$.

Performing Bartlett decomposition [45, 49], we get $\mathbf{W} = \mathbf{A}\mathbf{A}^\dagger$, where $\mathbf{A}$ is a square lower triangular matrix (left triangular matrix) in the form of

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p,1} & a_{p,2} & a_{p,3} & \cdots & a_{p,p} \end{bmatrix}, \tag{2.78}$$

where $a_{\ell,k} \sim \mathcal{CN}(0,1), \ell \neq k$, and $a_{\ell,\ell}^2, \forall \ell$ are independent gamma-distributed with scale 1 and shape $n - \ell + 1$. Clearly, $\det \mathbf{W} = \det \mathbf{A} \times \det \mathbf{A}^\dagger = \prod_{\ell=1}^{p} a_{\ell,\ell}^2$.

Therefore, the maximum throughput is

$$\begin{aligned} \mathcal{R}_s^m &= \max_s \Pr\left\{\det\left(\frac{P}{n_t}\mathbf{W}\right) \geq Ps\right\} \ln(1+Ps) \\ &= \max_s \Pr\left\{\det \mathbf{W} \geq \frac{n_t^p s}{P^{p-1}}\right\} \ln(1+Ps) \\ &= \max_s \Pr\left\{\prod_{\ell=1}^{p} a_{\ell,\ell}^2 \geq \frac{n_t^p s}{P^{p-1}}\right\} \ln(1+Ps). \end{aligned} \tag{2.79}$$

From Eqs. (2.69) to (2.76), the throughput can also be written as

$$\begin{aligned} \mathcal{R}_s^m &= \max_z \mathcal{Q}(z)\left(\sigma(n_t, n_r) z + \mu(n_t, n_r)\right) \\ &= \max_z \mathcal{Q}(z)\left(z\sqrt{\sum_{\ell=0}^{p-1} F'(n-\ell)} + p\ln\left(\frac{P}{n_t}\right) + \sum_{\ell=0}^{p-1} F(n-\ell)\right) \end{aligned}$$

$$= \max_{z} \mathcal{Q}(z)\left(z\sqrt{\frac{\pi^2}{6}p - \sum_{k=0}^{p-1}\sum_{\ell=1}^{n-k-1}\frac{1}{\ell^2}} + p\left(\digamma(1) + \ln\left(\frac{P}{n_t}\right)\right) + \sum_{k=0}^{p-1}\sum_{\ell=1}^{n-k-1}\frac{1}{\ell}\right). \quad (2.80)$$

□

Remark 2.8. *Since in asymptotically high SNR regime, the outage probability is Schur-convex with respect to the channel covariance matrix [43], the maximum throughput is a Schur-concave function of the channel covariance matrix, i.e., channel correlation decreases the maximum throughput.*

2.5.3 Asymptotically Large Number of Antennas

Here, two asymptotic results for large number of transmit antennas and large number of receive antennas are presented. As pointed out earlier, we can restrict our attention to diagonal transmit covariance matrices. To prove by contradiction, first we assume that the optimum transmit covariance matrix is $\mathbf{Q}^o = \frac{P}{l_t}\mathbf{I}_{l_t}$ based on Telatar's conjecture [77]; next, we shall show that the maximum throughput increases with the number of transmit antennas and hence, $\mathbf{Q}^o = \frac{P}{n_t}\mathbf{I}_{n_t}$. Finally, we formulate the maximum throughput.

In following, Theorems 2.5 and 2.6 yield the maximum throughput of asymptotically large number of transmit antennas and asymptotically large number of receive antennas, respectively. In the proof of both theorems, we use the results presented by Hochwald, Marzetta, and Tarokh [40] which provide us with approximations for mean and variance of the instantaneous mutual information in the large number of transmit antennas and large number of receive antennas asymptotes.

Theorem 2.5. *In the MIMO channel with asymptotically large number of transmit antennas, the optimum transmit covariance matrix which maximizes the throughput is* $\mathbf{Q}^o = \frac{P}{n_t}\mathbf{I}_{n_t}$.

The maximum throughput of the channel is given by

$$\mathcal{R}_s^m = \max_z \mathcal{Q}(z) \left(\sqrt{\frac{n_r}{n_t}} \frac{P}{\sqrt{1+P^2}} z + n_r \ln (1+P) \right). \tag{2.81}$$

Proof. According to the results provided in [40], we have

$$l_t \to \infty \Rightarrow \begin{cases} \mu(l_t, n_r) \approx n_r \ln(1+P), \\ \sigma^2(l_t, n_r) \approx \frac{n_r P^2}{l_t (1+P^2)}. \end{cases} \tag{2.82}$$

From Eq. (2.82) and noting the $\mathcal{Q}$-function's Chernoff bound, i.e., $\mathcal{Q}(z) \le \frac{1}{2} e^{-\frac{z^2}{2}},\ z \ge 0$, we have for $z \ge 0$,

$$\begin{aligned} & -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \left(\sigma(l_t, n_r) z + \mu(l_t, n_r) \right) + \sigma(l_t, n_r) \mathcal{Q}(z) \\ & \le -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma(l_t, n_r) \left(z + \frac{\mu(l_t, n_r)}{\sigma(l_t, n_r)} - \sqrt{\frac{\pi}{2}} \right) \overset{(a)}{<} 0, \end{aligned} \tag{2.83}$$

where (a) comes from the fact that for $z \ge 0$,

$$\begin{aligned} z + \frac{\mu(l_t, n_r)}{\sigma(l_t, n_r)} - \sqrt{\frac{\pi}{2}} & \ge \frac{\mu(l_t, n_r)}{\sigma(l_t, n_r)} - \sqrt{\frac{\pi}{2}} \\ & = \sqrt{n_r l_t} \sqrt{1 + \frac{1}{P^2}} \ln(1+P) - \sqrt{\frac{\pi}{2}} \overset{l_t \to \infty}{>} 0. \end{aligned} \tag{2.84}$$

Comparing Eqs. (2.49) and (2.83), we have $z^o < 0$. Since $\mu(l_t, n_r)$ does not depend on l_t, $\sigma(l_t, n_r)$ is a strictly decreasing functions with respect to l_t, and $z^o < 0$, one can conclude that $\mathcal{R}_s^m = \mathcal{Q}(z^o) \left(\sigma(l_t, n_r) z^o + \mu(l_t, n_r) \right)$ is a strictly increasing function with respect to l_t. Thus, $\mathbf{Q}^o = \frac{P}{n_t} \mathbf{I}_{n_t}$.

□

Theorem 2.6. *In the MIMO channel with asymptotically large number of receive antennas, the optimum transmit covariance matrix which maximizes the throughput is* $\mathbf{Q}^o = \frac{P}{n_t}\mathbf{I}_{n_t}$. *The maximum throughput of the channel is given by*

$$\mathcal{R}_s^m = \max_z \mathcal{Q}(z)\left(\sqrt{\frac{n_t}{n_r}}z + n_t \ln\left(1+\frac{n_r}{n_t}P\right)\right). \tag{2.85}$$

Proof. As the number of receive antennas goes to infinity, the mean and variance of the channel mutual information obey [40]

$$n_r \to \infty \Rightarrow \begin{cases} \mu\left(l_t, n_r\right) \approx l_t \ln\left(1+\frac{n_r}{l_t}P\right), \\ \sigma^2\left(l_t, n_r\right) \approx \frac{l_t}{n_r}. \end{cases} \tag{2.86}$$

From Eqs. (2.47) and (2.82), the maximum throughput is

$$\begin{aligned} \mathcal{R}_s^m &= \max_z \mathcal{Q}(z)\left(\sqrt{\frac{l_t}{n_r}}z + l_t \ln\left(1+\frac{n_r}{l_t}P\right)\right) \\ &\overset{(a)}{\geq} \mathcal{Q}(-\sqrt{n_r})\left(-\sqrt{l_t} + l_t \ln\left(1+\frac{n_r}{l_t}P\right)\right) \\ &\overset{(b)}{>} \mathcal{Q}(-\sqrt{n_r})\left(-\ln\left(1+\frac{n_r P}{l_t-1}\right)\right. \\ &\qquad \left. -l_t \ln\left(1-\frac{1}{l_t}\right) + l_t \ln\left(1+\frac{n_r}{l_t}P\right)\right) \\ &\overset{(c)}{>} \mathcal{Q}(-\sqrt{n_r})\left((l_t-1)\ln\left(1+\frac{n_r}{l_t-1}P\right)\right) \\ &\overset{(d)}{\geq} \left(1-\frac{1}{2}e^{-\frac{n_r}{2}}\right)\left((l_t-1)\ln\left(1+\frac{n_r}{l_t-1}P\right)\right) \\ &\overset{(e)}{\underset{n_r\to\infty}{\longrightarrow}} (l_t-1)\ln\left(1+\frac{n_r}{l_t-1}P\right) \\ &\overset{(f)}{\geq} \max_z \mathcal{Q}(z)\left(\sqrt{\frac{l_t-1}{n_r}}z + (l_t-1)\ln\left(1+\frac{n_r}{l_t-1}P\right)\right), \end{aligned} \tag{2.87}$$

where (a) follows from choosing $z = -\sqrt{n_r}$ instead of its optimum value, (b) follows form $\sqrt{l_t} + l_t \ln\left(\frac{l_t}{l_t-1}\right) < \ln\left(1 + \frac{n_r P}{l_t-1}\right)$ for large values of n_r, (c) follows from algebraic simplifications, (d) follows from the $\mathcal{Q}$-function's Chernoff bound, (e) follows from

$$\lim_{n_r \to \infty} e^{-\frac{n_r}{2}} \ln\left(1 + \frac{n_r}{l_t - 1} P\right) = 0,$$

and (f) follows from the fact that the maximum throughput is always less than or equal to the ergodic capacity based on Proposition 2.1.

Equation (2.87) proves that $\mathcal{R}_s^m$ is a strictly increasing function with respect to l_t, and hence, $\mathbf{Q}^o = \frac{P}{n_t}\mathbf{I}_{n_t}$. □

2.6 Two-Transmitter Distributed Antenna Systems

There has been some research in assumption of perfect cooperation between base stations, and consequently treat them as distributed antennas of one base station [35, 36]. Here, we investigate a block Rayleigh fading system wherein two uninformed single-antenna transmitters want to transmit a common message to a single-antenna receiver. Let h_1 and h_2 denote the fading coefficients of the first transmitter-receiver link and second transmitter-receiver link, respectively. We assume that h_1 and h_2 are independent i.i.d. complex Gaussian random variables, each with zero-mean and equal variance real and imaginary parts ($h_1, h_2 \sim \mathcal{CN}(0, 1)$). We also assume that h_1 and h_2 are constant during two consecutive transmission blocks.

We propose a practical distributed algorithm that provides all instantaneous mutual information distributions which are achievable by treating the transmitters as antennas of one composed element. Theorem 2.7 proves that the outage probability in a MISO channel with two transmit antennas is also achievable in this channel.

Theorem 2.7. *The outage probability in a MISO channel with two transmit antennas and total power constraint P is achievable in a distributed antenna system with two single-antenna transmitters and one single-antenna receiver, where the total power constraint at each transmitter is $\frac{P}{2}$.*

Proof. To prove the statement, first, a general expression for the outage probability in a 2×1 MISO channel is derived. Afterwards, we shall show that this expression is achievable in the two-transmitter distributed antennas system.

In the 2×1 MISO channel, the outage probability for transmission rate R is expressed as

$$\mathcal{P}_{\text{out}} = \Pr\left\{\ln\left(1 + \vec{h}\mathbf{Q}\vec{h}^{\dagger}\right) < R\right\}, \tag{2.88}$$

where $\mathbf{Q}$ is the transmit covariance matrix. Since $\mathbf{Q}$ is non-negative definite, one can write it as $\mathbf{Q} = \mathbf{U}\mathbf{D}\mathbf{U}^{\dagger}$, where $\mathbf{D}$ is non-negative diagonal and $\mathbf{U}$ is unitary. As h_1 and h_2 are independent complex Gaussian random variables, each with independent zero-mean and equal variance real and imaginary parts, the distribution of $\vec{h}\mathbf{U}$ is the same as that of $\vec{h}$ [77]. Thus, Eq. (2.88) is simplified to

$$\begin{aligned}\mathcal{P}_{\text{out}} &= \Pr\left\{\ln\left(1 + \left(\vec{h}\mathbf{U}\right)\mathbf{D}\left(\vec{h}\mathbf{U}\right)^{\dagger}\right) < R\right\} \\ &= \Pr\left\{\ln\left(1 + \vec{h}\mathbf{D}\vec{h}^{\dagger}\right) < R\right\}.\end{aligned} \tag{2.89}$$

Since $\mathbf{U}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}$ is unitary, the distribution of $\vec{h}\mathbf{U}_0$ is also the same as that of $\vec{h}$.

Inserting into Eq. (2.89) yields

$$\begin{aligned}\mathcal{P}_{\text{out}} &= \Pr\left\{\ln\left(1+\left(\vec{h}\mathbf{U}_0\right)\mathbf{D}\left(\vec{h}\mathbf{U}_0\right)^{\dagger}\right) < R\right\}\\ &= \Pr\left\{\ln\left(1+\vec{h}\left(\mathbf{U}_0\mathbf{D}\mathbf{U}_0\right)\vec{h}^{\dagger}\right) < R\right\}.\end{aligned} \tag{2.90}$$

Since $\operatorname{tr}(\mathbf{Q}) = \operatorname{tr}(\mathbf{D})$, the total power constraint can be written as $\operatorname{tr}(\mathbf{D}) \leq P$. Without loss of generality, let us define $\mathbf{D} \triangleq P\begin{bmatrix}\delta & 0\\ 0 & \overline{\delta}\end{bmatrix}$, where $0 \leq \delta \leq 1$ and $\overline{\delta} = 1-\delta$. Inserting into Eq. (2.90) yields

$$\mathcal{P}_{\text{out}} = \Pr\left\{\ln\left(1+\vec{h}\frac{P}{2}\begin{bmatrix}1 & 2\delta-1\\ 2\delta-1 & 1\end{bmatrix}\vec{h}^{\dagger}\right) < R\right\}. \tag{2.91}$$

Defining $\rho \triangleq 2\delta - 1$, we get

$$\begin{aligned}\mathcal{P}_{\text{out}} &= \Pr\left\{\ln\left(1+\vec{h}\frac{P}{2}\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\vec{h}^{\dagger}\right) < R\right\}\\ &= \Pr\left\{\ln\left(1+\left(|h_1|^2+|h_2|^2+2\rho\Re(h_1h_2^*)\right)\frac{P}{2}\right) < R\right\}.\end{aligned} \tag{2.92}$$

Note that as $0 \leq \delta \leq 1$, we have $-1 \leq \rho \leq 1$.

We shall now show that the outage probability in Eq. (2.92) is achievable in the two-transmitter distributed antenna system with power constraint $\frac{P}{2}$ at each transmitter.

The transmission strategy in two consecutive time slots is as follows. In time slot t, the first (resp. second) transmitter sends $X(t)$ (resp. $\rho X(t) + \sqrt{(1-\rho^2)}X(t+1)$). Note that $X(t)$ and $X(t+1)$ are independent, each with power $\frac{P}{2}$. In time slot $t+1$, the first (resp. second) transmitter sends $-X^*(t+1)$ (resp. $-\rho X^*(t+1) + \sqrt{(1-\rho^2)}X^*(t)$). Assuming $\mathbb{E}\left(|X|^2\right) = \frac{P}{2}$, the power consumption per time slot in each transmitter is $\frac{P}{2}$.

The received signal at the receiver is

$$\begin{cases} Y(t) = h_1 X(t) + h_2\Big(\rho X(t) + \sqrt{(1-\rho^2)}X(t+1)\Big) + Z(t), \\ Y(t+1) = -h_1 X^*(t+1) + h_2\Big(-\rho X^*(t+1) + \sqrt{(1-\rho^2)}X^*(t)\Big) + Z(t+1). \end{cases} \tag{2.93}$$

In matrix form,

$$\begin{bmatrix} Y(t) \\ -Y^*(t+1) \end{bmatrix} = \mathbf{G} \begin{bmatrix} X(t) \\ X(t+1) \end{bmatrix} + \begin{bmatrix} Z(t) \\ -Z^*(t+1) \end{bmatrix}, \tag{2.94}$$

where

$$\mathbf{G} \triangleq \begin{bmatrix} h_1 + h_2\rho & h_2\sqrt{(1-\rho^2)} \\ -h_2^*\sqrt{(1-\rho^2)} & h_1^* + h_2^*\rho \end{bmatrix}. \tag{2.95}$$

By multiplying $\mathbf{G}^\dagger$ to the both sides of Eq. (2.94), two parallel channels are separated as

$$\begin{aligned} \begin{bmatrix} \tilde{Y}(t) \\ \tilde{Y}(t+1) \end{bmatrix} = \mathbf{G}^\dagger \begin{bmatrix} Y(t) \\ -Y^*(t+1) \end{bmatrix} &= \Big(|h_1 + h_2\rho|^2 + |h_2|^2 \left(1-\rho^2\right) \Big)\mathbf{I}_2 \begin{bmatrix} X(t) \\ X(t+1) \end{bmatrix} \\ &+ \mathbf{G}^\dagger \begin{bmatrix} Z(t) \\ -Z^*(t+1) \end{bmatrix} = h\mathbf{I}_2 \begin{bmatrix} X(t) \\ X(t+1) \end{bmatrix} + \begin{bmatrix} \tilde{Z}(t) \\ \tilde{Z}(t+1) \end{bmatrix}, \end{aligned} \tag{2.96}$$

where $h \triangleq |h_1 + h_2\rho|^2 + |h_2|^2 (1-\rho^2)$, and $\tilde{Z}(t)$ and $\tilde{Z}(t+1)$ are independent zero mean complex Gaussian random variables with power equal to $\mathbb{E}\left(\left|\tilde{Z}\right|^2\right) = h$. Thus, the received signal-to-noise ratio (SNR) at the receiver is

$$\begin{aligned}\frac{h^2 \frac{P}{2}}{\mathbb{E}\left(\left|\tilde{Z}\right|^2\right)} &= \left(|h_1 + h_2\rho|^2 + |h_2|^2\left(1-\rho^2\right)\right)\frac{P}{2} \\ &= \left(|h_1|^2 + |h_2|^2 + 2\rho\Re\left(h_1 h_2^*\right)\right)\frac{P}{2}. \end{aligned} \tag{2.97}$$

Therefore, the outage probability in the proposed scheme is given by

$$\mathcal{P}_{\text{out}} = \Pr\left\{\ln\left(1 + \left(|h_1|^2 + |h_2|^2 + 2\rho\Re(h_1 h_2^*)\right)\frac{P}{2}\right) < R\right\}. \tag{2.98}$$

Equation (2.92) together with Eq. (2.98) shows that the outage probability in a 2×1 MISO channel with any transmit covariance matrix is also achievable in the two-transmitter distributed antenna system.

$\square$

Remark 2.9. *To achieve the minimum outage probability in Theorem 2.7, the optimum solution to* δ *is either* 1 *or* $\frac{1}{2}$*, depending on* R *and* P*. Equivalently, in the two-transmitter distributed antennas, the optimum value of* ρ *is either* 1 *or* 0*. This remark is an special case of [47].*

Note that for $\rho = 0$, the proposed transmission scheme in the two-transmitter distributed antenna system is equivalent to the Alamouti code [7].

Remark 2.10. *Since the outage probability is the CDF of the instantaneous mutual information, one concludes that any achievable instantaneous mutual information distribution in the* 2×1 *MISO channel is also achievable in this two-transmitter distributed antenna system.*

Remark 2.11. *Based on Theorem 2.7, the maximum throughput in the two-transmitter* *distributed antenna system with total power constraint $\frac{P}{2}$ at each transmitter is the same as that of a 2×1 MISO channel with total power constraint P. By substituting $n_t = 2$ in Eq. (2.12), the maximum throughput is given by*

$$\mathcal{R}_s^m = \max_{0<s<1} (1+2s)e^{-2s} \ln(1+Ps)\,. \tag{2.99}$$

Remark 2.12. *In a similar approach, it can be shown that the maximum expected-rate as well as the ergodic capacity of this two-transmitter distributed antenna system and the 2×1 MISO channel are the same.*

Based on Theorem 2.7 and recall from Theorem 2.3 with $n_t = 2$, we come up with the following Corollary.

Corollary 2.2. *The maximum continuous-layer expected-rate of the distributed antenna system with two transmitters each with total power $\frac{P}{2}$ is*

$$\mathcal{R}_c^m = 3\mathrm{E}_1(s_0) + (1-s_0)e^{-s_0} - 3\mathrm{E}_1(s_1) - (1-s_1)e^{-s_1}, \tag{2.100}$$

where $s_1 = \frac{1+\sqrt{5}}{2}$, and $s_0 = \sqrt[3]{\sqrt{A^2-B^3}+A} + \frac{B}{\sqrt[3]{\sqrt{A^2-B^3}+A}} - \frac{2}{3P}$ with $A = \frac{1}{P} - \frac{2}{3P^2} - \frac{8}{27P^3}$ and $B = \frac{2}{3P} + \frac{4}{9P^2}$.

From Proposition 2.3, the ergodic capacity in this channel is

$$C_{\text{erg}} = 1 + \left(1 - \frac{2}{P}\right) e^{\frac{2}{P}} \mathrm{E}_1\left(\frac{2}{P}\right). \tag{2.101}$$

Maximum throughput (red dashed-dotted line), maximum two-layer expected-rate (blue dashed line), maximum continuous-layer expected-rate (black solid line), and ergodic capacity (purple circle-marked line) in the two-transmitter distributed antenna system are depicted in Fig. 2.3.

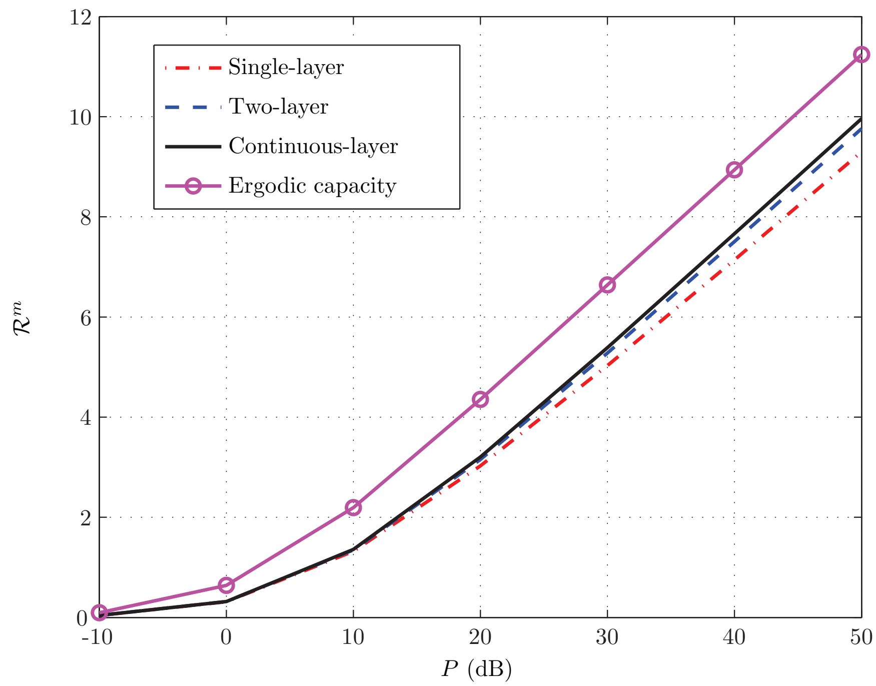

Figure 2.3: Maximum throughput, maximum two-layer expected-rate, maximum continuous-layer expected-rate, and ergodic capacity (all in *nats*) in the two-transmitter distributed antenna system with total power constraint $\frac{P}{2}$ at each transmitter.

Chapter 3

A Broadcast Approach to the Diamond Channel

3.1 Network Model

Let us first restate the network model. As Fig. 3.1 shows, the destination receives data via two parallel relays and there is no direct link between the source and the destination. The source transmits a signal X subject to the total power constraint P_s, i.e., $\mathbb{E}\left(|X|^2\right) \leq P_s$, and the received signal at the ℓ'th relay is denoted by

$$Y_{r_\ell} = h_{r_\ell} X + Z_{r_\ell}, \quad \ell = 1, 2 \tag{3.1}$$

The i.i.d. AWGN at the ℓ'th relay is represented by $Z_{r_\ell} \sim \mathcal{CN}(0, 1)$, and $h_{r_\ell} \sim \mathcal{CN}(0, 1)$ is the channel coefficient from the source to the ℓ'th relay. The ℓ'th relay forwards a signal X_{r_ℓ} to the destination under the total power constraint P_r, i.e., $\mathbb{E}\left(|X_{r_\ell}|^2\right) \leq P_r,\ \ell = 1, 2$.

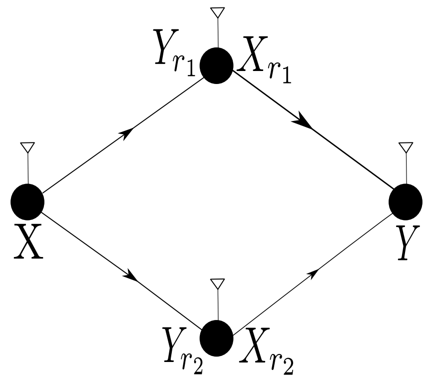

Figure 3.1: Network model of dual-hop transmission from a single-antenna source to single-antenna destinations via two single-antenna relays.

The received signal at the destination is

$$Y = h_1 X_{r_1} + h_2 X_{r_2} + Z, \tag{3.2}$$

where $Z \sim \mathcal{CN}(0,1)$ is the i.i.d. AWGN and $h_\ell \sim \mathcal{CN}(0,1)$ is the channel coefficient from the ℓ'th relay to the destination. All h_{r_ℓ} and h_ℓ are assumed to be constant during two consecutive transmission blocks. Obviously, channel gains $a_\ell = |h_\ell|^2$ and $a_{r_\ell} = |h_{r_\ell}|^2$ have exponential distribution.

Note that the source as well as both relays and the destination are equipped with one antenna. We assume that the relays operate in a full-duplex mode and they are not capable of buffering data over multiple coding blocks or rescheduling tasks. Since there is no link between the relays, the half-duplex mode is a direct result of the full-duplex mode with frequency or time division [91]. Throughout this chapter, we assume that $\mathbb{E}\left(|X_i|^2\right) = 1, \ \forall i$.

3.2 Decode-Forward Relays

In order to enhance the lucidity of this section, single-layer coding is studied first. The idea is then extended to multi-layer coding. Since the continuous-layer expected-rate of this scheme is a seemingly intractable problem, a finite-layer coding scenario is analyzed in Section 3.2.2.

3.2.1 Single-Layer Coding

In single-layer coding, a signal $X = \gamma X_1$ with power P_s and rate $R = \ln(1 + P_s s)$ is transmitted, where $\gamma^2 = P_s$. The ℓ'th relay decodes and forwards the received signal in case $a_{r_\ell} \geq s$. If $a_{r_\ell} < s$, then a_{r_ℓ} is replaced by zero. The coding scheme at the relays is a distributed block space-time code in the Alamouti code sense [7]. At time t, the first relay sends $\alpha X_1(t)$ while the other relay sends $\beta X_1(t+1)$. To satisfy the relays power constraint, it is required that $\alpha^2 = \beta^2 = P_r$. At time $t+1$, the first and the second relays send $-\alpha X_1^*(t+1)$ and $\beta X_1^*(t)$, respectively. The relay with $a_{r_\ell} < s$ simply sends nothing. Applying the Alamouti decoding procedure and decomposing into two parallel channels, the throughput is given by

$$\begin{aligned}\mathcal{R}_{D,s} = \Bigg[& \Pr\left\{a_{r_1} \geq s\right\} \Pr\left\{a_{r_2} \geq s\right\} \Pr\left\{a_1 + a_2 \geq s\frac{P_s}{P_r}\right\} \\ & + \Pr\left\{a_{r_1} \geq s\right\} \Pr\left\{a_{r_2} < s\right\} \Pr\left\{a_1 \geq s\frac{P_s}{P_r}\right\} \\ & + \Pr\left\{a_{r_1} < s\right\} \Pr\left\{a_{r_2} \geq s\right\} \Pr\left\{a_2 \geq s\frac{P_s}{P_r}\right\} \Bigg] \ln(1 + P_s s). \end{aligned} \tag{3.3}$$

The first term in the right-hand-side of Eq. (3.3) represents the case of decoding the signal at both relays and the destination. The second and third terms represent the probability of decoding the signal at only one relay and the destination. Substituting the channel gain CDFs in (3.3), the throughput is given by

$$\mathcal{R}_{D,s} = \left(\frac{P_s}{P_r} s e^{-s} - e^{-s} + 2\right) e^{-s\left(\frac{P_s}{P_r}+1\right)} \ln(1 + P_s s). \tag{3.4}$$

Theorem 3.1 proves the optimality of the above scheme and presents the maximum throughput of the channel.

Theorem 3.1. *In the proposed single-layer DF, the maximum throughput is achieved by sending uncorrelated signals on the relays. the maximum throughput is given by*

$$\mathcal{R}_{D,s}^{m} = \max_{0<s<s_t} \left(\frac{P_s}{P_r} s e^{-s} - e^{-s} + 2 \right) e^{-s\left(\frac{P_s}{P_r}+1\right)} \ln(1 + P_s s), \tag{3.5}$$

where $s_t = \min\left\{2\frac{P_r}{P_s}, 1.212\right\}$.

Proof. Without loss of generality, consider $\mathbf{Q} \triangleq P_r \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ as the relays transmit covariance matrix. Therefore, $\mathbb{E}\left(X_{r_1} X_{r_2}^*\right) = \rho P_r$. In the following, we shall show that $\rho^o = 0$. Let us define $\overline{F}(s)$ as follows

$$\begin{aligned} \overline{F}(s) \triangleq & \Pr\left\{a_{r_1} \geq s\right\} \Pr\left\{a_{r_2} \geq s\right\} \Pr\left\{a \geq s\frac{P_s}{P_r}\right\} \\ & + \Pr\left\{a_{r_1} \geq s\right\} \Pr\left\{a_{r_2} < s\right\} \Pr\left\{a_1 \geq s\frac{P_s}{P_r}\right\} \\ & + \Pr\left\{a_{r_1} < s\right\} \Pr\left\{a_{r_2} \geq s\right\} \Pr\left\{a_2 \geq s\frac{P_s}{P_r}\right\}, \end{aligned} \tag{3.6}$$

where $a \triangleq \frac{1}{P_r} \vec{h} \mathbf{Q} \vec{h}^\dagger$ and $\vec{h} \triangleq \begin{bmatrix} h_1 & h_2 \end{bmatrix}$. The maximum throughput of the diamond channel in general form is

$$\mathcal{R}_{D,s}^{m} = \max_{s, -1 \leq \rho \leq 1} \overline{F}(s) \ln(1 + P_s s). \tag{3.7}$$

The only term in $\overline{F}(s)$ which depends on ρ is $\Pr\left(a \geq s\frac{P_s}{P_r}\right)$. Since $\mathbf{Q}$ is non-negative definite, one can write it as $\mathbf{Q} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger$, where $\mathbf{D} = P_r \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix}$ is non-negative diagonal and

$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is unitary. Since h_1 and h_2 are independent complex Gaussian random variables, each with independent zero-mean and equal variance real and imaginary parts, the distribution of $\vec{h}\mathbf{U}$ is the same as that of $\vec{h}$ [77]. Thus,

$$\begin{aligned}
\Pr\left\{a \geq s\frac{P_s}{P_r}\right\} &= \Pr\left\{\vec{h}\mathbf{Q}\vec{h}^{\dagger} \geq P_s s\right\} \\
&= \Pr\left\{\left(\vec{h}\mathbf{U}\right)\mathbf{D}\left(\vec{h}\mathbf{U}\right)^{\dagger} \geq P_s s\right\} \\
&= \Pr\left\{\vec{h}\mathbf{D}\vec{h}^{\dagger} \geq P_s s\right\}. \qquad (3.8)
\end{aligned}$$

The last expression in Eq. (3.8) corresponds to the complementary CDF in MISO channels. Abbe et al. [3] proved that in an uncorrelated MISO channel with no CSI at the transmitter, but perfect CSI at the receiver, for every transmission rate, the optimal transmit strategy minimizing the outage probability is to use a fraction of all available transmit antennas and perform equal power allocation with uncorrelated signals. Therefore, the solution of $\max_{-1\leq\rho\leq 1}\Pr\left\{\ln\left(1+\vec{h}'\mathbf{D}\vec{h}^{\dagger}\right) \geq \ln(1+P_s s)\right\}$ is $\rho = 0$ or $\rho = 1$.

Defining

$$s_c \triangleq -\left(2\mathcal{W}_{-1}\left(\frac{-1}{2\sqrt{e}}\right)+1\right)\frac{P_r}{P_s} \approx 2.5129\frac{P_r}{P_s}, \qquad (3.9)$$

where $\mathcal{W}_{-1}(\cdot)$ is the -1 branch of the Lambert W-function [17, 18], one can show that if $s \leq s_c$, then

$$\overline{F}_{\rho=0}(s) \geq \overline{F}_{\rho=1}(s). \qquad (3.10)$$

In the remainder of the proof, we shall show that in case $\rho = 1$, $s^o \leq s_c$. Then, as $\forall s \leq s_c$, $\overline{F}_{\rho=0}(s^o) \geq \overline{F}_{\rho=1}(s^o)$, it implies $\rho^o = 0$, i.e., the optimum correlation coefficient between the relay signals maximizing the throughput of DF diamond channel is zero.

Assume that s^o maximizes $\mathcal{R}(s) = \overline{F}_{\rho=1}(s)\ln(1+P_s s)$. Hence, $\mathcal{R}'(s^o) = 0$. Defining $f_{\rho=1}(s) = -\overline{F}'_{\rho=1}(s)$, we get

$$\mathcal{R}'(s) = \overline{F}_{\rho=1}(s)\frac{P_s}{1+P_s s} - f_{\rho=1}(s)\ln(1+P_s s). \tag{3.11}$$

Let us define $g(s,P_s) = \ln(1+P_s s)^{\frac{1+P_s s}{P_s}}$ and $r(s) = \frac{\overline{F}_{\rho=1}(s)}{f_{\rho=1}(s)}$. As such, we get

$$\begin{cases} \mathcal{R}'(s) > 0 \quad \text{iff} \quad r(s) > g(s,P_s), \\ \mathcal{R}'(s) = 0 \quad \text{iff} \quad r(s) = g(s,P_s), \\ \mathcal{R}'(s) < 0 \quad \text{iff} \quad r(s) < g(s,P_s). \end{cases} \tag{3.12}$$

Noting $\overline{F}_{\rho=1}(s) = \left(e^{-s} + 2\left(1-e^{-s}\right)e^{-\frac{P_s}{P_r}\frac{s}{2}}\right)e^{-\left(1+\frac{P_s}{2P_r}\right)s}$, we have

$$r(s) = \frac{e^{-s} + 2(1-e^{-s})e^{-s\frac{P_s}{2P_r}}}{\left(2+\frac{P_s}{2P_r}\right)e^{-s} + 2\left(1+\frac{P_s}{P_r}\right)e^{-s\frac{P_s}{2P_r}} - 2\left(2+\frac{P_s}{P_r}\right)e^{-s}e^{-s\frac{P_s}{2P_r}}}. \tag{3.13}$$

It can be shown that as far as $s \geq s_t = \min\left\{2\frac{P_r}{P_s}, 1.212\right\}$, we have

$$r(s) < s, \quad \forall s \geq s_t. \tag{3.14}$$

The derivative of $g(s,P_s)$ over P_s is

$$\frac{\partial g(s,P_s)}{\partial P_s} = \frac{sP_s - \ln(1+sP_s)}{P_s^2} = \frac{1}{P_s^2}\ln\left(1 + \frac{1}{1+sP_s}\sum_{k=2}^{\infty}\frac{(sP_s)^k}{k!}\right) \geq 0. \tag{3.15}$$

Therefore, $g(s,P_s)$ is a monotonically increasing function of P_s and its minimum is in

$P_s = 0$. As a result,

$$g(s, P_s) > \lim_{P_s \to 0} \ln (1 + P_s s)^{\frac{1+P_s s}{P_s}} = s. \tag{3.16}$$

Comparing Eq. (3.14), Eq. (3.16), $r(0) = \frac{2P_r}{P_s} > 0$ and $g(0, P_s) = 0$ yields

$$\begin{cases} r(s) > g(s, P_s) & s = 0, \\ r(s) < g(s, P_s) & s \geq s_t. \end{cases} \tag{3.17}$$

Applying Eq. (3.17) to Eq. (3.12) gives

$$\begin{cases} \mathcal{R}'(s) > 0 & s = 0, \\ \mathcal{R}'(s) < 0 & s \geq s_t. \end{cases} \tag{3.18}$$

As $\mathcal{R}(s)$ is a continuous function, according to Eq. (3.18), $0 < s^o < s_t$. Noting $s_t < s_c$, Eq. (3.10) yields $\overline{F}_{\rho=0}(s^o) > \overline{F}_{\rho=1}(s^o)$ and as a result, $\rho^o = 0$ and $a = a_1 + a_2$. Substituting the channel gain CDFs in Eq. (3.6), the maximum throughput of the DF diamond channel is given by Eq. (3.5), which is achievable by applying the aforementioned distributed space-time code.

□

3.2.2 Finite-Layer Coding

For the lucidity of this section, the encoding and decoding procedures are presented sparately.

Encoding Procedure

The transmitter sends a K-layer code $X = \sum_{i=1}^{K} \gamma_i X_i$ to the relays, where γ_i^2 represents the power allocated to the i'th layer with rate

$$R_i = \ln\left(1 + \frac{\gamma_i^2 s_i}{1 + \sum_{j=i+1}^{K} \gamma_j^2 s_i}\right). \tag{3.19}$$

The relays start decoding the received signal from the first layer up to the layer that their backward channel conditions allow. Then, the relays re-encode and forward the decoded layers to the destination. To design the transmission strategy, we first state Theorem 3.2.

Theorem 3.2. *In multi-layer DF, if the layers' power distribution in the first relay is equal to that of the second relay, the relay signals must be uncorrelated in order to achieve the maximum expected-rate.*

Proof. Analogous to the proof of Theorem 3.1, let us define

$$\begin{aligned}\mathcal{P}_i \triangleq\ & \overline{F}_{\mathrm{a}_{r_1}}(s_i)\overline{F}_{\mathrm{a}_{r_2}}(s_i)\mathcal{P}_{i,1,2} \\ &+ \overline{F}_{\mathrm{a}_{r_1}}(s_i)F_{\mathrm{a}_{r_2}}(s_i)\mathcal{P}_{i,1} + F_{\mathrm{a}_{r_1}}(s_i)\overline{F}_{\mathrm{a}_{r_2}}(s_i)\mathcal{P}_{i,2},\end{aligned} \tag{3.20}$$

where $\mathcal{P}_{i,1,2}$, $\mathcal{P}_{i,1}$, and $\mathcal{P}_{i,2}$ are the probability of decoding the i'th layer at the destination successfully when both relays, only the first relay, and only the second relay decode the signal, respectively. The expected-rate of the i'th layer can be written as

$$\mathcal{R}_i(s) = \mathcal{P}_i \ln\left(1 + \frac{\gamma_i^2 s_i}{1 + \sum_{j=i+1}^{K} \gamma_j^2 s_i}\right). \tag{3.21}$$

The only term in Eq. (3.20) which depends on the transmit strategy at the relays is $\mathcal{P}_{i,1,2}$. We denote $\mathbf{Q}_i$ as the transmit covariance matrix of the relays in the i'th layer. So that,

$$\mathcal{P}_{i,1,2} = \Pr\left\{1 + \frac{\vec{h}\mathbf{Q}_i\vec{h}^\dagger}{1 + \vec{h}\sum_{j=i+1}^{K}\mathbf{Q}_j\vec{h}^\dagger} \geq e^{R_i}\right\}. \tag{3.22}$$

Analogous to the proof of Theorem 3.1, by decomposing $\mathbf{Q}_i$ and $\sum_{j=i+1}^{K}\mathbf{Q}_j$, and noting the fact that multiplying $\vec{h}$ by any unitary matrix does not change the distribution of $\vec{h}$, we get

$$\mathcal{P}_{i,1,2} = \Pr\left\{1 + \frac{P_i\vec{h}\begin{bmatrix}1+\rho_i & 0\\ 0 & 1-\rho_i\end{bmatrix}\vec{h}^\dagger}{1 + I_i\vec{h}\begin{bmatrix}1+\hat{\rho}_i & 0\\ 0 & 1-\hat{\rho}_i\end{bmatrix}\vec{h}^\dagger} \geq e^{R_i}\right\}. \tag{3.23}$$

It can be shown that the optimum solutions for ρ and $\hat{\rho}$ to minimize $\mathcal{P}_{i,1,2}$ in Eq. (3.23) is either $\rho_i = \hat{\rho}_i = 0$ or $\rho_i = \hat{\rho}_i = 1$. We shall now show that the optimum solution is $\rho_i^o = \hat{\rho}_i^o = 0$. Towards this, we follow the same general outline to the proof of Theorem 3.1.

Let us define the following functions,

$$g\left(s_i, P_i, I_i\right) = \frac{\left(1 + I_i s_i\right)\left(1 + \left(I_i + P_i\right)s_i\right)}{P_i}\ln\left(1 + \frac{P_i s_i}{1 + I_i s_i}\right), \tag{3.24}$$

$$r(s_i) = -\frac{\mathcal{P}_i}{\frac{\mathrm{d}\mathcal{P}_i}{\mathrm{d}s_i}}. \tag{3.25}$$

One can simply show that Eqs. (3.12) and (3.14) still hold by redefining the functions as above, and with s replaced by s_i.

Defining $\hat{P} \triangleq \frac{P_i}{1+I_i s_i}$, from Eq. (3.16) and noting $I_i s_i \geq 0$, we have

$$g\left(s_i, P_i, I_i\right) = \left(1 + I_i s_i\right)\frac{\left(1 + \frac{P_i s_i}{1+I_i s_i}\right)}{\frac{P_i}{1+I_i s_i}}\ln\left(1 + \frac{P_i s_i}{1 + I_i s_i}\right)$$

$$
\begin{aligned}
&\geq \frac{\left(1+\frac{P_i s_i}{1+I_i s_i}\right)}{\frac{P_i}{1+I_i s_i}} \ln\left(1+\frac{P_i s_i}{1+I_i s_i}\right) \\
&= \ln\left(1+\hat{P}s_i\right)^{\frac{(1+\hat{P}s_i)}{\hat{P}}} > s_i.
\end{aligned} \tag{3.26}
$$

Therefore, Eqs. (3.17) and (3.18) still hold with the above functions, and then, $0 < s_i^o < s_t$. Noting $s_t < s_c$ results because as pointed out earlier $\mathcal{P}_{i,\rho=0}(s_i^o) > \mathcal{P}_{i,\rho=1}(s_i^o)$.

□

With respect to Theorem 3.2, the following transmission scheme is proposed. Assume that the first and the second relays decode M and N layers out of the whole K transmitted layers, respectively, according to their corresponding backward channel. As the relays do not know the channel of the other relay, and hence, do not know the layers' power distribution in the other relay, its code construction is based on a similar power distribution assumption for the other relay. Theorem 3.2 demonstrates that uncorrelated signals must be transmitted over the relays. For this purpose, the following scheme is proposed. At time t, the first relay sends $\sum_{i=1}^{K} \alpha_i X_i(t)$ while the other relay sends $\sum_{i=1}^{K} \beta_i X_i(t+1)$. At time $t+1$, the first and the second relays send $\sum_{i=1}^{K} -\alpha_i X_i^*(t+1)$ and $\sum_{i=1}^{K} \beta_i X_i^*(t)$, respectively. Note that $\sum_{i=1}^{M} \alpha_i^2 = P_r$, $\alpha_i = 0$ for $i = M+1, ..., K$ and $\sum_{i=1}^{N} \beta_i^2 = P_r$, $\beta_i = 0$ for $i = N+1, ..., K$.

The received signal at the destination is

$$
\begin{cases}
Y(t) = h_1 \sum_{i=1}^{K} \alpha_i X_i(t) + h_2 \sum_{i=1}^{K} \beta_i X_i(t+1) + Z(t), \\
Y(t+1) = -h_1 \sum_{i=1}^{K} \alpha_i X_i^*(t+1) + h_2 \sum_{i=1}^{K} \beta_i X_i^*(t) + Z(t+1).
\end{cases} \tag{3.27}
$$

One may express a matrix representation for Eq. (3.27) as

$$\begin{bmatrix} Y(t) \\ -Y^*(t+1) \end{bmatrix} = \sum_{i=1}^{K} \begin{bmatrix} h_1\alpha_i & h_2\beta_i \\ -h_2^*\beta_i & h_1^*\alpha_i \end{bmatrix} \begin{bmatrix} X_i(t) \\ X_i(t+1) \end{bmatrix} + \begin{bmatrix} Z(t) \\ -Z^*(t+1) \end{bmatrix}. \tag{3.28}$$

Decoding procedure

The destination starts decoding the code layers in order, from the first layer up to the highest layer that is decodable. To decode the i'th layer, after decoding the first $i-1$ layers, the channels are separated into two parallel channels by multiplying both sides of Eq. (3.28) by $\begin{bmatrix} h_1^*\alpha_i & -h_2\beta_i \\ h_2^*\beta_i & h_1\alpha_i \end{bmatrix}$. Therefore,

$$\begin{aligned} \begin{bmatrix} \tilde{Y}(t) \\ \tilde{Y}(t+1) \end{bmatrix} &= \begin{bmatrix} a_1\alpha_i^2 + a_2\beta_i^2 & 0 \\ 0 & a_1\alpha_i^2 + a_2\beta_i^2 \end{bmatrix} \begin{bmatrix} X_i(t) \\ X_i(t+1) \end{bmatrix} \\ &+ \sum_{j=i+1}^{K} \begin{bmatrix} h_1^*\alpha_i & -h_2\beta_i \\ h_2^*\beta_i & h_1\alpha_i \end{bmatrix} \begin{bmatrix} h_1\alpha_j & h_2\beta_j \\ -h_2^*\beta_j & h_1^*\alpha_j \end{bmatrix} \begin{bmatrix} X_j(t) \\ X_j(t+1) \end{bmatrix} + \begin{bmatrix} \tilde{Z}(t) \\ \tilde{Z}(t+1) \end{bmatrix}. \end{aligned} \tag{3.29}$$

$\tilde{Z}(t)$ and $\tilde{Z}(t+1)$ are two independent i.i.d AWGN, each with power $a_1\alpha_i^2 + a_2\beta_i^2$.

The interference power caused by upper layers while decoding the i'th layer is

$$\begin{aligned} I_i &= \sum_{j=i+1}^{K} \left((a_1\alpha_i\alpha_j + a_2\beta_i\beta_j)^2 + a_1a_2\left(\alpha_i\beta_j - \alpha_j\beta_i\right)^2 \right) \\ &= \left(a_1\alpha_i^2 + a_2\beta_i^2\right) \sum_{j=i+1}^{K} \left(a_1\alpha_j^2 + a_2\beta_j^2\right). \end{aligned} \tag{3.30}$$

Thus, the probability that the i'th layer can be successfully decoded at the destination is

$$\mathcal{P}_i = \Pr\left\{\frac{a_1\alpha_i^2 + a_2\beta_i^2}{1+\sum_{j=i+1}^{K}\left(a_1\alpha_j^2 + a_2\beta_j^2\right)} \geq \frac{\gamma_i^2 s_i}{1+\sum_{j=i+1}^{K}\gamma_j^2 s_i}\right\}. \tag{3.31}$$

Hence, the achievable expected-rate using this scheme can be written as

$$\mathcal{R}_{D,f} = \sum_{i=1}^{K}\mathcal{P}_i \ln\left(1+\frac{\gamma_i^2 s_i}{1+\sum_{j=i+1}^{K}\gamma_j^2 s_i}\right). \tag{3.32}$$

To summarize, we have shown the following.

Theorem 3.3. *In the diamond channel, the above result implies that the following expected-rate is achievable.*

$$\mathcal{R}_{D,f}^{m} = \max_{s_i,\gamma_i,\alpha_i,\beta_i}\sum_{i=1}^{K}\mathcal{P}_i \ln\left(1+\frac{\gamma_i^2 s_i}{1+\sum_{j=i+1}^{K}\gamma_j^2 s_i}\right), \tag{3.33}$$

with $\mathcal{P}_i = \Pr\left\{\frac{|h_1|^2\alpha_i^2+|h_2|^2\beta_i^2}{1+\sum_{j=i+1}^{K}\left(|h_1|^2\alpha_j^2+|h_2|^2\beta_j^2\right)} \geq \frac{\gamma_i^2 s_i}{1+\sum_{j=i+1}^{K}\gamma_j^2 s_i}\right\}$. *The maximization is subject to* $\sum_{i=1}^{K}\gamma_i^2 = P_s$, $\sum_{i=1}^{K}\alpha_i^2 = \sum_{i=1}^{K}\beta_i^2 = P_r$, *where* α_i *and* β_i *are zero for the layers which are not decoded at the relays. Note that* $\alpha_i s$ *and* $\beta_i s$ *are optimized separately.*

Remark 3.1. *One important feature of the proposed scheme is that the layers being decoded at both relays are added coherently at the destination although each relay has no information about the number of layers being successfully decoded by the other relay.*

3.3 Amplify-Forward Relays

A simple but efficient relaying solution for the diamond channel is to amplify and forward the received signals. In order for the destination to coherently decode the signals, it employs a distributed space-time code permutation along with the threshold-based ON/OFF power scheme, which has been shown that improves the performance of AF relaying [41]. According to the ON/OFF concept, any relay whose backward channel gain is less than a pre-determined threshold, namely a_{th}, is silent. In this scheme, the relays transmit the signals to the destination in two consecutive time slots. In time slot t, the first (resp. second) relay transmits $c_1 Y_{r_1}(t)$ (resp. $c_2 Y_{r_2}(t+1)$). In time slot $t+1$, the first (resp. second) relay transmits $-c_1 Y_{r_1}^*(t+1)$ (resp. $c_2 Y_{r_2}^*(t)$) with the *backward channel phase compensation* [41]. To satisfy the relays' power constraint, it is required that $c_\ell = \sqrt{\frac{\mathcal{U}\left(a_{r_\ell} - a_{th}\right) P_r}{a_{r_\ell} P_s + 1}}$, $\ell = 1, 2$, where $\mathcal{U}(\cdot)$ is the unit step function. At the destination, the channels are parallelized using the Alamouti decoding procedure [7]. The received signal at the destination is

$$\begin{cases} Y(t) = c_1 h_1 Y_{r_1}(t) + c_2 h_2 Y_{r_2}(t+1) + Z(t), \\ Y(t+1) = -c_1 h_1 Y_{r_1}^*(t+1) + c_2 h_2 Y_{r_2}^*(t) + Z(t+1). \end{cases} \tag{3.34}$$

As the destination accesses the backward channels, after compensating the phases of h_{r_1} and h_{r_2} into $h_{r_1}^*$ and $h_{r_2}^*$ in time slot $t+1$, we get

$$\begin{aligned} \begin{bmatrix} Y(t) \\ -Y^*(t+1) \end{bmatrix} &= \begin{bmatrix} h_{r_1} h_1 c_1 & h_{r_2} h_2 c_2 \\ -h_{r_2}^* h_2^* c_2 & h_{r_1}^* h_1^* c_1 \end{bmatrix} \begin{bmatrix} X(t) \\ X(t+1) \end{bmatrix} \\ &\quad + \begin{bmatrix} c_1 h_1 Z_{r_1}(t) + c_2 h_2 Z_{r_2}(t+1) + Z(t) \\ c_1 h_1^* Z_{r_1}(t+1) - c_2 h_2^* Z_{r_2}(t) - Z^*(t+1) \end{bmatrix}. \end{aligned} \tag{3.35}$$

Multiplying $\begin{bmatrix} h_{r_1} h_1 c_1 & h_{r_2} h_2 c_2 \\ -h_{r_2}^* h_2^* c_2 & h_{r_1}^* h_1^* c_1 \end{bmatrix}^\dagger$ to both sides of Eq. (3.35), two channels are parallelized, and the source-destination instantaneous mutual information is

$$\mathcal{I}(X;Y) = \ln\left(1 + \frac{\left(|h_{r_1} h_1|^2 c_1^2 + |h_{r_2} h_2|^2 c_2^2\right) P_s}{1 + |h_1|^2 c_1^2 + |h_2|^2 c_2^2}\right), \tag{3.36}$$

which is equivalent to a point-to-point channel with the following channel gain,

$$a_{AF,2} \triangleq \frac{\frac{P_r}{a_{r_1}P_s+1}a_{r_1}a_1 + \frac{P_r}{a_{r_2}P_s+1}a_{r_2}a_2}{1 + \frac{P_r}{a_{r_1}P_s+1}a_1 + \frac{P_r}{a_{r_2}P_s+1}a_2}. \tag{3.37}$$

If one relay is silent and only one relay transmits, let say the ℓ'th relay, by replacing zero instead of one of the channel gains into Eq. (3.37), we get

$$a_{AF,1} \triangleq \frac{a_{r_\ell}a_\ell P_r}{1 + a_{r_\ell}P_s + a_\ell P_r}. \tag{3.38}$$

The expected value of the optimum ON/OFF threshold in which $a_{AF,2} > a_{AF,1}$ is given by

$$a_{th} = \frac{P_r}{1 + P_s + P_r}. \tag{3.39}$$

Proposition 3.1 yields the maximum achievable throughput in this method.

Proposition 3.1. *The maximum achievable throughput in the above AF scheme is specified by*

$$\begin{aligned}\mathcal{R}^m_{A,s} = \max_s \;\; & e^{-\frac{P_r}{1+P_s+P_r}} \left(e^{-\frac{P_r}{1+P_s+P_r}} \overline{F}_{\mathrm{a}_{AF,2}}(s) \right. \\ & \left. +2\left(1 - e^{-\frac{P_r}{1+P_s+P_r}}\right) \overline{F}_{\mathrm{a}_{AF,1}}(s) \right) \ln(1 + P_s s),\end{aligned} \tag{3.40}$$

where $F_{\mathrm{a}_{AF,2}}(\cdot)$ and $F_{\mathrm{a}_{AF,1}}(\cdot)$ are the CDFs of $a_{AF,2}$ and $a_{AF,1}$ from Eqs. (3.37) *and* (3.38)*, respectively.*

The maximum continuous-layer expected-rate of the above AF relaying is presented in Theorem 3.4.

Theorem 3.4. *The maximum achievable expected-rate in the above AF relaying is given by*

$$\mathcal{R}_{A,c}^{m} = e^{-\frac{P_r}{1+P_s+P_r}} \left(2 - e^{-\frac{P_r}{1+P_s+P_r}}\right) \int_{s_0}^{s_1} \overline{F}(s) \left(\frac{2}{s} + \frac{f'(s)}{f(s)}\right) \mathrm{d}s, \tag{3.41}$$

with

$$F(s) \triangleq \frac{2\left(e^{\frac{P_r}{1+P_s+P_r}} - 1\right) F_{\mathrm{a}_{AF,1}}(s) + F_{\mathrm{a}_{AF,2}}(s)}{2e^{\frac{P_r}{1+P_s+P_r}} - 1}, \tag{3.42}$$

$$f(s) \triangleq F'(s). \tag{3.43}$$

The integration limits are the solutions to $\overline{F}(s_0) = s_0(1 + P_s s_0) f(s_0)$ *and* $\overline{F}(s_1) = s_1 f(s_1)$, *respectively.*

Proof. The maximum achievable expected-rate at the destination can be expressed by

$$\begin{aligned}\mathcal{R}_{A,c}^{m} &= 2e^{-a_{th}}\left(1 - e^{-a_{th}}\right)\mathcal{R}_1^m + e^{-2a_{th}}\mathcal{R}_2^m \\ &= e^{-a_{th}}\left(2 - e^{-a_{th}}\right)\left(\frac{2\left(1 - e^{-a_{th}}\right)}{2 - e^{-a_{th}}}\mathcal{R}_1^m + \frac{e^{-a_{th}}}{2 - e^{-a_{th}}}\mathcal{R}_2^m\right),\end{aligned} \tag{3.44}$$

where $\mathcal{R}_1^m$ and $\mathcal{R}_2^m$ are the maximum expected-rates when only one relay is active and both relays are active, respectively. According to [67], $\mathcal{R}_1^m$ and $\mathcal{R}_2^m$ are given by

$$\begin{aligned}\mathcal{R}_1^m &= \max_{I(s)} \int_0^\infty \overline{F}_{\mathrm{a}_{AF,1}}(s) \frac{-sI'(s)}{1 + sI(s)} \mathrm{d}s, \\ \mathcal{R}_2^m &= \max_{I(s)} \int_0^\infty \overline{F}_{\mathrm{a}_{AF,2}}(s) \frac{-sI'(s)}{1 + sI(s)} \mathrm{d}s.\end{aligned} \tag{3.45}$$

Substituting the above equations in Eq. (3.44), we get

$$\mathcal{R}_{A,c}^{m} = \max_{I(s)} e^{-a_{th}}\left(2 - e^{-a_{th}}\right) \int_0^\infty \left(1 - \frac{2\left(1 - e^{-a_{th}}\right)}{2 - e^{-a_{th}}} F_{\mathrm{a}_{AF,1}}(s)\right.$$

$$- \frac{e^{-a_{th}}}{2-e^{-a_{th}}} F_{\mathrm{a}_{AF,2}}(s) \Big) \frac{-sI'(s)}{1+sI(s)} \mathrm{d}s. \tag{3.46}$$

Defining

$$F(s) \triangleq \frac{2\left(1-e^{-a_{th}}\right)}{2-e^{-a_{th}}} F_{\mathrm{a}_{AF,1}}(s) + \frac{e^{-a_{th}}}{2-e^{-a_{th}}} F_{\mathrm{a}_{AF,2}}(s), \tag{3.47}$$

the maximum expected-rate of the proposed AF scheme is found by

$$\mathcal{R}^m_{A,c} = \max_{I(s)} e^{-a_{th}} \left(2-e^{-a_{th}}\right) \int_0^\infty \overline{F}(s) \frac{-sI'(s)}{1+sI(s)} \mathrm{d}s. \tag{3.48}$$

Substituting a_{th} by $\frac{P_r}{1+P_s+P_r}$ and maximizing over $I(s)$ by solving the corresponding Eüler equation [33], we come up with the maximum expected-rate as

$$\mathcal{R}^m_{A,c} = e^{-\frac{P_r}{1+P_s+P_r}} \left(2-e^{-\frac{P_r}{1+P_s+P_r}}\right) \int_{s_0}^{s_1} \overline{F}(s) \left(\frac{2}{s} + \frac{f'(s)}{f(s)}\right) \mathrm{d}s, \tag{3.49}$$

where s_0 and s_1 are the solutions to $\overline{F}(s_0) = s_0(1+P_s s_0)f(s_0)$ and $\overline{F}(s_1) = s_1 f(s_1)$, respectively, and $f(s) \triangleq F'(s)$.

□

Remark 3.2. *In the above results, the power constraint P_r has been applied only to the time slots when the relays are ON. Alternatively, one can assume that the relays have the ability to save their power while working in the OFF state and consume it in the ON state. In this case, all the above calculations in Theorem 3.4 hold except for the integration limit s_0 which is now the solution to $\overline{F}(s_0) = s_0(1 + e^{\frac{P_r}{1+P_s+P_r}} P_s s_0)f(s_0)$.*

3.4 Hybrid Decode-Amplify-Forward Relays

In this section, we propose a DAF relaying strategy which takes advantage of amplifying the layers that could not be decoded at the relays in the DF scheme. Specifically, each relay tries to decode as many layers as possible and forward them by spending a portion of its power budget. The remaining power is dedicated to amplifying and forwarding the rest of the layers. This method is indeed completely different from another method with similar name in [10, 15].

In order to enhance the lucidity of this section, single-layer coding is studied first. The idea is then extended to multi-layer coding. As the continuous-layer expected-rate of this scheme is a seemingly intractable problem, a finite-layer coding scenario is analyzed.

3.4.1 Maximum Throughput

Here, a single-layer code $X = \gamma X_1$ with power P_s, i.e., $\gamma^2 = P_s$, and rate $R = \ln(1 + P_s s)$ is transmitted. If $a_{r_\ell} \geq s$, then the ℓ'th relay decodes the signal and forwards it, otherwise, it amplifies and forwards the signal to the destination. In time slot t, the first (resp. second) relay transmits $X_{r_1}(t)$ (resp. $X_{r_2}(t+1)$). In time slot $t+1$, the first (resp. second) relay transmits $-X^*_{r_1}(t+1)$ (resp. $X^*_{r_2}(t)$) with the *backward channel phase compensation* on the amplified layers. There are three possibilities:

1. $a_{r_1} \geq s$ and $a_{r_2} \geq s$: both relays decode the signal. In this case DAF is simplified to DF in Section 3.2.

2. $a_{r_1} < s$ and $a_{r_2} < s$: none of the relays decodes the signal. This case is simplified to AF in Section 3.3.

3. $a_{r_1} \geq s, a_{r_2} < s$ or $a_{r_1} < s, a_{r_2} \geq s$: only one relay decodes the signal.

In the third case, without loss of generality, assume that the first relay decodes the signal and the second relay does not decode it, i.e., $a_{r_1} \geq s, a_{r_2} < s$. Hence, $X_{r_1}(t) = \alpha X_1(t)$ and $X_{r_2}(t) = c_2 Y_{r_2}(t) = c_2\left(h_{r_2}\gamma X_1(t) + Z_{r_2}(t)\right)$, where $\alpha^2 = P_r$ and $c_2 = \sqrt{\frac{P_r}{a_{r_2}P_s+1}}$. At the destination, we have

$$\begin{cases} Y(t) = h_1\alpha X_1(t) + h_2c_2h_{r_2}\gamma X_1(t+1) + h_2c_2Z_{r_2}(t+1) + Z(t), \\ Y(t+1) = -h_1\alpha X_1^*(t+1) + h_2c_2h_{r_2}^*\gamma X_1^*(t) + h_2c_2Z_{r_2}^*(t) + Z(t+1). \end{cases} \tag{3.50}$$

After compensating the phase of h_{r_2} into $h_{r_2}^*$ in time slot $t+1$, we get

$$\begin{bmatrix} Y(t) \\ -Y^*(t+1) \end{bmatrix} = \begin{bmatrix} h_1\alpha & h_{r_2}h_2c_2\gamma \\ -h_{r_2}^*h_2^*c_2\gamma & h_1^*\alpha \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_1(t+1) \end{bmatrix} + \begin{bmatrix} c_2h_2Z_{r_2}(t+1) + Z(t) \\ -c_2h_2^*Z_{r_2}(t) - Z^*(t+1) \end{bmatrix}. \tag{3.51}$$

Multiplying $\begin{bmatrix} h_1\alpha & h_{r_2}h_2c_2\gamma \\ -h_{r_2}^*h_2^*c_2\gamma & h_1^*\alpha \end{bmatrix}^\dagger$ to both sides of Eq. (3.51), two channels are parallelized, and the source-destination instantaneous mutual information is

$$\mathcal{I}(X;Y) = \ln\left(1 + \frac{(|h_1|^2\alpha^2 + |h_{r_2}h_2|^2c_2^2\gamma^2)}{1 + |h_2|^2c_2^2}\right). \tag{3.52}$$

A comparison of this method and the DF scheme reveals that if $a_{r_2} > \frac{P_r}{P_s}a_1$, then DAF outperforms DF, otherwise, we switch to DF, that is the second relay becomes silent. Since the relays have no information about the other relay and thereby, do not know the relaying protocol, the threshold value is optimized. As a result, the amplification coefficient of DAF can be written as $c_\ell = \sqrt{\frac{\mathcal{U}\left(a_{r_\ell} - a_{th}\right)P_r}{a_{r_\ell}P_s+1}}$. It can be shown that the maximum throughput of this scheme is given by the following proposition.

Proposition 3.2. *The maximum throughput of the proposed hybrid decode-amplify-forward* *relaying is given by*

$$\mathcal{R}_{DA,s}^{m} = \max_{s,a_{th}} \Big[\left(e^{-a_{th}} - e^{-s}\right)^2 \overline{F}_{\mathrm{a}_{AF,2}}(s) + 2\left(e^{-a_{th}} - e^{-s}\right)\left(1 - e^{-a_{th}}\right)\overline{F}_{\mathrm{a}_{AF,1}}(s)$$
$$+ 2e^{-s}\left(e^{-a_{th}} - e^{-s}\right)\overline{F}_{\mathrm{a}_{DAF}}\left(s\frac{P_s}{P_r}\right) + 2\left(1 - e^{-a_{th}}\right)e^{-s\left(1+\frac{P_s}{P_r}\right)}$$
$$+ \left(1 + s\frac{P_s}{P_r}\right)e^{-s\left(2+\frac{P_s}{P_r}\right)}\Big] \ln(1 + P_s s), \tag{3.53}$$

where $a_{DAF} \triangleq \frac{a_1 + a_{r_2} P_s (a_1 + a_2)}{1 + a_{r_2} P_s + a_2 P_r}$, *and* $a_{AF,2}$ *and* $a_{AF,1}$ *are from Eqs.* (3.37) *and* (3.38)*, respectively.*

3.4.2 Maximum Finite-Layer Expected-Rate

Since continuous-layer coding for DAF relaying can not be directly solved by variations methods, we choose a finite-layer code and proceed as follows. In the finite-layer broadcast approach, the source transmits a K layer code $X = \sum_{i=1}^{K} \gamma_i X_i$ to the relays, where γ_i^2 represents the power allocated to the i'th layer with rate

$$R_i = \ln\left(1 + \frac{\gamma_i^2 s_i}{1 + \sum_{j=i+1}^{K} \gamma_j^2 s_i}\right). \tag{3.54}$$

Each relay decodes its received signal from the first layer up to the layer that its backward channel conditions allow and forwards them to the destination. Afterwards, each relay amplifies and forwards the remaining undecoded layers.

Suppose that the first and second relays allocate portions ξP_r and ζP_r of their power to the decoded layers, respectively. Also, assume that the first and second relays respectively decode M and N layers out of the K transmitted layers. Without loss of generality, assume $M \geq N$. Denote by α_i^2 (resp. β_i^2) the power allocated to the i'th layer at the first (resp. second) relay. The amplifying coefficients are $c_1 = \sqrt{\frac{\bar{\xi} P_r}{a_{r_1} \sum_{i=M+1}^{K} \gamma_i^2 + 1}}$ for the first relay and

$c_2 = \sqrt{\frac{\bar{\zeta} P_r}{a_{r_2} \sum_{i=N+1}^{K} \gamma_i^2 + 1}}$ for the second relay. Note that $\bar{\xi} = 1 - \xi$ and $\bar{\zeta} = 1 - \zeta$. Let us define $\alpha_i \triangleq h_{r_1} c_1 \gamma_i$ for $i = M+1, ..., K$ and $\beta_i \triangleq h_{r_2} c_2 \gamma_i$ for $i = N+1, ..., K$. The coding scheme is as follows. At time t, the first relay sends $\sum_{i=1}^{K} \alpha_i X_i(t) + c_1 Z_{r_1}(t)$ while the other relay sends $\sum_{i=1}^{K} \beta_i X_i(t+1) + c_2 Z_{r_2}(t+1)$. At time $t+1$, the first and the second relays send $\sum_{i=1}^{K} -\alpha_i X_i^*(t+1) - c_1 Z_{r_1}^*(t+1)$ and $\sum_{i=1}^{K} \beta_i X_i^*(t) + c_2 Z_{r_2}^*(t)$, respectively.

The received signal at the destination is

$$\begin{cases} Y(t) = h_1 \sum_{i=1}^{K} \alpha_i X_i(t) + h_2 \sum_{i=1}^{K} \beta_i X_i(t+1) \\ \qquad\qquad +h_1 c_1 Z_{r_1}(t) + h_2 c_2 Z_{r_2}(t+1) + Z(t), \\ Y(t+1) = -h_1 \sum_{i=1}^{K} \alpha_i X_i^*(t+1) + h_2 \sum_{i=1}^{K} \beta_i X_i^*(t) \\ \qquad\qquad -h_1 c_1 Z_{r_1}^*(t+1) + h_2 c_2 Z_{r_2}^*(t) + Z(t+1). \end{cases} \tag{3.55}$$

One may express a matrix representation for Eq. (3.55) as

$$\begin{aligned} \begin{bmatrix} Y(t) \\ -Y^*(t+1) \end{bmatrix} &= \sum_{i=1}^{K} \begin{bmatrix} h_1 \alpha_i & h_2 \beta_i \\ -h_2^* \beta_i^* & h_1^* \alpha_i^* \end{bmatrix} \begin{bmatrix} X_i(t) \\ X_i(t+1) \end{bmatrix} \\ &\quad + \begin{bmatrix} h_1 c_1 Z_{r_1}(t) + h_2 c_2 Z_{r_2}(t+1) + Z(t) \\ h_1^* c_1 Z_{r_1}(t+1) - h_2^* c_2 Z_{r_2}(t) - Z^*(t+1) \end{bmatrix}. \end{aligned} \tag{3.56}$$

The destination starts decoding the code layers in order, from the first layer up to the highest layer that is decodable. To decode the i'th layer, after decoding the first $i-1$ layers, the channels are separated into two parallel channels by multiplying both sides of Eq. (3.56) by $\begin{bmatrix} h_1\alpha_i & h_2\beta_i \\ -h_2^*\beta_i^* & h_1^*\alpha_i^* \end{bmatrix}^\dagger$. Therefore,

$$\begin{aligned}\begin{bmatrix} \tilde{Y}(t) \\ \tilde{Y}(t+1) \end{bmatrix} &= \begin{bmatrix} a_1|\alpha_i|^2 + a_2|\beta_i|^2 & 0 \\ 0 & a_1|\alpha_i|^2 + a_2|\beta_i|^2 \end{bmatrix} \begin{bmatrix} X_i(t) \\ X_i(t+1) \end{bmatrix} \\ &+ \sum_{j=i+1}^{K} \begin{bmatrix} h_1^*\alpha_i^* & -h_2\beta_i \\ h_2^*\beta_i^* & h_1\alpha_i \end{bmatrix} \begin{bmatrix} h_1\alpha_j & h_2\beta_j \\ -h_2^*\beta_j^* & h_1^*\alpha_j^* \end{bmatrix} \begin{bmatrix} X_j(t) \\ X_j(t+1) \end{bmatrix} + \begin{bmatrix} \tilde{Z}(t) \\ \tilde{Z}(t+1) \end{bmatrix}. \end{aligned} \tag{3.57}$$

$\tilde{Z}(t)$ and $\tilde{Z}(t+1)$ are two independent i.i.d. AWGN, each with power:

$$\left(a_1|\alpha_i|^2 + a_2|\beta_i|^2\right)\left(1 + a_1c_1^2 + a_2c_2^2\right).$$

The interference power caused by upper layers while decoding the i'th layer is

$$\begin{aligned} I_i &= \sum_{j=i+1}^{K} \left(\left|a_1\alpha_i^*\alpha_j + a_2\beta_i\beta_j^*\right|^2 + a_1a_2\left|\alpha_i^*\beta_j - \alpha_j^*\beta_i\right|^2 \right) \\ &= \left(a_1|\alpha_i|^2 + a_2|\beta_i|^2\right) \sum_{j=i+1}^{K} \left(a_1|\alpha_j|^2 + a_2|\beta_j|^2\right). \end{aligned} \tag{3.58}$$

Thus, the probability that the i'th layer can be correctly decoded at the destination is

$$\mathcal{P}_i = \Pr\left\{ \frac{a_1|\alpha_i|^2 + a_2|\beta_i|^2}{1 + a_1c_1^2 + a_2c_2^2 + \sum_{j=i+1}^{K}\left(a_1|\alpha_j|^2 + a_2|\beta_j|^2\right)} \geq \frac{\gamma_i^2 s_i}{1 + \sum_{j=i+1}^{K}\gamma_j^2 s_i} \right\}, \tag{3.59}$$

Hence, the expected-rate at the destination using this scheme can be written as

$$\mathcal{R}_{DA,f} = \sum_{i=1}^{K} \mathcal{P}_i \ln\left(1 + \frac{\gamma_i^2 s_i}{1 + \sum_{j=i+1}^{K}\gamma_j^2 s_i}\right). \tag{3.60}$$

To summarize, we have shown the following.

Theorem 3.5. *The maximum achievable expected-rate in the proposed DAF relaying is given by*

$$\mathcal{R}^{m}_{DA,f} = \max_{\xi,\zeta,s_i,\gamma_i,\alpha_i,\beta_i} \sum_{i=1}^{K} \mathcal{P}_i \ln\left(1 + \frac{\gamma_i^2 s_i}{1 + \sum_{j=i+1}^{K} \gamma_j^2 s_i}\right), \tag{3.61}$$

where

$$\mathcal{P}_i = \Pr\left\{\frac{a_1|\alpha_i|^2 + a_2|\beta_i|^2}{1 + a_1 c_1^2 + a_2 c_2^2 + \sum_{j=i+1}^{K} (a_1|\alpha_j|^2 + a_2|\beta_j|^2)} \geq \frac{\gamma_i^2 s_i}{1 + \sum_{j=i+1}^{K} \gamma_j^2 s_i}\right\}, \tag{3.62}$$

and $\alpha_i = h_{r_1}\sqrt{\frac{\bar{\xi}P_r}{a_{r_1}\sum_{j=M+1}^{K}\gamma_j^2+1}}\gamma_i,\ i = M+1, ..., K,$ *and* $\beta_i = h_{r_2}\sqrt{\frac{\bar{\zeta}P_r}{a_{r_2}\sum_{j=N+1}^{K}\gamma_j^2+1}}\gamma_i,\ i = N+1, ..., K.$ *The power constraints are* $\sum_{i=1}^{K}\gamma_i^2 = P_s$, $\sum_{i=1}^{M}\alpha_i^2 = \xi P_r$, *and* $\sum_{i=1}^{N}\beta_i^2 = \zeta P_r$. *Note that* $(\alpha_1, \alpha_2, \ldots, \alpha_M, \xi)$ *and* $(\beta_1, \beta_2, \ldots, \beta_N, \zeta)$ *are real positive values and optimized separately.*

3.5 Compress-Forward Relays

In CF relaying, the relays quantize their received signals using an optimal Gaussian quantizer with minimum mean-square error (MSE) criterion [21], and then forward the quantized signals. With respect to the correlation between the relays signals, Wyner-Ziv compression method [85] is applied. In this scheme, the relays do not decode the signal and hence, the latency and complexity is lower in comparison with DF and DAF. Also, the relays do not need to access the source codebook; however, the source-relay channel gains must be available at the destination.

Denote by q_{r_1} and q_{r_2} the quantized signals at the first and second relays, respectively.

One can write the following equations on $q_{r_\ell},\ \ell = 1, 2$,

$$Y_{r_\ell} = q_{r_\ell} + n_{r_\ell}, \tag{3.63}$$

and

$$q_{r_\ell} = \theta_\ell Y_{r_\ell} + \tilde{n}_{r_\ell}, \tag{3.64}$$

where $n_{r_\ell} \sim \mathcal{CN}(0, D_\ell)$ and $\tilde{n}_{r_\ell} \sim \mathcal{CN}(0, \theta_\ell D_\ell)$ are the equivalent quantization noises independent of q_{r_ℓ}, $\theta_\ell \triangleq 1 - \frac{D_\ell}{1+a_{r_\ell}P_s}$, and D_ℓ is the quantizer distortion at the ℓ'th relay [11].

If the destination decodes q_{r_1} and q_{r_2}, and the transmission rate is below $\mathcal{I}(X; q_{r_1}, q_{r_2})$, the signal is successfully decodable. For simplicity, let us assume that the optimum value of the quantizer distortion D_ℓ^o and the optimum value of the relays rate $R_{r_\ell}^o$ are selected independent of the source-relays channel gains. Hence, with respect to the network symmetry, $D_1^o = D_2^o$ and $R_{r_1}^o = R_{r_2}^o$, and therefore, they are simply denoted by D and R_r, respectively.

To decoded the quantized signals at the destination, based on the multiple-access capacity region [21] in the second-hop, the following inequalities must be satisfied,

$$\begin{aligned}
R_r &< \mathcal{I}(X_{r_1}; Y|X_{r_2}) = \ln\left(1 + a_1 P_r\right), \\
R_r &< \mathcal{I}(X_{r_2}; Y|X_{r_1}) = \ln\left(1 + a_2 P_r\right), \\
2R_r &< \mathcal{I}(X_{r_1}, X_{r_2}; Y) = \ln\left(1 + (a_1 + a_2) P_r\right).
\end{aligned} \tag{3.65}$$

For compression of the quantized signals, based on the Wyner-ziv rate region [85], we have the following inequalities,

$$R_r \geq \mathcal{I}(q_{r_1}; Y_{r_1} | q_{r_2}), \tag{3.66}$$

$$R_r \geq \mathcal{I}(q_{r_2}; Y_{r_2} | q_{r_1}), \tag{3.67}$$

$$2R_r \geq \mathcal{I}(q_{r_1}, q_{r_2}; Y_{r_1}, Y_{r_2}). \tag{3.68}$$

For the problem in consideration, Eq. (3.68) is

$$\begin{aligned}\mathcal{I}(q_{r_1}, q_{r_2}; Y_{r_1}, Y_{r_2}) &= \ln\left(\frac{\det \mathbf{R}_{Y_1Y_2}}{\det \mathbf{R}_{Y_1Y_2|q_{r_1},q_{r_2}}}\right) \\ &= \ln\left(\frac{1+(a_{r_1}+a_{r_2})P_s}{D^2}\right),\end{aligned} \tag{3.69}$$

where $\mathbf{R}_{Y_1Y_2}$ and $\mathbf{R}_{Y_1Y_2|q_{r_1},q_{r_2}}$ represent the covariance matrix and conditional covariance matrix of random variables, respectively. In order to derive a closed form expression for Eqs. (3.66) and (3.67), let us first estate the following lemmas.

Lemma 3.1. *The mutual information between the source signal and the relays quantized signals can be expressed by*

$$\mathcal{I}(q_{r_1}, q_{r_2}; X) = \ln\left(1 + a_{CF}P_s\right), \tag{3.70}$$

where

$$a_{CF} \triangleq \frac{a_{r_1}}{1+\frac{\theta_2+D}{\theta_2+1}\frac{D}{\theta_1}} + \frac{a_{r_2}}{1+\frac{\theta_1+D}{\theta_1+1}\frac{D}{\theta_2}}. \tag{3.71}$$

Proof. The mutual information between the source signal and the relays quantized signals can be expressed by

$$\mathcal{I}(q_{r_1}, q_{r_2}; X) = \ln\left(\frac{\det \mathbf{R}_{q_{r_1} q_{r_2}}}{\det \mathbf{R}_{q_{r_1} q_{r_2}|X}}\right), \tag{3.72}$$

where

$$\begin{aligned}\det \mathbf{R}_{q_{r_1} q_{r_2}} &= \theta_1^2\theta_2^2 a_{r_1} P_s + \theta_1^2\theta_2 D a_{r_1} P_s + \theta_1^2\theta_2^2 a_{r_2} P_s + \theta_1^2\theta_2^2 \\ &\quad + \theta_1^2\theta_2 D + \theta_1\theta_2^2 a_{r_2} P_s D + \theta_1\theta_2^2 D + \theta_1\theta_2 D^2,\end{aligned} \tag{3.73}$$

and

$$\det \mathbf{R}_{q_{r_1} q_{r_2}|X} = \theta_1^2\theta_2^2 + \theta_1^2\theta_2 D + \theta_1\theta_2^2 D + \theta_1\theta_2 D^2. \tag{3.74}$$

Thus,

$$\begin{aligned}\mathcal{I}(X; q_{r_1}, q_{r_2}) &= \ln\left(1 + \frac{\theta_1\theta_2 a_{r_1} + \theta_1 D a_{r_1} + \theta_1\theta_2 a_{r_2} + \theta_2 a_{r_2} D}{\theta_1\theta_2 + \theta_1 D + \theta_2 D + D^2} P_s\right) \\ &= \ln\left(1 + \left(a_{r_1}\frac{\theta_1\theta_2 + \theta_1 D}{\theta_1\theta_2 + \theta_1 D + \theta_2 D + D^2}\right.\right. \\ &\quad \left.\left. + \, a_{r_2}\frac{\theta_1\theta_2 + \theta_2 D}{\theta_1\theta_2 + \theta_1 D + \theta_2 D + D^2}\right) P_s\right) \\ &= \ln\left(1 + \left(\frac{a_{r_1}}{1 + \frac{\theta_2 D + D^2}{\theta_1\theta_2 + \theta_1}} + \frac{a_{r_2}}{1 + \frac{\theta_1 D + D^2}{\theta_1\theta_2 + \theta_2}}\right) P_s\right) \\ &= \ln\left(1 + \left(\frac{a_{r_1}}{1 + \frac{\theta_2 + D}{\theta_2 + 1}\frac{D}{\theta_1}} + \frac{a_{r_2}}{1 + \frac{\theta_1 + D}{\theta_1 + 1}\frac{D}{\theta_2}}\right) P_s\right).\end{aligned} \tag{3.75}$$

Equation (3.71) together with Eq. (3.75) results. $\square$

Lemma 3.2. *In the problem of interest, we have*

$$\mathcal{I}(q_{r_1}; Y_{r_1}|q_{r_2}) = \ln\left(1 + a_{CF}P_s\right) + \ln\left(\frac{\left(\theta_1 + D\right)\left(\theta_2 + D\right)}{D\left(1 + a_{r_2}P_s\right)}\right). \tag{3.76}$$

Proof.

$$\begin{aligned}
\mathcal{I}(q_{r_1}; Y_{r_1}|q_{r_2}) &= \mathcal{I}(q_{r_1}; X, Y_{r_1}|q_{r_2}) - \mathcal{I}(q_{r_1}; X|Y_{r_1}, q_{r_2}) \\
&\overset{(a)}{=} \mathcal{I}(q_{r_1}; X, Y_{r_1}|q_{r_2}) \\
&= \mathcal{I}(q_{r_1}, q_{r_2}; X, Y_{r_1}) - \mathcal{I}(q_{r_2}; X, Y_{r_1}) \\
&= \mathcal{I}(q_{r_1}, q_{r_2}; X, Y_{r_1}) - \mathcal{I}(q_{r_2}; Y_{r_1}|X) - \mathcal{I}(q_{r_2}; X) \\
&\overset{(b)}{=} \mathcal{I}(q_{r_1}, q_{r_2}; X, Y_{r_1}) - \mathcal{I}(q_{r_2}; X) \\
&= \mathcal{I}(q_{r_1}, q_{r_2}; X) + \mathcal{I}(q_{r_1}, q_{r_2}; Y_{r_1}|X) - \mathcal{I}(q_{r_2}; X) \\
&= \mathcal{I}(q_{r_1}, q_{r_2}; X) + \mathcal{I}(q_{r_1}; Y_{r_1}|X) \\
&+ \mathcal{I}(q_{r_2}; Y_{r_1}|q_{r_1}, X) - \mathcal{I}(q_{r_2}; X) \\
&\overset{(c)}{=} \mathcal{I}(q_{r_1}, q_{r_2}; X) + \mathcal{I}(q_{r_1}; Y_{r_1}|X) - \mathcal{I}(q_{r_2}; X) \\
&= \mathcal{I}(q_{r_1}, q_{r_2}; X) + \mathcal{H}(q_{r_1}|X) \\
&- \mathcal{H}(q_{r_1}|Y_{r_1}, X) - \mathcal{H}(q_{r_2}) + \mathcal{H}(q_{r_2}|X) \\
&\overset{(d)}{=} \mathcal{I}(q_{r_1}, q_{r_2}; X) + \mathcal{H}(q_{r_1}|X) \\
&- \mathcal{H}(q_{r_1}|Y_{r_1}) - \mathcal{H}(q_{r_2}) + \mathcal{H}(q_{r_2}|X) \\
&= \ln\left(1 + a_{CF}P_s\right) + \ln\left(1 + \frac{\theta_1}{D}\right) \\
&- \ln\left(1 + \frac{\theta_2 a_{r_2} P_s}{\theta_2 + D}\right) \\
&= \ln\left(1 + a_{CF}P_s\right) \\
&+ \ln\left(\frac{\left(\theta_1 + D\right)\left(\theta_2 + D\right)}{D\left(D + \theta_2\left(1 + a_{r_2}P_s\right)\right)}\right)
\end{aligned}$$

$$= \ln\left(1 + a_{CF} P_s\right) + \ln\left(\frac{(\theta_1 + D)(\theta_2 + D)}{D(1 + a_{r_2} P_s)}\right). \tag{3.77}$$

(a) and (d) follow from the fact that $X \longmapsto Y_{r_1} \longmapsto q_{r_1}$ is a Markov chain, and hence $\mathcal{I}(q_{r_1}; X|Y_{r_1}, q_{r_2}) = 0$ and $\mathcal{H}(q_{r_1}|Y_{r_1}, X) = \mathcal{H}(q_{r_1}|Y_{r_1})$. (b) and (c) follow from $\mathcal{I}(q_{r_2}; Y_{r_1}|X) = 0$ and $\mathcal{I}(q_{r_2}; Y_{r_1}|q_{r_1}, X) = 0$, respectively, with respect to the Markov chain $q_{r_2} \longmapsto X \longmapsto Y_{r_1}$. □

With respect to the network symmetry and based on Lemma 3.2, one can express

$$\mathcal{I}(q_{r_2}; Y_{r_2}|q_{r_1}) = \ln\left(1 + a_{CF} P_s\right) + \ln\left(\frac{(\theta_1 + D)(\theta_2 + D)}{D(1 + a_{r_1} P_s)}\right). \tag{3.78}$$

In order to have a successful transmission, the destination must first decode the relays signals and then X. From Eqs. (3.65) to (3.69), (3.76) and (3.78), to decode the relays signals at the detination, the following inequalities must be satisfied.

$$\begin{aligned}
&\ln\left(1 + a_{CF} P_s\right) + \ln\left(\frac{(\theta_1 + D)(\theta_2 + D)}{D(1 + a_{r_2} P_s)}\right) \leq R_r < \ln\left(1 + a_1 P_r\right),\\
&\ln\left(1 + a_{CF} P_s\right) + \ln\left(\frac{(\theta_1 + D)(\theta_2 + D)}{D(1 + a_{r_1} P_s)}\right) \leq R_r < \ln\left(1 + a_2 P_r\right),\\
&\ln\left(\frac{1 + (a_{r_1} + a_{r_2}) P_s}{D^2}\right) \leq 2R_r < \ln\left(1 + (a_1 + a_2) P_r\right).
\end{aligned} \tag{3.79}$$

Therefore, the probability of decoding the relays signals at the destination successfully is expressed as follows,

$$\mathcal{P}_C = \Pr\left\{ \max\left\{ \ln\left(\frac{\sqrt{1 + (a_{r_1} + a_{r_2}) P_s}}{D}\right), \right.\right.$$
$$\left.\ln\left(1 + a_{CF} P_s\right) + \ln\left(\frac{(\theta_1 + D)(\theta_2 + D)}{D(1 + a_{r_{min}} P_s)}\right)\right\}$$

$$< R_r <$$

$$\min\left\{\ln\left(\sqrt{1+(a_1+a_2)\,P_r}\right),\ln\left(1+a_{min}P_r\right)\right\}\Bigg\}, \tag{3.80}$$

where $a_{r_{min}} \triangleq \min\{a_{r_1}, a_{r_2}\}$ and $a_{min} \triangleq \min\{a_1, a_2\}$.

After decoding the relays signals at the destination, the source signal is decoded subject to

$$R \leq \mathcal{I}(q_{r_1}, q_{r_2}; X) = \ln\left(1 + a_{CF}P_s\right), \tag{3.81}$$

where $R = \ln(1 + P_s s)$ is the source transmission rate.

To summarize, we have shown the following.

Theorem 3.6. *The maximum throughput in the proposed CF scheme is expressed by*

$$\mathcal{R}^m_{C,s} = \max_{s,D,R_r} \mathcal{P}_C \overline{F}_{\mathrm{a}_{CF}}(s) \ln\left(1 + P_s s\right), \tag{3.82}$$

where a_{CF} and $\mathcal{P}_C$ are given by Eqs. (3.71) *and* (3.80)*, respectively.*

Analogously, Eq. (3.83) *yields the maximum continuous-layer expected-rate in this scheme.*

$$\mathcal{R}^m_{C,c} = \max_{D,R_r} \mathcal{P}_C \int_{s_0}^{s_1} \overline{F}_{\mathrm{a}_{CF}}(s) \left(\frac{2}{s} + \frac{f'_{\mathrm{a}_{CF}}(s)}{f_{\mathrm{a}_{CF}}(s)}\right) \mathrm{d}s. \tag{3.83}$$

The integration limits are the solutions to $\overline{F}_{\mathrm{a}_{CF}}(s_0) = s_0\left(1 + P_s s_0\right) f_{\mathrm{a}_{CF}}(s_0)$ and $\overline{F}_{\mathrm{a}_{CF}}(s_1) = s_1 f_{\mathrm{a}_{CF}}(s_1)$, respectively.

It turns out from the numerical results that the proposed CF scheme outperforms DAF and consequently, DF and AF, when the relays power to the source power ratio is higher than a threshold. This is in contrast to the full-duplex AWGN diamond channel in which CF is always worse than either DF or AF in terms of channel capacity [65].

Remark 3.3. *If $P_s \to \infty$, Eq.* (3.71) *is simplified to* $a_{CF} \approx \frac{a_{r_1}+a_{r_2}}{1+\frac{D(D+1)}{2}}$. *If $P_r \to \infty$, then $\mathcal{P}_C \to 1$ and $a_{CF} \approx a_{r_1} + a_{r_2}$. In high SNR asymptote at the relays, Eq.* (3.83) *meets the cutset-bound in Proposition 3.3 in Section 3.6.1, and is optimum.*

3.6 Upper-Bounds

3.6.1 Cutset Bound

The network cutset bound is the minimum of the maximum throughput and maximum expected-rate of the first-hop and the second-hop which lends itself to a closed form expression. The first-hop cutset is equivalent to a point-to-point SIMO channel with two receive antennas. The second-hop cutset is equivalent to a MISO channel with two transmit antennas. The throughput cutset bound is the minimum of the maximum throughput in these two cutsets, that is

$$\mathcal{R}^m_{CS,s} = \max_s e^{-s}(1+s)\ln\left(1+Ps\right), \tag{3.84}$$

where $P \triangleq \min\{P_s, P_r\}$.

Similarly, the maximum expected-rate of the diamond channel is upper-bounded by the minimum of the maximum expected-rates of those two cutsets, which is summarized below.

Proposition 3.3. *In the diamond channel, the cutset bound on the maximum expected-rate is specified by*

$$\mathcal{R}^m_{CS,c} = 3\mathrm{E}_1(s_0) - 3\mathrm{E}_1(s_1) - (s_0-1)e^{-s_0} + (s_1-1)e^{-s_1}, \tag{3.85}$$

where $s_1 = \frac{1+\sqrt{5}}{2}$, *and* $s_0 = \sqrt[3]{\sqrt{A^2 - B^3} + A} + \frac{B}{\sqrt[3]{\sqrt{A^2-B^3}+A}} - \frac{1}{3P}$ *with* $P = \min\{P_s, P_r\}$, $A = \frac{1}{2P} - \frac{1}{6P^2} - \frac{1}{27P^3}$, *and* $B = \frac{1}{3P} + \frac{1}{9P^2}$.

Proof. According to [67], the maximum continuous-layer expected-rate is given by

$$\mathcal{R}^m_{CS,c} = \max_{I(s)} \int_0^\infty \overline{F}_{\mathrm{a}}(s) \frac{-sI'(s)}{1+sI(s)} \mathrm{d}s. \tag{3.86}$$

Noting $\overline{F}_{\mathrm{a}}(s) = (1+s)\, e^{-s}$ based on Theorem 2.2, we have

$$\mathcal{R}^m_{CS,c} = \max_{I(s)} \int_0^\infty \frac{-s\,(1+s)\, e^{-s} I'(s)}{1+sI(s)} \mathrm{d}s. \tag{3.87}$$

The optimization solution to Eq. (3.87) with respect to $I(s)$ under the total power constraint $P = \min\{P_s, P_r\}$ is found using variation methods [33]. By solving the corresponding Eüler equation [33], we come up with the final solution as follows,

$$\mathcal{R}^m_{CS,c} = \int_{s_0}^{s_1} e^{-s}\,(1+s) \left(\frac{3}{s} - 1\right) \mathrm{d}s, \tag{3.88}$$

where boundaries s_0 and s_1 are the solutions to $Ps_0^3 + s_0^2 - s_0 - 1 = 0$ and $s_1^2 - s_1 - 1 = 0$, respectively. Therefore, $s_1 = \frac{1+\sqrt{5}}{2}$, and $s_0 = \sqrt[3]{\sqrt{A^2 - B^3} + A} + \frac{B}{\sqrt[3]{\sqrt{A^2-B^3}+A}} - \frac{1}{3P}$ with $A = \frac{1}{2P} - \frac{1}{6P^2} - \frac{1}{27P^3}$, and $B = \frac{1}{3P} + \frac{1}{9P^2}$. The indefinite integral (antiderivative) of Eq. (3.88) is

$$\int e^{-s}\,(1+s) \left(\frac{3}{s} - 1\right) \mathrm{d}s = (s-1)e^{-s} - 3\mathrm{E}_1(s). \tag{3.89}$$

Applying the integration limits completes the proof.

□

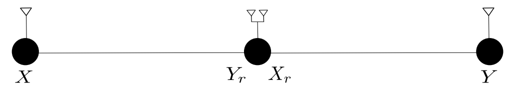

Figure 3.2: The *upper-bound model.*

3.6.2 Relay-Cooperation (RC) Bound

Here, a tighter upper-bound based on a full-cooperation between the relays is proposed. Let us define an *upper-bound model* by considering a full cooperation and power cooperation between the relays in the problem of interest. The *upper-bound model* is equivalent to a dual-hop single-relay channel with two antennas at the relay (see Fig. 3.2). The following presents the throughput of this *upper-bound model.*

Proposition 3.4. *the maximum throughput in the above upper-bound model is given by*

$$\mathcal{R}^{m}_{RC,s} = \max_{s} (1+s)\left(1+s\frac{P_s}{P_r}\right) e^{-s\left(1+\frac{P_s}{P_r}\right)} \ln(1+P_s s). \tag{3.90}$$

Proof. The optimum relaying strategy for dual-hop single-relay channels is DF. In the same general outline to the proof of Theorem 3.1, let $\overline{F}(s)$ denote

$$\overline{F}(s) = \Pr\{a_r \geq s\} \Pr\left\{a \geq \frac{P_s}{P_r} s\right\}, \tag{3.91}$$

where $a \triangleq \frac{1}{P_r}\vec{h}\mathbf{Q}\vec{h}^{\dagger}$, $\vec{h} \triangleq \begin{bmatrix} h_1 & h_2 \end{bmatrix}$, and $\mathbf{Q}$ is the transmit covariance matrix at the relays. The maximum throughput in general can be expressed as

$$\mathcal{R}_{RC,s} = \max_{s} \overline{F}(s) \ln(1+P_s s). \tag{3.92}$$

Analogously to the proof of Theorem 3.1, we can restrict our attention to $\rho = 0$ or $\rho = 1$, where ρ is the correlation coefficient between the signals transmitted from two relay antennas. To prove by contradiction, first we assume that $\rho^o = 1$; next, we shall show that $\overline{F}_{\rho=0}(s^o) > \overline{F}_{\rho=1}(s^o)$, which implies a contradiction and concludes $\rho^o = 0$. Defining

$$g\left(s, P_s\right) \triangleq \ln\left(1 + P_s s\right)^{\frac{1+P_s s}{P_s}}, \tag{3.93}$$

$$r(s) \triangleq \frac{\overline{F}_{\rho=1}(s)}{f_{\rho=1}(s)}, \tag{3.94}$$

Eqs. (3.10) and (3.12) hold.

Noting

$$\overline{F}_{\rho=1}(s) = (1+s)\, e^{-s\left(1+\frac{P_s}{2P_r}\right)}, \tag{3.95}$$

we have

$$r(s) = \frac{1+s}{(1+s)\left(1+\frac{P_s}{2P_r}\right) - 1}. \tag{3.96}$$

It can be shown that

$$r(s) < s, \quad \forall s \geq s_t, \tag{3.97}$$

where

$$s_t \triangleq \frac{\sqrt{P_s^2 + 4P_sP_r + 20P_r^2} - P_s + 2P_r}{2P_s + 4P_r}. \tag{3.98}$$

Hence, Eqs. (3.17) and (3.18) still hold by redefining $r(s)$ and s_t as above.

As $\mathcal{R}(s)$ is a continuous function, one can conclude that $0 < s^o < s_t$. Noting

$$s_t < s_c = -\left(2\mathcal{W}_{-1}\left(\frac{-1}{2\sqrt{e}}\right)+1\right)\frac{P_r}{P_s} \approx 2.5129\frac{P_r}{P_s} \tag{3.99}$$

and

$$\overline{F}_{\rho=0}(s) > \overline{F}_{\rho=1}(s), \quad \forall s < s_c \tag{3.100}$$

yields $\overline{F}_{\rho=0}(s^o) > \overline{F}_{\rho=1}(s^o)$ and thereby, $\rho^o = 0$ and $a = a_1 + a_2$. Substituting the channel gain CDFs in Eqs. (3.91) and (3.92), the maximum throughput of the DF diamond channel is given by Eq. (3.90).

□

The highest expected-rate of dual-hop single-relay channels has been studied in [70]. Here, only the final solution is mentioned as

$$\mathcal{R}^m_{RC,c} = \max_{\substack{I_s(a_r)\\ I_r(a|a_r)}} \int_0^\infty \int_0^\infty f_{a_r}(t)\overline{F}_a(s)\frac{-sI'_r(s|a_r=t)\mathrm{d}s}{1+sI_r(s|a_r=t)}\mathrm{d}t. \tag{3.101}$$

The power constraints at the transmitter and the relay are

$$I_s(0) = P_s,\ I_r(0|a_r=t) = P_r. \tag{3.102}$$

Note that in the *upper-bound model*, the power constraint at the relay is $2P_r$; however, the factor 2 is absorbed in the channel CDF. As the maximum transmission rate of the relay can not exceed its successfully decoded rate, the constraint on rate is

$$\int_0^\infty \frac{sI'_r(s|a_r=t)\mathrm{d}s}{1+sI_r(s|a_r=t)} = \int_0^t \frac{sI'_s(s)\mathrm{d}s}{1+sI_s(s)}, \quad \forall t. \tag{3.103}$$

The optimization problem of Eq. (3.101) can be solved numerically using the algorithm proposed in [59].

Following a similar outline in the proof of Theorem 3.2 and Proposition 3.4, one can show that the optimum transmission strategy at the relay is to transmit uncorrelated equal power signals from both of the relay antennas at each layer. Thus, $\overline{F}_{\mathrm{a}_r}(s) = \overline{F}_{\mathrm{a}}(s) = (1+s)e^{-s}$. Substituting in Eq. (3.101), we come up with the upper-bound as follows, which does not lend itself to a closed form formulation.

Proposition 3.5. *In the diamond channel, the maximum expected-rate at the destination is bounded by*

$$\mathcal{R}^{m}_{RC,c} = \max_{\substack{I_s(a_r)\\ I_r(a|a_r)}} \int_0^\infty te^{-t} \int_0^\infty \frac{-s(s+1)e^{-s} I_r'(s|a_r = t)}{1 + sI_r(s|a_r = t)} \mathrm{d}s\mathrm{d}t, \tag{3.104}$$

subject to the power and rate constraints Eqs. (3.102) *and* (3.103)*, respectively.*

3.6.3 DF-Upper-Bounds

As pointed out earlier, the continuous-layer coding for DF relaying can not be directly solved by variations methods. Here, two upper-bounds for the maximum continuous-layer expected-rate in DF scheme are obtained. Let us define a *DF-upper-bound model* as a diamond channel with uninformed transmitters, wherein the channel gains of the source-relay links are both $\max\{a_{r_1}, a_{r_2}\}$, and those of the relays-destination links are a_1 and a_2, respectively. This channel can be modeled by a dual-hop single-relay channel with the channel gains $a_r = \max\{a_{r_1}, a_{r_2}\}$ and a for the source-relay link and the relay-destination link, respectively. Clearly, the maximum expected-rate of this model yields an upper-bound on the maximum expected-rate of DF relaying.

The optimum relaying strategy in the *DF-upper-bound model* is DF, and is given by

Eq. (3.101). Analogous to Section 3.6.2, it can be shown that the optimum transmission strategy at the relays is to transmit uncorrelated equal power signals from the relays at each layer. Hence, substituting $\overline{F}_{\mathrm{a}_r}(s) = e^{-s}\left(2 - e^{-s}\right)$ and $\overline{F}_{\mathrm{a}}(s) = (1+s)e^{-s}$ in Eq. (3.101), we come up with the upper-bound as follows, which does not lend itself to a closed form formulation.

Proposition 3.6. *In the DF diamond channel, the maximum expected-rate at the destination is bounded by*

$$\mathcal{R}^m_{RC,D} = \max_{\substack{I_s(a_r)\\ I_r(a|a_r)}} 2\int_0^\infty e^{-t}\left(e^{-t}-1\right)\int_0^\infty \frac{s(s+1)e^{-s}I_r'(s|a_r=t)}{1+sI_r(s|a_r=t)}\mathrm{d}s\mathrm{d}t, \tag{3.105}$$

subject to the power and rate constraints Eqs. (3.102) *and* (3.103), *respectively.*

The cutset bound of the *DF-upper-bound model* results in a closed form expression. The results are summarized below.

Proposition 3.7. *The cutset bound of the DF-upper-bound model is specified by* $\mathcal{R}^m_{CS,D} = \min\{\mathcal{R}_1, \mathcal{R}_2\}$, *where*

$$\begin{aligned}\mathcal{R}_1 &= 4\mathrm{E}_1(s_1) - 2\mathrm{E}_1(2s_1) + e^{-2s_1} - 3e^{-s_1} - \ln\left(1-e^{-s_1}\right) - 0.1157,\\ \mathcal{R}_2 &= 3\mathrm{E}_1(s_2) - (s_2-1)e^{-s_2} - 0.1296.\end{aligned} \tag{3.106}$$

s_1 *is the solution to* $\frac{2-e^{-s_1}}{2s_1\left(1-e^{-s_1}\right)} = 1 + P_s s_1$ *and* $s_2 = \sqrt[3]{\sqrt{A^2-B^3}+A} + \frac{B}{\sqrt[3]{\sqrt{A^2-B^3}+A}} - \frac{1}{3P_r}$ *with* $A = \frac{1}{2P_r} - \frac{1}{6P_r^2} - \frac{1}{27P_r^3}$, *and* $B = \frac{1}{3P_r} + \frac{1}{9P_r^2}$.

Proof. The bound on the second hop, i.e., $\mathcal{R}_2$, is a direct result of Proposition 3.3.

Noting $\overline{F}_{\mathrm{a}_r}(s) = e^{-s}\left(2 - e^{-s}\right)$ in the first hop, analogous to the proof of Proposition 3.3, we have

$$\mathcal{R}_1 = \max_{I(s)} \int_0^\infty \frac{-se^{-s}\left(2-e^{-s}\right)I'(s)}{1+sI(s)}\mathrm{d}s. \tag{3.107}$$

The optimization solution to Eq. (3.107) with respect to $I(s)$ under the total power constraint P_s is found by solving the associated Eüler equation [33], which leads to

$$\mathcal{R}_1 = \int_{s_1}^{s_3} e^{-s}\left(2-e^{-s}\right)\left(\frac{2}{s}+\frac{2e^{-s}-1}{1-e^{-s}}\right)\mathrm{d}s, \tag{3.108}$$

where boundaries s_1 and s_3 are the solutions to $\frac{2-e^{-s_1}}{2s_1\left(1-e^{-s_1}\right)} = 1 + P_s s_1$ and $\frac{2-e^{-s_3}}{2s_3\left(1-e^{-s_3}\right)} = 1$, respectively. The indefinite integral (antiderivative) of Eq. (3.108) is

$$\begin{aligned}&\int e^{-s}\left(2-e^{-s}\right)\left(\frac{2}{s}+\frac{2e^{-s}-1}{1-e^{-s}}\right)\mathrm{d}s = \\ &-4\mathrm{E}_1(s)+2\mathrm{E}_1(2s)-e^{-2s}+3e^{-s}+\ln\left(1-e^{-s}\right).\end{aligned} \tag{3.109}$$

Applying the integration limits completes the proof.

□

3.7 Numerical Results

The achievable throughput, two-layer expected-rate, and continuous-layer expected-rate in the proposed multi-layer relaying schemes are shown respectively in Figs. 3.3 to 3.5 for $P_s = 0$ dB and -10 dB $\leq P_r \leq 60$ dB. Note that the rates are expressed in *nats*. When $\frac{P_r}{P_s}$, namely *powers ratio*, is less than 25 dB, DAF is the best scheme. In higher values of the *powers ratio*, CF is the superior. AF has the worst performance for $\frac{P_r}{P_s} > 10$ dB, but ON/OFF based AF, outperforms DF for $\frac{P_r}{P_s} > 30$ dB. When P_r goes to infinity, CF meets the upper-bounds, which is consistent with remark 3.3. As pointed out earlier, these results

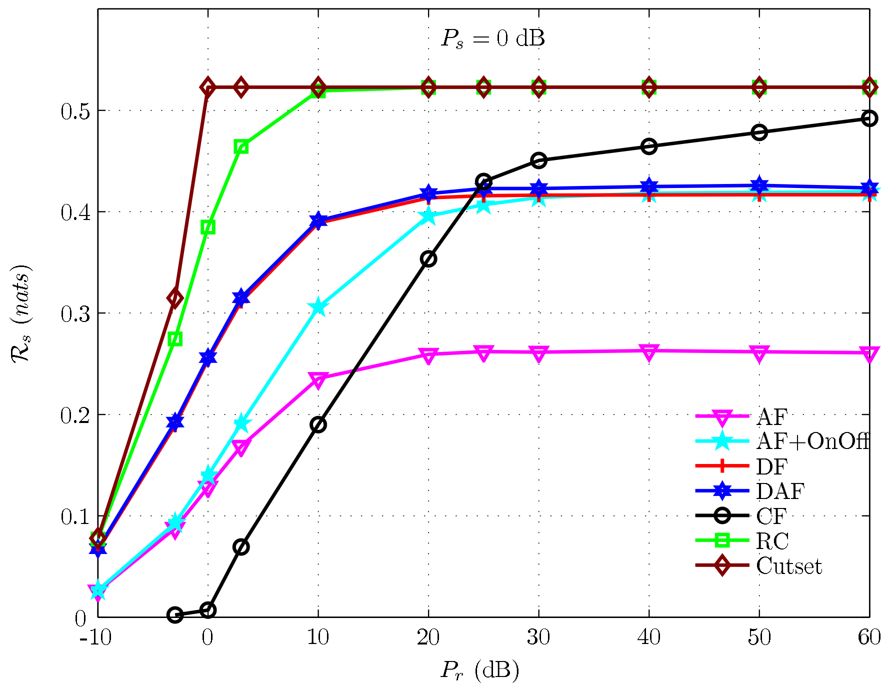

Figure 3.3: Throughput in the diamond channel.

are in contrast to the full-duplex AWGN diamond channel in which CF is never the best option [65].

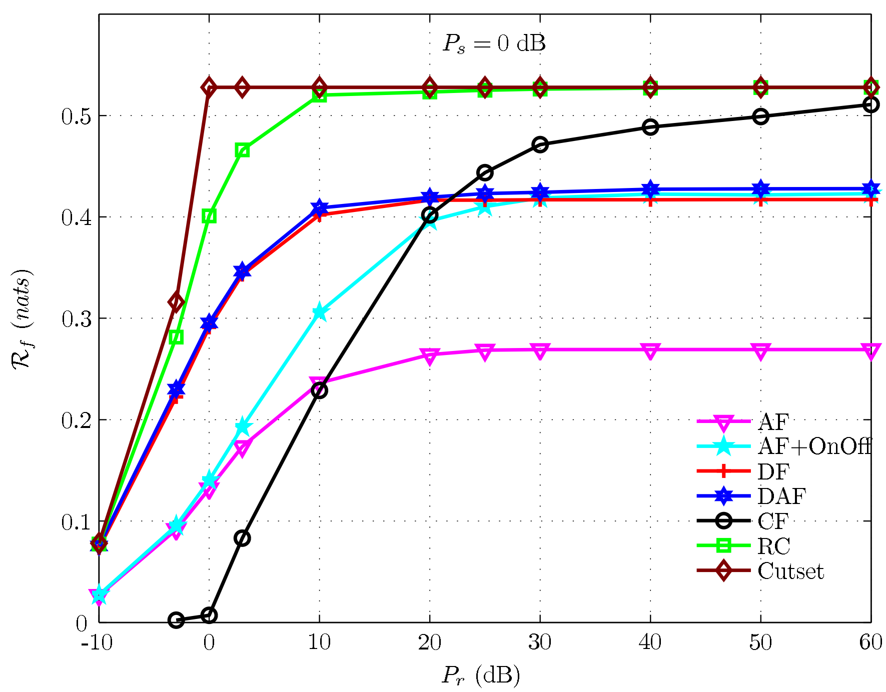

Figure 3.4: Two-layer expected-rate in the diamond channel.

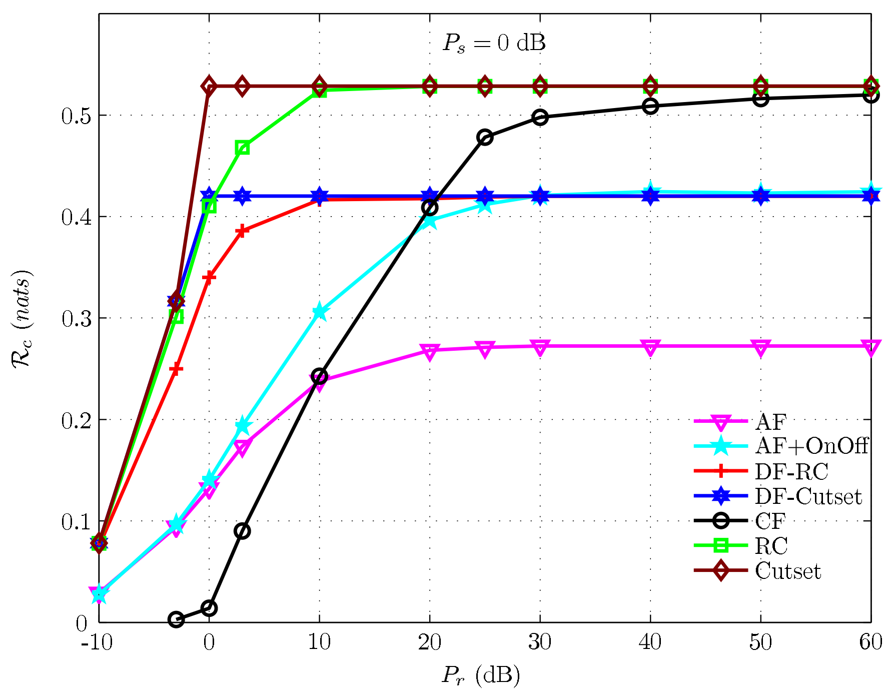

Figure 3.5: Continuous-layer expected-rate in the diamond channel.

Chapter 4

Conclusions and Future Directions

In chapter 2, the throughput and expected-rate maximization of point-to-point multiple-antenna channels are addressed in Rayleigh block fading environments, in which the transmitter does not access the CSI. It is established that, in order to achieve the maximum throughput, one has to transmit uncorrelated circularly symmetric zero mean equal power Gaussian signals on all the transmit antennas. This indeed yields the same transmit covariance matrix that achieves the ergodic capacity.

In point-to-point uncorrelated MISO channels, in contrast to using a fraction of antennas which is optimum for outage capacity, the throughput is maximized by sending uncorrelated equal power signals on all transmit antennas. The maximum expected-rate is analyzed using multi-layer codes. It is proved that in each layer, sending uncorrelated signals with equal powers on all available antennas is optimum. The continuous-layer expected-rate of the channel is then derived in closed form.

The optimum transmit strategy maximizing the throughput is obtained for point-to-point uncorrelated MIMO channels. Since the PDF of the MIMO instantaneous mutual information is not tractable in general, four asymptotic cases are considered: low SNR regime, high SNR regime, large number of transmit antennas, and large number of receive antennas. In each case, the maximum throughput of the MIMO channel is derived.

Afterwards, a distributed antenna system with two single-antenna transmitters and one single-antenna receiver is investigated. It is proved that any achievable instantaneous mutual information distribution in the 2×1 MISO channel is also achievable in the two-transmitter distributed antenna system. Hence, both systems achieve the same maximum throughput and expected-rate.

The problem of maximum throughput and maximum expected-rate in the general MIMO channel can be investigated in future. Another future extension is to investigate the problem of maximum average achievable rate instead of maximum expected-rate.

In chapter 3, simple, efficient, and practical relaying schemes are proposed in order to increase the average achievable rate in dual-hop networks with two parallel relays, Rayleigh block fading links, and uninformed transmitters. To this end, different relaying schemes, in conjunction with the broadcast approach, were proposed. The performance of the proposed schemes were derived and numerically compared with obtained upper-bounds.

Our results in this chapter are restricted to two relays. In a more general scenario, the number of relays may be increased in future works. Increasing the number of hops is also of practical relevance. it would be interesting to consider the problem of asymptotically large number of relays and propose optimal coding schemes. In this thesis, we considered Rayleigh distributed fading links. Investigating different fading distributions such as Rician and Nakagami are other possible extensions.

APPENDICES

Appendix A

Proof of Proposition 2.3

The ergodic capacity of a $1 \times n_r$ SIMO channel is given by

$$C_{\text{erg}} = \int_0^\infty \frac{x^{n_r-1}e^{-x}}{(n_r-1)!} \ln(1+Px)\,\mathrm{d}x. \tag{A.1}$$

Applying the integration by parts rule on Eq. (A.1) leads to

$$\begin{aligned} C_{\text{erg}} = & \left[-e^{-x} \sum_{\ell=0}^{n_r-1} \frac{x^\ell}{\ell!} \ln(1+Px) \right]_0^\infty \\ & + \int_0^\infty e^{-x} \sum_{\ell=0}^{n_r-1} \frac{x^\ell}{\ell!} \frac{P}{1+Px} \mathrm{d}x. \end{aligned} \tag{A.2}$$

One can simply show that the first part on the right-hand-side in Eq. (A.2) is zero by repeatedly applying l'Hôpital's rule. With $t \triangleq 1 + Px$, Eq. (A.2) yields

$$C_{\text{erg}} = \int_1^\infty e^{-\frac{t-1}{P}} \sum_{\ell=0}^{n_r-1} \frac{1}{t\ell!} \left(\frac{t-1}{P} \right)^\ell \mathrm{d}t. \tag{A.3}$$

From $(t-1)^{\ell} = \sum_{\imath=0}^{\ell} \binom{\ell}{\imath} t^{\imath} (-1)^{\ell-\imath}$, where $\binom{\ell}{\imath}$ is the binomial coefficient, we get

$$\begin{aligned} C_{\mathrm{erg}} &= e^{\frac{1}{P}} \int_{1}^{\infty} e^{-\frac{t}{P}} \sum_{\ell=0}^{n_r-1} \frac{1}{P^{\ell} \ell! t} \sum_{\imath=0}^{\ell} \binom{\ell}{\imath} t^{\imath} (-1)^{\ell-\imath} \mathrm{d}t \\ &= e^{\frac{1}{P}} \sum_{\ell=0}^{n_r-1} \frac{(-1)^{\ell}}{P^{\ell} \ell!} \int_{1}^{\infty} \frac{e^{-\frac{t}{P}}}{t} \mathrm{d}t \\ &+ e^{\frac{1}{P}} \sum_{\ell=1}^{n_r-1} \frac{1}{P^{\ell} \ell!} \sum_{\imath=1}^{\ell} (-1)^{\ell-\imath} \binom{\ell}{\imath} \int_{1}^{\infty} e^{-\frac{t}{P}} t^{\imath-1} \mathrm{d}t. \end{aligned} \tag{A.4}$$

With $u \triangleq \frac{t}{P}$, we have

$$\begin{aligned} \int_{1}^{\infty} e^{-\frac{t}{P}} t^{\imath-1} \mathrm{d}t &= P^{\imath} \int_{\frac{1}{P}}^{\infty} e^{-u} u^{\imath-1} \mathrm{d}u \\ &= (\imath-1)! P^{\imath} e^{-\frac{1}{P}} \sum_{m=0}^{\imath-1} \frac{1}{m!} \left(\frac{1}{P}\right)^{m}. \end{aligned} \tag{A.5}$$

Inserting Eq. (A.5) into Eq. (A.4), we obtain

$$\begin{aligned} C_{\mathrm{erg}} &= e^{\frac{1}{P}} \mathrm{E}_1\left(\frac{1}{P}\right) \sum_{\ell=0}^{n_r-1} \frac{(-1)^{\ell}}{P^{\ell} \ell!} \\ &+ \sum_{\ell=1}^{n_r-1} \frac{1}{P^{\ell}} \sum_{\imath=1}^{\ell} \frac{(-1)^{\ell-\imath}}{\imath (\ell-\imath)!} P^{\imath} \sum_{m=0}^{\imath-1} \frac{1}{m!} \frac{1}{P^{m}}. \end{aligned} \tag{A.6}$$

Let $k \triangleq \ell - \imath$, the above leads to

$$\begin{aligned} C_{\mathrm{erg}} &= e^{\frac{1}{P}} \mathrm{E}_1\left(\frac{1}{P}\right) \sum_{\ell=0}^{n_r-1} \frac{(-1)^{\ell}}{P^{\ell} \ell!} \\ &+ \sum_{\ell=1}^{n_r-1} \sum_{k=0}^{\ell-1} \frac{(-1)^{k}}{(\ell-k)\, k!} \sum_{m=0}^{\ell-k-1} \frac{1}{m! P^{k+m}}. \end{aligned} \tag{A.7}$$

From [77], the ergodic capacity in an $n_t \times 1$ MISO channel with total power constraint

P equals the ergodic capacity in a $1 \times n_t$ SIMO channel with total power constraint $\frac{P}{n_t}$. Hence, we obtain Eq. (2.10) by replacing P with $\frac{P}{n_t}$ and n_r with n_t in Eq. (A.7).

Appendix B

Part of the Proof of Theorem 2.3

The indefinite integral (antiderivative) of Eq. (2.42) can be written as

$$\begin{aligned}
\mathcal{R}(s) &= \int e^{-s}\left(\frac{n_t+1}{s}-1\right)\sum_{\ell=0}^{n_t-1}\frac{s^{\ell}}{\ell!}\mathrm{d}s \\
&= (n_t+1)\int\frac{e^{-s}}{s}\mathrm{d}s + (n_t+1)\int e^{-s}\sum_{\ell=0}^{n_t-1}\frac{s^{\ell-1}}{\ell!}\mathrm{d}s \\
&- \int e^{-s}\sum_{\ell=0}^{n_t-1}\frac{s^{\ell}}{\ell!}\mathrm{d}s \\
&= (n_t+1)\int\frac{e^{-s}}{s}\mathrm{d}s + \sum_{\ell=0}^{n_t-1}\frac{1}{\ell!}\Bigg((n_t+1)\int s^{\ell-1}e^{-s}\mathrm{d}s \\
&- \int s^{\ell}e^{-s}\mathrm{d}s\Bigg). \qquad \text{(B.1)}
\end{aligned}$$

The definite integral of $\mathcal{R}(s)$ over the interval $[s_0 \ \ \infty]$ is given by

$$[\mathcal{R}(s)]_{s_0}^{\infty} = (n_t+1)\int_{s_0}^{\infty}\frac{e^{-s}}{s}\mathrm{d}s + \sum_{\ell=0}^{n_t-1}\frac{1}{\ell!}\Bigg($$

$$
\begin{aligned}
&(n_t+1)\int_{s_0}^{\infty} s^{\ell-1}e^{-s}\mathrm{d}s - \int_{s_0}^{\infty} s^{\ell}e^{-s}\mathrm{d}s\Bigg)\\
&= (n_t+1)\,\mathrm{E}_1\left(s_0\right) + \sum_{\ell=0}^{n_t-1}\frac{1}{\ell!}\Bigg(\\
&(n_t+1)\,(\ell-1)!e^{-s_0}\sum_{k=0}^{\ell-1}\frac{s_0^k}{k!} - \ell!e^{-s_0}\sum_{k=0}^{\ell}\frac{s_0^k}{k!}\Bigg)\\
&= (n_t+1)\,\mathrm{E}_1\left(s_0\right) - e^{-s_0} + e^{-s_0}\sum_{\ell=1}^{n_t-1}\frac{1}{\ell!}\Bigg(\\
&- s_0^{\ell} + (n_t+1-\ell)\,(\ell-1)!\sum_{k=0}^{\ell-1}\frac{s_0^k}{k!}\Bigg).
\end{aligned}
\tag{B.2}
$$

The definite integral of $\mathcal{R}(s)$ over the interval $[s_0 \;\; s_1]$ can be written as $[\mathcal{R}(s)]_{s_0}^{\infty} - [\mathcal{R}(s)]_{s_1}^{\infty}$. Therefore, defining

$$
\begin{aligned}
\mathcal{R}(s) \triangleq{}& -(n_t+1)\,\mathrm{E}_1\left(s\right) + e^{-s}\\
&+ e^{-s}\sum_{\ell=1}^{n_t-1}\frac{1}{\ell!}\left(s^{\ell} - (n_t+1-\ell)\,(\ell-1)!\sum_{k=0}^{\ell-1}\frac{s^k}{k!}\right),
\end{aligned}
\tag{B.3}
$$

and inserting into Eq. (B.2) leads to the conclusion that

$$
[\mathcal{R}(s)]_{s_0}^{s_1} = \mathcal{R}(s_1) - \mathcal{R}(s_0).
\tag{B.4}
$$

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