Cyclic Sieving Phenomenon of Promotion on Rectangular Tableaux

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Cyclic sieving phenomenon (CSP) is a generalization by Reiner, Stanton, White of Stembridge's $q = -1$ phenomenon. When CSP is exhibited, orbits of a cyclic action on combinatorial objects show a nice structure and their sizes can be encoded by one polynomial.

In this thesis we study various proofs of a very interesting cyclic sieving phenomenon, that jeu-de-taquin promotion on rectangular Young tableaux exhibits CSP. The first proof was obtained by Rhoades, who used Kazhdan-Lusztig representation. Purbhoo's proof uses Wronski map to equate tableaux with points in the fibre of the map. Finally, we consider Petersen, Pylyavskyy, Rhoades's proof on 2 and 3 row tableaux by bijecting the promotion of tableaux to rotation of webs.

This thesis also propose a combinatorial approach to prove the CSP for square tableaux. A variation of jeu-de-taquin move yields a way to count square tableaux which has minimal orbit under promotion. These tableaux are then in bijection to permutations. We consider how this can be generalized.

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Chapter 1

Background

The purpose of this thesis is to give an overview of current understanding of promotion of rectangular tableaux and to see how the cyclic sieving phenomenon of the promotion can be shown.

1.1 Introduction

Cyclic sieving phenomenon was motivated by $q = -1$ phenomenon which Stembridge introduced in 1993 [22]. Suppose we have a finite set of combinatorial objects S, a natural involution $* : S \to S$, and an associated generating function $F(q)$. We say that the $q = -1$ phenomenon occurs if $F(-1)$ evaluates the number of fixed points in S by $*$. This not only yields nice closed expressions for the number of self-dual objects, but it also provides explanations (using representation theory) for previously known formulae. Stembridge found that this phenomenon was exhibited in many instances, such as:

1. Minuscule posets with order reversing involution and the generating function for multichains of order ideals [23].

2. Plane partitions with complementation and generating function for the size [22].

3. Set of tableaux of shape λ with evacuation and Schur function $s_\lambda(q^0, q^1, \cdots, q^{n-1})$, which is the generating function for content [24].

In 2004, Reiner, Stanton, and White [26] defined cyclic sieving phenomenon (CSP), generalizing the $q = -1$ phenomenon. Suppose we have a finite set of combinatorial objects S, a cyclic action c on S, and an associated generating function $F(q)$. We say that the cyclic sieving phenomenon occurs if $F(\zeta^k)$ evaluates the number of fixed points in S by c^k where ζ is a nth root of unity.

This phenomenon was also found to be exhibited in many instances, such as:

1. k-multisets of $[n]$ with cycling of elements and q-binomial coefficient $\binom{n+k-1}{k}_q$.
2. k-subsets of $[n]$ with cycling of elements and q-binomial coefficient $\binom{n}{k}_q$.
3. Rooted plane trees of size n with cycling of subtrees at the root and q-binomial coefficient $\binom{2n-1}{n}_q$.
4. N-colorings of set S with cycling of colors and generating function for the number of occurrences of color in an orbit.
5. Dissections of convex n-gon with rotation and generating function for the major index of Young tableaux of certain shape.
6. Non-crossing partitions with rotations and q-analog of Narayana number.

CSP was then further generalized to study number of fixed points of any group action with one polynomial. In 2005, it was conjectured by Abuzzahab, Korson, Li, and Meyer in their undergraduate research that actions of promotion and evacuation may exhibit dihedral sieving phenomenon [10]. This has now been resolved by Rhoades in 2010 and by Purbhoo in 2011 for rectangular shape. Cyclic sieving of promotion in particular is our focus for this thesis, although we may briefly touch on evacuations sometimes.

Rest of the chapter will be quick introductions to prerequisites, such as tableaux theory, representation theory, and cyclic sieving phenomenon. See [16] [4] [13] [17] for more thorough study of these subjects.

1.2 Symmetric Groups

We start off with a brief review of symmetric groups and their Coxeter presentation.

1.2.1 Symmetric Group

Definition 1.2.1. *A* ***symmetric group*** *of degree $n \in \mathbb{N}$ is a set of bijections $\sigma : [n] \to [n]$ with function composition as its group operation, where $[n] := \{1, 2, 3, \cdots, n\}$. It is denoted as S_n. The elements of S_n are called* ***permutation*** *of n.*

There are $n! := n \cdot (n-1) \cdot \cdots \cdot 1$ permutations in S_n.

Permutations can be written in multiple ways. In one line notation, we simply write down $\sigma(1)\sigma(2)\cdots\sigma(n)$. For cycle notation, the permutation is written as product of cycles. **Cycles** are permutations with one non-trivial orbit, i.e. the permutation maps $a_1 \mapsto a_2 \mapsto \cdots \mapsto a_k \mapsto a_1$ and fixes the rest. This cycle is denoted as $(a_1\ a_2\ \cdots\ a_k)$. All permutations decompose into disjoint cycles. We call a cycle of length 2 **transposition**.

1.2.2 Coxeter Presentation

The symmetric group S_n is generated by the tranpositions:

$$s_1 := (1\ 2), s_2 := (2\ 3), \dots, s_{n-1} := (n-1\ n).$$

The group S_n has a following presentation with these transpositions as its generators, called the **Coxeter presentation of** S_n.

Definition 1.2.2. *Let s_i be the same as above, then*

1. *s_i commutes with s_j if $|i-j| > 1$.*
2. *$s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$.*
3. *s_i^2 is 1 for all i.*

All $\sigma \in S_n$ can be written as a product of these generators:

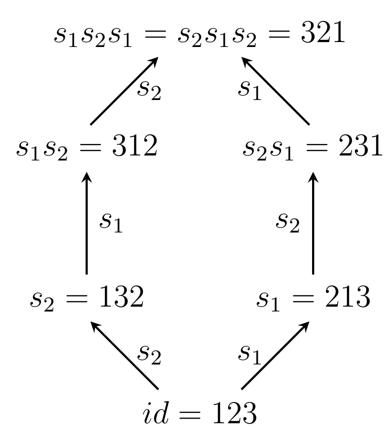
$$\sigma = s_{a_1} s_{a_2} \dots s_{a_k}.$$

Definition 1.2.3. *We say that the **length** of σ, $l(\sigma)$, is the minimal number of generators needed to get σ as their product. Such minimal length products $s_{a_1} s_{a_2} \dots s_{a_k}$ are called **reduced words** of σ.*

This Coxeter presentation comes with a poset structure called **Bruhat ordering**.

Definition 1.2.4. *Let $\sigma, \tau \in S_n$ then we say $\sigma \leq \tau$ in the Bruhat ordering if σ and τ can be written as reduced words such that the first is the subword of the second. Here, a subword of $\tau_1 \tau_2 \cdots \tau_n$ is defined as a word of form $\tau_{a_1} \tau_{a_2} \cdots \tau_{a_k}$ for some $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n$.*

Example 1.2.5. *Below is the Hasse diagram of the Bruhat order in S_3.*



Lastly, we will name some special permutations: the permutation $123 \cdots n \in S_n$ is called the **identity permutation**; a cycle of size n in S_n is called a **long cycle**, and the permutation $n, n-1, \cdots, 1 \in S_n$ is called the **long element**. The length of the long element is $\binom{n}{2}$. This is the unique maximal element in the Bruhat order.

1.3 Young Tableaux

1.3.1 Partitions

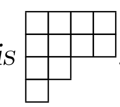
Definition 1.3.1. *A* ***partition*** *is an weakly decreasing set of positive integers* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. *It is denoted as* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)$. *It is often useful to write partitions using multiplicities of parts. Thus, we define* $a_1^{b_1} a_2^{b_2} \cdots a_k^{b_k}$ *to be the partition*

$$(\underbrace{a_1, a_1, \cdots, a_1}_{b_1}, \underbrace{a_2, a_2, \cdots, a_2}_{b_2}, \cdots, \underbrace{a_k, a_k, \cdots, a_k}_{b_k}).$$

We say that λ *partitions* $n = \lambda_1 + \lambda_2 + \cdots + \lambda_d$ *and write* $\lambda \vdash n$ *or* $|\lambda| = n$.

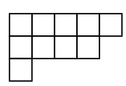
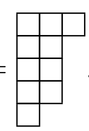
Partitions are often presented in a Ferrers diagram. The **Ferrers diagram** (also called **Young diagram** or **partition diagram**) of λ is a set of unit squares in integer lattice grids such that there are λ_1 squares horizontally in the top row, λ_2 squares in the second top row, and so on.

Example 1.3.2.

The Ferrers diagram of $\lambda = (4, 4, 2, 1) = 4^2 2^1 1^1$ *is* .

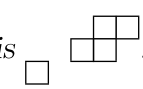
Partitions can be **conjugated** by simply mirroring the Ferrers diagram at the diagonal. This yields a partition of the same size.

Example 1.3.3.

The conjugate of $\lambda = (5, 4, 1) =$  *is* $\lambda' = (3, 2, 2, 2, 1) =$ .

We say λ contains μ and write $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all i. A **skew shape** is a pair of partition λ and μ such that $\lambda \supseteq \mu$. It is denoted by λ/μ. The Ferrers diagram of λ/μ is the Ferrers diagram of λ with the squares belonging to μ taken away.

Example 1.3.4.

The Ferrers diagram of the skew shape $(5, 4, 1)/(3, 2)$ *is* .

Thoughout this thesis, we will be considering partitions that are rectangular i.e. has the form a^b. We will write $\square$ to denote such partitions.

An another useful poset structure one can define on partitions is the **dominance order**. We say λ dominates μ if $\sum_{i=1}^{k} \lambda_i \geq \sum_{j=1}^{k} \mu_i$ for all $k = 1, 2, \cdots$ and $|\lambda| = |\mu|$. This is denoted as $\lambda \trianglerighteq \mu$.

1.3.2 Young Tableaux

A combinatorial object that is of extreme importance to us is Young tableaux. Young tableaux were introduced by Alfred Young in 1900 and since then has been used and appear in various fields such as representation theory, geometry, and algebra. For a very brief introduction, see [27]. For more complete introduction of tableaux theory, see [4] [21].

Definition 1.3.5. *Let λ and μ be partitions such that $\lambda \supseteq \mu$.*

- *A* ***Young tableau*** *(or simply* ***tableau****) of shape λ/μ is a filling of the Ferrers diagram with positive integers such that the entries weakly increase from left to right and weakly increase from top to bottom.*
- *A Young tableau is of* ***normal*** *shape λ if μ is the empty partition $\varnothing$. The tableau is* ***skew*** *otherwise.*
- *A Young tableau is* ***semistandard*** *if the entries strictly increase from top to bottom.*
- *A Young tableau is* ***standard*** *if the entires are $1, 2, 3, \cdots, n$ and they strictly increase from left to right, from top to bottom.*
- *The* ***size*** *of a tableau is the size of the shape.*
- *The* ***content*** *of a tableau is $(c_1, c_2, \cdots)$ where c_k is the number of k's in the Young Tableau. This is not necessarily a partition.*
- *The* ***word*** *of a tableau is a reading of the entries from left to right and bottom to top.*

Plural of 'tableau' is 'tableaux'. Shape of tableau T is denoted as $\operatorname{sh} T$.

Example 1.3.6. *An example of a semistandard Young tableau of shape $\lambda = (4, 4, 2, 1)$ is*

2	2	4	5
3	4	5	6
6	6		
8			

which has the size 11, *content* $(0, 2, 1, 2, 2, 3, 0, 1, 0, 0, \cdots)$*, and reading* $8, 6, 6, 3, 4, 5, 6, 2, 2, 4, 5$.

An example of a standard Young tableau of shape λ is

1	2	7	8
3	6	9	10
4	11		
5			

.

An example of a semistandard skew tableau of shape $\lambda/\mu = (4,4,2,1)/(3,2,2)$ *is*

			1
		2	3
2			

.

One can conjugate a tableau U to get a tableau of same size U', as we did to conjugate partitions.

1.3.3 Ribbon Tableaux

Ribbon tableaux are more general form of the standard Young tableaux. An r**-ribbon** (or rim-hook) is a connected skew shape that does not contain 2×2 block. Another way to say this is that any north-west to south-east diagonal hits at most one box. Given $|\lambda/\mu| = rl$, a r**-ribbon tableau** of shape λ/μ is a filling of the Ferrers diagram λ/μ with entries $1, 2, \cdots, l$ such that the shape determined by the same entries are r-ribbons.

Example 1.3.7.
A 4-ribbon tableau of shape $(6,5,5,4,2)/(2)$ *is*

		1	4	5	5
1	1	1	4	5	
2	2	3	4	5	
2	3	3	4		
2	3				

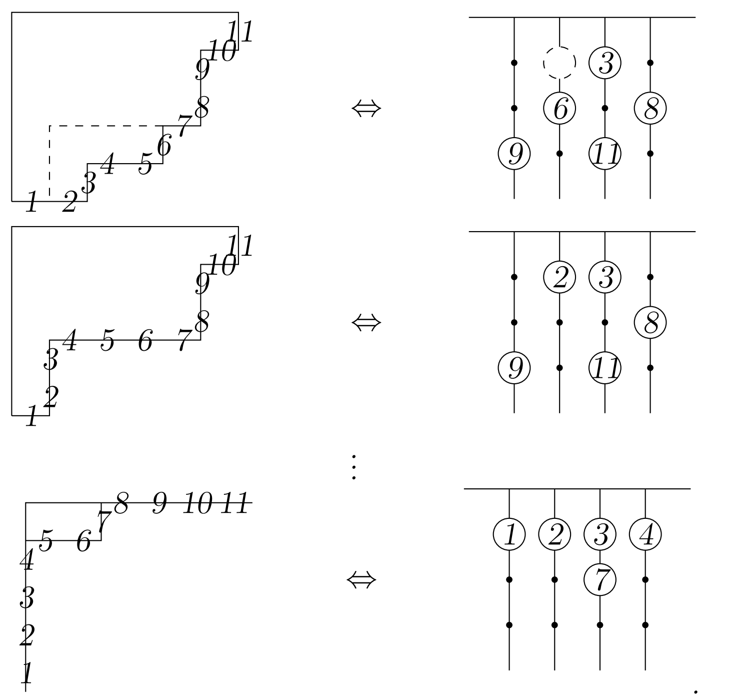
.

Given a partition λ, we define r**-core** of λ as the unique partition obtained as one removes r-ribbons from the partition until one cannot do so.

The uniqueness of r-core can be seen in the r**-abacus diagram** of the partition. r-abacus consists of r columns with beads on some of them. We label the edges of the partition diagram with $1, 2, \cdots,$ as we traverse from bottom left to top right. For each vertical edge labeled with $qr + p$ for some integer $0 < p \leq r$, we place a bead in the pth column at the qth position from the top in the abacus.

A removal of r-ribbon corresponds to decreasing a vertical label by r, which corresponds to shifting of a bead on the abacus one step up. When we shift all the beads to the top, we end up with the r-core.

Example 1.3.8. *Let $r = 4$ and $\lambda = (6, 5, 5, 4, 2)$. Then removal of r-ribbons from λ looks like this:*



Thus, the r-core of λ is □□.

1.3.4 Littlewood Richardson Tableaux

An important class of tableaux that arises while studying symmetric polynomials is the Littlewood-Richardson Tableaux. It first appeared in 1934 to claim Littlewood Richardson Rule [9], which was proven by Thomas and Schützenberger four decades later. It will come to be useful to us when we use jeu-de-taquin moves on tableaux.

Definition 1.3.9. *A **Littlewood-Richardson Tableau** is a skew semistandard Young tableau T if there exists a Young tableau U of a partition such that the number of i's in row j of T is equal to the number of j's in row i of U.*

Example 1.3.10.

T=

					1	1	1
					2	2	
				2	3		
			1				
		1	2				
1	1	3					

, U=

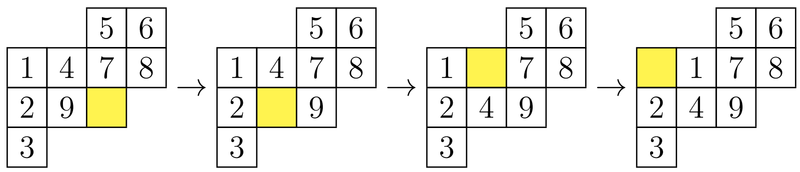
1	1	1	4	5	6	6
2	2	3	5			
3	6					

.

1.3.5 Jeu-de-taquin

Jeu-de-taquin moves are operations on Young tableaux. Given a tableau of shape λ/μ, pick a square that can be added to λ to get a new partition. Then repeat the following: from the chosen square shift the entry above down, or the entry to the left right, whichever is bigger (if it is same, take the vertical move). Continue this until there are no entries above and to the left of the empty square. We now have a new tableau. This operation can be done in reverse: pick a square that can be removed from μ and move it out. This operation on tableaux is called **jeu-de-taquin** sliding.

Example 1.3.11. *Slide in the yellow square:*



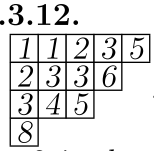
Given a skewed tableau, one can perform sliding repeatedly until we obtain a normal tableau. This process is called **rectification**. Skewed tableaux become rectified to an unique normal tableau, no matter what the order of the empty squares is. Also note that Littlewood Richardson tableaux rectify to the shape that is same as its content, ending up with only 1s on the first row, 2s on the second row, etc.

A related operation is row insertion. Given a normal tableau T, one **inserts** an entry x by following the pseudocode:

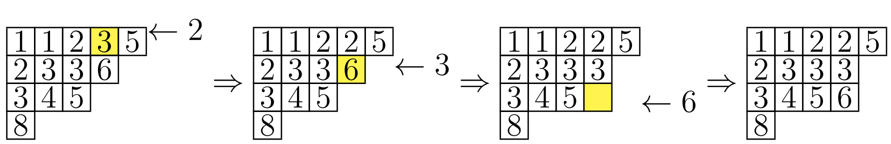
1. Start at row $r = 1$ and entry x.
2. Find the leftmost entry y on row r that is bigger than x.
3. If it exists, swap x and y. Increase r by 1. Go to 2
4. Otherwise, add x to the end of the row and stop.

The new tableau is denoted $T \leftarrow x$.

Example 1.3.12.

Let T be  .

Then $T \leftarrow 2$ is obtained as follows:



The following is an useful lemma:

Lemma 1.3.13. *If skew tableau T has word $W = w_1, w_2, \cdots, w_n$, then rectification of T is equal to the inserting w_i's into an empty tableau in order.*

1.3.6 Robinson Schensted Correspondence

Robinson Schensted Correspondence (RSK from now) is a bijective correspondence between permutations of size n and pairs of standard tableaux of shape $\lambda \vdash n$. This correspondence has many applications and is extremely important in tableaux theory.

The bijection is defined as follows: given $\sigma \in S_n$, begin with empty tableaux P_0 and Q_0. P_k is obtained from P_{k-1} by inserting σ_k. To get Q_k, we add a square to Q_{k-1} so that $\mathrm{sh}(P_k) = \mathrm{sh}(Q_k)$ and then fill the square with k. We will end up with two standard tableaux $P := P_n$ and $Q := Q_n$. One can see that this is reversible from any pair of standard tableaux of same shape, so this is indeed a bijection. We write $\sigma = \mathrm{RSK}(P, Q)$, $P = P(\sigma)$, and $Q = Q(\sigma)$ to denote this.

Example 1.3.14. *Let $\sigma = 351264 \in S_6$. Then*

$$P_1 = \begin{array}{|c|}\hline 3\\\hline\end{array}, \qquad Q_1 = \begin{array}{|c|}\hline 1\\\hline\end{array}$$

$$P_2 = \begin{array}{|c|c|}\hline 3 & 5\\\hline\end{array}, \qquad Q_2 = \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}$$

$$P_3 = \begin{array}{|c|c|}\hline 1 & 5\\\hline 3 \\\cline{1-1}\end{array}, \qquad Q_3 = \begin{array}{|c|c|}\hline 1 & 2\\\hline 3 \\\cline{1-1}\end{array}$$

$$P_4 = \begin{array}{|c|c|}\hline 1 & 2\\\hline 3 & 5\\\hline\end{array}, \qquad Q_4 = \begin{array}{|c|c|}\hline 1 & 2\\\hline 3 & 4\\\hline\end{array}$$

$$P_5 = \begin{array}{|c|c|c|}\hline 1 & 2 & 6\\\hline 3 & 5 \\\cline{1-2}\end{array}, \qquad Q_5 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5\\\hline 3 & 4 \\\cline{1-2}\end{array}$$

$$P = P_6 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4\\\hline 3 & 5 & 6\\\hline\end{array}, \qquad Q = Q_6 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5\\\hline 3 & 4 & 6\\\hline\end{array}.$$

We say that two permutations are **Knuth equivalent** if their insertion tableaux P are the same. Here are some facts about Knuth equivalence (most are easy to check using RSK):

1. There exists exactly one permutation such that it is also a reading of a normal standard tableau.

2. The word of the tableau stays in the same Knuth equivalence class under jeu-de-taquin slides.

3. Two skew standard tableaux with words in the same Knuth equivalence class rectifies to the same normal tableau.

From RSK, we know the following:

Lemma 1.3.15.

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

where f_λ is the number of standard tableaux of shape λ.

1.3.7 Hook Length Formula

It is natural to ask what f_λ is. The hook length formula yields a simple way to calculate f_λ.

Theorem 1.3.16.

$$f_\lambda = \frac{n!}{\prod_{\alpha \in \lambda} hook(\alpha)}$$

where $\alpha \in \lambda$ means that α is a square in the Ferrers diagram of λ and $hook(\alpha)$ is the number of squares that are either below or to the right of α (including α).

Example 1.3.17. *Let $\lambda = (4, 4, 2, 1)$. Then the hook lengths are:* $\begin{array}{|c|c|c|c|}\hline 7&5&3&2\\\hline 6&4&2&1\\\hline 3&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$.

and $f_\lambda = \frac{11!}{7\cdot5\cdot3\cdot2\cdot6\cdot4\cdot2\cdot1\cdot3\cdot1\cdot1} = 1320.$

1.4 Promotion

1.4.1 Definition of Promotion and Evacuation

Promotion is an operation on standard tableaux. It works as follows: given a standard tableau T, take the square with 1 and slide it out. Then, cycle all the entries such that $k+1$ becomes k and 1 becomes n. The resulting filling is also a tableau, denoted as $j(T)$.

Example 1.4.1. *Let* $T = \begin{array}{|c|c|c|c|}\hline 1&2&4&10\\\hline 3&5&8&11\\\hline 6&7\\\cline{1-2} 9\\\cline{1-1}\end{array}$. *Then*

$$\begin{array}{|c|c|c|c|}\hline \ &2&4&10\\\hline 3&5&8&11\\\hline 6&7\\\cline{1-2} 9\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|c|}\hline 2&\ &4&10\\\hline 3&5&8&11\\\hline 6&7\\\cline{1-2} 9\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|c|}\hline 2&4&\ &10\\\hline 3&5&8&11\\\hline 6&7\\\cline{1-2} 9\\\cline{1-1}\end{array}$$

$$\to \begin{array}{|c|c|c|c|}\hline 2&4&8&10\\\hline 3&5&\ &11\\\hline 6&7\\\cline{1-2} 9\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|c|}\hline 2&4&8&10\\\hline 3&5&11&\ \\\hline 6&7\\\cline{1-2} 9\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|c|}\hline 1&3&7&9\\\hline 2&4&10&11\\\hline 5&6\\\cline{1-2} 8\\\cline{1-1}\end{array} = j(T)$$

The set of boxes traveled by the empty box during the computation of promotion is called the **promotion path**. During promotion, all entries shift along the promotion path and then every entry decrease by 1.

A very closely related operation is evacuation: given a standard tableau $T = T_1$, slide k out from T_k to get T_{k+1}. Then we have a chain of shapes: $\text{sh}(T_1) \geq \text{sh}(T_2) \geq \cdots \geq \text{sh}(t_{n+1}) = \phi$. This defines a normal standard tableau of the same shape and we call this evacuation on T, denoted as $e(T)$.

Example 1.4.2. *Let* $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$. *Then*

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array},\ T_2 = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline\end{array},\ T_3 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array},$$

$$T_4 = \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline\end{array},\ T_5 = \begin{array}{|c|}\hline 5 \\ \hline\end{array},\ T_6 = \phi.$$

Thus, $e(T) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$.

We can define dual promotion by taking the largest entry of the tableau and sliding the empty square in, cycling the entries the other way to making a new tableau. Dual evacuation is defined similarly: take the biggest entry left in the skew tableau T_k and slide it in. The recording tableau is the result of dual evacuation. We denote these operations by j^* and e^*.

1.4.2 Properties of Promotion and Evacuation

We have the following theorem by Schützenberger [18] [19].

Theorem 1.4.3. *The following properties are true:*

1. e^2 *is the identity operation.*
2. $j^{-1} = j^*$.
3. $j^n = ee^*$ *on tableaux of size* n.
4. $je = ej^*$.

If the shape of the tableau T is rectangular (i.e. $\text{sh}(T) = \square$) then the evacuation has a simpler description: rotate the diagram 180° and then re-label each k by $(n+1) - k$. This also implies that $e = e^*$. Thus we have $e^2 = id$, $j^n = id$, and $je = ej$, obtaining a dihedral group generated by e and j.

A key observation regarding promotion is that promotion cycles the descent set. The **descent set** (often called extended descent set) of standard tableau T of size n is

$$D(T) = \begin{cases} S & \text{if } n-1 \text{ is at right of } n \text{ in } j(T) \\ S \cup \{n\} & \text{if } n-1 \text{ is on top of } n \text{ in } j(T) \end{cases}$$

where $S = \{k \in [n-1] : k+1 \text{ is in lower row than } k\}$.

Lemma 1.4.4. *We have* $i \in D(T)$ *if and only if* $i+1 \in D(j(T))$.

Lastly, we note that evacuation interacts nicely with RSK.

Lemma 1.4.5. *Let* U *and* V *be normal standard tableau of same shape* $\lambda \vdash n$. *Suppose* $\mathrm{RSK}(U, V) = \sigma \in S_n$ *and* $\omega_0 \in S_n$ *be the long element. Then*

1. $\sigma^{-1} = \mathrm{RSK}(V, U)$.
2. $\omega_0 \sigma = \mathrm{RSK}(U', e(V)')$.
3. $\sigma \omega_0 = \mathrm{RSK}(e(U)', V')$.
4. $\omega_0 \sigma \omega_0 = \mathrm{RSK}(e(U), e(V))$

1.5 Representations of the Symmetric Group

Most proofs of CSP relies on representation theory. We give a brief overview of for representations of the symmetric group. See [28], [13] for quick introduction and [16] for complete introduction.

1.5.1 Representations

Let V be a complex vector space. The group of invertible linear maps from V to itself is denoted as $GL(V)$. A group homomorphism $[\cdot] : G \to GL(V)$ is called **representation**.

Having a representation is equivalent to taking V as a $\mathbb{C}[G]$-module (i.e. algebraic representation of $\mathbb{C}[G]$). We will call representation V of G as G**-module**.

We say a representation is **irreducible** if there is no proper, non-trivial (i.e. $[\cdot]$ is not a zero map) subrepresentation. Note that we can make new representations by direct summing two representations. With this definition and observation, we can state the following fundamental theorem for finite group representations:

Theorem 1.5.1. *(Maschke's theorem)*

If V *is a representation of a finite group* G*, then*

1. V *can be written as the direct sum of irreducible representations of* G.
2. *The* ***regular representation*** *of* G, $\mathbb{C}[G]$, *is isomorphic to* $\oplus_i V_i^{\oplus \dim V_i}$, *where* V_i *are all irreducible representations of* G.

Lastly, given a representation V of group G, we can also make a representation for a subgroup $H \leq G$ by simply restricting the $[\cdot]$ map to H. We denote this by $\mathrm{Res}^H_G V$. We call this **restricted representation**. This can be done in reverse in a natural way: given a representation $\rho : V \to H$, we define **induced representation** for a group $G \geq H$ as:

$$\mathrm{Ind}^G_H V = \{f : G \to V : f(hg) = \rho(h)f(g) \text{ for all } g \in G, h \in H\}$$

with action $(g(f))(h) := f(hg)$.

1.5.2 Characters

Given a choice of basis B on V, $[g]$ can be written in a matrix form. Define the **character** of a representation to be $\chi : G \to \mathbb{C}$ such that $\chi(g) = \mathrm{tr}[g]$. Since the trace is independent of choice of basis, $\chi(g)$ is well-defined.

Characters are very crucial when one studies representations. Here are some important facts:

1. Two representations of a group G are isomorphic if and only if their characters are equal.

2. The character of the direct sum of the two representations of G is the sum of the characters of the two representations.

3. The character has the same value for conjugate elements. For symmetric groups, this means that permutations with same cycle structure has the same character value.

1.5.3 Specht Modules

We are interested in studying representations of the symmetric group. Specht Modules are the irreducible representations of the symmetric group.

A **tabloid** of shape $\lambda \vdash n$ is a filling of the Ferrers diagram of λ with $1, 2, \cdots, n$ where the order of the entries in the same row does not matter (e.g. $\begin{smallmatrix}\overline{12}\\ \overline{3}\end{smallmatrix} = \begin{smallmatrix}\overline{21}\\ \overline{3}\end{smallmatrix}$). Tabloids are drawn without vertical lines to show this equivalence.

The symmetric group S_n acts on tabloids of shape $\lambda \vdash n$ by simply permuting the entries (e.g. $(123)\begin{smallmatrix}\overline{135}\\ \overline{24}\end{smallmatrix} = \begin{smallmatrix}\overline{215}\\ \overline{34}\end{smallmatrix}$). This action extends to the vector space whose basis is the set of tabloids of shape $\lambda \vdash n$, and thus it is a representation of S_n. Let's call this vector space M^λ.

Given $\lambda \vdash n$, define

$$e_T := \sum_{\pi} \mathrm{sgn}(\pi)\pi(T)$$

where T is a filling of λ and the sum is over permutations that fix the columns of T. e_T lies in M^λ.

The action of π on e_T results in $e_{\pi T}$. We now define S^λ to be a subspace of M^λ spanned by e_T. This is a representation of S_n and we will call it **Specht Module**.

This explicit construction of representations are all possible irreducible representations of S_n. We list some facts regarding Specht module. For proofs of them, see the references listed at the beginning of this section.

Lemma 1.5.2.

$$\dim(S^\lambda) = f^\lambda.$$

The previously proven (in the RSK section) theorem again follows from this lemma and the second part of the Maschke's theorem:

Corollary 1.5.3.

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!.$$

Finally, we state the branching rule, which tells us what happens when we restrict or induce a Specht module to a smaller or bigger symmetric group.

Theorem 1.5.4. *(Branching Rule) Let $\lambda \vdash n$ then*

$$Res_{S_n}^{S_{n-1}} S^\lambda \cong \oplus_\mu S^\mu$$

where the sum is over $\mu \vdash n-1$ obtained by removing one box from λ and

$$Ind_{S_n}^{S_{n+1}} S^\lambda \cong \oplus_\mu S^\mu$$

where the sum is over $\mu \vdash n+1$ obtained by adding one box to λ.

1.6 Cyclic Sieving Phenomenon

1.6.1 Definition of CSP

Suppose that we have a cyclic group C, generated by c, acting on a set X. In combinatorics, it is natural to ask the number of fixed points: $|X^g| = |\{x \in X : gx = x\}|$. In their 2004 paper, Reiner, Stanton, White describe a way to encode all these data in one polynomial [26]. This is called cyclic sieving phenomenon.

Definition 1.6.1. *Let $C = \{1, c, c^2, \cdots, c^{n-1}\}$ be a finite cyclic group acting on a finite set X. Let $\zeta = e^{\frac{2\pi i}{n}} \in \mathbb{C}$ be the root of unity of order n and let $f(q)$ be a polynomial with rational coefficients. We say that the triple $(X, C, f(q))$* ***exhibits the cyclic sieving phenomenon (CSP)*** *if for any nonnegative integer d, we have that the fixed point set cardinality $|X^{c^d}|$ is equal to the polynomial evaluation $f(\zeta^d)$.*

We closely follow this introduction article [14], which is based on Sagan's survey paper [17].

It is easy to see that $f(q)$ is unique up to the cyclotomic polynomial $\Phi_n(q)$. If $(X, C, f_1(q))$ and $(X, C, f_2(q))$ exhibit the CSP, then $f_1(q) - f_2(q)$ is a polynomial that has $1, \zeta, \zeta^2, \cdots, \zeta^{n-1}$ as roots, so $\Phi_n(q)$ divides $f_1(q) - f_2(q)$.

If $f(q) = \sum_{k=0}^{n-1} a_k q^k$ where a_k is the number of C-orbits in X with stabilizer order dividing k, then

$$f(q) = \sum_{k=0}^{n-1} a_k q^k = \sum_{\text{orbit } O} 1 + q^{|C|/|O|} + q^{2|C|/|O|} + \cdots + q^{(|O|-1)|C|/|O|}.$$

If $f(q)$ is evaluated at $q = \zeta^d$, $1 + q^{|C|/|O|} + q^{2|C|/|O|} + \cdots + q^{(|O|-1)|C|/|O|}$ is $|O|$ if the order of ζ^d divides $|C|/|O|$ and 0 otherwise. Therefore, $f(\zeta^d) = |X^{g^d}|$ and $(X, C, f(q))$ exhibits the CSP.

Note that in many situations, these polynomials are naturally associated to the combinatoric structure. It is often a generating function associated to X.

1.6.2 q-analogs

A q-analog is an identity in the variable q that gives back a familiar identity in the limit as q approaches 1. Define the **q-analog of the number** n as $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. Let $[n]_q! = [1]_q[2]_q \cdots [n]_q$ be the **q-analog of** $n!$ and $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ be the **q-analog of** $\binom{n}{k}$. These q-analogs are all polynomials in q and are our ordinary numbers, factorials, and binomial coefficients as q approaches 1.

The q-binomial coefficient adds extra information on each object the binomial coefficient is counting, making it a generating function. If $\binom{n}{k}$ counts the number of monotonic lattice path inside $(n-k) \times k$ box, then its q-analog also takes the number of boxes that are above the path into account.

We note that $f(1) = |X|$ when $(X, C, f(q))$ exhibits the CSP. In many cases, f turns out to be the q-analog of the number of elements in X.

1.6.3 Canonical example of the multisets

The canonical example of the CSP is of the multisets. Let positive integers n and k be fixed. A k-multiset of $[n] := \{1, 2, \cdots, n\}$ is an unordered family of k elements of $[n]$ where repetitions are allowed. For example, the set of 3-multisets on [3] is $\{111, 222, 333, 112, 113, 221, 223, 331, 332, 123\}$. Let X be the set of k-multisets on $[n]$.

Let C be a cyclic subgroup of S_n that is generated by the cycle $c = (1, 2, \cdots, n)$. C acts naturally on X: if $M = m_1 m_2 \cdots m_k$, then $gM = g(m_1)g(m_2)\cdots g(m_k)$ where $g \in C$. For example, $(1, 2, 3)223 = 331$.

It is well known that there are $\binom{n+k-1}{k}$ such multisets. For our polynomial, we take the q-analog. Define $f(q)$ as $\binom{n+k-1}{k}_q$.

Theorem 1.6.2. *The triple $(X, C, f(q))$ defined as above exhibits the cyclic sieving phenomenon.*

Example 1.6.3. *Let $n = 3$ and $k = 3$. $f(q)$ is*

$$\binom{3+3-1}{3}_q = \frac{(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}{1(1+q)} = 1+q+2q^2+2q^3+2q^4+q^5+q^6.$$

Clearly, $|X^{id}| = |X|$ is 10. Since $f(\zeta^0) = f(1) = 10$, $|X^{id}| = f(\zeta^0)$.

Only 123 is in the fixed set, so $|X^{(1,2,3)}|$ is 1. The value of $f(\zeta^1)$ is $1 + \zeta + 2\zeta^2 + 2\zeta^3 + 2\zeta^4 + \zeta^5 + \zeta^6 = 4 + 3\zeta + 3\zeta^2 = 1$ since $\zeta^3 = 1$ and $1 + \zeta + \zeta^2 = 0$. Therefore, $|X^{(1,2,3)}| = f(\zeta)$.

1.6.4 Proof by direct evaluation

One can prove the above theorem by simply evaluating both $|X^{c^d}|$ and $f(\zeta^d)$ explicitly.

Lemma 1.6.4. *Let o be the order of c^d. Then $|X^{c^d}|$ is $\binom{n/o+k/o-1}{k/o}$ if $o|k$ and 0 otherwise.*

Proof. If $g = c_1 c_2 \cdots c_t \in C$ is the cycle decomposition of $g \in C$, then $x \in X$ is fixed under the action by g if and only if x can be written as disjoint union of the cycles c_i with repetition allowed. For example, if $g = (1, 4)(2, 3, 5)$, then the multisets fixed by g have the form $1^a 4^a 2^b 3^b 5^b$. This claim is easy to check.

The permutation c^d decomposes into n/o cycles of size o, so no k-multiset is fixed if o does not divide k. If it does, then one must choose k/o cycles with repetition allowed to form a multiset of size k that is fixed under the action by c^d. Therefore, $|X^{c^d}| = \binom{n/o+k/o-1}{k/o}$. □

Lemma 1.6.5. *Let o be the order of ζ^d. Then $f(\zeta^d)$ is $\binom{n/o+k/o-1}{k/o}$ if $o|k$ and 0 otherwise.*

Proof. Note that $\frac{[ao+k]_q}{[bo+k]_q}$ is

$$\frac{1+q+q^2+\cdots+q^{ao+k-1}}{1+q+q^2+\cdots+q^{bo+k-1}}.$$

When $q=\zeta^d$, this is equal to

$$\frac{1+q^o+q^{2o}+\cdots+q^{(a-1)o}}{1+q^o+q^{2o}+\cdots+q^{(b-1)o}}=\frac{a}{b}$$

if $k=0$. Otherwise, it is

$$\frac{1+q+q^2+\cdots+q^{k-1}}{1+q+q^2+\cdots+q^{k-1}}=1.$$

If o does not divide k, then $f(\zeta^d)=\binom{n+k-1}{k}_{\zeta^d}$ has more zeros in the numerator then the denominator, so $f(\zeta^d)=0$. Otherwise,

$$\begin{aligned} f(\zeta^d)=\binom{n+k-1}{k}_{\zeta^d} &= \frac{[n]_q}{[k]_q}\frac{[n+1]_q}{[1]_q}\cdots\frac{[n+k-1]_q}{[k-1]_q} \\ &= \frac{n}{k}\cdot 1\cdot 1\cdots 1\cdot\frac{n+o}{o}\cdot 1\cdot 1\cdots 1\cdot\frac{n+2o}{2o}\cdots \\ &= \frac{n/o}{k/o}\cdot\frac{n/o+1}{1}\cdot\frac{n/o+2}{2}\cdots \\ &= \binom{n/o+k/o-1}{k/o} \end{aligned}$$

which is what we wanted. □

By these two lemmas, we conclude that $|X^{c^d}|=f(\zeta^d)$. Therefore, $(X,C,f(q))$ exhibits the cyclic sieving phenomenon.

1.6.5 Proof by representation theory

The previous proof is elementary, but it does not tell us much about why the equality holds. We now present an another proof that uses representation theory and it will provide more insight to our situation.

Given a set $S=\{s_1,s_2,\cdots,s_n\}$, we define a complex vector space $\mathbb{C}S$ as $\{c_1s_1+c_2s_2+\cdots+c_ns_n|c_i\in\mathbb{C}\}$. $g\in S_n$ acts naturally on $\mathbb{C}[n]$: $g(c_1\mathbf{1}+c_2\mathbf{2}+\cdots+c_n\mathbf{n})=c_1g(\mathbf{1})+c_2g(\mathbf{2})+\cdots+c_ng(\mathbf{n})$. Define $\mathrm{Sym}_k(n)$ as $\mathbb{C}X$ where X is the set of multisets of $[n]$ of size k.

Clearly, X is a standard basis of $\mathrm{Sym}_k(n)$. The diagonal entry of $[g]_X$ is going to be 1 if multiset $M\in X$ is fixed by g and 0 otherwise. Therefore, $\chi(g^d)=\mathrm{tr}[g^d]_X=|X^{g^d}|$.

Example 1.6.6. *If $n = 3$ and $k = 3$ as before and $g = (1, 2, 3)$, then*

$$\begin{aligned} &g(111) = 222, \quad g(222) = 333, \quad g(333) = 111, \quad g(112) = 223, \quad g(113) = 221, \\ &g(221) = 332, \quad g(223) = 331, \quad g(331) = 112, \quad g(332) = 113, \quad g(123) = 123. \end{aligned}$$

So $[g]_{\{111,222,333,112,113,221,223,331,332,123\}}$ is equal to

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Only one diagonal entry is 1 because only 123 is fixed under the action g. Therefore, $\chi(g) = 1$.

We want another way of evaluating this character $\chi(g)$, so that it yields $f(\zeta^d)$.

Let $c = (1, 2, 3, \cdots, n) \in S_n$. The characteristic polynomial of c is $x^n - 1$, which has n distinct roots: $x_1 = 1, x_2 = \zeta, x_3 = \zeta^2, \cdots, x_n = \zeta^{n-1}$. So there must be a basis $B = \{b_1, b_2, \cdots, b_n\}$ of $\mathbb{C}[n]$ such that the representation of c in $GL(\mathbb{C}[n])$ is diagonalized to $\operatorname{diag}(x_1, x_2, \cdots, x_n)$ by B. $[c^d]_B$ is $\operatorname{diag}(x_1^d, x_2^d, \cdots, x_n^d)$. Let B' be the set of k-multisets of B, then B' is another basis for $\operatorname{Sym}_k(n)$.

We now evaluate the character of $g = c^d$ with this basis. We have

$$g(b_{i_1} b_{i_2} \cdots b_{i_k}) = g(b_{i_1}) g(b_{i_2}) \cdots g(b_{i_k}) = x_{i_1}^d x_{i_2}^d \cdots x_{i_k}^d b_{i_1} b_{i_2} \cdots b_{i_k}.$$

So it follows that the diagonal entries of $[g]_{B'}$ is $x_{i_1}^d x_{i_2}^d \cdots x_{i_k}^d$ and the trace of $[g]$ is $\sum_{1 \le i_1 \le \cdots \le i_k \le n} x_{i_1}^d x_{i_2}^d \cdots x_{i_k}^d$. This polynomial is called **complete homogeneous symmetric polynomial** and is denoted $h_k(x_1^d, x_2^d, \cdots, x_n^d)$.

Example 1.6.7. *Again picking $n = 3$, $k = 3$, and $g = (1, 2, 3)$,*

$$\begin{array}{rrr} g(b_1b_1b_1) = x_1^3 b_1b_1b_1, & g(b_2b_2b_2) = x_2^3 b_2b_2b_2, & g(b_3b_3b_3) = x_3^3 b_3b_3b_3, \\ g(b_1b_1b_2) = x_1^2x_2 b_1b_1b_2, & g(b_1b_1b_3) = x_1^2x_3 b_1b_1b_3, & g(b_2b_2b_1) = x_2^2x_1 b_2b_2b_1, \\ g(b_2b_2b_3) = x_2^2x_3 b_2b_2b_3, & g(b_3b_3b_1) = x_3^2x_1 b_3b_3b_1, & g(b_3b_3b_2) = x_3^2x_2 b_3b_3b_2, \\ g(b_1b_2b_3) = x_1x_2x_3 b_1b_2b_3. & & \end{array}$$

So $[g]_{\{b_1b_1b_1, b_2b_2b_2, b_3b_3b_3, b_1b_1b_2, b_1b_1b_3, b_2b_2b_1, b_2b_2b_3, b_3b_3b_1, b_3b_3b_2, b_1b_2b_3\}}$ *is equal to*

$$\begin{bmatrix} x_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1^2x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1^2x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2^2x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2^2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3^2x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3^2x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1x_2x_3 \end{bmatrix}.$$

Therefore, $\chi(g) = x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_1x_2x_3 = h_3(x_1, x_2, x_3)$.

It remains to show that $h_k(1, q, q^2, \cdots, q^{n-1})$ is equal to $\binom{n+k-1}{k}_q$. One can check that $h_k(1, q, \cdots, q^{n-1}) = h_k(1, q, \cdots, q^{n-2}) + q^{n-1}h_{k-1}(1, q, \cdots, q^{n-1})$ and $\binom{n+k-1}{k}_q = \binom{n+k-2}{k}_q + q^{n-1}\binom{n+k-2}{k-1}_q$. Since the recursions are identical, the equality can be proved using induction.

1.7 Outline

The rest of the thesis is organized as follows: in chapter 2, we look at Rhoades' original proof of CSP of promotion. We first develop a bit of theory of Hecke algebra and its Kazhdan-Lusztig basis. This theory will allow us an alternative way to construct all irreducible representations of S_n. Specifically, the irreducible representation of S_n of shape $\square$ will be generated by basis elements that are labeled with standard tableaux of shape $\square$. In this set-up, the promotion operator is simply representation of the long cycle.

In chapter 3, we discuss the second proof of CSP of promotion by Purbhoo. We review some basics of Schubert calculus and then introduce the Wroskian map: a map from the Grassmanian $\mathrm{Gr}(d, \mathbb{C}_{n-1}[z])$ to $\mathbb{P}(\mathbb{C}_{n-1}(z))$. The Wronskian map has a finite number of points in its fibre, each of which can be labeled by tableaux of shape $\square$ by identifying its degeneration to Richardson varieties. The rotation of the roots of $p(z) \in \mathbb{P}(\mathbb{C}_{n-1}(z))$ then corresponds to the promotion of the points of the fibre, proving that the number of tableaux fixed by j^k is the number of points in the fibre of the generic polynomial $(x^{n/k} + a_1)(x^{n/k} + a_2) \cdots (x^{n/k} + a_k)$ fixed by k rotations of the roots, which can be alternatively counted as the number of ribbon tableaux of shape $\square$.

In chapter 4, we consider the result of Petersen, Pylyavskyy, and Rhoades, where the promotion of rectangular tableaux with 2 or 3 rows is visualized as rotation of combinatorial objects called webs. The bijection from tableaux to webs is quite elementary, but it can also be formulated algebraically as invariant tensor of the special linear group. The vector space generated by these webs is called web space. We realize these web spaces as irreducible module of S_n, then the rotation is just the representation of the long cycle.

In chapter 5, we give a new approach to this problem. We show a bijection between square tableaux fixed by j^n to permutation of size n where promotion corresponds to the long cycle. This bijection is a simple modification of standard jeu-de-taquin moves. We also discuss how this could be generalized to prove CSP for square tableaux.

Chapter 2

Proof of Cyclic Sieving Using Kahzdan-Lusztig Theory

We first look at Rhoades' proof of CSP of promotion on rectangular tableaux. Rhoades' proof uses Kahzdan-Lusztig theory, which yields a different construction of Specht module, as seen in [6]. We start with a brief introduction of Kahzdan-Lusztig basis.

2.1 Hecke Algebra and Kahzdan-Lusztig Basis

The Hecke algebra is a one parameter deformation of the group algebra $\mathbb{C}[S_n]$.

Definition 2.1.1. *Define the **Hecke Algebra** $H_n(q)$ over the field of Laurent polynomials $\mathbb{C}(q^{1/2})$ as follows:*

1. *The basis elements are T_σ for all $\sigma \in S_n$.*
2. *$T_\sigma T_\tau$ is $T_{\sigma\tau}$ if $l(\sigma) + l(\tau) = l(\sigma\tau)$.*
3. *$(T_{s_i} + 1)(T_{s_i} - q) = 0$ for all i.*

Note that specializing $q = 1$ yields the group algebra $\mathbb{C}[S_n]$.

From the third property, we have $((T_{s_i} - (1-q))/q) \cdot T_{s_i} = 1$. Thus T_{s_i} is invertible. Since all $\sigma \in S_n$ can be written as product of s_i's in a reduced word, T_σ is invertible for all $\sigma \in S_n$.

One can define an involution D on the Hecke algebra by letting $D(q^{1/2}) = q^{-1/2}$, $D(T_\sigma) = T_{\sigma^{-1}}^{-1}$.

Other than this standard basis $\{T_\sigma\}_{\sigma \in S_n}$, we also have Kahzdan-Lusztig basis, defined from the following theorem:

Theorem 2.1.2. *There exists an unique basis*

$$\left\{ C'_\sigma(q) = (q^{-l(\sigma)/2}) \sum_{\tau \in S_n} P_{\sigma,\tau}(q) T_\nu \mid \sigma \in S_n \right\}$$

of $H_n(q)$ with the following properties:

1. *C'_σ is invariant under D.*
2. *$P_{\sigma,\tau}$ is a polynomial in q with integer coefficient.*
3. *$P_{\sigma,\sigma} = 1$.*
4. *$P_{\sigma,\tau} = 0$ if $\sigma \not\leq \tau$.*
5. *The degree of $P_{\sigma,\tau}$ is at most $(l(\tau) - l(\sigma) - 1)/2$.*

The basis $\{C'_\sigma\}_{\sigma \in S_n}$ is called **Kazhdan-Lusztig basis** and $P_{\sigma,\tau}$ are called **Kazhdan-Lusztig polynomials**.

By specializing to $q = 1$, we get a new basis for $\mathbb{C}[S_n]$. To simplify some calculations, we throw in some signs.

$$C'_\sigma(1) = \sum_{\tau \in S_n} (-1)^{l(\tau)-l(\sigma)} P_{\sigma,\tau}(1) T_\nu.$$

2.2 μ Function

To help with calculations involving KL polynomials, we introduce the μ function.

Definition 2.2.1. *The function $\mu(\sigma, \tau) = [q^{(l(\tau)-l(\sigma)-1)/2}] P_{\sigma,\tau}(q)$ is a statistic on a pair of permutation. This is the coefficient of the maximal allowed degree term.*

Note that $\mu(\sigma, \tau) = 0$ unless $\sigma \leq \tau$ and $l(\tau) - l(\sigma)$ is odd.

KL μ function is important as it gives us a way to compute the KL polynomials recursively: If $\sigma \leq \tau$ and $s_i\tau < \tau$ then

$$P_{\sigma,\tau}(q) = q^{1-c} P_{s_i\sigma, s_i\tau} + q^c P_{\sigma, s_i\tau} - \sum_{s_i\nu < \nu} q^{(l(\tau)-l(\nu))/2} \mu(\nu, s_i\tau) P_{\sigma,\nu}(q)$$

where $c = 1$ if $s_i\sigma < \sigma$ and $c = 0$ otherwise. Note that the second index of the KL polynomials on the right hand side are all smaller than τ in Bruhat order.

The μ function behaves nicely with the multiplication by long word $\omega_0 \in S_n$. Below is a lemma that will be useful later:

Lemma 2.2.2. *For any* $\sigma, \tau \in S_n$,

$$\mu(\sigma, \tau) = \mu(\omega_0\sigma, \omega_0\tau) = \mu(\sigma\omega_0, \tau\omega_0) = \mu(\omega_0\sigma\omega_0, \omega_0\tau\omega_0).$$

Finally, we define a symmetrized version of μ: $\mu[\sigma, \tau] := \max(\mu(\sigma, \tau), \mu(\tau, \sigma))$.

This symmetric version only depends on one half of RSK, if one fixes the other half. In other words, if $S_1, T_1, S_2, T_2 \in \mathrm{SYT}(\lambda)$ then

$$\mu[\mathrm{RSK}(S_1, T_1), \mathrm{RSK}(S_1, T_2)] = \mu[\mathrm{RSK}(S_2, T_1), \mathrm{RSK}(S_2, T_2)] \text{ and}$$

$$\mu[\mathrm{RSK}(S_1, T_1), \mathrm{RSK}(S_2, T_1)] = \mu[\mathrm{RSK}(S_1, T_2), \mathrm{RSK}(S_2, T_2)].$$

Given this fact, we can now define μ function on a pair of standard Young tableaux with same shape.

Definition 2.2.3. *Given standard Young tableaux* S, T *of the same shape,* $\mu[S, T]$ *is defined as* $\mu[\mathrm{RSK}(S, U), \mathrm{RSK}(T, U)]$ *for any standard Young tableau* U *of the same shape.*

2.3 Kahzdan-Lusztig Representation

We now move on to construct representations of S_n using this basis.

Let $\lambda \vdash n$ be a partition, T be a standard Young tableau of shape λ.

We have a left action by S_n on

$$S_0^{\lambda,T} := \bigoplus_{\sigma} \mathbb{C}\{C'_\sigma(1)\}$$

where the sum is over permutations σ such that the shape of σ dominates the shape of λ or $P(\sigma)$ (first half of RSK) is T.

This action is invariant on a submodule

$$S_1^{\lambda,T} := \bigoplus_{\sigma} \mathbb{C}\{C'_\sigma(1)\}$$

where the sum is over permutations σ such that the shape of σ dominates the shape of λ.

Define $S^{\lambda,T}$ as $S_0^{\lambda,T}/S_1^{\lambda,T}$. We have a basis identified with $\mathrm{SYT}(\lambda)$: for each $S \in \mathrm{SYT}(\lambda)$, take the image of $C'_{\mathrm{RSK}(T,S)}(1)$.

It turns out that for $T, S \in \mathrm{SYT}(\lambda)$, $S^{\lambda,T}$ and $S^{\lambda,S}$ are isomorphic up to some reordering of the basis and Coxeter generators s_i acts identically on both modules. In particular, the action is given by the following formula:

Lemma 2.3.1. *If* $T \in \mathrm{SYT}(\lambda)$,

$$s_i T = \begin{cases} -T & \text{if } i \in D(T) \\ T + \sum_{i \in D(S)} \mu[T,S]S & \text{if } i \notin D(T) \end{cases}$$

where $D(T)$ *is the descent set of* T.

We define the $\mathbb{C}[S_n]$ module S^λ as $S^{\lambda,T}$ for any $T \in \mathrm{SYT}(\lambda)$. This is isomorphic to the Specht module indexed by λ.

2.4 μ Function and Tableau Operators

In this second half of the chapter, we now move forward to the proof of CSP. We will closely follow Rhoades' paper for this proof [15].

From here, c_n denotes long cycle $(1\ 2\ 3\ \ldots\ n) \in S_n$. $\square := b^a \vdash n$, $\square^{\lrcorner} := b^{a-1}(b-1) \vdash n-1$.

In this section, we will look at how tableau operators nicely interact with the μ function and ultimately understand how promotion interacts with it.

Definition 2.4.1. *The* ***deletion operator*** $d : \mathrm{SYT}(\square) \to \mathrm{SYT}(\square^{\lrcorner})$ *deletes the bottom leftmost square from the rectangular standard tableau.*

The ***creation operator*** $c : \mathrm{SYT}(\square^{\lrcorner}) \to \mathrm{SYT}(\square)$ *adds a cell with entry* N *at the missing corner.*

Note that promotion can be written as *eced* for rectangular tableaux, where e is the evacuation operator.

We will now show that μ function interacts well with these operators.

Lemma 2.4.2. *Let* $S, T \in \mathrm{SYT}(\square)$*, then* $\mu[S,T] = \mu[d(S), d(T)]$

Proof. Let the column superstandard tableau $\mathrm{CSS}(\lambda)$ be defined as a standard tableau obtained by filling top to bottom and from left to right (e.g. $\begin{smallmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & \end{smallmatrix}$). Let $\sigma = \mathrm{RSK}(d(S), \mathrm{CSS}(\square^{\lrcorner}))$ and $\tau = \mathrm{RSK}(d(T), \mathrm{CSS}(\square^{\lrcorner}))$. Then

$$\mathrm{RSK}(S, \mathrm{CSS}(\square)) = \sigma_1 \sigma_2 \cdots \sigma_{n-a} n \sigma_{n-a+1} \cdots \sigma_{n-1} \text{ and}$$

$$\mathrm{RSK}(T, \mathrm{CSS}(\square)) = \tau_1 \tau_2 \cdots \tau_{n-a} n \tau_{n-a+1} \cdots \tau_{n-1}.$$

Brenti's result tells us that we can append n at the end of the permutation and get the same KL polynomial, that is $P_{\sigma,\tau} = P_{\sigma n, \tau n}$ where the permutations are in one line notation. We can get to $\mathrm{RSK}(S, \mathrm{CSS}(\square))$ from σn by using transpositions $s_{n-1}, s_{n-2}, \cdots, s_{n-a}$ and the same holds for τ.

Let $\sigma^{(k)}$ be $s_{n-k}\cdots s_{n-1}(\sigma n)$ and $\tau^{(k)}$ be $s_{n-k}\cdots s_{n-1}(\tau n)$. It is sufficient to show that $P_{\sigma^{(k)},\tau^{(k)}} = P_{\sigma^{(k+1)},\tau^{(k+1)}}$ for $k = 0, 1, \cdots, a-1$, since this implies $P_{\sigma,\tau} = P_{\sigma n,\tau n} = P_{\sigma^{(0)},\tau^{(0)}} = P_{\sigma^{(a-1)},\tau^{(a-1)}} = P_{\mathrm{RSK}(S,\mathrm{CSS}(\Box)),\mathrm{RSK}(T,\mathrm{CSS}(\Box))}$.

We use the recursive formula of KL polynomial with $c = 1$.

$$
\begin{aligned}
&P_{\sigma^{(k+1)},\tau^{(k+1)}}(q) = \\
&P_{\sigma^{(k)},\tau^{(k)}}(q) + qP_{\sigma^{(k+1)},\tau^{(k)}}(q) - \sum_{s_k\nu<\nu} q^{(l(\tau^{(k+1)})-l(\nu))/2}\mu(\nu,\tau^{(k)})P_{\sigma^{(k+1)},\nu}(q).
\end{aligned}
$$

We observe that $\sigma^{(k+1)} \not\geq \tau^{(k)}$ as $\sigma^{(k+1)}$ maps $n-k$ to n but $\tau^{(k)}$ maps $n-k+1$ to n. A non-zero term in the sum must have $\sigma^{(k+1)} < \nu \leq \tau^{(k)}$, which means $\nu = \tau^{(k)}$. Therefore, only $qP_{\sigma^{(k+1)},\tau^{(k)}}(q)$ term survives in the sum on the right hand side and this shows that $P_{\sigma^{(k+1)},\tau^{(k+1)}}(q) = P_{\sigma^{(k)},\tau^{(k)}}(q)$. □

As a direct corollary, we get $\mu[S,T] = \mu[c(S), c(T)]$ for any $S, T \in \mathrm{SYT}(\Box)$. We can now show that μ is also invariant under evacuation and promotion.

Lemma 2.4.3. *Let $S, T \in \mathrm{SYT}(\lambda)$, then $\mu[S,T] = \mu[e(S), e(T)]$.*

Proof. For any $U \in \mathrm{SYT}(\lambda)$, we have

$$
\begin{aligned}
\mu(S,T) &= \mu(\mathrm{RSK}(S,U), \mathrm{RSK}(T,U)) \\
&= \mu(\omega_0 \cdot \mathrm{RSK}(S,U)\cdot \omega_0, \omega_0 \cdot \mathrm{RSK}(T,U)\cdot\omega_0) \\
&= \mu(\mathrm{RSK}(e(S), e(U)), \mathrm{RSK}(e(T), e(U))) \\
&= \mu(e(S), e(T)).
\end{aligned}
$$

□

Theorem 2.4.4. *Let $S, T \in \mathrm{SYT}(\Box)$, then $\mu[S,T] = \mu[j(S), j(T)]$.*

Proof. By the previous two lemmas, $\mu[j(S), j(T)] = \mu[eced(S), eced(T)] = \mu[S,T]$. □

2.5 Promotion in Kahzdan-Lusztig Representation

Before we go on to the proof of CSP, we will need the next technical lemma.

Lemma 2.5.1. *Let $c_n = (1\ 2\ \cdots\ n) \in S_n$ be the long cycle. Recall that $C'_\omega(1) = \sum_{\nu\in S_n}(-1)^{l(\omega)-l(\mu)}P_{\nu,\omega}(1)\nu$ is the specialization of KL basis for $\mathbb{C}[S_n]$. The coefficient of $C'_{\mathrm{RSK}(j(\mathrm{CSS}(\Box)),\mathrm{CSS}(\Box))}(1)$ in the expansion of $C'_{\mathrm{RSK}(\mathrm{CSS}(\Box)),\mathrm{CSS}(\Box))}(1)\cdot c_n$ is $(-1)^{a-1}$.*

Proof. Denote RSK(CSS($\square$), CSS($\square$)) by σ and RSK(j(CSS($\square$)), CSS($\square$)) by τ. Then σ and τ in one-line notation is:

$$\begin{aligned}\sigma =& a(a-1)(a-2)\ldots 1\\ &(2a)(2a-1)(2a-2)\ldots(a+1)\\ &\ldots\\ &(ba)(ba-1)(ba-2)\ldots((b-1)a+1)\end{aligned}$$

$$\begin{aligned}\tau =&(a+1)a(a-1)(a-2)\ldots 31\\ &(2a+1)(2a)(2a-1)(2a-2)\ldots(a+3)2\\ &\ldots\\ &(ba)(ba-1)\ldots((b-1)a+3)((b-1)a+2)((b-2)a+2).\end{aligned}$$

Both of them are 3412, 4231 pattern avoiding, so by smoothness, $P_{\nu,\sigma}(q)=1$ for $\nu\in S_{a^b}$. Thus, $C'_\sigma(1)$ is

$$\sum_{\nu\in S_{a^b}}(-1)^{l(\sigma)-l(\nu)}\nu$$

and $C'_\sigma(1)c_n$ is

$$\sum_{\nu\in S_{a^b}}(-1)^{l(\sigma)-l(\nu)}\nu c_n.$$

The Bruhat maximal element $\nu_0 c_n$ is

$$\begin{aligned}&(a+1)a(a-1)(a-2)\ldots 2\\ &(2a+1)(2a)(2a-1)\ldots(a+2)\\ &\ldots\\ &(ba)(ba-1)(ba-2)\ldots((b-1)a+2)1,\end{aligned}$$

so ν_0 is

$$\begin{aligned}&a(a-1)(a-2)\ldots 1\\ &(2a)(2a-1)(2a-2)\ldots(a+1)\\ &\ldots\\ &(ba-1)(ba-2)\ldots((b-1)a+1)(ba).\end{aligned}$$

We observe that ν_0 is one a-cycle away from σ, so

$$C'_\sigma(1)c_n = \sum_{\nu \leq \nu_0 c_n} a_\nu C'_\nu(1)$$

for some $a_\nu \in \mathbb{C}$ and $a_{\nu_0 c_n} = (-1)^{a-1}$. The value we want is a_τ.

Note that the permutations between τ and $\nu_0 c_n$ in the Bruhat order must be of the form

$$\begin{aligned}
&(a+1)a(a-1)(a-2)\ldots 3x_1\\
&(2a+1)(2a)(2a-1)\ldots(a+3)x_2\\
&\ldots\\
&(ba)(ba-1)(ba-2)\ldots((b-1)a+2)x_b,
\end{aligned}$$

where $x_1 \in \{1,2\}$, $x_2 \in \{2, a+2\}$, $x_{b-1} \in \{(b-3)a+2, (b-2)a+2\}$, $x_b \in \{(b-2)a+2, 1\}$. These permutations form a Boolean lattice of rank $b-1$ and they are all 3412, 4231 avoiding. Thus $C'_\nu(1)$ can be written as $\sum_{\omega \leq \nu}(-1)^{l(\nu)-l(\omega)}\omega$. By inclusion-exclusion principle, this is inverted to $\nu_0 c_n = \sum_{\nu \leq \nu_0 c_n} C'_\nu(1)$.

No $\nu \in [\tau, \nu_0 c_n]$ is in $S_{a^b} c_n$ other than $\nu_0 c_n$. Thus, if we consider $C'_\sigma(1)c_n$ in the subalgebra of $\mathbb{C}[S_n]$ where $\nu = 0$ if $\nu \notin [\tau, \nu_0 c_n]$, it must be equal to $(-1)^{a-1}\nu_0 c_n = \sum_{\tau \leq \nu \leq \nu_0 c_n}(-1)^{a-1}C'_\nu(1)$. Therefore, $a_\tau = (-1)^{a-1}$. □

We now show the key theorem. It shows that promotion is the action by long cycle $(1\ 2\ \ldots\ n) \in S_n$ in KL representation up to some sign.

Theorem 2.5.2. *Let $\rho : S_n \to GL(S^{\square})$ be the associated KL representation, with basis identified with* SYT($\square$)*. Define $J : S^{\square} \to S^{\square}$ by extending promotion j $\mathbb{C}$-linearly. Then,*

$$\rho(c_n) = (-1)^{a-1}J.$$

Proof.

We would like to show that the action of S_n commutes with the operator $J^{-1}\rho(c_n)$. Then by Schur's Lemma, $J^{-1}\rho c_n$ is cI for some $c \in \mathbb{C}$. To show this, it is sufficient to show that the Coxeter generators s_i commutes with it.

We approach this problem in two steps: first, we show that s_i commutes with $J^{-1}\rho(c_n)$ for all $i = 1, 2, \ldots, n-2$. Then we will show that this implies that s_{n-1} also commutes.

Clearly, $c_n \cdot s_i = s_{i+1}c_n$, so $J^{-1}\rho(c_n)\rho(s_i) = J^{-1}\rho(s_{i+1})\rho(c_n)$. Thus it is sufficient to show that $J^{-1}\rho(s_{i+1}) = \rho(s_i)J^{-1}$.

For some $T \in \mathrm{SYT}(\square)$, we have

$$\begin{aligned} J^{-1}\rho(s_{i+1})(T) &= \begin{cases} -j^{-1}(T) & \text{if } i+1 \in D(T) \\ j^{-1}(T) + \sum_{i+1 \in D(S)} \mu[T,S] j^{-1}(S) & \text{if } i+1 \notin D(T) \end{cases} \\ &= \begin{cases} -j^{-1}(T) & \text{if } i \in D(j^{-1}(T)) \\ j^{-1}(T) + \sum_{i \in D(j^{-1}S)} \mu[T,S] j^{-1}(S) & \text{if } i \notin D(j^{-1}(T)) \end{cases} \\ &= \begin{cases} -j^{-1}(T) & \text{if } i \in D(j^{-1}(T)) \\ j^{-1}(T) + \sum_{i \in D(j^{-1}S)} \mu[j^{-1}(T), j^{-1}(S)] j^{-1}(S) & \text{if } i \notin D(j^{-1}(T)) \end{cases} \\ &= \rho(s_i) J^{-1}(T). \end{aligned}$$

Since $\square$ has a unique corner, $S^{\square} \downarrow^{S_n}_{S_{N-1}}$ is $S^{\square'}$ by the Branching rule. So our representation is irreducible when restricted to S_{n-1}. Since the operator commutes with S_{n-1}, it also commutes with S_n.

Consider the basis element indexed by $U := \mathrm{CSS}(\square)$ in $S^{\square} \cong S^{\square, U}$, which is $C'_{U,U}(1)$. The action of c_n will yield $C'_{U,U}(1)c_n$ and $J(C'_{U,U}(1))$ is $C'_{j(U),U}(1)$. Therefore, $\rho(c_n) = (-1)^{a-1}J$ by the previous lemma. □

This brings us to the main theorem.

Theorem 2.5.3. $(\mathrm{SYT}(\square), j, f^{\square}(q))$ *exhibits the cyclic sieving phenomenon.*

Proof.

We use the representation theoretic proof to show this CSP. Let $\zeta = e^{2\pi i/n}$. Consider the KL representation $\rho : S_n \to \mathrm{GL}(S^{\square})$.

We turn to the theory of finite reflection groups, subgroups of the group generated by linear transformations that has all but one eigenvalue 1. Springer's result allows evaluation of the character at a regular element, a group element that has an eigenvector that is not fixed by any non-identity element [20]. The character at c_n^k is evaluated to the q-analog of the hook length formula with scalar $\zeta^{kba(a-1)/2} = (-1)^{k(a-1)}$.

Clearly, $c_n^k = ((-1)^{a-1}J)^k$ is a permutation matrix with sign. The character at $((-1)^{a-1}J)^k$ counts the number of fixed tableaux under j^k with sign $(-1)^{k(a-1)}$. Therefore, we have our cyclic sieving phenomenon. □

Chapter 3

Proof of Cyclic Sieving Using Wronskian Map

In this chapter, we follow Purbhoo's proof of the CSP of promotion by solving the inverse Wroskian problem [11] [12]. To count the rectangular standard tableaux fixed by some iteration of promotions, we show a one to one correspondence to the inverse image of a certain point. We can also count the number of points in the inverse image in terms of ribbon tableaux. Then the result follows right away, as the number of ribbon tableaux is well studied [3].

As the focus of the thesis is on promotion, the last two sections on ribbon tableaux will be less detailed. See [11] [12] for the complete proof.

3.1 Schubert Calculus

Before we go on to the main content of the chapter, let us briefly review some Schubert calculus. Specifically, we will define what Schubert variety and Richardson variety are and also look at these variety in terms of Plücker coordinates.

Schubert calculus takes place in Grassmannian,

Definition 3.1.1. *A* ***Grassmannian*** $\mathrm{Gr}(d, V)$ *of a* m*-dim vector space* V *of field* $\mathbb{F}$ *and* $0 \leq d \leq m$ *is a set of* d*-dim subspaces of* V*.*

It is easy to see that Grassmannian is a topological manifold. When $d = 1$, the Grassmanian is just the projective space $\mathbb{P}(V)$.

For this section, we will fix V to have a standard basis $\{e_1, e_2, \cdots, e_m\}$. In this basis, one can represent a point in $\mathrm{Gr}(d, V)$ by a d-by-m matrix of rank d; the d-dim subspace it represents will be the row space of the matrix.

Definition 3.1.2. *The* ***Plücker map*** *is*

$\mathrm{Gr}(d, V) \to \mathbb{P}(\wedge^d(\mathbb{F})) : span(v_1, v_2, \cdots, v_d) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_d.$

This map lets one define Grassmannian as a projective variety.

If we look at this map in terms of the standard basis, the map will send

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{dm} \end{bmatrix} \mapsto \sum_{I \subseteq [m], |I| = d} p_I(A) \wedge_{k=1}^{d} e_{I_k}$$

where $p_I(A)$ is the I-minor of A, $\det(A_I) = \det\left(\begin{bmatrix} a_{1I_1} & a_{1I_2} & \cdots & a_{1I_m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{dI_1} & a_{dI_2} & \cdots & a_{dI_m} \end{bmatrix}\right)$.

Definition 3.1.3.

The ***Plücker coordinates*** *of $x \in \mathrm{Gr}(d, V)$ is the $\binom{m}{d}$-tuple $(p_I)_{I \subseteq [m]}(x)$.*

Example 3.1.4. *Let's denote the set of complex polynomials of degree at most m as $\mathbb{C}_m[z]$. Let $x \in \mathrm{Gr}(2, \mathbb{C}_3(z))$, where x is spanned by $1 + z + z^2$ and $2 + 2x^2 - z^3$. Then x is the row space of*

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & -1 \end{bmatrix}$$

in the standard basis of $\mathbb{C}_3(z)$, $\{1, z, z^2, z^3\}$. Then

$$p_{\{1,2\}} = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2, \quad p_{\{1,3\}} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0, \quad p_{\{1,4\}} = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1$$

$$p_{\{2,3\}} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2, \quad p_{\{2,4\}} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1, \quad p_{\{3,4\}} = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1.$$

Thus, the Plücker coordinates of x is $(-2, 0, -1, 2, -1, -1)$.

As we noted before, this is in the projective space and the coordinates are well defined up to scaling by non-zero complex number.

Another observation is that d-subsets of $[m]$ is in bijection with partitions λ that fits in d-by-$(m-d)$ rectangle (add/subtract $d + 1 - i$ from ith entry). So we can also index the Plücker coordinates with these partitions. e.g. Instead of

$$(p_{\{1,2\}}, p_{\{1,3\}}, p_{\{1,4\}}, p_{\{2,3\}}, p_{\{2,4\}}, p_{\{3,4\}}),$$

one can write

$$(p_{\varnothing}, p_{\square}, p_{\square\square}, p_{\substack{\square\\\square}}, p_{\substack{\square\square\\\square}}, p_{\substack{\square\square\\\square\square}}).$$

We would like to define Schubert variety, which lives inside Grassmanian. In order to do so, we first define what flags are.

Definition 3.1.5. *A **complete flag** of m-dimensional vector space V is a chain of subspaces*

$$F : F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m$$

where F_k is a k-dimensional subspace.

*Given a standard basis $\{e_1, e_2, \cdots, e_m\}$ of V, the **standard flag** $F_\bullet$ is defined as*

$$(F_\bullet)_k := span(e_1, e_2, \cdots, e_k).$$

*The **opposite flag** $\tilde{F}_\bullet$ is defined as*

$$(\tilde{F_\bullet})_k := span(e_m, e_{m-1}, \cdots, e_{m-k+1}).$$

Definition 3.1.6. *Let λ be a partition contained in $\square$. Given a complete flag F of V, the **Schubert variety** is the subset of the Grassmannian $\mathrm{Gr}(d, V)$:*

$$X_\lambda(F) := \left\{ x \in Gr(d, V) : \dim(x \cap F_{i+(m-d)-\lambda_i}) \geq i \right\}.$$

This definition may seem unnatural at a glance. One easy way to visualize the set is to look at row reduced echelon form of the matrix representation. The shape of the leading zeros before ones (flip it upside down to get a partition shape) contains the shape $(\lambda_i + i - 1)_i$ if and only if the subspace is in $X_\lambda(F_\bullet)$.

One can also see Schubert variety in terms of Plücker coordinates.

Theorem 3.1.7. $X_\lambda(F_\bullet) = \{x \in \mathrm{Gr}(d, V) : p_\mu(x) = 0$ *for* $\mu \leq \lambda\}$.

Finally, we define Richardson variety.

Definition 3.1.8. *Given a standard basis of a vector space V, The **Richardson variety** $X_{\lambda/\mu}$ is $X_\lambda(F_\bullet) \cap X_\mu(\tilde{F_\bullet}) \subseteq \mathrm{Gr}(d, V)$.*

From the previous theorem, the following lemma follows.

Lemma 3.1.9. $X_{\lambda/\mu} = \{x \in \mathrm{Gr}(d, V) : p_\nu(x) = 0$ *for* $\nu \notin \Lambda_{\lambda/\mu}\}$, *where* $\Lambda_{\lambda/\mu} = \{\nu : \mu \leq \nu \leq \lambda\}$.

3.2 Wronski Map

We fix $0 < d \leq m$. Let $f_1(z), f_2(z), \cdots, f_d(z) \in \mathbb{C}_{m-1}(z)$.

Definition 3.2.1. *The **Wronskian** of $f_1, f_2, \cdots, f_d \in \mathbb{C}_{m-1}(z)$ is the complex polynomial*

$$\mathrm{Wr}_{f_1,f_2,\cdots,f_d}(z) := \begin{vmatrix} f_1(z) & f_2(z) & \cdots & f_d(z) \\ f_1'(z) & f_x'(z) & \cdots & f_d'(z) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(d-1)}(z) & f_2^{(d-1)} & \cdots & f_d^{(d-1)}(z) \end{vmatrix}.$$

We note some trivial properties of Wronskian:

Remark 3.2.2.

1. $Wr_{f_1, f_2, \cdots, f_d}(z) = 0 \Leftrightarrow f_1, f_2, \cdots, f_d$ *are linearly dependent.*
2. *If span*$(f_1, f_2, \cdots, f_d) =$ *span*$(g_1, g_2, \cdots, g_d)$*, then* $Wr_{f_1, f_2, \cdots, f_d}(z)$ *is equal to* $Wr_{g_1, g_2, \cdots, g_d}(z)$ *up to scaling by a non-zero complex number.*

These observations lead us to consider Wronski map to be defined as a map on a Grassmannian $\mathrm{Gr}(d, \mathbb{C}_{m-1}(z))$ by taking any basis of the d-dimensional subspace and evaluating the Wronskian. The answer we get will be up to a scalar, so it lives in $\mathbb{P}(\mathbb{C}_{m-1}(z))$.

The inverse Wronskian problem asks that given $h(z) \in \mathbb{P}(\mathbb{C}_{m-1}(z))$, find all $x \in \mathrm{Gr}(d, \mathbb{C}_{m-1}(z))$ such that $\mathrm{Wr}(x) = h(z)$. We will denote the set of solutions to this as $X(h(z))$.

The Wronski map can be written in terms of Plücker coordinates.

Lemma 3.2.3. *Let* $x \in \mathrm{Gr}(d, \mathbb{C}_{m-1}(z))$*, then*

$$Wr(x) = \sum_{\lambda \subseteq \square} q_\lambda p_\lambda(x) z^{|\lambda|}$$

where $\square := d^{n-d}$ *and* q_λ *is the Vandermonde determinant:*

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1+\lambda_d & 2+\lambda_{d-1} & 3+\lambda_{d-2} & \cdots & d+\lambda_1 \\ (1+\lambda_d)^2 & (2+\lambda_{d-1})^2 & (3+\lambda_{d-2})^2 & \cdots & (d+\lambda_1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1+\lambda_d)^{d-1} & (2+\lambda_{d-1})^{d-1} & (3+\lambda_{d-2})^{d-1} & \cdots & (d+\lambda_1)^{d-1} \end{vmatrix}$$
$$= \prod_{1 \le i \le j \le d} (j + \lambda_{d+1-j} - i - \lambda_{d+1-i}).$$

Proof. Let A be a d-by-m matrix of $\mathbb{C}$ such that x is the row space of A in the standard basis $1, z, z^2, \cdots, z^{m-1}$. Let B be a d-by-m matrix such that $B_{i,j} = (\frac{d}{dz^{i-1}})z^{j-1}$. Then the Wronskian $\mathrm{Wr}(x)$ is simply equal to $|BA^t|$, which can be calculated with Cauchy-Binet formula. The maximal minors of A are just the Plücker coordinates, $p_\lambda(x)$. The maximal minors of B are $q_\lambda z^{|\lambda|}$. The lemma follows from here. $\square$

Now that we translated everything to Plücker coordinates, we can prove a key theorem:

Theorem 3.2.4. *Every point in* $X(h(z))$ *lies in an unique Richardson variety such that* $|\lambda| = \deg h(z)$ *and* $|\mu| =$ *mindeg* $h(z)$*.*

Proof. For $x \in X(h(z))$, let λ be the unique maximal partition such that $p_\lambda(x) \neq 0$ and μ be the unique minimal partition such that $p_\mu(x) \neq 0$. Because of the Plücker coordinates characterization of the Richardson variety, x is in the Richardson variety $X_{\lambda/\mu}$.

Because of the Plücker coordinates characterization of the Wronskian, we know that coefficients of z^k terms are 0 if $k > |\lambda|$ or $k < |\mu|$. However, the coefficient of $z^{|\lambda|}$ is $q_\lambda p_\lambda(x) \neq 0$ and the coefficient of $z^{|\mu|}$ is $q_\mu p_\mu(x) \neq 0$. Therefore, $|\lambda| = \deg h(z)$ and $|\mu| = \text{mindeg } h(z)$.

If x is in two distinct Richardson variety, then it must be in their intersection. However, this intersection is a proper Richardson subvariety, which is a contradiction. Therefore, x lies in an unique Richardson variety.

3.3 Roots on a Circle

Let $r = \{r_1, r_2, \cdots, r_n\}$ be a multiset where $r_1, r_2, \cdots, r_n \in \mathbb{RP}^1$. We denote the polynomial $(x + r_1)(x + r_2)\cdots(x + r_n)$ by r. In particular, $X((x + r_1)(x + r_2)\cdots(x + r_n))$ is denoted as $X(r)$.

A very important result on Wronski map is the theorem by Mukhin, Tarasov, and Varchenko [2], formally Shapiro-Shapiro conjecture.

Theorem 3.3.1. *(Mukhin, Tarasov, Varchenko)*

If $r_1, r_2, \cdots, r_n$ are real, then $X(r)$ is reduced and is real (i.e. it has a basis with real coefficients).

We will use this theorem to associate a tableau $T_x \in \text{SYT}(\square)$ to a point in the fibre of the Wronski map $x \in X(r)$.

Put a total order on $\mathbb{RP}^1$ as follows: $a \leq b$ if and only if $a = b$ or $|a| < |b|$ or $0 < a = -b < \infty$. When r is a set, we have $r_1 < r_2 < \cdots < r_n$ without loss of generality.

Define $r_k(t) = \{tr_1, tr_2, \cdots, tr_k, r_{k+1}, \cdots, r_n\}$ for $k = 0, 1, 2, \cdots, n$ and $t \in [0, 1]$. If $t \neq 0$, $r_k(t)$ is still a set, so by Mukhin-Tarasov-Varchenko, $X(r_k(t))$ is reduced. Then there is an unique lifting of path $r_k(t)$, $t \in [0, 1]$ to a path $x_k(t) \in X(r_k(t))$.

The polynomial $r_k(0)$ has mindeg k. So $x_k(0) \in X(r_k(0))$ lies in an unique Richardson variety X_{ν_k/λ_k} where $\lambda_k \vdash k$. These λ_k's form a chain: $\varnothing = \lambda_0 \subset \lambda_1 \subset \cdots \subset \lambda_n = \square$. This defines a standard young tableau of shape $\square$. Call this tableau T_x.

By construction, if a path $r(t)$ stays in the same total order for all of $t \in [0, 1]$ and $x(t) \in X(r(t))$ is a lifting of $r(t)$, then $T_{x_0} = T_{x_1}$.

To understand the situation when $r(t)$ does not stay in the same order, we look at the case where two adjacent elements swap their place. Once we can handle this, we can concatenate these "transposes" and get the general behavior.

Theorem 3.3.2. *Let $r(t)$ be a path such that $r_1(t) < r_2(t) < \cdots < r_k(t)$ and $r_{k+1}(t) < r_{k+2}(t) < \cdots < r_N(t)$ for all $t \in [0,1]$. Let $r_k(0) < r_{k+1}(0)$ and $r_k(1) > r_{k+1}(1)$. Let $x(t) \in X(r(t))$ be a lifting of $r(t)$. Then*

1. *If $r_k(0)$ and $r_{k+1}(0)$ have the same sign, then $T_{x(0)} = T_{x(1)}$.*
2. *If k and $k+1$ are in the same row or column in $T_{x(0)}$, then $T_{x(0)} = T_{x(1)}$.*
3. *Else, $T_{x(1)}$ is obtained by switching k and $k+1$ from $T_{x(0)}$.*

This operation is best visualized by real valued tableaux: tableaux obtained by replacing k's in T_x by r_k.

Example 3.3.3. *Let $d = 2$ and $n = 5$. Let $r(t) = \{r_1(t), 3, 7, 18, 42, 61\}$ where $r_1(0) = 0$ and $r_1(1) = \infty$, $r_1(t)$ moving along the negative numbers. Assume that our tableau is $T_{x(0)}(r(0)) = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}$.*

As t goes from 0 to 1, the tableau $T_{x(t)}(r(t))$ changes as follows:

$$\underbrace{\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}}_{r_1(t)=-1} \to \underbrace{\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}}_{r_1(t)=-4} \to \underbrace{\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}}_{r_1(t)=-12} \to \underbrace{\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}}_{r_1(t)=-27} \to \underbrace{\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}}_{r_1(t)=-50} \to \underbrace{\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}}_{r_1(t)=-98}$$

As a real valued tableaux, this operation looks like:

$$\underbrace{\begin{array}{|c|c|c|}\hline -1&3&42\\\hline 7&18&61\\\hline\end{array}}_{r_1(t)=-1} \to \underbrace{\begin{array}{|c|c|c|}\hline 3&-4&42\\\hline 7&18&61\\\hline\end{array}}_{r_1(t)=-4} \to \underbrace{\begin{array}{|c|c|c|}\hline 3&-12&42\\\hline 7&18&61\\\hline\end{array}}_{r_1(t)=-12} \to \underbrace{\begin{array}{|c|c|c|}\hline 3&18&42\\\hline 7&-27&61\\\hline\end{array}}_{r_1(t)=-27} \to \underbrace{\begin{array}{|c|c|c|}\hline 3&18&42\\\hline 7&-50&61\\\hline\end{array}}_{r_1(t)=-50} \to \underbrace{\begin{array}{|c|c|c|}\hline 3&18&42\\\hline 7&61&-98\\\hline\end{array}}_{r_1(t)=-98}$$

It is very clear that following this path results in promotion of the associated tableau. So the following theorem is true.

Theorem 3.3.4. *Let $0 < r_1(0) < r_2(0) < r_3(0) < \cdots < r_N(0) < \infty$. Take a path such that $r_k(t)$ decreases from $r_k(0)$ to r_{k-1} for all $k \neq 1$ and $r_1(t)$ decreases to 0, goes along the negative numbers to ∞ and then down to $r_n(0)$ (i.e. the r_k's are rotated once) and let $x(t)$ be a lifting of this path. Then $T(0) = j(T(1))$.*

We remind ourselves of projective general linear group. This will come to be useful to us, since this rotation of roots can be neatly written as a $\mathrm{PGL}_2(\mathbb{C})$ action. The group $\mathrm{PGL}_2(\mathbb{C})$ (quotient group of the general linear group by non-zero scalings) acts on $\mathbb{CP}^1$ by Möbius transformation: $\phi = \left[\begin{smallmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{smallmatrix}\right]$ acts on $a \in \mathbb{CP}^1$ as

$$\phi(a) = \frac{\phi_{11}a + \phi_{12}}{\phi_{21}a + \phi_{22}}.$$

This restricts to $\mathrm{PGL}_2(\mathbb{R})$ on $\mathbb{RP}^1$.

Let's define $r = \{r_1, r_2, r_3, \cdots, r_n\}$ to be a subset of $\mathbb{RP}^1$ such that

$$\phi(r) := \{\phi(r_1), \phi(r_2), \cdots, \phi(r_n)\}$$

is also a subset of $\mathbb{RP}^1$ for some $\phi \in (\mathrm{PGL}_2(\mathbb{R}))$. If $x \in X(r)$ then $\phi(x) \in X(\phi(r))$ since the Wronskian map and Möbius transformations commute. From here, we can define the action of $\mathrm{PGL}_2(\mathbb{R})$ on real valued tableaux: $\phi(T_x(r)) := T_{\phi(x)}(\phi(r))$.

If ϕ is in the same connected component as $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, then $\phi(T)$ can be computed by the previous theorem, simply by lifting the path from $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ to ϕ.

If $\phi(r) = r$, then ϕ acting on $T_x(r)$ can be viewed as an operator on $\mathrm{SYT}(\square)$.

Lemma 3.3.5. *Let $\phi \in PGL_2(\mathbb{R})$, $\phi(r) = r$ where $0 < r_1 < r_2 < \cdots < r_n < \infty$. If ϕ is in the same component as $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, then the operation on $\mathrm{SYT}(\square)$ corresponds to the action of j^k for some integer k.*

This immediately follows from the previous theorem, since a path $r(t)$ lifted from $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ to ϕ must look like rotation of r_i's by k steps.

To get our main theorem, we will transform the $\mathbb{RP}^1$ circle into the unit circle $S^1 \subset \mathbb{CP}^1$. This with a specific choice of points will give us a simple realization of j in $\mathrm{PGL}_2(\mathbb{R})$.

Let $\psi := \begin{bmatrix} 1+\bar{\eta} & -(1+\eta) \\ 1+\eta & -(1+\bar{\eta}) \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{C})$ where $\eta := e^{i\pi/2n}$. With this map, we can transform all our work in $\mathbb{RP}^1$ to S^1. The group $\mathrm{PGL}_2(S^1) := \psi \mathrm{PGL}_2(\mathbb{R}) \psi^{-1}$ acts on subset of S^1. We now have $c := \begin{bmatrix} e^{-i\pi/n} & 0 \\ 0 & e^{i\pi/n} \end{bmatrix} \in \mathrm{PGL}_2(S^1)$ fixes

$$s := \{e^{i\pi/(2n)}, e^{3i\pi/(2n)}, \cdots, e^{(2n-1)i\pi/(2n)}\}.$$

Not only that, since it rotates the points in s, it operates on $\mathrm{SYT}(\square)$ as a promotion.

We can finally count the number of tableaux fixed by j^k.

Theorem 3.3.6. *Let $h(z) = (x^{n/k}+a_1)(x^{n/k}+a_2)\cdots(x^{n/k}+a_k)$ be a generic polynomial. Then the number of points in $X(h(z))$ fixed by the cyclic group generated by c^k is equal to the number of j^k-fixed tableaux in $\mathrm{SYT}(\square)$.*

Proof. The polynomial $s(z) = z^n + (-1)^n = (x + s_1)(x + s_2)\cdots(x + s_n)$ is fixed by c and is reduced. So this is a generic case. $x \in X(s(z))$ is c^k-fixed if and only if the associated tableau T_x is j^k-fixed. So the theorem follows. □

3.4 Components of X^r

Following from the theorem, we are now interested in $|X^r(h(z))|$ where $h(z) = (x^{n/k} + a_1)(x^{n/k} + a_2)\cdots(x^{n/k} + a_k)$ is a generic polynomial, where $X^r(h(z))$ is the set of points in the fibre of $h(z)$ fixed under the action c^k with order $r := n/k$. We will enumerate these points with ribbon tableaux. In order to do so, we introduce components of X^r, each of which corresponds to a r-core.

We have that c^k, considered as an element of $\mathrm{SL}_2(\mathbb{C})$, acts on $\mathbb{C}_{n-1}[z]$ such that $c^k(z^l) = \zeta^{n-1-2l}z^l$ where $\zeta := e^{2ki\pi/n}$. The ζ^{n-1-2l}-eigenspace for $l = 0, 1, \cdots, r-1$ is generated by $\{z^l, z^{r+l}, \cdots, z^{(k-1)r+1}\}$.

If $x \in X^r$, the c^k action on $\mathbb{C}_{n-1}[z]$ induces an action on x. Define $\mathrm{spec}^r(x) := (\mathrm{spec}^r_0(x), \mathrm{spec}^r_1(x), \cdots, \mathrm{spec}^r_{r-1}(x))$ where $\mathrm{spec}^r_l(x)$ is the multiplicity of the eigenvalue ζ^{n-1-2l} for the action c^k on x. We know that $\mathrm{spec}^r_l(x) \le k$ and that $\sum_l \mathrm{spec}^r_l(x) = d$.

For $s = (s_0, s_1, \cdots, s_{r-1})$ with $s_l \le k$ and $\sum_l s_l = d$, we define

$$X^s := \{x \in X^r \,|\, \mathrm{spec}^r(x) = s\}$$

to be the components of X^r.

Example 3.4.1. *Let $d = 3$, $m = 7$, $n = d(m-d) = 12$, $k = 4$, $r = n/k = 3$, and $s = (2,0,1)$ and consider $X^{(2,0,1)}$ component of X^r. The points in $X^{(2,0,1)}$ are direct sum of a 2-dimensional subspace of span($\{1, z^3, z^6\}$) and the 1-dimensional subspace of span($\{z^2, z^5\}$). In other words, they are row spaces of matrices of the form*

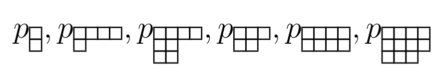
$$\begin{bmatrix} * & 0 & 0 & * & 0 & 0 & * \\ * & 0 & 0 & * & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & * & 0 \end{bmatrix}$$

with z^l as its basis.

There are only $\binom{3}{2} \cdot \binom{2}{1} = 6$ Plücker coordinates that could be non-zero. They are:

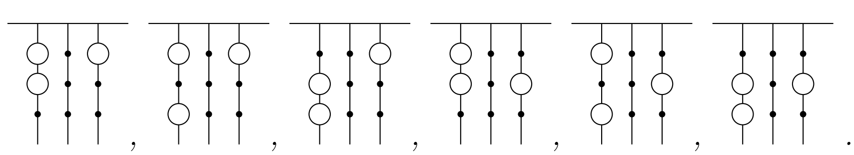
$$p_{134}, p_{137}, p_{347}, p_{146}, p_{167}, p_{467}.$$

In partition notation, they are:



We see that these partitions all have the same r-core. In fact, these are all possible partitions with the same r-core in the shape □.

In r-abacus notation, these partitions look like:



Note that the number of beads on lth column is s_l.

These observations lead us to the next lemma:

Lemma 3.4.2. *Let X^s be a component of X^r. There exists an unique r-core κ such that for every $x \in X^s$ and $\lambda \subseteq \square$ whose r-core is not κ, we have $p_\lambda(x) = 0$.*

Proof. We pick such r-core κ by putting s_k beads in the kth column in r-abacus and taking its corresponding r-core. Let $\lambda \subseteq \square$ be a partition whose r-core is $\kappa' \neq \kappa$, the r-core corresponding to $s' \neq s$. For some choice of l, $s'_l > s_l$. Then $p_\lambda(x) = 0$ for $x \in X^s$ because of the dimension of the ζ^{n-1-2l}-eigenspace. □

3.5 Ribbon Tableaux

We note that we are free to work in any algebraically closed fields, as this does not change the number of points in a fibre. For the rest of the chapter, we now use the field of puiseux series $\mathbb{K} := \mathbb{F}\{\{u\}\} := \cup_{n \geq 1} \mathbb{F}((u^{1/n}))$ where $\mathbb{F}$ is an algebraically closed field of characteristic zero. This change of field gives us an advantage since $\mathbb{K}$ is a complete valuation ring.

We define **valuation** of $g(u) = c_l + \sum_{r>l} c_r u^r \in \mathbb{K}^\times$ to be $\mathrm{val}(g(u)) := l$ and the **leading term** to be $\mathrm{LT}(g(u)) := c_l u^l$.

Given $a_i \in \mathbb{K}^\times$ such that $\mathrm{val}(a_1) > \mathrm{val}(a_2) > \cdots > \mathrm{val}(a_n)$, we wish to associate a tableau $U_x \in \mathrm{SYT}(\square)$ to each $x \in X(a)$ (this will be $r = 1$ case of 1-ribbon tableaux corresponding to points in the fibre fixed by the trivial action $c^n = id$). Note that U_x is not related with T_x. Since it is known that $|X(a)| = |\,\mathrm{SYT}(\square)|$, we work the other way: given $U \in \mathrm{SYT}(\square)$, we find $x \in X(a)$ such that $U = U_x$.

Theorem 3.5.1. *Let $U \in \mathrm{SYT}(\square)$ be a tableau defined by the chain of partitions $\varnothing = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_n = \square$. Then, there exist $x \in X(a)$ such that*

$$\mathrm{LT}(p_\mu(x)) = \prod_i \left(a_i \frac{q_{\lambda_{i+1}}}{q_{\lambda_i}} \right)$$

for all $\mu \subseteq \square$ where the product is over the entries of U that are not in the shape μ and where q_{λ_i} is the Vandermonde determinant defined in lemma 3.2.3.

Since $|X(a)| = |\,\mathrm{SYT}(\square)|$, such x is unique. Also note that $\mathrm{val}(p_{\lambda_k}(x)) \geq \mathrm{val}(p_\mu(x))$ for all $\mu \vdash k$.

We are interested in the more general case where $a = h(z) = (z^r+h_1)(z^r+h_2)\cdots(z^r+h_k)$. In generic case, we have $\mathrm{val}(h_1) > \mathrm{val}(h_2) > \cdots > \mathrm{val}(h_k)$. If we sort the roots by valuation, we have

$$\begin{aligned}
&\mathrm{val}(a_1) = \mathrm{val}(a_2) = \cdots = \mathrm{val}(a_r)\\
>&\,\mathrm{val}(a_{r+1}) = \mathrm{val}(a_{r+2}) = \cdots = \mathrm{val}(a_{2r})\\
>&\cdots\\
>&\,\mathrm{val}(a_{(k-1)r+1}) = \mathrm{val}(a_{(k-1)r+2}) = \cdots = \mathrm{val}(a_n)
\end{aligned}$$

The above theorem does not apply here, but one can prove a similar version of the theorem with a bit more work.

Theorem 3.5.2. *Let $h(z)$ be as above. Then for $x \in X^r(h(z))$, there exists a chain of partitions $\varnothing = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_n = \square$ (this may not be unique) such that*

$$\mathrm{LT}(p_{\lambda_{lr}}(x)) = \prod_{i=lr+1}^{n} \left(\mathrm{LT}(a_i) \frac{q_{\lambda_{i+1}}}{q_{\lambda_i}} \right)$$

for $l = 0, 1, \cdots, k-1$ and

$$\mathrm{val}(p_{\lambda_{lr}}(x)) > \mathrm{val}(p_\mu(x))$$

for all $\mu \vdash lr$ and $\mu \neq \lambda_{lr}$.

Although the corresponding chain of partitions may not be unique, the second condition identifies the unique partition λ_{lr} for $l = 0, 1, \cdots, k-1$. This yields us a uniquely corresponding tableau of $\square$ with content $(r, r, \cdots, r)$.

Suppose $x \in X^s$. Then by the lemma 3.4.2, there exists an r-core κ such that $p_\lambda(x) = 0$ for all $\lambda \subseteq \square$ whose r-core is not κ. We know that $p_{\lambda_{lr}}(x)$ are all non-zero. Therefore, the r-core of λ_{lr} are all κ and the chain of partitions results in a r-ribbon tableau.

Once we show that $X^r(h(z))$ is reduced and that this correspondence is surjective, we can conclude the following theorem.

Theorem 3.5.3. *Let $h(z) = (x^{n/k}+a_1)(x^{n/k}+a_2)\cdots(x^{n/k}+a_k)$ be a generic polynomial. Then the number of points in $X(h(z))$ fixed by the cyclic group generated by c^k is equal to the number of n/k-ribbon tableaux of shape $\square$.*

Together with theorem 3.3.6, we have that the number of j^k-fixed tableaux is equal to the number of n/k-ribbon tableaux. The number of ribbon tableaux is well studied [3] [1]. In particular, we know that the number is the Kostka-Foulkes polynomial evaluated at a root of unity, which turns out to be the q-analog of the hook length formula when the shape is a rectangle. Hence, the CSP of promotion of rectangular tableaux is proven again.

Chapter 4

Proof of Cyclic Sieving Using Webs

In this chapter, we look at another proof of the main theorem. The proof realizes tableaux as combinatorial objects called webs. In this realization, the promotion corresponds to the rotation. Currently, only 2 and 3-row tableaux can be studied this way, as the webs corresponding to tableaux with more rows are yet to be found with combinatorial interpretation, but this approach yields a very strong way to visualize the rotating behavior of the promotion of rectangular tableaux.

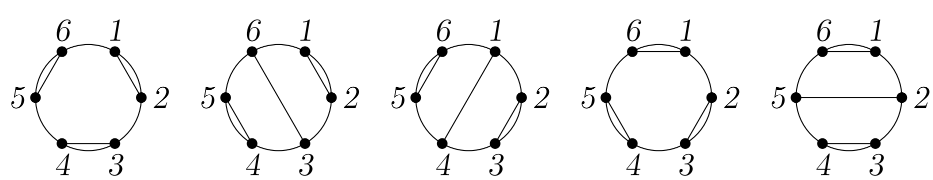
4.1 2-Row Tableaux and Non-Crossing Matching

We start by looking at the web corresponding to 2-row tableaux. This web, $A1$-web, is more well known as non-crossing matching.

Definition 4.1.1. *A* ***non-crossing matching*** *of size* $2n$ *is a perfect matching with vertex set* $[2n]$ *such that for any two edges* (a, b) *and* (c, d)*, we cannot have* $a < c < b < d$*.*

The condition is equivalent to saying that if the vertices are arranged in a circle in order, then no two edges cross.

Example 4.1.2. *There are 5 non-crossing matchings of size* $6 = 2 \cdot 3$*:*

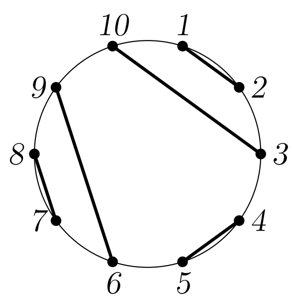


It is well-known that the number of non-crossing matchings of size $2n$ is the nth Catalan number, $\frac{1}{n+1}\binom{2n}{n}$.

We know that the number of 2-by-n tableaux are also nth Catalan number, so we suspect that there is a bijection between these two objects. There does exist such bijection: given a non-crossing matching $\{(a_i, b_i) : i \in [n], a_i < b_i\}$ (let's call vertices a_i's **initial** and b_i's **final**), construct a $\{1, 2\}$-word of length $2n$ by inserting 1s at a_ith positions and 2s at b_ith positions. This is a lattice word, so we have a corresponding 2-row tableau. One can trivially reverse the process and obtain the non-crossing matching corresponding to a 2-row tableau. This bijection can be stated in much simpler terms, but we leave it like this to draw parallel to the 3 row case.

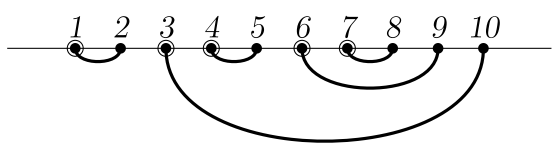
Example 4.1.3.

We start with a non-crossing matching:



.

Cut between 1 and 10, the diagram looks like (initials are circled):



.

Then the corresponding lattice word is:

$$1211211222.$$

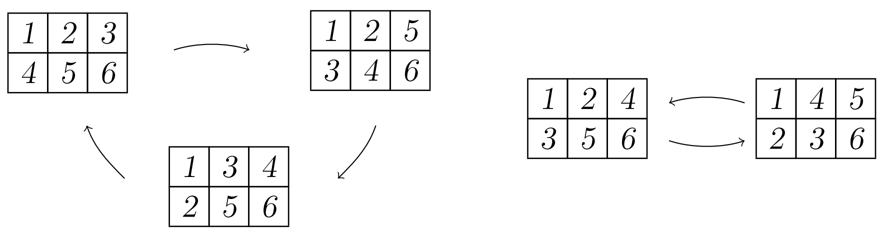
And the resulting tableau is:

1	3	4	6	7
2	5	8	9	10

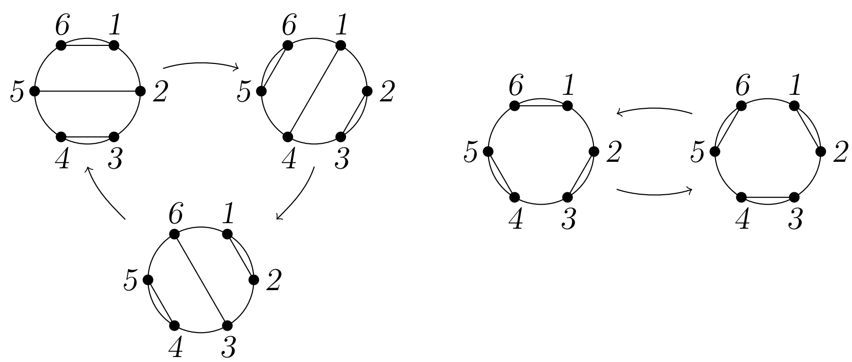
.

Let us consider the promotion under this bijection.

Example 4.1.4. *The orbits of the 5 2-by-3 standard tableaux by promotion are:*



Under the bijection, this becomes:



From this example, we can observe the following main theorem.

Theorem 4.1.5. *The promotion of a 2-row standard tableaux corresponds with the rotation of a non-crossing matching under the bijection.*

Proof. Suppose that 1 is connected to $2k$ in a non-crossing matching, then $2k$ is the kth entry in the second row in the corresponding tableau. Consider the sub-non-crossing matching $\{(a_i, b_i) : i \in [2, 3, \cdots, 2k-1], a_i < b_i\}$. Since the jth biggest initial is smaller than the jth biggest final, we conclude that $j+1$th entry in the first row is smaller than the jth entry in the second row. Thus the promotion path on the tableau follows the first row up to kth column, go down, and then takes the rest of the second row. This means that promotion will change the set of initials by removing 1, adding $2k$, and decreasing all elements by 1.

The rotation can be described as re-labelling 1 as $n+1$ and then decreasing all the label by 1. Thus set of initials change exactly the same under rotation. □

4.2 3-Row Tableaux and A2-Webs

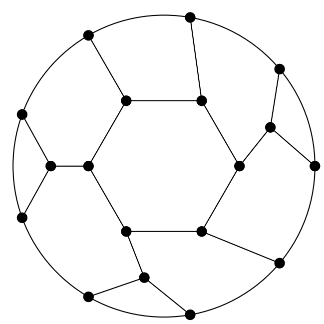
We now generalize this notion of rotation of non-crossing matching for 3-row tableaux.

Definition 4.2.1. *An **A2-web** is a planar, bipartite graph embedded on a disk, satisfying the following conditions:*

1. *The boundary vertices (on the boundary of the disk) have degree 1.*
2. *The internal vertices (not on the boundary) have degree 3.*

*An A2-web is **irreducible** (or elliptic) if all internal faces have at least 6 sides.*

Example 4.2.2. *An example of size 9 irreducible A2-web is:*



.

Note that the number of boundary vertices must be a multiple of 3. We will label them with $[3n]$ going clockwise.

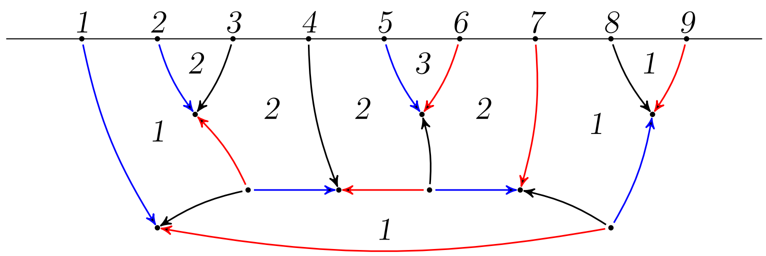
There exists a natural bijection between 3-row tableaux and irreducible A2-webs where all boundary vertices are in the same part in the bipartition. We now describe the bijection.

First, we embed the graph such that the disk is now the half plane, cut between 1 and $3n$. Label the faces by the distance to the outer face in the dual graph. In other words, these labels indicate the minimum number of edges one needs to cross to reach the outer face. We also label each edges with

$$\begin{cases} 1 & \text{if the label of the face on the left side is greater than the label of the face} \\ & \text{on the right side as one travels on the edge (blue in the example diagram)} \\ 2 & \text{if the labels are the same (black in the example diagram)} \\ 3 & \text{if the right label is greater (red in the example diagram)} \end{cases}$$

with direction of the edge given from the bipartition, edges pointing away from the partition with the boundary vertices. Looking at the labels of the edges incident to the boundary vertices, we get a lattice word of length $3n$, from which we get our 3-row tableau.

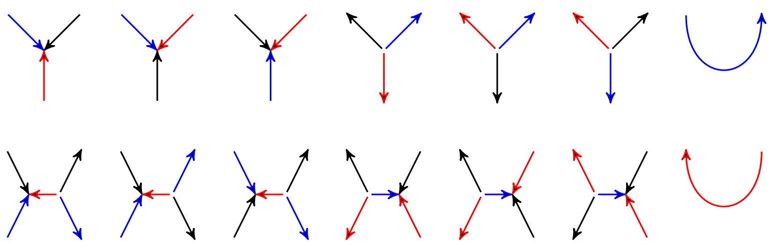
Example 4.2.3. *The A2-web from the previous example becomes:*



and the lattice word is 112213323, yielding the tableau

1	*2*	*5*
3	*4*	*8*
6	*7*	*9*

The inverse is mostly straight forward, except for the part where we have to construct the A2-web from the labeled edges. This uses the growth rule (shown below in a diagram), which tells us local construction given uncompleted neighbouring edges. It is proven by Khovanov and Kuperberg [8] that the end result does not depend on the choices one makes in the growth rule algorithm.



A2-web gives an excellent picture of behavior of promotion on 3-row tableaux: its equivalent action is simply a rotation. We have the main theorem for A2-webs.

Theorem 4.2.4. *The promotion of a 3-row standard tableaux corresponds with the rotation of a A2-webs under the bijection.*

The proof can be found in [25]. The proof is elementary and is similar to the 2-row case. To show the correspondence, keep track of how the lattice word changes under rotation. The two key vertices, which represent the two column changes in the promotion path, can be determined from two alternating paths one can follow in the A2 webs. These two paths divide the web into 3 sections and the change of depth in each section after the rotation can be understood to be in agreement with the promotion.

4.3 Web Space

We now look at algebraic interpreatation of these webs. Our goal here is to see the motivation for webs and its reduction rules and to get enough algebra to prove the cyclic sieving. Thus we will not dive into the invariant theory in depth. We will only provide definitions and theorems without proofs and see how it relates to our combinatorial object.

A classic problem from the invariant theory is the following:

Given irreducible representations of group G, $V_1, V_2, \cdots, V_n$, characterize the subspace of $\{f : V_1 \times V_2 \times \cdots \times V_n \to \mathbb{F}\}$ which is invariant under the action of G (i.e. $f(v_1, v_2, \cdots, v_n) = f(gv_1, gv_2, \cdots, gv_n)$ for any $g \in G$).

This is to characterize the invariant tensor: $Inv(V_1 \otimes V_2 \otimes \cdots \otimes V_n)$.

Consider $G = SL_2(\mathbb{C})$ (the A_1 Lie group) , and $V = \mathbb{C}^2$ as the irreducible representation of G, acting in the obvious way. The map $V \otimes V \to \mathbb{C} : (x_1, y_1) \otimes (x_2, y_2) \mapsto x_1y_2 - x_2y_1$ is an invariant contraction under any basis, since the action by $A \in SL_2(\mathbb{C})$ yields

$$\begin{aligned}
&(A_{11}x_1 + A_{12}y_1)(A_{21}x_2 + A_{22}y_2) - (A_{11}x_2 + A_{12}y_2)(A_{21}x_1 + A_{22}y_1) \\
=&det(A)(x_1y_2 - x_2y_1) \\
=&x_1y_2 - x_2y_1.
\end{aligned}$$

One can also think of the contraction as the natural map of $V \otimes V^* \to \mathbb{C}$ since V is self-dual.

Therefore, we have a contraction operation $\text{Inv}(V \otimes V^* \otimes W) \to \text{Inv}(W)$ where $W = V^{\otimes n}$. In addition, we have cyclic permutation $\text{Inv}(V \otimes W) \to \text{Inv}(W \otimes V)$ and join operation $\text{Inv}(V) \otimes \text{Inv}(W) \to \text{Inv}(V \otimes W)$ defined naturally.

It turns out that this is closely related to non-crossing matchings. With non-crossing matchings, we also have contraction (adding a loop to adjacent pair of vertices), cyclic permutation, and join operation. The following theorem makes this connection.

Theorem 4.3.1. *(Rumer, Teller, and Weyl [5])*

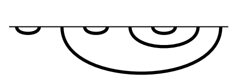
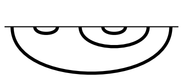
There exists an unique isomorphism ϕ between the space of formal linear combination of non-crossing matchings and $\bigcup_n Inv(V^{\otimes 2n})$ where

1. *The join operator $\bowtie_{n,m}$: $Inv(V^{\otimes 2n}) \otimes Inv(V^{\otimes 2m}) \to Inv(V^{\otimes 2(n+m)})$ corresponds to the merging of two non-crossing matching.*

2. *The cyclic operator $\rho_n : Inv(V^{\otimes 2n}) \to Inv(V^{\otimes 2n})$ corresponds to the rotation of the non-crossing matching.*

3. *The contraction operator $\sigma_n : Inv(V^{\otimes 2n}) \to Inv(V^{\otimes 2n-2})$ corresponds to contraction of the non-crossing matching.*

Say we want to construct such isomorphism $\phi_{2n} : W_{2n} \to \mathrm{Inv}(V^{\otimes 2n})$. If one can pick $\phi_2(\;)$ to be r, then all other values can be deduced from here. For instance,

$\phi_{10}\Big($  $\Big) = \phi_2(\;) \otimes \phi_8\Big($  $\Big)$

$= r \otimes \rho_8\Big(\phi_8\big($  $\big)\Big)$

$$= r \otimes \rho_8(\phi_2(\;) \otimes \phi_4(\;) \otimes \phi_2(\;))$$
$$= r \otimes \rho_8(r \otimes \rho_4(r \otimes r) \otimes r).$$

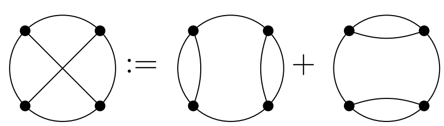
Note that contracting vertices 2 and 3 of yields . In invariant tensor setting, this is $(I \otimes \sigma \otimes I)(r \otimes r) = r$. One can now solve for σ. The most natural choice is $r = e_1 \otimes e_2 - e_2 \otimes e_1$ and $\sigma = (e_2 \otimes e_1)^* - (e_1 \otimes e_2)^*$. Regardless of the choice of r and σ, $\sigma(r) = -2$. This means that a loop (result of contraction of vertices 1 and 2 of) should count as -2 scalar.

Lastly, let's consider what crossing should evaluate to. Sticking to our choice of $\phi_2(\;) = e_1 \otimes e_2 - e_2 \otimes e_1$, we have

$$\begin{aligned}\phi_4(\;) &= \text{swap 2nd and 3rd component of } r \otimes r\\ &= e_{1122} - e_{1221} - e_{2112} + e_{2211}\\ \phi_4(\;) &= r \otimes r\\ &= e_{1212} - e_{1221} - e_{2112} + e_{2121}\\ \phi_4(\;) &= \rho(r \otimes r)\\ &= -e_{2121} + e_{1122} + e_{2211} - e_{1212}\end{aligned}$$

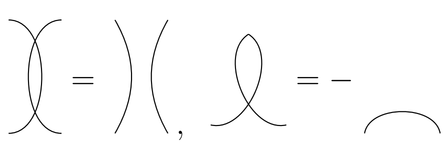
where $e_{ijkl} := e_i \otimes e_j \otimes e_k \otimes e_l$.

Since, $\phi_4(\;) = \phi_4(\;) + \phi_4(\;)$, we define



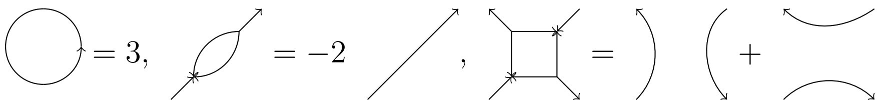
in the A_1 web space.

It is helpful to introduce crossing to our language of webs. It can be used to express some complicated webs in a simpler way. It also allows us to have these knot theory-like 'Reidemeister' moves to 'unknot' the web.



This correspondence between the invariant tensor and linear space of A_1 webs can be generalized one rank higher: there exists a canonical correspondence between $\mathrm{Inv}(V^{\otimes n})$ where $V = \mathbb{C}^3$ and $\mathbb{C}$-linear combination of irreducible A_2 webs such that join, rotation, and stitching operations are respected.

Reducible webs are reduced by the following rules:



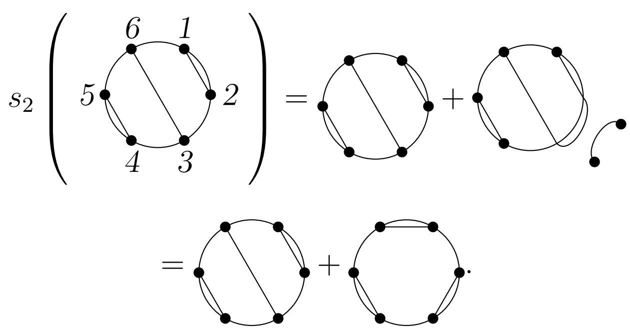
This allows us to reduce any A_2 webs to linear combinations of irreducible webs.

4.4 Proof of CSP

Finally, we put together our understanding of web space and its connection to promotion to prove the cyclic sieving of promotion. We will proceed by proving the 2-row case, while noting the differences that arises when 3-row proof is carried on.

We begin by defining the action of S_{2n} on $W_n^{(2)}$. Given a Coxeter generator s_i of S_{2n}, define $s_i(D)$ by the sum of D with $)\,($ added at the vertices i and $i+1$ (For rank 3 case, add $\rangle\!-\!\langle$ instead of $)\,($).

Example 4.4.1.

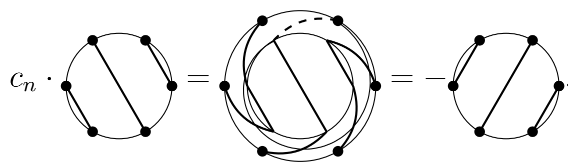


It is easy to check that this satisfy the Coxeter rules, because it is simply crossing ith and $i+1$th vertices. Therefore, this extends to S_{2n} acting on $W_n^{(2)}$.

We note that the long cycle $c_n \in S_{2n}$ acts on $W_n^{(2)}$ as rotation.

Lemma 4.4.2. *For any* $D \in W_n^{(2)}$, $p(D) = -c_n \cdot D$.

Proof. Clearly, $c_n = s_{n-1}s_{n-2}\cdots s_1$, so adding the crossings, we get this swirl:



By performing Reidemeister type moves from the earlier section, we can transform the line from 1 to n into the dashed line, with one sign change. Therefore, the action by the long cycle is equivalent to rotating the web clockwise by one step. □

One can check that $W_n^{(2)}$ is an irreducible S_{2n}-module of shape (n, n) (and $W_n^{(3)}$ is an irreducible S_{3n}-module of shape (n, n, n)). This can be shown by the fact that 1. $W_n^{(2)}$ factors through Temperley-Lieb algebra, showing that the irreducible components have at most two rows 2. we can construct an action that does not act trivially, showing that the irreducible components must have shape containing (n, n) 3. the dimension of $W_n^{(2)}$ is $\frac{1}{2n+1}\binom{2n}{n}$, showing that $W_n^{(2)}$ is the irreducible module of shape (n, n).

Finally, we are at the same situation as we were in the Rhoades' proof. We have a irreducible representation of S_n of shape (n, n) and the action of c_n corresponds to the promotion of the basis elements up to some sign. The same calculation of characters yields CSP.

Chapter 5

Square Tableaux and Promotion

In this chapter, we see a combinatorial way of enumerating n-by-n square tableaux fixed by n promotions. There are $n!$ such tableaux, so it is natural to ask whether there exists a bijection to permutations. We define a procedure, which is a variation of jeu-de-taquin rectification, that constructs such tableaux from a permutation. This map can be shown to be an one-to-one correspondence. We also consider how this could be generalized to show CSP for square tableaux. The material in this chapter is a work in progress with K. Purbhoo.

5.1 Three Equivalent Procedures

Before we begin, let's define some terminology: given a square shape $\square := n^n$, a diagonal refers to the set of cells in row i and column $k-i$ for some fixed k, $i = 1, 2, \cdots, n$. The **main diagonal** is a diagonal with $k = n+1$. **Upper/lower triangular part** will mean the set of cells that are weakly north-west/south-east of a cell in the main diagonal (i.e. upper triangular part will be the staircase shape $\lrcorner := (n, n-1, \cdots, 1)$). We denote the set of tableaux of shape λ fixed by k iterations of promotion as $\mathrm{SYT}_k(\lambda)$.

We define a **sliding procedure** which takes $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ as its input and outputs $T \in \mathrm{SYT}$. We first compute the upper triangular part. Begin by filling the main diagonal with $\sigma_1, \sigma_2, \cdots, \sigma_n$ from bottom left to top right.

Choose an empty box and slide it out until it reaches the main diagonal . Fill the empty box on the main diagonal by a number that is n bigger than the previous entry. Repeat the sliding until the shape becomes normal. We will denote the result of this procedure by $s(\sigma) \in SYT(\lrcorner)$.

Example 5.1.1. *Let* $\sigma = 3124 \in S_4$. *Then*

$$
\begin{array}{|c|c|c|c|}\hline \ & \ & \ & 4 \\ \hline \ & \blacksquare & 2 \\ \cline{1-3} \ & 1 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}
\to
\begin{array}{|c|c|c|c|}\hline \ & \ & \ & 4 \\ \hline \ & 1 & 2 \\ \cline{1-3} \blacksquare & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}
\to
\begin{array}{|c|c|c|c|}\hline \ & \ & \blacksquare & 4 \\ \hline \ & 1 & 2 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}
\to
\begin{array}{|c|c|c|c|}\hline \ & \blacksquare & 2 & 4 \\ \hline \ & 1 & 6 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}
\to
\begin{array}{|c|c|c|c|}\hline \ & 1 & 2 & 4 \\ \hline \blacksquare & 5 & 6 \\ \cline{1-3} 3 & 9 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}
\to
\begin{array}{|c|c|c|c|}\hline \blacksquare & 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \cline{1-3} 7 & 9 \\ \cline{1-2} 11 \\ \cline{1-1}\end{array}
\to
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 \\ \cline{1-3} 7 & 9 \\ \cline{1-2} 11 \\ \cline{1-1}\end{array}.
$$

There are other ways of computing $s(\sigma)$. Let $\mathrm{aug}(\sigma)$ be the sequence $\sigma_1, \sigma_1+n, \sigma_1+2n, \cdots, \sigma_1+n(n-1), \sigma_2, \sigma_2+n, \sigma_2+2n, \cdots, \sigma_2+n(n-1), \cdots, \sigma_1, \sigma_n+n, \sigma_n+2n, \cdots, \sigma_n+n(n-1)$. Inserting $\mathrm{aug}(\sigma)$ into the empty tableau will produce a tableau. Taking the staircase part of this tableau, we get $s(\sigma)$. We will call this **insertion procedure**.

Example 5.1.2. *Let* $\sigma = 3124 \in S_4$. *Then*

$$\mathrm{aug}(\sigma) = 3, 7, 11, 15, 1, 5, 9, 13, 2, 6, 10, 14, 4, 8, 12, 16,$$

$$
\mathrm{ins}(\mathrm{aug}(\sigma)) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 8 & 12 & 16 \\ \hline 3 & 5 & 6 & 10 & 14 \\ \cline{1-5} 7 & 9 & 13 \\ \cline{1-3} 11 & 15 \\ \cline{1-2}\end{array},
$$

$$
s(\sigma) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 \\ \cline{1-3} 7 & 9 \\ \cline{1-2} 11 \\ \cline{1-1}\end{array}.
$$

There is a third way of computing $s(\sigma)$, which we will call **rectification procedure**. Denote the tableau $\begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_n \\ \hline\end{array}$ by $T_k(\sigma)$ where $a_l = \sigma_k + n(l-1)$. Begin with $T_1(\sigma)$. Place $T_2(\sigma)$ right on top of $T_1(\sigma)$. If $\sigma_2 > \sigma_1$, shift $T_2(\sigma)$ right by one cell so that the resulting tableau is an young tableau. Continue in this manner until we get a skewed standard young tableau with entry from 1 to n^2. Call this tableau $T(\sigma)$. Now, rectify $T(\sigma)$ and look at the entries in the shape $\begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}$. This is $s(\sigma)$.

Example 5.1.3. *Let* $\sigma = 3124 \in S_4$. *Then*

$$
T(\sigma) = \begin{array}{cc|c|c|c|c|}\cline{3-6} & & 4 & 8 & 12 & 16 \\ \cline{2-6} & \multicolumn{1}{|c|}{2} & 6 & 10 & 14 \\ \hline \multicolumn{1}{|c|}{1} & 5 & 9 & 13 \\ \cline{1-4} \multicolumn{1}{|c|}{3} & 7 & 11 & 15 \\ \cline{1-4}\end{array},\ \mathrm{rec}(T(\sigma)) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 8 & 12 & 16 \\ \hline 3 & 5 & 6 & 10 & 14 \\ \cline{1-5} 7 & 9 & 13 \\ \cline{1-3} 11 & 15 \\ \cline{1-2}\end{array},\ s(\sigma) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline 3 & 5 & 6 \\ \cline{1-3} 7 & 9 \\ \cline{1-2} 11 \\ \cline{1-1}\end{array}.
$$

Since the word of the $T(\sigma)$ is $\mathrm{aug}(\sigma)$, the last two procedure yields the same result. However, the equivalence of the first procedure is not proven.

Conjecture 5.1.4. *The sliding procedure and the rectification procedure yield the same result.*

Our effort on this conjecture has been unsuccessful. The weaker claim that the sliding procedure is well defined also resisted our attempts. However, our computer search did not yield any counterexample. For the rest of the chapter, we will assume that this conjecture is true. The conjecture is used to assume that the diagonal of $s(\sigma)$ is distinct in mod n, and also in the proof of surjection of our correspondence. If the former is true, the later is also true as we know that the sets have the same cardinality by other results. It is therefore suspected that the bijection could be proven independent of the conjecture, but we did not pursue this approach.

Assuming the conjecture, this lemma follows:

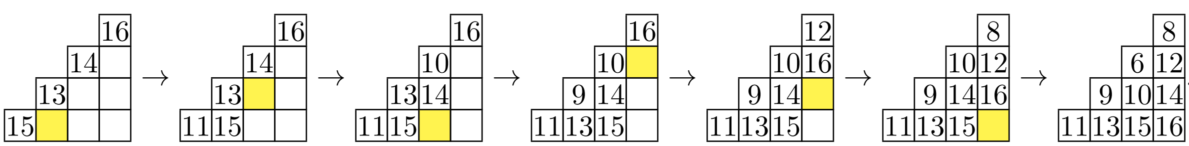
Lemma 5.1.5. *When the rectification/insertion procedure is terminated, the entries outside the upper triangular part is in form of $d+n, d+2n, \cdots, d+kn$ for each row.*

5.2 Dual Procedures

We can also define the dual of this procedure that takes $\sigma \in S_n$ as its input but outputs a tableau of shape ⌟ $:= (n, n, \cdots, n)/(n-1, n-2, \cdots, 1, 0)$. All three dual procedures are simply the previously defined procedures rotated 180 degrees and then performed with the revered ordering of the entries.

The dual sliding procedure will start with filling the main diagonal with $\sigma_1 + n(n-1), \sigma_2 + n(n-1), \cdots, \sigma_n + n(n-1)$ and boxes will slide out northwest instead of southeast. The main diagonal entries will be replaced by an entry that is n less than the previous entry. We will call the result of the dual procedure $s^*(\sigma)$.

Example 5.2.1. *Let $\sigma = 3124 \in S_4$. Then*



This procedure is also well defined.

The dual insertion procedure is to 'dual' insert $\operatorname{aug}^*(\sigma) := \sigma_1 + n(n-1), \sigma_1 + n(n-2), \cdots, \sigma_1, \sigma_2 + n(n-1), \sigma_2 + n(n-2), \cdots, \sigma_2, \cdots, \sigma_n + n(n-1), \sigma_n + n(n-2), \cdots, \sigma_n$.

Example 5.2.2. *Let $\sigma = 3124 \in S_4$. Then*

$\operatorname{aug}^*(\sigma) = 15, 11, 7, 3, 13, 9, 5, 1, 14, 10, 6, 2, 16, 12, 8, 4,$

$\operatorname{ins}^*(\operatorname{aug}^*(\sigma)) =$

				4	8
			2	6	12
	1	5	9	10	14
3	7	11	13	15	16

,

$s^*(\sigma) =$

			8
		6	12
	9	10	14
11	13	15	16

.

The dual rectification procedure is just 'dual' rectifying $T(\sigma)$ and looking at ⌟ part of it.

Example 5.2.3. *Let $\sigma = 3124 \in S_4$. Then*

$T(\sigma) =$

		4	8	12	16
	2	6	10	14	
1	5	9	13		
3	7	11	15		

, $\operatorname{rec}^*(T(\sigma)) =$

				4	8
			2	6	12
	1	5	9	10	14
3	7	11	13	15	16

, $s^*(\sigma) =$

			8
		6	12
	9	10	14
11	13	15	16

.

These three dual procedures produce the same result as well.

We would like to merge $s(\sigma)$ and $s^*(\sigma)$ together to construct a square standard young tableaux that corresponds to $\sigma \in S_n$. However, we need to show that the main diagonal of $s(\sigma)$ and $s^*(\sigma)$ are identical in order to define such square tableau.

Let μ be the shape of $\operatorname{rec}(T(\sigma))$. In a previous lemma, we saw that $\operatorname{rec}(T(\sigma))$ is composed of two parts: $s(\sigma)$ and the rest where the entries increase by n as we move right. The entries on the main diagonal of $s(\sigma)$ are congruent to $\sigma_1, \sigma_2, \cdots, \sigma_n$ in mod n (this is implied from our conjecture). Thus, it is possible to write the entries on the main diagonal of $s(\sigma)$ in terms of μ and σ.

Example 5.2.4. *Let $\sigma = 3124 \in S_4$ and $\mu = \operatorname{sh}(\operatorname{rec}(T(\sigma))) = (6, 5, 3, 2)$, then*

$\operatorname{rec}(T(\sigma)) =$

*	*	*	8	12	16
*	*	6	10	14	
*	9	13			
11	15				

$$\begin{aligned} 8 &= 4 \cdot (4 + 4 - \mu_1 - 1) + \sigma_4 \\ 6 &= 4 \cdot (4 + 3 - \mu_2 - 1) + \sigma_3 \\ 9 &= 4 \cdot (4 + 2 - \mu_3 - 1) + \sigma_2 \\ 11 &= 4 \cdot (4 + 1 - \mu_4 - 1) + \sigma_1 \end{aligned}.$$

In general, the entry in $s(\sigma)$ on the main diagonal and on the kth row is equal to $n(n + (n + 1 - k) - \mu_k - 1) + \sigma_{n+1-k}$.

Clearly, $\text{rec}^*(T(\sigma))$ has the same shape as $\text{rec}(T(\sigma))$, so we can also write the entries on the main diagonal of $s^*(\sigma)$ in terms of μ and σ. The entry in $s^*(\sigma)$ on the main diagonal and on the kth row is equal to $n \cdot (\mu_{n+1-k} - k) + \sigma_{n+1-k}$.

Therefore, in order to show that the main diagonal s of $s(\sigma)$ and $s^*(\sigma)$ is equal, it is sufficient to show that:

$$\begin{aligned} & n(n + (n+1-k) - \mu_k - 1) + \sigma_{n+1-k} = n(\mu_{n+1-k} - k) + \sigma_{n+1-k} \\ \Leftrightarrow \; & \mu_k + \mu_{n+1-k} = 2n. \end{aligned}$$

Let $(\nu + \Box)/\nu$ be the shape of $T(\sigma)$. This is determined by the descent set of σ. $T(\sigma)$ rectifies to the shape μ, so there exists a Littlewood-Richardson tableau of shape $(\nu + \Box)/\nu$ that rectifies to the shape λ, thus having the content λ.

A Littlewood-Richardson tableau could be constructed in the shape of $(\nu + \Box)/\nu$ by filling every column with entries $1, 2, 3, \cdots$. The content of this tableau dominates the content of any Littlewood-Richardson tableau of the same shape. Therefore, $(n + \nu_k - \nu_{n+1-k})_k$ dominates μ. This bounds μ from above in the dominance order.

On the other hand, we can use Greene's theorem [7] to bound μ from below.

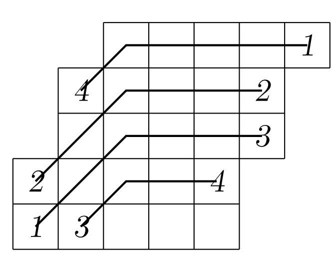
Theorem 5.2.5. *Let $\pi \in S_n$ and $\lambda = RSK(\pi)$. Then $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ is bigger or equal than the sum of lengths of k disjoint increasing subsequences of π.*

We are left with finding k disjoint increasing subsequence of $\text{aug}(\sigma)$. We noted that the entries increase as we traveling north-east in $T(\sigma)$. Therefore, a path that travels east or north-east in $T(\sigma)$ yields an increasing subsequence of $\text{aug}(\sigma)$.

We now construct k such disjoint paths as follows. Pick "initial" boxes on $T(\sigma)$ one by one by going up each column, starting from the left most column and going right. When a box is picked, cross off all boxes that are north-east of that box. Stop after k boxes are picked. Pick "final" boxes on T one by one by selecting the rightmost box on each row, starting from the top row going down. Stop after k boxes are picked.

Each diagonal has at most one initial boxes. By the construction of the tableau $T(\sigma)$, the leftmost diagonal will reach the top row, the second left most diagonal will reach the second row from the top, and so on. To construct the k paths, start from the k initial boxes, go along the k diagonals until they reach the top k rows and then go along these rows to reach the k final boxes.

Example 5.2.6.



It is easy to see that the sum of the lengths of these k paths are also $\sum_{i=1}^{k} n + \nu_i - \nu_{n+1-i}$ and $(n + \nu_i - \nu_{n+1-i})_i$ is dominated by μ. Therefore, $\mu_k = n + \nu_k - \nu_{n+1-k}$ and $\mu_k + \mu_{n+1-k} = 2n$. We can now conclude the following:

Proposition 5.2.7. *The main diagonal s of $s(\sigma)$ and $s^*(\sigma)$ are equal.*

To show that $t(\sigma) \in \text{SYT}$, it remains to show that the set of entries is exactly $\{1, 2, \cdots, n^2\}$. This is clear if one considers the sliding procedure. Consider the numbers in this set that are congruent to m mod n. Exactly one of the numbers in the set is on the main diagonal. If a number in the set is smaller than this entry, than it must show up in $s^*(\sigma)$, otherwise it must be in $s(\sigma)$. Therefore, all the numbers in the set is present in $t(\sigma)$.

Corollary 5.2.8. $t(\sigma)$ *is well defined and is in* SYT.

5.3 Connections to the Promotion

We now have a map from S_n to SYT. It is easy to see that it is injective: reading the main diagonal of $t(\sigma)$ in mod n retrieves $\sigma \in S_n$. We claim that t maps S_n to $\text{SYT}(n^n)_n$ and that this is an one to one correspondence.

Lemma 5.3.1. $j(t(\sigma)) = t(\sigma \cdot (n(n-1) \cdots 1))$

Proof. If we perform promotion after sliding procedure with σ, only watching the upper triangular part, we will see that we are taking out 1, sliding the now empty box up to the main diagonal (we don't know what number will fill this cell on the main diagonal, but let's not worry about this cell for now), and then decreasing all entries by 1. This is equivalent to starting the sliding procedure with $n+1$ instead of 1 and then decreasing all entries by 1, which is also equivalent to starting the sliding procedure with $\sigma \cdot (n(n-1) \cdots 1)$. This shows that $j(s(\sigma)) = s(\sigma \cdot (n(n-1) \cdots 1))$ if we ignore one cell on the diagonal. We similarly have $j(s^*(\sigma)) = s^*(\sigma \cdot (n(n-1) \cdots 1))$. From these two facts, we can conclude that $j(t(\sigma)) = t(\sigma \cdot (n(n-1) \cdots 1))$. □

Corollary 5.3.2. $t(\sigma) \in SYT(n^n)_n$

Proof. This follows since $\sigma \cdot (n(n-1) \cdots 1))^n = \sigma$. □

The above corollary shows that t is an injection to $\text{SYT}(n^n)_n$. It remains to show that it is also surjective.

Lemma 5.3.3. *Let $T \in SYT(n^n)_n$. Then,* $\text{diag}(T) \in S_n$ *and* $\text{diag}(j(T)) = \text{diag}(T) \cdot (n(n-1) \cdots 1)$*, where* $\text{diag}(T)$ *is the sequence of remainders mod n of the n entries in the main diagonal of T.*

Proof. Suppose that two entries on the main diagonal of T is congruent in mod n. Say that the difference between the two entries is mn. In $T = j^{mn}(T)$, the bigger entry could only have traveled up or left and is now mn smaller. This cannot be on a diagonal, so this is a contradiction. Therefore, $\text{diag}(T) \in S_n$.

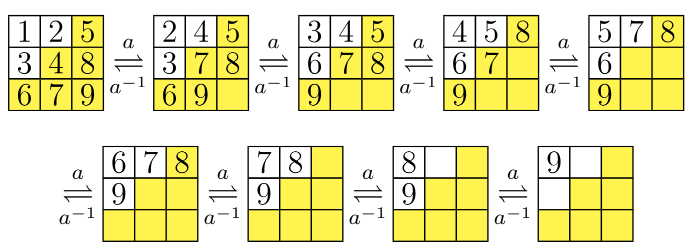
Consider the n promotion path as we compute $j^n(T)$. At least one of these path will go through a chosen box in $\square$; the box will contain an entry that is n smaller after the n promotions otherwise. There are n boxes on the main diagonal and each path can only go through one of them. Therefore, a box on the main diagonal is passed by exactly one promotion path. In other $n-1$ promotion steps, the entry will be decreased by 1. Since $j^n(T) = T$, the promotion step where the promotion path passes through the box must increase that box's entry by $n-1$. Therefore, all entries on the main diagonal is decreased by 1 in mod n in every promotion and this is equivalent to multiplying diag by $(n(n-1)\cdots 1)$ on the right side.

Theorem 5.3.4. *For each $T \in SYT(n^n)_n$, there exists $\sigma \in S_n$ such that $t(\sigma) = T$.*

Proof. As we mentioned before, we can find σ by taking $\text{diag}(T)$.

Starting from T, slide out the top left entry. We will call this procedure a. The inverse procedure a^{-1} is to perform RSK backwards with recording tableau $e(T)$. Focusing on the lower triangular part in reverse, we see that this is exactly like computing $s^*(T)$ using sliding procedure. All we need to make sure is that the replacement of the diagonal entries are done in the same way: it must be n less than the previous entry.

Example 5.3.5.



Disregarding the empty boxes, $a^k(T)$'s entries are same as $j^k(T)+k$'s entries (where the addition is to each entry), so

$$\begin{aligned}\text{diag}(a^k(T)) &= \text{diag}(j^k(T)+k)\\ &= \text{diag}(T)\cdot(n(n-1)\cdots 1)^k + k\\ &= \text{diag}(T) - k + k\\ &= \text{diag}(T)\end{aligned}$$

in mod n, disregarding the empty boxes. Therefore, the replacement is congruent to the previous entry in mod n.

However, it cannot be replaced by an entry with difference more than n, since the entry that is exactly n more must exist in the lower triangular part of the final result but has no more opportunity to enter. Therefore, computing the lower triangular part of $T = a^{-n^2}(\epsilon)$ is identical to computing $s^*(\sigma) = s^*(\text{diag}(T))$ using sliding procedure. The same holds for the upper triangular part. Therefore, $T = t(\sigma)$ and we are done. □

5.4 Generalization

We note a possible generalization of this bijection in order to show CSP for square tableaux. We remind ourselves that the number of n-by-n square tableaux fixed by n^2/k promotion is equal to the number of k-ribbon tableaux of the square shape. In order to work with ribbon tableaux, we define i**-diagonal** to mean the set of cells on row k and column $k+i$ (note that these new diagonals are in different orientation than the previously defined diagonals).

Example 5.4.1. *Labelling each box with i if it is in the i-diagonal, we get:*

0	*1*	*2*	*3*	*4*	*5*
-1	*0*	*1*	*2*	*3*	*4*
-2	*-1*	*0*	*1*	*2*	*3*
-3	*-2*	*-1*	*0*	*1*	*2*
-4	*-3*	*-2*	*-1*	*0*	*1*
-5	*-4*	*-3*	*-2*	*-1*	*0*

.

Let's k-color the Ferrers diagram of □ by coloring each i-diagonal by $i \pmod{k}$. A ribbon takes exactly one of each color. So if there exists a k-ribbon tableaux of shape □, then there must be equal number of cells of each color. This is not the case when $k \nmid n$. So we only concern ourselves with k that divides n.

Let ⧉$_k$ be the shape

$$n^m((k-1)m)^m((k-2)m)^m \cdots m^m/((k-1)m)^m((k-2)m)^m \cdots m^m$$

where $m = n/k$. It forms a diagonal chain of k (n/k)-by-(n/k) squares.

We make the following observation:

Lemma 5.4.2. *The number of n-by-n k-ribbon tableaux is equal to the number of standard tableaux of shape* ⧉$_k$.

Proof. One can see this using hook length formula and Kostka-Foulkes polynomial. However, we present a bijective proof.

Given a standard tableau of shape ⧉$_k$, we construct a k-ribbon tableaux by placing each entry from d-diagonal of ith square (from the bottom, although the order does not matter) to a ribbon that spans $i - k(d+1)$ to $i - kd - 1$-diagonal. Then the ribbon tableau is uniquely determined by sorting each i-diagonal. This will be much clearer after peaking at the example below. The inverse is also simple to carry out, showing that we have a bijection. □

Example 5.4.3. *Start with a tableau of shape* $_3$*:*

				4	7
				5	8
		2	6		
		9	12		
1	3				
10	11				

.

We list which i-diagonals the ribbons span:

i-diagonal						
5						7
4				6		7
3		3		6		7
2		3		6	4	8
1		3	2	12	4	8
0	1	11	2	12	4	8
-1	1	11	2	12	5	
-2	1	11	9		5	
-3	10		9		5	
-4	10		9			
-5	10					

.

Sorting this yields:

i-diagonal						
5	7					
4	6	7				
3	3	6	7			
2	3	4	6	8		
1	2	3	4	8	12	
0	1	2	4	8	11	12
-1	1	2	5	11	12	
-2	1	5	9	11		
-3	5	9	10			
-4	9	10				
-5	10					

,

which results in the 3-ribbon tableau:

1	2	3	3	6	7
1	2	3	4	6	7
1	2	4	4	6	7
5	5	5	8	8	8
9	9	9	11	11	12
10	10	10	11	12	12

.

We studied $k = n$ case and gave a bijection between $\mathrm{SYT}(\boxminus_n)$ and $\mathrm{SYT}_n(\square)$. In general, we would like a bijection between $\mathrm{SYT}(\boxminus_k)$ and $\mathrm{SYT}_{n^2/k}(\square)$. When $k = 1$, the bijection is trivial since $\boxminus_1$ and $\square$ is the same shape.

When $k = 2$, we can rectify $T \in \mathrm{SYT}(\boxminus_2)$ to get the first half of a tableaux in $\mathrm{SYT}_{n^2/2}(\square)$ and 'dual' rectify it to get the second half.

Example 5.4.4. *Let $n = 4$, $k = 2$, and*

$$T = \begin{array}{|c|c|c|c|} \hline & & 4 & 6 \\ \hline & & 5 & 8 \\ \hline 1 & 2 & & \\ \hline 3 & 7 & & \\ \hline \end{array} \in \mathrm{SYT}(\boxminus_2).$$

$$rect(T) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 8 & \\ \hline 7 & & & \\ \hline \end{array}, \quad rect^*(T) = \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline & 2 & 5 & 6 \\ \hline 1 & 3 & 7 & 8 \\ \hline \end{array}.$$

Combining the two tableaux, we get
$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 8 & 12 \\ \hline 7 & 10 & 13 & 14 \\ \hline 9 & 11 & 15 & 16 \\ \hline \end{array} \in \mathrm{SYT}_8(\square).$$

After $n(n/2)$ promotions on the end result, the second half rectifies to the first half. After $-n(n/2)$ promotions, the first half 'dual' rectifies to the second half. Since all square tableaux are fixed under $n(n/2) - (-n(n/2)) = n^2$ promotions, these two end results are the same tableau. Therefore, this tableaux is also same as the original tableau, which then is fixed by $n(n/2)$ promotions. Other details can be checked to see that this is indeed a bijection between $\mathrm{SYT}(\boxminus_2)$ and $\mathrm{SYT}_{n^2/2}(\square)$.

Under this bijection, the promotion of tableaux in $\mathrm{SYT}_{n^2/2}(\square)$ corresponds to the promotion of tableaux in $\mathrm{SYT}(\boxminus_2)$. This can also be said for cases where $k = 1$ and $k = n$. These special cases seem to imply that the sliding/rectification procedures could be generalized to prove CSP for square tableaux.

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