The two-space homogenization method

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we consider the two-space homogenization method, which produces macroscopic expressions out of descriptions of the behaviour of the microstructure. Specifically, we focus on its application to poroelastic media. After describing the method, we provide examples to demonstrate that the resultant expressions are equivalent to an explicit derivation, which might not always be possible, and to outline the method for proving that the expressions converge to their macroscopic equivalents. Upon providing the basis for this method, we follow Burridge and Keller's work for using this to prove the existence of Biot's consolidation equations for poroelastic media and to provide expressions for the derivation of the parameters of these equations from the microstructure [5]. We then discuss the benefits and challenges that arise from this formulation of Biot's consolidation equations.

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without whom I would not have made it this far.

Contents

Introduction

Mathematical modeling is always a balancing act of detail. Too much detail results in models that are too complicated to be useful, while insufficient detail results in inaccurate models. While in some problems it is fairly obvious which aspects are important to capture in the model, this is not always the case.

One kind of problem of this type involves properties that vary on multiple scales. One common form of multiple scales is that of two length scales: a microscopic scale and a macroscopic scale. This is to say that we examine problems where the properties of interest change significantly between points in space that are a microscopic distance apart as well as points that are different to a degree that is appreciable on a macroscopic level.

A simple example of such a problem would be determining the behaviour of a composite material formed of thin fibres of two different materials [12]. On a macroscopic scale, this composite material would seem to be homogeneous, but on a microscopic scale, it is possible to distinguish between the two materials. As such, on the macroscopic scale, properties dependent upon the composition of the materials would seem to vary slowly, while they would vary rapidly on the micoscopic scale as it changed between materials. Another example would be the treatment of acoustic waves through a turbulent fluid [12].

Several methods to deal with this problem have been developed, such as volume averaging and mixture theory (see [9]). These two methods have been used with some success in studying our application of interest – that of a porous, elastic solid that is saturated with fluid – and, in fact, some ideas that would later develop into the volume averaging method were used in Biot's study of the problem in the middle of the 20th century. Both of these methods consider a set of continuous points, but their approach to averaging is different. In the volume averaging method, we consider a heterogenous "representative volume element" (referred to as an RVE) which describes the microstructure around this point in order to determine the continuum properties at the point [9, 21]. On the other hand, in the mixture theory approach the point is treated as though it is occupied by each substance in the overall medium – in this case, fluid and solid. Rather than considering the microstructure around this point, we simply work with the flux through that point [9].

While this is a very simplified explanation for these methods, we will be concentrating on the method of homogenization for two spatial scales for the majority of this work and, as such, will not require a more thorough understanding of the other methods. In this method, we consider how these properties vary on both a small and large scale and then average variation on the small scale to obtain an understanding of the larger scale. A higher number of scales is possible (see chapter 1, section 8 in [1] for an example), but since our application of interest only uses two scales, we will focus on two scales. To assist in this averaging, we generally assume that the small-scale variation repeats periodically throughout the medium. We will consider this method in more detail in later sections.

In the first section, we introduce the homogenization method in general terms. In the second section, we derive an effective thermal conductivity for the one-dimensional heat equation through the homogenization method and compare it to the equivalent effective parameter determined explicitly to demonstrate the equivalency between homogenization results and explicit results for this example. In the third section, we prove that the functions derived from the method converge to the proper solution in the case of an elliptic differential equation. While no general proof is given, the proof may be altered for a variety of problems (some are given in [1] and [18]) and later results are given assuming this convergence. In the fourth section, we derive Biot's equations for a poroelastic medium from its microstructure and provide relations between the microscopic properties and the equations of the entire medium. In the final section, we discuss the advantages and shortcomings of the method.

CHAPTER 1

The two-space homogenization method

The two-space homogenization method is applied to problems where properties vary over two different length scales, where one is much smaller than the other. Typically, these scales are a microscopic scale where the space is clearly heterogeneous and a macroscopic scale over which the space appears approximately homogeneous (see [12]. However, despite using the term "microscopic," the length scale may not be strictly microscopic. Rather, it refers to a scale much smaller than the macroscopic scale and may be more accurately considered a mesoscopic scale. For example, we may require that continuity assumptions hold on the smaller scale, which therefore cannot be on an atomic level.

These scales may also be referred to as a fast scale and a slow scale, respectively (as in [5]). They are largely equivalent, as properties that vary on a microscopic scale will appear to vary rapidly from a macroscopic viewpoint and, similarly, macroscopic variations will seem slow from a microscopic viewpoint. We will use these terms interchangeably throughout the thesis.

We begin with an initial set of equations that are influenced by the heterogeneous microstructure; for example, a medium that contains a fluid phase might begin with the Navier-Stokes equations (as in [5] and [13]). However, in order to represent variations on these separate scales, properties of a medium in n dimensions are represented as functions of $2n$ dimensions, separated into two n-dimensional vectors (or scalars in one dimension), $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ [18]. These variables x and y are referred to as the slow (or macroscopic) variable and the fast (or microscopic) variable respectively,

allowing macroscopic variation to be represented in expressions of x and microscopic variation in y. Many works in homogenization quantify this fast variation in y by relating the two by the relation

$$y = \frac{x}{\varepsilon} \tag{1.1}$$

for a small, strictly positive parameter ε (for example, [1, 5, 12, 13, 18]). The precise definition of ε is not as significant as the assumption that it is small, but it is often considered to be approximately the ratio of the microscopic length scale to the macroscopic length scale.

This comes with some advantages. Most significantly, it justifies the definition of y as the "fast" variable by giving a quantifiable reasoning for its rate of change, since for a function f with a bounded derivative f',

$$\frac{d}{dx} f\left(\frac{x}{\varepsilon}\right) = \varepsilon^{-1} f'\left(\frac{x}{\varepsilon}\right)$$

which, for a small ε, is large even if f' is bounded [13]. This factor of ε^{-1} multiplied by f' in the derivative also provides a justification for replacing the derivatives in n dimensions in the initial set of equations for derivatives of the $2n$-dimensional functions used in the homogenization equation as

$$\nabla \to \nabla_x + \varepsilon^{-1} \nabla_y \tag{1.2}$$

where ∇_x and ∇_y are del operators with respect to x and y respectively [5].

However, defining y as in (1.1) is not particularly rigorous, as the homogenization method requires x and y to be treated as independent variables at some stages of the derivation of the homogenized equations. Rather, a more cautious approach is to define x as the variable of macroscopic variation and y as the respective microscopic variable, which allows for them to be independent. The derivative substitution (1.2) may be viewed as an assumption of the method rather than a result of relation (1.1). The relation (1.1) is then a diagonal in the $2n$-dimensional space of (x,y), which is made significant as the diagonal along which the physical solutions lie in the n-dimensional problem [5]. Specifically, the $2n$-dimensional functions $f(x,y)$ in the equations with the substitution (1.2) are expansions

from this diagonal of the n-dimensional functions of our initial set of equations $f(x)$ by [12]

$$f(x, \frac{x}{\varepsilon}) = f(x)$$

In both [12] and [5], this line of reasoning was used after the previous definition to justify the treatment of x and y as independent. Here, we use this as the definition to allow for true independence of the variables.

This definition causes an epsilon dependence in the original n-dimensional functions, which arises from a perturbation expansion of the functions in terms of epsilon. This is also used in the $2n$-dimensional functions such that

$$f(x,y) = f_0(x,y) + \varepsilon f_1(x,y) + \frac{\varepsilon^2}{2} f_2(x,y) + O(\varepsilon^3) \tag{1.3}$$

These expansions are substituted into the set of equations. Since this is an infinite series expansion, it is possible to equate the parts multiplied by ε^i for $i \in \mathbb{Z}$ and obtain equations for f_0, f_1, and so on.

The final assumption required for the homogenization method is the periodicity of the heterogeneous underlying microstructure, which is represented in periodic behaviour of the properties in y. We refer to a cell over which one period occurs as Y, which is repeated over the entirety of the domain of the medium in y, Ω [18]. This domain may be bounded, but calculation for even simple boundary conditions can become complicated [1] and thus many applications assume that the medium is infinite to simplify calculations (for example, in [5] and [20]). Since x and y are considered as different scales over the same medium, it is simplest to consider both domains as being infinite.

It is from this point that the specific steps vary dependent on these equations. The initial goal is to obtain expressions for the zeroth-order part of the epsilon expansion, $f_0(x,y)$, for each property, which is taken to be approximately equivalent to the property since ε is small. This may only require the zeroth and first order expansion terms (such as in [5]) or it may require higher-order terms (such as in [13]). The eventual goal will be to obtain expressions for these zeroth-order terms only in terms of the macroscopic variable x.

In some cases, the zeroth-order term is already only in terms of x. For those where the

zeroth-order term has a dependence on y, it becomes necessary to average over the microscopic variable y to provide an approximation to the macroscopic behaviour. If the shape of a period cell Y is explicitly determined, the average takes the form of an integral with respect to y over Y divided by the volume of Y [12]. If the specifics of the periodicity is not given, then we take the average over the whole domain of y divided by its volume. In the case where the domain is infinite, this may be accomplished by taking the integral over a ball of radius R divided by the volume of the ball and then taking the limit as R extends to infinity.

Whatever the form the average takes, this is used on the expressions for the zeroth-order terms that still depend upon y. These averaged expressions become the new equations for the macroscopic medium.

Since the flexibility of this method means that we cannot give a more exact description of the steps required to obtain these expressions, we will make this more explicit by providing a simple example. Specifically, we will demonstrate its use in a one-dimensional heat equation. We use the one-dimensional heat equation because an explicit treatment of the equation can provide an equivalent result as the one obtained by the method of homogenization. While this example cannot be used as proof of the method of homogenization, it does lend credence to it when used in circumstances where an explicit solution is not possible.

CHAPTER 2

The one-dimensional heat equation

In order to demonstrate the two-space homogenization method, we apply it to the one-dimensional heat equation along a metal rod and demonstrate that it is equivalent to the explicit solution. We largely follow the approach used by Keller in [13], rearranging and expanding the calculations as required for greater clarity.

However, despite its usefulness as an example of the two-space homogenization method, its use in practical applications is limited. In this case, the only new information obtained is a relation between the small scale behaviour of the parameters and the effective macroscopic behaviour of these parameters, so it is only useful when we can model this small scale behaviour. This requires a much more intimate knowledge of the impurities of a specific metal rod than is practical in real applications. This does not detract from its usefulness as a demonstration of the method, but it does suggest that one must be careful in considering the application of the method to real-life situations.

2.1 The two-space solution

The one-dimensional heat equation, which models heat flow along a rod, is a well-known application of the theory of ordinary differential equations. For a rod where the thermal conductivity, k, varies along its length, the equation for the steady state temperature

distribution is

$$\frac{d}{dx}\big(k(x)\frac{d}{dx}u(x)\big) = h(x) \qquad 0 \le x \le 1 \tag{2.1a}$$

where x represents the position along the rod, defined such that the ends of the rod are at 0 and 1, u represents the temperature along the rod, and h represents a heat source. For this example, we assume that one end of the rod is held at temperature 0 while the other end is insulated so that no heat flows out of the rod at that end. This results in the simple boundary conditions

$$\begin{aligned} u(0) &= 0 \\ \frac{du(1)}{dx} &= 0 \end{aligned} \tag{2.1b}$$

We also will assume that the thermal conductivity k is both positive and bounded to prevent unphysical conditions. This assumption will also prove useful later in the application of the two-space homogenization method.

However, for this method, it is necessary to write the equations in terms of a slow variable x for changes over a macroscopic scale and a fast variable y for changes over the microscopic scale, turning it into a partial differential equation. A small parameter ε, independent of x and y, is also introduced so that

$$u(x,y,\varepsilon) = u_0(x,y) + \varepsilon u_1(x,y) + \frac{\varepsilon^2}{2}u_2(x,y) + O(\varepsilon^3) \tag{2.2}$$

and so that the derivative in (2.1a) may be replaced by the partial derivatives

$$\frac{d}{dx} \to \frac{\partial}{\partial x} + \varepsilon^{-1}\frac{\partial}{\partial y}$$

For the sake of convenience, we will use ∂_x and ∂_y to represent these partial derivatives for the rest of this section. From this, (2.1a) becomes

$$\big(\varepsilon^{-2}\partial_y k(x,y)\partial_y + \varepsilon^{-1}(\partial_x k(x,y)\partial_y + \partial_y k(x,y)\partial_x) + \partial_x k(x,y)\partial_x\big)u(x,y,\varepsilon) = h(x,y) \tag{2.3}$$

Using the epsilon expansion (2.2) and equating like powers of ε, the lowest ε terms result

in three equations

$$O(1): \quad \partial_y\big(k(x,y)\partial_y u_0(x,y)\big) = 0 \tag{2.4}$$

$$O(\varepsilon): \quad \partial_y\big(k(x,y)\partial_y u_1(x,y)\big) = -\Big(\partial_x\big(k(x,y)\partial_y\big) + \partial_y\big(k(x,y)\partial_x\big)\Big)u_0(x,y) \tag{2.5}$$

$$O(\varepsilon^2): \quad \frac{1}{2}\partial_y\big(k(x,y)\partial_y u_2(x,y)\big) = -\Big(\partial_x\big(k(x,y)\partial_y\big) + \partial_y\big(k(x,y)\partial_x\big)\Big)u_1(x,y) - \partial_x\big(k(x,y)\partial_x u_0(x,y)\big) + h(x,y) \tag{2.6}$$

Solving (2.4) in the most straightforward manner possible, we obtain

$$\begin{aligned} \partial_y k(x,y)\partial_y u_0(x,y) &= 0 \\ k(x,y)\partial_y u_0(x,y) &= f(x) \\ \partial_y u_0(x,y) &= f(x)k^{-1}(x,y) \\ u_0(x,y) &= f(x)\int_{y_0}^{y} k^{-1}(x,y')dy' + g(x) \end{aligned}$$

for an arbitrary y_0. In the absence of boundary conditions for $u_0(x,y)$, the unknown functions f and g must be determined in terms of the functions at this same y_0. To determine g, we set $y = y_0$, which makes the integral zero, giving

$$u_0(x,y_0) = g(x)$$

Similarly for f, we have

$$f(x) = k(x,y_0)\big(\partial_y u_0(x,y_0)\big)$$

Thus, the solution for arbitrary y_0 can be expressed as

$$u_0(x,y) = u_0(x,y_0) + k(x,y_0)\big(\partial_y u_0(x,y_0)\big)\int_{y_0}^{y} k^{-1}(x,y')dy' \tag{2.7}$$

However, since k is strictly positive and bounded, its reciprocal never approaches zero and thus the integral does not reach a finite limit as the arbitrary y_0 increases. While the slow variable x is bounded, the domain of the fast variable along the physical diagonal

$y = \varepsilon^{-1}x$ may be made arbitrarily large. So, under the assumption that the heat function u_0 is bounded, $\partial_y u_0(x,y_0)$ must be zero, since the integral is nonzero when $y \neq y_0$ and k is strictly positive. Since y_0 is arbitrary, this shows that the derivative of u_0 with respect to y is 0 at every point. So u_0 is a function of the slowly varying term x only.

Using this information in (2.5) to eliminate a derivative with respect to y, we obtain

$$\partial_y k(x,y)\big(\partial_y u_1(x,y) + \partial_x u_0(x)\big) = 0 \tag{2.8}$$

We can solve (2.8) in a similar way to (2.4), such that

$$\begin{aligned} k(x,y)\big(\partial_y u_1(x,y) + \partial_x u_0(x)\big) &= f(x) \\ \partial_y u_1(x,y) &= k^{-1}(x,y)f(x) - \partial_x u_0(x) \\ u_1(x,y) &= f(x)\int_{y_0}^{y} k^{-1}(x,y')dy' - \partial_x u_0(x)y + g(x) \end{aligned}$$

By the same method used to determine f and g in (2.4), we find

$$\begin{aligned} f(x) &= k(x,y_0)\big(\partial_y u_1(x,y_0) + \partial_x u_0(x)\big) \\ g(x) &= y_0\partial_x u_0(x) + u_1(x,y_0) \end{aligned}$$

and thereby obtain

$$u_1(x,y) = u_1(x,y_0) - (y-y_0)\partial_x u_0(x) + k(x,y_0)\big(\partial_y u_1(x,y_0) + \partial_x u_0(x)\big)\int_{y_0}^{y} k^{-1}(x,y')dy'$$

By rearranging and dividing by $y-y_0$, as before, it can be shown that

$$\partial_x u_0(x) = \frac{u_1(x,y_0) - u_1(x,y)}{y-y_0} + \frac{k(x,y_0)\big(\partial_y u_1(x,y_0) + \partial_x u_0(x)\big)\int_{y_0}^{y} k^{-1}(x,y')dy'}{y-y_0} \tag{2.9}$$

As with u_0, we assume that u_1 is a bounded function. Then, we define a function $k_0(x)$ as

$$k_0^{-1}(x) = \lim_{y\to\infty} \frac{1}{y-y_0} \int_{y_0}^{y} k^{-1}(x,y')dy' \tag{2.10}$$

We assume that this limit exists and is independent of y_0. With this, we can take the limit of (2.9) as y goes to infinity to find

$$\begin{aligned} \partial_x u_0(x) &= k(x,y_0)\big(\partial_y u_1(x,y_0) + \partial_x u_0(x)\big)k_0^{-1}(x) \\ k(x,y_0)\partial_y u_1(x,y_0) &= k_0(x)\partial_x u_0(x) - k(x,y_0)\partial_x u_0(x) \end{aligned} \tag{2.11}$$

Now, since k_0 is independent of both y and y_0 and (2.11) is only in terms of y_0, we may replace the arbitrary y_0 with the variable y in (2.11). This is simply a notational change, since both k and $\partial_y u_1$ are not affected by the symbol used for the second variable.

With this, we can substitute this expression into (2.6) and obtain

$$\begin{aligned} \partial_y k(x,y)\partial_y \frac{u_2(x,y)}{2} + \partial_y k(x,y)\partial_x u_1(x,y) &= -\partial_x\big(k_0(x)\partial_x u_0(x) - k(x,y)\partial_x u_0(x)\big) \\ &\quad - \partial_x k(x,y)\partial_x u_0(x) + h(x,y) \\ \partial_y\Big(k(x,y)\partial_y \frac{u_2(x,y)}{2} + k(x,y)\partial_x u_1(x,y)\Big) &= -\partial_x k_0(x)\partial_x u_0(x) + h(x,y) \end{aligned} \tag{2.12}$$

Integrating (2.12) with respect to y from y_0 to y results in

$$\Big(k(x,y)\partial_y \frac{u_2(x,y)}{2} + k(x,y)\partial_x u_1(x,y)\Big)\Big|_{y_0}^{y} = -(y-y_0)\partial_x k_0(x)\partial_x u_0(x) + \int_{y_0}^{y} h(x,y')dy' \tag{2.13}$$

We define the average of h with respect to y as

$$\overline{h}(x) = \lim_{y\to\infty} \frac{1}{y-y_0} \int_{y_0}^{y} h(x,y')dy' \tag{2.14}$$

Under the assumption that the expression on the lefthand side of (2.13) is bounded, we divide the equation by $(y-y_0)$ and take the limit as y tends to infinity. Thus, we obtain

$$\frac{d}{dx}\Big(k_0(x)\frac{du_0}{dx}(x)\Big) = \overline{h}(x) \tag{2.15}$$

This gives us an expression for the slowly varying, or macroscopic scale, properties of the heat equation. However, this is not simply a case of averaging the functions over the swiftly varying y, as k_0 is determined from (2.10). Since the heat equation is as simple as it is, we can show that this definition for k_0 is correct by explicitly determining the effective thermal conductivity.

2.2 Effective thermal conductivity

We begin with the same boundary value problem as described in (2.1a,b). This time, we ignore the rapid variation of the source term h since its homogenized equivalent is just an average over the rapid variation. We concentrate instead on the rapid variation of the thermal conductivity k by considering it in terms of the variable $\varepsilon^{-1}x$ for a small parameter ε. For a bounded derivative of k, k', this provides the rapid variation desired, as

$$\frac{dk(\varepsilon^{-1}x)}{dx} = \varepsilon^{-1}k'(\varepsilon^{-1}x)$$

means that the rate of change $\frac{dk(\varepsilon^{-1}x)}{dx}$ is large compared to the derivative of $k(x)$ when ε is small. To capture both rapid and slow variations, we write the thermal conductivity as $k(x,\varepsilon^{-1}x)$.

For the sake of computational convenience, we write $h(x)$ as the derivative of some function $g(x)$, where we assume $g(1) = 0$. This assumption on g does not affect the results, as only the rate of change of g is important in this source term. Thus, our steady-state equation becomes

$$\frac{d}{dx}\left(k(x,\varepsilon^{-1}x)\frac{d}{dx}u(x)\right) = \frac{d}{dx}g(x) \qquad 0 \le x \le 1 \tag{2.16}$$

with the boundary conditions remaining the same as in (2.1b). The solution of the equa-

tion is quite straightforward under these conditions, as

$$\frac{d}{dx}\left(k(x,\varepsilon^{-1}x)\frac{d}{dx}u(x)-g(x)\right)=0$$
$$k(x,\varepsilon^{-1}x)\frac{d}{dx}u(x)-g(x)=c_1$$

for some constant c_1. Setting $x=1$, by our assumption on g and the second boundary condition in (2.1b), it is clear that $c_1=0$. So

$$k(x,\varepsilon^{-1}x)\frac{d}{dx}u(x)-g(x)=0$$
$$\frac{d}{dx}u(x)=\frac{g(x)}{k(x,\varepsilon^{-1}x)}$$
$$u(x)=\int_0^x \frac{g(x')}{k(x',\varepsilon^{-1}x')}dx'+c_2$$

for some constant c_2, which is determined to also be equal to zero by setting $x=0$ and using the first boundary condition in (2.1b). From this, it becomes clear that u is also dependent on ε, so we write it as

$$u(x,\varepsilon)=\int_0^x \frac{g(x')}{k(x',\varepsilon^{-1}x')}dx' \tag{2.17}$$

This is to say that $u(x,\varepsilon)$ varies rapidly as a result of its dependence on k, which is to say that microscopic changes in the position along the rod result in large changes in the value of u. To eliminate this effect and consider the macroscopic variation in u, we wish to find the limit as ε tends to zero,

$$u_0(x)=\lim_{\varepsilon\to 0}u(x,\varepsilon) \tag{2.18}$$

To prove that this limit exists and that the result may be expressed in the same manner as $u_0(x)$ in (2.15), we prove the following theorem.

Theorem 2.1: [13] Let f be a function such that the derivative of $f(x,y)$ with respect to its first argument, f_x, exists and is continuous. If for some finite value B, $|f_x(x,\varepsilon^{-1}x)|\leq B$

for all values of ε and the limit

$$\overline{f}(x) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\Delta x} \int_{\varepsilon^{-1}x}^{\varepsilon^{-1}(x+\Delta x)} f(x,y)dy \tag{2.19}$$

exists uniformly in x, independently of Δx, then

$$\lim_{\varepsilon \to 0} \int_0^x f(x', \varepsilon^{-1}x')dx' = \int_0^x \overline{f}(x')dx' \tag{2.20}$$

Proof: For notational convenience, we define the integral $I(x,\varepsilon)$ as

$$I(x,\varepsilon) = \int_0^x f(x', \varepsilon^{-1}x')dx' = \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x', \varepsilon^{-1}x')dx' \tag{2.21}$$

where $x_j = \frac{jx}{N}$, $j = 0, 1, \ldots, N-1$ for some natural number N. We define the difference between adjacent x_j as $\Delta x = \frac{x}{N}$. Then, by the assumption that $|f_x| \leq B$ and the mean value theorem[23], we have

$$\begin{aligned} &|f(x', \varepsilon^{-1}x') - f(x_j, \varepsilon^{-1}x')| = |(x' - x_j) f_x(\tilde{x}_j, \varepsilon^{-1}x')|, \qquad x_j \leq \tilde{x}_j \leq x' \leq x_{j+1} \\ &|f(x', \varepsilon^{-1}x') - f(x_j, \varepsilon^{-1}x')| \leq B\Delta x \end{aligned} \tag{2.22}$$

From (2.21) and (2.22), it follows that

$$\begin{aligned} |I(x,\varepsilon) - \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx'| &= |\sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x', \varepsilon^{-1}x') - f(x_j, \varepsilon^{-1}x')dx'| \\ |I(x,\varepsilon) - \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx'| &\leq \sum_{j=0}^{N-1} |\int_{x_j}^{x_{j+1}} f(x', \varepsilon^{-1}x') - f(x_j, \varepsilon^{-1}x')dx'| \\ |I(x,\varepsilon) - \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx'| &\leq \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} |f(x', \varepsilon^{-1}x') - f(x_j, \varepsilon^{-1}x')|dx' \\ |I(x,\varepsilon) - \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx'| &\leq \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} B\Delta x dx' \\ |I(x,\varepsilon) - \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx'| &\leq \sum_{j=0}^{N-1} B(\Delta x)^2 \end{aligned}$$

$$|I(x,\varepsilon) - \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx'| \leq NB(\Delta x)^2$$
$$|I(x,\varepsilon) - \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx'| \leq \frac{Bx^2}{N} \tag{2.23}$$

Taking the limit of (2.23) as N tends to infinity, the righthand side tends to zero. Since the lefthand side is nonnegative, this means

$$\lim_{N\to\infty} |I(x,\varepsilon) - \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx'| = 0$$

Thus, as N tends to infinity, the relation between$I(x,\varepsilon)$ and $\sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx'$ approaches equality. That is to say,

$$I(x,\varepsilon) = \lim_{N\to\infty} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx' \tag{2.24}$$

If we take the limit of (2.24) as ε tends to zero, we find

$$\begin{aligned}
\lim_{\varepsilon\to 0} I(x,\varepsilon) &= \lim_{\varepsilon\to 0}\lim_{N\to\infty} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx' \\
&= \lim_{N\to\infty} \sum_{j=0}^{N-1} \lim_{\varepsilon\to 0} \int_{x_j}^{x_{j+1}} f(x_j, \varepsilon^{-1}x')dx' \\
&= \lim_{N\to\infty} \sum_{j=0}^{N-1} \lim_{\varepsilon\to 0} \varepsilon \int_{\varepsilon^{-1}x_j}^{\varepsilon^{-1}x_{j+1}} f(x_j, y)dy \quad (\textit{using } y = \varepsilon^{-1}x') \\
&= \lim_{N\to\infty} \sum_{j=0}^{N-1} \frac{\Delta x}{\Delta x} \lim_{\varepsilon\to 0} \varepsilon \int_{\varepsilon^{-1}x_j}^{\varepsilon^{-1}(x_j+\Delta x)} f(x_j, y)dy \\
&= \lim_{N\to\infty} \sum_{j=0}^{N-1} \Delta x \lim_{\varepsilon\to 0} \frac{\varepsilon}{\Delta x} \int_{\varepsilon^{-1}x_j}^{\varepsilon^{-1}(x_j+\Delta x)} f(x_j, y)dy \\
&= \lim_{N\to\infty} \sum_{j=0}^{N-1} \Delta x \overline{f}(x_j)
\end{aligned} \tag{2.25}$$

where on the final line we use the definition (2.19). The righthand side is in the form of the definite integral of $\overline{f}(x)$ from 0 to x. Thus, by (2.25) and (2.21),

$$\lim_{\varepsilon\to 0}\int_0^x f(x',\varepsilon^{-1}x')dx' = \int_0^x \overline{f}(x')dx'$$

as required. □

Using this theorem, we may now find $u_0(x)$ as defined in (2.18). If we assume that $u(x,\varepsilon)$ and its first derivative is bounded, then by using the solution to $u(x,\varepsilon)$ in (2.17) and applying theorem 2.1, we obtain

$$\begin{aligned} u_0(x) &= \lim_{\varepsilon\to 0} u(x,\varepsilon) \\ &= \lim_{\varepsilon\to 0}\int_0^x \frac{g(x')}{k(x',\varepsilon^{-1}x')}dx' \\ &= \int_0^x \lim_{\varepsilon\to 0}\frac{\varepsilon}{\Delta x}\int_{\varepsilon^{-1}x}^{\varepsilon^{-1}(x+\Delta x)} \frac{g(x')}{k(x',y)}\,dy\,dx' \\ &= \int_0^x g(x')\lim_{\varepsilon\to 0}\frac{\varepsilon}{\Delta x}\int_{\varepsilon^{-1}x}^{\varepsilon^{-1}(x+\Delta x)} \frac{dy}{k(x',y)}\,dx' \end{aligned} \tag{2.26}$$

If we define $k_0(x)$ as

$$\frac{1}{k_0(x)} = \lim_{\varepsilon\to 0}\frac{\varepsilon}{\Delta x}\int_{\varepsilon^{-1}x}^{\varepsilon^{-1}(x+\Delta x)} \frac{dy}{k(x,y)} \tag{2.27}$$

then u_0 is equal to

$$u_0(x) = \int_0^x \frac{g(x)}{k_0(x)}dx \tag{2.28}$$

By comparing u_0 to (2.17), it is clear that it is of a similar form to $u(x,\varepsilon)$. It is thereby simple to show that u_0 satisfies a similar equation to (2.16). Specifically, it satisfies the equation

$$\frac{d}{dx}\big(k_0(x)\frac{d}{dx}u_0(x)\big) = \frac{d}{dx}g(x) \qquad 0\le x\le 1 \tag{2.29}$$

From this equation, it is clear to see that $k_0(x)$ is an effective thermal conductivity for the steady-state temperature function $u_0(x)$. The definition for this effective conductivity when derived explicitly, (2.27), is equivalent to its definition when derived by the two-space homogenization method, as given in (2.10). Thus, the effective thermal conductivity

derived by the two-space homogenization method is equivalent to its explicit derivation.

Due to the simplicity of the steady-state heat equation, the two methods are roughly equivalent when it comes to deriving a macroscopic version of a differential equation that also includes microscopic effects. However, the two-space homogenization method may be applied to examples where the explicit solution may not be so obvious, as will be demonstrated in succeeding sections.

CHAPTER 3

Energy proof of the convergence of the homogenization method for a second-order elliptic equation

Since the final steps in the homogenization method depends heavily on the initial set of equations, no general proof of the convergence of the epsilon expansion has been developed. As such, we will follow the proof of one such equation, as given by Bensoussan, Lions, and Papanicolaou [1], supplementing the proof with Sanchez-Palencia's work on a similar problem [18]. Throughout this section, we will be using Einstein summation convention.

For a set of points y_j^0 in the direction of y_j, we define one periodic cell Y as [1]

$$Y = \prod_{j=1}^{n} [0, y_j^0] \subset \mathbb{R}^n \tag{3.1}$$

As might be clear from the term "period," this shape is repeated throughout the domain of y, Ω. In $\mathbb{R}^3$, this would take the form of a set of rectangular prisms filling Ω without overlapping except on the edges [18]. We will call a function Y-periodic if it repeats over these period cells. To avoid dealing with the period cells hitting the boundary, we assume Ω to be an infinite domain.

The problem we consider is a second-order elliptic equation. We define a set of bounded,

measurable functions [16] which are Y-periodic, $a_{ij}(y)$ for $i, j = 1, 2, \ldots, n$ – this is to say, these Y-periodic functions belong to $L^{\infty}(\mathbb{R}^n)$.

For this set of functions, we require that they satisfy the ellipticity condition such that there exists some $\alpha > 0$ so that

$$a_{ij}(y)\xi_i\xi_j \geq \alpha\xi_i\xi_i \qquad \forall \xi \in \mathbb{R}^n \tag{3.2}$$

almost everywhere in y.

In Bensoussan et al.'s work [1], these expressions are used to construct a family of operators dependent on x and ε, defining $y = \frac{x}{\varepsilon}$ (1.1) to represent the fast variation. However, since in our approach we are using the more careful definition of x and y as independent variables noted in chapter 1, we must begin by using y to represent the fast variation. However, since the fast variable is now independent of the slow variable x, we cannot use the derivative of x in the same manner as [1]. We instead introduce a variable that we will call z so that

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \varepsilon^{-1}\frac{\partial}{\partial y} \tag{3.3}$$

This is equivalent to the derivative substitution assumption (1.2) and will have the same result as in [1], since they make this kind of substitution at a later stage. Thus, for our purposes,

$$A_{\varepsilon} = -\frac{\partial}{\partial z_i}\Big(a_{ij}(y)\frac{\partial}{\partial z_j}\Big) \tag{3.4}$$

where in our derivation, the ε reliance suggested in the subscript is implicit through y and z. For this family of operators, equation (3.2) is an ellipticity condition [18], meaning that (3.4) is a family of second-order elliptic operators [1]. In the original derivation by Bensoussan et al., this family of operators includes a term $a_0(y)$ added to A_{ε}. However, this term is taken to be 0 everywhere in many later sections, even though it is originally defined to be strictly positive. For the sake of consistency, we do not include this term.

Using this, we consider the equation

$$A_{\varepsilon}u_{\varepsilon}(x,y) = f(x) \tag{3.5}$$

where we assume that u_ε is Y-periodic in y.

Due to this assumption and the assumption of an infinite domain Ω, we do not need to concern ourselves with boundary conditions. However, even in a bounded region, the boundary conditions are not very relevant, as claimed by Bensoussan et al. [1]. The reasoning for this relevant irrelevance is not explicitly given when they claim that the method will give what they refer to as the "right answer" regardless of the boundary conditions. Remarks in the chapter suggest that they may believe the differences in the final results based on different boundary conditions are relatively small compared to the increase in technical difficulty arising from keeping track of the boundary conditions. This technical difficulty and the reasoning behind the small changes in the result for differences in boundary conditions may both arise from the periodicity in the result throughout the domain, although, again, this is not explicitly stated.

We also assume that the source term f is only a function of the macroscopic variable x. Bensoussan et al. suggest in their convergence proof that this may be generalized to being in terms of both x and y [1]. However, in an earlier part of the homogenization method, they use its independence from y and thus we will also assume it to be independent of y.

From here, we seek a homogenized solution in the form of an operator A and a function $u(x)$ so that $u_\varepsilon(x,y)$ converges weakly to $u(x)$ as ε tends to zero and u satisfies

$$Au(x) = f(x) \tag{3.6}$$

We call A the homogenized operator of A_ε.

To begin, we follow the standard steps of the homogenization method. As such, we expand u_ε as an epsilon expansion

$$u_\varepsilon(x,y) = u_0(x,y) + \varepsilon u_1(x,y) + \frac{\varepsilon^2}{2} u_2(x,y) + O(\varepsilon^3) \tag{3.7}$$

where each function u_j is also Y-period in y. Using this and the derivative substitution (3.3), the ε-dependent operator A_ε is rewritten as

$$A_\varepsilon = \varepsilon^{-2} A_1 + \varepsilon^{-1} A_2 + A_3 \tag{3.8}$$

where

$$\begin{aligned} A_1 &= -\frac{\partial}{\partial y_i}\Big(a_{ij}(y)\frac{\partial}{\partial y_j}\Big) \\ A_2 &= -\frac{\partial}{\partial y_i}\Big(a_{ij}(y)\frac{\partial}{\partial x_j}\Big) - \frac{\partial}{\partial x_i}\Big(a_{ij}(y)\frac{\partial}{\partial y_j}\Big) \\ A_3 &= \frac{\partial}{\partial x_i}\Big(a_{ij}(y)\frac{\partial}{\partial x_j}\Big) \end{aligned} \tag{3.9}$$

We rewrite (3.5) with the expansions (3.7) and (3.8) so that

$$\varepsilon^{-2}A_1u_0 + \varepsilon^{-1}\Big(A_1u_1 + A_2u_0\Big) + \Big(A_1\frac{u_2}{2} + A_2u_1 + A_3u_0\Big) + \ldots = f$$
$$A_1u_0 + \varepsilon^1\Big(A_1u_1 + A_2u_0\Big) + \varepsilon^2\Big(A_1\frac{u_2}{2} + A_2u_1 + A_3u_0 - f\Big) + \ldots = 0$$

Since our definition of y is independent of ε, we equate the coefficients of the various orders of ε to zero. For the first three orders of ε, we obtain

$$O(1): \qquad A_1u_0 = 0 \tag{3.10}$$
$$O(\varepsilon): \qquad A_1u_1 + A_2u_0 = 0 \tag{3.11}$$
$$O(\varepsilon^2): \qquad A_1\frac{u_2}{2} + A_2u_1 + A_3u_0 = f \tag{3.12}$$

To solve these equations, we will need a lemma and the appropriate theoretical framework. We define a Hilbert space $H^1(Y)$ as the completion of the set

$$H^1(Y) = \Big\{v\Big|v, \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n} \in L^2(Y)\Big\} \tag{3.13}$$

under the norm

$$||v||^2_{H^1(Y)} = \int_Y \Big(v^2 + \big(\frac{\partial v}{\partial x_i}\big)^2\Big)dy \tag{3.14}$$

We also define a set

$$W(Y) = \{\phi|\phi \in H^1(Y), \phi \ \ Y\text{-}periodic\} \tag{3.15}$$

However, since ϕ is only considered over Y, this only means that ϕ takes the same value on opposite sides of Y. Using these, we write the following lemma.

Lemma 3.1 [1] For $\phi \in W(Y)$ and $F \in L^2(Y)$, the equation

$$A_1\phi(x,y) = F(x,y) \quad in\ Y \tag{3.16}$$

has a unique solution for ϕ up to the addition of a function of x if and only if

$$\int_Y F(x,y)dy = 0 \tag{3.17}$$

Proof: To show that (3.16) implies (3.17), we note that both a_{ij} and ϕ are Y-periodic. Thus, using the relation (3.16),

$$\int_Y F(x,y)dy = \int_Y A_1\phi(x,y) = -\int_Y \frac{\partial}{\partial y_i}\Big(a_{ij}(y)\frac{\partial\phi}{\partial y_j}(x,y)\Big)dy = 0$$

which is (3.17). So it is left for us to prove that (3.17) implies (3.16).

We assume that (3.17) is true. For arbitrary $\phi, \psi \in W(Y)$, we define an operator a_1 as

$$a_1(\phi,\psi) = \int_Y a_{ij}(y)\frac{\partial\phi}{\partial y_j}\frac{\partial\psi}{\partial y_i}dy \tag{3.18}$$

and an inner product

$$(F,\psi)_Y = \int_Y F(x,y)\psi(x,y)dy. \tag{3.19}$$

Equation (3.16) may be rewritten in an equivalent form for arbitrary $\psi \in W(y)$ by

$$\begin{aligned}
\int_Y \psi(x,y)A_1\phi(x,y)dy &= \int_Y \psi(x,y)F(x,y)dy \\
-\int_Y \psi(x,y)\frac{\partial}{\partial y_i}\Big(a_{ij}(y)\frac{\partial\phi}{\partial y_j}(x,y)\Big)dy &= \int_Y \psi(x,y)F(x,y)dy \\
-\int_Y \frac{\partial}{\partial y_i}\Big(\psi(x,y)a_{ij}(y)\frac{\partial\phi}{\partial y_j}(x,y)\Big) + \frac{\partial\psi}{\partial y_i}(x,y)a_{ij}(y)\frac{\partial\phi}{\partial y_j}(x,y)\Big)dy &= \int_Y \psi(x,y)F(x,y)dy \\
-\int_Y \frac{\partial}{\partial y_i}\Big(\psi a_{ij}\frac{\partial\phi}{\partial y_j}\Big)dy + \int_Y \frac{\partial\psi}{\partial y_i}a_{ij}\frac{\partial\phi}{\partial y_j}dy &= \int_Y \psi F dy
\end{aligned}$$

Because a_{ij}, ϕ, and ψ are all Y-periodic, the first integral is equal to zero. This leaves us with

$$a_1(\phi,\psi) = (F,\psi)_Y, \qquad \forall \psi \in W(y) \tag{3.20}$$

We now consider the subset of $W(y)$ not containing functions that are constant in y,

$$W^*(Y) = W(Y)\backslash\mathbb{R} \tag{3.21}$$

From our assumption that (3.17) holds, then for $\psi \in W^*(Y)$, $(F,\psi)_Y = (F,\psi+c)_Y$ $\forall c \in \mathbb{R}$. So we have a continuous linear form $\psi \to (F,\psi)_Y = (F,\psi+c)_Y$. By what Sanchez-Palencia refers to as the Friedrichs inequality [18], there exists $\gamma > 0$ such that

$$a_1(\phi,\phi) \geq \gamma ||\phi||^2_{L^2} \qquad \forall \phi \in W^*(Y)$$

By substituting $\xi_i = \frac{\partial \phi}{\partial y_i}$ into the ellipticity condition (3.2), we also obtain that

$$a_{ij}(y)\frac{\partial \phi}{\partial y_i}\frac{\partial \phi}{\partial y_j} \geq \alpha \frac{\partial \phi}{\partial y_i}\frac{\partial \phi}{\partial y_i}$$
$$\int_Y a_{ij}(y)\frac{\partial \phi}{\partial y_i}\frac{\partial \phi}{\partial y_j}dx \geq \alpha \int_Y \Big(\frac{\partial \phi}{\partial y_i}\Big)^2 dx \qquad \forall \phi \in W^*(Y)$$

Letting $c = \inf(\frac{\gamma}{2}, \frac{\alpha}{2})$, we obtain

$$a_1(\phi,\phi) \geq c||\phi||^2_{W^*(Y)} \qquad \forall \phi \in W^*(Y) \tag{3.22}$$

By (3.20), $a_1(\phi,\phi)$ is equal to our continuous linear form at ϕ, this relation (3.22) means that it is a strongly monotone operator and thus (3.20) has a unique solution in $W^*(Y)$ [25]. Since $W^*(Y)$ does not admit constants with respect to y, this means that the solution is only unique up to an additive function of x in $W(Y)$. □

We are now able to solve equations (3.10)-(3.12). We follow Bensoussan et al.'s derivation closely for these equations [1].

(3.10): Since A_1 includes a derivative with respect to y of u_0, a function that is independent of x is a solution to (3.10). Since (3.10) is of the form of (3.16) for $F = 0$, by Lemma 3.1,

this solution is unique up to an additive function of x, retaining its independence from y. Thus, u_0 is independent of y and may be written as

$$u_0(x,y) = u(x) \tag{3.23}$$

(3.11): Since u_0 is independent of y, (3.11) reduces to

$$A_1 u_1 = \Big(\frac{\partial a_{ij}}{\partial y_i}(y)\Big)\Big(\frac{\partial u}{\partial x_j}(x)\Big) \tag{3.24}$$

Because of this, we use separation of variables to resolve this. The operator A_1 is entirely in terms of y, so we only need to consider the y terms in the equation. For this, we define a Y-periodic function $\chi_j(y)$ as the solution to the equation

$$A_1\chi_j(y) = -\frac{\partial}{\partial y_i}a_{ij}(y) = A_1 y_j \tag{3.25}$$

Due to the periodic nature of $a_{ij}(y)$, $\int_Y A_1 y_j dy = 0$, so we find that $\chi_j(y)$ is a unique solution to (3.25) up to an additive function of x. So the solution of (3.24) may be written as

$$u_1(x,y) = -\chi_j(y)\frac{\partial u}{\partial x_j}(x) + \tilde{u}_1(x) \tag{3.26}$$

where $\tilde{u}_1(x)$ is a result of the additive function of x with χ_j being multiplied by $\frac{\partial u}{\partial x_j}$. Its exact form is not important for this discussion.

(3.12): By Lemma 3.1, we know that, for there to be a unique solution for u_2, the integral with respect to y of the terms other than $A_1\frac{u_2}{2}$ must be zero. So, for there to be a unique solution up to an additive function of x, we require

$$\int_Y \big(A_2 u_1(x,y) + A_3 u(x)\big)dy = \int_Y f(x)dy = |Y|f \tag{3.27}$$

where $|Y|$ is the measure of Y (in two dimensions, this is the area of Y; in three dimensions, this is the volume). Replacing the index j with k for later convenience, it is possible to

simplify the integral of A_2u_1 such that

$$\int_Y A_2u_1(x,y)dy = -\frac{\partial}{\partial x_i}\int_Y a_{ik}(y)\frac{\partial u_1}{\partial y_k}(x,y)dy$$

By substituting the solution to u_1, (3.26), into the above equation, we obtain

$$\int_Y A_2u_1(x,y)dy = -\frac{\partial^2 u}{\partial x_i \partial x_j}\int_Y a_{ik}(y)\frac{\partial \chi_j}{\partial y_k}(x,y)(y)dy$$

We substitute this into (3.27) and divide by $|Y|$, giving us

$$-\Big(\frac{1}{|Y|}\int_Y \big(a_{ij}(y) - a_{ik}(y)\frac{\partial \chi_j}{\partial y_k}(y)\big)dy\Big)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \tag{3.28}$$

This takes the form of the operator A that we were looking for in (3.6). For the sake of notational convenience, we define a parameter q_{ijk} as

$$q_{ijk} = \frac{1}{|Y|}\int_Y \big(a_{ij}(y) - a_{ik}(y)\frac{\partial \chi_j}{\partial y_k}(y)\big)dy \tag{3.29}$$

Thus, we may write A as

$$A = -q_{ijk}\frac{\partial^2}{\partial x_i \partial x_j} \tag{3.30}$$

which, by (3.28), satisfies (3.6).

It now remains to prove that u_ε converges to u as ε tends to zero.

3.1 Energy proof of convergence

Under the homogenization method, this is clear, since $u(x)$ is the zeroth-order term in the ε expansion. However, to demonstrate its viability, we demonstrate that this limit is equivalent when $y = \frac{x}{\varepsilon}$. To do so, we require the following spaces.

We have previously defined $H^1(Y)$ and its inner product related to the norm $||\cdot||^2_{H^1(Y)}$ in (3.13) and (3.14). This Hilbert space will be defined in the same way over the domain of x, which we call Ω_x. We define $H^1_0(\Omega_x)$ as a subspace of $H^1(\Omega_x)$, specifically as the

closure of the set of continuous, infinitely differentiable functions with compact support [11] in $H^1(\Omega_x)$. This space $H_0^1(\Omega_x)$ is equipped with the same $H^1(\Omega_x)$ inner product.

From this, we define V as a closed subspace of $H^1(\Omega_x)$ such that

$$H_0^1(\Omega_x) \subset V \subset H^1(\Omega_x) \tag{3.31}$$

We also define an operator a_ε such that, for $u, v \in H^1(\Omega_x)$,

$$a_\varepsilon(u,v) = \int_{\Omega_x} a_{ij}(\frac{x}{\varepsilon})\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx \tag{3.32}$$

and an operator a_2 such that

$$a_h(u,v) = \int_{\Omega_x} q_{ijk}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx \tag{3.33}$$

We also define an inner product $(\cdot,\cdot)$

$$(u,v) = \int_{\Omega_x} uvdx \tag{3.34}$$

Then for $y = \frac{x}{\varepsilon}$, equations (3.5) and (3.6) are rewritten in a similar manner to (3.20).

$$a_\varepsilon(u_\varepsilon, v) = (f,v) \qquad \forall v \in V \tag{3.35}$$

$$a_h(u,v) = (f,v) \qquad \forall v \in V \tag{3.36}$$

where $u_\varepsilon(x, \frac{x}{\varepsilon}, u(x) \in V$. Using these equivalent expressions, we now prove the convergence.

Theorem 3.1 [1] Let $u_\varepsilon(x, \frac{x}{\varepsilon})$ and $u(x)$ be functions satisfying (3.35) and (3.36). Then u_ε converges weakly to u in V as $\varepsilon \to 0$.

Proof: Following a similar method as in (3.22),

$$a_\varepsilon(v,v) \geq c||v||^2_{H^1(\Omega_x)} \qquad \forall v \in V \tag{3.37}$$

From Sanchez-Palencia [18],

$$\Big|\int_{\Omega_x} f v dx\Big| \leq ||f||_{L^2(\Omega_x)}||v||_{L^2(\Omega_x)} \leq C||v||_{H^1(\Omega_x)} \qquad \forall v \in V \tag{3.38}$$

for some constant C. Thus, by (3.22),

$$c||v||^2_{H^1(\Omega_x)} \leq a_\varepsilon(v,v) = (f,v) \tag{3.35}$$

$$c||v||^2_{H^1(\Omega_x)} \leq (f,v) \leq C||v||_{H^1(\Omega_x)} \tag{3.38}$$

$$||v||_{H^1(\Omega_x)} \leq \gamma \tag{3.39}$$

for $\gamma = \frac{C}{c}$. We define ξ_i^ε as

$$\xi_i^\varepsilon = a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j}(x, \frac{x}{\varepsilon}) \tag{3.40}$$

Then

$$||\xi_i^\varepsilon||_{L^2(\Omega_x)} \leq \gamma \tag{3.41}$$

So, by the Rellich theorem [18], there exists subsequences of u_ε and ξ_i^ε where

$$u_\varepsilon \to u_c \quad in\ V\ \ weakly \tag{3.42}$$

$$\xi_i^\varepsilon \to \xi_i \quad in\ L^2(\Omega_x)\ \ weakly \tag{3.43}$$

For notational convenience, we will refer to these subsequences by the name of their original sequences, u_ε and ξ_i^ε

We write (3.35) as

$$(\xi_i^\varepsilon, \frac{\partial v}{\partial x_i}) = (f,v) \qquad \forall v \in V \tag{3.44}$$

This expression converges weakly to

$$(\xi_i, \frac{\partial v}{\partial x_i}) = (f,v) \qquad \forall v \in V \tag{3.45}$$

It will thus suffice to show that (3.45) is equivalent to (3.36). This may be accomplished

through the use of adjoint operators. Letting A_1^* represent the adjoint of A_1 (3.9), we have

$$A_1^* = -\frac{\partial}{\partial y_i}\Big(a_{ij}^*(y)\frac{\partial}{\partial y_j}\Big), \qquad a_{ij}^* = a_{ji}. \tag{3.46}$$

Let w be a solution of

$$A_1^* w(y) = 0 \tag{3.47}$$

such that $-\hat{\chi} = w(y) - P(y)$ is Y-periodic for some homogeneous polynomial of degree 1, $P(y)$. If we apply the adjoint operator A_1^* to the definition of $\hat{\chi}$, we get

$$A_1^*\hat{\chi}(y) = A_1^* P(y) \tag{3.48}$$

We now define w_ε as [1]

$$w_\varepsilon(x) = \varepsilon w(\frac{x}{\varepsilon}) = P(x)\varepsilon\hat{\chi}(\frac{x}{\varepsilon}) \tag{3.49}$$

Bensoussan et al. claim that this is the solution to

$$(A_\varepsilon)^* w_\varepsilon(x) = 0 \tag{3.50}$$

We substitute ϕw_ε into v in (3.36) such that $\phi \in C^\infty(\Omega_x)$ and $\phi w_\varepsilon \in V$. Subtracting the scalar product of ϕu_ε and (3.50) from the resulting equation, we obtain

$$a_\varepsilon(u_\varepsilon, \phi w_\varepsilon) - a_\varepsilon(\phi u_\varepsilon, w_\varepsilon) = (f, \phi w_\varepsilon)$$

The lefthand side of this equation may be rewritten as

$$\Big(\xi_i^\varepsilon, \frac{\partial\phi}{\partial x_i} w_\varepsilon\Big) - \int_{\Omega_x} a_{ij}(\frac{x}{\varepsilon})\frac{\partial w_\varepsilon}{\partial x_i}\frac{\partial \phi u_\varepsilon}{\partial x_j} dx \tag{3.51}$$

In Bensoussan et al.'s original derivation, $\frac{\partial \phi u_\varepsilon}{\partial x_j}$ was written as $\frac{\partial\phi}{\partial x_j}u_\varepsilon$. The integral of this was later replaced by $\frac{\partial u_\varepsilon}{\partial x_j}\phi$. If we accept that $\frac{\partial\phi}{\partial x_j}u_\varepsilon$ is the correct derivation, then this replacement almost makes sense through integration by parts, but it is off by a factor of negative one. For now, we will follow their method and use $\frac{\partial\phi}{\partial x_j}u_\varepsilon$ [1]

Here, Bensoussan et al. [1] claim that (3.51) converges as $\varepsilon \to 0$ in such a way that $w_\varepsilon \to P$ in $L^2(\Omega_x)$ strongly and $\phi w_\varepsilon \to \phi P$ in $H^1(\Omega_x)$ weakly [1]. We also suppose that $a_{ij}(\frac{x}{\varepsilon})\frac{\partial w_\varepsilon}{\partial x_i} \to \overline{a_{ij}\frac{\partial w}{\partial y_i}}$, where $\bar{\cdot}$ represents an average over a periodic cell Y,

$$\overline{\psi} = \frac{1}{|Y|}\int_Y \psi dy \tag{3.52}$$

Then, using (3.45) to replace f, we obtain

$$\begin{gathered}\left(\xi_i, (\frac{\partial \phi}{\partial x_i})P\right) - \overline{a_{ij}\frac{\partial w}{\partial y_i}}\int \frac{\partial \phi}{\partial x_j}u_c dx = (f, \phi P) \\ \left(\xi_i, (\frac{\partial \phi}{\partial x_i})P\right) - \overline{a_{ij}\frac{\partial w}{\partial y_i}}\int \frac{\partial \phi}{\partial x_j}u_c dx = \left(\xi_i, \frac{\partial(\phi P)}{\partial x_i}\right)\end{gathered} \tag{3.53}$$

If we expand $\frac{\partial(\phi P)}{\partial x_i}$ in (3.53), it is possible to simplify this equation to

$$\left(\xi_i \frac{\partial P}{\partial x_i}, \phi\right) = \overline{a_{ij}\frac{\partial w}{\partial y_i}}\left(\frac{\partial u}{\partial x_j}, \phi\right) \quad \forall \phi \in C^\infty(\Omega_x)$$

Similarly to how we derived (3.35) and (3.36), this is equivalent to

$$\xi_i \frac{\partial P}{\partial x_i} = \overline{a_{ij}\frac{\partial w}{\partial y_i}}\frac{\partial u}{\partial x_j} \tag{3.54}$$

Since $P(y)$ is not a specified polynomial beyond being homogenous and first order, we take it to be equal to y_i. Then by (3.48),

$$A_1^* \hat{\chi}_i = A_1^* y_i \tag{3.55}$$

and by the definition of $\hat{\chi}$, $w = y_i - \hat{\chi}_i$ and thus (3.54) is

$$\xi_i = \overline{a_{kj}\frac{\partial}{\partial y_k}(y_i - \hat{\chi}_i)}\frac{\partial u}{\partial x_j} \tag{3.56}$$

We substitute (3.56) into (3.45) to obtain

$$\left(\overline{a_{ij} - a_{kj}\frac{\partial \hat{\chi}_i}{\partial y_k}}\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\right) = (f, v) \quad \forall v \in V \tag{3.57}$$

This equation is close to the form of (3.36). If we rearrange (3.36) to appear like (3.57), we find that they are equivalent if

$$\overline{a_{ik}\frac{\partial \hat{\chi}_j}{\partial y_k}} = \overline{a_{kj}\frac{\partial \hat{\chi}_i}{\partial y_k}}. \tag{3.58}$$

If we take the inner product of (3.55) with $\hat{\chi}_j$, we find

$$(A_1^*\hat{\chi}_i, \hat{\chi}_j)_Y = \int_Y a_{ik}\frac{\partial \chi_j}{\partial y_k} dy$$

Similarly, the inner product of (3.25) with $\hat{\chi}_i$ gives

$$(A_1\hat{\chi}_j, \hat{\chi}_i)_Y = \int_Y a_{kj}\frac{\partial \chi_i}{\partial y_k} dy$$

Due to the definition of an average $\bar{\cdot}$ and the preceding two equations, (3.58) is equivalent to

$$(A_1^*\hat{\chi}_i, \hat{\chi}_j)_Y = (A_1\hat{\chi}_j, \hat{\chi}_i)_Y \tag{3.59}$$

Due to the definition of an adjoint operator [11], this is true. Thus, (3.35) does converge to (3.36) as $\varepsilon \to 0$. This also shows that u_c, the function to which u_ε converges weakly, is equivalent to u, as desired. □

While such proofs of convergence have been done for many other types of differential equations, most works on specific applications assume this convergence exists and that their homogenized equations are legitimate. In the next chapter as we consider a specific application of this method, we will make this assumption [5].

CHAPTER 4

On *Poroelasticity equations derived from microstructure*

Poroelasticity equations model the mechanics solids which are riddled with fluid-filled pores – for simplicity, we will refer to these media as poroelastic solids. This type of solid, and thus these equations, appear in a number of applications, which include problems in the petroleum industry and biomechanics, among others [19]. Typically, the equations as derived by Biot have been used for these situations. However, while successfully used in many applications, their validity for general problems was questionable. In order to place these standard equations on a more theoretical footing, Burridge and Keller made use of the homogenization method to derive some of these equations – specifically, the equations for acoustic propagation through a poroelastic solid [4] – from the more theoretically sound linearized Navier-Stokes equations of fluids and linearized elasticity theory [5].

Certain problems have been raised regarding this derivation and it comes with certain limitations. These issues will be discussed more completely in a later section. However, Burridge and Keller's work has been used to justify the continued usage of Biot's equations under the required conditions [24] and it gives a possible direction for numerical determination of certain parameters from characteristic properties of the fluid and solid components, which has thus far been left to experimental determination (which is not always viable).

In the following sections, we follow the approach of Burridge and Keller [5] unless oth-

erwise stated. However, the details of their calculations have been considerably expanded for the sake of clarity.

4.1 Foundational arguments

For the sake of mathematical simplicity, the model of the poroelastic solid is simplified to a periodic medium. Specifically, we consider a typical section of the volume and construct an infinite volume by repeating this typical volume. The reasoning for this choice is not given in the original paper, but the likely reasoning is to allow for the use of transformation methods. In addition, this simplifies calculations by not requiring the consideration of boundary conditions, which have been previously noted to complicate calculations.

As in earlier work by Keller [12], the formulation begins with the definition of length scales h and H. In short, h is a typical microscopic length scale and H is a typical macroscopic length scale so that $h \ll H$. The microscopic length scale h in this problem is the length scale of the pore configuration, but there are several possible interpretations of the macroscopic length scale. Some examples suggested in the paper are the width of a sample of the medium and the wavelength of an acoustic wave propagating through the medium [5]. However, the specific scales are not so important as the ratio between them, denoted as $\varepsilon = \frac{h}{H}$. This ratio is very small due to the relation between h and H and thus forms the basis of a perturbation theory-like approach later in the work.

Before this can be used, the coordinate system must be defined. Some care must be taken here to ensure logical consistency, as the coordinates are used as independent and dependent variables depending on context. In either case, two separate three-dimensional coordinates are defined, $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$, where different properties of an arbitrary function, $f(\vec{x}, \vec{y})$, are expressed in terms of the different variables. In Burridge and Keller, $\vec{x}$ is the variable of slow or gradual change and $\vec{y}$ is the variable of rapid change [5]. Skotheim and Mahadevan use different notation, but the equivalent to $\vec{x}$ is considered to represent variation on the macroscopic scale and their $\vec{y}$ represents variation on the pore scale [20]. These are logically equivalent, as macroscopic variations would seem slower than changes on the pore scale if considered from the same scale.

However, both papers use these merely as descriptors, defining one variable in terms of the other. Continuing to follow Burridge and Keller's notation, the rapid variable $\vec{y}$ is defined as a stretching of the slow variable $\vec{x}$ – that is, $\vec{y} = \varepsilon^{-1}\vec{x}$. Under this definition, a small change in $\vec{x}$ results in a large change in $\vec{y}$, so it is consistent with the description of slow and rapid change [5]. But the method relies upon them being independent variables, so this definition has its flaws, as we noted in the first section.

As before, we consider the earlier descriptions of slow (or macroscopic scale) and rapid (or pore scale) variables as the definition for $\vec{x}$ and $\vec{y}$, which retains their independence.

Still, the relation between the variables, $\vec{y} = \varepsilon^{-1}\vec{x}$, has its own importance in the method and must be considered. One possible interpretation is the one used by Burridge and Keller when they consider $\vec{x}$ and $\vec{y}$ as independent variables. They claim that $\vec{y} = \varepsilon^{-1}\vec{x}$ may be considered a diagonal in the six-dimensional coordinate system formed by these variables. In this case, the physical solutions of the three-dimensional model being considered are said to lie on this diagonal [5]. This work, though, will follow the approach used in an earlier work by Keller [12]. Here, every function of a physical property, $g(\vec{x},\varepsilon)$, has an equivalent six-dimensional functional form, $g(\vec{x},\vec{y},\varepsilon)$, such that $g(\vec{x},\vec{y},\varepsilon)$ still satisfies the equations and $g(\vec{x},\varepsilon^{-1}\vec{x},\varepsilon) = g(\vec{x},\varepsilon)$.

In either view, the original physical solutions are associated with the $\vec{y} = \varepsilon^{-1}\vec{x}$ relation, which leads to the treatment of the del operator, ∇, for functions in x and y. Letting ∇_x and ∇_y represent the del operator with respect to x and y, respectively, it may be shown that

$$\nabla f(\vec{x},\varepsilon^{-1}\vec{x}) = \nabla_x f + \varepsilon^{-1}\nabla_y f.$$

Since the relation $\vec{y} = \varepsilon^{-1}\vec{x}$ represents the physical properties of the problem, ∇ is replaced with $\nabla_x + \varepsilon^{-1}\nabla_y$ for all functions of the form $f(\vec{x},\vec{y})$ (see [5]).

To simplify matters, however, time derivatives are removed by assuming that motions within the medium are time harmonic. This allows for time derivatives to be replaced by multiplication by $i\omega$, where ω is the angular frequency of the motions.

The final main assumption deals with the viscosity term, $\tilde{\mu}$, and has become customary in the literature [8]. On the pore scale, the dimensionless viscosity is defined using the fluid

density ρ_f, the angular frequency ω, and the microscopic length scale h as $\frac{\tilde{\mu}}{\omega\rho_f h^2}$. If we define the product of the frequency ω and the pore length scale h as the typical speed on the pore scale, then this is equivalent to the inverse of the Reynolds number [15] on the pore scale. Since the Reynolds number describes behaviour of the fluid and the fluid acts on the pore scale, we assume this is the true Reynolds number of the fluid and thus assume it to be of order 1 with respect to ε. This means that the inverse of the Reynolds number, which is equivalent to the dimensionless viscosity on the pore scale, is also of order 1. On the macroscopic scale, the dimensionless viscosity is defined using the macroscopic length scale H instead of h, leading to $\frac{\tilde{\mu}}{\omega\rho_f H^2}$ and can be related to a macroscopic version of the Reynolds number by defining a typical speed on the macroscopic scale similarly as ωH. Due to the relation between h and H, this macroscopic scale Reynolds number is of order ε^{-2} when the pore scale Reynolds number is of order 1, like we assume, which makes the macroscopic dimensionless viscosity of order ε^2. Since our eventual goal is to produce a description of macroscopic behaviour, the initial viscosity term $\tilde{\mu}$ is replaced by $\varepsilon^2\mu$ to represent this $O(\varepsilon^2)$ term [5].

Before getting into the derivation, it is important to formalize the underlying structure of the porous medium. The solid region is represented in the $\vec{x}$ and $\vec{y}$ coordinates as the domain D_s. Similarly, D_f represents the domain of the fluid region. These domains are then used to define an indicator function $\chi_s(\vec{x},\vec{y})$, such that χ_s has the value $\chi_s = 1$ in D_s and the value $\chi_s = 0$ outside of D_s. Since the pores are assumed to be filled with fluid, this means that it is equivalent to say $\chi_s = 0$ in D_f [5], and thus a fluid indicator function may be considered to be $1 - \chi_s$. While this function is not used directly, it may be useful when explicitly considering the functions which are commonly defined only over either the solid or fluid components.

4.2 Derivation of initial equations

Burridge and Keller's arguments follow from a set of three equations in the fluid domain, two equations in the solid domain, and two equations along the boundary. They are referred to as linearized Navier-Stokes equations, linearized equations of elasticity, and

linearized interface conditions, respectively, but they are given without reference [5]. Our first step will be to provide justification for these equations. It is important to note that the equations marked (3) in Burridge and Keller come before applying the assumptions listed in the previous section. This includes the six-dimensional coordinate system [5]. As such, the derivation will be in three dimensions and will occasionally use cartesian coordinate interpretations of such things as the del operator.

Beginning in the fluid domain, the fluid parameters and variables are as follows: fluid velocity, v; fluid pressure, p; fluid stress tensor, σ; fluid density, ρ_f; bulk modulus, κ; and fluid viscosity, $\tilde{\mu}$, which we replace with $\varepsilon^2\mu$ in later analysis.

The first equation is derived from the conservation of linear momentum, which in Einstein summation notation may be expressed as

$$\rho_f\Big(\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\Big)=\frac{\partial \sigma_{ij}}{\partial x_j}$$

Here, only linear terms are being considered. As such, the nonlinear $v_k\frac{\partial v_i}{\partial x_k}$ may be ignored. Additionally, by the previous assumption that motions are time harmonic, the time derivative of the velocity may be replaced by $i\omega$. As such,

$$i\omega\rho_f v_i=\frac{\partial \sigma_{ij}}{\partial x_j}$$

Since we are working in cartesian coordinates, this is equivalent to equation (3a) of Burridge and Keller, $i\omega\rho_f v=\nabla\cdot\sigma$ [5].

The second equation can be derived from the constitutive equation below. Letting ξ represent dilatational viscosity and η represent the shear viscosity,

$$\sigma_{ij}=(-p+\xi v_{k,k})\delta_{ij}+\eta(v_{i,j}+v_{j,i})$$

At this point, we use the Stokes relation, $\xi=-\frac{2}{3}\eta$, to get

$$\sigma_{ij}=-p\delta_{ij}+2\eta\left(\Big(\frac{1}{2}(v_{i,j}+v_{j,i}\Big)-\frac{1}{3}v_{k,k}\delta ij)\right)$$

This leads to a small problem. If it is assumed that the shear viscosity in the constitutive equation is equivalent the the fluid viscosity, $\tilde{\mu}$, as used by Burridge and Keller, this expression is nearly equivalent to (3b) in their work [5]. However, there is a factor of 2 in front of η in the above equation which is missing from Burridge and Keller. This is certainly an issue that needs to be resolved, but it will not affect this chapter's discussion, as this is the only place where the viscosity term appears and thus the factor of 2 may be lumped into the viscosity coefficient. This is largely a temporary measure, as knowledge of whether the coefficient is the true viscosity of the fluid or twice that amount is clearly of great importance in numerical simulation. However, despite some efforts in the literature, such as that by Zhou and Sheng [26], the method from Burridge and Keller has not been successfully used numerically beyond specialized results for specific parameters. For simplicity, this work will use the $\tilde{\mu}$ coefficient used in Burridge and Keller rather than the $2\tilde{\mu}$ obtained above.

The third equation is derived from the continuity equation for the fluid, where ρ_f represents the density, v the velocity, and p the pressure,

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f v_f) = 0$$
$$\frac{\partial \rho_f}{\partial t} + v_f \cdot \nabla \rho_f + \rho \nabla \cdot v_f = 0$$

Since our eventual goal is to obtain linearized equations, we drop the nonlinear term $v_f \cdot \nabla \rho_f$.

$$\frac{\partial \rho_f}{\partial t} + \rho \nabla \cdot v_f = 0$$

Now assuming an equation of state for the fluid, namely $\rho_f = \rho_f(p)$, the above equation becomes

$$\frac{\partial \rho_f}{\partial p}\frac{\partial p}{\partial t} + \rho \nabla \cdot v_f = 0$$

This may be rearranged as

$$\frac{\partial p}{\partial t} = -\frac{\rho_f}{\frac{\partial \rho_f}{\partial p}} \nabla \cdot v_f$$

By writing the density ρ_f in terms of component fluid mass, M_f, and fluid volume, V_f, this may be simplified down to

$$\frac{\partial p}{\partial t} = V_f \frac{\partial p}{\partial V_f} \nabla \cdot v_f$$
$$\frac{\partial p}{\partial t} = \kappa \nabla \cdot v_f$$

where the bulk modulus κ comes from the isothermal compressibility of thermodynamics, given by

$$\frac{1}{\kappa} = -\frac{1}{V_f}\frac{\partial V_f}{\partial p}.$$

Under our assumption that time derivatives may be replaced by $i\omega$, we obtain the third equation.

The next four equations are more common. On the boundary between the fluid and solid

domains, we use simple boundary conditions for equality of fluid and solid velocities,

$$v = i\omega u$$

and equality of normal stresses, using normal vectors $\hat{n}$ facing in towards the solid,

$$\hat{n} \cdot \sigma = \hat{n} \cdot \tau$$

where the linear elasticity stress tensor τ is given by the constitutive equation

$$\tau = C\nabla u$$

which is a generalized version of Hooke's law. The fourth-rank tensor $C(x,y)$ is left unspecified.

Finally, we require that the elastic solid obeys the Lamé equation for the displacement vector u. With our assumed replacement of the time derivative with $i\omega$, this takes the form of

$$-\omega^2 \rho_s u = \nabla \cdot \tau$$

The above results in the equations as given in Burridge and Keller's works, with the left-hand column showing the initial form of the equations and the righthand column showing the form of the equations after the application of the previously described hypotheses before the perturbation expansion. As the equations are only valid in certain regions, the centre column gives the region in which the equations are valid.

Initial equations	Domain of validity	Rearranged equations
$i\omega\rho_f v = \nabla \cdot \sigma$	D_f	$\nabla_y \cdot \sigma + \varepsilon(\nabla_x \cdot \sigma - i\omega\rho_f v) = 0$
$\sigma = -pI + \tilde{\mu} + \tilde{\mu} D\nabla v$	D_f	$\sigma + pI - \varepsilon\mu D\nabla_y v - \varepsilon^2 \mu D\nabla_x v = 0$
$i\omega p = -\kappa\nabla \cdot v$	D_f	$\kappa\nabla_y \cdot v + \varepsilon(\kappa\nabla_x \cdot v + i\omega p) = 0$
$v = i\omega u$	$\partial D_f = \partial D_s$	$v - i\omega u = 0$
$\hat{n} \cdot \sigma = \hat{n} \cdot \tau$	$\partial D_f = \partial D_s$	$hat n \cdot \sigma - \hat{n} \cdot \tau = 0$
$-\omega^2 \rho_s u = \nabla \cdot \tau$	D_s	$\nabla_y \cdot \tau + \varepsilon(\nabla_x \cdot \tau + \omega^2 \rho_s u) = 0$
$\tau = C\nabla u$	D_s	$C\nabla_y u + \varepsilon(C\nabla_x u - \tau) = 0$

After this, we apply a form of perturbation theory expansion to the field quantities: the fluid stress tensor, σ; fluid pressure, p; fluid velocity, v; solid displacement, u; and the solid stress tensor, τ. Specifically, we replace a field quantity, $f(\vec{x},\vec{y},\varepsilon)$, with

$$f(\vec{x},\vec{y},\varepsilon) = f_0(\vec{x},\vec{y}) + \varepsilon f_1(\vec{x},\vec{y}) + (\varepsilon^2/2) f_2(\vec{x},\vec{y}) + O(\varepsilon^3)$$

Since $\varepsilon \ll 1$, we can discard the higher order terms in the epsilon expansion and use $f_0(\vec{x},\vec{y})$ as an approximation to $f(\vec{x},\vec{y},\varepsilon)$. To determine their values, we consider the first two orders of epsilon – that is, first by equating epsilon to zero in one, and then taking the derivative of the equations with respect to epsilon and setting epsilon to zero. Since $f(\vec{x},\vec{y},0) = f_0(\vec{x},\vec{y})$ and $\partial_\varepsilon f(\vec{x},\vec{y},0) = f_1(\vec{x},\vec{y})$, this produces equations of the first two terms of the epsilon perturbation expansion, as shown below. For future reference, they have been labelled with the same numbering as in Burridge and Keller [5]. We will continue to use this labeling through this section and, as such, these labels may not follow strict numerical order as the order of the equations is changed for the needs of this work. A few equations will not be written in exactly the same form as given by Burridge and Keller for the same reason, but they will retain the same label as its equivalent in their work.

First order equations		Domain of validity
$\nabla_y \cdot \sigma_0 = 0$	(8a)	D_f
$\sigma_0 + p_0 I = 0$	(8b)	D_f
$\kappa \nabla_y \cdot v_0 = 0$	(8c)	D_f
$v_0 - i\omega u_0 = 0$	(8d)	$\partial D_f = \partial D_s$
$\hat{n} \cdot \sigma_0 - \hat{n} \cdot \tau_0 = 0$	(8e)	$\partial D_f = \partial D_s$
$\nabla_y \cdot \tau_0 = 0$	(8f)	D_s
$C \nabla_y u_0 = 0$	(8g)	D_s
Second order equations		**Domain of validity**
$\nabla_y \cdot \sigma_1 + \nabla_x \cdot \sigma_0 - i\omega\rho_f v_0 = 0$	(9a)	D_f
$\sigma_1 + p_1 I - \mu D \nabla_y v_0 = 0$	(9b)	D_f
$\kappa \nabla_y \cdot v_1 + \kappa \nabla_x \cdot v_0 + i\omega p_0 = 0$	(9c)	D_f
$v_1 - i\omega u_1 = 0$	(9d)	$\partial D_f = \partial D_s$
$\hat{n} \cdot \sigma_1 - \hat{n} \cdot \tau_1 = 0$	(9e)	$\partial D_f = \partial D_s$
$\nabla_y \cdot \tau_1 + \nabla_x \cdot \tau_0 + \omega^2 \rho_s u_0 = 0$	(9f)	D_s
$C \nabla_y u_1 + C \nabla_x u_0 - \tau_0 = 0$	(9g)	D_s

From this set of fourteen equations, expressions for the zeroth order terms of the epsilon expansion may be derived.

4.3 Solutions of the equations

In (8a), the divergence with respect to $\vec{y}$ of σ_0 is zero at every point. This would not be enough to prove independence from $\vec{y}$ for a second-order tensor, but by (8b), it is proportional to p_0 times an identity matrix. As such, by the definition of the divergence of a second-order tensor in cartesian coordinates [10], this results in the partial derivative of p_0 being 0 with respect to each component of $\vec{y}$. From this, both p_0 and σ_0 are independent of $\vec{y}$.

This is not true of most of the other properties with divergence 0, such as τ_0 or v_0. How-

ever, for (8g), the tensor C is a part of the generalized Hooke's Law, relating displacement to stresses, but here the stress tensor is replaced by a 0 tensor and the operator acts only on the $\vec{y}$ components. As such, the zeroth order displacement, $u_0(\vec{x},\vec{y})$, has a $\vec{y}$ component of motion under no stress, which, under the assumption of being bounded in $\vec{y}$, must be a rigid transformation in $\vec{y}$. Thus, u_0 is also independent of $\vec{y}$.

It is worth noting at this point that the paper by Biot that originally studied the poroelastic case did not use a fluid velocity. Instead, it considered the fluid displacement relative to the solid displacement. In order to compare them more efficiently, the solid displacement is expanded into the fluid region and a term for the fluid displacement relative to the solid, $w(\vec{x},\vec{y})$, is introduced. The expansion of u_0 into the fluid domain is not specified, but it is unlikely to matter so long as it is continuous across the boundary, as the eventual result will be considering macroscopic properties rather than rather than microscopic properties. The relative displacement term, w, is introduced through the relation between the first term of the solid displacement, u_0, and the first term of the fluid velocity, v_0, by

$$v_0(\vec{x},\vec{y}) = i\omega[u_0(\vec{x}) + w(\vec{x},\vec{y})], \qquad inD_f \tag{12}$$

using the replacement of time derivatives with $i\omega$.

From this, equations containing v_0 can be updated to be equations containing w. Specifically, (8c) and (8d) become equations of w only, and (9a) and (9b) into more complicated expressions. Though it also contains v_0, (9c) is ignored as it unnecessarily introduces a higher order term of velocity.

From (8c) in the fluid domain,

$$\begin{aligned} \kappa\nabla_y \cdot \big(i\omega[u_0(\vec{x}) + w(\vec{x},\vec{y})]\big) &= 0 \\ i\omega\kappa\big(\nabla_y \cdot u_0(\vec{x}) + \nabla_y \cdot w(\vec{x},\vec{y})\big) &= 0 \\ i\omega\kappa\nabla_y \cdot w(\vec{x},\vec{y}) &= 0 \\ \nabla_y \cdot w(\vec{x},\vec{y}) &= 0 \end{aligned} \tag{13}$$

and from (8d) in the boundary between domains,

$$\begin{aligned} i\omega[u_0(\vec{x}) + w(\vec{x},\vec{y})] - i\omega u_0(\vec{x}) &= 0 \\ i\omega w(\vec{x},\vec{y}) &= 0 \\ w(\vec{x},\vec{y}) &= 0 \end{aligned} \tag{14}$$

(9a) and (9b), both in the fluid domain, follow similarly, with (9a) only expanding terms and (9b) also taking advantage of the independence of u_0 from $\vec{y}$. As a result,

$$-\nabla_y \cdot \sigma_1(\vec{x},\vec{y}) + \omega^2 \rho_f w(\vec{x},\vec{y}) = \nabla_x \cdot \sigma_0(\vec{x}) + \omega^2 \rho_f u_0(\vec{x}) \tag{15a}$$

$$\sigma_1 = -p_1 I + i\omega\mu D\nabla_y w \tag{15b}$$

Before substituting (15b) into (15a), it is convenient to consider the term $\nabla_y \cdot D\nabla_y w$ separately. First, using the definition of the gradient of a vector [10], letting $\hat{e}_i$ represent the normal vectors in the y_i direction and using the Einstein summation convention,

$$\begin{aligned} D\nabla_y w &= \frac{1}{2}\big((\nabla_y w) + (\nabla_y w)^T - \frac{2}{3} I\, tr(\nabla_y w)\big) \\ &= \frac{1}{2}\big((\frac{\partial w_j}{\partial y_i}\hat{e}_i\hat{e}_j) + (\frac{\partial w_j}{\partial y_i}\hat{e}_i\hat{e}_j)^T - \frac{2}{3} I\, tr(\frac{\partial w_j}{\partial y_i}\hat{e}_i\hat{e}_j)\big) \\ &= \frac{1}{2}\big((\frac{\partial w_j}{\partial y_i}\hat{e}_i\hat{e}_j) + (\frac{\partial w_i}{\partial y_j}\hat{e}_i\hat{e}_j) - \frac{2}{3} I\, (\frac{\partial w_i}{\partial y_i})\big) \\ &= \frac{1}{2}\big((\frac{\partial w_j}{\partial y_i} + \frac{\partial w_i}{\partial y_j})\hat{e}_i\hat{e}_j - \frac{2}{3} I\, (\nabla_y \cdot w)\big) \end{aligned}$$

From this, it is possible to write the components as in Zhou and Sheng [26]:

$$[D\nabla_y w]_{ij} = \frac{1}{2}\big(\frac{\partial w_j}{\partial y_i} + \frac{\partial w_i}{\partial y_j} - \frac{2}{3}\delta_{ij}\nabla_y \cdot w\big)$$

However, in this case, it is known from (13) that $\nabla_y \cdot w = 0$ and thus the last term may be

dropped. So, from here,

$$\begin{aligned}
\nabla_y \cdot D\nabla_y w &= \frac{\partial}{\partial y_i}\Big(\frac{1}{2}\big(\frac{\partial w_j}{\partial y_i}+\frac{\partial w_i}{\partial y_j}\big)\Big) \\
&= \frac{1}{2}\big(\frac{\partial^2 w_j}{\partial {y_i}^2}+\frac{\partial^2 w_i}{\partial y_i \partial y_j}\big) \\
&= \frac{1}{2}\big(\frac{\partial^2 w_j}{\partial {y_i}^2}+\frac{\partial}{\partial y_j}\big(\frac{\partial w_i}{\partial y_i}\big)\big) \\
&= \frac{1}{2}\big(\frac{\partial^2 w_j}{\partial {y_i}^2}+\frac{\partial}{\partial y_j}(\nabla_y \cdot w)\big) \\
&= \frac{1}{2}\frac{\partial^2 w_j}{\partial {y_i}^2} \\
&= \frac{1}{2}\nabla^2 w
\end{aligned}$$

It is also important to show that

$$\begin{aligned}
\nabla_y \cdot p_1 I &= \frac{\partial}{\partial y_i}[p_1 I]_{ij}\hat{e}_j \\
&= \frac{\partial}{\partial y_i}\big(\delta_{ij} p_1\big)\hat{e}_j \\
&= \nabla p_1
\end{aligned}$$

Thus, substituting (15b) into (15a) gives

$$\nabla_y p_1(\vec{x},\vec{y}) - i\omega\frac{\mu}{2}\nabla_y^2 w(\vec{x},\vec{y}) + \omega^2\rho_f w(\vec{x},\vec{y}) = \nabla_x \cdot \sigma_0(\vec{x}) + \omega^2\rho_f u_0(\vec{x})$$

This equation, along with (13) and (14), functions as a linear set of equations for w and p_1 with $\nabla_x \cdot \sigma_0(\vec{x}) + \omega^2\rho_f u_0(\vec{x})$ as the inhomogeneous term. The solutions for these terms can be written as linearly dependent on this inhomogenous term. The solution to w is unique if it is required to be bounded in $\vec{y}$, but since p_1 only appears as a gradient of $\vec{y}$, the bounded solution is unique only up to the addition of some function of $\vec{x}$, $f(\vec{x})$. So,

$$w(\vec{x},\vec{y}) = W(\vec{x},\vec{y})\big(\nabla_x \cdot \sigma_0(\vec{x}) + \omega^2\rho_f u_0(\vec{x})\big) \tag{16a}$$

$$p_1(\vec{x},\vec{y}) = P(\vec{x},\vec{y})\big(\nabla_x \cdot \sigma_0(\vec{x}) + \omega^2\rho_f u_0(\vec{x})\big) + f(\vec{x}) \tag{16b}$$

for some matrix W and some vector P. The solution for σ_1 can be found by substituting into (15b), but the final goal only requires the first term of the epsilon expansion for the various properties. As such, only the solution to w is important for this particular work.

A similar method may be used to determine u_1 and τ_0. Specifically, using

$$\hat{n}\cdot\sigma_0(\vec{x}) - \hat{n}\cdot\tau_0(\vec{x},\vec{y}) = 0 \qquad \partial D_s = \partial D_f \tag{8e}$$

$$\nabla_y\cdot\tau_0(\vec{x},\vec{y}) = 0 \qquad D_s \tag{8f}$$

$$C(\vec{x},\vec{y})\nabla_y u_1(\vec{x},\vec{y}) + C(\vec{x},\vec{y})\nabla_x u_0(\vec{x}) - \tau_0(\vec{x},\vec{y}) = 0 \qquad D_s \tag{9g}$$

it is possible to determine u_1 as a unique solution up to an additive function of $\vec{x}$ under the assumption that u_1 is bounded in $\vec{y}$. To avoid the undetermined function, we solve instead for $\nabla_y u_1$, in terms of inhomogeneous terms $C\nabla_x u_0(\vec{x})$ and $p_0(\vec{x})$. It is worth noting that p_0 does not explicitly appear in the above equations. However, it is directly related to σ_0 and thus may be replaced into (8e). From this, we obtain

$$\nabla_y u_1(\vec{x},\vec{y}) = Q(\vec{x},\vec{y})p_0(\vec{x}) + L(\vec{x},\vec{y})C(\vec{x},\vec{y})\nabla_x u_0(\vec{x}) \tag{17a}$$

Substituting this back into (9g),

$$\begin{aligned}\tau_0(\vec{x},\vec{y}) &= C(\vec{x},\vec{y})\big(Q(\vec{x},\vec{y})p_0(\vec{x}) + L(\vec{x},\vec{y})C(\vec{x},\vec{y})\nabla_x u_0(\vec{x})\big) + C(\vec{x},\vec{y})\nabla_x u_0(\vec{x}) \\ \tau_0(\vec{x},\vec{y}) &= \big(C(\vec{x},\vec{y}) + CLC(\vec{x},\vec{y})\big)\nabla_x u_0(\vec{x}) + CQ(\vec{x},\vec{y})p_0(\vec{x})\end{aligned} \tag{17b}$$

At this point, there are expressions for the first term in each epsilon expansion. However, both τ_0 and w (and thereby v_0) are in terms of both $\vec{x}$ and $\vec{y}$. Since the goal is to obtain expressions for the slow, macroscopic variation, it is necessary to remove the dependence on the fast, microscopic variable $\vec{y}$. Generally, this is done by averaging those quantities with respect to $\vec{y}$, as shown in the following section.

4.3.1 Averaging over $\vec{y}$

The first step is to consider how to define the average over $\vec{y}$. For an arbitrary function $f(\vec{x},\vec{y})$ defined over the domain of the fluid, D_f, we begin by considering it over a finite sphere of radius R, B_R. The function f is integrated with respect to $\vec{y}$ over the intersection between this sphere and the fluid domain and this integral is divided by the volume of the sphere B_R. Since the medium is assumed to be infinite, we can take the limit of this average over the sphere as the radius R goes to infinity. This limit is defined as the average of the function, $\overline{f}(\vec{x})$. For a function defined over the solid instead, the same principle applies by replacing D_f with D_s.

Before using this defined average on the equations determined above, it is important to note that the average of the indicator function of the solid, χ_s, is equal to the volume fraction of the solid, $V_s(\vec{x})$, since the indicator function is equal to one at every point of the solid domain. Similarly, $V_f(\vec{x})$ represents the volume fraction of the fluid. These are used when averaging functions dependent only on $\vec{x}$, since it would be equivalent to the function multiplied by the indicator function of that particular domain. Thus, the average of a function dependent only on $\vec{x}$ is equal to that function multiplied by the particular volume fraction.

With this averaging method, it is now possible to obtain expressions for averaged versions of τ_0 and w. To begin, we average the solutions previously obtained in (16a) and (17b) to obtain

$$\overline{w}(\vec{x}) = \overline{W}(\vec{x})\left(\omega^2\rho_f u_0(\vec{x}) - \nabla_x p_0(\vec{x})\right) \tag{24}$$

$$\overline{\tau}_0 = \left(\overline{C}(\vec{x}) + \overline{CLC}(\vec{x})\right)\nabla_x u_0(\vec{x}) + \overline{CQ}(\vec{x})p_0(\vec{x}) \tag{25}$$

However, these equations are insufficient to properly express $\overline{w}$ and $\overline{\tau}_0$, as it leaves p_0 and u_0 undefined. As such, it will be necessary to provide two more equations in order to determine these next two unknowns.

We substitute the definition of v_0 with w – that is, (12) – into equation (9a), which results in

$$\nabla_x \cdot \sigma_0(\vec{x}) + \omega^2\rho_f u_0(\vec{x}) + \omega^2\rho_f w(\vec{x},\vec{y}) = -\nabla_y \cdot \sigma_1(\vec{x},\vec{y})$$

Averaging this gives

$$V_f(\vec{x})\left(\nabla_x\cdot\sigma_0(\vec{x})+\omega^2\rho_f u_0(\vec{x})\right)+\omega^2\rho_f\overline{w}(\vec{x}) = -\lim_{R\to\infty}\frac{1}{\frac{4}{3}\pi R^3}\int_{D_f\cap B_R}\nabla_y\cdot\sigma_1(\vec{x},\vec{y})d\vec{y}$$
$$V_f(\vec{x})\left(\nabla_x\cdot\sigma_0(\vec{x})+\omega^2\rho_f u_0(\vec{x})\right)+\omega^2\rho_f\overline{w}(\vec{x}) \overset{*}{=} -\lim_{R\to\infty}\frac{1}{\frac{4}{3}\pi R^3}\int_{\partial(D_f\cap B_R)}\sigma_1(\vec{x},\vec{y})\cdot\hat{n}d\vec{y} \tag{20}$$

where in the step marked (*) the divergence theorem was used to convert the volume integral into a surface integral for reasons that will become apparent later. A similar use of the divergence theorem is used when averaging (9f). However, as the normal vector $\hat{n}$ is pointing into the solid, the normal vector to be used in the solid region is $-\hat{n}$, resulting in

$$\overline{\nabla_x\cdot\tau_0}(\vec{x})+\omega^2\overline{\rho}_s u_0(\vec{x}) = \lim_{R\to\infty}\frac{1}{\frac{4}{3}\pi R^3}\int_{\partial(D_f\cap B_R)}\tau_1(\vec{x},\vec{y})\cdot\hat{n}d\vec{y} \tag{21}$$

Since the solid density, ρ_s, has not been noted to be dependent on $\vec{y}$, it is likely that the averaged quantity, $\overline{\rho}_s$, is equivalent to $\rho_s V_s$. However, even if there were a $\vec{y}$ dependence, this would not change the current results, as it only appears in (9f), which is only used here. Even if it required epsilon expansion, then the ρ_s used here would be the first term of the expansion. However, the same cannot be applied to ρ_f, which must be constant.

It is also important to make similar uses of the divergence theorem with u_1 and v_1. In Burridge and Keller's work, they use a specialized gradient version of the divergence theorem [5], but later take the trace of it in order to obtain the average of the divergence. As such, this work will begin with the divergence in the first place, giving a slightly modified (22) and (23) as follows:

$$\overline{\nabla\cdot v_1}(\vec{x}) = \lim_{R\to\infty}\frac{1}{\frac{4}{3}\pi R^3}\int_{\partial(D_f\cap B_R)}v_1(\vec{x},\vec{y})\cdot\hat{n}d\vec{y} \tag{22}$$
$$\overline{\nabla\cdot u_1}(\vec{x}) = \lim_{R\to\infty}\frac{1}{\frac{4}{3}\pi R^3}\int_{\partial(D_f\cap B_R)}u_1(\vec{x},\vec{y})\cdot\hat{n}d\vec{y} \tag{23}$$

The surfaces used on the integrals, $\partial(D_f\cap B_R)$ and $\partial(D_s\cap B_R)$, can be split into two (possibly overlapping) components. In the case of D_f, these are $(\partial D_f)\cap B_R$ and $D_f\cap(\partial B_R)$. The latter portion represents the part of the surface of the sphere B_R lying inside

the domain of the fluid D_f. Since the integrands are independent of R and the total surface of B_R is $4\pi R^2$, the integral may be bounded by an equation of order R^2. As the integral is divided by R^3, the limit as R tends to infinity will be zero. As such, only the former portion, $\partial D_f \cap B_R$, is necessary for the integrals in (20) and (22). Similarly, $\partial(D_s \cap B_R)$ may be replaced by $\partial D_s \cap B_R$ in (21) and (23).

By (9d) and (9e), however, $v_1 = i\omega u_1$ and $\sigma_1 \cdot \hat{n} = \tau_1 \cdot \hat{n}$ over the border of the fluid and solid domains. Thus, with the replacement of surfaces to a subset of this border in these equations, the integrals of (20) and (21) are equal to one another and may be equated to one another to give

$$\begin{aligned} -V_f(\vec{x})\big(\nabla_x \cdot \sigma_0(\vec{x}) + \omega^2 \rho_f u_0(\vec{x})\big) - \omega^2 \rho_f \overline{w}(\vec{x}) &= \overline{\nabla_x \cdot \tau_0}(\vec{x}) + \omega^2 \overline{\rho}_s u_0(\vec{x}) \\ -\omega^2(\overline{\rho}_s + V_f(\vec{x})\rho_f)u_0(\vec{x}) - \omega^2 \rho_f \overline{w}(\vec{x}) &= \overline{\nabla_x \cdot \tau_0}(\vec{x}) + V_f(\vec{x})\nabla_x \cdot \sigma_0(\vec{x}) \\ -\omega^2 \overline{\rho} u_0(\vec{x}) - \omega^2 \rho_f \overline{w}(\vec{x}) &= \overline{\nabla_x \cdot \tau_0}(\vec{x}) - V_f(\vec{x})\nabla_x p_0(\vec{x}) \end{aligned} \tag{27}$$

where in the final step, σ_0 was replaced by $-p_0 I$ as from (8b) and the term $\overline{\rho}$ was introduced as a notational shorthand for

$$\overline{\rho} = \overline{\rho}_s + \overline{\rho}_f = \overline{\rho}_s + V_f(\vec{x})\rho_f u_0(\vec{x}) \tag{28}$$

Similarly, equations (22) and (23) may be combined to give

$$i\omega \overline{\nabla_y \cdot u_1} + \overline{\nabla_y \cdot v_1} = 0 \tag{29}$$

Next, replacing v_0 in equation (9c) by its definition in (12) and averaging, we obtain

$$\begin{aligned} \kappa \nabla_y \cdot v_1(\vec{x},\vec{y}) + \kappa \nabla_x \cdot \big(i\omega[u_0(\vec{x}) + w(\vec{x},\vec{y})]\big) + i\omega p_0(\vec{x}) &= 0 \\ \kappa \overline{\nabla_y \cdot v_1}(\vec{x}) + i\omega\kappa V_f \nabla_x \cdot u_0(\vec{x}) + i\omega\kappa \overline{\nabla_x \cdot w}(\vec{x}) + i\omega V_f p_0(\vec{x}) &= 0 \end{aligned} \tag{26}$$

Using (29) in (26) to replace v_1 with u_1 and dividing the equation by $i\omega$,

$$\kappa \overline{\nabla_y \cdot u_1}(\vec{x}) + \kappa V_f \nabla_x \cdot u_0(\vec{x}) + \kappa \overline{\nabla_x \cdot w}(\vec{x}) + V_f p_0(\vec{x}) = 0 \tag{30}$$

While on the surface this appears to have only swapped one unknown term for another, taking the trace of (17a) results in changing the expression for the gradient of u_1 into an expression for the divergence of u_1,

$$\nabla_y \cdot u_1(\vec{x},\vec{y}) = tr\big(Q(\vec{x},\vec{y})\big)\, p_0(\vec{x}) + tr\big(L(\vec{x},\vec{y})C(\vec{x},\vec{y})\nabla_x u_0(\vec{x})\big) \tag{18}$$

Taking the average of (18) and inserting the result into (30) gives

$$-\big(V_f - \kappa tr\big(\overline{Q}(\vec{x})\big)\big)\, p_0(\vec{x}) = \kappa\big(V_f \nabla_x \cdot u_0(\vec{x}) - tr\big(\overline{LC}(\vec{x})\nabla_x u_0(\vec{x})\big) + \overline{\nabla_x \cdot w}(\vec{x})\big) \tag{31}$$

This can be further simplified using the formula for arbitrary function $f(\vec{x},\vec{y})$

$$\nabla_x \int_{D_f \cap B_R} f(\vec{x},\vec{y})d\vec{y} = \int_{D_f \cap B_R} \nabla_x f(\vec{x},\vec{y})d\vec{y} + \int_{\partial(D_f \cap B_R)} q(\vec{x},\vec{y})\, f(\vec{x},\vec{y})d\vec{y} \tag{33}$$

where the vector q is dependent on the variation with x of the boundary. As before, the boundary may be divided up into $\partial D_f \cap B_R$ and $D_f \cap \partial B_R$, but since B_R is independent of $\vec{x}$, there is no contribution from ∂B_R. As such, the last integral may be replaced by $\partial D_f \cap B_R$. Thus, if w is used in place of f, then by equation (14), $w = 0$ in this region and the last integral can be dropped. Taking the trace to convert the gradient into a divergence operator, we find

$$\nabla_x \cdot \int_{D_f \cap B_R} w(\vec{x},\vec{y})d\vec{y} = \int_{D_f \cap B_R} \nabla_x \cdot w(\vec{x},\vec{y})d\vec{y} \tag{34}$$

Dividing by $\frac{4}{3}\pi R^3$ and taking the limit as R tends to infinity, we find that

$$\overline{\nabla_x \cdot w}(\vec{x}) = \nabla_x \cdot \overline{w}(\vec{x}) \tag{32}$$

This may now be used in (31). This, along with (24), (25), and (27), form a set of four equations for the four functions $u_0(\vec{x})$, $\overline{\tau}_0(\vec{x}$, $\overline{w}(\vec{x})$, and $p_0(\vec{x})$, which describe all the properties of the poroelastic medium on the macroscopic scale since the fluid stress tensor, σ_0 may be determined from the pressure p_0 in (8b). To simplify the notation, we assume

that $\overline{W}$ is invertible and we introduce

$$M(\vec{x}) = \kappa(V_f - \kappa tr\overline{Q}(\vec{x}))^{-1} \tag{37}$$

With these assumptions, we can rewrite the four equations and group them between equations of motion,

$$-\omega^2\overline{\rho}u_0(\vec{x}) - \omega^2\rho_f\overline{w}(\vec{x}) = \overline{\nabla_x\cdot\tau_0}(\vec{x}) - V_f\nabla_x p_0(\vec{x}) \tag{35a}$$
$$-\omega\rho_f u_0 + \overline{W}^{-1}(\vec{x})\overline{w}(\vec{x}) = -\nabla_x p_0 \tag{35b}$$

and constitutive equations

$$\overline{\tau}_0(\vec{x}) = \big(\overline{C}(\vec{x}) + \overline{CLC}(\vec{x})\big)\nabla_x u_0(\vec{x}) + \overline{CQ}(\vec{x})p_0(\vec{x}) \tag{36a}$$
$$p_0(\vec{x}) = -M(\vec{x})\big(V_f\nabla_x\cdot u_0(\vec{x}) - tr\big(\overline{LC}(\vec{x})\nabla_x u_0(\vec{x})\big) + \nabla_x\cdot\overline{w}\big). \tag{36b}$$

4.4 Biot's equations

The equations given by Biot are for a uniform medium [4], which is to say a *macroscopically* uniform medium. This may be portrayed by letting the domains D_f and D_s be independent of the macroscopic variable $\vec{x}$. As such, in (33), there is no variation of x in the boundary and thus q is zero. Therefore, as in (32),

$$\overline{\nabla_x f}(\vec{x}) = \nabla_x\cdot\overline{f}(\vec{x}) \tag{38}$$

for any function $f(\vec{x},\vec{y})$. The averaged coefficients in the four equations are also no longer dependent on $\vec{x}$ due to the uniformity, rendering them constants [5]. With this, equation (35a) may be simplified, leaving it and (35b) as

$$-\omega^2\overline{\rho}u_0(\vec{x}) - \omega^2\rho_f\overline{w}(\vec{x}) = \nabla_x\cdot\big(\overline{\tau}_0(\vec{x}) - V_f p_0(\vec{x})I\big) \tag{39a}$$
$$-\omega\rho_f u_0(\vec{x}) + \overline{W}^{-1}\overline{w}(\vec{x}) = -\nabla_x p_0(\vec{x}) \tag{39b}$$

For reasons of comparison, $V_f p_0(\vec{x})I$ is subtracted from either side of equation (36a), using the definition of p_0 from (36b) to remove p_0 from the right side of the equation. After rearranging both this altered (36a) and the original (36b), this results in

$$\begin{aligned}\overline{\tau}_0(\vec{x}) - V_f p_0(\vec{x})I = &\left[\overline{C} + \overline{CLC} + \left(V_f I - \overline{CQ}\right)M\, tr\left(V_f - \overline{LC}\right)\right]\nabla_x u_0(\vec{x}) \\ &+ \left(V_f I - \overline{CQ}\right)M\nabla_x \cdot \overline{w}(\vec{x}) \qquad (40a)\end{aligned}$$

$$p_0(\vec{x}) = -M\, tr\left(V_f - \overline{LC}\right)\nabla_x u_0(\vec{x}) - M\nabla \cdot \overline{w}(\vec{x}) \qquad (40b)$$

These equations may be compared to the equations in [4] marked (2.2), (5.2), and (5.1), which are

$$\frac{\partial \tau_{ij}}{\partial x_j} = \rho \ddot{u}_i + \rho_f \ddot{w}_i \qquad (41a)$$

$$-\frac{\partial p_f}{\partial x_i} - \rho_f \ddot{u}_i = \overline{Y}_{ij}(p)\dot{w}_j \qquad (41b)$$

$$\tau_{ij} = A_{ij}^{\mu\nu} e_{\mu\nu} + Mij\zeta \qquad (42a)$$

$$p_f = M_{ij} e_{ij} + M\zeta \qquad (42b)$$

Equations (39) and (40) from Burridge and Keller can be shown to agree with equations (41) and (42) from Biot by identifying the differences between notation. Some are obvious, including some which have identical notation, while others are quite complicated. The differences are summarized in the following table.

Burridge & Keller [5]	Biot [4]	
$\overline{\rho}, \rho_f, M$	ρ, ρ_f, M	(43a)
$\overline{C} + \overline{CLC} + \left(V_f I - \overline{CQ}\right)M\, tr\left(V_f - \overline{LC}\right)$	$[A_{ij}^{\mu\nu}]$	(43b)
$i\omega$	$i\omega = p = \frac{d}{dt}$	(43c)
$\overline{W}^{-1}$	$[\overline{Y}(p)]\left(\frac{d}{dt}\right)$	(43d)
$u_0, \overline{w}, p_0$	u, w, p_f	(43e)
$M(V_f I - \overline{CQ}$	$(-M_{ij})$	(43f)
$\overline{\tau}_0 - V_f p_0 I$	τ_{ij}	(43g)
$\frac{1}{2}[\nabla_x u_0 + (\nabla_x u_0)^T], \nabla \cdot \overline{w}$	$e_{ij}, -\zeta$	(43h)

These quantities in Biot's derivation are phenomenological, but Burridge and Keller's

quantities are calculated from the continuum mechanics of fluid flow and elasticity theory. Thus, the parameters in Biot's work could be potentially calculated from Burridge and Keller's work given the viscosity of the component fluid and the lamé constants of the component elastic solid. However, in practice, this is not so simple, which will be discussed in the next section.

4.4.1 Special cases

Burridge and Keller [5] went on in their work to consider special cases of these equations, of which we will demonstrate two.

(1) The absence of fluid: Here, we consider an elastic porous solid with empty pores. In this case, expressions arising from the fluid portion are removed. In particular, W, which is a part of the solution to the relative motion of the fluid w, is not defined. This leaves us with adjusted versions of (39a) and (36a). That is,

$$-\omega^2\overline{\rho}u_0(\vec{x}) = \nabla_x \cdot \overline{\tau}_0(\vec{x}) \tag{44a}$$

$$\overline{\tau}_0(\vec{x}) = \big(\overline{C}(\vec{x}) + \overline{CLC}(\vec{x})\big)\nabla_x u_0(\vec{x}) \tag{44b}$$

These are of the same form as the equations for the elastic solid [5] that we derived in section 4.2, as may have been expected. □

(2) Fluid in a rigid porous solid: We assume that the porous solid is completely rigid. In this case, the solid displacement $u = u_0 + \varepsilon u_1 + \ldots$ is 0 everywhere. In such a case, we find that (24) is

$$\overline{w}(\vec{x}) = -\overline{W}(\vec{x})\big(\nabla_x p_0(\vec{x})\big) \tag{45a}$$

which is equivalent to Darcy's law for the fluid [5]. Similarly, (25) gives that

$$p_0(\vec{x}) = -M(\vec{x})\nabla_x \cdot \overline{w} \tag{45b}$$

where M is defined as in (37) with $Q = 0$ [5], since Q arises from the definition of u_1, which is equal to 0 everywhere. □

Both of these cases reduce to well-known relations for the solid and the fluid, respectively. Burridge and Keller also looked at the isotropic case by taking the averaged tensors to be also isotropic. It is also possible to look at the case where the Reynolds number is of order unity on the macroscopic scale, which results in viscoelastic equations. However, this latter case requires the derivation process to begin again from the initial equations.

CHAPTER 5

Conclusions

The derivation of the poroelastic equations through the homogenization method was helpful in demonstrating that Biot's equations, derived by phenomenological arguments, had a firm basis in theory when the Reynolds number (or dimensionless viscosity) was of order one. At the time, their validity was being called into question [5], but this derivation is now used to justify the use of Biot's equations [24]

This method is not the only method to derive Biot's poroelastic equations. For example, mixture theory has also been used to derive these equations [9]. However, the homogenization method has some advantages over other continuum methods, as well as some challenges.

The primary advantage of the homogenization method is that it results in expressions for the parameters in terms of the microstructure, as seen in the previous sections. Other continuum approaches often require macroscopic estimation for the parameter values for the specific material, but parameters derived from the homogenization method can theoretically be determined for any material for which we understand the solid and fluid properties [19].

This can be invaluable in some circumstances. For example, it is clearly difficult to measure properties of the human brain in vivo. Attempts have been made to estimate these through measuring the same properties in other mammalian brains, such as monkeys [14] and cats [17], and assuming that the properties are similar to the human brain. This has resulted in a wide variance in the estimated results: Metz et al. suggest a range of 10 to 20

kPa for the shear modulus [14], while Tenti et al. conclude that it is closer to 7.5 kPa [22]. Since the poroelasticity equations may be used to describe the fluid-solid interactions in hydrocephalus [24], a method to produce the parameters for such a model would clear up this issue.

We consider some examples of results produced in the literature below.

Skotheim and Mahadevan [20] consider a poroelastic medium at a low Reynolds number and in an incompressible fluid. One area of biological research in which the flows are characterized by low Reynolds number is the motion of microscopic organisms [15], which makes it a natural assumption for their work on filaments in the microscopic pores [20]. In such a region, properties are approximately time-independent and thus the time derivatives may usually be ignored [15]. As a result, they use the same initial equations as in Burridge and Keller after setting the time derivative terms to zero, although they still retain the boundary condition relating the fluid velocity to the derivative of the solid displacement, which is equivalent to (3d) in Burridge and Keller [5].

From these simplified equations, they derive an expression for the stress tensor which may be written in the notation used in the previous chapter as

$$\overline{\tau}_0(\vec{x}) = 2\hat{\mu}\left(\frac{1}{2}[\nabla_x u_0 + (\nabla_x u_0)^T]\right) + \lambda I \nabla_x \cdot \overline{w}(\vec{x}) - (V_f - \gamma) p_0 I \tag{5.1}$$

where $\hat{\mu}$ and λ are the effective Lamé coefficients and γ is a constant derived in Appendix A of Skotheim and Mahadevan [20]. This is of a similar form as (40a) and (42a) in Burridge and Keller [5]. However, their coefficient of the linearized strain, $\frac{1}{2}[\nabla_x u_0 + (\nabla_x u_0)^T]$, is a scalar rather than a tensor, and their coefficient to the pressure is of opposite parity to that in (40a). This does not indicate that the derivation in Skotheim and Mahadevan is inconsistent with Burridge and Keller [5] or Biot [4], as tensor coefficients may reduce to scalars in certain circumstances [26]. For example, Burridge and Keller demonstrated that the expression for the stress tensor in the case where the medium is isotropic on the macroscopic scale also results in $\hat{\mu}$ as a coefficient of the same term [5], although the equations are not equivalent. Instead, we would have to consider only the isotropic equations (2.1) in Biot's work [4].

The derivation in Skotheim and Mahadevan is incomplete from the point of view of one seeking a low Reynolds number version of Biot's poroelastic equations. As they were seeking to describe a specific related question, they did not derive any other equations that corresponded to the other poroelastic equations.

Zhou and Sheng [26] relate the tensor W of Burridge and Keller's work [5] to the dynamic permeability function, $\kappa(\omega)$. However, rather than considering the tensor version of κ, they focus on the scalar version, which occurs when the microstructure is isotropic, uniaxial, or simple-cubic [26]. This allows them to calculate the dynamic permeability numerically, but these are quite strong assumptions on the microstructure, reducing it to much simpler cases. In particular, the microscopic isotropy condition may restrict its use to a special case of macroscopic isotropy discussed at the end of Burridge and Keller, as the poroelasticity equations discussed in the previous section are for anisotropic media. It is worth noting that microscopic isotropy is not required for macroscopic isotropy, so this does not cover the entirety of the macroscopic isotropy problem [5].

The problem was also studied for a medium with an isotropic microstructure by Chapman and Higdon [7]. Here, the fluid stress is considered isotropic and all properties of the solid phase are also isotropic. They estimated the microstructure as a three-dimensional grid of overlapping spheres in a simple cubic lattice, with the fluid occupying the remaining space. On this restricted domain, they were able to compute expressions for the stress and some other related properties.

One may begin to notice a pattern of incomplete results for restricted cases of the equations. This is a pattern through the literature working on this problem of calculating the parameters to the poroelasticity equations. There has been extensive study in the case of a porous elastic solid without fluid and the case of fluid flow through a rigid solid [6], but no solution has been formulated for the general case [3].

Indeed, the greatest strength of the homogenization method when compared to other upscaling methods is directly related to its greatest weakness. While the parameters derived by the homogenization method may theoretically be calculated from the microscopic properties, this is not always simple in practice. In the case of the poroelasticity equations, this has not been solved even numerically in the thirty years since Burridge and

Keller first derived the expressions for these parameters.

It is worth noting that the other methods for deriving homogeneous macroscopic properties from a heterogeneous microstructure also require the parameters to be discovered phenomenologically. Thus, on the surface, it would seem that this lack of calculation from theory is no more of an issue than for other continuum methods. However, the issue similarly arises out of the method's strengths. As seen in the comparison to Biot's equations in Burridge and Keller [5], the relation between the parameters derived in the homogenization method and their respective macroscopic equivalents are not obvious (see (43b) in Burridge and Keller or the previous section for a particularly non-obvious example). Thus, a proper identification of the parameters may require comparison with an equivalent model. This is to say that we may be unable to understand what these homogenized parameters mean if we do not already have an equivalent model for the behaviour being studied, so we either already have a more useful model or we are limited to theoretical exploration of parameter space without a sense for the limitations on the parameters.

This might appear at first to obviate most reasons for using the homogenization method, but there remain reasons to make use of this method. As was the original intention of Burridge and Keller's work, the method allows one to place phenomenological models on a more firm theoretical footing. While Biot's poroelatic equations were successfully used in many applications, their validity was questionable. In applying the homogenization method, Burridge and Keller not only proved that these equations are valid, and further clarified the conditions under which they are valid – specifically, when the Reynolds number length scale is on the order of the microstructure. When it is of order unity on the macroscopic scale, the equations are equivalent to those for a viscoelastic material [5].

The lack of a generally applicable method to calculate the parameters is also not an insurmountable setback. In terms of the method, this one setback does not mean that all such problems will encounter these issues and it remains a valid approximation method to try and tackle problems involving heterogeneous microstructures. More specifically there has been some work has been done to correct this intractability for the case of the poroelastic equations. For example, more recent work by Clopeau et al. [8] has attempted to rederive the poroelasticity equations through the homogenization method in a some-

what different manner than Burridge and Keller, with more of an emphasis on the specific problem of acoustic waves in a seabed. They claim that their results are also equivalent to Biot's poroelastic equations, but in a form that may prove more tractable to numerical methods. While a full exploration of this possibility has not been conducted, some initial attempts to apply a finite element approach have shown results for a simple model of the microstructure [2].

In the end, while the method has its challenges, the two-space homogenization method is a worthwhile technique for treating problems that include rapid or microscopic variation in problems where a macroscopic description is desired. While the calculation of parameters is not always sufficiently straightforward as to allow for an exact calculation in terms of the microscopic properties, it still provides an understanding of the general shape of a macroscopic model and may be used to justify a more phenomenological approach. In cases where an exact or numerical solution of the parameters may be derived, it provides a more general framework to determine the properties of the model without resorting to heuristic experiments.

More specifically to the poroelastic application, the homogenization method has provided a firm theoretical basis for Biot's theory, including an indication of where it is valid. While the problem of calculating the parameters remains unsolved in all but some simplified cases, there remains a possibility of obtaining these numerically. If this is done, then it may provide a solution to the problem of measuring the properties of difficult materials, such as biological tissues.

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