Optimization Models for Applications in Portfolio Management and Advertising Industry

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Optimization problems in two different application fields are investigated: the first one is the popular portfolio optimization problem and the second one is the newly developed online display advertising problem.

The portfolio optimization problem has two main concerns: an appropriate statistical input data, which is improved with the use of factor model and, the inclusion of the transaction cost function into the original objective function. Two methods are applied to solve the optimization problem, namely, the conditional value at risk (CVaR) method and the reliability based (RB) method.

Asset allocation problem in finance continues to be of practical interest because decisions as to where to invest must be made to maximize the total return and minimizing the risk of not attaining the target return. However, the commonly used Markowitz method, also known as the mean-variance approach, uses historic stock prices data and has been facing problems of parameter estimation and short sample errors. An alternative method that attempts to overcome this problem is the use of factor models. This thesis will explain this model in addition to explaining the basic portfolio optimization problem.

Conditional value at risk and the reliability based optimization method are applied to solve the portfolio optimization problem with the consideration of transaction costs in the objective function. They are applied and evaluated by simulation in terms of their convergence, efficiency and results.

The online display advertising problem extends a normal deterministic revenue optimization model to a stochastic allocation model. The incorporation of randomness makes it more realistic for the estimation of demand, supply and market price. Revenues are considered as a combination of gains from guaranteed contracts and unguaranteed spot market. The objective is not only to maximize the revenue but also to consider the quality of ads, so that the whole market obtains long-term benefits and stability. The thesis accomplishes in solving the online display advertising allocation problem in a stochastic case with the measure of conditional value at risk algorithm.

Acknowledgements

I would like to give my sincere thanks to my supervisor Kumaraswamy Ponnambalam for his guidance and help on improving my thesis work. He has been patiently helping me with background study, optimization fundamentals, programming, writing and so on. I've learned from him the attitude for research and study: curiosity, diligence and critical thinking based on a solid knowledge of the research area.

I would also like to express my gratitude to my colleagues in the lab. They are always helpful in discussions and creative ideas.

Dedication

I would like to dedicate my thesis to my family and friends, who have been helping all the way through my master studies.

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Chapter 1
Introduction

1.1 Introduction to Optimization

Optimization is all around in our life, from industry supply chain operation to business investment, from applications in Engineering to Finance. We need optimization techniques to make a better life either to maximize our investment or minimize the use of resources.

Many of the engineering methods are being applied to financial areas because they share some similar characteristics such as the need for modeling and design optimization. When the input variables and model are deterministic, the solution is certain. While if the system becomes more complex involving multiple objectives or several constraints to meet at the same time, and most importantly if it has uncertainty in it, it is difficult to get an optimal decision with only a deterministic implementation of the problem.

Depending on the features of decision variables, uncertainties, and the objective of the optimization problem, optimization models can be classified into linear programming, dynamic programming, integer programming, stochastic programming and so on.

1.2 Optimization Steps

In short, optimization is a systematic decision making process. (Diwekar, 2008)

According to (Beightler, Phillips, & Wilde, 1979), the optimization process can be summarized in 3 steps:

1) Get to know the background of the system, including all information of inputs, and model the system using mathematical notations.

2) Define a measure of system effectiveness to this model, that is, determine the objective function and constraints

3) Apply an appropriate optimization algorithm to solve the problem

The whole process can be intuitively described as in Figure 1.1:

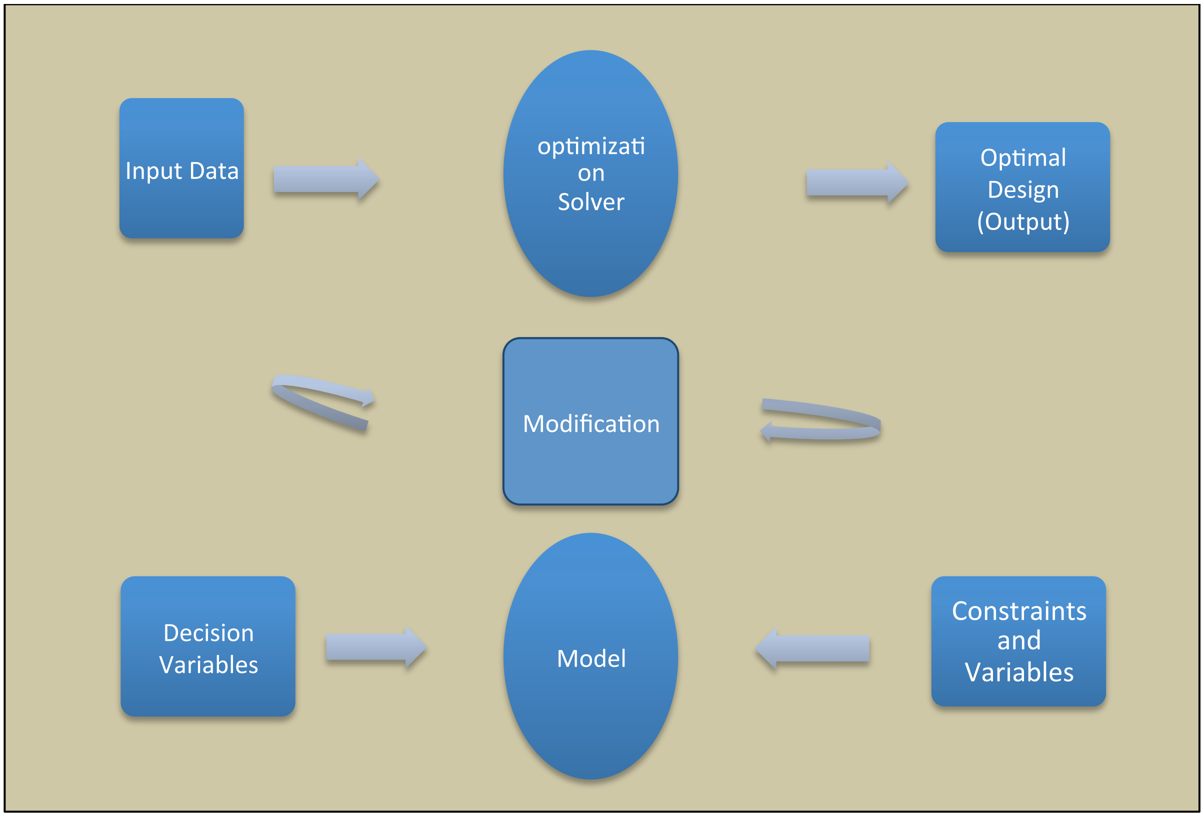


Figure 1.1 Optimization Process (Diwekar, 2008)

1.3 **Problem Statement**

We have two different applications to deal within this thesis. The first one is in portfolio optimization and the other one in advertising allocation.

There has been a lot of research done in portfolio optimization and theories have been developed to speed up computation and ensure better accuracy. Portfolio optimization is a decision-making problem in how we allocate our funding to different possible investment options so that we can get the maximum return. Both Conditional-Value-at-risk (CVaR), an advanced measure of risk technique and the reliability method (RBO) , where the chances of failure in the system is low, will be applied for investment allocation and results will be compared between two techniques..

The application in advertising allocation targets a more specific field and needs more background in advertising marketing. Ad space, ad relevance and prices have to be taken into consideration instead of return rate directly. The model is basically developed for the service providers of advertising exchange trading system. The service providers share a certain percentage of return from the publishers, who obtain cash inflows from the advertising opportunities, and thus all three parties- the publishers, the advertisers and the service provider who offers the trading system, gain from the system, either from the aspect of promoting business or increasing income.

The objectives for the two cases are about the same: maximize return. While the factors that affect return on investment are quite diversified. They share some similarities in terms of optimization but vary in modeling.

1.4 Contribution

The main contribution of the thesis is:

- Formalization of investment allocation model with transaction costs
- Optimization application with CVaR and RBO methods
- Transferring inputs into a Factor model
- Modeling of online display advertising
- Optimization formulation for the advertising problem with CVaR
- Experimental evaluation of proposed techniques

1.5 Content organization

The thesis is composed of four chapters. The first chapter gives a general idea of what an optimization problem is and how to deal with it.

Chapter 2 deals with the portfolio optimization problem. It starts with a background introduction and a problem statement. Section 3 in that chapter introduces the definition of transaction costs used. After that, in Section 5, the basis of Value-at-risk (VaR), CVaR, RBO and factor models are defined. Section 6 explains how CVaR, RBO and factor model apply in the asset allocation problem. Then in the next section, a specific example is given implementing and comparing both methods from the aspects of data analysis, efficiency, result analysis and convergence proof. The final conclusion is summarized in Section 2.11.

Chapter 3 presents the online display advertising problem. Description of types of advertising, advertising goals, revenue models, guaranteed and unguaranteed contracts are included. Uncertainty in the problem is also defined. Then the model is set up based on case study 2 described in the previous sections and CVaR is applied to solve this problem.

The last chapter is a summary and conclusion of all thesis work done so far as well as expected future work.

Chapter 2
Case Study 1: Portfolio optimization with transaction costs

2.1 Introduction

Change is certain, future is uncertain. –Bertrand Russell (Diwekar, 2008)

This is especially true with the financial market. The volatility of the market makes it interesting as well as challenging to researchers and investors. The future of any of those instruments in the market cannot be perfectly predicted but instead should be considered random or uncertain. Stochastic programming applications refer to this branch of optimization where there are uncertainties involved in the data (inputs) or the model.

Because the asset allocation problem has its practical relevance in the financial industry, it has aroused intense interest and focus for years and will continue to do so, in coming decades. Researchers from both educational and financial institutions aim at setting up a model designed to maximize the benefits of investments. The more efficient the forecast is, the better. Because of randomness in return, many ways of approximation have been tried to consider uncertainty.

2.2 Statement of Problem

The problem of interest can be generalized as follows:

$$\begin{gathered}\max_{x,t} \quad t \\ \textit{Subject to}: \\ Prob\{g(x,c) \geq t|x,t\} \geq 1-\alpha \quad (1) \\ e^T x = 1 \\ x \geq 0\end{gathered}$$

in which $x \in R^n$ is the vector of decision variables, i.e., the percentage of asset allocations; $c \in R^n$ is the vector of returns of the uncertain assets. The vector e is defined as:

$$e = (1,1, \ldots 1)^T \quad (2)$$

The objective function that is maximized is g(x,c) and t is the desired target of function g. The risk level set by users is designed by α.

The goal is to maximize the target return under such probabilistic constraint.

2.3 **Defining Transaction Costs**

The transaction cost (Burghardt, 2008) (Markowitz, 1952) involved in this thesis is the contracting cost, which primarily means buying and selling expenses related to the purchase and sale of trading instruments, excluding interest income. We assume that it is nonlinear with respect to x, the percentage holdings of assets. We define the transaction cost function named h(x) next.

The transaction cost function h(x) will later be used as an addition to the loss function.

2.3.1 **Two-Part**

This type of transaction cost consists of two parts: a base constant rate as well as a floating fee depending upon the amount traded.

$$h(x) = \begin{cases} c + px & when\ x \succ 0 \\ 0 & when\ x = 0 \end{cases} \qquad (3)$$

2.3.2 **Two-Block**

A threshold criterion is held for this 'two-block' type. The fee rate differs after the trading amount exceeds a certain amount q, but remains the same for the part smaller than the threshold value.

$$h(x) = \begin{cases} p_1 x & when\ 0 \preccurlyeq x \preccurlyeq q \\ p_1 q + p_2(x - q) & when\ x \succ q \end{cases} \qquad (4)$$

2.3.3 **All Units Quantity Discount**

The fee rate depends upon the volume executed and, thus, two different rates are used, depending on whether it exceeds the threshold or not. This type of transaction cost function is especially introduced and practiced in the example in section 2.6.

$$h(x) = \begin{cases} p_1 x & when\ 0 \leq x < q \\ p_2 x & when\ x \geq q \end{cases} \qquad (5)$$

2.3.4 **With Caps and Floors**

The way of calculating transaction costs in this case is much more complex: several threshold criteria are used and a maximum constant value is set for all trading activities.

$$h(x) = \begin{cases} 0 & when\ x = 0 \\ p_1 q_f & when\ 0 < x \leq q_f \\ p_1 x & when\ q_f < x \leq q \\ p_1 q + p_2(x - q) & when\ q < x \leq q_c \\ constant & when\ x > q_c \end{cases} \quad (6)$$

2.4 **Literature Review**

The theory of portfolio optimization has come a long way from the Mean-Variance theory of Markowitz (Markowitz, 1952) who first introduced his mathematical model in 1951 It regards expected return as a desirable thing and variance of return as an undesirable thing, or in other words, risk. Despite its pioneering importance to modern portfolio theory, it suffers some limitations in practice. In mean variance analysis, only the first two moments are considered in the portfolio model. Furthermore, the expected return μ is hard to estimate. The measure of risk by variance places equal weight on upside deviations and downside deviations (HKUST), but volatility that makes the prices increase is good. This idea suggests that it may be more appropriate to minimize downside risk only for a long position.

Evaluating investments using expected return and variance of return is a simplification because returns do not simply follow a normal distribution; it has a distribution that is negatively skewed and with greater kurtosis than a normal distribution.

Next, value-at-risk (VaR), a widely used performance measure came on stage and answers the question: what is the maximum loss with a specified confidence level. Value at Risk (VaR) is a widely used measure of the risk of loss on a specific portfolio of financial assets. For a given portfolio, probability and time horizon, VaR is defined as a threshold value such that the probability that the mark-to-market loss on the portfolio over the given time horizon exceeds this value (assuming normal markets and no trading in the portfolio) which is the given probability level.

An alternative to VaR, is the Conditional Value at risk (CVaR). Rockafellar and Uryasev (Rockafellar & Uryasev, 2000) propose this new technique for portfolio optimization. It calculates VaR and optimizes CVaR simultaneously. CVaR comes with attractive properties such as transition-equivariant, positively homogenous and convex, which are absent from VaR. But this kind of scenario-based stochastic programming method becomes inefficient when dimension gets larger, or in other words, the number of assets grows.

The other proposed reliability method (Hanafizadeh & Ponnambalam, 2009) separates the space of decision variables from the space of random returns and thus forms a two step recursive optimization problem.

2.5 **Basic Theory of Techniques applied**

One underlying assumption underlying modern portfolio theory and the capital asset pricing model is that investors have homogeneous expectations, which means they have the same estimates and thus face the same efficient frontiers of risky portfolios and will all have the same optimal risky portfolio.

2.5.1 **VaR (Value at Risk)**

Let f(x,y) be the loss associated with the decision vector x of R^n and the random vector y in R^m.

The underlying probability distribution of y in R^m will be assumed for convenience to have probability density p(y).

The probability of f(x,y), not exceeding a threshold $\boldsymbol{\alpha}$, is then given by

$$\Psi(x,\alpha) = \int_{f(x,y)\leq\alpha} p(y)\, dy \qquad (7)$$

As a function of α for fixed x, Ψ is the cumulative distribution function for the loss associated with x. It completely determines the behavior of this random variable and is fundamental in defining VaR and CVaR.

$\Psi(x,\alpha)$ is nondecreasing with respect to α and continuous from the right.

The β-VaR is then given by

$$\alpha_\beta(x) = \min\{\alpha \in R: \Psi(x,\alpha) \geq \beta\} \qquad (8)$$

β is the given probability level.

2.5.2 **CVaR (Conditional Value at Risk)**

Although VaR is a very popular measure of risk, and has been applied in the financial industry, there does exist some undesirable features such as a lack of sub-additivity and convexity (Artzner, Delbaen, Eber, & Heath, 1997) (Artzner, Delbaen, Eber, & Heath, 1999). Sub-additivity and convexity are especially important in the study of optimization problems. In mathematics, sub-additivity is a property of a function that evaluating the function for the sum of two elements of the domain always returns something less than or equal to the sum of the function's values at each element, which is essential when it comes to the computation of the optimization problem. Convexity brings about a number of convenient properties, where particularly, a convex function on an open set has no more than one minimum.

CVaR is based on VaR, which can be regarded as an extension to the notion of the worst case (Quaranta & Zaffaroni, 2008) . It produces a portfolio based on a tail of the mean loss distribution (Zhu, Coleman, & Li, 2009).

The β-CVaR is given by

$$\phi_\beta(x) = (1-\beta)^{-1} \int_{f(x,y)\geq \alpha_\beta(x)} f(x,y)p(y)\,dy \qquad (9)$$

Define the auxiliary function:

$$F_\beta(x,\alpha) = \alpha + (1-\beta)^{-1} \int_{y\in R^m} [f(x,y)-\alpha]^+ p(y)\,dy \qquad (10)$$

Where

$$[t]^+ = \begin{cases} t, & when\ t > 0 \\ 0, & when\ t \leq 0 \end{cases} \qquad (11)$$

The β-CVaR of the loss associated with any x∈X can be determined from

$$\phi_\beta(x) = \min_{\alpha\in R} F_\beta(x,\alpha) \qquad (12)$$

2.5.3 **Reliability based optimization method (RBO)**

This method takes the first two statistical moments of a linear approximation of the performance function and attempts to find the minimal distance from the given nominal point to the tangent hyper-plane. This distance provides a measure of the yield. (Seifi, Ponnambalam, & Vlach, 1999)

Let c* be the reference point at the minimal distance from the nominal point $\bar{c}$ where $g(c^*|x) = t$, then linearize g(c|x) about the reference point c*:

$$g^L(c|x) = g(c^*|x) + (c-c^*)^T \nabla_c g(c^*|x) \qquad (13)$$

The first and second moment of $g^L(c|x)$ can then be computed as:

$$E(g^L) = t + (\bar{c}-c^*)^T \nabla_c g(c^*|x) \quad (14)$$

$$var(g^L) = \nabla_C g(c^*|x)^T C \nabla_C g(c^*|x) \quad (15)$$

Assume that the random vector c follows Gaussian distribution, and then rewrite the original problem into two separate but combined optimization problems.

The so called outer optimization problem is solved in the space of decision variables x and t, when c* is assumed to be known and is defined as follows:

$$\begin{gathered} \max_{x,t} g(x, c^*) \\ s.t. \\ (\bar{c} - c^*)^T \nabla_C g(c^*|x) \geq \Phi^{-1}(1-\alpha) \nabla_C g(c^*|x)^T C \nabla_C g(c^*|x)^{\frac{1}{2}} \\ e^T x = 1 \\ x \geq 0 \end{gathered} \quad (16)$$

The inner optimization problem tries to find the value of c* assuming x and t are given.

It is defined as:

$$\beta = \left\{ \min_c [(c-\bar{c})^T (C)^{-1} (c-\bar{c})]^{\frac{1}{2}} \middle| g(c|x) = t \right\} \quad (17)$$

The final optimum set is obtained through iteration of these two optimization problems.

2.5.4 **Factor Model**

2.5.4.1 The definition of a factor model

The factor model is a way of decomposing the forces that influence a security's rate of return into market and firm-specific influences (Harvey, 2009).

2.5.4.2 Input data issue

There are basically two problems resulting from input data in optimization models.

1) The number of estimates needed for mean-variance analyses
    a. Generally, with N different assets, we require a total of (N^2+3*N)/2 different estimates
2) The use of historic data

First, historic data must be smoothed to try to focus on underlying relationships that are more likely to be true in the future and to ignore deviations from those relationships that are more likely to be due to random noise or errors. The tools used most often to accomplish this are factor models. (Sharpe, 2012)

N	T	Available/Estimated
10	60	9.23
100	60	1.17
1000	60	0.12
10	120	18.46
100	120	2.33
1000	120	0.24
10	840	129.23
100	840	16.31
1000	840	1.68
10000	840	0.17

Table 2.1 : Input data number comparison (Sharpe, 2012)

The variable N in the table stands for the number of samples; T is the number of sampling time.

The table above is a specific example showing comparative ratios of parameter estimates available divided by needed given different sample levels.

As N, the number of samples increases as large as 1000, the number of data available divided by the number of estimates we need is smaller than 1, which means we are short of data. This is demonstrated by the case in T=60 and T=120. As for the case in T=840, the shortage becomes a problem when N reaches 10000.

2.5.4.3 The need for Factor model

- Problems involving large numbers of assets require a great many estimates.
- It's too difficult to estimate each of the required values explicitly.

2.5.4.4 Framework

The Linear Factor Model can be written mathematically as (Sharpe):

$$R_i = b_{i1} * f_1 + b_{i2} * f_2 + \cdots + b_{im} * f_m + e_i \quad (18)$$

Variable	Definition
R_i	return of asset i
f_m	value of factor m
b_{im}	factor loadings
M	number of factors
e_i	portion of the return on asset i not related to the m factors

Table 2.2 Definition of variables in factor model

Factor models are also capable of transferring into matrix forms. The matrix representation of factor model is (Sharpe):

$$R = B * F + E \quad (19)$$

Variable	Definition
R	N*T matrix, where R(i,t) is the return on asset i in realization t
B	N*m matrix, where B(i,j) is the exposure of asset i to factor j
F	m*T matrix, where F(j,t) is the value of factor j in realization t
E	N*T matrix, where E(i,t) is the residual return on asset i in realization t

Table 2.3 Definition of variables in matrix form factor model

2.5.4.5 Factor based portfolio

Factor model can make up a portfolio in the way of a return model (Stubbs, 2012).

As of the matrix form of the factor model shown in equation (19), the expected return model can be derived as:

$$E[R] = B * E[F] \quad (20)$$

The risk model is:

$$Var[R] = B * E[FF^T] * B^T + E[EE^T] = B\Omega B + \Delta \quad (21)$$

2.5.4.6 Summary

Our factor models are used to estimate the expected returns and variances on risky assets based on specific factors. For each asset, we need to estimate the sensitivity to each specific factor. In this way we transform the return data into a basket multiplication of factors and its factor loadings.

Factors that explain asset returns can be classified as macroeconomic, fundamental and statistical factors. We would go further into that in the next section.

2.6 Application Problem

2.6.1 CVaR

Let $\mu \in R^n$ be the vector of the mean returns of n risky assets. Let x_i, $1 \leq i \leq n$ denote the percentage holding of the i^{th} asset. A portfolio allocation is considered to be efficient if it has the minimum risk for the given level of expected return. Furthermore, the integral in (10) of F can be approximated in various ways (Krokhmal, Palmquist, & Uryasev, 1999). We take advantage of the historical data obtained from the TSX market recorded on the Yahoo! Finance website as samples for the distribution of the mean return.

Then the corresponding approximation to F is

$$\tilde{F}_\beta(x,\alpha) = \alpha + \frac{1}{q(1-\beta)}\sum_{k=1}^{q}[f(x,y_k)-\alpha]^+ \quad (22)$$

In this case, let $f(x,y) = -\mu^T x + h(x)$, where the transaction cost function h(x) is also taken into account.

Rewrite as follows:

$$\tilde{F}_\beta(x,\alpha) = \alpha + \frac{1}{q(1-\beta)}\sum_{k=1}^{q}[-\mu^T x + h(x) - \alpha]^+ \quad (23)$$

The above conclusions are made under following assumptions:

- The underlying probability distribution of y in R^m are assumed for convenience to have probability density p(y).
- We also assume that the probability distribution Ψ(x,α) is non-decreasing with respect to α and such that no jumps occur, or in other words that Ψ(x,α) is everywhere continuous with respect to α.
- As a function of α for fixed x, F(x,α) is convex and continuously differentiable.
- $\tilde{F}_\beta(x,\alpha)$ is convex and piecewise linear with respect to α.

2.6.2 **Reliability based optimization method**

In this case, let $g(x,c) = \mu^T x - h(x)$, in which the transaction cost function h(x) is also taken into account.

In for our case, g(x,c) is a linear function with respect to c(i.e. $\nabla_c g(c|x) = x$), then the outer optimization problem does not depend on the reference point c^*. Thus, we do not need to solve the inner optimization problem.

The corresponding deterministic counterpart of the uncertain inequality is

$$(\bar{c} - c^*)^T x \geq \Phi^{-1}(1-\alpha)(x^T C x)^{\frac{1}{2}} \qquad (24)$$

Then the asset allocation problem is simplified to

$$\begin{gathered} \max_{x,t} t \\ s.t. \\ \bar{c}^T x - h(x) - \Phi^{-1}(1-\alpha)(x^T C x)^{\frac{1}{2}} \geq t \\ e^T x = 1 \\ x \geq 0 \end{gathered} \qquad (25)$$

2.6.3 **The Factor Model**

The methodology of setting up a factor model can be summarized into several steps.

1) Range of selection for factors
2) Determining number of factors
3) Regression, parameters estimates
4) Model set up

In the factor model, the choices of factors are determined based on two concerns:

1) The economic approach
   - macroeconomic and financial market variables (Chen, Roll, & Ross, 1986)
   - Characteristics of firms (Fama & French, 1993) (Fama & French, 1992)
2) The statistical approach includes principal component analysis and factor analysis.

For instance, we determine the factor portfolio model for GE company from a range of factors including gold price, 3-month treasury bill price, unemployment rate, earnings per share of GE, commodity food & beverage index, consumer price index-oil, export price, book value of GE and consumer price index all inclusive. Our example would be to use regression analysis to estimate the relationship between return and these factors. Research has found that stock returns are related to known economic fundamentals such as interest rates and dividend yields. This is expected to occur in efficient markets.

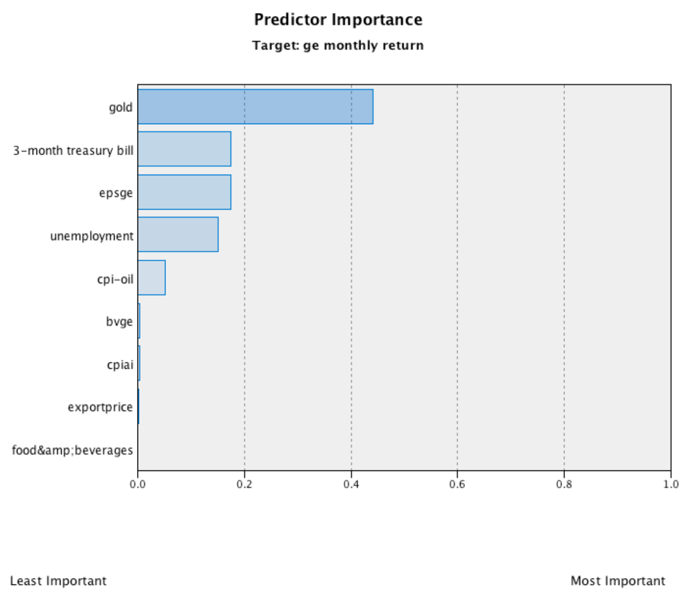


Figure 2.1 Predictor importance for GE returns (exported from software: SPSS)

After conducting a regression analysis in the SPSS statistical software, we can obtain the output shown above. Then the five most influential factors: gold price, 3-month treasury bill price, earnings per share, unemployment rate and cpi-oil price should be put in the factor model for GE monthly return model. All applications for the other four equities are shown in appendix A.

Considering the factor model for a portfolio, which is composed of 8 different assets, we would need to pick the factors that are essential overall. Gold price, unemployment rate and book value for each company are selected as key factors. In that way, the monthly return of every company, which is part of the portfolio, are written in the form of a linear factor model with three factors.

Detailed coefficients of GE company are shown in the table below. Bvge stands for book value of GE company.

Coefficients[a]

Model	Unstandardized Coefficients		Standardized Coefficients	t	Sig.
	B	Std. Error	Beta		
(Constant)	.056	.045		1.254	.212
Gold	-.302	.191	-.156	-1.579	.117
unemployment	-.008	.009	-.083	-.831	.408
Bvge	-.001	.003	-.031	-.292	.771

a. Dependent Variable: ge monthly return

Table 2.4 Coefficients of GE factor model (exported from software: SPSS)

All other coefficients are shown in appendix B.

The factor model can be applied to get the data inputs we need for the model. But the application of factor model for generating data for the portfolio optimization problem is not included in this thesis.

2.6.4 **Example Application**

Both of the methods (CVaR and RBO) are presented to find optimal solutions for the asset allocation problem. The results from both methods are compared in terms of efficiency and optimum return levels. Calculation, simulation, and test are realized via Matlab R2009b in a PC (intel core i5 processor 2.26GHZ). The percentage holding of each asset within the n-asset portfolio is denoted by $x=(x_1,\ldots x_n)^T$.

$$0 \le x_j \le 1 \; for \; j = 1, \ldots, n, with \sum_{j=1}^{n} x_j = 1 \quad (26)$$

The risk levels are assigned the three most possible values: 0.9, 0.95 and 0.99. The number of sample data tested is 1000,2000,3633. 3633 is the largest sample we can get ever since the objectives are listed companies in the market.

2.7 **Data Analysis**

An example is provided in which the optimal portfolio is composed of five equities from Toronto Stock Exchange Market: Royal Bank of Canada, Suncor Energy Inc., Bank of Nova Scotia, Teck Resources ltd, Canadian Natural Resources Ltd (Yahoo! Finance). There is diversity in the equities in the sense that the components of the portfolio are of different industries, e.g. Finance as well as Resources and Energy. In addition, all of them have been active in the Toronto Stock Exchange Market ever since 1995. We use daily return data on these five stocks as sources of μ, to set up the program for different risk levels.

In the following chapters, we will be using the short forms of the equities for simplicity, as shown in the table below.

Equity	Code
Suncor Energy	su
Royal Bank of Canada	ry
Canadian Natural Resources	cnq
Bank of Nova Scotia	bns
Teck Resources	tck-bo

Table 2.5 equity code list

The mean and covariance information are shown in Table 2.5 and Table 2.6 below, respectively.

According to Table 2.5, all of the five stocks are price gainers, or more specifically, equities that have positive daily return. Moreover, all of the entries in the covariance matrix are non-zero.

Equity	mean return
Suncor Energy	0.0010
Royal Bank of Canada	0.0007
Canadian Natural Resources	0.0011
Bank of Nova Scotia	0.0008
Teck Resources	0.0008

Table 2.6 mean return (4124 samples)

Furthermore, all of the entries in the correlation coefficient matrix are non-zero according to Table 2.8. These all show that there is correlation (linear dependence) among these five stocks.

All in all, all five equities with correlation set up the targeted portfolio.

	su	ry	cnq	bns	tck-bo
su	0.0005	0.0001	0.0004	0.0001	0.0003
ry	0.0001	0.0002	0.0001	0.0002	0.0002
cnq	0.0004	0.0001	0.0006	0.0001	0.0004
bns	0.0001	0.0002	0.0001	0.0003	0.0002
tck-bo	0.0003	0.0002	0.0004	0.0002	0.0010

Table 2.7 covariance matrix (4124 samples)

	su	ry	cnq	bns	tck-bo
su	1.0000	-0.9102	0.7374	-0.8729	0.3924
ry	-0.9102	1.0000	-0.8136	0.7038	-0.2737
cnq	0.7374	-0.8136	1.0006	-0.7804	0.3681
bns	-0.8729	0.7038	-0.7804	1.0003	-0.2736
tck-bo	0.3924	-0.2737	0.3681	-0.2736	1.0010

Table 2.8 correlation coefficient (4124 samples)

One other alternative for data entry is to use the factor model. The factor model can be applied to the get the data inputs we need for the model. The application of factor model for generating data for the optimization problem is not included in this thesis.

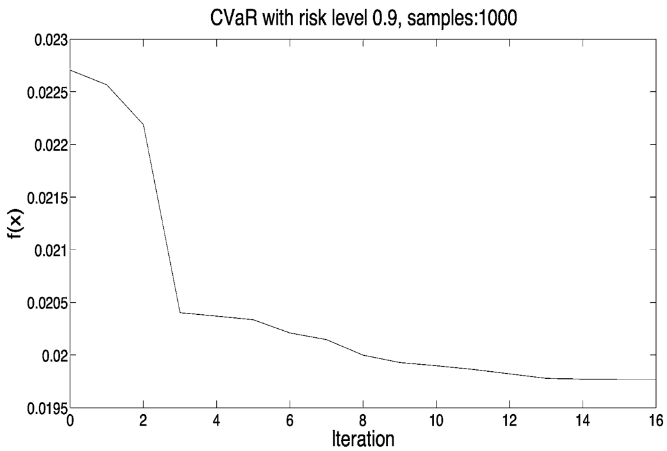
2.8 **Convergence Proof**

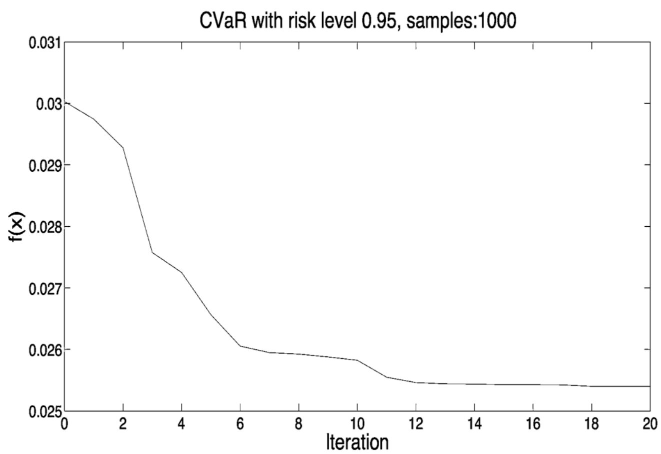
To carry out the convergence study of the RBO method and the CVaR when they are applied to this asset allocation problem, tests for different risk levels are conducted.

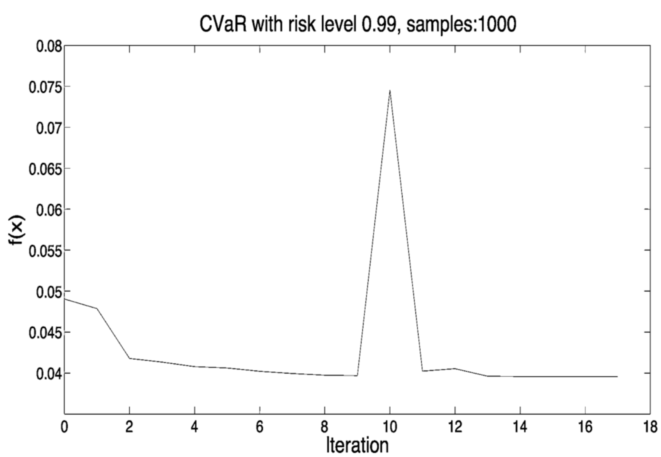
Figure 1 and 2 in the next few pages show the trend of f(x) as the algorithm iterates. All of them do converge after a small number of iterations, with little difference in different risk levels. Data are obtained through Matlab's internal computation process. For the CVaR method, the y-label f(x) displayed in the curve stands for the $\tilde{F}_\beta(x, \alpha)$ in equation (23), but not the loss function mentioned before; In the RBO method, f(x) stands for $\left[-\bar{c}^T x + h(x) + \Phi^{-1}(1-\alpha)(x^T C x)^{1/2}\right]$.

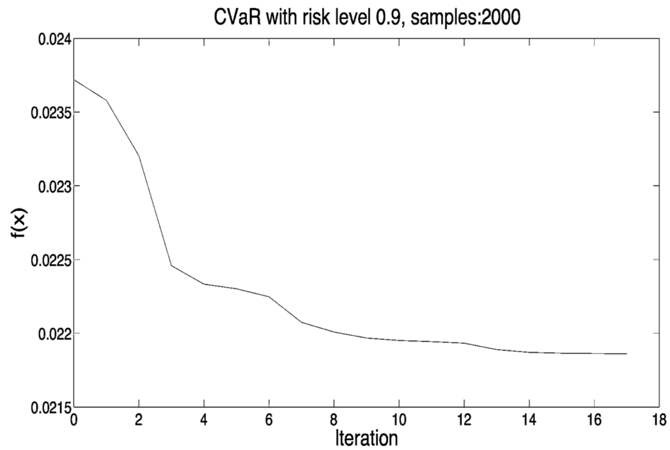
In comparison, the RBO reliability method converges much more quickly than the CVaR method after approximately four to five iterations. By examining the convergence curve for both methods, we can determine that, in some cases, oscillations are introduced into the CVaR method. As the number of sample data increases, oscillations seem to be more obvious and magnified. Another finding is that RBO arrives at a faster convergence rate when the optimization problem turns out to be in high dimension. This is good news for its value in industrial applications.

2.8.1 **Case1: CVaR**

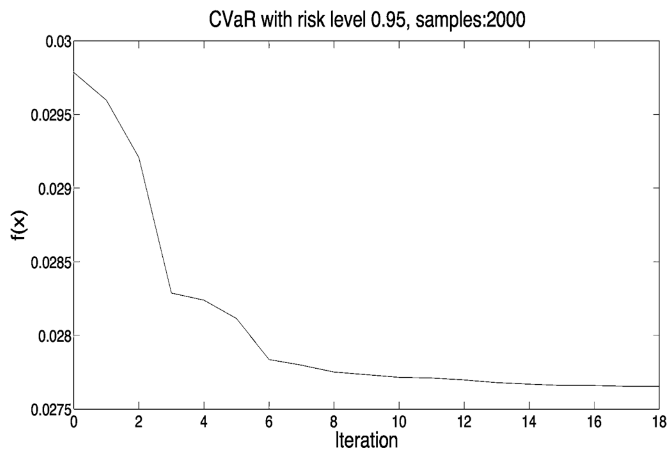




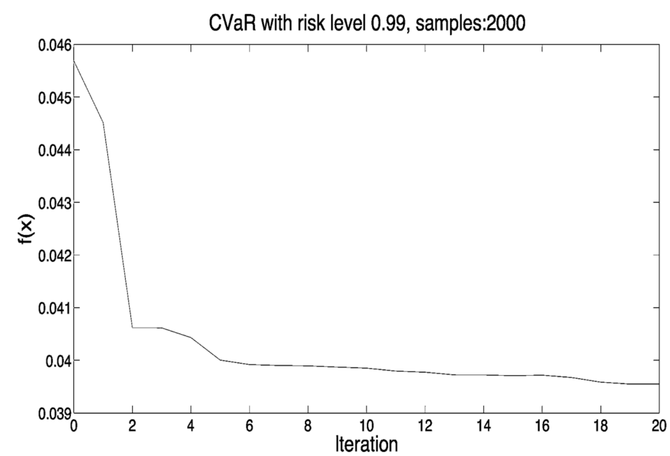




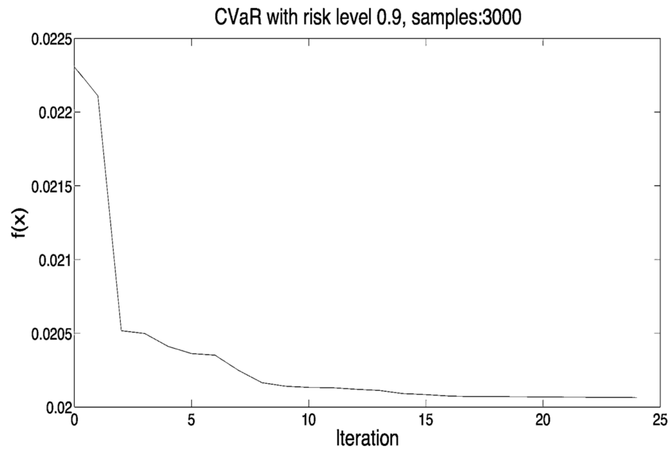
CVaR with risk level 0.9, samples:2000
0.024
0.0235
0.023
0.0225
0.022
0.0215
f(x)
0
2
4
6
8
10
12
14
16
18
Iteration

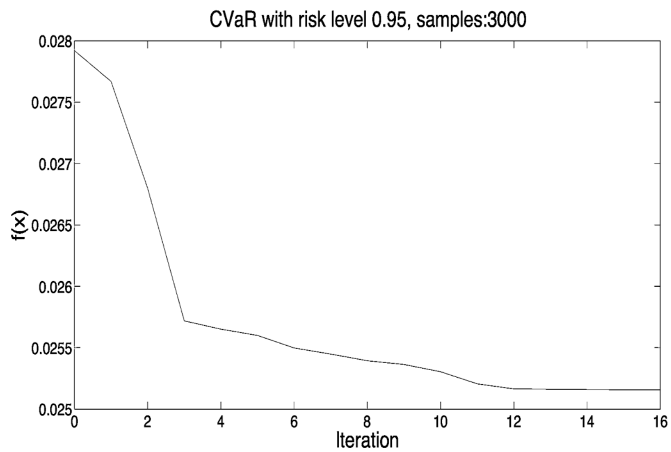


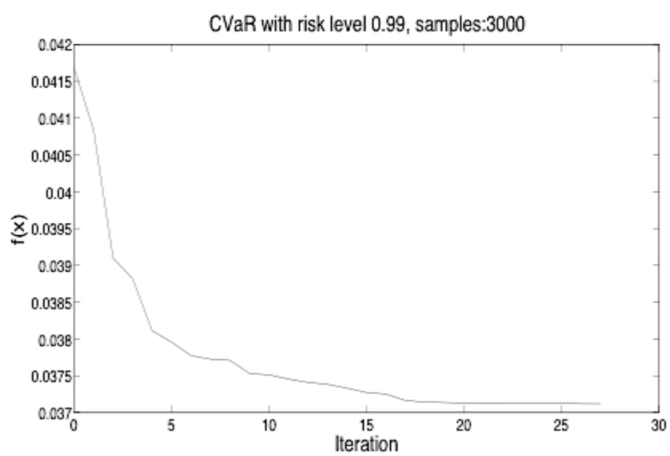
CVaR with risk level 0.95, samples:2000
0.03
0.0295
0.029
0.0285
0.028
0.0275
f(x)
0
2
4
6
8
10
12
14
16
18
Iteration



CVaR with risk level 0.99, samples:2000
0.046
0.045
0.044
0.043
0.042
0.041
0.04
0.039
f(x)
0
2
4
6
8
10
12
14
16
18
20
Iteration


CVaR with risk level 0.9, samples:3000
f(x)
Iteration
0.0225
0.022
0.0215
0.021
0.0205
0.02
0
5
10
15
20
25


CVaR with risk level 0.95, samples:3000
f(x)
Iteration
0.028
0.0275
0.027
0.0265
0.026
0.0255
0.025
0
2
4
6
8
10
12
14
16


CVaR with risk level 0.99, samples:3000
f(x)
Iteration
0.042
0.0415
0.041
0.0405
0.04
0.0395
0.039
0.0385
0.038
0.0375
0.037
0
5
10
15
20
25
30

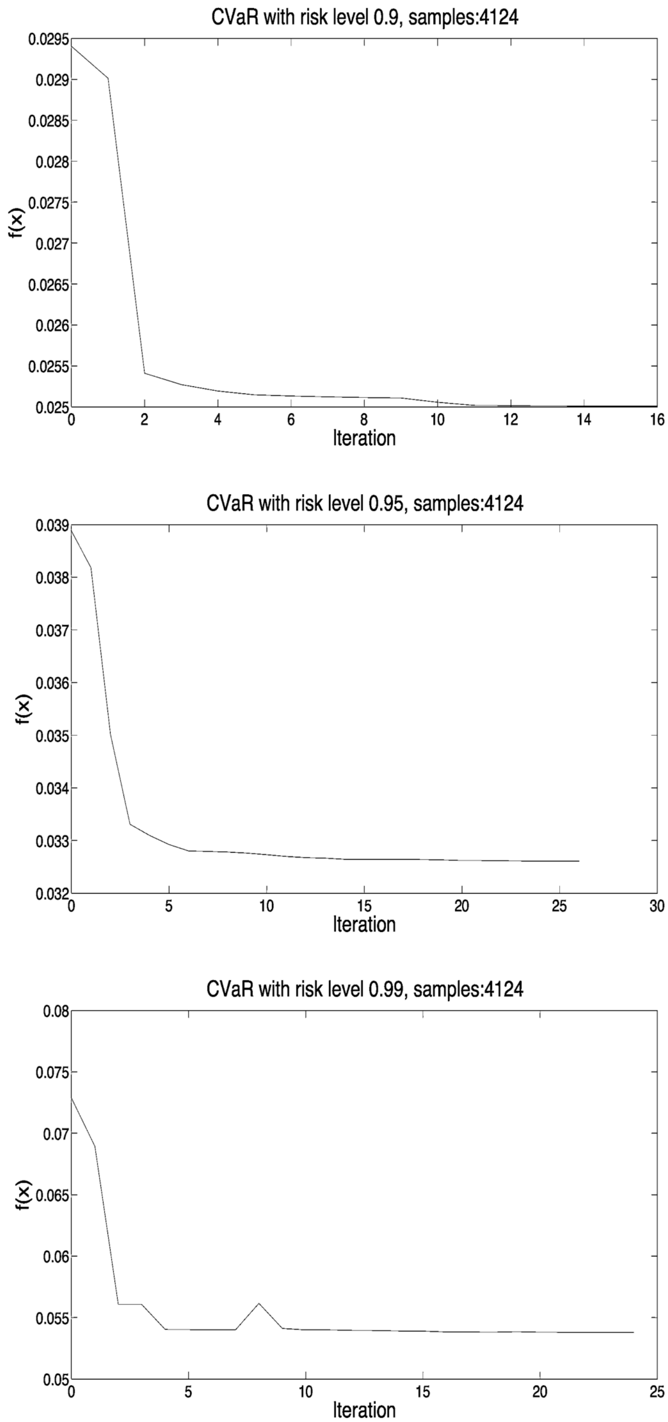


Figure 2.2 Convergence demonstration of CVaR with differnet risk levels and samples

2.8.2 **Case2: RBO**

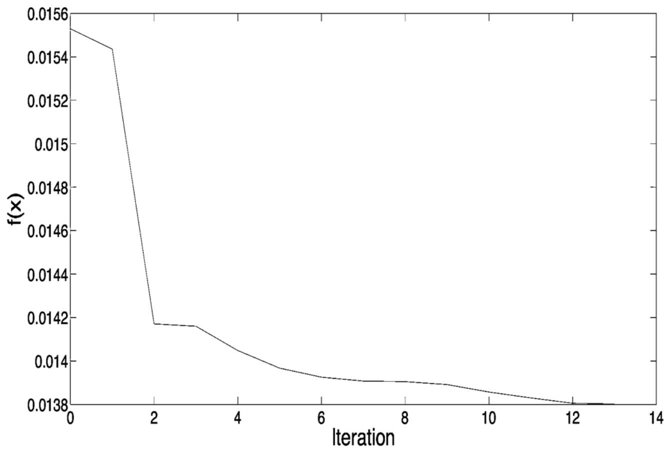


Figure 2.3-(1) RBO with risk level 0.9, samples: 1000

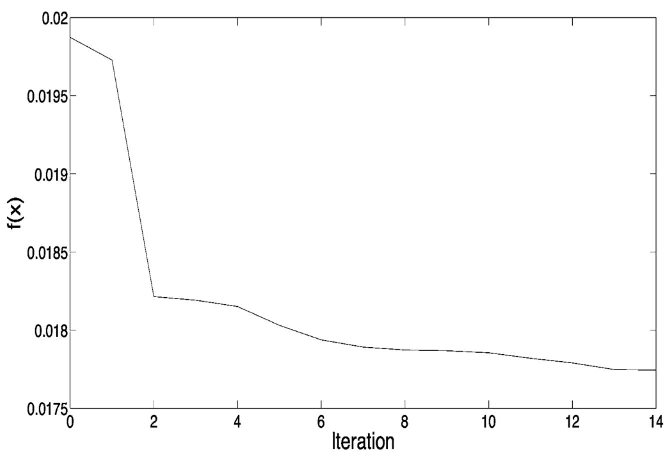


Figure 2.3-(2) RBO with risk level 0.95, samples: 1000

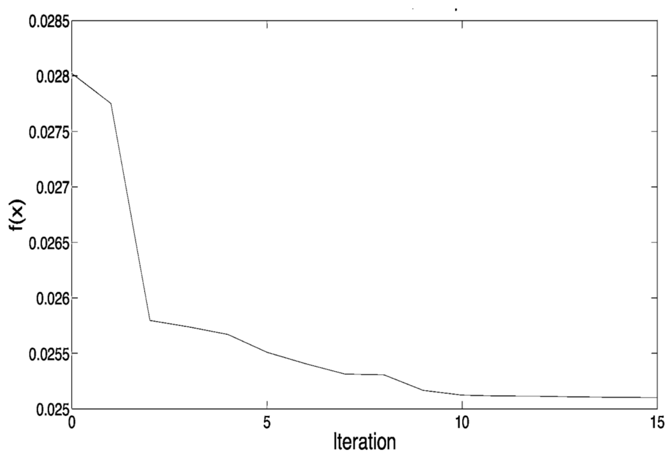


Figure 2.3-(3) RBO with risk level 0.99, samples: 1000

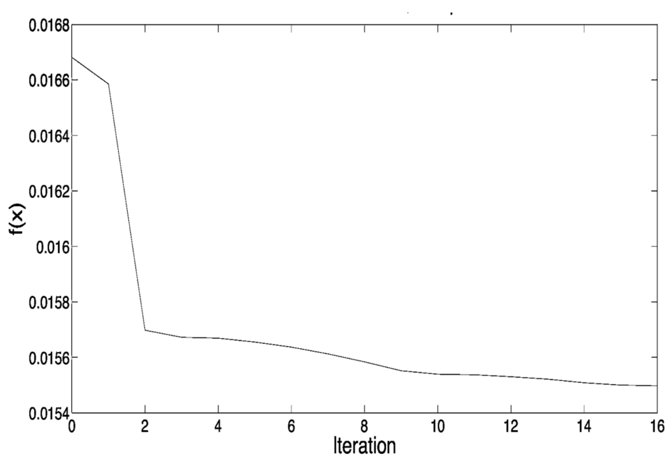


Figure 2.3-(4) RBO with risk level 0.9, samples: 2000

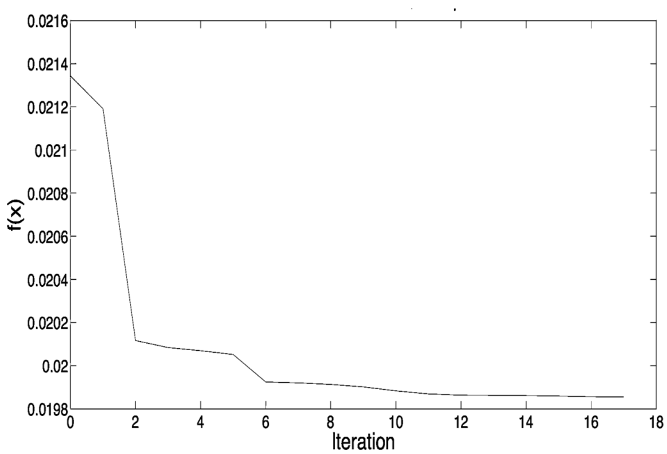


Figure 2.3-(5) RBO with risk level 0.95, samples: 2000

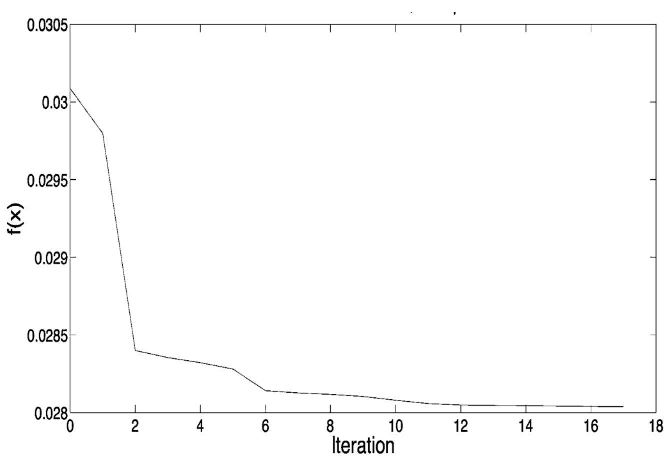


Figure 2.3-(6) RBO with risk level 0.99, samples: 2000

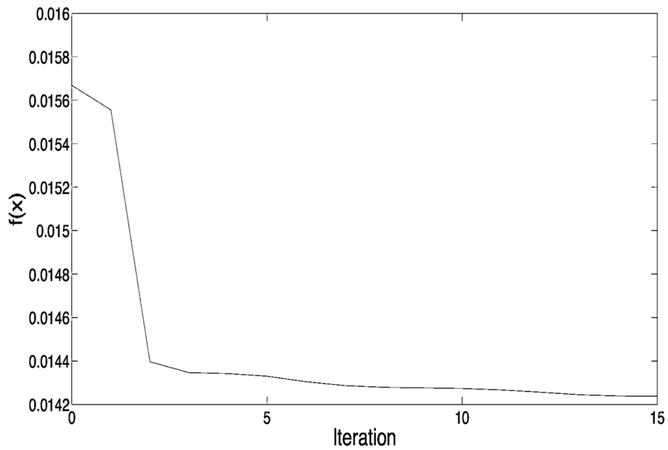


Figure 2.3-(7) RBO with risk level 0.9, samples: 3000

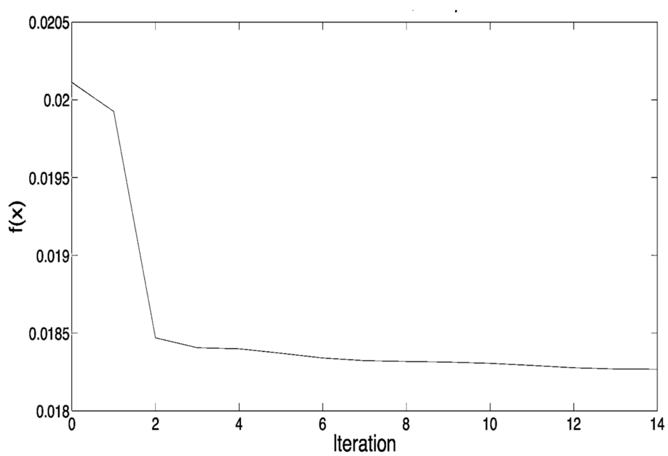


Figure 2.3-(8) RBO with risk level 0.95, samples: 3000

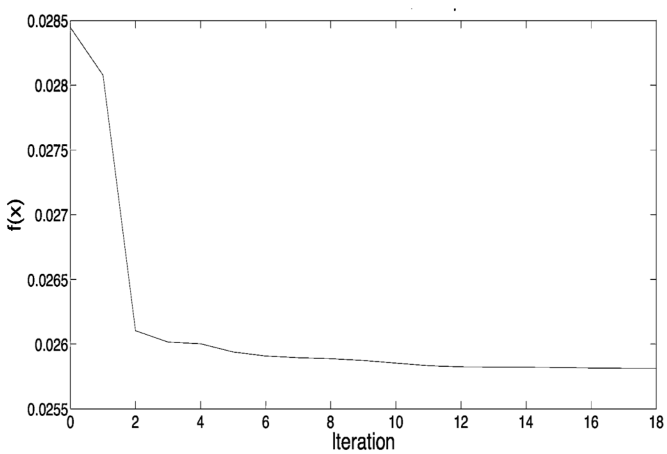


Figure 2.3-(9) RBO with risk level 0.99, samples: 3000

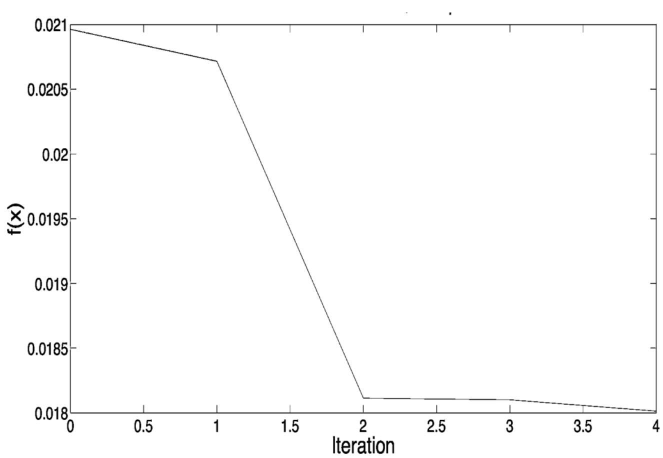


Figure 2.3-(10) RBO with risk level 0.9, samples: 4124

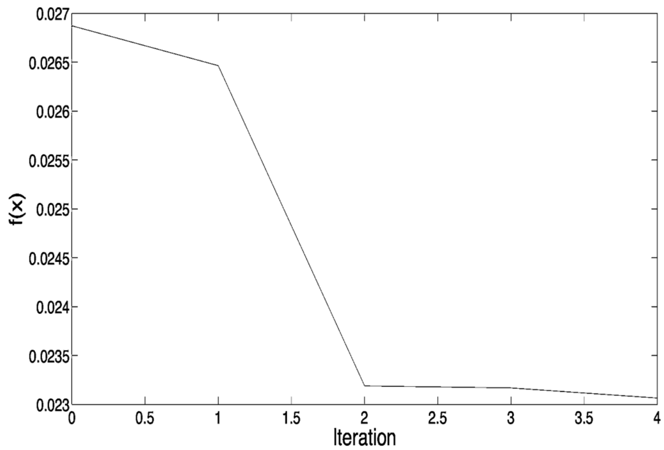


Figure 2.3-(11) RBO with risk level 0.95, samples: 4124

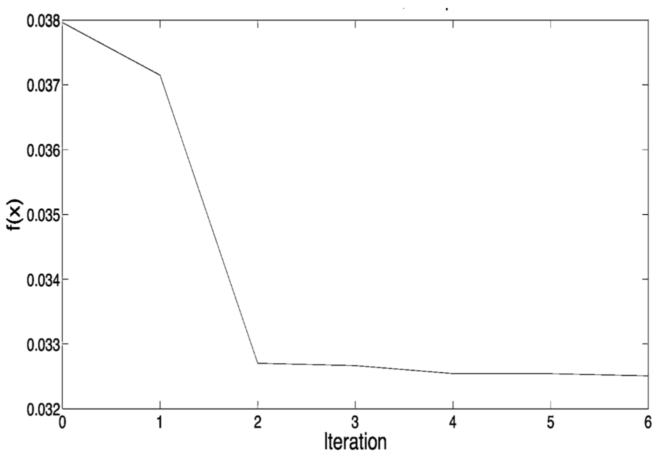


Figure 2.3-(12) RBO with risk level 0.99, samples: 4124

Figure 2.3 Convergence demonstration of RBO with different risk levels and samples

2.9 **Result Analysis**

2.9.1 **Case1: CVaR**

VaR is obtained as a byproduct of this optimization problem programmed in Matlab. The unique solutions for the optimal portfolio $\mathbf{x}^*$, VaR as well as CVaR for three different risk levels, are displayed in the tables below (Tables 2.9-2.12).

For a specific risk level, VaR and CVaR differ only slightly depending upon the number of samples.

For different risk levels, 0.9, 0.95 and 0.99 in the same scale of data, as the value of the risk level increases, the corresponding VaR and CVaR also increase. This finding coincides with the fact that the risk level naturally corresponds to an investor's tolerance to estimation risk.

	β=0.9	β=0.95	β=0.99
		x^*	
su	0.5446	0.5197	0.5769
ry	0.4393	0.4751	0.4231
cnq	0.0058	0.0000	0.0000
bns	0.0000	0.0000	0.0000
tck-bo	0.0102	0.0052	0.0000
VaR	0.0115	0.0172	0.0318
CVaR	0.0198	0.0254	0.0396

Table 2.9 data samples=1000, CVaR

	β=0.9	β=0.95	β=0.99
		x^*	
su	0.2883	0.3310	0.3206
ry	0.4802	0.4402	0.5290
cnq	0.0955	0.0505	0.0813
bns	0.0843	0.1330	0.0691
tck-bo	0.0518	0.0452	0.0000
VaR	0.0139	0.0190	0.0332
CVaR	0.0219	0.0277	0.0395

Table 2.10 data samples = 2000, CVaR

	β=0.9	β=0.95	β=0.99
	x^*		
su	0.2437	0.2584	0.3325
ry	0.5015	0.4730	0.5211
cnq	0.0644	0.0256	0.0928
bns	0.1447	0.1889	0.0536
tck-bo	0.0457	0.0541	0.0000
VaR	0.0129	0.0176	0.0315
CVaR	0.0201	0.0252	0.0371

Table 2.11 data samples = 3000, CVaR

	β=0.9	β=0.95	β=0.99
	x^*		
su	0.1992	0.1499	0.0448
ry	0.5205	0.5334	0.5984
cnq	0.0243	0.0467	0.0000
bns	0.2560	0.2700	0.3568
tck-bo	0.0000	0.0000	0.0000
VaR	0.0142	0.0211	0.0393
CVaR	0.0250	0.0326	0.0538

Table 2.12 data samples = 4124, CVaR

2.9.2 **Case2: RBO**

In this method, t is the target of the total investment, or the net return of the portfolio.

The unique solutions for the optimal portfolio **x**, as well as t for three different risk levels and three different data dimensions, are displayed in the tables below (Table 2.13-2.16). For different risk levels, 0.9, 0.95 and 0.99 in the same scale of data, as the value of the risk level increases, the corresponding t also increases greatly. For a specific risk level, t differs only slightly depending upon the number of samples.

	β=0.9	β=0.95	β=0.99
	x^*		
su	0.4121	0.4093	0.3955
ry	0.4848	0.4795	0.4865
cnq	0.0852	0.0854	0.0752
bns	0.0000	0.0000	0.0000
tck-bo	0.0180	0.0259	0.0428
t	0.0138	0.0177	0.0251

Table 2.13 Data samples = 1000, RBO

	β=0.9	β=0.95	β=0.99
	x^*		
su	0.2955	0.2778	0.2753
ry	0.4042	0.4202	0.4217
cnq	0.1327	0.1322	0.1313
bns	0.1223	0.1215	0.1167
tck-bo	0.0452	0.0482	0.0551
t	0.0155	0.0199	0.0280

Table 2.14 Data samples = 2000, RBO

	β=0.9	β=0.95	β=0.99
	x^*		
su	0.2378	0.2361	0.2160
ry	0.4176	0.4211	0.4397
cnq	0.0875	0.0868	0.0960
bns	0.2090	0.2073	0.1957
tck-bo	0.0481	0.0487	0.0526
t	0.0142	0.0183	0.0258

Table 2.15 Data samples = 3000, RBO

	β=0.9	β=0.95	β=0.99
	x^*		
su	0.1530	0.1527	0.1840
ry	0.3867	0.3881	0.4140
cnq	0.0942	0.0928	0.0668
bns	0.3661	0.3664	0.3352
tck-bo	0.0000	0.0000	0.0000
t	0.0180	0.0231	0.0325

Table 2.16 Data samples = 4124, RBO

2.10 Computational Efficiency

The following table shows exactly how much time each algorithm takes in terms of data scale and risk levels.

CVaR takes more time and space, especially when the dimension grows since CVaR is a kind of scenario-based stochastic programming method. However, the number of decision variables in RBO remains the same irrespective of number of samples, so it is more efficient.

When we consider accuracy, the better solution must always be traded-off with higher computing costs.

As Table 2.18 shows, as samples increase from 1000, 2000 to 4124, CVaR has a larger growth in time, which implies difficulties for large-scale problems solving in the real financial market. The RBO reliability method is so efficient that its speed remains about the same. This method takes just a few seconds as the sample doubles. Furthermore, CVaR needs to store the whole data matrix in the process of computation. On the other hand, RBO needs to obtain only the mean and variance vectors on hand before calculation.

2.10.1 Case1: CVaR

	β=0.9	β=0.95	β=0.99
Sample number	cputime(s)		
1000	7.3008	6.3960	9.7657
2000	21.6685	22.7605	26.4578
3000	26.8166	19.7653	32.6510
4124	20.8573	37.0502	30.3734

Table 2.17 Efficiency: CVaR

2.10.2 Case2: RBO

	β=0.9	β=0.95	β=0.99
Sample number	cputime(s)		
1000	4.9296	3.4944	3.9156
2000	4.9920	5.1480	5.1168
3000	5.8032	5.6316	8.3773
4124	3.8064	4.0560	4.5084

Table 2.18 Efficiency: RBO

2.11 **Conclusion**

This Chapter presented methods for solving the portfolio optimization problem in which the investors pay a transaction cost as a function of the trading volume of the risky assets. The main contribution goes to the extension of both the Conditional Value at Risk method and the reliability based optimization method, with an application in asset allocation considering nonlinear transaction costs. The RBO method is faster especially in higher dimensions; The CVaR risk measurement can be more accurate since it is entirely based on historical data.

Chapter 3
Case study 2: Online Display Advertising Allocation Problem

3.1 Introduction

As is shown in the Actual +2011 Estimated Canadian online Advertising Revenue Survey detailed report supported by IAB Canada, in 2010, online ad revenues surpassed Daily Newspaper ad revenues. As a result, the Internet is now second only to Television in terms of share of total Canadian media advertising revenue (15.9%). This is a convincible fact showing the critical role of online advertising in the advertising industry. (IAB Canada, 2012)

Moreover, the potential expansion of business in online advertising is inevitable. Online advertising's 23% increase from 2009 to 2010 also bested other major media, all but one experiencing only single-digit growth rates during this time. Online advertising growth as is surprisingly high, which we can see clearly in the table below.

Total 2010 Online Advertising Revenue			
	2009	2010	%growth
Millions($)	1822	2232	23

Table 3.1 Online Advertising Revenue

Nobody could resist this "big tasty cake". Canadian Online Advertising Revenues for 2010 exceeded budgeted expectations of $2.1 billion and grew by 23% to $2.23 billion for 2010, while it still remains underdeveloped (IAB Canada, 2012). Algorithms as well as techniques need advancement and attention of mathematicians, financial engineers and IT specialists.

The automation platform for the online media exchange system boosts the values of the publishers' remnant inventory and tries to produce the most competitive outcomes for both parties, advertisers and publishers, through the allocation process.

3.2 Types of Advertising

Normally, advertisements are grouped into three different categories: display advertising, networking and affiliation advertising and search-based advertising. What we are focusing on in this thesis is the first type: display advertising, and more specifically, online display advertising. The “online” feature indicates how it differs from traditional media in the advertising industry. Meanwhile, “display” shows that advertisement could be shown in different formats, such as text, picture, music, video and etc.

Search advertising continues to lead in terms of share of dollars booked by Online Publishers ($907 million/41%), followed by Display ($688 million/31%) and Classifieds ($587 million/26%). Together, these three advertising vehicles represent 98% of all online advertising booked in Canada.

Online advertising eliminates transportation cost and at the same time enjoys all convenience of online business. The advancement of information technology now enables and guarantees easy access to advertisements at anytime anywhere to any web users. The immediate publishing of information is not limited by geography or time (Hanafizadeh, Online Advertising and Promotion: Modern Technologies For Marketing, 2012).

It’s also user-friendly as it offers several options to users. For example, the ads could be opened or closed, clicked or expanded, paused or downloaded according to user’s preferences.

There are a series of targeting tools available including contextual targeting, placement targeting, remarketing, demographic targeting and interest categories that matches contents of ads with contents of websites to the right people. “Right” here mean audiences with the same age, gender, interests or region. By design, the system uses cookie and browser history to determine geographic and interests.

3.3 **Goals of Publishers and Advertisers**

On the one hand, the advertisers try to put their ads on the publisher’s website with the lowest possible cost. On the other hand, the publishers are seeking competitive revenues for all their available resources. This involves the basic demand-supply economic relationships between publishers and advertisers.

Besides this, advertisers has certain goals to accomplish, whether it’s to generate brand awareness, target certain customer groups or promote direct purchases, there are different models to support each mission.

- If you want to generate traffic to your website, focusing on clicks could be ideal for you. Cost-per-click (CPC) bidding, manual or automatic, may be right for your campaign.

- If you want to increase brand awareness, not driving traffic to your site, focusing on impressions may be your strategy. You can use cost per thousand impressions (CPM) bidding to put your message in front of customers.
- If you want customers to take a direct action on your site, and you're using conversion tracking, then it may be best to focus on conversions. The advanced bidding option, namely, the cost-per-acquisition (CPA) bidding allows for such a possibility (Google Inc, 2012)

3.4 **The Revenue Model**

3.4.1 **CPM**

Cost per impression, often abbreviated to CPI or CPM (Cost per mille) are terms used in online advertising and marketing related to web traffic. They refer to the cost of internet marketing campaigns where advertisers pay for every time their ad is displayed, usually in the form of a banner ad on a website (Wiki).

An impression is the display of an ad to a user while viewing a web page. A single web page may contain multiple ads. In such cases, a single page view would result in one impression for each ad displayed. In order to count the impressions served as accurately as possible and prevent fraud, an ad server may exclude certain non-qualifying activities such as page-refreshes or other user actions from counting as impressions. When advertising rates are described as CPM or CPI, this is the amount paid for every thousand qualifying impressions served.

Cost per mille is one of the most common marketing practices used on the internet along with CPC and CPA described below.

3.4.2 **CPC**

Pay per click (PPC) (also called Cost per click) is an internet advertising model used to direct traffic to websites, where advertisers pay the publisher (typically a website owner) when the ad is clicked.

There are two primary models for determining cost per click: flat rate and bid-based. In both cases the advertiser must consider the potential value of a click from a given source. This value is based on the type of individual the advertiser is expecting to receive as a visitor to his or her website, and what the advertiser can gain from that visit, usually revenue, both in the short term as well as in the long term. As with other forms of advertising targeting is key, and factors that often play into PPC campaigns include the target's interest, intent (e.g., to purchase or not), location and the day and time that they are browsing (Wiki, 2012).

3.4.3 **CPA**

Cost Per Action or CPA (sometimes known as Pay Per Action or PPA) is an online advertising pricing model, where the advertiser pays for each specified action (a purchase, a form submission, and so on) linked to the advertisement (Wiki, 2012).

3.5 **Statement of Problem**

The functioning process of the system can be described as follows: in general, there are two basic types of buying and selling: guaranteed and unguaranteed. All advertisers and publishers could exchange and trade either in the guaranteed contract system or the unguaranteed (spot) market or both. Advertisers may manage their ads at the beginning of each trading period by setting up budgets and bid types. Normally, advertisers are allowed to set up daily budget, monthly budget, bi-monthly budget or for an even longer period. These budgets can be represented in terms of monetary value or numbers of advertisements. The options of bidding types range from cost-per-click (CPC), cost-per-view (CPV), cost-per-acquisition (CPA) and so on. Trading periods vary from one day, one month, and two months to a longer time period and it is related to the advertisers' preferences.

At the beginning of each trading period, all advertisers who are willing to conduct financial transactions in the guaranteed contract system would send a request to the trading system platform announcing how many advertisements they would like to purchase for their personalized contracts.

Meanwhile, there are a lot of activities going on from the publishers' side (seller's side). When a visitor visits a publisher's web site, a new "session" begins and there are one or several iterations of the following sequences of events:

The visitor requests a certain page to the web server (via its URL), and then the requested page is displayed to this visitor with an advertisement embedded in it within a very short period of time.

The visitor clicks on the advertisement with probability ctr_{bc} where b denotes the user profile of the visitor (i.e. a Bernoulli trial with success probability $p_{i,k}$) and c denotes the feature of the advertiser; this probability is usually called the click-through rate and the click-through rate (CTR) is summarized and updated right after each page view (impression) occurred. If there is a click, then the revenue associated with the advertisement, that is $price_{bc}$, is obtained. After a certain number of page requests, the visitor leaves the web site and the session terminates. The website will keep recording all the statistics of click-through-rate as well as the number of impressions and clicks.

At the beginning of the transaction, the publishers will make an estimate of how many advertisements they are supposed to exchange with the advertisers, which is probably going to be the amount of transaction signed for the contract. Then the system helps to match both the advertisers' need and the publishers' supply with a reasonable contract that clearly identifies duties, trading amount, trading value, maturity date and any other restricted elements so that they could maximize their revenues.

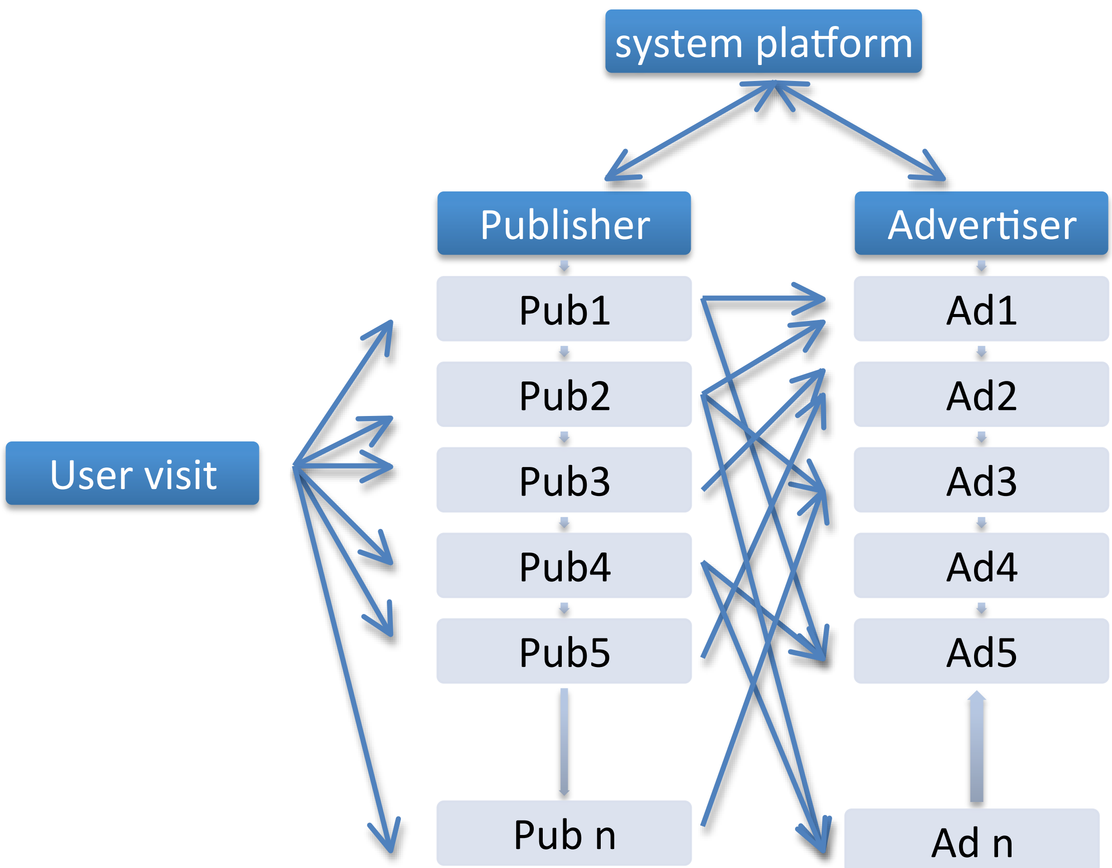


Figure 3.1 Functional process of the trading system

3.6 Guaranteed contracts and unguaranteed contracts

3.6.1 Guaranteed Contracts

Guaranteed contracts are contracts signed at different points of time before they start. It is a standardized contract between two parties issued at a fixed rate agreed today with a specified amount of trading volume guaranteed to deliver during a predetermined period of time, i.e., the publishers guarantee certain number of impressions, clicks or actions according to the signed contract before the contract terminates and the advertisers agree to make payment at the beginning of the period.

Advertiser’s inventory and audience preferences are diverse; therefore it’s hard to determine demand categorization.

3.6.2 Unguaranteed Contracts

Unguaranteed contracts refer to those occurred in the spot market, they are operated by auction through exchange. The prices are flexible and volatile, which is similar to other trading systems, and the trading volume varies among different trading activities.

3.7 Uncertainty

There is bias coming from variations in advertiser inventory requirements and noise from changes in current economy, seasonality and management decisions.

The uncertainty in this problem lies in the random nature of demand and supply, but we do not need to concern about the changes in demand because the spot market price is quite unpredictable. There are basically two problems involving the supply-demand relationship: How much inventory is available? What is the cost for advertiser? The first question varies by seasonal effects, user growth and economic environment.

3.8 Literature Review

Google Adwords, Yahoo! search marketing, Google Adsense and Microsoft adCenter are popular network systems that are most competitive in the ad market and they enable ads to be shown on relevant web pages or alongside search results.

In the research paper by Roles and Fridgeirsdottir (Roels & Fridgeirsdottir, 2009), the authors propose dynamic optimization model for web publishers to maximize their revenue from online display advertising. Similar to airline revenue management, in this model, the authors propose methods for web publishers to decide whether or not to accept an advertising request. Also, certainty equivalent heuristic is proposed to solve dynamic optimization problem.

Most of the past work simply uses strictly deterministic models or linear multi objective programming (Yang, et al., 2010) (Ahmed & Kwon, 2012) which neglects the fact that there's uncertainty in the problem.

3.9 **Remodel for media selection problem**

The model solving the allocation problem among all advertisers and publishers will be going from a deterministic case to a stochastic case.

The deterministic model, based on the allocation model that is commonly used, has been modified and can be described as:

$$\begin{gathered}\max_{s.j.} \sum_{i,j} rank_{ij} * price_{ij} * x_{ij} + \sum_i spot_i * \max(z_i, 0) \; + h * \sum_i min(z_i, 0) \\ \sum_j x_{ij} + z_i \le s_i \\ \sum_i x_{ij} \le d_j \\ x_{ij} \ge 0 \end{gathered} \tag{27}$$

The variables are defined in Table 3.2 below. It indeed combines both the revenues gained from the guaranteed contracts and the unguaranteed part, and deducts a penalty value if existing.

But this deterministic model regards supply and demand of advertisements in the future as a constant value; it also ignores the uncertain nature of the parameters rank, price and spot.

Moving forward to the stochastic model, the problem of online display advertising can be generalized as follows:

$$\begin{gathered}\max_{s.j.} t \\ Pr\{g(x,z) \ge t\} \ge 1 - \theta \\ g(x,z) = \sum_{i,j} rank_{ij} * price_{ij} * x_{ij} + \sum_i spot_i * \max(z_i, 0) \; + h * \sum_i min(z_i, 0) \\ Pr\left\{\sum_j x_{ij} + z_i \le s_i\right\} \ge 1 - \alpha \\ \sum_i x_{ij} \le d_j \\ x_{ij} \ge 0 \end{gathered} \tag{28}$$

The probability of gaining a maximum revenue at specific optimal x and z is set by θ. Because of the randomness in contract prices and spot prices, we cannot say for sure that we can obtain a maximized income every time we set the output x and z as our selection. Instead, we can guarantee that with probability (1- θ) we can reach our target.

The other addition to the model is the stochastic form of supply.

The definitions of parameters are also summarized as:

Parameter	Definition
i	i=1..n, the subscript representing the i^{th} publisher
j	j=1..m, the subscript representing the j^{th} advertiser
θ	risk ratio
rank	combines both quality and price
price	contract price per click for guaranteed contracts
x	the decision variable, i.e., the number of impressions allocated to guaranteed contracts
spot	spot price offers on trading system for thousand impressions
z	the number of impressions displayed for unguaranteed spot market
s	the number of user visits (impressions) available for the i_{th} ad unit
α	ratios for chance constraints
d	current market demand for guaranteed contracts for the j_{th} ad opportunity
h	penalty ratio per unit

Table 3.2 Definition of parameters

x_{ij}, z_i are the decision variables and they can be combined into one when setting z as the last row in the matrix formulation of x.

The system needs to decide upon how to allocate ads to publishers so that the whole system, including all advertisers and publishers, could obtain a maximum return.

The thesis tries to match ad opportunities with each ad unit (space) available. They are not in a one-by-one relationship. Each advertiser could sign contracts with different publishers for displaying their ads. The publishers could also take advantage of available user visits to allocate to different ads if possible.

All publishers would be trading off among guaranteed contracts (represented by the number of x) and spot markets (represented by the number of z). We also assume that there is always enough ad opportunities to fill out each ad space.

3.9.1 **Ad Rank**

The ad rank parameter used in our model is not simply the same with click through rate. Ideally, it is composed of click through rate as well as ad quality. Moreover, ad quality refers to the relevance of an ad to the user.

Click-through rate of an advertisement is defined as the number of clicks on an ad divided by the number of times the ad is shown (impressions), expressed as a percentage

So ad rank can be represented by a factor model, which looks like:

$$Ad\ rank = \alpha * ctr + \beta * relevance + \gamma * bid\ price + \cdots \quad (29)$$

Ctr in the equation (29) is the click-through-rate. Advertisers would only bid what an ad is worth to them. But ad price is only one part of the story. A more important measure for advertisers large and small is the return on investment of their advertising dollar. The ad rank which includes ad relevance will help advertisers convert more clicks into customers by showing more relevant ads on publishers' website, giving advertisers a better return for every dollar they invest.

Unlike other systems that are advertiser-driven or publisher-driven, this system deals with it all in a whole and there are no conflicts in earning more revenues. It will allow publishers to show more ads on pages where they previously showed no ads or only a few ads. Furthermore, advertisers will get more clicks on ads because the quality and relevance of those ads will be better. As is true today, advertisers are ultimately in control of how much they spend because they only pay what an ad is worth to them. So consumers will see more relevant ads and advertisers will attract more customers as a result.

The ad rank in the model, which is used as an affecting factor in decision-making, helps ensure that users see the most relevant ads not just the most expensive. It is a formula that reflects which ads consumers prefer based on how they respond to the ads. By using ad rank in addition to ad price in our advertising system, smaller companies can more effectively compete with larger businesses by creating highly relevant ads and websites.

3.10 **The optimization problem**

The main idea of the selection process is to pay the lowest amount possible for the highest position you can get given your quality score and bid price.

We've already known from the CVaR part that if we are minimizing F in Equation (30), it is equivalent to minimizing the inverse of our original objective that is represented in equation (9).

The approximation to F is

$$\tilde{F}_{\beta}(x,\alpha) = \alpha + \frac{1}{q(1-\beta)}\sum_{k=1}^{q}[g(x,z)-\alpha]^{+} \qquad (30)$$

In this case, let

$g(x,z) = -\sum_{i,j} rank_{ij} * price_{ij} * x_{ij} - \sum_i spot_i * \max(z_i, 0) \; - h * \sum_i min(z_i, 0),$

Rewrite as follows:

$$\tilde{F}_{\beta}(x,\alpha) = \alpha + \frac{1}{q(1-\beta)}\sum_{k=1}^{q}\left[-\sum_{i,j} rank_{ij} * price_{ij} * x_{ij} - \sum_i spot_i * \max(z_i,0) \; - h * \sum_i min(z_i,0) - \alpha\right]^{+} \qquad (31)$$

Then we could solve the online display advertising problem taking advantage of the CVaR method.

3.11 **Reasonable data**

We consider the case that has five samples, five publishers and five advertisers.

The demand for guaranteed contracts of each advertiser participating in the market is assumed to be d=mu2= [3500,3500,3500,3500,3500].

Besides, the page-view (supply) of each publisher is considered to follow a normal distribution whose mean value is μ=mu1=[2800, 3000, 3000, 3400, 2700] and the variance is σ=var1=[300,100,200,200,150] for the normal random variable s.

Using the third constraint in the optimization problem, it can be further stated as:

$$Pr\left\{(s-\mu)/\sigma \geq \left(\sum_j x_{ij} + z_i - \mu\right)/\sigma\right\} \geq 1-\alpha \quad (32)$$

Using the cumulative density function of the standard normal random variable, it can be simplified as:

$$1 - \Phi\{(\sum_j x_{ij} + z_i - \mu)/\sigma\} \geq 1 - \alpha \qquad (33)$$

where Φ is the inverse normal distribution function.

This can be further simplified as:

$$\Phi\{(\sum_j x_{ij} + z_i - \mu)/\sigma\} \leq \Phi(-K_\alpha) \qquad (34)$$

The chance constraint can now be transformed into a deterministic constraint as:

$$\sum_j x_{ij} + z_i \leq \mu - \sigma K_\alpha \qquad (35)$$

Using the same method, the second constraint in the optimization problem can also be simplified as:

$$\sum_i x_{ij} \leq \mu_2 - \sigma_2 K_\beta \qquad (36)$$

Data entries for input parameters are assumed as follows:

rank=unifrnd(0.1,0.3,5,5)

unifrnd is a built-in function in Matlab to produce continuous uniform distribution, so that ad rank is randomly distributed between the range of [0.1%,0.3%], which matches market statistics report: The average click-through rate of 3% in the 1990s declined to 0.1%-0.3% by 2011. The contract price is considered identical and follows normal distribution with randomly chosen mean value of [5,6,6,7,8] and variance [0.3,0.1,0.2,0.3,0.3]. The spot price in the unguaranteed market is also assumed to be following normal distribution with mean [0.9,1,1.1,1.3,1.5] and variance [0.1,0.1,0.1,0.1,0.1]. The penalty ratio h is chosen as 1.2. There exist a relationship among the value of contract price, spot price and penalty h:

Spot price<contract price<penalty value

This makes sense because business contracts involve more risk and promised duties, and that's exactly why contract price is greater than spot price in the exchange market. Besides this, the penalty value is also set above the contract price, which is intended to constraint publishers from breaking contracts so easily.

One another option of generating data is to use the factor model, which requires in-depth research in the relationship of supply, demand, market price with.major economic and statistical factors. But this application of data inputs with the use of factor model is not included in this thesis.

3.12 **Optimization Techniques**

In principle, a stochastic programming approach under the current assumptions is a little more computationally difficult than the deterministic model. In this thesis, we are applying CVaR to this online advertising case.

3.13 **Results Analysis**

3.13.1 **Case1: Sensitivity to the number of sample used**

1) sample number N=10

Advertiser/Publisher	1	2	3	4	5
1		145	1641	83	
2	1549	34	652	718	502
3		41	655	1426	1378
4	1521	2		1425	552
5		541	303		1215
Spot/penalty	114	2365	5	4	-755

Table 3.3 Allocation result-sample number N=10

Objective=19443.60416

2) sample number N=20

Advertiser/Publisher	1	2	3	4	5
1	852	33	157		
2	292	503	2411		294
3	1939	400	32	338	
4	329	70		184	893
5		705	426	2	
Spot/penalty	-228	1417	230	3132	1705

Table 3.4 Allocation result- sample number N=20

Objective=16419.28355

3) sample number N=30

Advertiser/Publisher	1	2	3	4	5
1	586	78		2	
2	1422	838		645	580
3	281	162		1213	
4	893	324	4	1	
5		433	586		274
Spot/penalty	2	1293	2666	1795	2038

Table 3.5 Allocation result- sample number N=30

Objective=15779.24828

From Figure 3.2 we can see that the revenue of the total market does not vary much as the sample numbers go up. It only slightly decreases as more samples show more statistical characteristics of both publishers and advertisers.

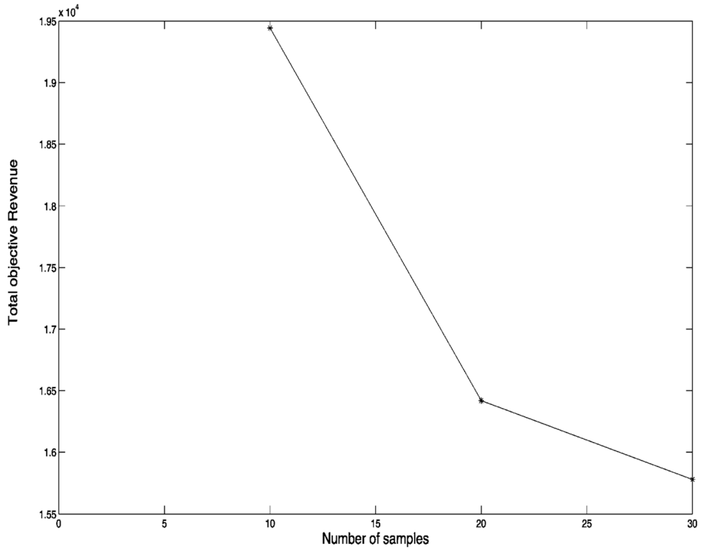


Figure 3.2 Total Revenue vs. sample number

3.13.2 **Case1: Sensitivity to competitor numbers**

1) publisher number N=5

Publisher/Advertiser	1	2	3	4	5	Spot/penalty
1	852	292	1939	329		-228
2	33	503	400	70	705	1417
3	157	2411	32		426	230
4			338	184	2	3132
5		294		893		1705

Table 3.6 Allocation result- publisher number N=5

Objective=16419.28355

2) publisher number N=10

Publisher/Advertiser	1	2	3	4	5	Spot/penalty
1		1806			747	631
2	2057		68			1003
3	1	1068	360	848	380	599
4		239	1	306		3110
5			6			2886
6	4	387		1425	413	955
7					1200	1928
8			1138		571	1547
9				569		3087
10				352	189	2351

Table 3.7 Allocation result- publisher number N=10

Objective=40857.65939

3) publisher number N=20

Publisher/Advertiser	1	2	3	4	5	Spot/penalty
1	245		1		981	1957
2	691	1	1	1		2434
3	1	1073	1	384	1	1796
4		8				3648
5	1	1	1	1	1	2887
6						3184
7	404	1	1	38	1	2683
8	1					3255
9	2		816	1	487	2350
10				1		2891
11	1079				2027	78
12						3128
13		2416		841		-1
14						3656
15						2892
16	180					3004
17	896			2231		1
18						3256
19						3656
20			2679	1	2	210

Table 3.8 Allocation result- publisher number N=20

Objective=73303.62082

As we keep adding more publishers, and introducing more competition into the market, the supply goes up and the output of the objective function shows that our total revenue for both advertisers and publishers dramatically increases. This can be shown clearly from the figure below:

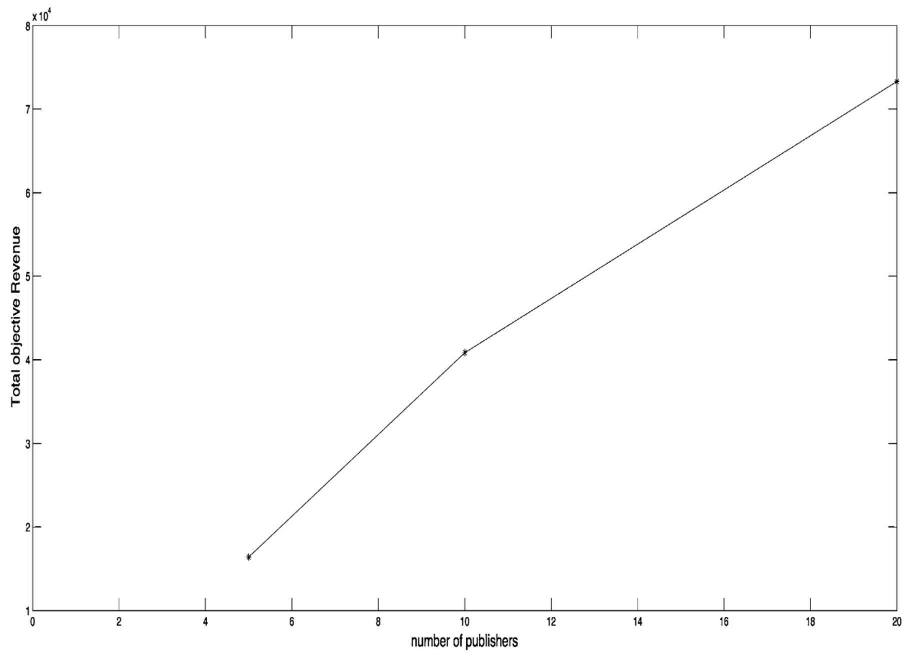


Figure 3.3 Total Revenue versus the number of publishers

One more interesting fact is that the system tends to give more availability to the spot market instead of signing guaranteed contracts. The three tables for publishers with the number of 5, 10 and 20 show that the number of allocation for spot contracts is obviously increasing for each publisher.

3.13.3 Case3: Sensitivity to risk ratio

1) θ=0.8

Advertiser/Publisher	1	2	3	4	5
1	1		2085	547	
2	546	864	8	393	858
3	2085	1970	4	129	1022
4	547	120	38	622	1162
5				1320	1
Spot/penalty	5	1005	130	1714	1571

Table 3.9 Allocation result-risk ratio θ=0.8

Objective=19057.75126

2) θ=0.9

Advertiser/Publisher	1	2	3	4	5
1	852	33	157		
2	292	503	2411		294
3	1939	400	32	338	
4	329	70		184	893
5		705	426	2	
Spot/penalty	-228	1417	230	3132	1705

Table 3.10 Allocation result-risk ratio θ=0.9

Objective=16419.28355

3) θ=0.95

Advertiser/Publisher	1	2	3	4	5
1	1896	145	1142		
2		1249	64		251
3		1953			
4	1		2	83	
5				599	
Spot/penalty	1	1564	1303	3570	2293

Table 3.11 Allocation result-risk ratio θ=0.95

Objective=12539.04

4) θ=0.99

Advertiser/Publisher	1	2	3	4	5
1	2831		1343		
2	1	200	358	33	
3		825	367		339
4	78	741	330	475	
5		971	239		
Spot/penalty	-990		1492	1187	1682

Table 3.12 Allocation result-risk ratio θ=0.99

Objective=12963.25672

The risk ratio θ stands for the probability of gaining a maximum revenue, or the preference of the investor's risk acceptance. The more θ reaches 1, that is 100% probability, the more safe and secure the investment is. On the other hand, if θ goes far below 1, the investment will be regarded as quite risky.

The figure shows exactly how our advertising allocation goes when θ changes.

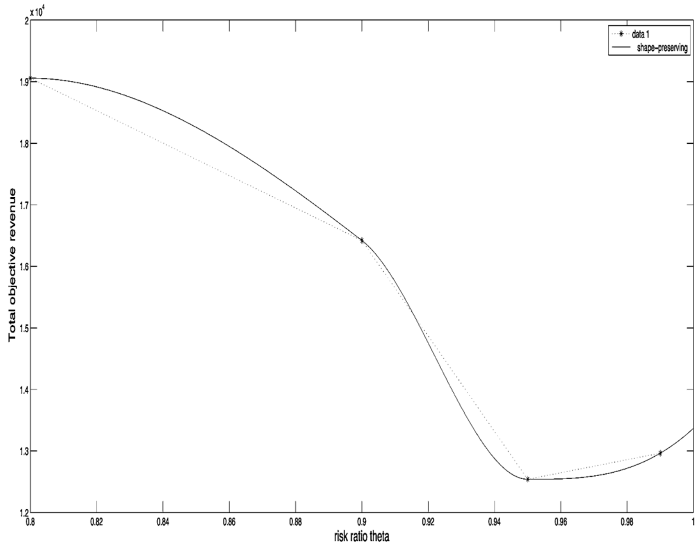


Figure 3.4 Total Revenue versus risk ratio

3.13.4 Case4: Sensitivity to penalty value

1) h=0.8

Advertiser/Publisher	1	2	3	4	5
1	8	881			663
2		3		619	497
3	3	260	44	33	105
4	60	23	149	107	29
5				23	337
Spot/penalty	1352	30	1128		1128

Table 3.13 Allocation result-penalty value h=0.8

Objective=4047.241051

2) h=1.2

Advertiser/Publisher	1	2	3	4	5
1	852	33	157		
2	292	503	2411		294
3	1939	400	32	338	
4	329	70		184	893
5		705	426	2	
Spot/penalty	-228	1417	230	3132	1705

Table 3.14 Allocation result-penalty value h=1.2

Objective=16419.28355

3) h=1.5

Advertiser/Publisher	1	2	3	4	5
1	342		998	1663	
2		723			620
3	1133	33	460		1
4			1266	394	
5	157	662	773		285
Spot/penalty	1	1615	1629	454	1014

Table 3.15 Allocation result-penalty value h=1.5

Objective=5691.287371

Comparing the objective function values of the three cases of different penalty values, we can see that total revenue reaches the maximum at h=1.2, but drops when it gets smaller to 0.8 or goes up to 1.5.

The decrease in revenue is resulted from a low spot price when compared to contract price and penalty terms. So even if penalty gets smaller, gain from spot market is not as profitable as it is in the guaranteed contract market. For the second case, that is penalty gets higher and strict, the market will tend to be more cautious, and hence limits the growth of revenue.

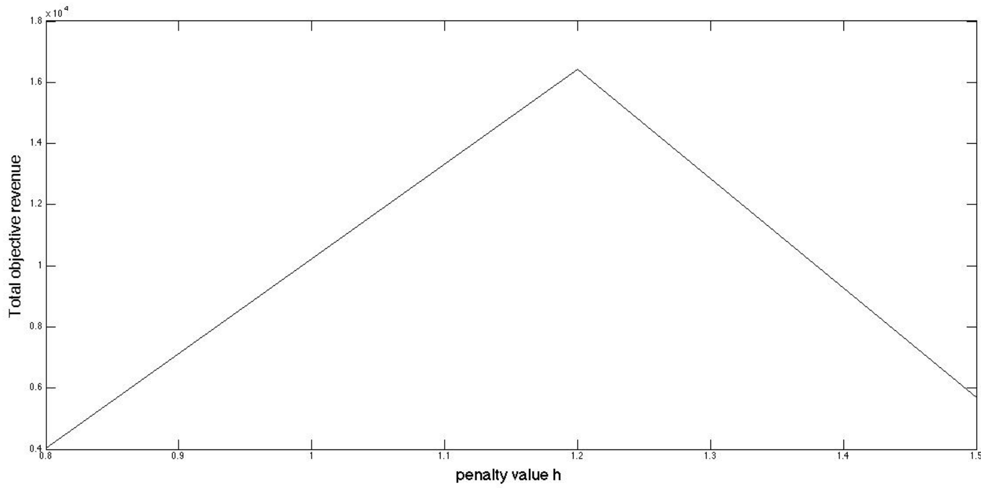


Figure 3.5 Total Revenue versus penalty value

3.14 **Conclusion**

This chapter deals with the online display advertising problem in which publishers and advertisers engage in the online display advertising trading system. The main contribution here is the stochastic formulation of the online display advertising model, the optimization formulation of the advertising problem with CVaR and the experimental evaluation of proposed techniques. The simulations are conducted under scenarios with different parameter values. The model, which incorporates uncertainty into it, has the ability to respond to volatility in the trading market.

Chapter 4 Conclusion and Future Work

4.1 Summary of work

In this study, we have presented stochastic formulations of optimization models utilizing advanced mathematical and statistical techniques for problems in Finance. One of the case studies is a portfolio optimization; the other one is an online display advertising case.

In chapter 2, the objective of the portfolio optimization model is set up with a transaction cost function. This is derived from the fact that we cannot make our investment decisions solely on the basis of our preferences of return and risk levels. From a more realistic aspect, the final optimal result will also be affected by transaction costs and taxes.

Both the Conditional Value at Risk and reliability based optimization method are applied to solve the optimization problem and including a risk measure.

The use of factor model to replace original return data is also addressed for the optimization problem. It overcomes the bias in historical data, which may not be a perfect representative for the future. Moreover, it makes up for the shortage of availability of data resources.

In Chapter 3, the thesis contributes in constructing the modeling of online display advertising. This approach puts uncertainty into the supply, demand and price volatilities into the model, which makes it a random complex problem to solve. The algorithm of Conditional Value at Risk is applied to the optimization formulation for the advertising problem for the first time in literature. Experimental evaluations of the proposed techniques are applied to test the efficiency and reliability of the system.

From the result of the portfolio optimization problem we could see that The RBO method comes to a faster solution especially in higher dimension. The CVaR risk measurement can be more accurate for a specific case since it is entirely based on every single historical data. From the experimental result of the online display advertising problem, we obtained the breakeven point for the penalty ratio which goes to maximum total revenue at this point.

4.2 Future approach

1. The CVaR and the reliability based optimization method are both applied for a single period transaction. We may extend the model to a multi-period problem so that it carries on as a series of investment decisions under a long-term investment plan.

2. It’s also possible to go from a linear factor model to time-varying factor model which is dynamic; another option is an augmented risk model which adds one additional risk factor to the original factor model which captures the effect of the missing factors.
3. Data resources of online display advertising are limited as a result of its commercial privacy constraints from service providers.

Appendix A

Predictor Importance Figures

Equity	Code
Union pacific corporation	UNP
Apple Inc.	AAPL
ArthroCare Corporation	ARTC
Exxon Mobil Corporation	XOM
Princeton National Bancorp Inc.	PNBC
ING Groep NV	ING
Wells Fargo & Company	WFC

Table A 1 code denotation for equity used

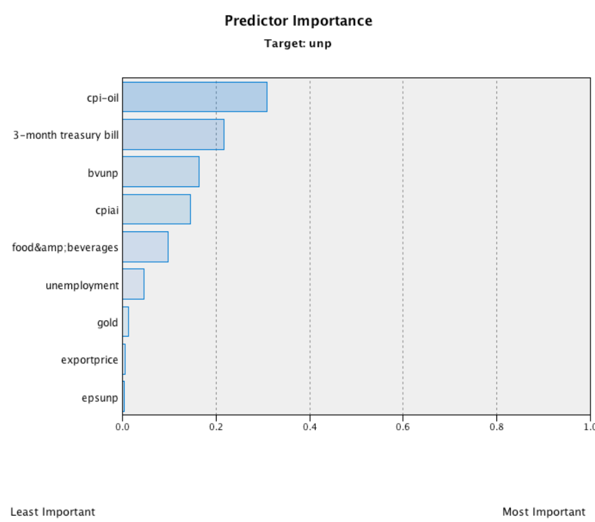


Figure A 1 Predictor importance figure for UNP

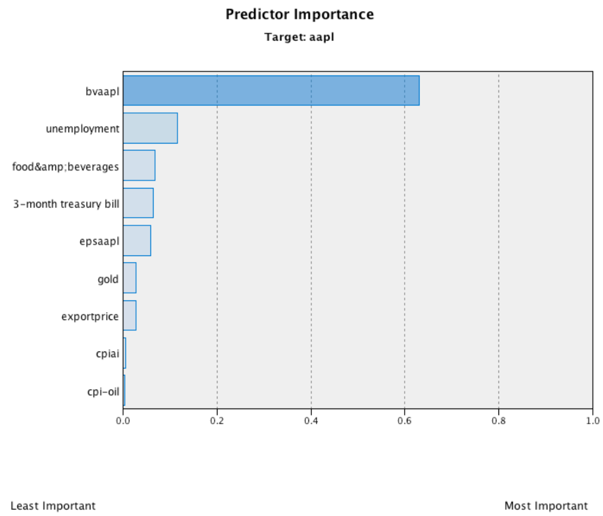


Figure A 2 predictor importance figure for AAPL

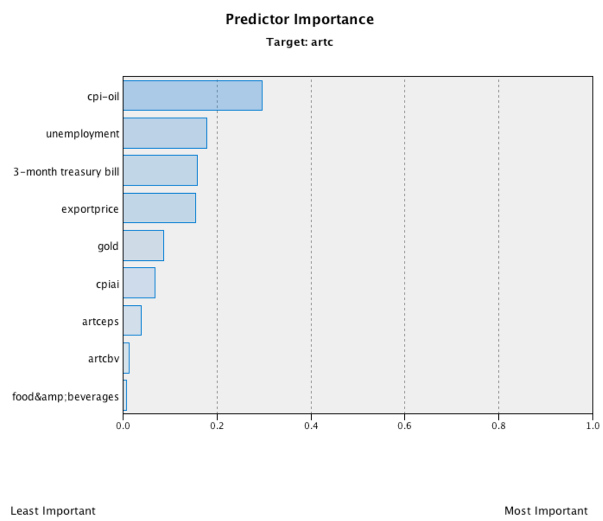


Figure A 3 Predictor importance figure for ARTC

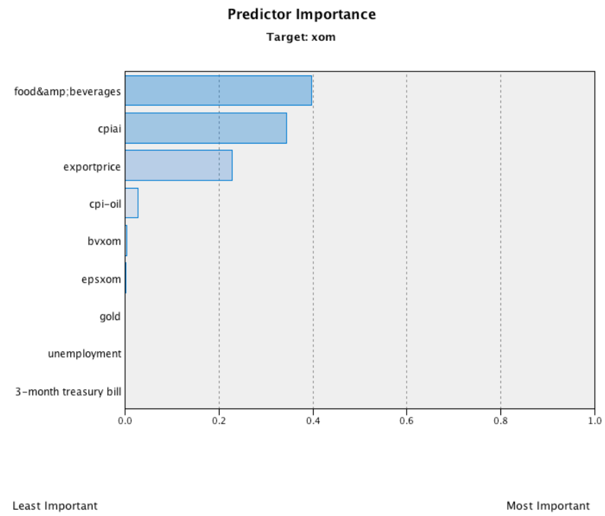


Figure A 4 Predictor importance figure for XOM

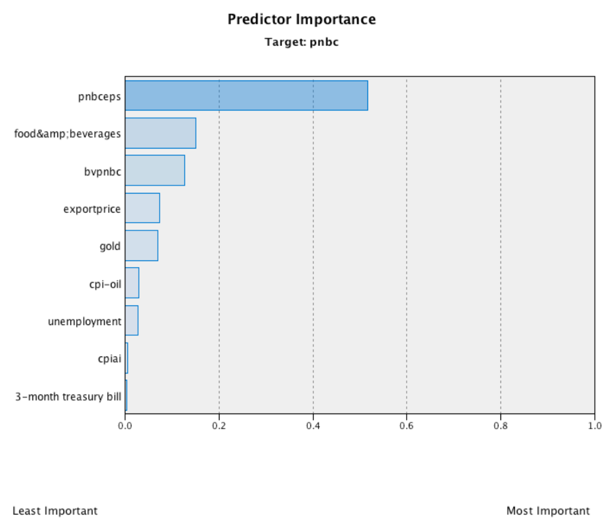


Figure A 5 Predictor importance figure for PNBC

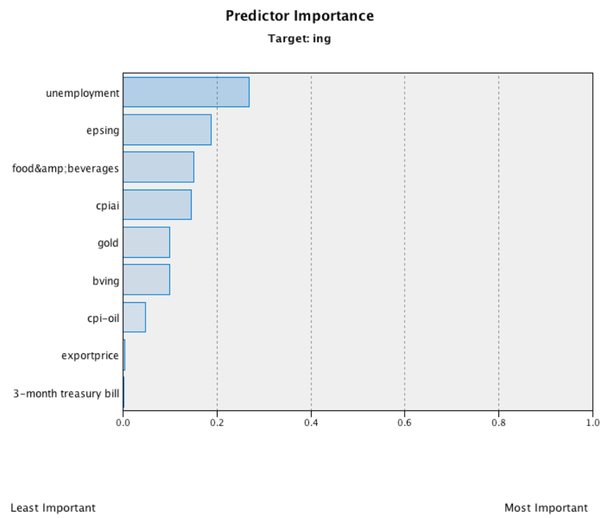


Figure A 6 Predictor importance figure for ING

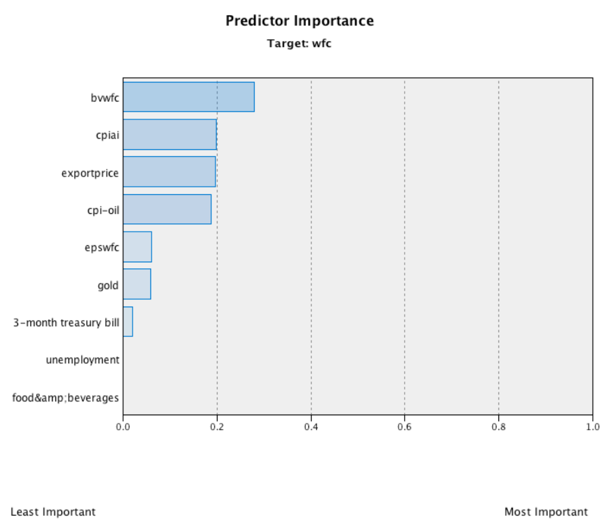


Figure A 7 Predictor importance figure for WFC

Appendix B Coefficients table for 7 assets

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	.021	.049		.420	.675
	gold	.173	.196	.087	.879	.381
	unemployment	-.009	.011	-.096	-.867	.388
	bvunp	.001	.001	.082	.710	.479

a. Dependent Variable: unp

Table B 1 Coefficients of UNP factor model

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	-.038	.103		-.375	.708
	gold	.566	.436	.126	1.299	.197
	unemployment	.035	.022	.159	1.559	.122
	bvaapl	-.021	.010	-.213	-2.007	.047

a. Dependent Variable: aapl

Table B 2 Coefficients of AAP factor model

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	.137	.129		1.062	.291
	gold	.479	.557	.086	.860	.392
	unemployment	-.013	.027	-.049	-.492	.624
	artcbv	-.007	.008	-.098	-.924	.357

a. Dependent Variable: artc

Table B 3 Coefficients of ARTC factor model

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	.001	.036		.026	.979
	gold	-.084	.148	-.057	-.564	.574
	unemployment	.000	.007	.006	.064	.949
	bvxom	.001	.002	.041	.391	.696

a. Dependent Variable: xom

Table B 4 Coefficients of XOM factor model

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	-.041	.043		-.953	.343
	gold	.126	.183	.070	.692	.491
	unemployment	.013	.010	.154	1.387	.168
	bvpnbc	-.001	.003	-.043	-.366	.715

a. Dependent Variable: pnbc

Table B 5 Coefficients of PNBC factor model

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	.034	.131		.262	.794
	gold	-.073	.264	-.027	-.278	.781
	unemployment	-.005	.017	-.039	-.302	.763
	bving	.001	.004	.016	.125	.900

a. Dependent Variable: ing

Table B 6 Coefficients of ING factor model

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	.054	.047		1.144	.255
	gold	.058	.204	.029	.285	.776
	unemployment	-.004	.010	-.044	-.430	.668
	bvwfc	-.001	.002	-.083	-.761	.449

a. Dependent Variable: wfc

Table B 7 Coefficients of WFC factor model

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