Stability for Systems with Unknown Time Delays

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Time delays are of long-standing interest in the study of control systems since they appear in many practical control problems and tend to degrade overall system performance. In this thesis, we consider two distinct problems involving uncertain time delays.

The first problem that we consider is the achievable delay margin problem, which is determining the longest delay for which stability can be maintained when using a linear time invariant (LTI) controller. This problem has been considered in continuous-time, where bounds (often tight) have been found for plants with non-zero right half plane poles. In this work, we consider the discrete-time case, where we prove that an LTI controller exists which stabilizes the plant and the plant with a one step delay if and only if the plant has no negative, real unstable poles.

The second problem that we consider is stabilizing any continuous-time single-input single-output LTI plant with an arbitrarily large time delay and gain. To solve this problem, we propose a simple generalized hold whose resulting discretized system is amenable to adaptive control. Furthermore, by exploiting the structure of the resulting discretized system, we propose purpose built estimators for the unknown gain and delay, which allows us to not only provide bounded-input bounded-output (BIBO) closed-loop stability, but also guarantees the exponential decay of any plant initial conditions, robustness to un-modelled dynamics, and tolerance to occasional, possibly persistent, jumps in the gain and delay. Furthermore, for the case of a first order plant, a similar, but suitably modified controller is shown to tolerate continuous variation of the unknown delay while still providing BIBO closed-loop stability.

Acknowledgements

I would like to thank my supervisor, Prof. Daniel Miller, for all his help in editing the mathematics that appear throughout this thesis. I would also like to thank Dr. Julie Vale for her great help in making my thinking more precise.

Dedication

This is dedicated to my father may he rest in peace.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

This thesis is in the field of control systems engineering. The most basic (and common) problem in control systems engineering is to regulate the behaviour of a dynamic system, often referred to as a plant. One such plant is an airplane, where a control objective is for the autopilot to maintain a constant speed, altitude and course. While this seems simple, there are many effects which work to degrade the desired behaviour; for example, wind may blow on the airplane, sending it off course, and noise affects the sensors on the plane, resulting in imperfect measurements. A well designed controller can be built to mitigate these problems, and thereby allow the plane to travel as close as possible to its desired path. To design this controller, a nominal model of the plant is found, and then a controller is designed based on the model.

Unfortunately, regardless of the time or money spent modelling a system, the physical system will deviate from the nominal model, and without care, the control objective may no longer be met. As such, it is important to be able to design a controller that will maintain performance despite this uncertainty, and thereby, make the controller robust to plant uncertainty. It is also important to ask: can we make the system behave in the way that we want it to? Intuitively, it is not realistic to demand that an oil tanker behave like a butterfly, and in fact, it can be shown through performance limitations that such behaviour, with a linear controller, is impossible due to the dynamics of the oil tanker.

In this thesis, we consider two distinct problems involving systems with an unknown time delay. The first problem is in the area of fundamental performance limitations for the class of linear time invariant (LTI) controllers; specifically, given a single-input single-output (SISO) plant with an unknown time delay, we ask what is the maximum duration of the unknown time delay for which stability can be maintained using a single LTI controller?

The second is a robust control problem: given any SISO LTI plant with an unknown time delay and gain with arbitrarily large known upper bounds, the goal is to provide a controller design algorithm which stabilizes the plant for every allowable gain and delay; we also consider a variant of this problem for a first order plant in which the delay is allowed to vary continuously. We describe both of these problems in more detail later in the introduction.

The remainder of this chapter is organized as follows. In Section 1.1, we provide some background material on time delays, with a primary focus on systems with uncertain time delays, in Sections 1.2 and 1.3 we elaborate on the two aforementioned problems, and finally, in Section 1.4 we provide an outline for the remainder of this thesis.

1.1 Time Delays

Time delays were first studied in the 1700s by famous mathematicians such as Euler, Bernoulli and Condercat in their attempts to better understand and solve differential equations. This was a natural area of study since a continuous-time delay is one of the simplest types of infinite dimensional systems, a class of systems which required new mathematical machinery in order to solve. Furthermore, infinite dimensional systems arise from a wide range of physical problems, such as in the study of electromagnetics, heat transfer, beams, making them of critical importance for many engineering problems. In the context of more modern control systems, time delays remain an important area of study since they occur in many control systems, with various causes ranging from measurement delays, signal transmission delays, computational delays, inherent process delays, etc.

While there are a few applications where time delays can be intentionally added to the controller in order to improve system performance, e.g., see [41] and [44], a delay more typically degrades system performance, harms robustness, and in a loose sense destabilizes the system e.g., see [20] and [19]. As such, quantifying and trying to overcome the negative impact of time delays has garnered much interest from control researchers. Work in this area dates back to at least the 1950's with the well known Smith Predictor - see [47].

In this thesis, we consider time delays in both the continuous-time and discrete-time settings; delays in these settings have very different representations and introduce different problems despite having similar physical meanings and causes. To see this difference, consider the transfer function representation of a fixed time delay in both time settings. For a continuous-time delay of τ seconds, the transfer function is given by

$$e^{-s\tau},$$

which is not a real rational transfer function, and is infinite dimensional. For a discrete-time delay of n samples, the transfer function is given by

$$\frac{1}{z^n};$$

unlike the continuous-time delay, this is a real rational transfer function, and the resulting system is finite dimensional; in fact, it is merely n poles at the origin, which means that the resulting model order increases as the length of the delay increases. In addition to the different model orders, it is also important to note that a continuous-time delay can be any positive real number, whereas a discrete-time delay can only be a positive integer. As a result of these two fundamental differences, techniques that work for a continuous-time delay do not necessarily transfer over to a discrete-time delay, and vice versa.

Despite the differences in the mathematical representation of discrete-time delays and continuous-time delays, the control problems associated with them are quite similar. One area of particular interest are time delays of an uncertain duration; this is a very natural problem since it is often the case that the exact length of the delay may not be known *a priori*. For example, a signal transmission delay may vary with geographic location or other traffic on a computer network. So, instead of a specific, known delay, there is instead a range of possible delays for the control engineer to consider when designing a stabilizing controller. Fortunately, in many of these cases, an upper bound on the maximum delay duration may be known, for example, a network protocol can be designed so that it guarantees a maximum transmission delay, and this knowledge can be invaluable for designing a controller. Naturally, there has been considerable research on systems with uncertain but bounded delays including synthesizing controllers, e.g., see [33], analyzing the stability of such systems, e.g., see [42], and combining an uncertain delay with other plant uncertainties, often using complicated linear matrix inequalities (LMI), e.g., see [38, 31].

Communication delays are often unknown and time varying; this is especially true when using a network to transmit information as interference from other network traffic can result in a time varying delay. Since a network operates using digital equipment, it is most naturally modeled in discrete-time; however, since networks often operate with a much shorter period than the control system, many papers consider the network delay to be a continuous-time delay [22, 49, 13]. Many results on time varying continuous-time delays consider state feedback controllers, see [23], with much of the recent research using LMI's to prove their results [8, 43, 12, 21, 51, 20]. While LMI's are able to handle additional plant uncertainty and time variations of the plant parameters, the results tend to be difficult to interpret, are only solvable via computer analysis and they provide little to no intuition.

1.2 The Achievable Delay Margin Problem

Chapter 3 is focused on the *Delay Margin* problem, which is a robust stabilization problem with uncertainty in the length of a time delay. There are two types of delay margin problems, controller dependent and controller independent (which we will also refer to as the *Achievable Delay Margin* problem). While we focus on the more fundamental controller independent problem, it is extremely useful to first consider the controller dependent problem.

The controller dependent delay margin problem is simple; we describe the problem in continuous-time, though the problem statement easily translates to discrete-time as well. Given a plant P and any stabilizing controller C, we are interested in the smallest delay for which the system becomes unstable. This is a very useful quantity, since if $\bar{\tau}$ is the smallest τ such that C no longer stabilizes $Pe^{-s\bar{\tau}}$, then we know that C stabilizes $Pe^{-s\tau}$ for all $\tau \in [0, \bar{\tau})$.

For a SISO LTI continuous-time system, it is well known that the solution to this problem can be found using a simple Nyquist argument. Given a SISO LTI plant $P(s)$ and a SISO LTI stabilizing controller $C(s)$, let $\{\omega_1, ..., \omega_q\}$ denote the frequencies where

$$|P(j\omega)C(j\omega)| = 1,$$

and then let $\phi_i \in (0, 2\pi)$ denote the phase margin of each ω_i (i.e., $P(j\omega_i)C(j\omega_i) = e^{j[-\pi+\phi_i)]}$). The delay margin of this plant controller combination is then

$$\min\{\frac{\phi_i}{\omega_i} : i = 1, ..., q\}.$$

This is easily seen by observing that for a delay of $\tau_i := \frac{\phi_i}{\omega_i}$ seconds that

$$P(j\omega_i)C(j\omega_i)e^{j\tau_i\omega_i} = -1, \qquad \text{for all } i = 1, ..., q.$$

Hence, for every delay τ_i, the Nyquist plot of $P(s)C(s)e^{-s\tau_i}$ passes through -1, ensuring that $P(s)C(s)e^{-s\tau_i}$ is unstable, so clearly the controller dependent delay margin is less than or equal to $\min\{\tau_i : i = 1, ..., q\}$. Finally, since the added delay produces a continuous deformation of the Nyquist plot and $P(s)C(s)$ is closed loop stable, $P(s)C(s)e^{-s\tau}$ must be stable for all

$$\tau \in [0, \min\{\tau_i : i = 1, ..., q\}).$$

Unfortunately, this elegant solution for the controller dependent delay margin does not translate to discrete-time systems (due to the lack of continuity in the delay variation), nor to continuous delays in a sampled data system (due to the delay also modifying the magnitude of the system). These differences makes extending the controller independent delay margin results of [35] to the corresponding discrete-time and sampled data problems very difficult; this is discussed in more detail in Chapter 7 where we present some open problems.

While the controller dependent delay margin is a useful quantity to know after a controller has been designed, it does not provide any insight into what is achievable for a class of controllers. For the answer to this question, we consider the controller independent (or achievable) delay margin problem. For this problem, given a plant P, the question is: given a class of controllers (linear time invariant, non-linear time varying, etc.), what is the maximum possible delay margin achievable by a controller of this class? In [34], it is proven that there exists a fundamental delay margin limitation for a SISO LTI plant with an open right half plane pole when using static state feedback, with an explicit bound only provided for a first order plant. In [35], it is proven that there exists a fundamental delay margin limitation for a SISO LTI plant with a non-zero unstable pole when using LTI dynamic output feedback. In that work, explicit bounds (often tight) are found for various combinations of unstable poles and right half plane zeros; for example, a plant with a single unstable pole p has an achievable LTI delay margin of $\frac{2}{p}$. Turning to time varying controllers, in [37], it was proven that a linear periodic output feedback controller can provide an arbitrarily large delay margin.

In Chapter 3, we consider the achievable delay margin problem for a LTI controller in the discrete-time setting. Unfortunately, as discussed above, the proof method of [35] breaks down when applied to the discrete-time case. Fortunately, we will be able to partially solve the problem in discrete-time by using a classic result on simultaneous stabilization for two plants, where we find a necessary and sufficient condition for an LTI discrete-time plant to have a non-zero delay margin. This also highlights the difference between the discrete-time and continuous-time problems, since any LTI continuous-time plant P and any LTI stabilizing controller C have a non-zero delay margin; as a result, the achievable delay margin is always non-zero in continuous-time which is not the case in discrete-time.

1.3 Stabilizing any LTI Plant with an Arbitrarily Large Unknown Gain and Delay

Due to difficulties solving the general achievable delay margin problem in both the discrete-time and sampled data settings, we turned our attention to a controller synthesis problem for any continuous-time LTI plant with an unknown, upper bounded continuous-time delay. More formally, the problem is as follows: given any $\overline{\tau} > 0$ and any LTI plant $P(s)$, the goal is to design a controller which stabilizes $P(s)e^{-s\tau}$ for every $\tau \in [0, \overline{\tau}]$. Clearly, from the results of [35], we know that if $P(s)$ has a non-zero unstable pole, that this problem is not solvable using an LTI controller for $\overline{\tau}$ sufficiently large; however, in [37] this problem was solved for an arbitrarily large $\overline{\tau}$ through the synthesis of a linear periodic output feedback controller. However, the controller proposed in [37] has some drawbacks, namely its complexity increases as the desired delay margin increases, the gain can be extremely large, even for simple cases, and it requires a period at least two times longer than the maximum length of the unknown delay $\overline{\tau}$;[1] that being said, it does demonstrate that using non-LTI control can have benefits for increasing the maximum allowable uncertain delay for which stability can be maintained.

Because [37] solves the problem when the only uncertainty is the unknown delay, in order to increase the difficulty of the problem we also consider an unknown, arbitrarily large gain. Problems involving uncertain gains have a very long history in control systems engineering, and in fact, it was one of the first robustness measures considered, with work dating back to before World War II with the development of feedback amplifiers and the work of Bode [3] and Nyquist [40] themselves. Like the unknown delay, this problem can easily be stated on its own: given any $\overline{g} \geq 1$ and any LTI plant $P(s)$, the goal is to design a controller which stabilizes $gP(s)$ for every $g \in [1, \overline{g}]$.

If we only consider an uncertain gain, it is well known that an LQR-optimal state feedback controller provides infinite[2] gain margin in the continuous-time setting [24, 54]; however this result does not translate to discrete-time plants [46]. In the case of output feedback, [26] shows that when using LTI control there is a fundamental gain margin limitation for non-minimum phase continuous-time plants; furthermore, through the use of the bilinear transformation, it can be easily seen that a similar limitation exists for all strictly proper discrete-time plants. However, numerous papers have shown that a linear time varying output feedback controller exists which provides an arbitrarily large gain margin for continuous-time plants, e.g. [53, 45, 52], and similarly, in [25] it was shown that a linear periodic output feedback controller exists which provides an arbitrarily large gain margin for bi-proper discrete-time systems.

In Chapters 4, 5 and 6, we consider the combined problem, that of an uncertain gain and uncertain delay: given any $\overline{\tau} > 0$ and $\overline{g} > 1$, and any SISO LTI plant $P(s)$, the

[1] This creates performance problems since the large period means that the potentially unstable system runs in open loop for at least $2\overline{\tau}$ seconds.

[2] Note that this is a stronger result than arbitrarily large, as no knowledge of an upper bound of the unknown gain is required a priori.

goal is to design a controller which stabilizes $gP(s)e^{-s\tau}$ for every $\tau \in [0, \overline{\tau}]$ and $g \in [1, \overline{g}]$. This combined problem poses new difficulties and creates certain trade-offs. For example, consider [35], which designed a first order LTI controller, parameterized by $\epsilon > 0$, for the plant $\frac{1}{s-p}$, $p > 0$; this controller provides a delay margin within ϵ of the maximum achievable delay margin of $\frac{2}{p}$, but as observed by the authors, as $\epsilon \to 0$, the gain margin provided by this controller tends to zero. As often occurs in control systems engineering, maximizing performance in one area tends to come with trade-offs in other areas.

Various robust control techniques have been used in order to synthesize controllers for systems with uncertain delays and various forms of plant uncertainty, such as H_∞ techniques [32], linear matrix inequalities [39, 13], and adaptive control [50, 7, 6, 4]. However, as far as we are aware, there is currently only one controller that can solve the combined problem with an arbitrarily large unknown gain and delay [4]-[6]. In that work, using the backstepping methodology, an infinite dimensional, non-linear, time varying adaptive controller is proposed and shown to achieve asymptotic tracking for any LTI plant with an unknown arbitrarily long delay and with unknown A and B matrices lying in a known, linearly parameterized observable/controllable set; however, since [6] uses many classical adaptive control techniques, it is unlikely to provide bounded-input bounded-output (BIBO) stability, handle un-modelled dynamics or tolerate jumps in the unknown gain and delay.

In Chapter 4, we use a generalized hold to convert the problem under discussion into one amenable to classical adaptive control techniques; while our controller is finite dimensional, it suffers from many of the same deficiencies as [4]-[6]. To combat these deficiencies, in Chapter 5, using a similar hold to the one in Chapter 4, we design a purpose built controller which will handle noise, un-modelled dynamics and occasional jumps in the unknown gain and delay. In Chapter 6 we again consider the combined gain/delay problem, but we allow the time delay to be time varying and the goal is to find explicit bounds on the allowable time variation of the delay in terms of the pole location and the maximum length of the unknown delay.

1.4 Outline

The remainder of this thesis is organized as follows. In Chapter 2 we define the notation used throughout this thesis and provide a definition and discussion of pathological sampling. In Chapter 3 we consider the achievable delay margin problem in discrete-time. In Chapters 4 - 6, we consider the problem of stabilizing an LTI plant with an arbitrarily large uncertain gain and delay. We start with Chapter 4, where we present a generalized hold which allows us to weakly stabilize the system using an off-the-shelf adaptive controller. In Chapter 5, using a similar generalized hold, we design a new adaptive controller from scratch, yielding a controller which not only provides BIBO stability, but also guarantees the exponential decay of the plant initial conditions. In Chapter 6, we allow the delay to be continuously varying, and then design a controller which BIBO stabilizes the system if the time variation satisfies an explicit bound in terms of the maximum length of the delay, the rate of change of the delay, and the location of the plant pole. In Chapter 7 we present some unsolved achievable delay margin problems in both the discrete-time and sampled data settings. Finally, in Chapter 8 we present some concluding remarks and propose some future work.

Chapter 2

Preliminaries

2.1 Notation

We first define various sets of numbers, to this end, we let:

- $\mathbf{R}$ denote the set of real numbers,
- $\mathbf{R}^+$ denote the set of non-negative real numbers,
- $\mathbf{Z}$ denote the set of integers,
- $\mathbf{Z}^+$ denote the set of non-negative integers,
- $\mathbf{N}$ denote the set of natural numbers,
- $\mathbf{C}$ denote the set of complex numbers,
- $\mathbf{C}^+$ denote the set of complex numbers with a positive real part,
- $\mathbf{C}^-$ denote the set of complex numbers with a negative real part,
- $\mathbf{D}$ denote the set of complex numbers with magnitude less than one,
- $\overline{\mathbf{D}}$ denote the set of complex numbers with magnitude greater than one.

We now turn to defining our vector norm; since we almost always want to use the infinity norm, we define the norm of a vector $x \in \mathbf{R}^n$ as $\|x\| = \max_{i=1,2,\cdots,n} |x_i|$ and the corresponding induced norm of a matrix $A \in \mathbf{R}^{n\times m}$ is given by $\|A\| = \max\{\|Ax\| : x \in \mathbf{R}^m,\ \ \|x\| = 1\}$. With the vector norm defined, we want to define the various sets of sequences and functions used throughout this thesis. We let $\ell(\mathbf{R}^n)$ denote the set of $\mathbf{R}^n$ valued sequences on $\mathbf{Z}^+$, and we will always denote a sequence in $\mathbf{R}^n$ using a Greek letter and a $[\cdot]$ for the argument (i.e., $\psi[k]$). We define the infinity norm of a sequence $\psi \in \ell(\mathbf{R}^n)$ as $\|\psi\|_\infty = \sup_{k\geq 0} \|\psi[k]\|$, and we say that a sequence $\psi \in \ell_\infty(\mathbf{R}^n)$ if $\|\psi\|_\infty < \infty$. We let $PC(\mathbf{R}^n)$ denote the set of piecewise continuous functions from $\mathbf{R}$ to $\mathbf{R}^n$ and $AC(\mathbf{R}^n)$

denote the set of absolutely continuous functions from $\mathbf{R}$ to $\mathbf{R}^n$; we write a function in $PC(\mathbf{R}^n)$ ($AC(\mathbf{R}^n)$) using a Roman letter and $(\cdot)$ for the argument (i.e., $y(t)$). We define the infinity norm of a function $f \in PC(\mathbf{R}^n)$ ($f \in AC(\mathbf{R}^n)$) as $\|f\|_\infty = \sup_{t\geq 0} \|f(t)\|$, and we say that a function $f \in PC_\infty$ (or $f \in AC_\infty$) if $f \in PC$ ($f \in AC$) and $\|f\|_\infty < \infty$. To simplify notation, we will often drop the dimensions from our spaces. To further distinguish between continuous-time and discrete-time, we adopt the standard convention of using solid lines to denote continuous-time signals and dashed lines to denote discrete-time signals.

In order to discuss stability of a non-linear time varying system, we require the notion of the gain of such a system starting with zero initial conditions at time zero. To this end, the gain of $G : PC_\infty \to PC$ is defined by

$$\|G\| := \sup \left\{ \frac{\|Gu\|_\infty}{\|u\|_\infty} \ : \ u \in PC_\infty, \|u\|_\infty \neq 0 \right\}.$$

We will also require a special version[1] of the "sign" function which maps $\mathbf{R}$ to $\{-1, 1\}$:

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Finally, we let $H_\infty^{n\times m}(\mathbf{D})$ denote the set of $n \times m$ complex-valued functions that are bounded and analytic on $\mathbf{D}$, $H_\infty^{n\times m}(\overline{\mathbf{D}})$ denote the set of $n \times m$ complex-valued functions that are bounded and analytic on $\overline{\mathbf{D}}$ and $H_\infty^{n\times m}(\mathbf{C}^+)$ denote the set of $n \times m$ complex-valued functions that are bounded and analytic on $\mathbf{C}^+$. We let $RH_\infty^{n\times m}(\mathbf{D})$ denote the subset consisting of the real rational elements of $H_\infty^{n\times m}(\mathbf{D})$, $RH_\infty^{n\times m}(\overline{\mathbf{D}})$ denote the subset consisting of the real rational elements of $H_\infty^{n\times m}(\overline{\mathbf{D}})$ and $RH_\infty^{n\times m}(\mathbf{C}^+)$ denote the subset consisting of the real rational elements of $H_\infty^{n\times m}(\mathbf{C}^+)$. We will often drop the $\mathbf{D}$, $\overline{\mathbf{D}}$ or $\mathbf{C}^+$ since the correct space should be clear from the context, and since the use of these spaces normally occurs when analyzing SISO systems, we are normally interested in $H_\infty^{1\times 1}$, so when the dimensions are 1×1 we simply write H_∞ (or RH_∞).

2.2 Stability

In this thesis, we have many different time settings and notions of stability. In particular, Chapter 3 considers the stability of an LTI discrete-time system, Chapter 4 considers the asymptotic regulation of a non-linear time varying continuous-time system's output to zero,

[1] Normally $sgn(0)$ is defined to be zero.

Chapters 5 and 6 consider bounded-input bounded-output stability of a non-linear time varying continuous-time system, and Chapter 7 considers both discrete-time and sampled data systems. Due to these different time paradigms and notions of stability, we will define the notion of stability at an appropriate place in each chapter.

2.3 Pathological Sampling

Throughout this thesis, we often consider the stability of a continuous-time system which is controlled via a digital controller and a zero order hold. To analyze such a system, it is common practice to create an equivalent discrete-time model through analysis of the system behaviour at integer multiples of the sampling period. We would like to say that if this resulting discrete-time model is stable, then so too is the original sampled data system. It turns out from [11] that so long as we choose an 'appropriate' sampling time, that if the discretized system is stable, then the original sampled data system is stable as well.

To define these 'appropriate' sampling times, consider a SISO continuous-time plant P, with a state space representation given by

$$\dot{x} = Ax + Bu \tag{2.1}$$

$$y = Cx + Du \tag{2.2}$$

with (A, B) controllable and (C, A) observable.

Definition 2.1. *A sampling period T is* non-pathological *(with respect to A) if, whenever $\mu \in \mathbf{C}$ is an eigenvalue of A, none of the points $\left\{\mu + \frac{j2\pi k}{T} : k \in \mathbf{N}, k \neq 0\right\}$ is an eigenvalue of A. Otherwise, we say that the sampling period T is* pathological *(with respect to A).*

To see the benefits of choosing a non-pathological sampling period T, consider the solution to the state equation (2.1) with a constant input $\nu[k]$ starting from time kT and ending at time $(k+1)T$ as well as the solution to the output equation (2.2) at time kT, i.e., the solution at the sampling points when a zero-order-hold is applied at the plant input:

$$\begin{aligned} x((k+1)T) &= \underbrace{e^{AT}}_{=:A_d} x(kT) + \underbrace{\int_{kT}^{(k+1)T} e^{A(kT+T-v)} B dv}_{=:B_d} \nu[k] \\ y(kT) &= Cx(kT) + D\nu[k]. \end{aligned}$$

If T is non-pathological with respect to A, then from [11], it follows that (A, B) controllable implies that (A_d, B_d) is controllable, and (C, A) observable implies that (C, A_d) is observable. In Chapters 4 and 5, we use a hold that is very similar to a zero-order-hold, so the concept of non-pathological sampling is still of critical importance.

Chapter 3

Achievable Delay Margin

In this chapter, we consider the achievable delay margin problem in the discrete-time setting, which is that of determining the maximum allowable length of an unknown delay for which a single LTI controller can maintain stability. In continuous-time, [35] found upper bounds on the achievable delay margin problem for many cases, many of which were tight bounds; of particular interest is the case of a plant with a single, real unstable pole where a tight upper bound was found. Our original goal was to extend these results to the discrete-time setting; unfortunately, this has mainly proved unsuccessful, as the methods employed in [35] do not translate from the continuous-time setting to the discrete-time setting[1]. However, as published in [17], we have been able to prove a necessary and sufficient condition for when the discrete-time achievable delay margin is non-zero, which highlights the difference between the continuous-time case (where the achievable delay margin is always non-zero) and the discrete-time case.

This chapter is organized as follows. We start by formally stating the discrete-time delay margin problem and then we present some important results on simultaneous stabilization. Using the simultaneous stabilization results, we then ascertain when the achievable delay margin is non-zero, as well as show that those tools provide no further insight into the problem for longer delays.

[1]We will revisit this issue in great detail in Chapter 7 where we present numerous open achievable delay margin problems in both the discrete-time and sampled data settings.

3.1 Problem Setup

This section formally defines the discrete-time achievable delay margin problem. We start with a discrete-time SISO finite-dimensional linear time invariant (FDLTI) plant G_0 with a state space representation given by

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k], \end{aligned} \tag{3.1}$$

with A, B, C, and D of appropriate dimensions. We make a standing assumption throughout this chapter that (A, B) is stabilizable and that (C, A) is detectable. We also require versions of this plant with an input delay: with $n \in \mathbf{N}$, let G_n have the following state space representation:

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k-n] \\ y[k] &= Cx[k] + Du[k-n]. \end{aligned}$$

We also need transfer function representations of G_0 and G_n, which we would normally write in the z-domain; however, it will be easier to prove the main theorem if the unstable region is the interior of the closed unit disk. To that end, define $\lambda := z^{-1}$, which leads naturally to the following transfer functions:

$$G_0[\lambda] = \lambda C(I - \lambda A)^{-1}B + D, \tag{3.2}$$

$$G_n[\lambda] = \lambda^n(\lambda C(I - \lambda A)^{-1}B + D) = \lambda^n G_0[\lambda], \quad n \in \mathbf{N}. \tag{3.3}$$

To define stability in this chapter, we consider the feedback setup shown in Figure 3.1 and say that a controller K *stabilizes* a plant G if the transfer function from

$$\begin{bmatrix} d \\ w \end{bmatrix} \to \begin{bmatrix} y \\ u \end{bmatrix}$$

belongs to $H_\infty^{2\times 2}(\mathbf{D})$, i.e.,

$$(1 + KG)^{-1}, K(1 + KG)^{-1}, G(1 + KG)^{-1}, KG(1 + KG)^{-1} \in H_\infty(\mathbf{D}).$$

We now formally define the delay margin problem, adopting the notation from [35]. If a controller K stabilizes a plant G_0, then the *delay margin* is

$$DM(G_0, K) := \max\{n \geq 0 : K \text{ stabilizes } G_0, G_1, \cdots, G_n\}.$$

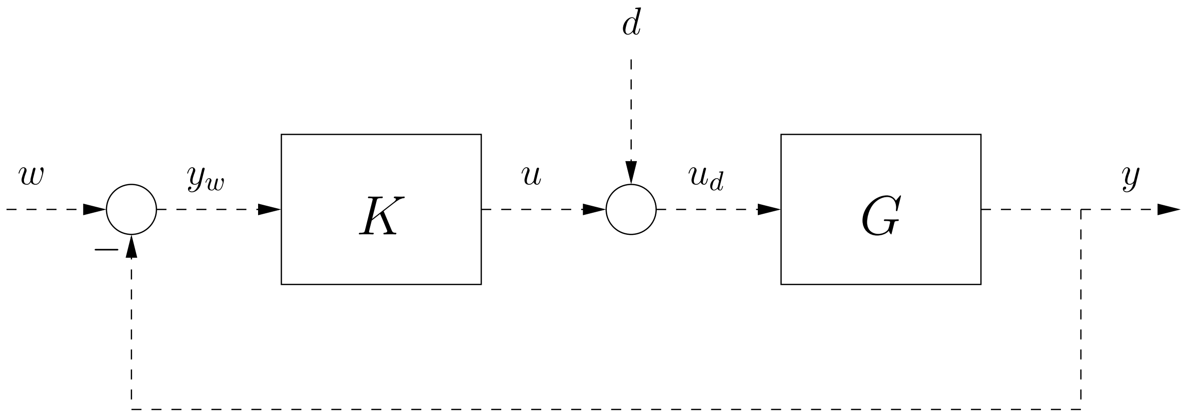

Figure 3.1: The discrete-time delay margin problem setup.

While $DM(G_0, K)$ is a useful quantity to know about a particular plant/controller combination, a more fundamental property of the plant is

$$DM(G_0) := \max\{DM(G_0, K) : K \text{ is FDLTI and stabilizes } G_0\},$$

which is simply the maximum achievable delay margin when using a stabilizing FDLTI controller.

We would like to easily compute $DM(G_0)$. To proceed, it is helpful to note that $DM(G_0)$ is actually a classic *simultaneous stabilization* problem.

Definition 3.1. *We say that the set of LTI plants $\mathcal{G}$ is* simultaneously stabilizable *if there exists a real, rational, proper transfer function $K[\lambda]$ that stabilizes every $G[\lambda] \in \mathcal{G}$.*

For example, to see if $DM(G_0) \geq 3$, one is really asking the question of does there exist a single FDLTI controller K that stabilizes G_0, G_1, G_2, and G_3. Unfortunately, while the simultaneous stabilization problem is well studied, a tractable test that provides necessary and sufficient conditions for determining when *any* set of plants $\mathcal{G}$ is simultaneously stabilizable only exists when $\mathcal{G}$ has two elements (e.g. see [48] and [9]), while it has been shown [2] that the problem is "rationally undecidable" for the case when $\mathcal{G}$ has three or more elements[2]. Hence, we first focus on trying to determine when $DM(G_0) > 0$.

[2]Of course, if the elements of $\mathcal{G}$ have rich structure, as they do here, perhaps more can be proven.

3.2 The Approach

The problem of determining if $DM(G_0)$ is non-zero is simply a two plant simultaneous stabilization problem, since

$$DM(G_0) > 0 \Leftrightarrow G_0 \text{ and } G_1 \text{ are simultaneously stabilizable.}$$

We will use the standard two plant simultaneous stabilization results presented in [48], which converts the simultaneous stabilization problem to a strong stabilization problem.

Definition 3.2. *A plant $G[\lambda]$ is* strongly stabilizable *if there exists a stable controller $D[\lambda] \in H_\infty^{1\times 1}(\mathbf{D})$ that stabilizes $G[\lambda]$ (in which case we say that $D[\lambda]$ strongly stabilizes $G[\lambda]$).*

We now require a stable coprime factorization [9] of $G_0[\lambda]$, so we choose polynomials[3] $N_0[\lambda]$, $M_0[\lambda]$, $X_0[\lambda]$ and $Y_0[\lambda]$ satisfying

$$G_0[\lambda] = \frac{N_0[\lambda]}{M_0[\lambda]} \quad \text{and} \quad N_0[\lambda]X_0[\lambda] + M_0[\lambda]Y_0[\lambda] = 1. \tag{3.4}$$

Observe that $M_0[\lambda]$ and $N_0[\lambda]$ are clearly coprime and that $M_0[0] \neq 0$.

We can now state the two lemmas that give necessary and sufficient conditions for $G_0[\lambda]$ and $G_n[\lambda] = \lambda^n G_0[\lambda]$, $n \in \mathbf{N}$ to be simultaneously stabilizable:[4]

Lemma 3.1. *$G_0[\lambda]$ and $G_n[\lambda] = \lambda^n G_0[\lambda]$ are simultaneously stabilizable if and only if*

$$\frac{(\lambda^n - 1)M_0[\lambda]N_0[\lambda]}{\lambda^n N_0[\lambda]X_0[\lambda] + M_0[\lambda]Y_0[\lambda]} \tag{3.5}$$

is strongly stabilizable.

Proof. Using the coprime factorization given by (3.4), we apply Theorem 5.4.2 from [48], with $N_n[\lambda] = \lambda^n N_0[\lambda]$ and $M_n[\lambda] = M_0[\lambda]$. Since the system is SISO, the left and right coprime factorizations are the same, so we define

$$A[\lambda] := \lambda^n N_0[\lambda]X_0[\lambda] + M_0[\lambda]Y_0[\lambda],$$

[3]Note that we can choose polynomials here instead of the typical proper transfer functions for the coprime factorization due to our use of λ instead of z.

[4]Note that we present this result for an n sample delay; for the question of when is the achievable delay margin non-zero, we will of course use these results with $n = 1$.

$$B[\lambda] := (\lambda^n - 1)M_0[\lambda]N_0[\lambda].$$

We then use the remark immediately following Theorem 5.4.2 of [48], which implies that $G_0[\lambda]$ and $G_n[\lambda]$ are simultaneously stabilizable if and only if

$$\frac{B[\lambda]}{A[\lambda]} = \frac{(\lambda^n - 1)M_0[\lambda]N_0[\lambda]}{\lambda^n N_0[\lambda]X_0[\lambda] + M_0[\lambda]Y_0[\lambda]}$$

is strongly stabilizable, as desired. □

Lemma 3.2. *A discrete-time FDLTI plant $G[\lambda]$ is strongly stabilizable if and only if $G[\lambda]$ has an even number of real poles (counting multiplicities) between every pair of consecutive real zeros that lie in $[-1, 1]$.*

Proof. The result is given for continuous-time in Corollary 3.2.2 of [48]. An extension to discrete-time (in particular for $\lambda = z^{-1}$)[5] using the bilinear transform $s = \frac{1-\lambda}{1+\lambda}$ is stated in the last paragraph of Section 3.2 of [48]. □

Lemma 3.2 allows us to convert the simultaneous stabilization problem of two plants into a strong stabilization problem of a single "new" plant which is easily solvable. To make solving the strong stabilization problem even easier, we require a result relating the sign of a real polynomial when evaluated at two points to the number of real zeros between those two points, which we do in the following result:

Lemma 3.3. *Given a real polynomial f and two real numbers $a < b$ for which $f(a) \neq 0$ and $f(b) \neq 0$, we have that f has an even number of real zeros (counting multiplicities) in the interval $[a, b]$ if and only if $f(a)$ and $f(b)$ have the same sign.*

Proof. Let a, $b \in \mathbf{R}$ be arbitrary and satisfy $a < b$, $f(a) \neq 0$ and $f(b) \neq 0$.

First suppose that f has no zeros over $[a, b]$. By the continuity of f, it follows that the sign of f can not change on $[a, b]$, so $f(a)$ and $f(b)$ have the same sign.

Now suppose f has zeros on $[a, b]$. Using the Fundamental Theorem of Algebra, we can write f as a product of two real polynomials g and h, with g monic and with zeros

[5] Note that the author of [48] defines z to be the same as the λ used here.

only in $[a, b]$ and h with no zeros in $[a, b]$. Hence, there exists an integer $k \in \mathbf{N}$, and $z_i, \ldots, z_k \in (a, b)$ (possibly repeated) so that g has the form

$$g(z) = \prod_{i=1}^{k} (z - z_i).$$

Since h has no zeros on $[a, b]$, it is clear that $f(a)$ and $f(b)$ have the same sign iff $g(a)$ and $g(b)$ have the same sign, or equivalently iff $g(a)g(b) > 0$. However, clearly $g(b) > 0$, so

$$\begin{aligned}
f(a)f(b) > 0 &\Leftrightarrow g(a) > 0 \\
&\Leftrightarrow \prod_{i=1}^{k} (a - z_i) > 0 \\
&\Leftrightarrow (-1)^k > 0 \\
&\Leftrightarrow k \text{ is even} \\
&\Leftrightarrow g \text{ has an even number (counting multiplicities) of zeros on } [a, b] \\
&\Leftrightarrow f \text{ has an even number (counting multiplicities) of zeros on } [a, b].
\end{aligned}$$

□

3.3 When is the Achievable Discrete Time Delay Margin Non-Zero?

We now have all the tools required to prove when the discrete-time achievable delay margin is non-zero.

Theorem 3.1. *A discrete-time plant G_0 with a state space representation given by (3.1) has $DM(G_0) = 0$ if and only if A has a real eigenvalue in $(-\infty, -1]$.*

Proof. Recall from (3.2) and (3.3) that G_0 and G_1 have transfer functions given by

$$G_0[\lambda] := \lambda C(I - \lambda A)^{-1} B + D,$$

$$G_1[\lambda] := \lambda(\lambda C(I - \lambda A)^{-1} B + D) = \lambda G_0[\lambda].$$

Next, observe that $DM(G_0) > 0$ if and only if $G_0[\lambda]$ and $\lambda G_0[\lambda]$ are simultaneously stabilizable, so with N_0, M_0, X_0 and Y_0 polynomials satisfying (3.4), by Lemma 3.1 $G_0[\lambda]$ and $\lambda G_0[\lambda]$ are simultaneously stabilizable if and only if

$$G[\lambda] := \frac{(\lambda - 1)M_0[\lambda]N_0[\lambda]}{\lambda N_0[\lambda]X_0[\lambda] + M_0[\lambda]Y_0[\lambda]} \tag{3.6}$$

is strongly stabilizable.

We will now use Lemma 3.2, which states that $G[\lambda]$ is strongly stabilizable if and only if there is an even number of real poles (counting multiplicities) between every pair of consecutive real zeros of $G[\lambda]$ for $\lambda \in [-1, 1]$; furthermore, we will use Lemma 3.3 to check the condition required by Lemma 3.2. To that end, we define the numerator and denominator polynomials of G:

$$n[\lambda] := (\lambda - 1)N_0[\lambda]M_0[\lambda], \tag{3.7}$$
$$d[\lambda] := \lambda N_0[\lambda]X_0[\lambda] + M_0[\lambda]Y_0[\lambda]. \tag{3.8}$$

Recalling that $M_0[0] \neq 0$ since G is causal, it is easy to check that n and d are coprime, so the poles of G and the zeros of G are given by the zeros of d and n respectively. Let λ_i, $i = 1, ..., k$, denote the real zeros of $n[\lambda]$, ignoring multiplicities, which lie in $[-1, 1]$,[6] and assume, without loss of generality, that $\lambda_1 < \lambda_2 < ... < \lambda_k$. There are three ways that such a λ_i can arise:

Case 1: $\lambda_k = 1$.

Case 2: $N_0[\lambda_i] = 0$, which corresponds to a real zero of $G_0[\lambda]$ in $\mathbf{D}$.

Case 3: $M_0[\lambda_i] = 0$, which corresponds to a real pole of $G_0[\lambda]$ in $\mathbf{D}$.

In order to apply Lemma 3.3 we need to determine the sign of $d[\lambda]$ at each λ_i. Evaluating $d[\lambda]$ for each λ_i, and making use of (3.4), gives the following results:

Case 1: At $\lambda_k = 1$,

$$\begin{aligned} d[\lambda_k] &= \lambda_k N_0[\lambda_k]X_0[\lambda_k] + M_0[\lambda_k]Y_0[\lambda_k] \\ &= 1. \end{aligned}$$

[6]Note that $\{\lambda_i\} \neq \emptyset$ since $n[\lambda]$ will always have a zero at $\lambda = 1$.

Case 2: If $N_0[\lambda_i] = 0$ then

$$\begin{aligned} d[\lambda_i] &= \lambda_i N_0[\lambda_i] X_0[\lambda_i] + M_0[\lambda_i] Y_0[\lambda_i] \\ &= \lambda_i N_0[\lambda_i] X_0[\lambda_i] + 1 - N_0[\lambda_i] X_0[\lambda_i] \\ &= 1. \end{aligned}$$

Case 3: If $M_0[\lambda_i] = 0$ then

$$\begin{aligned} d[\lambda_i] &= \lambda_i N_0[\lambda_i] X_0[\lambda_i] + M_0[\lambda_i] Y_0[\lambda_i] \\ &= \lambda_i \left(1 - M_0[\lambda_i] Y_0[\lambda_i]\right) + M_0[\lambda_i] Y_0[\lambda_i] \\ &= \lambda_i \qquad (3.9) \\ &\neq 0 \text{ (since } G_0 \text{ is causal).} \end{aligned}$$

Note from Cases 1-3 that for $d[\lambda_i] < 0$, λ_i must satisfy Case 3 and $\lambda_i \in [-1, 0)$.

($\Rightarrow$) Suppose $DM(G_0) = 0$. By Lemma 3.2 there exists an integer $i \in \{1, 2, \ldots, k-1\}$ so that the interval $(\lambda_i, \lambda_{i+1})$ contains an odd number of real poles of G, or equivalently an odd number of real zeros of d. By Lemma 3.3, this means that $d[\lambda_i]$ and $d[\lambda_{i+1}]$ must have different signs; hence, exactly one of λ_i or λ_{i+1} is both negative and satisfies Case 3; let $\overline{\lambda}$ denote that quantity, and note that $\overline{\lambda} \in [-1, 0)$. Therefore, by Case 3 $M_0[\overline{\lambda}] = 0$, which means that G_0 has a pole at $\overline{\lambda}$; since (A, B) is stabilizable and (C, A) detectable, A has an eigenvalue at $1/\overline{\lambda} \in (-\infty, -1]$.

($\Leftarrow$) Suppose that A has a real eigenvalue at $v \in (-\infty, -1]$. Since (A, B) is stabilizable and (C, A) detectable by hypothesis, $M_0(\lambda)$ has a zero at $1/v \in [-1, 0)$. Hence, $1/v$ is a zero of n and therefore an admissible λ_i; furthermore, observe that

$$d[1/v] = 1/v < 0.$$

Observe that n also has a zero at one, which we label λ_k, and that $d[\lambda_k] = d[1] = 1$. Hence, there must exist an integer $j \in \{1, 2, \ldots, k-1\}$ for which $d[\lambda_j]$ and $d[\lambda_{j+1}]$ have opposite signs. From Lemma 3.3, d has an odd number of zeros on the interval $(\lambda_j, \lambda_{j+1})$. By Lemma 3.2 it follows that G is not strongly stabilizable, so by Lemma 3.1 it follows that G_0 and G_1 are not simultaneously stabilizable and hence $DM(G_0) = 0$. □

Theorem 3.1 can be thought of as a performance limitation for any plant with a negative real unstable pole, but it also has important ramifications when designing a controller for a discrete-time system with an unknown delay, as described by the following Corollary.

Corollary 3.1. *If a FDLTI controller $K[\lambda]$ has a pole in $[-1, 0)$ and stabilizes $G_0[\lambda]$, then*

$$DM(G, K) = 0.$$

Proof. Suppose that K has the required properties. Since K stabilizes G_0, it follows immediately that G_0 stabilizes K; by Theorem 3.1 this means that

$$DM(K) = 0,$$

which means that

$$DM(G_0, K) = DM(K, G_0) = 0.$$

□

3.3.1 Continuous-Time Plant with a Discrete-Time Delay

Theorem 3.1 has an interesting implication for a sampled data system (employing a sampler and zero-order-hold with the same period) with a discrete-time delay, as shown in Figure 3.2. Such a setup (a continuous-time plant with a discrete-time delay) can arise from many practical problems, for example, controlling a vehicle using complicated image processing or controlling an airplane using a sensor network. Consider a SISO continuous-time plant P, with state space representation given by

$$\begin{aligned} \dot{x} &= A_c x + B_c u \\ y &= C_c x + D_c u, \end{aligned}$$

with (A_c, B_c) controllable and (C_c, A_c) observable. From [11], we can analyze the stability of the sampled data system by determining the stability of an appropriate discrete-time (discretized) system so long as the sampling period T is non-pathological[7] with respect to A_c. To that end, following the procedure from [11], we apply the hold to the plant P and solve at the sampling points, yielding a discretized (and discrete-time) plant G. Doing so, we get that the A matrix of the discretized plant G is given by

$$A = e^{A_c T};$$

hence, for any non-pathological sampling period T, it follows that all real eigenvalues of A_c will become positive real eigenvalues of A and any non-real eigenvalues of A_c will become

[7] See Section 2.3 for a definition of pathological sampling.

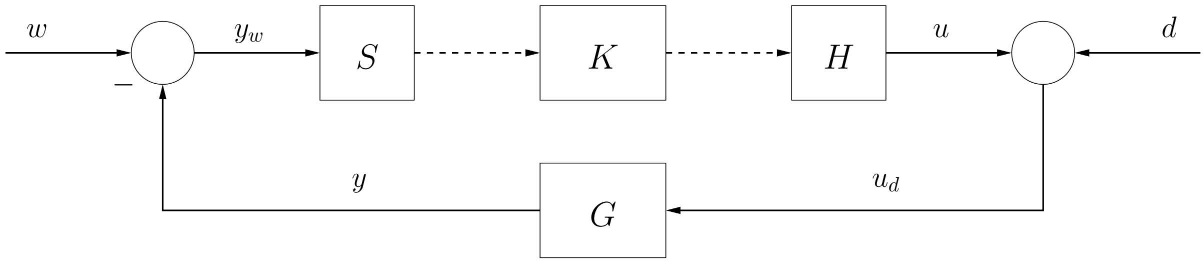

Figure 3.2: The sampled data with a discrete-time delay problem setup.

non-real eigenvalues of A. As such, it immediately follows that for any non-pathological sampling period T that A has no negative real eigenvalues. Hence, by Theorem 3.1 there will always exist a controller $K[\lambda]$ that can stabilize $G[\lambda]$ and $\lambda G[\lambda]$ for any non-pathological sampling period. This means that, for sampled data systems with a discrete-time delay, so long as the sampling period is non-pathological, we get a result that is analogous to continuous-time, namely, that the achievable delay margin is always strictly greater than zero.

In Chapter 7 we will consider the other sampled data problem, namely, controlling a continuous-time plant with a continuous-time delay using a discrete-time controller. In contrast to the problem of a continuous-time plant with a discrete-time delay, which is essentially a special case of the pure discrete-time problem, the problem of a continuous-time plant with a continuous-time delay with a discrete-time controller bears little resemblance to the pure discrete-time problem.

3.4 Longer Delays

Now that we know when the discrete-time achievable delay margin is non-zero, the next step is to try to determine an upper bound on the delay margin for a general discrete-time plant, with the seemingly most natural approach being to continue within the simultaneous stabilization framework. Unfortunately, as shown by [2], this problem is, in general, rationally undecidable, i.e., we can't convert the problem into an equivalent easily solvable problem as was the case for two plants. However, it is certainly the case that for three or more plants to be simultaneously stabilizable, it must be that every pairwise combination of plants must be simultaneously stabilizable, so some insight may be gained using the same technique that we used to prove when the achievable delay margin is non-zero. Unfortunately, as shown in the following theorem, if a discrete-time plant has no negative, real, unstable poles, then any two delayed versions of the plant are always simultaneously stabilizable, i.e., a pairwise application of the simultaneous stabilization result does not help find an upper bound on the achievable delay margin.

Theorem 3.2. *For any discrete-time plant G_0 with a state space representation given by (3.1) and such that A has no negative real eigenvalues in $(-\infty, -1]$, it follows that for every $m \in \mathbf{Z}^+$ and $n \in \mathbf{N}$ that $G_m[\lambda] = \lambda^m G_0[\lambda]$ and $G_n[\lambda] = \lambda^n G_0[\lambda]$ are simultaneously stabilizable.*

Proof. Let G_0 given by (3.1) be such that A has no negative real eigenvalues in $(-\infty, -1]$. Since we can create a $\tilde{G}_0[\lambda] := \lambda^m G_0[\lambda]$ with the state space representation of $\tilde{G}_0[\lambda]$ also having no negative real eigenvalues in $(-\infty, -1]$, without loss of generality, we set $m = 0$ and let $n \in \mathbf{N}$ be arbitrary.

We employ the same method as the proof of Theorem 3.1, namely, we use (3.4) to get a co-prime factorization of $G_0[\lambda]$, and then we apply Lemma 3.1 to convert the simultaneous stabilization problem of $G_0[\lambda]$ and $G_n[\lambda]$ into a strong stabilization problem for the plant

$$G[\lambda] := \frac{(\lambda^n - 1)M_0[\lambda]N_0[\lambda]}{\lambda^n N_0[\lambda]X_0[\lambda] + M_0[\lambda]Y_0[\lambda]}. \tag{3.10}$$

We will now use Lemma 3.2, which states that $G[\lambda]$ is strongly stabilizable if and only if there is an even number of real poles (counting multiplicities) between every pair of consecutive real zeros of $G[\lambda]$ for $\lambda \in [-1, 1]$; furthermore, we will use Lemma 3.3 to check the condition required by Lemma 3.2. To that end, we define the numerator and denominator polynomials of G:

$$n[\lambda] := (\lambda^n - 1)N_0[\lambda]M_0[\lambda], \tag{3.11}$$
$$d[\lambda] := \lambda^n N_0[\lambda]X_0[\lambda] + M_0[\lambda]Y_0[\lambda]. \tag{3.12}$$

Recalling that $M_0[0] \neq 0$ since G is causal, it is easy to check that n and d are coprime, so the poles of G and the zeros of G are given by the zeros of d and n respectively. Let λ_i, $i = 1, ..., k$, denote the real zeros of $n[\lambda]$, ignoring multiplicities, which lie in $[-1, 1]$,[8] and assume, without loss of generality, that $\lambda_1 < \lambda_2 < ... < \lambda_k$. There are three ways that such a λ_i can arise:

[8]Note that $\{\lambda_i\} \neq \emptyset$ since $n[\lambda]$ will always have a zero at $\lambda = 1$.

Case 1: $\lambda^n - 1 = 0$; if n is odd, then there is one λ_i, namely, $\lambda_k = 1$, if n is even, then there are two λ_i, namely, $\lambda_1 = -1$ and $\lambda_k = 1$.

Case 2: $N_0[\lambda_i] = 0$, which corresponds to a real zero of $G_0[\lambda]$ in $\mathbf{D}$.

Case 3: $M_0[\lambda_i] = 0$, which corresponds to a real pole of $G_0[\lambda]$ in $\mathbf{D}$.

In order to apply Lemma 3.3 we need to determine the sign of $d[\lambda]$ at each λ_i. Evaluating $d[\lambda]$ for each λ_i, and making use of (3.4), gives the following results:

Case 1: If $\lambda^n - 1 = 0$ then we always have $\lambda_k = 1$ and

$$\begin{aligned} d[\lambda_k] &= \lambda_k^n N_0[\lambda_k] X_0[\lambda_k] + M_0[\lambda_k] Y_0[\lambda_k] \\ &= 1. \end{aligned}$$

In addition, if n is even, then we have $\lambda_1 = -1$ and

$$\begin{aligned} d[\lambda_1] &= \lambda_1^n N_0[\lambda_1] X_0[\lambda_1] + M_0[\lambda_1] Y_0[\lambda_1] \\ &= 1. \end{aligned}$$

Case 2: If $N_0[\lambda_i] = 0$ then

$$\begin{aligned} d[\lambda_i] &= \lambda_i^n N_0[\lambda_i] X_0[\lambda_i] + M_0[\lambda_i] Y_0[\lambda_i] \\ &= \lambda_i^n N_0[\lambda_i] X_0[\lambda_i] + 1 - N_0[\lambda_i] X_0[\lambda_i] \\ &= 1. \end{aligned}$$

Case 3: If $M_0[\lambda_i] = 0$ then

$$\begin{aligned} d[\lambda_i] &= \lambda_i^n N_0[\lambda_i] X_0[\lambda_i] + M_0[\lambda_i] Y_0[\lambda_i] \\ &= \lambda_i^n \left(1 - M_0[\lambda_i] Y_0[\lambda_i]\right) + M_0[\lambda_i] Y_0[\lambda_i] \\ &= \lambda_i^n \qquad (3.13) \\ &\neq 0 \text{ (since } G_0 \text{ is causal).} \end{aligned}$$

Observe from Cases 1-3 that for $d[\lambda_i] < 0$, λ_i must satisfy Case 3, with $\lambda_i \in [-1, 0)$ and n odd. Since A has no negative real eigenvalues in $(-\infty, -1]$, it follows that $G_0[\lambda]$ has no poles in $[-1, 0)$, so there does not exist a $\lambda_i \in [-1, 0)$ such that $M_0[\lambda_i] = 0$, which means that, for every $i = 1, 2, \cdots, k$ we have that $d[\lambda_i] > 0$. It then immediately follows from Lemma 3.3 that there is an even number of real poles between each pair of real zeros of G, so by Lemma 3.1 it follows that G is strongly stabilizable. Hence, from Lemma 3.1, G_0 and G_n are simultaneously stabilizable. □

There is one other interesting application of the simultaneous stabilization results for discrete-time plants with a negative, real unstable pole.

Theorem 3.3. *For any discrete-time plant G_0 with a state space representation given by (3.1) it follows for any $m \in \mathbf{Z}^+$ and $n \in \mathbf{N}$ such that $n - m$ is even, we have that $G_m[\lambda] = \lambda^m G_0[\lambda]$ and $G_n[\lambda] = \lambda^n G_0[\lambda]$ are simultaneously stabilizable.*

Proof. Let G_0 given by (3.1) be arbitrary. Since we can create a $\tilde{G}_0[\lambda] := \lambda^m G_0[\lambda]$, without loss of generality, we set $m = 0$ and let $n \in \mathbf{N}$ be an arbitrary even number.

We employ the same method as the proof of Theorem 3.2, namely, we use (3.4) to get a co-prime factorization of $G_0[\lambda]$, and then we apply Lemma 3.1 to convert the simultaneous stabilization problem of $G_0[\lambda]$ and $G_n[\lambda]$ into a strong stabilization problem for the plant

$$G[\lambda] := \frac{(\lambda^n - 1)M_0[\lambda]N_0[\lambda]}{\lambda^n N_0[\lambda]X_0[\lambda] + M_0[\lambda]Y_0[\lambda]}. \tag{3.14}$$

We will now use Lemma 3.2, which states that $G[\lambda]$ is strongly stabilizable if and only if there is an even number of real poles (counting multiplicities) between every pair of consecutive real zeros of $G[\lambda]$ for $\lambda \in [-1, 1]$; furthermore, we will use Lemma 3.3 to check the condition required by Lemma 3.2. To that end, we define the numerator and denominator polynomials of G:

$$n[\lambda] := (\lambda^n - 1)N_0[\lambda]M_0[\lambda], \tag{3.15}$$
$$d[\lambda] := \lambda^n N_0[\lambda]X_0[\lambda] + M_0[\lambda]Y_0[\lambda]. \tag{3.16}$$

Recalling that $M_0[0] \neq 0$ since G is causal, it is easy to check that n and d are coprime, so the poles of G and the zeros of G are given by the zeros of d and n respectively. Let λ_i, $i = 1, ..., k$, denote the real zeros of $n[\lambda]$, ignoring multiplicities, which lie in $[-1, 1]$,[9] and assume, without loss of generality, that $\lambda_1 < \lambda_2 < ... < \lambda_k$. There are three ways that such a λ_i can arise:

Case 1: $\lambda^n - 1 = 0$; since n is even, we have two λ_i's, namely $\lambda_1 = -1$ and $\lambda_k = 1$.

Case 2: $N_0[\lambda_i] = 0$, which corresponds to a real zero of $G_0[\lambda]$ in $\mathbf{D}$.

[9]Note that $\{\lambda_i\} \neq \emptyset$ since $n[\lambda]$ will always have a zero at $\lambda = 1$.

Case 3: $M_0[\lambda_i] = 0$, which corresponds to a real pole of $G_0[\lambda]$ in $\mathbf{D}$.

In order to apply Lemma 3.3 we need to determine the sign of $d[\lambda]$ at each λ_i. Evaluating $d[\lambda]$ for each λ_i, and making use of (3.4), gives the following results:

Case 1: If $\lambda^n - 1 = 0$, we have $\lambda_k = 1$, so

$$\begin{aligned} d[\lambda_k] &= \lambda_k^n N_0[\lambda_k] X_0[\lambda_k] + M_0[\lambda_k] Y_0[\lambda_k] \\ &= 1, \end{aligned}$$

and we have $\lambda_1 = -1$, yielding

$$\begin{aligned} d[\lambda_1] &= \lambda_1^n N_0[\lambda_1] X_0[\lambda_1] + M_0[\lambda_1] Y_0[\lambda_1] \\ &= 1. \end{aligned}$$

Case 2: If $N_0[\lambda_i] = 0$ then

$$\begin{aligned} d[\lambda_i] &= \lambda_i^n N_0[\lambda_i] X_0[\lambda_i] + M_0[\lambda_i] Y_0[\lambda_i] \\ &= \lambda_i^n N_0[\lambda_i] X_0[\lambda_i] + 1 - N_0[\lambda_i] X_0[\lambda_i] \\ &= 1. \end{aligned}$$

Case 3: If $M_0[\lambda_i] = 0$ then

$$\begin{aligned} d[\lambda_i] &= \lambda_i^n N_0[\lambda_i] X_0[\lambda_i] + M_0[\lambda_i] Y_0[\lambda_i] \\ &= \lambda_i^n \left(1 - M_0[\lambda_i] Y_0[\lambda_i]\right) + M_0[\lambda_i] Y_0[\lambda_i] \\ &= \lambda_i^n \qquad (3.17) \\ &> 0 \text{ (since } G_0 \text{ is causal and } n \text{ is even).} \end{aligned}$$

Observe from Cases 1-3 that for every $i = 1, 2, \cdots, k$ we have that $d[\lambda_i] > 0$. It then immediately follows from Lemma 3.3 that there is an even number of real poles between each pair of real zeros of G, so by Lemma 3.1 it follows that G is strongly stabilizable. Hence, from Lemma 3.1, G_0 and G_n are simultaneously stabilizable if n is even. $\square$

The result shown in Theorem 3.3 may appear quite odd at first. However, intuition into why this is occurring can be found by considering the step responses of a plant with a

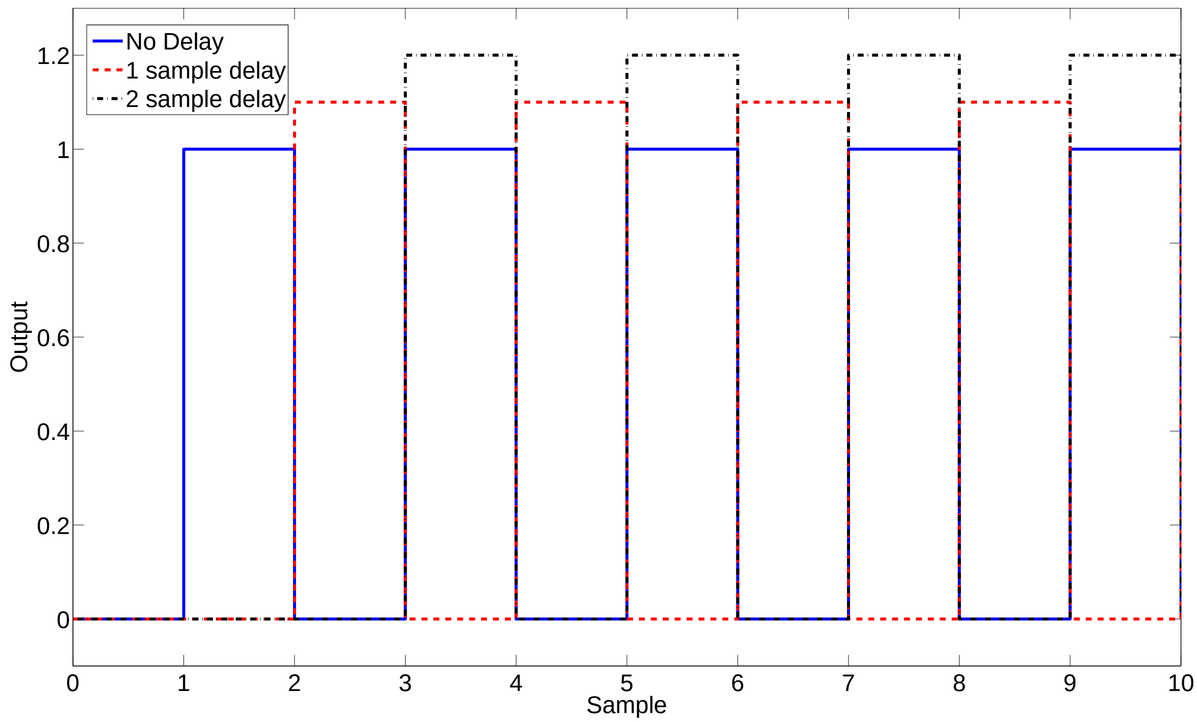

Figure 3.3: Step response of a discrete-time plant with a pole at -1 for various delays.

negative, real unstable pole. For example, in Figure 3.3, we plot the step response of the plant

$$G[z] = \frac{1}{z+1}$$

along with the step response for a one sampled delay version $G[z]z^{-1}$ and a two sampled delayed version, $G[z]z^{-2}$ (note that we introduce scaling factors of 1.1 and 1.2 respectively to separate the various step responses shown in Figure 3.3). It is easy to see that when the delay is odd that the plant output is completely out of phase with the nominal (un-delayed) version of the plant, whereas when the delay is even, the only difference is the initial startup period, after which, the response is exactly in phase. So, intuitively, stabilizing $G[z]$ and $G[z]z^{-1}$ appears more difficult than stabilizing $G[z]$ and $G[z]z^{-2}$, which is borne out by the results of Theorem 3.1 and Theorem 3.3.

3.5 Conclusions and Future Work

In this chapter, we set up the discrete-time delay margin problem, noted that it is a classical simultaneous stabilization problem, and using a classical result in that area, we determined when the achievable discrete-time delay margin is non-zero. We were also able to show that the most obvious extension of the simultaneous stabilization results to cases with longer delays provides no further insight into the problem. The key result of this chapter, namely Theorem 3.1, was reported in [17].

Since necessary and sufficient conditions for the general n-plant simultaneous stabilization problem has been proven to be rationally undecidable [2], we abandoned this approach in favour of trying to apply the tools developed in [35] for the continuous-time problem to the discrete-time (and sampled data) problems. Unfortunately, while these tools provide great insight into the possible solution for the problem, we have been unable to prove anything using them; a brief description of this work is placed in the penultimate chapter.

As a result of the difficulties encountered with solving the achievable delay margin problem in both the discrete-time and sampled data settings, in the next three chapters, we turn our attention to a different problem, that of stabilizing a plant with an arbitrarily large uncertain time delay and an arbitrarily large uncertain gain. However, we would still like to solve the discrete-time achievable delay margin problem, and it is an area for future research.

Chapter 4

Gain and Delay Margin - Adaptive Control

In this chapter, we formally state the problem of stabilizing any SISO LTI continuous-time plant with an arbitrarily long unknown delay and unknown gain. Unlike in the next two chapters where we must know the sign of the unknown gain a priori, in this chapter, we merely require an upper and lower bound on the magnitude of the unknown gain and require no knowledge of its sign. To solve this problem, we propose the use of a simple generalized hold together with a sampler which, when applied to a continuous-time plant, converts the problem into a classical discrete-time adaptive control problem. Unfortunately, due to the lack of convexity of the set of admissible discretized plant models, the simplest adaptive controllers (for example, a classical pole placement adaptive controller) can not guarantee any form of stability, but more advanced adaptive controllers can; to that end, we will prove that the output asymptotically approaches zero through the use of our hold, a sampler and an adaptive controller proposed in [30].

This combined gain and delay margin problem has only one current solution in the literature, [5, 4], which considers a linearly parameterized uncertainty set in the A and B state space matrices along with an unknown but upper bounded time delay; however, that controller is infinite dimensional and does not consider noise. While we don't consider noise in this chapter, we achieve a similar stability result to [5, 4], namely asymptotic stability, except we do so using a finite dimensional controller. In Chapters 5 and 6, using a nearly identical generalized hold, we design finite dimensional controllers which provide a much stronger notion of stability, namely BIBO stability.

Since we will be using off the shelf adaptive controllers in this chapter (in particular the adaptive controller from [30]), we require a forward (and backward) shift operator. To this end, following the notation of [18] we define q as the forward shift operator and q^{-1} as the backward shift operator, i.e., $q\nu[k] = \nu[k+1]$ and $q^{-1}\nu[k] = \nu[k-1]$.

This chapter is organized as follows. In Section 4.1 we formally state the problem. In Section 4.2, we give our generalized hold and sampler. In Section 4.3 we discretize the plant and show that the resulting discrete-time model is observable and controllable for all non-pathological sampling periods. In Section 4.4 we state the adaptive controller and show that it weakly stabilizes the system, in Section 4.5 we provide some simulations and finally, in Section 4.6 we provide some conclusions and discuss some future work.

4.1 The Problem

Our SISO LTI plant G is described by

$$\left.\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + gBu(t-\tau) \\ y(t) & = & Cx(t), \end{array}\right\} \tag{4.1}$$

with $x(t) \in \mathbf{R}^n$ the plant state, $u(t) \in \mathbf{R}$ the plant input, and $y(t) \in \mathbf{R}$ the plant output. Here, τ represents an uncertain delay which lies in $[0, \overline{\tau}]$ and g an uncertain gain which lies in $[-\overline{g}, -1] \cup [1, \overline{g}]$.[1] Because of the presence of the delay, the initial condition of the plant is not only on the state, but also on the input; more specifically, the plant initial conditions are $x(0) = x_0$ and $u(\theta) = u_0(\theta)$ for $\theta \in [-\overline{\tau}, 0)$. For simplicity of exposition, in this chapter we assume that $u_0(\theta) = 0$ for $\theta \in [-\overline{\tau}, 0)$;[2] we remove this restriction when we return to this problem in Chapters 5 and 6. We also assume that (A, B) is controllable and that (C, A) is observable. We define $G_0(s) := C(sI - A)^{-1}B$, which is the plant transfer function with no delay and no extra gain. Our set of uncertainty is

$$\mathcal{G} := \left\{ ge^{-s\tau}G_0(s) : \;\; \tau \in [0, \overline{\tau}], g \in [-\overline{g}, -1] \cup [1, \overline{g}] \right\}.$$

In this chapter, we consider a weak notion of stability: for each $G \in \mathcal{G}$, we have that there are no unbounded signals and that

$$\{u(t), y(t), x(t)\} \to 0 \quad \text{as} \quad t \to \infty.$$

If we design a controller K which achieves this, then we say that K *weakly stabilizes* $\mathcal{G}$. The goal of this chapter is to design a controller K which weakly stabilizes $\mathcal{G}$.

[1] We can always write the unknown gain interval in this form by making the positive and negative intervals symmetric and then absorbing the magnitude of the lower bound into B.

[2] We do this to avoid having a different discretization for the first period of the system.

4.2 The Sampler and Hold

Our control scheme relies on using a generalized 'pulse' hold as shown in Figure 4.1 b). In contrast to the normal zero order hold shown in Figure 4.1 a) which outputs a constant value during each period, the hold we propose does the same over the first part of the period, but then outputs zero for the remainder of the period. As a result, our hold depends on two quantities of time, $T_2 > 0$ and $T_3 > \overline{\tau}$,[3] with an overall period of $T = T_2 + T_3$ seconds. The quantity $T_2 > 0$ denotes the length of time the hold is 'on', and can be chosen freely, although there is an inherent trade-off between the performance of the system, which depends on the overall period T, and the size of the control signal, which grows as $T_2 \to 0$. The quantity $T_3 > \overline{\tau}$ must be chosen to be longer than the maximum possible delay; this is done so that during each period, the plant receives exactly one pulse regardless of $\tau \in [0, \overline{\tau}]$; this eliminates any dependence on the delay in the discretized A matrix that occurs when using a normal zero order hold with a time delay, e.g., [1] or equations (7.7)-(7.8). Finally, we must impose a restriction on the overall period T; namely, that T is non-pathological[4] with respect to A. Since $T_2 > 0$ and $T_3 > \overline{\tau}$ are otherwise free and since A has a finite number of eigenvalues, we can always choose T_2 and T_3 such that $T = T_2 + T_3$ is non-pathological with respect to A.

We define the hold $H : \ell(\mathbf{R}) \to PC(\mathbf{R})$ by

$$(H\nu)(t) := \begin{cases} \nu[k] & t \in [kT, kT + T_2] \\ 0 & t \in (kT + T_2, kT + T) \end{cases} \tag{4.2}$$

for $k \geq 0$ and the sampler $S : PC(\mathbf{R}) \to \ell(\mathbf{R})$ by

$$(Sy)[k] := y(kT), \quad k \geq 0. \tag{4.3}$$

4.3 Discretizing the Plant

We would now like to discretize the plant (4.1) using the sampler (4.3) and hold (4.2) for $k \geq 0$; doing so yields:

[3] We adopt this unusual notation in preparation for the next chapter where we use a similar hold which requires an additional quantity of time (to be labeled T_1) at the beginning of each period.

[4] See Section 2.3 for a definition of pathological sampling.

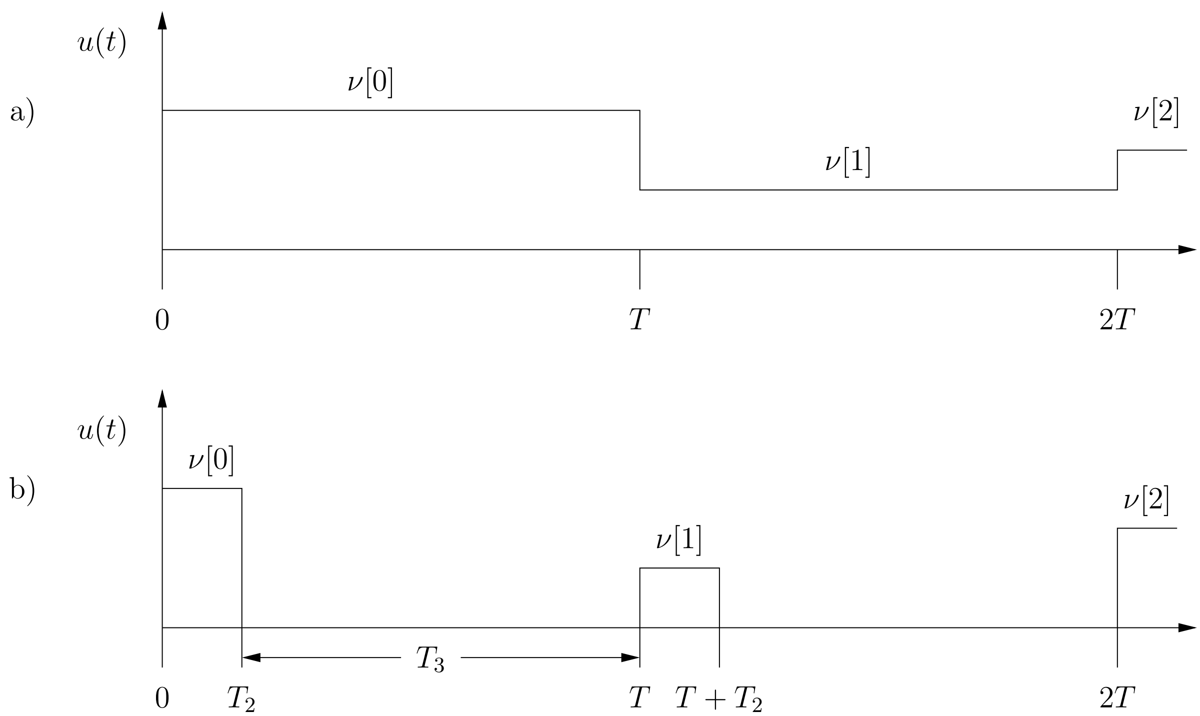

Figure 4.1: Showing the output of a) a normal zero order hold, b) the ‘pulse’ hold used in this chapter.

$$\left.\begin{aligned}\underbrace{x(kT+T)}_{=:\chi[k+1]} &= \underbrace{e^{AT}}_{=:A_d}\underbrace{x(kT)}_{=:\chi[k]} + \underbrace{\left[e^{AT}e^{-A\tau}\int_0^{T_2} e^{-Av}gBdv\right]}_{=:B_d(\tau,g)}\nu[k] \\ \underbrace{y_n(kT)}_{=:\psi[k]} &= Cx(kT),\end{aligned}\right\} \tag{4.4}$$

with an initial condition $\chi[0] = x_0$. We will also require a representation of the discretized plant using our shift operator q^{-1}, so to this end, we define

$$G_{(\tau,g)}[q^{-1}] := q^{-1}C(I - q^{-1}A_d)B_d(\tau, g). \tag{4.5}$$

We know by assumption that (A, B) is controllable and (C, A) is observable, but is $(A_d, B_d(\tau, g))$ controllable and (C, A_d) observable? Before answering this question, first consider what happens when we apply a zero-order-hold with period T, as shown in Figure 4.1 a), to the plant (4.1) with $g = 1$ and $\tau = 0$.

Proposition 4.1. *[11] If T is non-pathological with respect to (4.1), then (C, e^{AT}) is observable and $(e^{AT}, \int_0^T e^{Av}Bdv)$ is controllable.*

For our case, since T was chosen so that it is non-pathological with respect to A, it clearly follows from Proposition 4.1 that (C, A_d) is observable. However, we must perform more analysis to determine if $(A_d, B_d(\tau, g))$ is controllable for all $\tau \in [0, \overline{\tau}]$ and $g \in [-\overline{g}, -1] \cup [1, \overline{g}]$; we do so in the following lemma:

Lemma 4.1. *$(A_d, B_d(\tau, g))$ is controllable and (C, A_d) is observable for all $\tau \in [0, \overline{\tau}]$ and $g \in [-\overline{g}, -1] \cup [1, \overline{g}]$.*

Proof. Let $\tau \in [0, \overline{\tau}]$ and $g \in [-\overline{g}, -1] \cup [1, \overline{g}]$ be arbitrary, and recall that T was chosen to be non-pathological with respect to A. Since τ and g are fixed, to simplify notation in the proof, we write B_d instead of $B_d(\tau, g)$.

Since $A_d = e^{AT}$, and T is non-pathological, clearly by Proposition 4.1 (C, A_d) is observable.

For controllability, we prove the result using the Jordan form of the original continuous-time system. Since (A, B) (and hence (A, gB)) is controllable by hypothesis and u is scalar valued, there must be only one Jordan Block per eigenvalue. Let the $\overline{n}$ distinct eigenvalues of A be denoted by $\lambda_1, \lambda_2, \cdots, \lambda_{\overline{n}}$, with λ_i having multiplicity m_i for each $i = 1, 2, \cdots, \overline{n}$. Then there exists an invertible matrix $X \in \mathbf{C}^{n\times n}$ such that

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & J_{\overline{n}} \end{bmatrix} = X^{-1}AX,$$

with each $J_i \in \mathbf{C}^{m_i\times m_i}$ of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in \mathbf{R}^{m_i\times m_i}.$$

Applying the similarity transform X to (4.1) and defining $\tilde{x} := X^{-1}x$ yields

$$\begin{aligned} \dot{\tilde{x}}(t) &= X^{-1}AX\tilde{x}(t) + X^{-1}gBu(t-\tau) \\ y(t) &= CX\tilde{x}(t). \end{aligned}$$

It will be useful to segment $X^{-1}gB$, so to that end, with $\hat{B}_i \in \mathbf{C}^{m_i}$ we can write

$$X^{-1}gB = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \vdots \\ \hat{B}_{\overline{n}} \end{bmatrix};$$

furthermore, we can segment B_i: for each $i = 1, 2, \cdots, \overline{n}$, we write

$$\hat{B}_i = \begin{bmatrix} \hat{b}_{i1} \\ \hat{b}_{i2} \\ \vdots \\ \hat{b}_{im_i} \end{bmatrix}.$$

Since (A, B) is controllable, it follows from the Popov-Belevitch-Hautus (PBH) test that each $\hat{b}_{im_i}$, $i = 1, 2, \cdots, \overline{n}$ is non-zero.

Now define

$$\tilde{A}_d := e^{JT}$$

and

$$\tilde{B}_d := \int_0^{T_2} e^{J(T-\tau-v)} g X^{-1} B dv;$$

clearly (A_d, B_d) is controllable if and only if $(\tilde{A}_d, \tilde{B}_d)$ is controllable. Examining $\tilde{A}_d$ more closely yields

$$\tilde{A}_d = \begin{bmatrix} e^{J_1 T} & 0 & \cdots & 0 \\ 0 & e^{J_2 T} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{J_{\overline{n}} T} \end{bmatrix},$$

with

$$e^{J_i T} = \begin{bmatrix} e^{\lambda_i T} & T e^{\lambda_i T} & \frac{T^2}{2!} e^{\lambda_i T} & \cdots & \frac{T^{m_i - 1} e^{\lambda_i T}}{(m_i - 1)!} \\ 0 & e^{\lambda_i T} & T e^{\lambda_i T} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & e^{\lambda_i T} & T e^{\lambda_i T} \\ 0 & 0 & \cdots & 0 & e^{\lambda_i T} \end{bmatrix}.$$

It will also be useful to segment $\tilde{B}_d$, which we do in the natural way: with $\tilde{B}_i \in \mathbf{C}^{m_i}$ for each $i = 1, 2, \cdots, \overline{n}$, we write

$$\tilde{B}_d = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \vdots \\ \tilde{B}_{\overline{n}} \end{bmatrix};$$

we further partition $\tilde{B}_i$: for each $i = 1, 2, \cdots, \overline{n}$ we write

$$\tilde{B}_i = \begin{bmatrix} \tilde{b}_{i1} \\ \tilde{b}_{i2} \\ \vdots \\ \tilde{b}_{im_i} \end{bmatrix}.$$

Since T is non-pathological with respect to A, the eigenvalues of each $e^{J_i T}$, namely $e^{\lambda_1 T}$, $e^{\lambda_2 T}$, $\cdots$, $e^{\lambda_{\overline{n}} T}$ differ from each other. Furthermore, since $\tilde{A}_d$ is block diagonal, $(\tilde{A}_d, \tilde{B}_d)$ is controllable if and only if every $(e^{J_i T}, \tilde{B}_i)$ is controllable for every $i = 1, 2, \cdots, \overline{n}$.

Using the PBH test, it follows for each $i = 1, 2, \cdots, \overline{n}$ that

$$(e^{J_iT}, \tilde{B}_i) \text{ is controllable} \qquad \Leftrightarrow \qquad \text{rank}[\lambda_i I - e^{J_iT} \ \ \tilde{B}_i] = m_i.$$

Since

$$e^{\lambda_i T} I - e^{J_iT} = \begin{bmatrix} 0 & -Te^{\lambda_i T} & \frac{-T^2}{2!}e^{\lambda_i T} & \cdots & \frac{-T^{m_i-1}e^{\lambda_i T}}{(m_i-1)!} \\ 0 & 0 & -Te^{\lambda_i T} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & -Te^{\lambda_i T} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

clearly has rank $m_i - 1$ with an empty bottom row, it follows that $\text{rank}[\lambda_i I - e^{J_iT} \ \ \tilde{B}_i] = m_i$ if and only if $\tilde{b}_{im_i}$ is non-zero. Therefore, $(\tilde{A}_d, \tilde{B}_d)$ is controllable if and only if $\tilde{b}_{im_i}$ is non zero for every $i = 1, 2, \cdots \overline{n}$.

For $\lambda_i \neq 0$, J_i is invertible, and

$$\tilde{B}_i = -e^{J_iT} e^{-J_i\tau}(e^{-J_iT_2} - I)J_i^{-1} g B_i. \tag{4.6}$$

We are interested in $\tilde{b}_{im_i}$, which is the bottom entry of $\tilde{B}_i$. We can compute this easily since all the matrices in (4.6) are upper triangular, so $\tilde{b}_{im_i}$ is merely the product of all the bottom right entries, namely,

$$\tilde{b}_{im_i} = -e^{\lambda_i T} e^{-\lambda_i \tau}(e^{-\lambda_i T_2} - I)\lambda_i^{-1} g \hat{b}_{im_i},$$

which is non-zero since T and T_2 are greater than zero and finite, $\hat{b}_{im_i} \neq 0$ and $g \in [-\overline{g}, -1] \cup [1, \overline{g}]$.

For $\lambda_i = 0$, J_i is not invertible, so we are interested in the bottom term ($\tilde{b}_{im_i}$) of

$$\tilde{B}_i = -e^{J_iT} e^{-J_i\tau} \int_0^{T_2} e^{-J_i v} dv g B_i,$$

which is exactly

$$\tilde{b}_{im_i} = -e^{\lambda_i T} e^{-\lambda_i \tau} T_2 g \hat{b}_{im_i},$$

which again is clearly non-zero since T and T_2 are greater than zero and finite, $\hat{b}_{im_i} \neq 0$, and $g \in [-\overline{g}, -1] \cup [1, \overline{g}]$. $\square$

4.4 The Controller

With our discretized plant being controllable and observable for all $\tau \in [0, \overline{\tau}]$ and $g \neq 0$, we now want to design an adaptive controller K which weakly stabilizes $\mathcal{G}$. To do so, we will use the adaptive controller from Kreisselmeier [30], and for simplicity, we adopt the notation from that work. We will discuss why we chose the Kreisselmeier [30] controller after showing that for each pair $(\tau, g) \in [0, \overline{\tau}] \times [-\overline{g}, 1] \cup [1, \overline{g}]$, the resulting discretized plant satisfies the assumptions required by [30]:

Lemma 4.2. *For every $\tau \in [0, \overline{\tau}]$ and $g \in [-\overline{g}, -1] \cup [1, \overline{g}]$ we have that there exists polynomials*

$$
\begin{aligned}
A^*[q^{-1}] &= 1 + a_1^* q^{-1} + a_2^* q^{-2} + \cdots + a_n^* q^{-n} \\
B^*[q^{-1}] &= b_0^* + b_1^* q^{-1} + b_2^* q^{-2} + \cdots + b_n^* q^{-n}
\end{aligned}
$$

such that

$$
G_{(\tau,g)}[q^{-1}] = \frac{B^*[q^{-1}]}{A^*[q^{-1}]},
$$

and $q^n A^*[q^{-1}]$ and $q^n B^*[q^{-1}]$ are coprime.

Proof. Let $\tau \in [0, \overline{\tau}]$ and $g \in [-\overline{g}, -1] \cup [1, \overline{g}]$ be arbitrary.

From Lemma 4.1, it follows immediately that $(A_d, B_d(\tau, g))$ is controllable and (A_d, C) is observable. Hence, the poles of $G_{(\tau,g)}[q]$ are exactly the eigenvalues of A_d, and the zeros of $G_{(\tau,g)}[q]$ must not be eigenvalues of A_d. So, $A^*[q^{-1}]$ and $B^*[q^{-1}]$ are coprime, as desired. □

Lemma 4.2 shows that after applying our sampler and hold that we have a very natural problem for adaptive control. Unlike most adaptive control problems, we actually know the coefficients of the polynomial $A^*[q^{-1}] = q^{-1} \det[I - q^{-1} e^{AT}]$, so we only have a maximum of $n+1$ unknown parameters (as opposed to $2n+1$ for the general problem). Unfortunately, since the general problem contains plants that are non-minimum phase (and whose resulting discretizations are also non-minimum phase), for the general problem we can not use a model reference adaptive controller (MRAC), though this does not rule out the use of a classical pole placement adaptive controller. Furthermore, from Lemma 4.2 we know that for all $\tau \in [0, \overline{\tau}]$ and $g \in [-\overline{g}, -1] \cup [1, \overline{g}]$ that $A^*[q^{-1}]$ and $B^*[q^{-1}]$ are coprime; however, in general we can not construct a convex set which contains these models and for which

$A^*[q^{-1}]$ and $B^*[q^{-1}]$ are always coprime; recall that most classical adaptive algorithms require this in order to carry out parameter projections with the desirable features needed to prove that the controller works. To see our convexity problem, consider the plant

$$\left.\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(t) + g \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t-\tau) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}\right\} \tag{4.7}$$

with $\overline{g} = 10$ and $\overline{\tau} = 8.5$. The nominal version of this plant has a continuous-time transfer function given by

$$G_0(s) = \frac{s+1}{s^2+1},$$

and a resulting uncertainty set given by[5]

$$\mathcal{G} = \left\{ g e^{-s\tau} G_0(s) : \ \tau \in [0, 8.5], g \in [1, 10] \right\}.$$

With this uncertainty set, we set $T_2 = 0.05$ and $T_3 = 8.55$, and apply the hold (4.2) to various plants in $\mathcal{G}$; in particular, we vary the delay from zero to 8.5 with the gain at its lower and upper bounds. In Figure 4.2, we plot the resulting values for the discretized B matrix, namely

$$B_d = \begin{bmatrix} B_{d_1} \\ B_{d_2} \end{bmatrix},$$

with the grey region denoting the possible plant models. Clearly the smallest convex set which contains the grey region includes the origin, which corresponds to $B_d = 0$, so the corresponding $B^*[q^{-1}] = 0$, which means that $A^*[q^{-1}]$ and $B^*[q^{-1}]$ are not coprime.[6]

Because of the lack of a convex set of plants for which $A^*[q^{-1}]$ and $B^*[q^{-1}]$ are coprime, we must use an approach which does not require convexity. We adopt the approach of Kreisselmeier (see [30], [28], and [29]) wherein the controller is guaranteed to work without requiring any projections of the plant parameter estimates. For simplicity of proving the result, we will use the first controller of this type, the one presented in [30]; furthermore, while it should be possible to exploit our knowledge of the polynomial $A^*[q^{-1}]$ to improve performance, we will not exploit this knowledge here except in setting up the initial conditions on the plant estimator.

The controller topology from [30] is shown in Figure 4.3. Each of the four controller blocks will be described from right to left, starting with the two adaptive identifiers; the

[5] We restrict the gain to be positive only to aid in illustrating the convexity problem.

[6] It should be noted that if the plant only has real eigenvalues, then this problem does not occur.

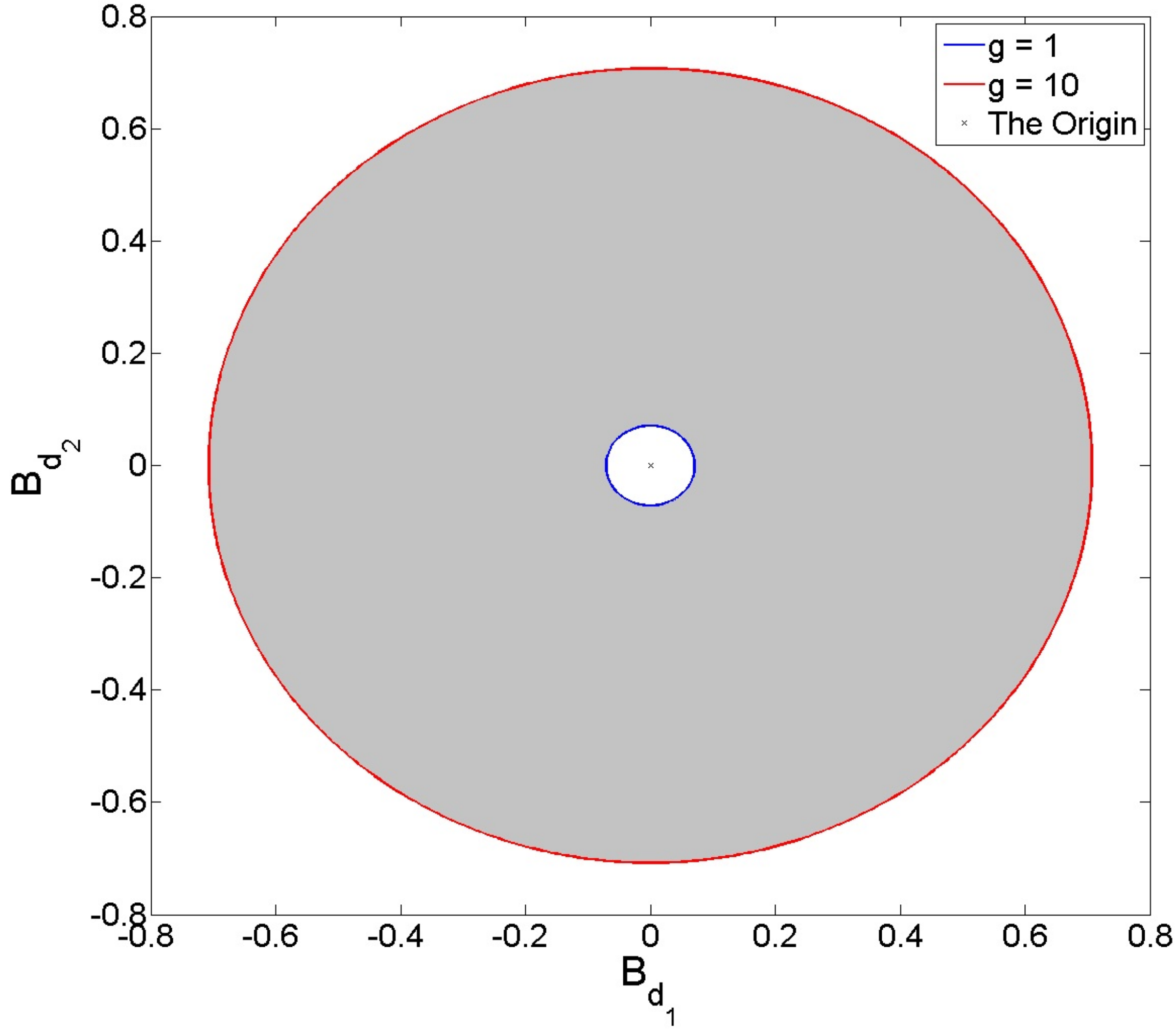

Figure 4.2: Showing the convexity problem that we have.

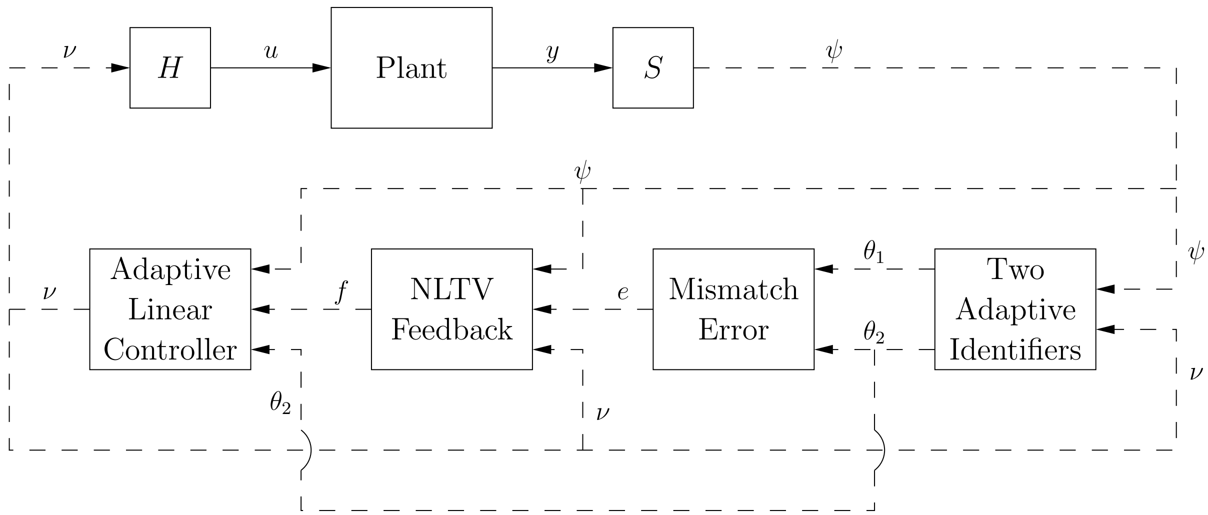

Figure 4.3: The controller and feedback setup of [30].

plant identification is described in Section 4.4.1 and the controller identification is described in Section 4.4.2. The mismatch error is then described in Section 4.4.3, followed by both the NLTV and linear control components in Section 4.4.4. Our overall control scheme is then given by those components, along with the hold (4.2) and sampler (4.3) given earlier in this chapter. To present this controller here, we adopt the notation and verbiage from [30].

4.4.1 Identification of the Plant

We start by defining a vector of the actual coefficients of $A^*[q^{-1}]$ and $B^*[q^{-1}]$:

$$\theta_1^* := \begin{bmatrix} -b_0^* \\ -b_1^* \\ \vdots \\ -b_n^* \\ a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{bmatrix};$$

clearly, the goal is to design an estimation scheme that estimates θ_1^*. To do so, at each time step we use information from the current and past plant input ν and output ψ; this requires us to use information before time 0 which we do not know, so for the purposes of setting up the controller, for $k < 0$ we replace $\nu[k]$ and $\psi[k]$ with zero.[7] We then define the following vector for $k \geq 0$:

$$v_1[k] := \begin{bmatrix} \nu[k] \\ q^{-1}\nu[k] \\ \vdots \\ q^{-n}\nu[k] \\ q^{-1}\psi[k] \\ q^{-2}\psi[k] \\ \vdots \\ q^{-n}\psi[k] \end{bmatrix}, \tag{4.8}$$

and note that

$$v_1[k]\theta_1^* + \psi[k] = 0.$$

With $a_i[k]$ an estimate of a_i^* and $b_i[k]$ an estimate of b_i^*, we define our estimate of θ_1^* for $k \geq 0$:

$$\theta_1[k] := \begin{bmatrix} -b_0[k] \\ -b_1[k] \\ \vdots \\ -b_n[k] \\ a_1[k] \\ a_2[k] \\ \vdots \\ a_n[k] \end{bmatrix}. \tag{4.9}$$

At each time step, and with an arbitrary initial condition on the estimate, i.e., $\theta_1[0] \in \mathbf{R}^{2n+1}$, we update these estimates recursively for $k \geq 0$ using

$$\theta_1[k+1] = \theta_1[k] - \frac{v_1[k]\left(v_1'[k]\theta_1[k] + \psi[k]\right)}{1 + v_1'[k]v_1[k]}. \tag{4.10}$$

For the initial condition on θ_1, since we know $A^*[q^{-1}]$ exactly, it is a reasonable idea to set the corresponding initial conditions of θ_1 accordingly, i.e., $a_1[0] = a_0^*$, etc.

[7] These are not the actual plant initial conditions; we choose these values so that we can run the algorithm starting from time zero.

4.4.2 Identification of the Controller

Here we list the required parts of the controller; for more insight into how this section of the controller works, the reader is encouraged to look at [30, 10]. Let

$$P[q^{-1}] := 1 + p_1 q^{-1} + p_2 q^{-2} + \cdots + p_{2n} q^{-2n}$$

be an arbitrary polynomial such that $q^{2n}P[q^{-1}]$ has all its zeros in the open unit disk. Since $A^*[q^{-1}]$ and $B^*[q^{-1}]$ are coprime, it follows that there exists unique polynomials R^*, S^*, C^* and D^*:

$$\begin{aligned}
R^*[q^{-1}] &:= 1 + r_1^* q^{-1} + r_2^* q^{-2} + \cdots + r_n^* q^{-n}, \\
S^*[q^{-1}] &:= s_1^* q^{-1} + s_2^* q^{-2} + \cdots + s_n^* q^{-n}, \\
C^*[q^{-1}] &:= c_1^* q^{-1} + c_2^* q^{-2} + \cdots + c_n^* q^{-n}, \\
D^*[q^{-1}] &:= d_1^* q^{-1} + d_2^* q^{-2} + \cdots + d_n^* q^{-n},
\end{aligned}$$

such that

$$\begin{aligned}
P &= A^*R^* - B^*S^*, \\
q^{-2n} &= A^*C^* + B^*D^*.
\end{aligned}$$

Our goal is to estimate the coefficients of the polynomials R^*, S^*, C^* and D^* using information from the plant input and output; this again requires us to use information before time zero which we do not know, so for the purposes of setting up the controller, for $k < 0$ we again replace $\nu[k]$ and $\psi[k]$ with zero. Define for $k \geq 0$:

$$\upsilon_2[k] := \begin{bmatrix} q^{-2n}\nu[k] \\ \vdots \\ q^{-3n+1}\nu[k] \\ q^{-2n}\psi[k] \\ \vdots \\ q^{-3n+1}\psi[k] \\ P\nu[k] \\ \vdots \\ q^{-n+1}P\nu[k] \\ P\psi[k] \\ \vdots \\ q^{-n+1}P\psi[k] \end{bmatrix}, \tag{4.11}$$

and

$$\theta_2^* := \begin{bmatrix} r_1^* \\ \vdots \\ r_n^* \\ -s_1^* \\ \vdots \\ -s_n^* \\ -c_1^* \\ \vdots \\ -c_n^* \\ -d_1^* \\ \vdots \\ -d_n^* \end{bmatrix}.$$

We would like to estimate θ_2^*; to that end, we define estimates of θ_2^*:

$$\theta_2 := \begin{bmatrix} r_1 \\ \vdots \\ r_n \\ -s_1 \\ \vdots \\ -s_n \\ -c_1 \\ \vdots \\ -c_n \\ -d_1 \\ \vdots \\ -d_n \end{bmatrix}. \tag{4.12}$$

At each time step, and with an arbitrary initial condition on the estimate, i.e., $\theta_2[0] \in \mathbf{R}^{4n}$, we update these estimates recursively for $k \geq 0$ using

$$\theta_2[k+1] = \theta_2[k] - \frac{v_2[k]\left(v_2'[k]\theta_2[k] + \nu[k-2n+1]\right)}{1 + v_2'[k]v_2[k]}. \tag{4.13}$$

4.4.3 Identification of the Mismatch Error

To avoid dealing with polynomials in the shift operator q^{-1}, we replace it by the complex variable λ and define the following polynomials in λ:

$$\begin{aligned} A(k,\lambda) &:= 1 + a_1[k]\lambda + a_2[k]\lambda^2 + \cdots + a_n[k]\lambda^n, \\ B(k,\lambda) &:= b_0[k] + b_1[k]\lambda + b_2[k]\lambda^2 + \cdots + b_n[k]\lambda^n, \\ R(k,\lambda) &:= 1 + r_1[k]\lambda + r_2[k]\lambda^2 + \cdots + r_n[k]\lambda^n, \\ S(k,\lambda) &:= s_1[k]\lambda + s_2[k]\lambda^2 + \cdots + s_n[k]\lambda^n, \end{aligned}$$

and let

$$\begin{aligned} Q(k,\lambda) &:= A(k,\lambda)R(k,\lambda) - B(k,\lambda)S(k,\lambda) - P(\lambda) \\ &= \sum_{i,j=0}^{n} \left(a_i[k]r_j[k] - b_i[k]s_j[k]\right)\lambda^{i+j} - \sum_{i=0}^{2n} p_i\lambda^i \\ &=: \sum_{i=1}^{2n} q_i[k]\lambda^i, \end{aligned}$$

where $a_0[k] = r_0[k] = p_0 = 1$ and $s_0[k] = 0$, and then define the identification mismatch error

$$e[k] := \left(\sum_{i=1}^{2n}(q_i[k])^2\right)^{\frac{1}{2}}. \tag{4.14}$$

4.4.4 The Control Law

The control law is given by

$$R(k,q^{-1})\nu[k] = S(k,q^{-1})\psi[k] + f(m[k],e[k],k) \tag{4.15}$$

$$m[k+1] = \sigma m[k] + |\nu[k]| + |\psi[k]| \tag{4.16}$$

where $m[0] \geq 0$ and $\sigma \in (0,1)$ are arbitrary and

$$f(m[k],e[k],k) := \begin{cases} (1+m[k])e[k] & \text{when } k = 0, 4n, 8n, \cdots \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

Our controller K is then given by the sampler (4.3), the adaptive identification of the plant given by (4.8), (4.9) and (4.10), the adaptive identification of the controller parameters given by (4.11), (4.12) and (4.13), the identification mismatch error given by (4.14), the control law given by (4.15), (4.16) and (4.17), and the the hold (4.2) as shown in Figure 4.3. This control scheme has initial conditions given by $\theta_1[0]$, $\theta_2[0]$, $m[0]$, and $\nu[-1] = \nu[-2] = \cdots = \nu[-3n+1] = \psi[-1] = \psi[-2] = \cdots = \psi[-3n+1] = 0$.

4.4.5 The Main Result

Theorem 4.1. *For the plant given by* (4.1) *and using the controller K in the feedback structure shown in Figure 4.3, we have for each $G \in \mathcal{G}$ that*

$$\sup_{k\geq 0} \|\nu[k]\| < \infty,$$
$$\sup_{k\geq 0} \|\psi[k]\| < \infty,$$
$$\sup_{k\geq 0} \|\theta_1[k]\| < \infty,$$
$$\sup_{k\geq 0} \|\theta_2[k]\| < \infty,$$
$$\sup_{k\geq 0} \|m[k]\| < \infty,$$

and asymptotic regulation is achieved, i.e.

$$\{u(t), y(t), x(t)\} \to 0 \qquad \text{as } t \to \infty.$$

Proof. Let $\tau \in [0, \overline{\tau}]$, $g \in [-\overline{g}, -1] \cup [1, \overline{g}]$ and $x_0 \in \mathbf{R}^n$ be arbitrary.

By Lemma 4.1, the discrete-time plant is observable and controllable with order n. Since the plant order n is known and the controller is identical to that of [30], using the result of Lemma 4.2 our problem satisfies the requirements required by [30], so we can apply the main theorem from that work: for arbitrary initial conditions, the trajectory

$$\{\nu[k], \psi[k], \theta_1[k], \theta_2[k], m[k]\}$$

of the adaptive control system is bounded uniformly in time, i.e.,

$$\sup_{k\geq 0} \|\nu[k]\| < \infty,$$
$$\sup_{k\geq 0} \|\psi[k]\| < \infty,$$
$$\sup_{k\geq 0} \|\theta_1[k]\| < \infty,$$
$$\sup_{k\geq 0} \|\theta_2[k]\| < \infty,$$
$$\sup_{k\geq 0} \|m[k]\| < \infty,$$

and asymptotic regulation is achieved on the discrete-time signals, i.e.

$$\{\nu[k], \psi[k]\} \to 0 \qquad \text{as } k \to \infty.$$

We now need to show that the continuous-time signals also go to zero as t gets large. From the hold (4.2), it clearly follows that since $\nu[k] \to 0$ as $k \to \infty$, that $u(t) \to 0$ as $t \to \infty$. Furthermore, since (C, A_d) is observable, it follows that

$$\begin{bmatrix} \psi[k] \\ \psi[k+1] \\ \vdots \\ \psi[k+n-1] \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA_d \\ \vdots \\ CA_d^{n-1} \end{bmatrix}}_{=:\mathcal{O}} \chi[k] + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ CB_d & 0 & \cdots & 0 \\ CA_dB_b & CB_d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA_d^{n-2}B_d & CA_d^{n-3}B_d & \cdots & CB_d \end{bmatrix} \begin{bmatrix} \nu[k] \\ \nu[k+1] \\ \vdots \\ \nu[k+n-2] \end{bmatrix};$$

since $\{\nu[k], \psi[k]\} \to 0$ as $k \to \infty$, and $\mathcal{O}$ is invertible, it follows that $\chi[k] \to 0$ as $k \to \infty$. Solving the continuous-time plant state equation (4.1) with a starting time of kT, it follows that

$$\|x(t)\| \leq e^{\|A\|t}\|\chi[k]\| + \left\| \int_{kT}^{t} e^{A(t-v)}Bdv \right\| |\nu[k]|, \quad t \in [kT, kT+T);$$

since $\{\nu[k], \chi[k]\} \to 0$ as $k \to \infty$, it follows that $x(t) \to 0$ as $t \to \infty$. Finally, since

$$y(t) = Cx(t),$$

it follows that $y(t) \to 0$ as $t \to \infty$. $\square$

Remark 1. *Theorem 4.1 states that we can provide (4.1) with an arbitrarily large delay margin, and an arbitrarily large gain margin with no requirement that the sign of the gain be known. However, BIBO stability is not proven.*

4.5 Simulations

We simulate the above controller for the plant

$$G_0 = \frac{1}{s-1},$$

with the uncertainty set

$$\mathcal{G} := \left\{ g e^{-s\tau} G_0(s) : \ \ \tau \in [0, 0.3], g \in [-20, -1] \cup [1, 20] \right\},$$

with a plant initial condition of $y(0) = 1$, and with controller initial conditions of $\psi(-1) = \psi(-2) = \nu(-1) = \nu(-2) = 0$, $\theta_1(0) = 0$ and $\theta_2(0) = 0$. We set $T_2 = 0.1$ and $T_3 = 0.35$ for an overall period $T = 0.45$, and then run the simulation with $\tau = 0.2$ and $g = -5$, with the results shown in Figure 4.4. As can be seen, despite the desired delay margin being much smaller than the achievable LTI delay margin of 2 and the lack of noise, the controller still struggles to weakly stabilize $\mathcal{G}$. Attempts to simulate this controller on higher order systems or with longer delays have for the most part failed due to numerical issues, and this controller, while theoretically sound, does not appear to work very well for our problem.

4.6 Conclusions and Future Work

In this chapter, we considered the problem of stabilizing an LTI continuous-time plant with an uncertain gain and delay. Since the continuous-time plant is infinite dimensional, trying to use classical continuous-time adaptive control is difficult. Instead, we use a simple generalized hold to convert the infinite dimensional continuous-time problem to a finite dimensional discrete-time problem of known order. Since we can not form a convex uncertainty set which contains only observable/controllable models, we need to use one of the more complicated adaptive controllers. Unfortunately, while the controller is proven to weakly stabilize the uncertain plant, the transient response is extremely poor and the algorithm is numerically sensitive. The key results of this chapter, Lemma 4.1 and Theorem 4.1 are based on work that we presented in [14].

In the next Chapter, using essentially the same hold, we will develop a new and better estimation scheme designed specifically for our problem which will allow us to estimate the two unknown parameters (τ and g) directly, which not only provides much better performance, but will also allow us to handle noise, guarantee the exponential decay of the plant initial conditions, tolerate jumps in the gain and delay, and will provide robustness to un-modelled dynamics.

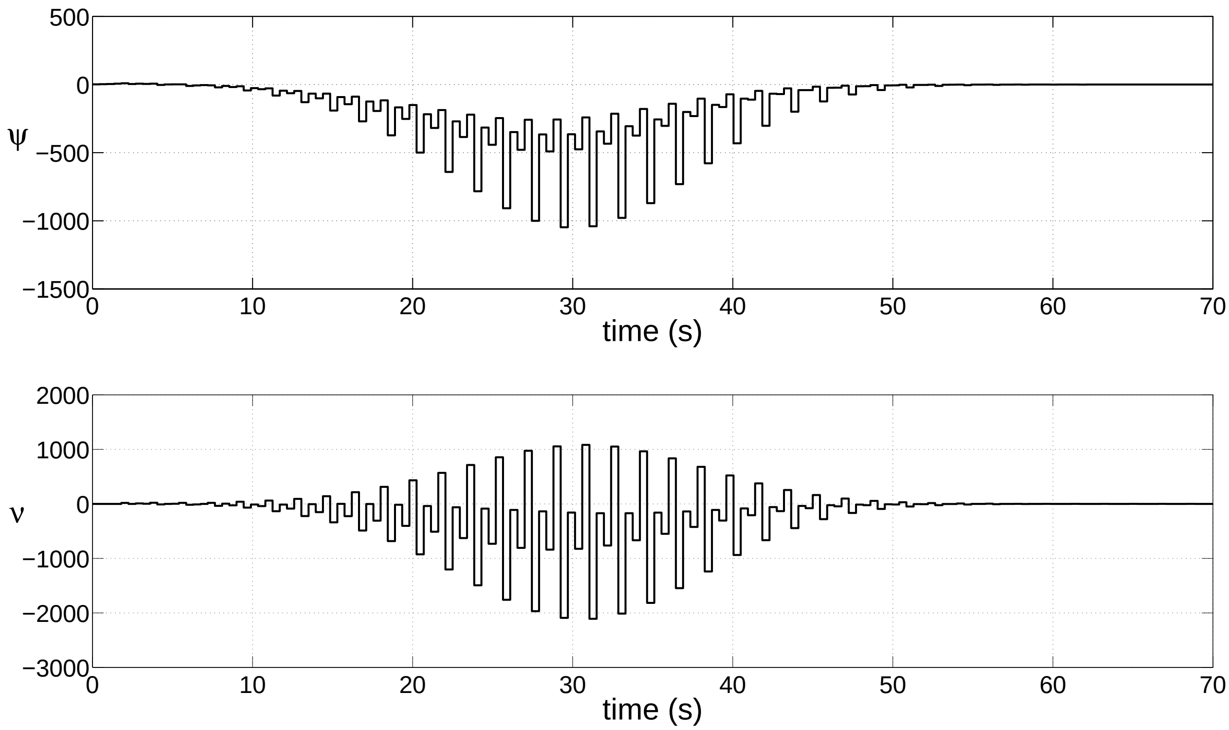

Figure 4.4: The output and control signal.

For future work, we would like to consider a larger uncertainty set, namely, uncertainty in the A and B matrices of the plant, which is the uncertainty set considered in [5, 4]. We would also like to consider different adaptive controllers in the hope of achieving better performance, especially in simulation.

Chapter 5

Stabilizing a Plant with an Arbitrarily Large Gain and Delay with a Novel Estimator

In this chapter, we consider a nearly identical problem to that of the previous chapter, namely, given a SISO LTI controllable/observable continuous-time plant and a desired gain and delay margin, find a controller which will achieve these margins. Unlike in the previous chapter, we require *a priori* knowledge of the sign of the unknown gain, but the controller we propose achieves a much stronger notion of stability; indeed, the proposed controller not only achieves BIBO stability, but it also guarantees the exponential decay of the plant initial conditions. Furthermore, in contrast to the previous chapter, the proposed controller is able to achieve uniform bounds on all important signals, can handle un-modeled dynamics and can tolerate switches in the unknown gain and delay.

The approach adopted here is motivated by the rudimentary approach to the first order case considered in our earlier work [14] and the result presented in Chapter 4. In [14], by applying an unimplementable impulse hold to a first order plant with an unknown but upper bounded time delay, we were able to derive a simple update equation which estimated the time delay at each time step. Using this estimate, we were then able to generate a control signal which guaranteed that the plant output decayed exponentially for an arbitrary initial condition with no noise. In Chapter 4 (which is also based on the work presented in [14]), we were able to use a 'pulse' hold which created a very natural discrete-time adaptive control problem; however, despite only having two unknown parameters, a standard adaptive controller requires estimates of at least n unknown parameters for an

n^{th} order plant, and due to problems with projecting, we potentially require estimates of up to $2n+1$ unknown parameters. The controller that we propose in this chapter combines these two concepts; namely, we use a 'pulse' hold to discretize the plant as done in Chapter 4, and then use that model to estimate the two unknown plant parameters directly at each time step, similar to the approach adopted in [14]. While the proposed controller that we present is (mildly) nonlinear and periodic, it has a number of desirable features:

- It has modest complexity, and mainly depends on solving very simple update equations; hence, the approach is practical to scale to systems with a high order.
- Despite being non-linear, it provides BIBO stability and guarantees the exponential decay of the plant initial conditions, even in the presence of noise.
- It is robust to un-modeled dynamics and other plant uncertainty.
- It tolerates infrequent but possibly persistent switches in the unknown gain and delay.
- It updates the estimate of the delay and gain at each time step, so it should be able to tolerate time variations in the gain and delay parameters.

This chapter is organized as follows. In Section 5.1, we formally state the problem. In 5.2, we briefly consider the first order case[1] to help provide intuition for Section 5.3, where we state our proposed controller. In Section 5.4 we prove that our proposed controller achieves an arbitrarily large delay and gain margin, that it is robust to un-modeled dynamics and that it can tolerate jumps in the unknown gain and delay. In Section 5.5, we provide many simulations, and finally in Section 5.6 we provide some conclusions and discuss some future work.

5.1 Problem Formulation

We start by solving the basic problem, i.e., no additional plant uncertainty and no jumps in the unknown gain and delay, so our SISO LTI plant G is described by

$$\left.\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + gBu(t-\tau) \\ y(t) & = & Cx(t), \end{array}\right\} \tag{5.1}$$

with $x(t) \in \mathbf{R}^n$ the plant state, $u(t) \in \mathbf{R}$ the plant input, and $y(t) \in \mathbf{R}$ the plant output. Here, τ represents an uncertain delay which lies in $[0, \overline{\tau}]$ and g an uncertain gain which lies in $[1, \overline{g}]$.[2] Because of the presence of the delay, the initial condition of the plant is not

[1]In Chapter 6 we will prove a result with a time varying delay for a first order plant.

[2]We can always write the unknown gain interval in this form by absorbing any lower gain bound and the sign into B.

only on the state, but also on the input; more specifically, the plant initial conditions are $x(0) = x_0$ and $u_d(\theta) = u_0(\theta)$ for $\theta \in [-\overline{\tau}, 0)$. We also assume that (A, B) is controllable and that (C, A) is observable and we define $G_0(s) := C(sI - A)^{-1}B$, resulting in the following uncertainty set:

$$\mathcal{G} := \left\{ ge^{-s\tau}G_0(s) : \ \tau \in [0, \overline{\tau}], g \in [1, \overline{g}] \right\}.$$

We consider the standard feedback structure: the controllers are input-output of the form

$$u = Ky.$$

To define stability, we introduce noise at the two plant/controller interfaces as shown in Figure 5.1. Due to the input delay, the input noise d has an initial condition $d(\theta) = d_0(\theta)$ for $\theta \in [-\overline{\tau}, 0)$, and we define the stacked noise vector[3]

$$\overline{w}(t) := \begin{bmatrix} w(t) \\ d(t - \overline{\tau}) \end{bmatrix}.$$

With zero initial conditions on the plant, i.e., $x_0 = 0$, $u_0(\theta) = d_0(\theta) = 0$ for $\theta \in [-\overline{\tau}, 0)$, we let $\Phi(\tau, g)$ be the closed loop map from $\begin{bmatrix} d \\ w \end{bmatrix} \to \begin{bmatrix} y \\ u \end{bmatrix}$.

Definition 5.1. *We say that K* stabilizes *$\mathcal{G}$ if $\Phi(\tau, g)$ is uniformly bounded, i.e.*

$$\sup_{\tau \in [0, \overline{\tau}], g \in [1, \overline{g}]} \|\Phi(\tau, g)\| < \infty.$$

The goal of this chapter is to design a controller K which stabilizes $\mathcal{G}$.

5.2 High Level Idea

To help provide intuition into the controller design philosophy, it is useful to briefly consider a first order plant. We will revisit this problem in the next chapter (Chapter 6) with the additional difficulty of allowing the delay to be time varying and with the explicit goal of finding a simple relationship between the unstable pole location, the upper bound on the unknown delay and the time variation of the delay for which stability is maintained. For now, we will simply provide an outline of the procedure, as it provides great insight into

[3] We delay d by $\overline{\tau}$ seconds so that d from time $-\overline{\tau}$ to ∞ is included in the norm of $\overline{w}$.

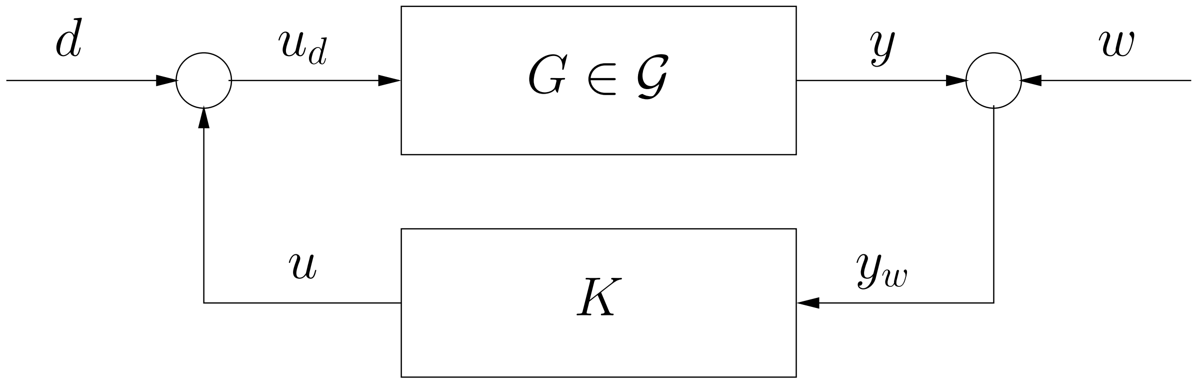

Figure 5.1: The feedback setup considered in this paper.

the higher order case. So, with $a > 0$, $b \neq 0$ and $c = 1$,[4] a first order plant in $\mathcal{G}$ (with the input and output noise added) is given by

$$\left.\begin{array}{rcl} \dot{x}(t) & = & ax(t) + gb(u(t-\tau) + d(t-\tau)) \\ y_w(t) & = & x(t) + w(t), \end{array}\right\} \tag{5.2}$$

with initial conditions of $x(0) = x_0$ and (for simplicity) $d(\theta) = u(\theta) = 0$ for $\theta \in [-\overline{\tau}, 0)$.[5]

Our control scheme relies on using a generalized "pulse" hold, so with $T_2 > 0$, $T_3 > \overline{\tau}$ and the period $T := T_2 + T_3$,[6] we define the hold $H : \ell(\mathbf{R}) \to PC(\mathbf{R})$ by

$$(H\nu)(t) := \begin{cases} \nu[k] & t \in [kT, kT + T_2] \\ 0 & t \in (kT + T_2, kT + T) \end{cases} \tag{5.3}$$

for $k \geq 0$ and the sampler $S : PC(\mathbf{R}) \to \ell(\mathbf{R})$ by

$$(Sy_w)[k] := y_w(kT), \quad k \geq 0. \tag{5.4}$$

Applying this hold and sampler to (5.2) and combining the state and output equations

[4]Without loss of generality we can assume that $c = 1$ since it can easily be absorbed into b.

[5]The input initial condition will be allowed to be non-zero for the general case as well as for the result presented in Chapter 6.

[6]The reasons for using the unusual nomenclature of T_2 and T_3 will become clear once we analyze the higher order case in the next section.

yields the following discretized plant:

$$\underbrace{y_w(kT+T)}_{=:\psi[k+1]} = \underbrace{e^{aT}}_{=:a_d}\underbrace{y_w(kT)}_{=:\psi[k]} + \underbrace{ge^{-a\tau}}_{=:\alpha^{-1}}\underbrace{ba^{-1}e^{aT}(1-e^{-aT_2})}_{=:b_d}\nu[k]+$$

$$\underbrace{w(kT+T)}_{=:\omega[k+1]} - e^{aT}\underbrace{w(kT)}_{=:\omega[k]} + \underbrace{bge^{aT}\int_0^T e^{-av}d(kT+v-\tau)dv}_{=:\zeta[k]}, \tag{5.5}$$

with initial condition $\psi[0] = x_0$. It will also be useful to define a single noise term, so we let $\eta[k] := \omega[k+1] - e^{aT}\omega[k] + \zeta[k]$, yielding

$$\psi[k+1] = a_d\psi[k] + \alpha^{-1}b_d\nu[k] + \eta[k]. \tag{5.6}$$

The basic idea is similar to what we used in [14], namely, we solve (5.6) at each time step for α under the hypothesis that there is no noise, and then use this solution to define an estimate of this quantity: with arbitrary initial conditions $\psi[-1] \in \mathbf{R}$ and $\nu[-1] \in \mathbf{R}$, for $k \geq 1$ define

$$\check{\alpha}[k] := \begin{cases} \frac{b_d\nu[k-1]}{\psi[k]-a_d\psi[k-1]} & \text{if } \nu[k-1] \neq 0 \text{ and } \psi[k]-a_d\psi[k-1] \neq 0 \\ e^{a\overline{\tau}} & \text{otherwise.} \end{cases} \tag{5.7}$$

Since there is noise, we saturate $\check{\alpha}[k]$ into the range of permissible values, namely, $[(\overline{g})^{-1}, e^{a\overline{\tau}}]$, yielding our final estimate of

$$\hat{\alpha}[k] = \begin{cases} \frac{1}{\overline{g}} & \check{\alpha}[k] < \frac{1}{\overline{g}} \\ \check{\alpha}[k] & \check{\alpha}[k] \in [\frac{1}{\overline{g}}, e^{a\overline{\tau}}] \\ e^{a\overline{\tau}} & \check{\alpha}[k] > e^{a\overline{\tau}}. \end{cases} \tag{5.8}$$

To design the control signal, we assume that there is no noise, which means that $\hat{\alpha}[k] = \alpha$, and then choose $\nu[k]$ to drive $\psi[k+1]$ to zero at the next time step:

$$\nu[k] := -\frac{a_d\hat{\alpha}[k]\psi[k]}{b_d}, \qquad k \in \mathbf{Z}^+. \tag{5.9}$$

When we apply the controller (5.7), (5.8), (5.9) when the noise is zero, the output $\psi[k]$ goes to zero in two time steps. When we apply the controller in the presence of noise, in general $\hat{\alpha}[k] \neq \alpha$, and the output $\psi[k]$ will typically not go to zero. While we won't do so here, fortunately, we can prove that:

1. When the state is large with respect to the noise, the estimate $\hat{\alpha}[k]$ is close to α.

2. There exists constants $\lambda \in (0, 1)$, $c_1 > 0$ so that when the estimate $\hat{\alpha}[k]$ is 'close enough' to α, we have $|\psi[k+1]| \leq \lambda|\psi[k]| + c_1\|\eta\|_\infty$.
3. Regardless of the estimate, the output grows by a bounded amount over a single time step.

With these three properties, we can than show that the controller K, consisting of the sampler given by (5.4), the hold given by (5.3), the estimator given by (5.7) and (5.8), and the control signal given by (5.9), stabilizes $\mathcal{G}$.

In the remainder of this chapter, we will apply a similar control scheme to higher order plants. However, a few complications arise when attempting to do so, namely:

- We no longer have full access to the state, so we use a state estimator; this is made difficult since the discretized B matrix is unknown due to the uncertainty in the delay and gain.
- In the first order case, $ge^{-a\tau}$ is a strictly positive real number, and as a result, we are able to estimate the combined quantity (or in fact, its inverse). In the higher order case, $ge^{-A\tau}$ is an $n \times n$ matrix, so we estimate g and τ separately.
- In the first order case, it was easy to construct a control signal using the estimate of $ge^{-a\tau}$; in the higher order case, we use pole placement.

Fortunately, these problems are not insurmountable, and in the next section we propose a control scheme that resolves all three of these issues, and thereby solves the problem.

5.3 The Controller

We now consider the higher order case. As in the first order case, if we incorporate the input and output noise into the plant model, we obtain

$$\left.\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + gB(u(t-\tau) + d(t-\tau)) \\ y_w(t) & = & Cx(t) + w(t), \end{array}\right\} \tag{5.10}$$

with initial conditions of $x(0) = x_0$, $u(\theta) = u_0(\theta)$ and $d(\theta) = d_0(\theta)$ for $\theta \in [-\overline{\tau}, 0)$. In addition, we require one further property from A, namely, that it has at least two distinct real eigenvalues, $p_1 \neq 0$ and $p_2 \neq 0$, with multiplicity one.

Remark 1. *If A does not have two real distinct eigenvalues with multiplicity one, then we can add a filter $W(s)$ at the plant output so that $G(s)W(s)$ has two real poles with* *multiplicity one. The simplest way to do this is to set*

$$W = \begin{cases} \dfrac{s+\lambda}{s+2\lambda} & \text{if } A \text{ has one real distinct eigenvalue} \\ \dfrac{(s+\lambda)(s+3\lambda)}{(s+2\lambda)(s+4\lambda)} & \text{if } A \text{ has no real distinct eigenvalues,} \end{cases} \tag{5.11}$$

and then choose $\lambda > 0$ so that $G(s)$ and W have no pole-zero cancellations.

Henceforth, without loss of generality, we partition A (by carrying out a similarity transformation if necessary) as follows:

$$A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & A_r \end{bmatrix}. \tag{5.12}$$

Our proposed control scheme is given in Figure 5.2; the controller is periodic of period T. We will describe each block of the controller in detail, but for now, we present the basic idea. We start with a generalized hold H and a higher frequency sampler S which only operates over the first part of the period, as described in more detail in Sub-section 5.3.1. In Sub-section 5.3.2, we apply the sampler and hold to the plant to yield a discretized model. Using this discretized plant model, in Sub-section 5.3.3 we analyze the state feedback gain associated with a pole placement problem, and in Sub-section 5.3.4 we design a state estimator. In Sub-section 5.3.5 we introduce our control law, which we use along with the discretized state model to construct a delay estimator in Sub-section 5.3.6 and a gain estimator in Sub-section 5.3.7; we then use the gain and delay estimates to construct the state feedback gain estimate (as introduced in Sub-section 5.3.3) in Sub-section 5.3.8. Finally, we summarize the control scheme in Sub-section 5.3.9.

5.3.1 The Sampler and Hold

We are interested in applying a generalized hold and an almost normal sampler to a plant in $\mathcal{G}$; a relevant question then is "Since (A, B) is controllable and (C, A) is observable, is the same true for the discretized system?" If we are using a zero-order hold and a normal sampler, then from [11], the answer is yes so long as the sampling period is 'non-pathological';[7] it turns out that we have a very similar requirement for our simple generalized hold H, which is parameterized by three quantities of time, T_1, T_2, and T_3, which satisfy the following:

[7] See Section 2.3 for a definition of pathological sampling.

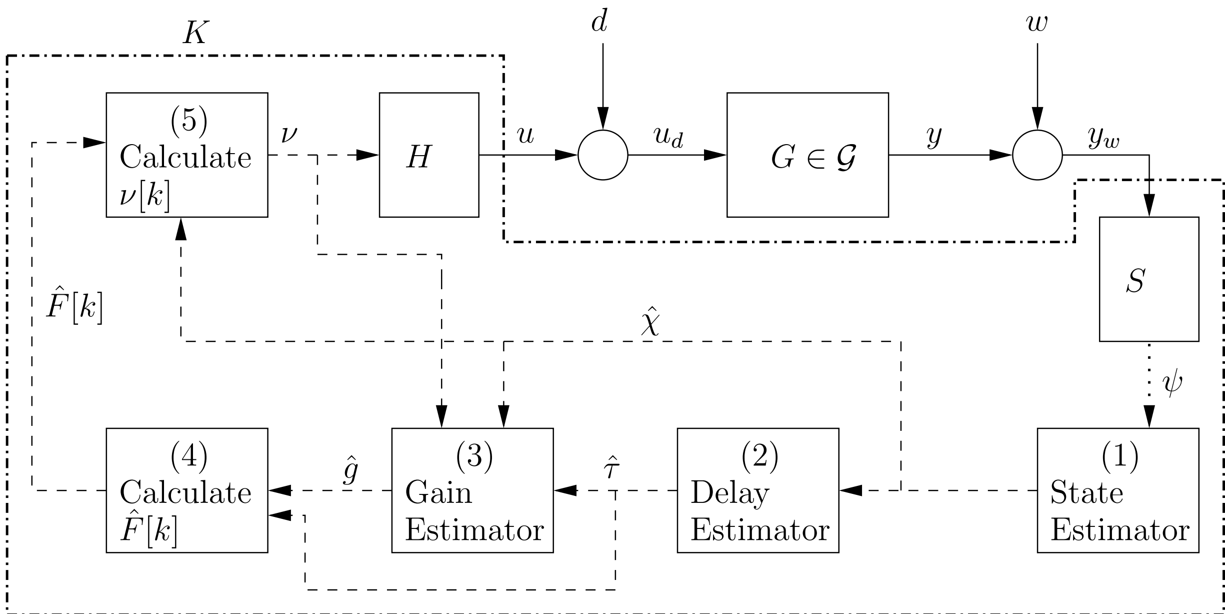

Figure 5.2: The feedback setup used for the higher order case.

i) T_1, T_2, and T_3 are all positive,
ii) $T_3 > \overline{\tau}$,
iii) $T = T_1 + T_2 + T_3$ is non-pathological (with respect to A).

We define the hold $H : \ell(\mathbf{R}) \to PC(\mathbf{R})$ by

$$(H\nu)(t) := \begin{cases} 0 & t \in [kT, kT + T_1) \\ \nu[k] & t \in [kT + T_1, kT + T_1 + T_2) \\ 0 & t \in [kT + T_1 + T_2, kT + T) \end{cases} \tag{5.13}$$

for $k \geq 0$. Figure 5.3 provides an illustration of the output of the hold over the first period.

The purpose of the sampler S is to allow us to estimate the plant state at time kT; to accomplish this goal, we run the sampler at a period h, and we impose the following three constraints on h:

i) $\frac{T}{h}$ must be an integer so that h is synchronous with T.
ii) h must be non-pathological with respect to A so that (C, e^{Ah}) is guaranteed to be observable.
iii) $h < \frac{T_1}{n}$ so that we can obtain at least n samples over the interval $[kT, kT + T_1)$.[8]

[8]Note that if there is no noise and zero initial conditions, then for every $\tau \in [0, \overline{\tau}]$, we have that

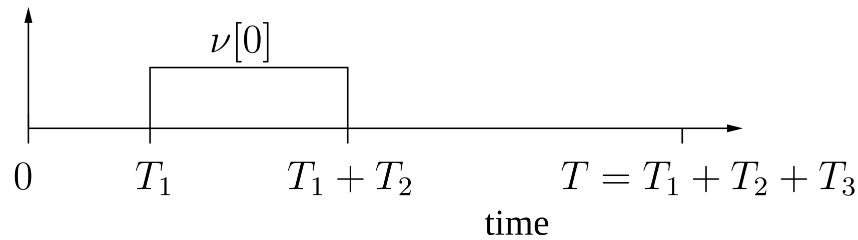

Figure 5.3: Showing the output from the hold given by (5.13).

With these restrictions in place, we define the sampler $S : PC(\mathbf{R}) \to \ell(\mathbf{R})$ by

$$(Sy_w)[jh] := y_w(jh), \qquad j \in \mathbf{Z}^+. \tag{5.14}$$

Remark 2. *While any combination of T_1, T_2, T_3, T, and h that satisfy the above constraints is acceptable for proving stability, these parameters do have an impact on the performance of the system.*

- T: *Since the system updates its estimate of the gain and delay every T seconds, the overall period should be made as small as possible. However, while we can theoretically make T as close as we want to $\overline{\tau}$, doing so will not normally provide the best performance.*
- T_3: *There is no drawback for making this as small as possible, so we can choose T_3 to be as close as we wish to $\overline{\tau}$.*
- T_2: *Larger values of T_2 produce a smaller control signal, which creates the obvious trade-off.*
- h: *The estimate of the state $x(kT)$ is obtained by multiplying the inverse of the observability matrix (associated with the pair (C, e^{Ah})) by a vector of samples of y; hence, the effect of the noise on the error of this estimate is reduced by having a large h, which creates an obvious trade-off.*
- T_1: *With h selected, there is no drawback for making this as small as possible, so we can choose T_1 as close as we wish to $n \times h$.*

$u(t-\tau) = 0$ for $t \in [kT, kT + T_1)$; this property is critical for the design of our state estimator.

5.3.2 Discretizing the Plant

We would now like to discretize the plant (5.10) using our sampler and hold:[9]

$$\left.\begin{aligned}
\underbrace{x(kT+T)}_{=:\chi[k+1]} &= \underbrace{e^{AT}}_{=:A_d}\underbrace{x(kT)}_{=:\chi[k]} + \underbrace{\left[e^{A(T-T_1)}e^{-A\tau}\int_0^{T_2} e^{-Av}gBdv\right]}_{=:B_d(\tau,g)}\nu[k]+ \\
&\quad \underbrace{\left[\int_0^T e^{A(T-v)}gBd(kT+v-\tau)dv\right]}_{=:\zeta[k]\in\mathbf{R}^{1\times n}} + \delta[k]\underbrace{\int_{-\tau}^0 e^{A(T-v)}gBu_0(v)dv}_{=:\phi} \\
\underbrace{y_w(kT)}_{=:\psi[k]} &= Cx(kT) + \underbrace{w(kT)}_{=:\omega[k]}.
\end{aligned}\right\} \tag{5.15}$$

Our discretized plant has the initial condition $\chi_0 = x_0$; it is also useful to note that

$$\|\phi\| \leq \underbrace{\overline{g}\overline{\tau}e^{\|A\|\overline{\tau}}\|B\|}_{=:c_\phi}\sup_{\theta\in[-\overline{\tau},0)}|u_0(\theta)|, \tag{5.16}$$

and recalling that $\overline{w}(t) := \begin{bmatrix} w(t) \\ d(t-\overline{\tau}) \end{bmatrix}$ it is easy to obtain the following bounds:

$$\|\omega\|_\infty \leq \|\overline{w}\|_\infty, \qquad \|\zeta\|_\infty \leq \underbrace{\overline{g}Te^{\|A\|T}\|B\|}_{=:c_\zeta}\|\overline{w}\|_\infty. \tag{5.17}$$

Since (C, A) is observable by assumption, and T is chosen to be non-pathological with respect to A, it follows from [11] that (C, A_d) is observable. Verifying that $(A_d, B(\tau, g))$ is controllable requires some work. To this end, define the set of all possible $B_d(\tau, g)$:

$$\mathcal{B} = \left\{ ge^{A(T-T_1-\tau)}\int_0^{T_2} e^{-Av}Bdv;\ g\in[1,\overline{g}], \tau\in[0,\overline{\tau}]\right\}.$$

Proposition 5.1. *(A_d, B_d) is controllable for every $B_d \in \mathcal{B}$.*

Proof. In Lemma 4.1, with T non-pathological with respect to A and $T_2 > 0$ it was proven that the pair $(A_d, ge^{AT}e^{-A\tau}\int_0^{T_2} e^{-Av}Bdv)$ is controllable for all $g \in [1, \overline{g}]$ and $\tau \in [0, T-T_2)$. In our case, we are interested in $(A_d, ge^{AT}e^{-A(T_1+\tau)}\int_0^{T_2} e^{-Av}Bdv)$; since $T_1+\tau \in [0, T-T_2)$, the result follows immediately. $\square$

[9] Due to the initial condition on the plant input, we have an extra term in the state equation, given by $\delta[k]\int_{-\tau}^0 e^{-av}gbu_0(v)dv$ (here $\delta[k]$ is the standard discrete-time pulse sequence).

5.3.3 The Pole Placement Problem

In the case of perfect information, we can use state feedback to provide closed-loop stability. To this end, fix $\beta[z]$ to be an n^{th} order monic Schur polynomial (with real coefficients), and then for $\tau \in [0, \overline{\tau}]$ and $g \in [1, \overline{g}]$, we let $F(\tau, g)$ represent the unique element of $\mathbf{R}^{1\times n}$ which satisfies

$$\det\left[zI - A_d - B_d(\tau, g)F(\tau, g)\right] = \beta[z].$$[10]

While A_d is known exactly, we only have running estimates of τ and g since in general, τ and g are unknown; as a result, we only have estimates of $B_d(\tau, g)$ and $F(\tau, g)$ as well. An important issue which will arise is the accuracy of our estimate of F in terms of the accuracy of our estimates of τ and g. To explore this issue, we define the set

$$\mathcal{F} := \left\{ F \in \mathbf{R}^{1\times n} : \text{characteristic polynomial of } A_d + B_d F \text{ is } \beta[z] \text{ for some } B_d \in \mathcal{B} \right\}.$$

Lemma 5.1. *$\mathcal{B}$ and $\mathcal{F}$ are compact and there exists a constant $c_1 > 0$ such that for all $\tau_1, \tau_2 \in [0, \overline{\tau}]$ and $g_1, g_2 \in [1, \overline{g}]$, we have that*

$$\|F(\tau_2, g_2) - F(\tau_1, g_1)\| \leq c_1 \left(|\tau_1 - \tau_2| + |g_1 - g_2|\right).$$

Proof. From the definition of $B_d(\tau, g)$, it is clear that it is an analytic function of $(\tau, g) \in [0, \overline{\tau}] \times [1, \overline{g}]$. Lemma 3 from [36] states that $F(\tau, g)$ is an analytic function of $B_d(\tau, g)$, which means that $F(\tau, g)$ is an analytic function of (τ, g) on the compact set $[0, \overline{\tau}] \times [1, \overline{g}]$. The result now follows immediately. $\square$

In later sections, we will provide estimation algorithms for τ, g, $B_d(\tau, g)$ and $F(\tau, g)$. We will label our estimate of the latter (at time k) by $\hat{F}[k]$, and restrict it to take values in $\mathcal{F}$. To this end, we make the following definition:

Definition 5.2. *$\hat{F} : \mathbf{Z}^+ \to \mathbf{R}^{1\times n}$ is* admissible *if its range is contained in $\mathcal{F}$.*

Last of all, we would like to upper bound $F \in \mathcal{F}$ and $B_d \in \mathcal{B}$; since both are compact, the following two quantities are finite:

$$\overline{f} := \sup\left\{\|F\| : F \in \mathcal{F}\right\}, \qquad \overline{b} := \sup\left\{\|B_d\| : B_d \in \mathcal{B}\right\}. \tag{5.18}$$

[10]By Proposition 5.1, we have that $(A_d, B_d(\tau, g))$ is controllable, which implies that $F(\tau, g)$ is unique.

5.3.4 Estimating the State χ

To estimate $\chi[k] = x[kT]$ we will follow the approach outlined in [37]. During the initial part of the period, i.e., $t \in [kT, kT + T_1)$, the sampler S samples the output $y_w(t)$ every h seconds, generating a total of n samples. Furthermore, we are guaranteed that the controller input to the plant (even with the unknown delay) is zero over this time period.[11]

Since we chose h to be non-pathological with respect to A, it follows that (C, e^{Ah}) is observable. Assuming no noise, we have

$$\underbrace{\begin{bmatrix} y_w(kT) \\ y_w(kT+h) \\ \vdots \\ y_w(kT+(n-1)h) \end{bmatrix}}_{=:\mathcal{Y}(kT)} = \underbrace{\begin{bmatrix} C \\ Ce^{Ah} \\ \vdots \\ C\left(e^{Ah}\right)^{n-1} \end{bmatrix}}_{=:\mathcal{O}_h} x(kT);$$

our observability assumption ensures that $\mathcal{O}_h$ is invertible, so we can solve for $x(KT)$, yielding $x(kT) = \chi[k] = \mathcal{O}_h^{-1}\mathcal{Y}(kT)$; of course, we have noise in our actual system, so we set

$$\hat{\chi}[k] = \mathcal{O}_h^{-1}\mathcal{Y}(kT), \quad k \geq 0, \tag{5.19}$$

and then define the state estimation error $\tilde{\chi}[k] := \hat{\chi}[k] - \chi[k]$.

Lemma 5.2. *There exists constants $c_2 > 0$ and $\bar{c}_2 > 0$ such that*

$$\|\hat{\chi}[k] - \chi[k]\| = \|\tilde{\chi}[k]\| \leq c_2\|\overline{w}\|_\infty + \delta[k]\bar{c}_2 \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|, \quad k \geq 0.$$

Proof. For $k \geq 1$ and $\ell \in \{0, 1, \cdots, n-1\}$, in the presence of noise, it follows from (5.10) and the fact that u is off during the first T_1 seconds of each period, that

$$y_w(kT + \ell h) = Ce^{A\ell h}\chi[k] + w(kT + \ell h) + C\int_0^{\ell h} e^{A(\ell h - v)} gBd(kT + v - \tau)dv,$$

[11] Due to the initial condition on the input u, this is only true for $k \geq 1$, and of course, the plant will still be affected by noise over this time frame.

so using (5.19) we have

$$\tilde{\chi}[k] = \mathcal{O}_h^{-1} \begin{bmatrix} \omega(kT) \\ \omega(kT+h) \\ \vdots \\ \omega(kT+(n-1)h) \end{bmatrix} + \mathcal{O}_h^{-1} \begin{bmatrix} 0 \\ C\int_0^h e^{A(h-v)} gBd(kT+v-\tau)dv \\ \vdots \\ C\int_0^{(n-1)h} e^{A((n-1)h-v)} gBd(kT+v-\tau)dv \end{bmatrix}.$$

Hence,

$$\begin{aligned} \|\tilde{\chi}[k]\| &\le \|\mathcal{O}_h^{-1}\| \times \|w\|_\infty + \|\mathcal{O}_h^{-1}\| \times \|C\|(n-1)he^{\|A\|(n-1)h}\overline{g}\|B\| \times \|d\|_\infty \\ &\le \underbrace{\|\mathcal{O}_h^{-1}\| \left(1 + \|C\|(n-1)he^{\|A\|(n-1)h}\overline{g}\|B\|\right)}_{=:c_2} \|\overline{w}\|_\infty. \end{aligned}$$

Now, consider the case of $k = 0$ with $\ell \in \{0, 1, \cdots, n-1\}$. By solving the system equation (5.10) starting from zero, $y_w(\ell h)$ has an extra term:

$$\begin{aligned} y_w(\ell h) = Ce^{A\ell h}\chi[0] + w(\ell h) + C\int_0^{\ell h} e^{A(\ell h - v)} gBd(kT+v-\tau)dv + \\ C\int_{-\tau}^{-\tau+\ell h} e^{A(-\tau+\ell h-v)} gBu_0(v)dv; \end{aligned}$$

it is easy to check that $\tilde{\chi}[0]$ is as above but with an extra term of

$$\mathcal{O}_h^{-1} \begin{bmatrix} 0 \\ C\displaystyle\int_{-\tau}^{-\tau+h} e^{A(-\tau+h-v)} gBu_0(v)dv \\ \vdots \\ C\displaystyle\int_{-\tau}^{-\tau+(n-1)h} e^{A(-\tau+(n-1)h-v)} gBu_0(v)dv \end{bmatrix},$$

whose norm is upper bounded by

$$\underbrace{\|\mathcal{O}_h^{-1}\| \times \|C\|\overline{\tau}\overline{g}e^{\|A\|\overline{\tau}}\|B\|}_{=:\overline{c}_2} \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|,$$

which completes the proof. $\square$

5.3.5 The Control Signal

In the case of perfect information, our ideal control law is clearly $\nu[k] = F(\tau, g)\chi[k]$. However, in order to estimate τ and g, we require an additional probing signal, yielding

$$\nu[k] = F(\tau, g)\chi[k] + \rho \mathrm{sgn}(F(\tau, g)\chi[k])\|\chi[k]\|. \tag{5.20}$$

We will now show that this stabilizes the plant if $\rho > 0$ is small enough;[12] in the following sub-sections, we will make (5.20) 'adaptive' by replacing τ and g by on-line estimates.

Applying (5.20) to (5.15), with $u_0(\theta) = 0$ for $\theta \in [-\overline{\tau}, 0)$, $d = 0$ and $w = 0$, results in the following closed loop system:

$$\chi[k+1] = (A_d + B_d(\tau, g)F(\tau, g))\chi[k] + \rho B_d(\tau, g)\mathrm{sgn}(F(\tau, g)\chi[k])\|\chi[k]\|. \tag{5.21}$$

We want to find the values of $\rho > 0$ so that for every $\tau \in [0, \overline{\tau}]$ and $g \in [1, \overline{g}]$ the origin of (5.21) is a globally exponentially stable equilibrium point[13]. To do so, first let $P(\tau, g)$ be the unique positive definite solution to the Lyapunov equation

$$(A_d + B_d(\tau, g)F(\tau, g))' P(\tau, g)(A_d + B_d(\tau, g)F(\tau, g)) - P(\tau, g) = -I. \tag{5.22}$$

Proposition 5.2. *There exists positive constants c_3 and c_4 such that for every $\tau \in [0, \overline{\tau}]$, $g \in [1, \overline{g}]$ and $\chi \in \mathbf{R}^n$ we have that*

$$\begin{array}{lrcl} (i) & c_3 I \leq & P(\tau, g) & \leq c_4 I \\ (ii) & c_3 \leq & \|P(\tau, g)\| & \leq c_4 \\ (iii) & c_3\|\chi\|^2 \leq & \chi' P(\tau, g)\chi & \leq c_4\|\chi\|^2. \end{array}$$

Proof. It is well known that the discrete-time Lyapunov equation can be rewritten as a linear equation [27], and that $P(\tau, g)$ is a continuous function of $B_d(\tau, g)F(\tau, g)$; since the latter is a continuous function of τ and g, it follows that $P(\tau, g)$ is also a continuous function of τ and g, so $\lambda_{\min}[P(\tau, g)]$ and $\lambda_{\max}[P(\tau, g)]$ are as well. Since $[0, \overline{\tau}] \times [1, \overline{g}]$ is a

[12] We choose $\rho > 0$ so that both terms on the RHS of (5.20) have the same sign; this means, in particular, that $\nu[k] = 0$ iff $\chi[k] = 0$.

[13] Although the end-goal of the paper is to prove that the adaptive version of (5.20) stabilizes $\mathcal{G}$, this related notion of stability is sufficient for the preliminary analysis.

compact set and $P(\tau, g)$ is positive definite (so $\lambda_{\min}[P(\tau, g)] > 0$) for every admissible τ and g, if we define

$$\underline{\gamma}_1 := \inf_{g \in [1, \overline{g}], \tau \in [0, \overline{\tau}]} \lambda_{\min}[P(\tau, g)] > 0 \qquad \text{and} \qquad \overline{\gamma}_1 := \sup_{g \in [1, \overline{g}], \tau \in [0, \overline{\tau}]} \lambda_{\max}[P(\tau, g)] < \infty,$$

then it follows that

$$\underline{\gamma}_1 I \leq P(\tau, g) \leq \overline{\gamma}_1 I.$$

Since this says that the Euclidean norm of $P(\tau, g)$ lies in the interval $[\underline{\gamma}_1, \overline{\gamma}_1]$ (which does not include zero), using the fact that all norms on $\mathbf{R}^{n \times n}$ are compatible, we conclude that there exists strictly positive constants $\underline{\gamma}_2$ and $\overline{\gamma}_2$ so that for all $\tau \in [0, \overline{\tau}]$ and $g \in [1, \overline{g}]$, we have

$$\underline{\gamma}_2 \leq \|P(\tau, g)\| \leq \overline{\gamma}_2.$$

Last of all, observe that for all $\chi \in \mathbf{R}^n$, we have

$$\underline{\gamma}_1 \chi^{'} \chi \leq \chi^{'} P(\tau, g) \chi \leq \overline{\gamma}_1 \chi^{'} \chi,$$

and since

$$\|\chi\|^2 \leq \chi^{'} \chi \leq n \|\chi\|^2,$$

it follows that

$$\underline{\gamma}_1 \|\chi\|^2 \leq \chi^{'} P(\tau, g) \chi \leq n \overline{\gamma}_1 \|\chi\|^2.$$

Defining

$$c_3 := \min\{\underline{\gamma}_1, \underline{\gamma}_2\}$$

and

$$c_4 := \max\{n \overline{\gamma}_1, \overline{\gamma}_2\}$$

completes the proof. $\square$

We would also like to bound $\|A_d + B_d(\tau, g) F(\tau, g)\|$; since $B_d(\tau, g) \in \mathcal{B}$ and $F(\tau, g) \in \mathcal{F}$ and $\mathcal{B}$ and $\mathcal{F}$ are compact sets, it follows that

$$\overline{a} := \sup_{\tau \in [0, \overline{\tau}], g \in [1, \overline{g}]} \|A_d + B_d(\tau, g) F(\tau, g)\| < \infty. \tag{5.23}$$

With these constants, we can define

$$\overline{\rho} := \frac{-n c_4 \overline{a} + \sqrt{n c_4 (n c_4 \overline{a}^2 + 1)}}{n c_4 \overline{b}}, \tag{5.24}$$

which leads to the following lemma:

Lemma 5.3. *If $\rho \in (0, \overline{\rho})$, then for all $\tau \in [0, \overline{\tau}]$ and $g \in [1, \overline{g}]$, the origin is a globally exponentially stable equilibrium point of (5.21).*

Remark 3. *The condition on ρ is equivalent to requiring that $\rho > 0$ and satisfies*

$$nc_4\rho^2\overline{b}^2 + 2nc_4\rho\overline{b}\overline{a} < 1. \tag{5.25}$$

Proof. Fix $\tau \in [0, \overline{\tau}]$ and $g \in [1, \overline{g}]$; for convenience, we will drop all (τ, g) arguments.

Let

$$V(\chi[k]) = \chi^{'}[k]P(\tau, g)\chi[k],$$

then

$$\begin{aligned}\Delta V(\chi[k]) &= V(\chi[k+1]) - V(\chi[k]) \\ &= \chi^{'}[k]\left((A_d + B_dF)^{'}P(A_d + B_dF) - P\right)\chi[k] + \\ &\quad \rho^2 \mathrm{sgn}(F\chi[k])^2 B_d^{'}PB_d\|\chi[k]\|^2 + 2\rho B_d^{'}P(A_d + B_dF)\chi[k]\mathrm{sgn}(F\chi[k])\|\chi[k]\|.\end{aligned}$$

Using the solution to the Lyapunov equation (5.22), using Proposition 5.2 to upper bound $\|P\|$, (5.23) to upper bound $\|A_d + B_dF\|$, and (5.18) to upper bound $\|B_d\|$, it follows that

$$\Delta V(\chi[k]) \leq (-1 + nc_4\rho^2\overline{b}^2 + 2nc_4\rho\overline{b}\overline{a})\|\chi[k]\|^2,$$

so clearly if $\rho > 0$ is such that $nc_4\rho^2\overline{b}^2 + 2nc_4\rho\overline{b}\overline{a} < 1$, then the origin is a globally exponentially stable equilibrium point of (5.21); using the quadratic formula and recalling that $\rho > 0$, it follows that we require

$$\rho \in \left(0, \frac{-nc_4\overline{a} + \sqrt{nc_4(nc_4\overline{a}^2 + 1)}}{nc_4\overline{b}}\right)$$

which is exactly equivalent to $\rho \in (0, \overline{\rho})$. □

We can now give our actual control law; it has the same general form as (5.20) except we replace the state with its estimate and we replace $F(\tau, g)$ with an admissible function $\hat{F}[k]$.[14] So, with $\rho \in (0, \overline{\rho})$ the control law is given by

$$\left.\begin{aligned}\nu[k] &= \hat{F}[k]\hat{\chi}[k] + \rho\mathrm{sgn}(\hat{F}[k]\hat{\chi}[k])\|\hat{\chi}[k]\|_\infty \\ u(t) &= (H\nu)(t) \qquad\qquad t \in [kT, (k+1)T),\ k \geq 0.\end{aligned}\right\} \tag{5.26}$$

[14]We will make the choice of $\hat{F}$ precise after we define the gain and delay estimators.

Observe that using (5.18) to bound $\|\hat{F}[k]\|$, we see that

$$|\nu[k]| \in [\rho\|\hat{\chi}[k]\|, (\rho + \overline{f})\|\hat{\chi}[k]\|], \qquad k \geq 0; \tag{5.27}$$

this means, in particular, that $\nu[k] = 0$ if and only if $\hat{\chi}[k] = 0$. Furthermore, using Lemma 5.2 to bound $\|\hat{\chi}[k]\|$, we can further refine (5.27), at least for $k \geq 1$:

$$|\nu[k]| \in [\rho(\|\chi[k]\| - c_2\|\overline{w}\|_\infty), (\rho + \overline{f})(\|\chi[k]\| + c_2\|\overline{w}\|_\infty)], \qquad k \geq 1. \tag{5.28}$$

5.3.6 Estimating τ with $\hat{\tau}$

To find $\hat{\tau}[k]$, we examine the update equation for the first two states (χ_1 and χ_2), and solve them for τ. To do so, we partition

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

define

$$\begin{aligned} \tilde{b}_1 &:= b_1 p_1^{-1}(1 - e^{-p_1 T_2})(e^{-p_1 T_1}), \\ \tilde{b}_2 &:= b_2 p_2^{-1}(1 - e^{-p_2 T_2})(e^{-p_2 T_1}), \end{aligned}$$

recall that $\chi[k] = \hat{\chi}[k] - \tilde{\chi}[k]$ and then write the first two state equations for $k \geq 2$ as[15]

$$e^{-p_1 T}(\hat{\chi}_1[k] - \tilde{\chi}_1[k] - \zeta_1[k-1]) = \hat{\chi}_1[k-1] - \tilde{\chi}_1[k-1] + \tilde{b}_1 g e^{-p_1\tau}\nu[k-1], \tag{5.29}$$
$$e^{-p_2 T}(\hat{\chi}_2[k] - \tilde{\chi}_2[k] - \zeta_2[k-1]) = \hat{\chi}_2[k-1] - \tilde{\chi}_2[k-1] + \tilde{b}_2 g e^{-p_2\tau}\nu[k-1]. \tag{5.30}$$

If $\nu[k-1] \neq 0$, we divide (5.29) by (5.30), yielding an equation for τ which is valid for $k \geq 2$:[16]

$$\tau = \frac{1}{p_2 - p_1} \ln\left(\frac{\tilde{b}_2(e^{-p_1 T}(\hat{\chi}_1[k] - \tilde{\chi}_1[k] - \zeta_1[k-1]) - \hat{\chi}_1[k-1] + \tilde{\chi}_1[k-1])}{\tilde{b}_1(e^{-p_2 T}(\hat{\chi}_2[k] - \tilde{\chi}_2[k] - \zeta_2[k-1]) - \hat{\chi}_2[k-1] + \tilde{\chi}_2[k-1])}\right). \tag{5.31}$$

[15] We restrict $k \geq 2$ to avoid the extra term in (5.15) that arises from the plant initial conditions.

[16] This restriction ensures that the denominator of the logarithm term in the equation which follows is non-zero as well.

Since we do not know $\tilde{\chi}$ or ζ, we set these terms to zero in the estimation algorithm; let $\hat{\chi}[-1] \in \mathbf{R}^n$ be arbitrary and then define $\check{\tau}[k]$ for $k \geq 0$ as follows[17]:

$$\check{\tau}[k] := \begin{cases} \frac{1}{p_2-p_1} \ln\left(\left|\frac{\tilde{b}_2(e^{-p_1 T}\hat{\chi}_1[k]-\hat{\chi}_1[k-1])}{\tilde{b}_1(e^{-p_2 T}\hat{\chi}_2[k]-\hat{\chi}_2[k-1])}\right|\right) & \text{if } e^{-p_2 T}\hat{\chi}_2[k]-\hat{\chi}_2[k-1] \neq 0 \\ \overline{\tau} & \text{if } e^{-p_2 T}\hat{\chi}_2[k]-\hat{\chi}_2[k-1] = 0, \end{cases} \tag{5.32}$$

which we then saturate into the range of possible delays:

$$\hat{\tau}[k] = \begin{cases} 0 & \check{\tau}[k] < 0 \\ \check{\tau}[k] & \check{\tau}[k] \in [0, \overline{\tau}] \\ \overline{\tau} & \check{\tau}[k] > \overline{\tau}. \end{cases} \tag{5.33}$$

Defining the delay estimate error as $\tilde{\tau}[k] := \hat{\tau}[k] - \tau$, we can now compare τ given by (5.31) and $\hat{\tau}[k]$ given by (5.32) and (5.33), which we do in the following Lemma.

Lemma 5.4. *There exist constants $c_5 > c_2$ and $c_6 > 0$ so that for every $g \in [1, \overline{g}]$, $\tau \in [0, \overline{\tau}]$, and admissible $\hat{F}$, when the control law (5.26) is applied to the plant (5.10), the closed loop system has the following property: for every $k \geq 2$,*

$$\|\chi[k-1]\| > c_5 \|\overline{w}\|_\infty \quad \Rightarrow \quad |\hat{\tau}[k] - \tau| = |\tilde{\tau}[k]| \leq \frac{c_6 \|\overline{w}\|_\infty}{\|\chi[k-1]\|}.$$

Remark 4. *If $\overline{w} = 0$, then Lemma 5.4 says that if the plant state at time $(k-1)T$ is non-zero, then the estimate $\hat{\tau}[k]$ of τ is exact.*

Remark 5. *If the noise is non-zero, i.e. $\|\overline{w}\|_\infty \neq 0$, then it will be useful to normalize signals by the noise: for a generic signal f we define*

$$\boldsymbol{f} := \frac{f}{\|\overline{w}\|_\infty}.$$

Remark 6. *Note that the only property of $\hat{F}$ used in Lemma 5.4 is that its range lies in $\mathcal{F}$.*

Proof. Let $\tau \in [0, \overline{\tau}]$, $g \in [1, \overline{g}]$, $\overline{w} \in PC_\infty$ and $k \geq 2$ be arbitrary.

[17] We use the magnitude inside the logarithm term to ensure that $\check{\tau}[k] \in \mathbf{R}$ when the system is affected by noise.

We first consider the case of $\overline{w} = 0$; it follows immediately that $\zeta = 0$. From Lemma 5.2, it is clear that

$$\hat{\chi}[k-1] = \chi[k-1]$$

and

$$\tilde{\chi}[k-1] = 0;$$

hence, if

$$\chi[k-1] \neq 0,$$

then

$$\nu[k-1] \neq 0$$

and it follows from (5.29) that

$$e^{-p_2 T}\hat{\chi}_2[k] - \hat{\chi}_2[k-1] \neq 0.$$

As a result, the right hand side of (5.32) is exactly equal to the right hand side of (5.31), so,

$$|\check{\tau}[k] - \tau| = |\hat{\tau}[k] - \tau| = 0.$$

Now we consider the case of $\overline{w} \neq 0$. Since

$$|\check{\tau}[k] - \tau| \geq |\hat{\tau}[k] - \tau|,$$

we will compare $\check{\tau}[k]$ to τ. To do so, we start by showing that if $\|\chi[k-1]\|$ is large enough, then

$$e^{-p_2 T}\hat{\chi}_2[k] - \hat{\chi}_2[k-1] \neq 0.$$[18]

To this end, we re-write (5.30) as

$$e^{-p_2 T}\hat{\chi}_2[k] - \hat{\chi}_2[k-1] = e^{-p_2 T}\tilde{\chi}_2[k] + e^{-p_2 T}\zeta_2[k-1] - \tilde{\chi}_2[k-1] + \tilde{b}_2 g e^{-p_2 \tau}\nu[k-1], \quad (5.34)$$

and then use Lemma 5.2 to bound $\|\tilde{\chi}[k]\|$ and $\|\tilde{\chi}[k-1]\|$ and (5.17) to bound $|\zeta_2[k-1]|$, yielding

$$\left|e^{-p_2 T}\hat{\chi}_2[k] - \hat{\chi}_2[k-1]\right| \geq -e^{-p_2 T}(c_2 + c_\zeta)\|\overline{w}\|_\infty - c_2\|\overline{w}\|_\infty + |\tilde{b}_2|e^{-|p_2|\overline{\tau}}|\nu[k-1]|. \quad (5.35)$$

If we substitute the lower bound on $|\nu[k-1]|$ given in (5.28) into equation (5.35), we obtain

$$\left|e^{-p_2 T}\hat{\chi}_2[k] - \hat{\chi}_2[k-1]\right| \geq (-e^{-p_2 T}(c_2 + c_\zeta) - c_2)\|\overline{w}\|_\infty + |\tilde{b}_2|e^{-|p_2|\overline{\tau}}\rho(\|\chi[k-1]\| - c_2\|\overline{w}\|_\infty). \quad (5.36)$$

[18] We do this so that $\check{\tau}[k]$ is given by the top line of (5.32).

It is easy to check that if

$$\|\chi[k-1]\| > \underbrace{\left[\left(e^{-p_2 T}(c_2 + c_\zeta) + c_2\right)|\tilde{b}_2^{-1}|e^{|p_2|\overline{\tau}}\rho^{-1} + c_2\right]}_{=:\gamma_1}\|\overline{w}\|_\infty,$$

then the right hand side of (5.36) is positive, which means that

$$e^{-p_2 T}\hat{\chi}_2[k] - \hat{\chi}_2[k-1] \neq 0.$$

Henceforth suppose that $k \geq 2$ is such that $\|\chi[k-1]\| > \gamma_1\|\overline{w}\|_\infty$; therefore,

$$\check{\tau}[k] = \frac{1}{p_2 - p_1}\ln\left(\left|\frac{\tilde{b}_2(e^{-p_1 T}\hat{\chi}_1[k] - \hat{\chi}_1[k-1])}{\tilde{b}_1(e^{-p_2 T}\hat{\chi}_2[k] - \hat{\chi}_2[k-1])}\right|\right). \tag{5.37}$$

We now normalize all the signals in (5.31) and (5.37) by $\|\overline{w}\|_\infty$, but we first define

$$\begin{aligned}
\boldsymbol{q}_1[k] &:= -e^{-p_1 T}\tilde{\boldsymbol{\chi}}_1[k] + \tilde{\boldsymbol{\chi}}_1[k-1] - e^{-p_1 T}\boldsymbol{\zeta}_1[k-1],\\
\boldsymbol{q}_2[k] &:= -e^{-p_2 T}\tilde{\boldsymbol{\chi}}_2[k] + \tilde{\boldsymbol{\chi}}_2[k-1] - e^{-p_2 T}\boldsymbol{\zeta}_2[k-1],
\end{aligned}$$

and note that we can use Lemma 5.2 to bound $\|\tilde{\boldsymbol{\chi}}[k]\|$ and (5.17) to bound $|\zeta_i|$ yielding:

$$|\boldsymbol{q}_i[k]| \leq \underbrace{c_2 e^{(\max\{|p_1|,|p_2|\}T)} + c_2 + c_\zeta e^{(\max\{|p_1|,|p_2|\}T)}}_{=:\gamma_2}, \qquad i = 1, 2. \tag{5.38}$$

We now normalize, and after some minor re-arranging we can re-write (5.31) as

$$e^{(p_2 - p_1)\tau} = \left(\frac{\tilde{b}_2}{\tilde{b}_1}\right)\left(\frac{e^{-p_1 T}\hat{\boldsymbol{\chi}}_1[k] - \hat{\boldsymbol{\chi}}_1[k-1] + \boldsymbol{q}_1[k]}{e^{-p_2 T}\hat{\boldsymbol{\chi}}_2[k] - \hat{\boldsymbol{\chi}}_2[k-1] + \boldsymbol{q}_2[k]}\right) \tag{5.39}$$

and (5.37) as

$$e^{(p_2 - p_1)\check{\tau}[k]} = \left|\left(\frac{\tilde{b}_2}{\tilde{b}_1}\right)\left(\frac{e^{-p_1 T}\hat{\boldsymbol{\chi}}_1[k] - \hat{\boldsymbol{\chi}}_1[k-1]}{e^{-p_2 T}\hat{\boldsymbol{\chi}}_2[k] - \hat{\boldsymbol{\chi}}_2[k-1]}\right)\right|. \tag{5.40}$$

Using (5.39) and (5.40), we can compare $e^{(p_2-p_1)\tau}$ and $e^{(p_2-p_1)\check{\tau}[k]}$ with the knowledge that if these two quantities are 'close', then τ and $\check{\tau}[k]$ will also be 'close' (to be formalized at the end of the proof); since $e^{(p_1-p_2)\tau} > 0$, it follows that[19]

$$\left|e^{(p_2-p_1)\check{\tau}[k]} - e^{(p_2-p_1)\tau}\right| \leq \left|\frac{e^{-p_1 T}\hat{\boldsymbol{\chi}}_1[k] - \hat{\boldsymbol{\chi}}_1[k-1]}{e^{-p_2 T}\hat{\boldsymbol{\chi}}_2[k] - \hat{\boldsymbol{\chi}}_2[k-1]} - \frac{e^{-p_1 T}\hat{\boldsymbol{\chi}}_1[k] - \hat{\boldsymbol{\chi}}_1[k-1] + \boldsymbol{q}_1[k]}{e^{-p_2 T}\hat{\boldsymbol{\chi}}_2[k] - \hat{\boldsymbol{\chi}}_2[k-1] + \boldsymbol{q}_2[k]}\right|\left|\frac{\tilde{b}_2}{\tilde{b}_1}\right|,$$

[19] Here we use the fact that that if $\gamma_3 > 0$ and $\gamma_4 \in \mathbf{R}$, then $||\gamma_4| - \gamma_3| \leq |\gamma_4 - \gamma_3|$.

so defining

$$f_1[k] := \left(e^{-p_1 T}\hat{\boldsymbol{\chi}}_1[k] - \hat{\boldsymbol{\chi}}_1[k-1]\right)\boldsymbol{q}_2[k] - \left(e^{-p_2 T}\hat{\boldsymbol{\chi}}_2[k] - \hat{\boldsymbol{\chi}}_2[k-1]\right)\boldsymbol{q}_1[k] \tag{5.41}$$

$$f_2[k] := \left(e^{-p_2 T}\hat{\boldsymbol{\chi}}_2[k] - \hat{\boldsymbol{\chi}}_2[k-1]\right)\left(e^{-p_2 T}\hat{\boldsymbol{\chi}}_2[k] - \hat{\boldsymbol{\chi}}_2[k-1] + \boldsymbol{q}_2[k]\right) \tag{5.42}$$

it follows that

$$\left|\frac{\tilde{b}_1}{\tilde{b}_2}\left(e^{(p_2-p_1)\check{\tau}[k]} - e^{(p_2-p_1)\tau}\right)\right| \leq \frac{|f_1[k]|}{|f_2[k]|}. \tag{5.43}$$

We now upper and lower bound f_1 and f_2 respectively. Substituting (5.34) into f_2 given by (5.42) yields

$$f_2[k] = \left(-\boldsymbol{q}_2[k-1] + g e^{-p_2\tau}\tilde{b}_2\boldsymbol{\nu}[k-1]\right)\left(g e^{-p_2\tau}\tilde{b}_2\boldsymbol{\nu}[k-1]\right).$$

Hence, using (5.38) to bound $|q_2[k]|$, and (5.28) to lower bound $\nu[k-1]$, it follows that

$$|f_2[k]| \geq \left[|\tilde{b}_2|e^{-|p_2|\overline{\tau}}\rho(\|\boldsymbol{\chi}[k-1]\| - c_2) - \gamma_2\right]\left||\tilde{b}_2|e^{-|p_2|\overline{\tau}}\rho(\|\boldsymbol{\chi}[k-1]\| - c_2)\right|, \tag{5.44}$$

so clearly if

$$\|\boldsymbol{\chi}[k-1]\| > \underbrace{\gamma_2|\tilde{b}_2^{-1}|e^{|p_2|\overline{\tau}}\rho^{-1} + c_2}_{=:\gamma_5 > c_2},$$

then $|f_2[k]| > 0$. Henceforth, suppose that $k \geq 2$ is such that

$$\|\boldsymbol{\chi}[k-1]\| \geq \max\{\gamma_1, \gamma_5\};$$

then there exists constants $\gamma_6 > 0$, γ_7, $\gamma_8 \in \mathbf{R}$, which are independent of τ and g, such that

$$|f_2[k]| \geq \gamma_6\|\boldsymbol{\chi}[k-1]\|^2 + \gamma_7\|\boldsymbol{\chi}[k-1]\| + \gamma_8. \tag{5.45}$$

We now turn to f_1 given by (5.41). Using

$$e^{-p_1 T}\hat{\chi}_1[k] - \hat{\chi}_1[k-1] = e^{-p_1 T}\tilde{\chi}_1[k] + e^{-p_1 T}\zeta_1[k-1] - \tilde{\chi}_1[k-1] + \tilde{b}_1 g e^{-p_1\tau}\nu[k-1], \tag{5.46}$$

its equivalent for the second state (5.34), and (5.28) to upper bound $\nu[k-1]$, it follows that

$$\begin{aligned}
f_1[k] &= (\tilde{b}_1 g e^{-p_1\tau}\boldsymbol{\nu}[k-1] + \boldsymbol{q}_1[k])\boldsymbol{q}_2[k] - (\tilde{b}_2 g e^{-p_2\tau}\boldsymbol{\nu}[k-1] + \boldsymbol{q}_2[k])\boldsymbol{q}_1[k] \\
&= (\tilde{b}_1 g e^{-p_1\tau}\boldsymbol{\nu}[k-1])(\boldsymbol{q}_2[k]) - (\tilde{b}_2 g e^{-p_2\tau}\boldsymbol{\nu}[k-1])(\boldsymbol{q}_1[k]) \\
\Rightarrow \qquad |f_1[k]| &\leq \underbrace{\gamma_2\overline{g}\left(|\tilde{b}_1|e^{|p_1|\overline{\tau}} + |\tilde{b}_2|e^{|p_2|\overline{\tau}}\right)(\overline{f} + \rho)}_{=:\gamma_9}(\|\boldsymbol{\chi}[k-1]\| + c_2) \\
\Rightarrow \qquad |f_1[k]| &\leq \gamma_9\|\boldsymbol{\chi}[k-1]\| + \underbrace{c_4\gamma_9}_{=:\gamma_{10}}.
\end{aligned} \tag{5.47}$$

Combining (5.43), (5.45) and (5.47) we have that

$$\begin{aligned}\left|\frac{\tilde{b}_1}{\tilde{b}_2}\left(e^{(p_2-p_1)\check{\tau}[k]} - e^{(p_2-p_1)\tau}\right)\right| &\leq \frac{|f_1[k]|}{|f_2[k]|} \\ &\leq \frac{\gamma_9\|\boldsymbol{\chi}[k-1]\| + \gamma_{10}}{\gamma_6\|\boldsymbol{\chi}[k-1]\|^2 + \gamma_7\|\boldsymbol{\chi}[k-1]\| + \gamma_8}.\end{aligned}$$

We now need to convert the bound given above to one on $|\hat{\tau}k] - \tau|$. First, note that

$$\left|e^{(p_2-p_1)\hat{\tau}[k]} - e^{(p_2-p_1)\tau}\right| \leq \left|e^{(p_2-p_1)\check{\tau}[k]} - e^{(p_2-p_1)\tau}\right|;$$

using the Fundamental Theorem of Calculus, we have that

$$\begin{aligned}\left|e^{(p_2-p_1)\hat{\tau}[k]} - e^{(p_2-p_1)\tau}\right| &\geq |\hat{\tau}[k] - \tau| \times \min_{\theta\in[0,\overline{\tau}]}\left[|p_2-p_1|e^{(p_2-p_1)\theta}\right] \\ &\geq |\hat{\tau}[k] - \tau| \times \underbrace{|p_2-p_1|e^{-|p_2-p_1|\overline{\tau}}}_{=:\gamma_{11}},\end{aligned}$$

so clearly

$$|\hat{\tau}[k] - \tau| \leq \left|\frac{\tilde{b}_2}{\tilde{b}_1\gamma_{11}}\right| \frac{\gamma_9\|\boldsymbol{\chi}[k-1]\| + \gamma_{10}}{\gamma_6\|\boldsymbol{\chi}[k-1]\|^2 + \gamma_7\|\boldsymbol{\chi}[k-1]\| + \gamma_8}.$$

Since the right hand side times $\|\boldsymbol{\chi}[k-1]\|$ tends to $\frac{\gamma_9}{\gamma_6} > 0$ as $\|\boldsymbol{\chi}[k-1]\| \to \infty$, we conclude that there exists constants $c_6 > 0$ and $c_5 > \gamma_5 > c_2$, which are independent of τ and g, so that

$$|\hat{\tau}[k] - \tau| \leq \frac{c_6}{\|\boldsymbol{\chi}[k-1]\|} = \frac{c_6\|\overline{w}\|_\infty}{\|\chi[k-1]\|} \quad \text{if } \|\chi[k-1]\| > c_5\|\overline{w}\|_\infty.$$

□

5.3.7 Estimating g with $\hat{g}$

We perform a similar analysis to estimate g with $\hat{g}$. Using (5.30) and assuming that $\nu[k-1] \neq 0$, we solve for g:

$$g = \frac{e^{-p_2 T}(\hat{\chi}_2[k] - \tilde{\chi}_2[k] - \zeta_2[k-1]) - \hat{\chi}_2[k-1] + \tilde{\chi}_2[k-1]}{\tilde{b}_2 e^{-p_2\tau}\nu[k-1]}, \quad k \geq 2. \qquad (5.48)$$

Since we do not know $\tilde{\chi}$, ζ or τ, in order to estimate g we set the first two terms to zero and replace τ with $\hat{\tau}$; we let $\nu[-1] \in \mathbf{R}$ be arbitrary, and recalling that $\chi[-1] \in \mathbf{R}^n$ is arbitrary from the delay estimator, we define for $k \geq 0$:

$$\check{g}[k] := \begin{cases} \frac{e^{-p_2 T}\hat{\chi}_2[k] - \hat{\chi}_2[k-1]}{\tilde{b}_2 e^{-p_2 \hat{\tau}[k]}\nu[k-1]} & \text{if } \nu[k-1] \neq 0 \\ \overline{g} & \text{if } \nu[k-1] = 0, \end{cases} \tag{5.49}$$

which we then saturate into the range of possible gains, yielding

$$\hat{g}[k] = \begin{cases} 1 & \check{g}[k] < 1 \\ \check{g}[k] & \check{g}[k] \in [1, \overline{g}] \\ \overline{g} & \check{g}[k] > \overline{g}. \end{cases} \tag{5.50}$$

Defining the gain estimate error as $\tilde{g}[k] := \hat{g}[k] - g$, we can now compare g given by (5.48) and $\hat{g}[k]$ given by (5.49) and (5.50), which we do in the following Lemma.

Lemma 5.5. *There exist constants $c_7 > c_5 > c_2$ and $c_8 > 0$ so that for every $g \in [1, \overline{g}]$, $\tau \in [0, \overline{\tau}]$, and admissible $\hat{F}$, when the control law (5.26) is applied to the plant (5.10), the closed loop system has the following property: for every $k \geq 2$,*

$$\|\chi[k-1]\| > c_7 \|\overline{w}\|_\infty \quad \Rightarrow \quad |\hat{g}[k] - g| = |\tilde{g}[k]| \leq \frac{c_8 \|\overline{w}\|_\infty}{\|\chi[k-1]\|}.$$

Remark 7. *If $\overline{w} = 0$, then Lemma 5.5 says that if the plant state at time $(k-1)T$ is non-zero, then the estimate $\hat{g}[k]$ of g is exact.*

Remark 8. *Note that the only property of $\hat{F}$ used in Lemma 5.5 is that its range lies in $\mathcal{F}$.*

Proof. Let $\tau \in [0, \overline{\tau}]$, $g \in [1, \overline{g}]$, $\overline{w} \in PC_\infty$ and $k \geq 2$ be arbitrary.

We first consider the case of $\overline{w} = 0$; it follows immediately that $\zeta = 0$. From Lemma 5.2 it is clear that

$$\hat{\chi}[k-1] = \chi[k-1]$$

and

$$\tilde{\chi}[k-1] = 0;$$

hence, if

$$\chi[k-1] \neq 0,$$

from (5.28) it is clear that

$$\nu[k-1] \neq 0,$$

and as a result, the right hand side of (5.49) is exactly equal to the right hand side of (5.48), so

$$|\check{g}[k] - g| = |\hat{g}[k] - g| = 0.$$

Now we consider the case of $\overline{w} \neq 0$. Since

$$|\hat{g}[k] - g| \leq |\check{g}[k] - g|,$$

if we can show the result for $\check{g}[k]$, it will hold for $\hat{g}[k]$ as well. If

$$\|\chi[k-1]\| > c_2 \|\overline{w}\|_\infty,$$

then from (5.28) it is clear that

$$\nu[k-1] \neq 0,$$

so $\check{g}[k]$ is given by the top line of (5.49) and g is given by (5.48); henceforth, we shall assume that

$$\|\chi[k-1]\| > c_2 \|\overline{w}\|_\infty.$$

We now normalize by $\|\overline{w}\|_\infty$, and re-write (5.48) as

$$g = \frac{e^{-p_2 T}(\hat{\boldsymbol{\chi}}_2[k] - \tilde{\boldsymbol{\chi}}_2[k] - \boldsymbol{\zeta}_2[k-1]) - \hat{\boldsymbol{\chi}}_2[k-1] + \tilde{\boldsymbol{\chi}}_2[k-1]}{\tilde{b}_2 e^{-p_2 \tau} \boldsymbol{\nu}[k-1]} \tag{5.51}$$

and (5.49) as

$$\check{g}[k] = \frac{e^{-p_2 T} \hat{\boldsymbol{\chi}}_2[k] - \hat{\boldsymbol{\chi}}_2[k-1]}{\tilde{b}_2 e^{-p_2 \hat{\tau}[k]} \boldsymbol{\nu}[k-1]}. \tag{5.52}$$

We now compare (5.51) and (5.52), which, with some minor re-arranging becomes

$$|\check{g}[k] - g| = \left| \frac{\left(e^{-p_2 T} \hat{\boldsymbol{\chi}}_2[k] - \hat{\boldsymbol{\chi}}_2[k-1]\right)\left(e^{-p_2 \tau} - e^{-p_2 \hat{\tau}[k]}\right)}{\tilde{b}_2 e^{-p_2 \hat{\tau}[k]} e^{-p_2 \tau} \boldsymbol{\nu}[k-1]} - \right.$$
$$\left. \frac{e^{-p_2 \hat{\tau}[k]}\left(\tilde{\boldsymbol{\chi}}_2[k-1] - e^{-p_2 T}(\tilde{\boldsymbol{\chi}}_2[k] + \boldsymbol{\zeta}_2[k-1])\right)}{\tilde{b}_2 e^{-p_2 \hat{\tau}[k]} e^{-p_2 \tau} \boldsymbol{\nu}[k-1]} \right|. \tag{5.53}$$

Next, note that the state equation (5.30) can be used to remove $\hat{\boldsymbol{\chi}}$ from (5.53):

$$|\check{g}[k] - g| = \left| \frac{\left(e^{-p_2 \tau} - e^{-p_2 \hat{\tau}[k]}\right)\left(e^{-p_2 T}(\tilde{\boldsymbol{\chi}}_2[k] + \boldsymbol{\zeta}_2[k-1]) - \tilde{\boldsymbol{\chi}}_2[k-1] + \tilde{b}_2 g e^{-p_2 \tau} \boldsymbol{\nu}[k-1]\right)}{\tilde{b}_2 g e^{-p_2(\tau + \hat{\tau}[k])} \boldsymbol{\nu}[k-1]} - \right.$$
$$\left. \frac{e^{-p_2 \hat{\tau}[k]}\left(\tilde{\boldsymbol{\chi}}_2[k-1] - e^{-p_2 T}(\tilde{\boldsymbol{\chi}}_2[k] + \boldsymbol{\zeta}_2[k-1])\right)}{\tilde{b}_2 g e^{-p_2(\tau + \hat{\tau}[k])} \boldsymbol{\nu}[k-1]} \right|.$$

Using Lemma 5.2 to bound $\tilde{\boldsymbol{\chi}}$, and using (5.28) to bound $|\boldsymbol{\nu}[k-1]|$, we have that

$$|\breve{g}[k]-g| \leq \left|\frac{(e^{-p_2\tau}-e^{-p_2\hat{\tau}[k]})\Big(e^{-p_2T}(c_2+c_\zeta)+c_2+|\tilde{b}_2|\overline{g}e^{|p_2|\overline{\tau}}(\rho+\overline{f})(\|\boldsymbol{\chi}[k-1]\|+c_2)\Big)}{|\tilde{b}_2|e^{-2|p_2|\overline{\tau}}\rho(\|\boldsymbol{\chi}[k-1]\|-c_2)}\right| + \left|\frac{e^{|p_2|\overline{\tau}}(e^{-p_2T}(c_2+c_\zeta)+c_2)}{|\tilde{b}_2|e^{-2|p_2|\overline{\tau}}\rho(\|\boldsymbol{\chi}[k-1]\|-c_2)}\right|. \tag{5.54}$$

Clearly, there exist constants $c_7 > c_5 > c_2$ and $\gamma_1 > 0$ such that if $\|\boldsymbol{\chi}[k-1]\| > c_7$,[20] we have

$$|\breve{g}[k]-g| \leq \left|e^{-p_2\tau}-e^{-p_2\hat{\tau}[k]}\right|\left(\frac{\gamma_1}{\|\boldsymbol{\chi}[k-1]\|-c_2}+\gamma_1\right)+\frac{\gamma_1}{\|\boldsymbol{\chi}[k-1]\|-c_2}.$$

But

$$\left|e^{-p_2\tau}-e^{-p_2\hat{\tau}[k]}\right| \leq e^{|p_2|\overline{\tau}}|\hat{\tau}[k]-\tau|,$$

so if $\|\boldsymbol{\chi}[k-1]\| > c_7$, we have

$$|\breve{g}[k]-g| \leq e^{|p_2|\overline{\tau}}\left(\frac{\gamma_1}{\|\boldsymbol{\chi}[k-1]\|-c_2}+\gamma_1\right)|\hat{\tau}[k]-\tau|+\frac{\gamma_1}{\|\boldsymbol{\chi}[k-1]\|-c_2};$$

using Lemma 5.4 to bound $|\hat{\tau}[k]-\tau|$, and recalling that $|\hat{g}[k]-g| \leq |\breve{g}[k]-g|$ yields that if

$$\|\boldsymbol{\chi}[k-1]\| > c_7$$

then

$$|\hat{g}[k]-g| \leq \frac{e^{|p_2|\overline{\tau}}c_6\gamma_1}{(\|\boldsymbol{\chi}[k-1]\|-c_2)\|\boldsymbol{\chi}[k-1]\|}+\frac{e^{|p_2|\overline{\tau}}c_6\gamma_1}{\|\boldsymbol{\chi}[k-1]\|}+\frac{\gamma_1}{\|\boldsymbol{\chi}[k-1]\|-c_2}.$$

Hence, there exists a constant $c_8 > 0$ so that if $\|\boldsymbol{\chi}[k-1]\| > c_7 > c_2$, then

$$|\hat{g}[k]-g| \leq \frac{c_8}{\|\boldsymbol{\chi}[k-1]\|} = \frac{c_8\|\overline{w}\|_\infty}{\|\chi[k-1]\|}. \qquad \square$$

[20]We restrict $c_7 > c_5$ so that we can apply the result of Lemma 5.4 later in the proof.

5.3.8 Estimating F with $\hat{F}[k]$

To find $\hat{F}[k]$, we use the estimates $\hat{\tau}[k]$ and $\hat{g}[k]$ and a simple two step procedure. At each time step, we start by finding our estimate, $\hat{B}_d[k]$, of B_d:

$$\hat{B}_d[k] := \hat{g}[k] e^{A(T-T_1)} e^{-A\hat{\tau}[k]} \int_0^{T_2} e^{-Av} B dv, \tag{5.55}$$

and then, since $\hat{B}_d[k] \in \mathcal{B}$, we find the unique choice of $\hat{F}[k] \in \mathcal{F}$ such that the eigenvalues of

$$A_d + \hat{B}_d[k]\hat{F}[k]$$

are at the zeros of $\beta[z]$.

Remark 9. *While any polynomial $\beta[z]$ with all zeros inside the open unit disk is sufficient for us to prove stability, the performance of the system can be improved by the choice of pole locations. One possible method is as follows:*

- *Given suitably chosen weighting matrices Q and R, let F_{LQR} be the LQR optimal feedback gain for*
$$\chi[k+1] = A_d\chi[k] + B_d(0,1)\nu[k].$$
- *Set $\beta[z] = \det\left(zI - (A_d - B_d(0,1)F_{LQR})\right)^{-1}$.*

It will be beneficial when proving the result to define a feedback error; recalling that $F(\tau, g) \in \mathcal{F}$ is the unique element of $\mathcal{F}$ so that the characteristic polynomial of

$$A_d + B_d(\tau, g)F(\tau, g)$$

are at the zeros of $\beta[z]$, we define the feedback gain error by

$$\tilde{F}[k] := \hat{F}[k] - F(\tau, g).$$

5.3.9 Summary of Proposed Controller K

A description of the controller K is given by the following algorithm, with each step corresponding to the same numbered block in Figure 5.2. Our controller runs for $k \in \mathbf{Z}^+$ and has an arbitrary initial condition $\hat{\chi}[-1] \in \mathbf{R}^n$ and $\nu[-1] \in \mathbf{R}$:

1. For $k \in \mathbf{Z}^+$, the state $x(kT) = \chi[k]$ is estimated using the sampler (5.14) and (5.19):
$$\hat{\chi}[k] = \mathcal{O}_h^{-1}\mathcal{Y}(kT). \tag{5.56}$$

2. Using the state estimate $\hat{\chi}[k]$, we find $\check{\tau}[k]$ using (5.32):

$$\check{\tau}[k] := \begin{cases} \frac{1}{p_2-p_1}\ln\left(\left|\frac{\tilde{b}_2(e^{-p_1T}\hat{\chi}_1[k]-\hat{\chi}_1[k-1])}{\tilde{b}_1(e^{-p_2T}\hat{\chi}_2[k]-\hat{\chi}_2[k-1])}\right|\right) & \text{if } e^{-p_2T}\hat{\chi}_2[k]-\hat{\chi}_2[k-1]\neq 0 \\ \overline{\tau} & \text{if } e^{-p_2T}\hat{\chi}_2[k]-\hat{\chi}_2[k-1]=0, \end{cases} \tag{5.57}$$

and saturate it to find $\hat{\tau}[k]$ using (5.33):

$$\hat{\tau}[k] = \begin{cases} 0 & \check{\tau}[k] < 0 \\ \check{\tau}[k] & \check{\tau}[k] \in [0,\overline{\tau}] \\ \overline{\tau} & \check{\tau}[k] > \overline{\tau}. \end{cases} \tag{5.58}$$

3. Using the state estimate $\hat{\chi}[k]$ and $\hat{\tau}[k]$, we find $\check{g}[k]$ using (5.49):

$$\check{g}[k] := \begin{cases} \dfrac{e^{-p_2T}\hat{\chi}_2[k]-\hat{\chi}_2[k-1]}{\tilde{b}_2 e^{-p_2\hat{\tau}[k]}\nu[k-1]} & \text{if } \nu[k-1]\neq 0 \\ \overline{g} & \text{if } \nu[k-1]=0, \end{cases} \tag{5.59}$$

and saturate it to find $\hat{g}[k]$ using (5.50)

$$\hat{g}[k] = \begin{cases} 1 & \check{g}[k] < 1 \\ \check{g}[k] & \check{g}[k] \in [1,\overline{g}] \\ \overline{g} & \check{g}[k] > \overline{g}. \end{cases} \tag{5.60}$$

4. Using $\hat{\tau}[k]$ and $\hat{g}[k]$, we find $\hat{B}_d[k]$ using (5.55):

$$\hat{B}_d[k] := \hat{g}[k]e^{A(T-T_1)}e^{-A\hat{\tau}[k]}\int_0^{T_2} e^{-Av}B\,dv, \tag{5.61}$$

and then using $\hat{B}_d[k]$, we find the unique choice of $\hat{F}[k]$ such that the eigenvalues of $A_d + \hat{B}_d[k]\hat{F}[k]$ are at the zeros of $\beta[z]$.
5. Using $\hat{F}[k]$ and $\hat{\chi}[k]$, we apply the control signal (5.26):

$$\left.\begin{aligned} \nu[k] &= \hat{F}[k]\hat{\chi}[k] + \rho\,\mathrm{sgn}(\hat{F}[k]\hat{\chi}[k])\|\hat{\chi}[k]\|_\infty \\ u(t) &= (H\nu)(t) \qquad\qquad t\in[kT,(k+1)T),\ k\geq 0. \end{aligned}\right\} \tag{5.62}$$

Our controller K is formally given by the sampler (5.14), the state estimator (5.56), the delay estimator (5.57) and (5.58), the gain estimator (5.59) and (5.60), the estimate of B_d given by (5.61) and the corresponding $\hat{F}[k]$, the control signal (5.62) and the hold (5.13)

5.4 The Main Result

Theorem 5.1.

(i) *K stabilizes $\mathcal{G}$.*

(ii) *There exist constants $c > 0$ and $\lambda < 0$ so that when the controller K is applied to the plant (5.10), for every $g \in [1, \overline{g}]$, $\tau \in [0, \overline{\tau}]$, plant initial conditions $x_0 \in \mathbf{R}^n$ and $u_0(\theta)$ satisfying $\sup_{\theta \in [-\overline{\tau}, 0)} |u_0| < \infty$, controller initial conditions $\hat{\chi}[-1] \in \mathbf{R}^n$ and $\nu[-1] \in \mathbf{R}$ and for every $\overline{w} \in PC_\infty$, we have that*

(a) $\hat{\tau}, \hat{g}, \hat{B}_d, \hat{F}$ *and* $\hat{\chi} \in \ell_\infty$.[21]

(b) $|\nu[k]| \leq ce^{\lambda k T}\left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)|\right) + c\|\overline{w}\|_\infty$, $k \geq 0$.

(c) $\|x(t)\| \leq ce^{\lambda t}\left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)|\right) + c\|\overline{w}\|_\infty$, $t \geq 0$.

Proof. Let $\tau \in [0, \overline{\tau}]$, $g \in [1, \overline{g}]$, $x_0 \in \mathbf{R}^n$, $\overline{w} \in PC_\infty$, $\hat{\chi}[-1] \in \mathbf{R}^n$, $\nu[-1] \in \mathbf{R}$ and $u_0(\theta)$ such that $\sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| < \infty$ be arbitrary. We start by proving some intermediary results.

Applying the control signal (5.62) to the discretized plant (5.15), recalling that $\hat{F}[k] = \tilde{F}[k] + F(\tau, g)$ and that $\hat{\chi}[k] = \tilde{\chi}[k] + \chi[k]$, the state of the closed loop system for $k \geq 2$ is given by

$$\chi[k+1] = A_d\chi[k] + B_d(\tau, g)\nu[k] + \zeta[k] \tag{5.63}$$

$$= (A_d + B_d(\tau, g)F(\tau, g))\chi[k] + \underbrace{B_d(\tau, g)\tilde{F}[k]\chi[k] + B_d(\tau, g)\hat{F}[k]\tilde{\chi}[k] + \zeta[k]}_{=:f_1[k]} + \underbrace{\rho B_d(\tau, g)\text{sgn}(\hat{F}[k](\chi[k] + \tilde{\chi}[k]))\|\chi[k] + \tilde{\chi}[k]\|}_{=:f_2[k]}. \tag{5.64}$$

Using the triangle inequality and Lemma 5.2 to bound $\|\tilde{\chi}[k]\|$, it follows that

$$\|f_2[k]\| \leq \rho\bar{b}\|\chi[k]\| + \rho\bar{b}c_2\|\overline{w}\|_\infty. \tag{5.65}$$

We would also like a similar sort of bound on f_1:

Claim 1. *There exists positive constants c_9 and c_{10} so that for every $k \geq 2$,*

$$\|\chi[k-1]\| > c_9\|\overline{w}\|_\infty \quad \Rightarrow \quad \|f_1[k]\| \leq c_{10}\|\overline{w}\|_\infty. \tag{5.66}$$

[21] Note that $\check{\tau}$ and $\check{g}$ may not belong to ℓ_∞; however both of these signals are intermediary in nature and are used in the description of K to enhance clarity.

Proof. Let $k \geq 2$ be arbitrary. Taking the norm of f_1 and using (5.17) and (5.18) yields

$$\|f_1[k]\| \leq \bar{b}\|\tilde{F}[k]\|\|\chi[k]\| + \bar{b}\|\hat{F}[k]\|\|\tilde{\chi}[k]\| + c_\zeta\|\overline{w}\|_\infty.$$

From Lemma 5.2, clearly $\|\tilde{\chi}[k]\| \leq c_2\|\overline{w}\|_\infty$, and since $\hat{F}[k] \in \mathcal{F}$, it follows from (5.18) that

$$\|f_1[k]\| \leq \bar{b}\|\tilde{F}[k]\|\|\chi[k]\| + \underbrace{(\overline{b}\overline{f}c_2 + c_\zeta)}_{=:\gamma_1}\|\overline{w}\|_\infty. \tag{5.67}$$

To finish proving the result, we need to bound $\|\tilde{F}[k]\|$; to do so, we use Lemma 5.1 which states that

$$\|\tilde{F}[k]\| \leq c_1(|\tilde{\tau}[k]| + |\tilde{g}[k]|),$$

and then we use Lemmas 5.4 and 5.5, which state that if

$$\|\chi[k-1]\| > c_7\|\overline{w}\|_\infty,$$

then

$$|\tilde{\tau}[k]| \leq \frac{c_6\|\overline{w}\|_\infty}{\|\chi[k-1]\|}$$

and

$$|\tilde{g}[k]| \leq \frac{c_8\|\overline{w}\|_\infty}{\|\chi[k-1]\|}.$$

Hence for $\|\chi[k-1]\| > c_7\|\overline{w}\|_\infty$ it follows that

$$\|\tilde{F}[k]\| \leq \frac{c_1(c_6 + c_8)\|\overline{w}\|_\infty}{\|\chi[k-1]\|},$$

so

$$\|f_1[k]\| \leq \left(\underbrace{\bar{b}c_1(c_6 + c_8)}_{=:\gamma_2}\frac{\|\chi[k]\|}{\|\chi[k-1]\|} + \gamma_1\right)\|\overline{w}\|_\infty. \tag{5.68}$$

Taking the norm of both sides of (5.63) (with k replaced by $k-1$), and using (5.28) to upper bound $|\nu[k-1]|$ it follows that

$$\|\chi[k]\| \leq (\|A_d\| + \bar{b}(\rho + \overline{f}))\|\chi[k-1]\| + (c_2\bar{b}(\rho + \overline{f}) + c_\zeta)\|\overline{w}\|_\infty; \tag{5.69}$$

so if $\|\chi[k-1]\| > c_7\|\overline{w}\|_\infty$, then

$$\frac{\|\chi[k]\|}{\|\chi[k-1]\|} \leq \underbrace{(\|A_d\| + \bar{b}(\rho + \overline{f})) + \frac{(c_2\bar{b}(\rho + \overline{f}) + c_\zeta)}{c_7}}_{=:\gamma_3},$$

so returning to (5.68), we have that if $\|\chi[k-1]\| > c_7\|\overline{w}\|_\infty$, then

$$\|f_1[k]\| \le \underbrace{(\gamma_2\gamma_3 + \gamma_1)}_{=:c_{10}}\|\overline{w}\|_\infty. \tag{5.70}$$

Finally, using (5.69), it is clear that if $\|\chi[k-1]\| \le c_7\|\overline{w}\|_\infty$, then

$$\|\chi[k]\| \le \underbrace{((\|A_d\| + \overline{b}(\rho + \overline{f}))c_7 + c_2\overline{b}(\rho + \overline{f}) + c_\zeta)}_{=:c_9}\|\overline{w}\|_\infty,$$

so, if $\|\chi[k]\| > c_9\|\overline{w}\|_\infty$, then $\|\chi[k-1]\| > c_7\|\overline{w}\|_\infty$, and (5.70) holds. ☐

Recalling that $P(\tau, g)$ is the unique positive definite solution to (5.22), we analyze the Lyapunov function $V(\chi[k]) := \chi^{'}[k]P(\tau, g)\chi[k]$.

Claim 2. *There exists constants $c_{11}, \delta > 0$ and $\lambda_1 \in [0, 1)$ so that for all $k \ge 2$,*

$$\sqrt{V(\chi[k+1])} \le \begin{cases} \lambda_1\sqrt{V(\chi[k])} + c_{11}\|\overline{w}\|_\infty, & \text{if } \sqrt{V(\chi[k])} > \delta\|\overline{w}\|_\infty \\ c_{11}\sqrt{V(\chi[k])} + c_{11}\|\overline{w}\|_\infty, & \text{if } \sqrt{V(\chi[k])} \le \delta\|\overline{w}\|_\infty \end{cases}$$

Proof. Let $k \ge 2$ be arbitrary. We will drop the τ and g arguments from most terms and functions going forward to enhance readability.

We start by bounding the growth of $\sqrt{V(\chi)}$ over a single time step; to do so, we take the norm of both sides of (5.63) and use (5.28) to upper bound $|\nu[k]|$, yielding:

$$\|\chi[k+1]\| \le \left(\|A_d\| + \overline{b}(\overline{f} + \rho)\right)\|\chi[k]\| + \left(\overline{b}c_2(\overline{f} + \rho) + c_\zeta\right)\|\overline{w}\|_\infty. \tag{5.71}$$

From Proposition 5.2, it is clear that

$$\sqrt{\frac{V(\chi[k])}{c_4}} \le \|\chi[k]\| \le \sqrt{\frac{V(\chi[k])}{c_3}},$$

so

$$\sqrt{V(\chi[k+1])} \le \underbrace{\left(\frac{\sqrt{c_4}}{\sqrt{c_3}}(\|A_d\| + \overline{b}(\overline{f} + \rho))\right)}_{=:\gamma_1}\sqrt{V([\chi[k])} + \underbrace{\left(\sqrt{c_4}\overline{b}c_2(\overline{f} + \rho) + c_\zeta\right)}_{=:\gamma_2}\|\overline{w}\|_\infty. \tag{5.72}$$

Next, we consider

$$\Delta V[k] := V[k+1] - V[k],$$

yielding:

$$\begin{aligned}\Delta V[k] = \chi^{'}[k]((A_d + B_d F)^{'} P(A_d + B_d F) - P)\chi[k] + f_1^{'}[k] P f_1[k] + f_2^{'}[k] P f_2[k] + \\ +2\chi^{'}[k](A_d + B_d F)^{'} P f_1[k] + 2\chi^{'}[k](A_d + B_d F)^{'} P f_2[k] + 2 f_1^{'}[k] P f_2[k].\end{aligned}$$

Taking the infinity norm on the right hand side, using Proposition 5.2 to upper bound $\|P\|$, and then using the Lyapunov equation (5.22) yields

$$\begin{aligned}\Delta V[k] \leq -\|\chi[k]\|^2 + nc_4\|f_1[k]\|^2 + nc_4\|f_2[k]\|^2 + 2nc_4\|(A_d + B_d F)\| \times \|\chi[k]\| \times \|f_2[k]\| + \\ 2nc_4\|f_1[k]\| \times \|f_2[k]\| + 2nc_4\|(A_d + B_d F)\| \times \|\chi[k]\| \times \|f_1[k]\|.\end{aligned} \tag{5.73}$$

We would like to use the bound on $\|f_1[k]\|$ from Claim 1; to this end, using Proposition 5.2 it follows that

$$\|\chi[k]\| \geq \frac{\sqrt{V(\chi[k])}}{\sqrt{c_4}},$$

so if

$$\sqrt{V(\chi[k])} > \sqrt{c_4} c_9 \|\overline{w}\|_\infty$$

then

$$\|\chi[k]\| > c_9 \|\overline{w}\|_\infty.$$

Defining

$$\delta := \sqrt{c_4} c_9,$$

henceforth, we restrict

$$\sqrt{V(\chi[k])} > \delta \|\overline{w}\|_\infty,$$

which means that the left hand side of (5.66) holds. Using (5.66) to upper bound $\|f_1[k]\|$, the upper bound on $\|f_2[k]\|$ from (5.65) and the upper bound on $\|A_d + B_d F\|$ from (5.23), from (5.73) we obtain

$$\begin{aligned}\Delta V[k] \leq (-1 + nc_4\rho^2\overline{b}^2 + 2nc_4\overline{a}\rho\overline{b})\|\chi[k]\|^2 + (nc_4 c_{10}^2 + nc_4\rho^2\overline{b}^2 c_2^2 + 2nc_4 c_{10}\rho\overline{b} c_2)\|\overline{w}\|_\infty^2 + \\ 2(nc_4\rho^2\overline{b}^2 c_2 + nc_4 c_{10}\overline{a} + nc_4\overline{a}\rho\overline{b} c_2 + nc_4 c_{10}\rho\overline{b})\|\chi[k]\|\|\overline{w}\|_\infty;\end{aligned} \tag{5.74}$$

observe that, using (5.25), the weight on $\|\chi[k\|^2$ is negative. To eliminate the cross terms from (5.74), we note that, for any $\epsilon > 0$:

$$(\epsilon\|\chi[k]\| - \epsilon^{-1}\|\overline{w}\|_\infty)^2 \geq 0 \qquad \Leftrightarrow \qquad 2\|\chi[k]\|\|\overline{w}\|_\infty \leq \epsilon^2\|\chi[k]\|^2 + \epsilon^{-2}\|\overline{w}\|_\infty^2,$$

so, returning to (5.74), it follows for any $\epsilon > 0$ that

$$\Delta V[k] \leq \left[-1 + nc_4(\rho^2\bar{b}^2 + 2\bar{a}\rho\bar{b} + \epsilon^2(\rho^2\bar{b}^2 c_2 + c_{10}\bar{a} + \bar{a}\rho\bar{b}c_2 + c_{10}\rho\bar{b}))\right] \|\chi[k]\|^2 + \\ nc_4(c_{10}^2 + \rho^2\bar{b}^2 c_2^2 + 2c_{10}\rho\bar{b}c_2 + \epsilon^{-2}(\rho^2\bar{b}^2 c_2 + c_{10}\bar{a} + \bar{a}\rho\bar{b}c_2 + c_{10}\rho\bar{b}))\|\overline{w}\|_\infty^2.$$

Recalling from Remark 3 that ρ was chosen so that

$$nc_4(\rho^2\bar{b}^2 + 2\bar{a}\rho\bar{b}) < 1,$$

it follows that by choosing $\epsilon > 0$ sufficiently small, we can define $\gamma_3 \in (0,1)$ and $\gamma_4 > 0$ so that

$$\Delta V[k] \leq -\gamma_3\|\chi[k]\|^2 + \gamma_4\|\overline{w}\|_\infty^2. \tag{5.75}$$

This, in turn, implies that

$$V(\chi[k+1]) \leq \left(1 - \frac{\gamma_3}{c_4}\right) V(\chi[k]) + \gamma_4\|\overline{w}\|_\infty^2.$$

Hence,

$$\sqrt{V(\chi[k+1])} \leq \underbrace{\max\{0, 1 - \frac{\gamma_3}{c_4}\}}_{=:\lambda_1 \in [0,1)} \sqrt{V(\chi[k])} + \sqrt{\gamma_4}\|\overline{w}\|_\infty, \qquad \sqrt{V(\chi[k])} > \delta\|\overline{w}\|_\infty. \tag{5.76}$$

Defining

$$c_{11} := \max\{\gamma_1, \gamma_2, \sqrt{\gamma_4}\}$$

and combining (5.72) and (5.76) completes the proof. $\square$

Claim 3. *There exists constants $c_{12} > 0$ and $\lambda_1 \in [0,1)$ so that*

$$\|\chi[k]\| \leq c_{12}\lambda_1^k \left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + c_{12}\|\overline{w}\|_\infty, \qquad k \geq 0.$$

Proof. From Claim 2, it is clear that for $k \geq 2$:

$$\begin{aligned} \sqrt{V(\chi[k+1])} &\leq \lambda_1\sqrt{V(\chi[k])} + \begin{cases} c_{11}\|\overline{w}\|_\infty, & \text{if } \sqrt{V(\chi[k])} > \delta\|\overline{w}\|_\infty \\ (c_{11} - \lambda_1)\sqrt{V(\chi[k])} + c_{11}\|\overline{w}\|_\infty, & \text{if } \sqrt{V(\chi[k])} \leq \delta\|\overline{w}\|_\infty \end{cases} \\ &\leq \lambda_1\sqrt{V(\chi[k])} + \begin{cases} c_{11}\|\overline{w}\|_\infty, & \text{if } \sqrt{V(\chi[k])} > \delta\|\overline{w}\|_\infty \\ [c_{11}\delta + c_{11}]\,\|\overline{w}\|_\infty, & \text{if } \sqrt{V(\chi[k])} \leq \delta\|\overline{w}\|_\infty \end{cases} \\ &\leq \lambda_1\sqrt{V(\chi[k])} + (c_{11}\delta + c_{11})\|\overline{w}\|_\infty, \qquad k \geq 2. \end{aligned} \tag{5.77}$$

Recursively applying (5.77) and noting that $|\lambda| < 1$, yields

$$\sqrt{V(\chi[k])} \leq \lambda_1^{k-k_0}\sqrt{V(\chi[2])} + (1-\lambda)^{-1}(c_{11}\delta + c_{11})\|\overline{w}\|_\infty \quad k \geq 2. \tag{5.78}$$

From Proposition 5.2, it is clear that

$$\sqrt{c_3}\,\|\chi[k]\| \leq \sqrt{V(\chi[k])} \leq \sqrt{c_4}\|\chi[k]\|,$$

so (5.78) yields

$$\|\chi[k]\| \leq \lambda_1^{k-2}\underbrace{\sqrt{\frac{c_4}{c_3}}}_{=:\gamma_1}\|\chi[2]\| + \underbrace{\left(\frac{c_{11}\delta + c_{11}}{\sqrt{c_3}(1-\lambda_1)}\right)}_{=:\gamma_2}\|\overline{w}\|_\infty, \qquad k \geq 2. \tag{5.79}$$

Using the discretized state equation (5.15), (5.28) to upper bound $|\nu[1]|$, (5.27) and Lemma 5.2 to upper bound $|\nu[0]|$, (5.16) to upper bound ϕ, and (5.17) to upper bound $\|\zeta\|_\infty$, we have that

$$\begin{aligned}\|\chi[2]\| &\leq \underbrace{(\|A_d\| + \overline{b}(\rho + \overline{f}))}_{=:\gamma_3}\|\chi[1]\| + \underbrace{(\overline{b}c_2(\rho + \overline{f}) + c_\zeta)}_{=:\gamma_4}\|\overline{w}\|_\infty,\\ \|\chi[1]\| &\leq \gamma_3\|x_0\| + \gamma_4\|\overline{w}\|_\infty + \underbrace{(\overline{b}\overline{c}_2(\rho + \overline{f}) + c_\phi)}_{=:\gamma_5}\sup_{\theta\in[-\overline{\tau},0)}|u_0(\theta)|.\end{aligned}$$

Since $\lambda_1^{k-2} \leq \lambda_1^{-2}$ for $k \geq 0$, it follows from (5.79) that

$$\begin{aligned}\|\chi[k]\| &\leq \lambda_1^{k-2}\gamma_1\gamma_3\|\chi[1]\| + (\lambda_1^{-2}\gamma_1\gamma_4 + \gamma_2)\|\overline{w}\|_\infty, & k \geq 1\\ &\leq \lambda_1^k\underbrace{\lambda_1^{-2}\gamma_1\gamma_3^2}_{=:\gamma_6}\|x_0\| + \lambda_1^k\underbrace{\lambda_1^{-2}\gamma_1\gamma_3\gamma_5}_{=:\gamma_7}\sup_{\theta\in[-\overline{\tau},0)}|u_0(\theta)| +\\ &\quad\underbrace{(\lambda_1^{-2}\gamma_1\gamma_4(1+\gamma_3) + \gamma_2)}_{=:\gamma_8}\|\overline{w}\|_\infty, & k \geq 0.\end{aligned}$$

Hence, defining

$$c_{12} := \max\{\gamma_6 + \gamma_7, \gamma_8\},$$

it follows that

$$\|\chi[k]\| \leq c_{12}\lambda_1^k(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)}|u_0(\theta)|) + c_{12}\|\overline{w}\|_\infty, \qquad k \geq 0. \qquad \square$$

We are now in a position to prove (ii), starting with (ii)-(a). Since, for all $k \geq 0$, we have that $\hat{\tau}[k] \in [0, \overline{\tau}]$ and $\hat{g}[k] \in [1, \overline{g}]$, it follows immediately that $\hat{\tau}, \hat{g} \in \ell_\infty$. Furthermore, since $\hat{B}_d[k] \in \mathcal{B}$ for all $k \in \mathbf{Z}^+$, it follows from (5.18) that $\hat{B}_d \in \ell_\infty$; similarly, since $\hat{F}[k] \in \mathcal{F}$ for all $k \in \mathbf{Z}^+$, it follows from (5.18) that $\hat{F} \in \ell_\infty$. Finally, from Lemma 5.2, we have for $k \geq 0$ that

$$\|\hat{\chi}[k]\| \leq \|\chi[k]\| + c_2\|\overline{w}\|_\infty + \delta[k]\overline{c}_2 \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|,$$

and since Claim 3 implies that $\chi \in \ell_\infty$, it follows that $\hat{\chi} \in \ell_\infty$, so we conclude that (ii)-(a) holds.

We now consider (ii)-(b) and (c). We start by bounding the control signal ν; from (5.28) we have that

$$|\nu[k]| \leq \underbrace{(\rho + \overline{f})}_{=:\gamma_1} \|x(kT)\| + \underbrace{c_2(\rho + \overline{f})}_{=:\gamma_2} \|\overline{w}\|_\infty, \qquad k \geq 1,$$

and using (5.27) and Lemma 5.2, it follows that

$$|\nu[0]| \leq \gamma_1\|x(kT)\| + \gamma_2\|\overline{w}\|_\infty + \gamma_1\overline{c}_2 \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|,$$

so

$$|\nu[k]| \leq \gamma_1\|x(kT)\| + \gamma_2\|\overline{w}\|_\infty + \delta[k]\gamma_1\overline{c}_2 \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|, \qquad k \geq 0.$$

Using Claim 3 to bound $\|x(kT)\|$ and observing that $\delta[k] \leq \lambda_1^k$ for $k \geq 0$, we obtain

$$|\nu[k]| \leq c_{12}\gamma_1\lambda_1^k \left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + \underbrace{(\gamma_1 c_{12} + \gamma_2)}_{=:\overline{\gamma}_2} \|\overline{w}\|_\infty + \lambda_1^k\gamma_1\overline{c}_2 \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|$$

for $k \geq 0$. So (ii)-(b) follows if we set

$$\overline{\gamma}_1 := \gamma_1(c_{12} + \overline{c}_2)$$

and

$$\lambda := \frac{1}{T}\ln(\lambda_1),$$

we have that

$$|\nu[k]| \leq \overline{\gamma}_1\lambda_1^k \left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + \overline{\gamma}_2\|\overline{w}\|_\infty \tag{5.80}$$

Now we turn to bounding the plant state. While Claim 3 provides a bound on $\|x(kT)\|$, we need a bound on the inter-sample behaviour. Solving (5.10) yields

$$x(t) = e^{A(t-kT)}x(kT) + \int_{kT}^{t} e^{A(t-\theta)}B[u(\theta-\tau)+d(\theta-\tau)]d\theta,\, t \in [kT, kT+T),$$

which implies that

$$\|x(t)\| \leq e^{\|A\|T}\|x(kT)\| + Te^{\|A\|T}\|B\| \sup_{\theta\in[kT,KT+T)} (|u(\theta-\tau)| + |d(\theta-\tau)|),\, t \in [kT, kT+T).$$

By the definition of $\overline{w}$, we have for all $k \geq 0$ that $\sup_{\theta\in[kT,KT+T)} |d(\theta-\tau)| \leq \|\overline{w}\|_\infty$, and clearly,

$$\sup_{\theta\in[kT,KT+T)} |u(\theta-\tau)| \leq |\nu[k]| + \delta[k] \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|,$$

so using Claim 3, (5.80), and noting that $\delta[k] \leq \lambda_1^k$ for $k \geq 0$, we have that there exists positive constants γ_3 and $\overline{\gamma}_4$ so that

$$\|x(t)\| \leq \gamma_3 e^{\lambda kT}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + \overline{\gamma}_4\|\overline{w}\|_\infty, \quad t \in [kT, KT+T),\ k \geq 0,$$

so it follows immediately that

$$\|x(t)\| \leq \underbrace{\gamma_3 e^{-\lambda T}}_{=:\overline{\gamma}_3} e^{-\lambda t}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + \overline{\gamma}_4\|\overline{w}\|_\infty, \quad t \geq 0. \tag{5.81}$$

Defining

$$c := \max\{\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3, \overline{\gamma}_4\}$$

and using (5.80) and (5.81) proves (ii)-(b) and (c) respectively.

Finally, part (i) follows immediately from parts (ii)-(b) and (ii)-(c). □

Using the results of Theorem 5.1, we can show that the controller K is both robust to some plant uncertainty and that it can tolerate infrequent but possibly persistent jumps in the unknown gain and delay, which we do in the following two sub-sections.

5.4.1 Robustness to Plant Uncertainty

We will now show that the controller K is robust to small plant uncertainties. In order to set up our plant uncertainty model, we first note that since the nominal plant G_0 is finite dimensional and linear that there exists stable transfer functions[22] M_0, N_0, X_0 and Y_0 such that

$$G_0 = N_0 M_0^{-1} = M_0^{-1} N_0 \quad \text{and} \quad N_0 X_0 + M_0 Y_0 = 1.$$

Using this co-prime factorization, our (quite general) uncertainty set on the nominal plant is given by

$$\mathcal{G}_0(\delta_N, \delta_M) := \left\{ (N_0 + \Delta_N)(M_0 + \Delta_M)^{-1} : \ \|\Delta_N\| \leq \delta_N, \|\Delta_M\| \leq \delta_M \right\},$$

resulting in an overall uncertainty set

$$\overline{\mathcal{G}}(\delta_N, \delta_M) := \left\{ g e^{-s\tau} G(s) : \ \tau \in [0, \overline{\tau}], g \in [1, \overline{g}], G \in \mathcal{G}_0(\delta_N, \delta_M) \right\}.$$

We will show that there exists constants $\delta_N > 0$ and $\delta_M > 0$ such that K stabilizes $\overline{\mathcal{G}}(\delta_N, \delta_M)$.

We proceed by denoting the controller (including the sampler and hold) by K and write the unknown gain and delay as its own block as shown in Figure 5.4. Defining $\overline{u}(t) := gu(t-\tau)$, and $\overline{d}(t) := gd(t-\tau)$,[23] we can convert the setup of Figure 5.4 to that of Figure 5.5 with the controller K_d incorporating K as well as the gain and delay. With zero initial conditions on the plant, i.e., $x_0 = 0$, $u_0(\theta) = d_0(\theta) = 0$ for $\theta \in [\overline{\tau}, 0)$ it follows that the map from $\begin{bmatrix} d \\ w \end{bmatrix} \to \begin{bmatrix} y \\ u \end{bmatrix}$ is uniformly bounded (with respect to $\tau \in [0, \overline{\tau}]$ and $g \in [1, \overline{g}]$) if and only if the map from $\begin{bmatrix} \overline{d} \\ w \end{bmatrix} \to \begin{bmatrix} y_w \\ \overline{u} \end{bmatrix}$ is uniformly bounded; hence we proceed by analyzing the latter map. From Theorem 5.1, we have that the map from $\begin{bmatrix} \overline{d} \\ w \end{bmatrix} \to \begin{bmatrix} y_w \\ \overline{u} \end{bmatrix}$ is uniformly bounded if $\overline{\mathcal{G}} = \mathcal{G}$, i.e., $\Delta_N = \Delta_M = 0$, so we start by using

[22] For brevity, we will not distinguish between a transfer function and its corresponding map from $PC_\infty \to PC$.

[23] Since $d(\theta) = 0$ for $\theta \in [-\overline{\tau}, 0)$ we do not lose any meaningful part of d with this transformation.

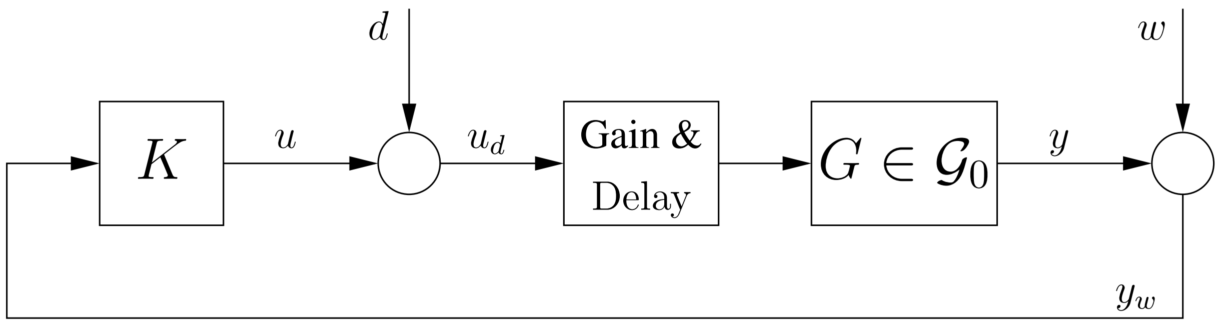

Figure 5.4: The basic feedback setup with the gain and delay separated from the plant.

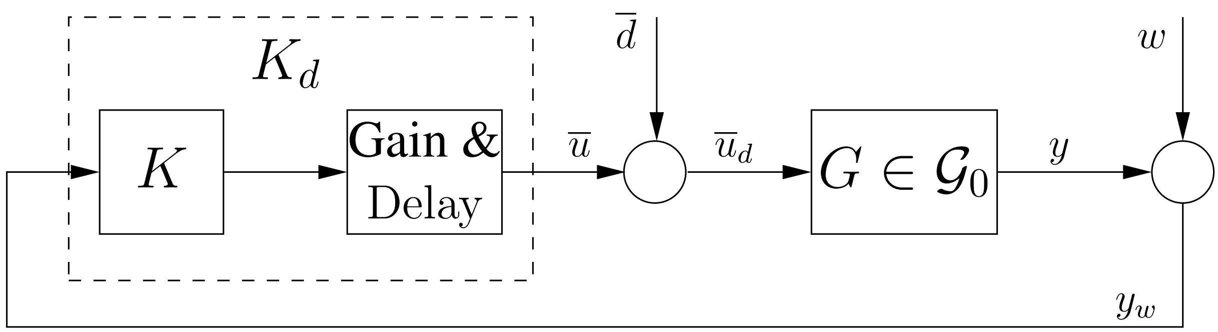

Figure 5.5: The converted feedback setup.

Figure 5.5 to write y_w and $\overline{u}$ for the nominal plant G_0; doing so yields

$$\begin{aligned}
& & y_w &= G_0\left(\overline{u}+\overline{d}\right)+w \\
&\Leftrightarrow & y_w &= G_0K_d(y_w)+G_0\overline{d}+w \\
&\Leftrightarrow & (1-G_0K_d)\,y_w &= G_0\overline{d}+w \\
&\Leftrightarrow & y_w &= (1-G_0K_d)^{-1}\left(G_0\overline{d}+w\right) \\
&\Leftrightarrow & y_w &= (1-M_0^{-1}N_0K_d)^{-1}(M_0^{-1}N_0\overline{d}+w) \\
&\Leftrightarrow & y_w &= \left[M_0^{-1}(M_0-N_0K_D)\right]^{-1}(M_0^{-1}N_0\overline{d}+w) \\
&\Leftrightarrow & y_w &= (M_0-N_0K_d)^{-1}(N_0\overline{d}+M_0w),
\end{aligned} \tag{5.82}$$

and $\overline{u}$ as

$$\begin{aligned}
& & \overline{u} &= K_dy_w \\
&\Leftrightarrow & \overline{u} &= K_d(M_0-N_0K_d)^{-1}(N_0\overline{d}+M_0w).
\end{aligned} \tag{5.83}$$

Since the map from $\begin{bmatrix} \overline{d} \\ w \end{bmatrix} \rightarrow \begin{bmatrix} y_w \\ \overline{u} \end{bmatrix}$ is uniformly bounded, it follows by analyzing (5.82) and (5.83) (first with $\overline{d}=0$ and w free, and then with $w=0$ and $\overline{d}$ free) that

$$(M_0-N_0K_d)^{-1}M_0, \tag{5.84}$$
$$(M_0-N_0K_d)^{-1}N_0, \tag{5.85}$$
$$K_d(M_0-N_0K_d)^{-1}M_0, \tag{5.86}$$
$$K_d(M_0-N_0K_d)^{-1}N_0, \tag{5.87}$$

are all uniformly bounded. Combining (5.84) and (5.85) it follows that

$$\begin{aligned}
& & (M_0-N_0K_d)^{-1}\begin{bmatrix} M_0 & N_0 \end{bmatrix} & \quad \text{is uniformly bounded} \\
&\Rightarrow & (M_0-N_0K_d)^{-1}\begin{bmatrix} M_0 & N_0 \end{bmatrix}\begin{bmatrix} Y_0 \\ X_0 \end{bmatrix} & \quad \text{is uniformly bounded} \\
&\Rightarrow & (M_0-N_0K_d)^{-1} & \quad \text{is uniformly bounded,}
\end{aligned} \tag{5.88}$$

and similarly, combining (5.86) and (5.87), it follows that

$$\begin{aligned}
& & K_d(M_0-N_0K_d)^{-1}\begin{bmatrix} M_0 & N_0 \end{bmatrix} & \quad \text{is uniformly bounded} \\
&\Rightarrow & K_d(M_0-N_0K_d)^{-1}\begin{bmatrix} M_0 & N_0 \end{bmatrix}\begin{bmatrix} Y_0 \\ X_0 \end{bmatrix} & \quad \text{is uniformly bounded} \\
&\Rightarrow & K_d(M_0-N_0K_d)^{-1} & \quad \text{is uniformly bounded.}
\end{aligned} \tag{5.89}$$

We now recompute the closed loop system equations for a plant in $\mathcal{G}_0(\delta_M, \delta_N)$; this yields

$$\begin{aligned}
y_w &= [M_0 + \Delta_M - (N_0 + \Delta_N)K_d]^{-1} [(N_0 + \Delta_N)\overline{d} + (M_0 + \Delta_M)w], \\
&= [M_0 - N_0 K_d + (\Delta_M - \Delta_N K_d)]^{-1} \left[(N + \Delta_N)\overline{d} + (M + \Delta_M)w\right] \\
&= \left\{\left[1 - (\Delta_M - \Delta_N K_d)(M_0 - N_0 K_d)^{-1}\right][M_0 - N_0 K_d]\right\}^{-1} \times \\
&\quad \left[(N + \Delta_N)\overline{d} + (M + \Delta_M)w\right] \\
&= (M_0 - N_0 K_d)^{-1} \left[1 - \Delta_M (M_0 - N_0 K_d)^{-1} + \Delta_N K_d (M_0 - N_0 K_d)^{-1}\right]^{-1} \times \\
&\quad \left[(N + \Delta_N)\overline{d} + (M + \Delta_M)w\right], \qquad (5.90)
\end{aligned}$$

and

$$\begin{aligned}
\overline{u} &= K_d (M_0 - N_0 K_d)^{-1} \left[1 - \Delta_M (M_0 - N_0 K_d)^{-1} + \Delta_N K_d (M_0 - N_0 K_d)^{-1}\right]^{-1} \times \\
&\quad \left[(N + \Delta_N)\overline{d} + (M + \Delta_M)w\right]. \qquad (5.91)
\end{aligned}$$

From (5.88) and (5.89) it follows that $(M_0 - N_0 K_d)^{-1}$ and $K_d(M_0 - N_0 K_d)^{-1}$ are uniformly bounded, so clearly, there exists $\delta_N > 0$ and $\delta_M > 0$ such that

$$\delta_M \sup_{\tau \in [0,\overline{\tau}], g \in [1,\overline{g}]} \|(M_0 - N_0 K_d)^{-1}\| + \delta_N \sup_{\tau \in [0,\overline{\tau}], g \in [1,\overline{g}]} \|K_d (M_0 - N_0 K_d)^{-1}\| < 1,$$

so that

$$\left[1 - \Delta_M (M_0 - N_0 K_d)^{-1} + \Delta_N K_d (M_0 - N_0 K_d)^{-1}\right]^{-1}$$

is uniformly bounded. Finally, since M_0, N_0, Δ_N and Δ_M are all bounded, it follows immediately for such a choice of δ_M and δ_N that the map from $\begin{bmatrix} \overline{d} \\ w \end{bmatrix} \to \begin{bmatrix} y_w \\ \overline{u} \end{bmatrix}$ is uniformly bounded, and hence, K stabilizes $\overline{\mathcal{G}}(\delta_M, \delta_N)$.

5.4.2 Jumps in the Gain and Delay

It turns out that the proposed controller has the very desirable property that it can tolerate a degree of variation in the gain and the delay; we consider the case of occasional, possibly persistent, jumps in the gain and delay. To this end, define $\mathcal{T}_{T_s}^{PC}$ to be the set of all piecewise constant functions from $\mathbf{R} \to [0, \overline{\tau}] \times [1, \overline{g}]$ for which discontinuities are at least $T_s > T$ seconds apart. This results in the following set of plants:

$$\mathcal{G}_{T_s}^{PC} := \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + g(t)Bu_d(t - \tau(t)) \\ y(t) & = & Cx(t) \end{array} \;\middle|\; (\tau, g) \in \mathcal{T}_{T_s}^{PC} \right\}.$$

A plant $G \in \mathcal{G}_{T_s}^{PC}$ with noise is given by

$$\left.\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + g(t)B(u(t-\tau(t)) + d(t-\tau(t))) \\ y_w(t) & = & Cx(t) + w(t), \end{array}\right\} \tag{5.92}$$

The following Proposition bounds the one period growth of the plant state and control signal for every $G \in \mathcal{G}_{T_s}^{PC}$:

Proposition 5.3. *There exists a constant $c_{13} > 0$ such that for every $T_s > T$, $G \in \mathcal{G}_{T_s}^{PC}$, plant initial conditions $x_0 \in \mathbf{R}^n$ and $u_0(\theta)$ satisfying $\sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)| < \infty$, controller initial conditions $\hat{\chi}[-1] \in \mathbf{R}^n$ and $\nu[-1] \in \mathbf{R}$, and every $\overline{w} \in PC_\infty$, then*

(i) $|\nu[k]| \leq c_{13}\|x(kT)\| + c_{13}\|\overline{w}\|_\infty, \quad k \geq 1$

(ii) $\|x(t)\| \leq c_{13}\|x(kT)\| + c_{13}\|\overline{w}\|_\infty, \quad t \in [kT, (k+1)T], \ \ k \geq 1$

(iii) $|\nu[0]| \leq c_{13}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + c_{13}\|\overline{w}\|_\infty$

(iv) $\|x(t)\| \leq c_{13}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + c_{13}\|\overline{w}\|_\infty, \quad t \in [0, T].$

Proof. Let $G \in \mathcal{G}_{T_s}^{PC}$, $x_0 \in \mathbf{R}^n$, $\hat{\chi}[-1] \in \mathbf{R}^n$, $\nu[-1] \in \mathbf{R}$, $\overline{w} \in PC_\infty$ and $u_0(\theta)$ such that $\sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)| < \infty$ be arbitrary.

We start by proving (i) and (ii); to do so, let $k \geq 1$ be arbitrary. Since the output of the hold is zero over the first T_1 seconds of each period[24], with minimal effort it can be proven that the claim of Lemma 5.2 still applies:

$$\|\hat{\chi}[k]\| \leq \|\chi[k]\| + \|\tilde{\chi}[k]\| \leq \|\chi[k]\| + c_2\|\overline{w}\|_\infty,$$

and (5.28) still applies:

$$|\nu[k]| \leq (\rho + \overline{f})(\|x(kT)\| + c_2\|\overline{w}\|_\infty).$$

Since $\nu[k]$ is the input to the hold (5.13), it follows that

$$|u_d(t-\tau(t))| \leq |\nu[k]| + \|\overline{w}\|_\infty, \quad t \in [kT, (k+1)T],$$

[24]This holds even if there are switches in the gain and delay.

so, regardless of any switches in the delay and gain, it follows for $t \in [kT,(k+1)T]$ that

$$
\begin{aligned}
\|x(t)\| &\leq \left\|e^{AT}\right\| \|x(kT)\| + \left\| \int_0^t e^{A(t-\theta)} g B u_d(\theta - \tau) d\theta \right\| \\
&\leq \left\|e^{AT}\right\| \|x(kT)\| + \left\| \int_0^t e^{A(t-\theta)} g B (|\nu[k]| + \|\overline{w}\|_\infty) d\theta \right\| \\
&\leq \left\|e^{AT}\right\| \|x(kT)\| + \overline{g} T e^{\|A\|T} \|B\| (|\nu[k]| + \|\overline{w}\|_\infty) \\
&\leq \left[\left\|e^{AT}\right\| + \overline{g} T e^{\|A\|T} \|B\| (\rho + \overline{f})\right] \|x(kT)\| + \overline{g} T e^{\|A\|T} \|B\| (c_2(\rho + \overline{f}) + 1) \|\overline{w}\|_\infty .
\end{aligned}
$$

Defining

$$
\overline{c}_{13} := \max\{\left\|e^{AT}\right\| + \overline{g} T e^{\|A\|T} \|B\| (\rho + \overline{f}), g T e^{\|A\|T} \|B\| (c_2(\rho + \overline{f}) + 1), \rho + \overline{f}, c_2(\rho + \overline{f})\}
$$

yields

$$
\begin{aligned}
|\nu[k]| &\leq \overline{c}_{13} \|x(kT)\| + \overline{c}_{13} \|\overline{w}\|_\infty \quad k \geq 1 \\
\|x(t)\| &\leq \overline{c}_{13} \|x(kT)\| + \overline{c}_{13} \|\overline{w}\|_\infty \quad t \in [kT,(k+1)T], \ \ k \geq 1.
\end{aligned}
$$

We now turn to proving (iii) and (iv); to do so, let $k = 0$. Since the output of the hold is zero over the first T_1 seconds of each period[25], with minimal effort it can be proven that the claim of Lemma 5.2 still applies:

$$
\|\hat{\chi}[0]\| \leq \|x_0\| + c_2 \|\overline{w}\|_\infty + \overline{c}_2 \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)|,
$$

and (5.27) still applies:

$$
|\nu[0]| \leq (\rho + \overline{f})(\|x_0\| + c_2 \|\overline{w}\|_\infty + \overline{c}_2 \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)|).
$$

Since $\nu[0]$ is the input to the hold (5.13), it follows that

$$
|u_d(t - \tau(t))| \leq |\nu[0]| + \|\overline{w}\|_\infty + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)|, \quad t \in [0, T],
$$

[25] This holds since regardless of any switches in the gain and delay, $\tau \in [0, \overline{\tau}]$.

so, regardless of any switches in the delay and gain, it follows that for $t \in [0, T]$:

$$\begin{aligned}
\|x(t)\| &\leq \left\|e^{AT}\right\| \|x_0\| + \left\| \int_0^t e^{A(t-\theta)} g B u_d(\theta - \tau) d\theta \right\| \\
&\leq \left\|e^{AT}\right\| \|x_0\| + \overline{g} T e^{\|A\| T} \|B\| (|\nu[0]| + \|\overline{w}\|_\infty + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)|) \\
&\leq \left[\left\|e^{AT}\right\| + \overline{g} T e^{\|A\| T} \|B\| (\rho + \overline{f})\right] \|x_0\| + \overline{g} T e^{\|A\| T} \|B\| ((\rho + \overline{f}) c_2 + 1) \|\overline{w}\|_\infty + \\
&\quad \overline{g} T e^{\|A\| T} \|B\| (1 + (\rho + \overline{f}) \overline{c}_2) \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| \\
&\leq \overline{c}_{13} \|x_0\| + \overline{c}_{13} \|\overline{w}\|_\infty + [(\rho + \overline{f}) \overline{c}_2 + 1](\overline{g} T e^{\|A\| T} \|B\|) \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)|.
\end{aligned}$$

Defining

$$c_{13} := \max \left\{ \overline{c}_{13} + [(\rho + \overline{f}) \overline{c}_2 + 1](\overline{g} T e^{\|A\| T} \|B\|), (\rho + \overline{f})(1 + \overline{c}_2) \right\}$$

completes the proof. $\square$

Before proving that our controller can tolerate switches in (τ, g), we first need to find a minimum time between these switches. To do so, suppose that (τ, g) has a discontinuity on $[0, T)$ but none on $[T, T_s)$ and that $d = w = u_0 = 0$. For $t \in [0, T)$, from Proposition 5.3 we have that $\|x(t)\| \leq c_{13} \|x_0\|$, while for $t \in [T, T_s]$, we can apply Theorem 5.1, so $\|x(t)\| \leq c e^{\lambda(t-T)} \|x(T)\|$. Combining these two intervals, it follows that

$$\begin{aligned}
&\|x(t)\| \leq c c_{13} e^{-\lambda T} e^{\lambda t} \|x_0\| \qquad t \in [0, T_s], \\
\Rightarrow \quad &\|x(T_s)\| \leq c c_{13} e^{-\lambda T} e^{\lambda T_s} \|x_0\|,
\end{aligned}$$

so $\|x(T_s)\|$ will be less than $\|x_0\|$ if

$$c c_{13} e^{-\lambda T} e^{\lambda T_s} < 1;$$

so we fix such a T_s, and for convenience we choose it to be an integer multiple of T. We now define $c_s := c c_{13} e^{-\lambda T}$ and then choose $\overline{\lambda} < 0$ so that

$$\begin{aligned}
& c_s e^{\lambda T_s} e^{-\overline{\lambda} T_s} = 1 \\
\Rightarrow \qquad & c_s e^{\lambda T_s} = e^{\overline{\lambda} T_s}.
\end{aligned} \tag{5.93}$$

Theorem 5.2.

(i) *K stabilizes $\mathcal{G}^{PC}_{T_s}$.*

(ii) *There exists constants $\overline{c} > 0$ and $\overline{\lambda} \in [0, 1)$ so that when the controller K is applied to the plant (5.92), for every $(\tau, g) \in \mathcal{T}^{PC}_{T_s}$, plant initial conditions $x_0 \in \mathbf{R}^n$, and $u_0(\theta)$ satisfying $\sup_{\theta\in[-\overline{\tau},0)} |u_0| < \infty$, controller initial conditions $\hat{\chi}[-1] \in \mathbf{R}^n$ and $\nu[-1] \in \mathbf{R}$ and for every $\overline{w} \in PC_\infty$, we have that*

(a) *$\hat{\tau}, \hat{g}, \hat{B}_d, \hat{F}$ and $\hat{\chi} \in \ell_\infty$.*[26]

(b) $|\nu[k]| \leq \overline{c}e^{\overline{\lambda}kT} \left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + \overline{c}\|\overline{w}\|_\infty$, $k \geq 0$.

(c) $\|x(t)\| \leq \overline{c}e^{\overline{\lambda}t} \left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + \overline{c}\|\overline{w}\|_\infty$, $t \geq 0$.

Proof. Let $(\tau, g) \in \mathcal{T}^{PC}_{T_s}$, $x_0 \in \mathbf{R}^n$, $\hat{\chi}[-1] \in \mathbf{R}^n$, $\nu[-1] \in \mathbf{R}$, $\overline{w} \in PC_\infty$ and $u_0(\theta)$ such that $\sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)| < \infty$ be arbitrary.

We start by formally stating the switching times for (τ, g); to do so, we must consider two cases, that of a finite number of switches and that of a infinite number. To handle this in the simplest way possible, if there is an finite number of switches, we define new switches[27] after the final switch to create an infinite number of switching times for both cases. To this end, if there is a finite number of switches, let κ denote the number of switches, let t_κ denote the time of the final switch and then define new switching times

$$t_{i+1} := t_i + T_s, \quad i \geq \kappa,$$

So, regardless of the number of switches, we define the monotonically increasing sequence of switching times

$$\{t_i\}_{i=1}^{\infty}.$$

It will also be useful to denote the last integer multiple of the period T before each switch occurs, so to that end, we define

$$k_i = \left\lfloor \frac{t_i}{T} \right\rfloor, \quad i \in \mathbf{N}.$$

We start by proving (ii)-(c). To do so, we first bound the state for $t \in [0, k_2T]$ and then consider two cases for the first period:

[26] Note that $\check{\tau}$ and $\check{g}$ may not belong to ℓ_∞; however both of these signals are intermediary in nature and are used in the description of K to enhance clarity.

[27] Of course, these new switches do not actually alter the value of (τ, g).

Case 1: $k_1 = 0$ (Switch in first period)

We split the interval $[0, k_2T]$ into two intervals, $t \in [0, T]$ and $t \in [T, k_2T]$. For $t \in [0, T]$, we apply Proposition 5.3 (iv), yielding

$$\begin{aligned}
\|x(t)\| &\leq c_{13}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + c_{13}\|\overline{w}\|_\infty, & t \in [0, T] \\
&\leq c_{13}e^{-\lambda T}e^{\lambda t}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + c_{13}\|\overline{w}\|_\infty, & t \in [0, T] \quad (5.94)
\end{aligned}$$

$$\|\chi[1]\| \leq c_{13}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + c_{13}\|\overline{w}\|_\infty. \quad (5.95)$$

For $t \in [T, k_2T]$, since our controller is periodic with period T,

$$u(\theta) = 0 \quad \text{for} \quad \theta \in [T - \overline{\tau}, T),$$

and

$$\sup_{\theta\in[k_iT-\overline{\tau},k_iT)} |d(\theta)| \leq \|\overline{w}\|_\infty$$

so we can apply Theorem 5.1 starting at time T:

$$\|x(t)\| \leq ce^{\lambda(t-T)}\|\chi[1]\| + c\|\overline{w}\|_\infty, \quad t \in [T, k_2T]. \quad (5.96)$$

Substituting (5.95) into (5.96) and recalling the definition of c_s yields

$$\begin{aligned}
\|x(t)\| &\leq cc_{13}e^{-\lambda T}e^{\lambda t}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + (cc_{13}e^{-\lambda T}e^{\lambda t} + c)\|\overline{w}\|_\infty, t \in [T, k_2T] \\
&\leq c_se^{\lambda t}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + (c_s + c)\|\overline{w}\|_\infty, \qquad t \in [T, k_2T] \quad (5.97)
\end{aligned}$$

and combining (5.97) and (5.94) yields for $t \in [0, k_2T]$

$$\|x(t)\| \leq (c_s + c_{13}e^{-\lambda T})e^{\lambda t}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + (c_s + c + c_{13})\|\overline{w}\|_\infty. \quad (5.98)$$

Case 2: $k_1 \geq 1$ (No switch in first period)

For $t \in [0, k_1 T]$, we apply Theorem 5.1, so

$$\|x(t)\| \le ce^{\lambda t}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + c\|\overline{w}\|_\infty, \qquad t \in [0, k_1 T]. \tag{5.99}$$

$$\|\chi[k_1]\| \le ce^{\lambda k_1 T}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + c\|\overline{w}\|_\infty. \tag{5.100}$$

For $t \in [k_1 T, k_2 T]$, we use the same logic as Case 1, namely, we split the interval into two parts, $t \in [k_1 T, (k_1+1)T]$ and $t \in [(k_1+1)T, k_2 T]$. For $t \in [k_1 T, (k_1+1)T]$, we apply Proposition 5.3 (ii) yielding

$$\begin{aligned} \|x(t)\| &\le c_{13}\|\chi[k_1]\| + c_{13}\|\overline{w}\|_\infty, && t \in [k_1 T, (k_1+1)T] \\ &\le c_{13}e^{-\lambda T} e^{\lambda(t-k_1 T)}\|\chi[k_1]\| + c_{13}\|\overline{w}\|_\infty && t \in [k_1 T, (k_1+1)T] \qquad (5.101) \\ \|\chi[k_1+1]\| &\le c_{13}\|\chi[k_1]\| + c_{13}\|\overline{w}\|_\infty, && \qquad (5.102) \end{aligned}$$

and for $t \in [(k_1+1)T, k_2 T]$ since,

$$u(\theta) = 0 \quad \text{for} \quad \theta \in [(k_1+1)T - \overline{\tau}, (k_1+1)T),$$

and

$$\sup_{\theta\in[k_i T-\overline{\tau}, k_i T)} |d(\theta)| \le \|\overline{w}\|_\infty$$

we can apply Theorem 5.1 starting at time $(k_1+1)T$:

$$\|x(t)\| \le ce^{\lambda(t-(k_1+1)T)}\|\chi[k_1+1]\| + c\|\overline{w}\|_\infty, \quad t \in [(k_1+1)T, k_2 T]. \tag{5.103}$$

Substituting (5.102) into (5.103) and recalling the definition of c_s yields

$$\begin{aligned} \|x(t)\| &\le cc_{13}e^{-\lambda T} e^{\lambda(t-k_1 T)}\|\chi[k_1]\| + (cc_{13}e^{-\lambda T} + c)\|\overline{w}\|_\infty, && t \in [(k_1+1)T, k_2 T] \\ &\le c_s e^{\lambda(t-k_1 T)}\|\chi[k_1]\| + (c_s + c)\|\overline{w}\|_\infty, && t \in [(k_1+1)T, k_2 T]. \qquad (5.104) \end{aligned}$$

Combining (5.104) and (5.101) yields

$$\|x(t)\| \le c_s e^{\lambda(t-k_1 T)}\|\chi[k_1]\| + (c_s + c + c_{13})\|\overline{w}\|_\infty, \quad t \in [k_1 T, k_2 T]. \tag{5.105}$$

Substituting (5.100) into (5.105) yields for $t \in [k_1 T, k_2 T]$

$$\|x(t)\| \le cc_s e^{\lambda t}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)} |u_0(\theta)|\right) + (cc_s + c_s + c + c_{13})\|\overline{w}\|_\infty. \tag{5.106}$$

Defining

$$\overline{c}_s := \max\{cc_s, (c_s + c_{13}e^{-\lambda T}), (cc_s + c_s + c + c_{13})\},$$

we combine (5.106) and (5.99) yielding

$$\|x(t)\| \leq \overline{c}_s e^{\lambda t} \left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| \right) + \overline{c}_s \|\overline{w}\|_\infty, \quad t \in [0, k_2 T] \tag{5.107}$$

By noting the definition of $\overline{c}_s$ and the fact that $\overline{\lambda} \leq \lambda$ we can combine both cases by merging (5.107) and (5.98); we thus conclude that in both cases

$$\|x(t)\| \leq \overline{c}_s e^{\overline{\lambda} t} \left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| \right) + \overline{c}_s \|\overline{w}\|_\infty, \qquad t \in [0, k_2 T], \tag{5.108}$$

$$\|\chi[k_2]\| \leq \overline{c}_s e^{\overline{\lambda} k_2 T} \left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| \right) + \overline{c}_s \|\overline{w}\|_\infty. \tag{5.109}$$

For $t \geq k_2 T$, we perform a similar analysis. For each of $i = 2, 3, \cdots$ we split the interval into two parts, $t \in [k_i T, (k_1 + 1)T)$ and $t \in [(k_1 + 1)T, k_{i+1} T)$. For $t \in [k_i T, (k_1 + 1)T)$, we apply Proposition 5.3 yielding

$$\begin{aligned} \|x(t)\| &\leq c_{13} \|\chi[k_i]\| + c_{13} \|\overline{w}\|_\infty \\ &\leq c_{13} e^{\lambda(t - (k_i + 1)T)} \|\chi[k_i]\| + c_{13} \|\overline{w}\|_\infty, \quad t \in [k_i T, (k_i + 1)T], \ \ i \geq 2. \end{aligned} \tag{5.110}$$

In particular, this means that

$$\|\chi[(k_i + 1)T]\| \leq c_{13} \|\chi[k_i]\| + c_{13} \|\overline{w}\|_\infty, \quad i \geq 2. \tag{5.111}$$

For $t \in [(k_1 + 1)T, k_{i+1} T)$, since our controller is periodic with period T,

$$u(\theta) = 0 \quad \text{for} \quad \theta \in [k_i T - \overline{\tau}, k_i T),$$

and

$$\sup_{\theta \in [k_i T - \overline{\tau}, k_i T)} |d(\theta)| \leq \|\overline{w}\|_\infty,$$

we can apply Theorem 5.1 starting at time $(k_i + 1)T$; doing so for $i \geq 2$ yields

$$\|x(t)\| \leq c e^{\lambda(t - (k_i + 1)T)} \|\chi[k_i + 1]\| + c \|\overline{w}\|_\infty, \quad t \in [(k_i + 1)T, k_{i+1} T]. \tag{5.112}$$

Substituting (5.111) into (5.112) yields for $i \geq 2$

$$\begin{aligned} \|x(t)\| &\leq cc_{13}e^{-\lambda T}e^{\lambda(t-k_iT)}\|\chi[k_i]\| + (cc_{13}e^{-\lambda T} + c)\|\overline{w}\|_\infty \\ &\leq c_s e^{\lambda(t-k_iT)}\|\chi[k_i]\| + (c_s + c)\|\overline{w}\|_\infty, \qquad t \in [(k_i+1)T, k_{i+1}T]. \end{aligned} \tag{5.113}$$

Since $c \geq 1$ (so $c_{13}e^{-\lambda T} \leq c_s$), we can combine (5.110) and (5.113), yielding for $i \geq 2$

$$\|x(t)\| \leq c_s e^{\lambda(t-k_iT)}\|\chi[k_i]\| + (c_s + c)\|\overline{w}\|_\infty, \quad t \in [k_iT, k_{i+1}T]. \tag{5.114}$$

Evaluating (5.114) at $t = k_{i+1}T$ and noting that $(c_s + c) \leq \overline{c}_s$ yields

$$\|\chi[k_{i+1}]\| \leq c_s e^{\lambda(k_{i+1}-k_i)T}\|\chi[k_i]\| + \overline{c}_s\|\overline{w}\|_\infty, \quad i \geq 2. \tag{5.115}$$

Applying (5.115) recursively from $i = 2$ and noting that

$$c_s e^{\lambda(k_{i+1}-k_i)T} \leq e^{\overline{\lambda}(k_{i+1}-k_i)T} \leq e^{\overline{\lambda}T_s}, \ \ i \geq 2,$$

yields

$$\begin{aligned} \|\chi[k_3]\| &\leq c_s e^{\lambda(k_3-k_2)T}\|\chi[k_2]\| + \overline{c}_s\|\overline{w}\|_\infty, \\ \|\chi[k_4]\| &\leq c_s e^{\lambda(k_4-k_3)T}\|\chi[k_3]\| + \overline{c}_s\|\overline{w}\|_\infty \\ &\leq c_s^2 e^{\lambda(k_4-k_2)T}\|\chi[k_2]\| + \overline{c}_s(c_s e^{\lambda(k_4-k_3)T} + 1)\|\overline{w}\|_\infty \\ &\leq c_s^2 e^{\lambda(k_4-k_2)T}\|\chi[k_2]\| + \overline{c}_s(e^{\overline{\lambda}T_s} + 1)\|\overline{w}\|_\infty, \\ \|\chi[k_5]\| &\leq c_s e^{\lambda(k_5-k_4)T}\|\chi[k_4]\| + \overline{c}_s\|\overline{w}\|_\infty \\ &\leq c_s^3 e^{\lambda(k_5-k_2)T}\|\chi[k_2]\| + \overline{c}_s(c_s^2 e^{\lambda(k_5-k_3)T} + c_s e^{\lambda(k_5-k_4)T} + 1)\|\overline{w}\|_\infty \\ &\leq c_s^3 e^{\lambda(k_5-k_2)T}\|\chi[k_2]\| + \overline{c}_s(e^{2\overline{\lambda}T_s} + e^{\overline{\lambda}T_s} + 1)\|\overline{w}\|_\infty, \\ &\ \ \vdots \qquad\qquad\qquad\qquad \vdots \\ \|\chi[k_i]\| &\leq c_s^{i-2} e^{\lambda(k_i-k_2)T}\|\chi[k_2]\| + \overline{c}_s\left(\sum_{j=0}^{i-3}(e^{\overline{\lambda}T_s})^j\right)\|\overline{w}\|_\infty, \qquad i \geq 2 \\ &\leq e^{\overline{\lambda}(k_i-k_2)T}\|\chi[k_2]\| + \underbrace{\left(\frac{\overline{c}_s}{1 - e^{\overline{\lambda}T_s}}\right)}_{=:\gamma_1}\|\overline{w}\|_\infty, \qquad i \geq 2. \end{aligned} \tag{5.116}$$

Substituting (5.109) into (5.116) yields

$$\|\chi[k_i]\| \leq \overline{c}_s e^{\overline{\lambda}k_iT}\left(\|x_0\| + \sup_{\theta\in[-\overline{\tau},0)}|u_0(\theta)|\right) + \underbrace{(\overline{c}_s + \gamma_1)}_{:=\overline{c}_{15}}\|\overline{w}\|_\infty, \quad i \geq 2, \tag{5.117}$$

and substituting (5.117) into (5.114) for $t \in [k_i T, k_{i+1} T]$, $i \geq 2$ yields

$$\begin{aligned}\|x(t)\| &\leq c_s e^{\overline{\lambda}(t-k_i T)} \|\chi[k_i]\| + (c_s + c)\|\overline{w}\|_\infty \\ &\leq c_s e^{\overline{\lambda}(t-k_i T)} \overline{c}_{15} e^{\overline{\lambda} k_i T} \left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| \right) + (c_s e^{\lambda(t-k_i T)} \overline{c}_{15} + c_s + c)\|\overline{w}\|_\infty \\ &\leq c_s \overline{c}_{15} e^{\overline{\lambda} t} \left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| \right) + (c_s \overline{c}_{15} + c_s + c)\|\overline{w}\|_\infty, \end{aligned} \tag{5.118}$$

then defining

$$\overline{\gamma} := \max\{c_s \overline{c}_{15} + c_s + c, \overline{c}_s\},$$

we can combine the ranges of (5.118) and (5.108) yielding

$$\|x(t)\| \leq \overline{\gamma} e^{\overline{\lambda} t} \left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| \right) + \overline{\gamma} \|\overline{w}\|_\infty, \quad t \geq 0. \tag{5.119}$$

Using Lemma 5.3 (i) and (iii), it follows for $k \geq 0$ that

$$|\nu[k]| \leq c_{13} \|\chi[k]\| + \delta[k] c_{13} \left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| \right) + c_{13} \|\overline{w}\|_\infty,$$

and evaluating (5.119) at time kT, we have for $k \geq 0$ that

$$\|\chi[k]\| \leq \overline{\gamma} e^{\overline{\lambda} k T} \left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| \right) + \overline{\gamma} \|\overline{w}\|_\infty,$$

hence

$$|\nu[k]| \leq c_{13}(1 + \overline{\gamma}) e^{\overline{\lambda} k T} \left(\|x_0\| + \sup_{\theta \in [-\overline{\tau}, 0)} |u_0(\theta)| \right) + c_{13}(1 + \overline{\gamma}) \|\overline{w}\|_\infty.$$

Defining $\overline{c} := c_{13}(1 + \overline{\gamma})$, it is clear that b) (ii) and (iii) hold; furthermore, parts a) and b) (i) follow immediately from parts b) (ii) and (iii), completing the proof. □

5.5 Simulations

We will now perform numerous simulations demonstrating the effectiveness of our proposed controller. We start by considering the nominal plant

$$G_0 = \frac{s-2}{s^2-1}, \tag{5.120}$$

which has a state space representation given by

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} \quad C = \begin{bmatrix} -1.0607 & -0.3535 \end{bmatrix}.$$

If we use an LTI controller, then from Theorem 15 of [35], this plant has an LTI delay margin less than one, and from [26], it has an LTI gain margin less than four. We will perform numerous simulations on this plant with different ranges for the unknown gain and delay, jumps in the unknown gain and delay, and with plant uncertainty. To perform the simulations on this plant, we set $\rho = 10^{-7}$, place our eigenvalues at $0.05 \pm 0.05j$, set the plant initial conditions to $u_0 = d_0 = 0$, $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, set the controller initial conditions to $\nu[-1] = 1$, and $\chi[-1] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. For each simulation, we will state the uncertainty set, state the time parameters T_1, T_2, T_3, T and h that define our sampler and hold, the value of τ and g used, the magnitude of the noise and how we introduce it.

Example 5.1. We start by performing a simulation at double the LTI delay margin but with a known gain, i.e., our uncertainty set is given by

$$\mathcal{G} = \left\{ g e^{-s\tau} G_0(s) : \ \tau \in [0, 2], g = 1 \right\}.$$

To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.095$, $T_3 = 2.005$, resulting in an overall period $T = 2.4$, we fix $\tau = 1$ and we run the simulation with no noise for the first 30 periods (a total of 72 seconds), then we add random noise with a maximum magnitude of 10^{-2} as shown in Figure 5.6. As can be seen, the controller quickly drives the output to zero with no noise, and clearly stabilizes the system with reasonable performance even when subject to noise.

Example 5.2. Using the plant (5.120), we now consider providing ten times the LTI delay margin, so our uncertainty set is given by:

$$\mathcal{G} = \left\{ g e^{-s\tau} G_0(s) : \ \tau \in [0, 10], g = 1 \right\}.$$

To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.195$, $T_3 = 10.005$, resulting in an overall period $T = 10.5$, we fix $\tau = 8$ and we run the simulation with random noise starting at time zero and with a maximum magnitude of 10^{-2} with the results shown in 5.7. While the controller continues to stabilize the system, there is a significant degradation in the performance when compared to Example 5.1. However, this performance degradation is

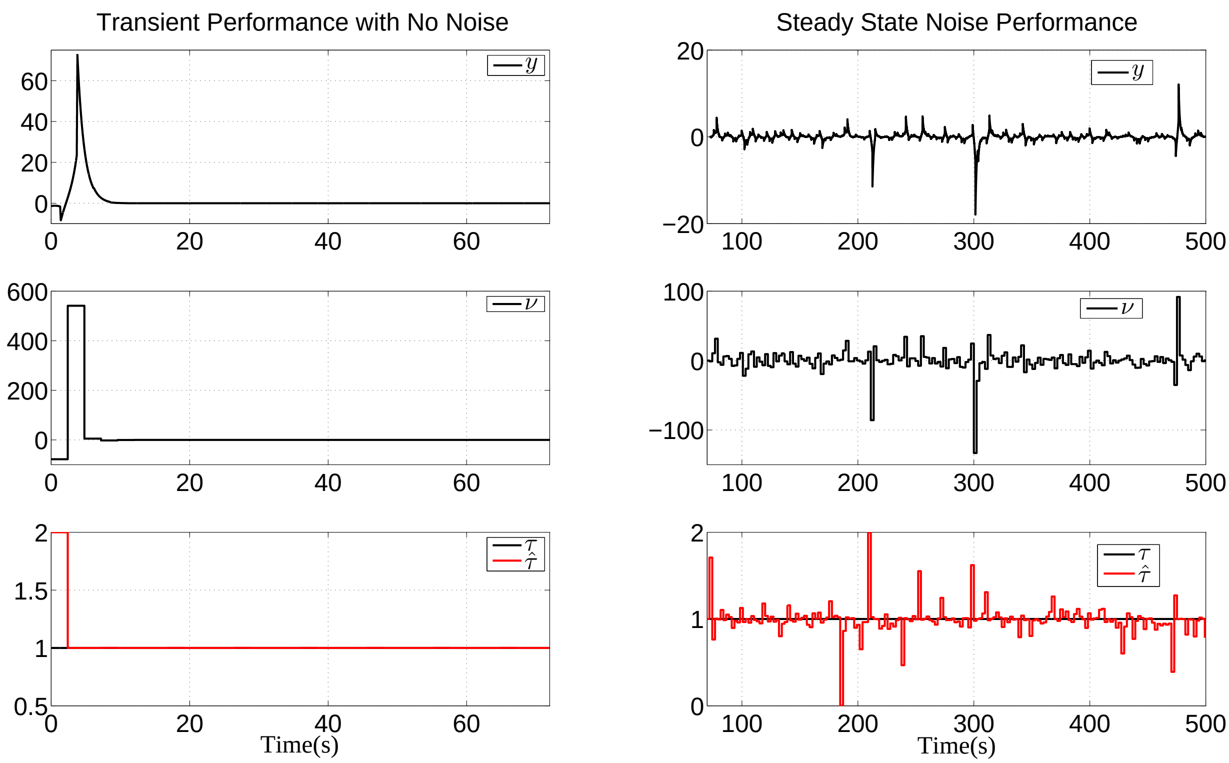

Figure 5.6: Example 5.1: the output, control signal and estimation errors for a simulation at double the LTI delay margin.

to be expected, which can easily be seen by considering the response to the initial condition after the first period which is given by

$$x(T) = e^{AT}x_0;$$

in particular, using our initial condition, for Example 5.1, we have that

$$x(T) = \begin{bmatrix} 0.0907 \\ 11.02 \end{bmatrix}$$

and for this case, we have that

$$x(T) = \begin{bmatrix} 2.756 \times 10^{-5} \\ 36136 \end{bmatrix}.$$

Due to the effect of the initial condition on our estimators, our control signal is not 'good' until the start of the third period; as a result, it takes approximately $2T + T_1 + \tau$ seconds for the plant to receive a stabilizing control signal. If we assume that the control signal and noise are zero for the first two periods, [28] we have that

$$x(2T + T_1 + \tau) = \begin{bmatrix} 1.884 \times 10^{-13} \\ 5.3067 \times 10^{12} \end{bmatrix},$$

which is slightly worse than what we observe in Figure 5.7. Since T is close to the same size as $\overline{\tau}$, even if it only takes two or three times the length of the unknown delay to estimate it accurately, the transient performance will be poor due to the difficulty of the problem.

Example 5.3. Using the same plant (5.120), we now consider the case of only having an unknown gain. Since the LTI gain margin is four [26], we perform the first simulation at double this value, so our uncertainty set is given by

$$\mathcal{G} = \left\{ g e^{-s\tau} G_0(s) : \ \tau = 0, g \in [1, 8] \right\}.$$

To perform the simulation, we set $h = 0.25$, $T_1 = 0.25$, $T_2 = 0.245$, $T_3 = 0.005$, resulting in an overall period $T = 0.5$, we fix $g = 4.5$ and we run the simulation with no noise for the first 30 periods (a total of 15 seconds), then we add random noise with a maximum magnitude of 10^{-2} as shown in Figure 5.8. Despite also being at twice the LTI margin (like the simulation of Figure 5.6), we can see that the performance is significantly improved from Example 5.1, which is mainly due to the reduction of the controller period from 2.4 seconds to 0.5 seconds; since it still takes approximately three periods for our controller to lock into the unknown gain or delay, this means that the plant has far less time to 'blow up' when we only have an unknown gain.

[28] Of course, our control signal has an impact on the plant, so our state does not get quite as large as this simple calculation predicts.

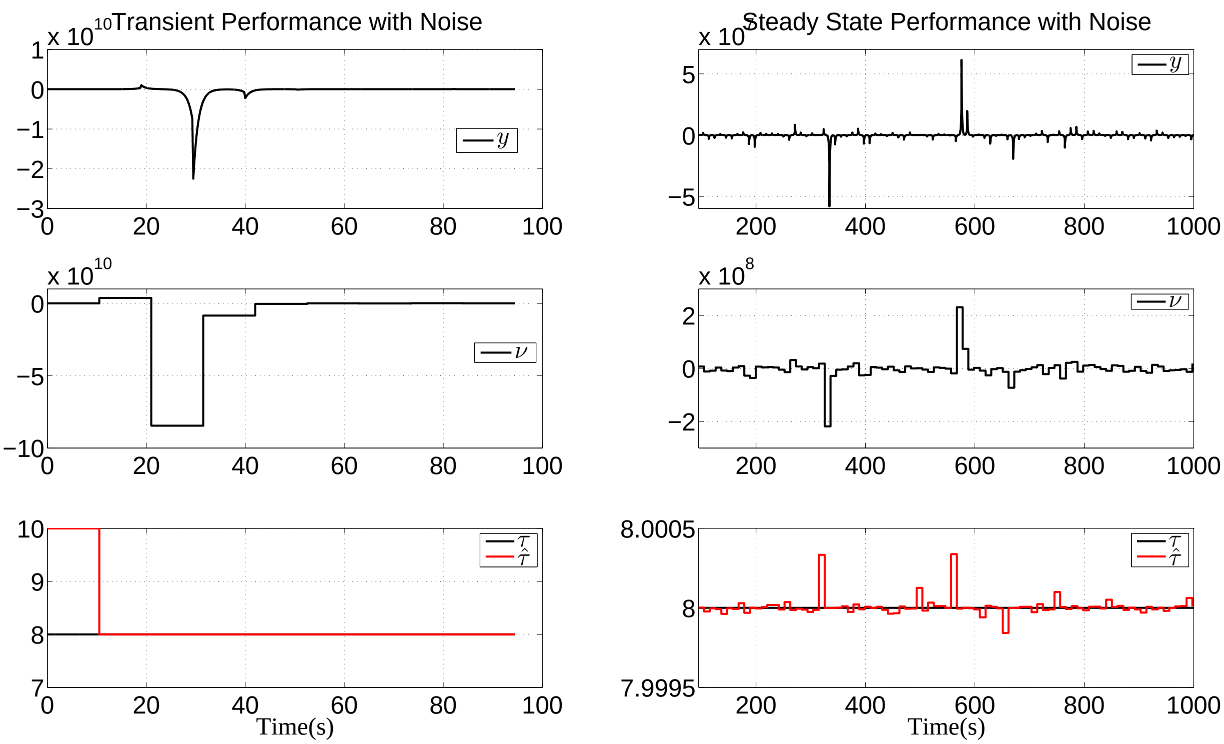

Figure 5.7: Example 5.2: the output, control signal and estimation errors for a simulation at ten times the LTI delay margin.

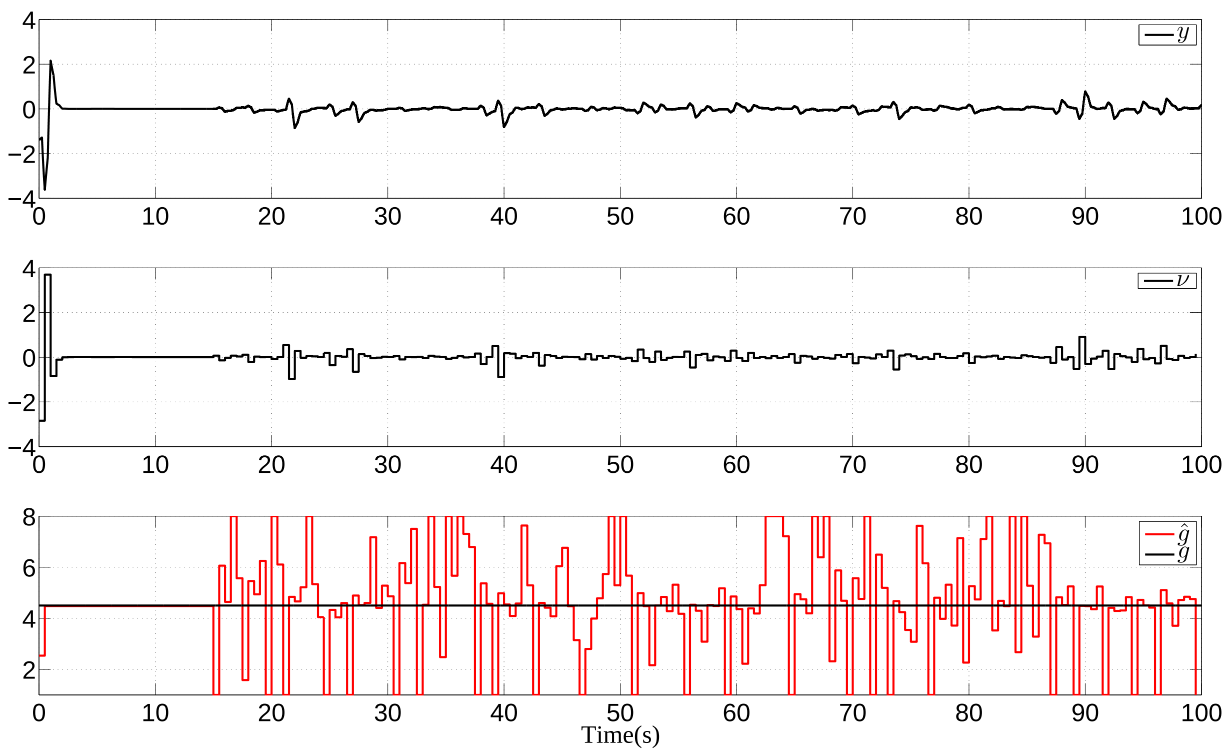

Figure 5.8: Example 5.3: the output, control signal and estimation errors for a simulation at twice the LTI gain margin.

Example 5.4. Using the same plant (5.120), we now consider a more extreme uncertainty set for our unknown gain, namely, one thousand times the LTI gain margin, so our uncertainty set is given by

$$\mathcal{G} = \left\{ g e^{-s\tau} G_0(s) : \ \tau = 0, g \in [1, 4000] \right\}.$$

To perform the simulation, we set $h = 0.25$, $T_1 = 0.25$, $T_2 = 0.245$, $T_3 = 0.005$, resulting in an overall period $T = 0.5$, we fix $g = 4000$ and we run the simulation with no noise for the first 30 periods (a total of 15 seconds), then we add random noise with a maximum magnitude of 10^{-2} as shown in Figure 5.9. Comparing the results to that of the previous example the maximum size of the output is approximately 1000 times worse while our desired margin increased by 500. So the larger size of uncertainty had a negative impact on the performance which was greater than the increase in margin, but not by a large amount. It should also be noted that the nature of our controller can also be clearly seen in the output spikes; we zoom in on one such spike in Figure 5.10. As can be seen at the start of Figure 5.10, the output gets small, $y(86) = -0.385$, and the noise then causes the controller to get a poor estimate of g; this in turn creates a poor control signal, leading to the output quickly increasing in size. Two periods later ($t = 87$), the previous output (and hence the previous state) is large, and from Lemma 5.5, our estimate of the unknown gain is greatly improved, which produces a control which rapidly drives the output back towards zero.

Example 5.5. Using the same plant (5.120), we now consider the case when we have both an unknown gain and delay. We start with a simulation at double the LTI delay margin of two [35] and the LTI gain margin of eight [26] simultaneously, so our uncertainty set is given by

$$\mathcal{G} = \left\{ g e^{-s\tau} G_0(s) : \ \tau \in [0, 2], g \in [1, 8] \right\}.$$

To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.095$, $T_3 = 2.005$, resulting in an overall period $T = 2.4$, we fix $\tau = 1$, $g = 4.5$ and we run the simulation with no noise for the first 30 periods (a total of 72 seconds), then we add random noise with a maximum magnitude of 10^{-2} as shown in Figure 5.11. As can be seen, the initial condition is quickly driven back towards zero, and the controller maintains stability once the noise is added in. It is interesting to note the differences between the results of Figure 5.11 to those of Figure 5.6, which is the same delay margin and operates at the same period, except with a known gain. As can be seen, the transient performance is actually improved when the gain is added in (in particular because the control signal is smaller due to the initial estimate of the gain), but the noise performance is worse by approximately the size of the unknown gain, i.e., a factor of 4.5.

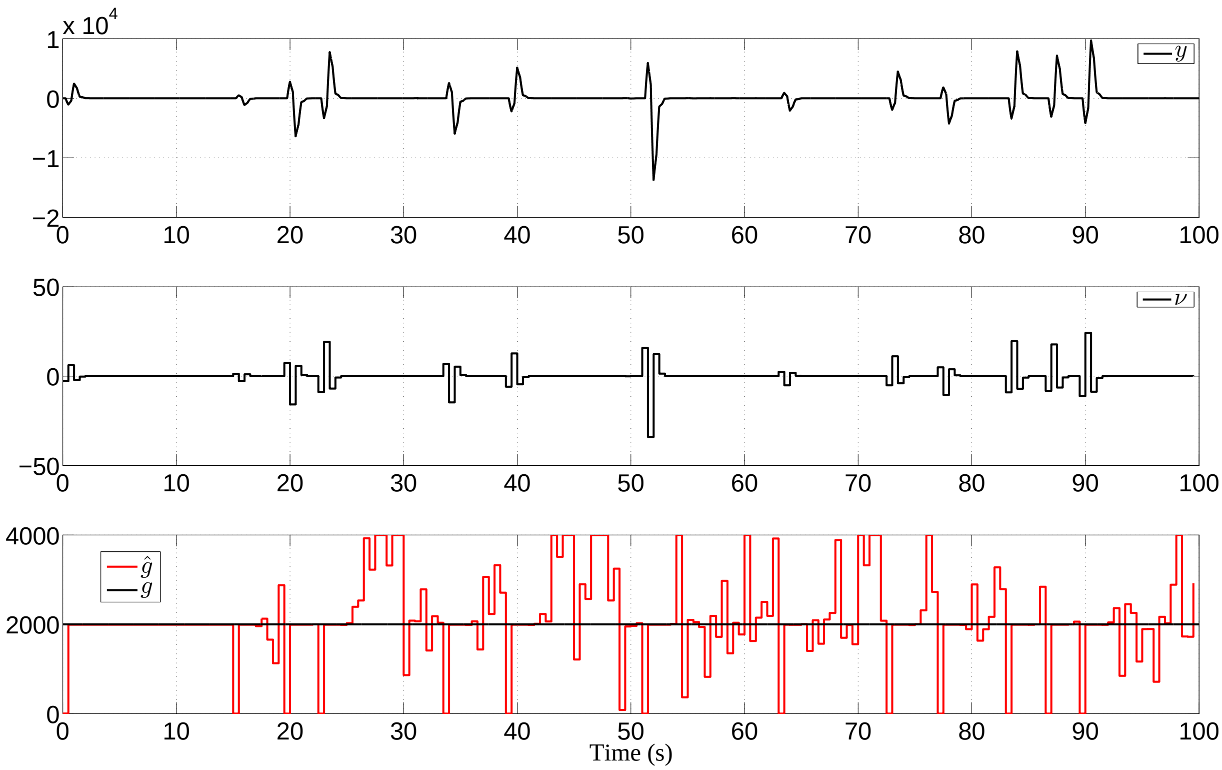

Figure 5.9: Example 5.4: the output, control signal and estimation errors for a simulation at one thousand times the LTI gain margin.

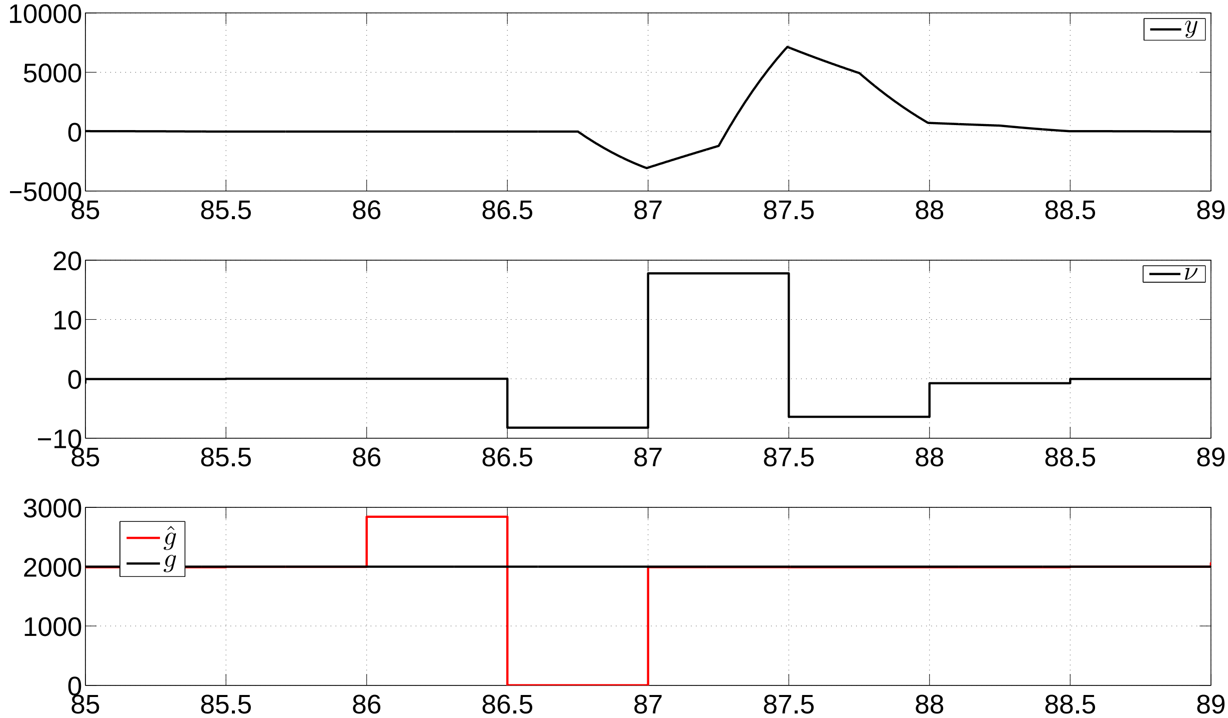

Figure 5.10: Examining the responsiveness of the controller once the output gets large.

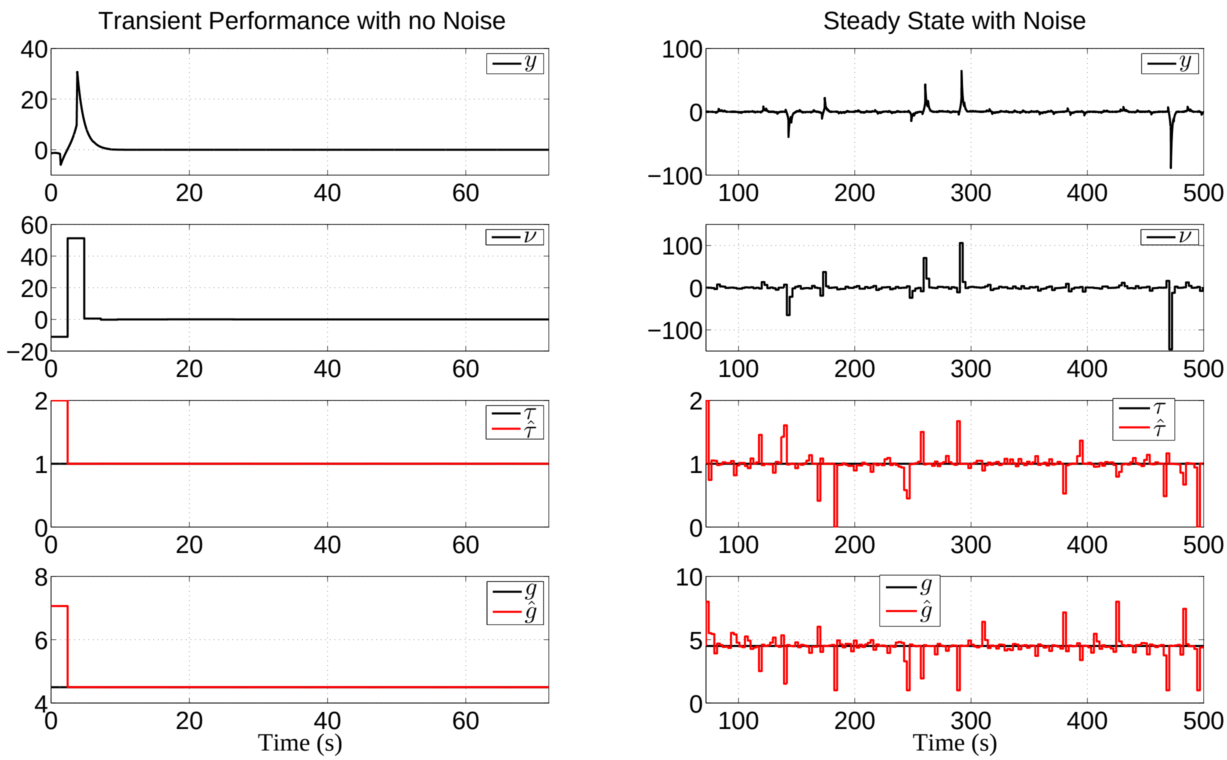

Figure 5.11: Example 5.5: the output, control signal and estimation errors for a simulation at twice the LTI delay and gain margins.

Example 5.6. Using the same plant (5.120), we consider a much more extreme uncertainty set of ten times the LTI delay margin and one hundred times the LTI gain margin, resulting in the uncertainty set:

$$\mathcal{G} = \left\{ge^{-s\tau}G_0(s): \quad \tau \in [0, 10], g \in [1, 400]\right\}.$$

To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.195$, $T_3 = 10.005$, resulting in an overall period $T = 10.5$, we fix $\tau = 8$, $g = 200$, and we run the simulation with random noise starting at time zero and with a maximum magnitude of 10^{-2} with the results shown in 5.12. It is interesting to compare the results shown here to those of Example 5.2 shown in Figure 5.7; while adding in the unknown gain degraded the performance, the degradation is essentially equal to the size of the unknown gain, i.e., the maximum output in Figure 5.12 is essentially 200 times larger than the maximum output in Figure 5.2. This re-enforces the point that while the unknown gain affects the performance, the primary driver of the performance is the maximum length of the unknown delay.

The next set of examples now consider switches in the unknown gain and delay.

Example 5.7. Still using the nominal plant (5.120), we start by performing a simulation with only an unknown delay at twice the LTI delay margin of two seconds and with a switch every 20 seconds, resulting in an uncertainty set for the delay and gain of

$$\mathcal{T}_{T_s}^{PC} = \{\tau \in [0, 2], g = 1 : T_s = 20\},$$

and an overall uncertainty set

$$\mathcal{G}_{T_s}^{PC} := \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + g(t)Bu_d(t - \tau(t)) \\ y(t) & = & Cx(t) \end{array} \;\middle|\; (\tau, g) \in \mathcal{T}_{T_s}^{PC} \right\}.$$

To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.095$, $T_3 = 2.005$, resulting in an overall period $T = 2.4$, and we run the simulation with noise starting from time zero and with a maximum magnitude of 10^{-2} as shown in Figure 5.13. Despite the fact that our switching time of 20 seconds is probably far smaller than what is required to apply Theorem 5.2, the controller easily handles the switches, with performance that is approximately ten times worse than Example 5.1 which had the same desired delay margin and controller settings except with a fixed delay. In particular, the switching can cause difficulties as it can extend the poor estimate phase by switching the unknown delay just as the estimate converges to the previous delay value; however, our controller works very well at quickly estimating the delay after a switch, resulting in very reasonable performance.

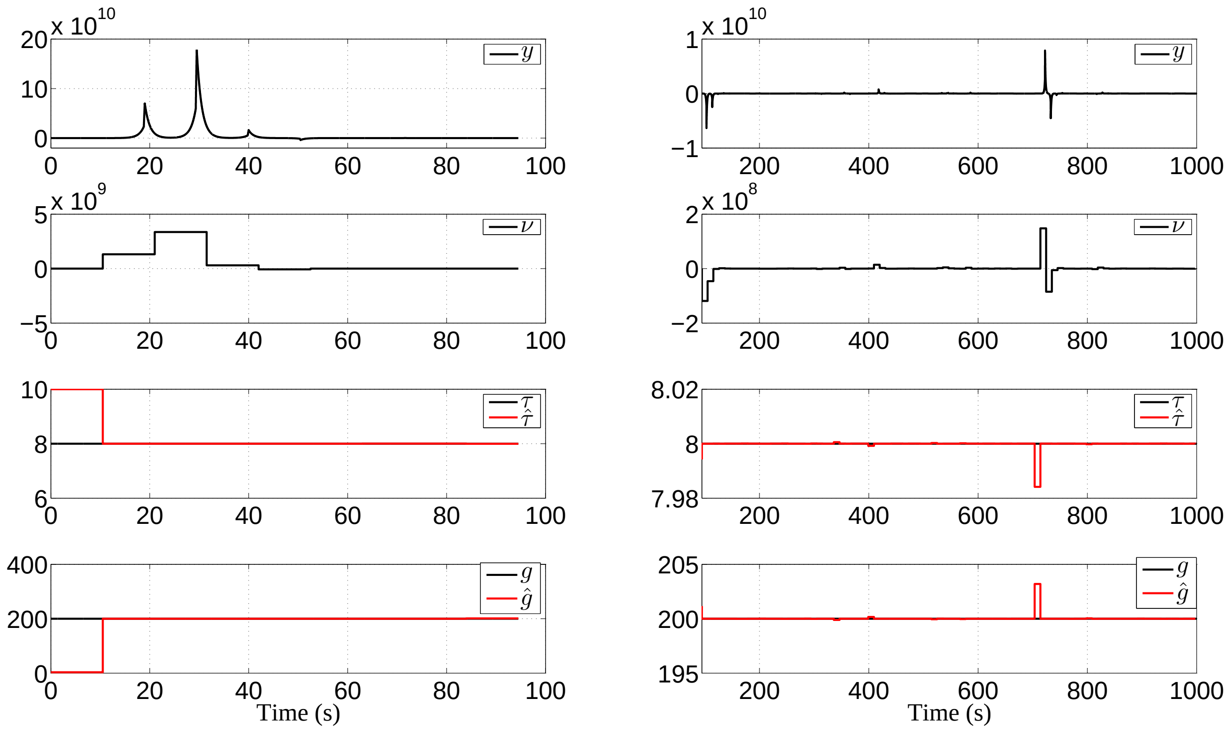

Figure 5.12: Example 5.6: the output, control signal and estimation errors for a simulation at ten times the LTI delay margin and 100 times the LTI gain margin.

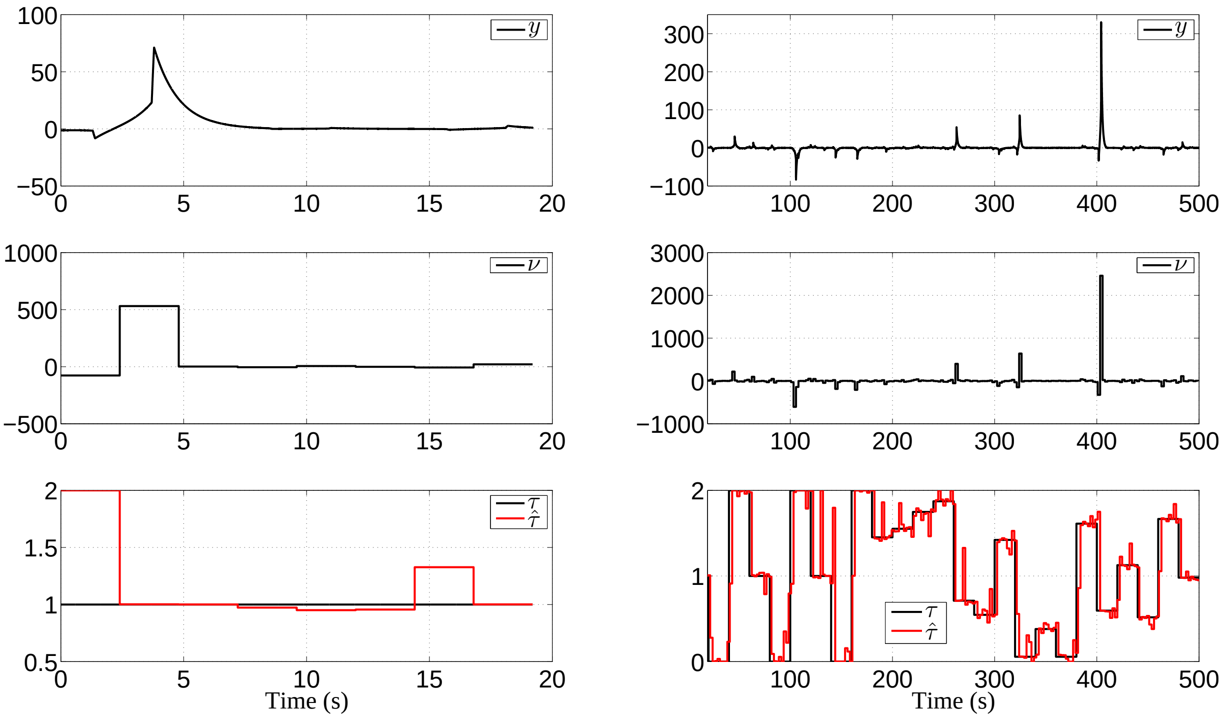

Figure 5.13: Example 5.7: the output, control signal and estimation errors for a simulation at two times the LTI delay margin with jumps every 20 seconds.

Example 5.8. Still using the nominal plant (5.120), we perform a simulation with only an unknown gain at twice the LTI gain margin of eight and with a switch every 2.1 seconds, resulting in an uncertainty set for the delay and gain of

$$\mathcal{T}_{T_s}^{PC} = \{\tau = 0, g \in [1, 8] : T_s = 2.1\},$$

and an overall uncertainty set

$$\mathcal{G}_{T_s}^{PC} := \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + g(t)Bu_d(t - \tau(t)) \\ y(t) & = & Cx(t) \end{array} \;\middle|\; (\tau, g) \in \mathcal{T}_{T_s}^{PC} \right\}.$$

To perform the simulation, we set $h = 0.25$, $T_1 = 0.25$, $T_2 = 0.245$, $T_3 = 0.005$, resulting in an overall period $T = 0.5$, and we run the simulation with no noise starting at time zero and with a maximum magnitude of 10^{-2} as shown in Figure 5.14. Despite the fact that our switching time of 2.1 seconds is probably far smaller than what is required to apply Theorem 5.2, as can be seen, the controller easily handles the switches, with performance that is again only moderately worse than Example 5.3 which had the same desired gain margins except with a fixed gain.

Example 5.9. Still using the nominal plant (5.120), we perform a simulation with both an unknown gain and delay at twice their respective LTI margins and with a switch every 20 seconds, resulting in an uncertainty set for the delay and gain of

$$\mathcal{T}_{T_s}^{PC} = \{\tau \in [0, 2], g \in [1, 8] : T_s = 20\},$$

and an overall uncertainty set

$$\mathcal{G}_{T_s}^{PC} := \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + g(t)Bu_d(t - \tau(t)) \\ y(t) & = & Cx(t) \end{array} \;\middle|\; (\tau, g) \in \mathcal{T}_{T_s}^{PC} \right\}.$$

To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.095$, $T_3 = 2.005$, resulting in an overall period $T = 2.4$, and we run the simulation with noise starting from time zero and with a maximum magnitude of 10^{-2} as shown in Figure 5.15. Once again, our controller tolerates these switches, and the performance is comparable to that of Examples 5.7 where we only considered an unknown delay with switching; furthermore, the performance is only moderately worse than Example 5.5 where the delay and gain were fixed.

Example 5.10. Still using the nominal plant (5.120), we again perform a simulation with both an unknown gain and delay at twice their respective LTI margins; however, this time we consider much more frequent jumps in the gain and delay by setting $T_s = 7$ seconds, resulting in an uncertainty set for the delay and gain of

$$\mathcal{T}_{T_s}^{PC} = \{\tau \in [0, 2], g \in [1, 8] : T_s = 7\},$$

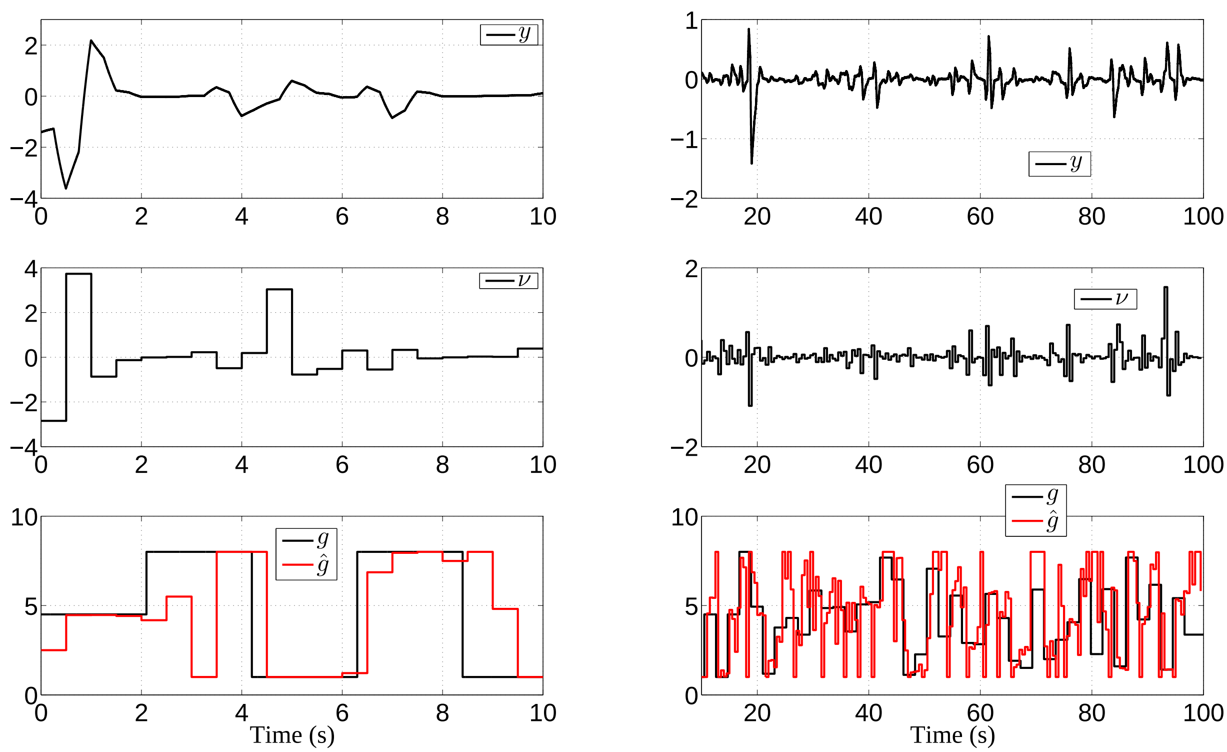

Figure 5.14: Example 5.8: the output, control signal and estimation errors for a simulation at two times the LTI gain margin with jumps every 2.1 seconds.

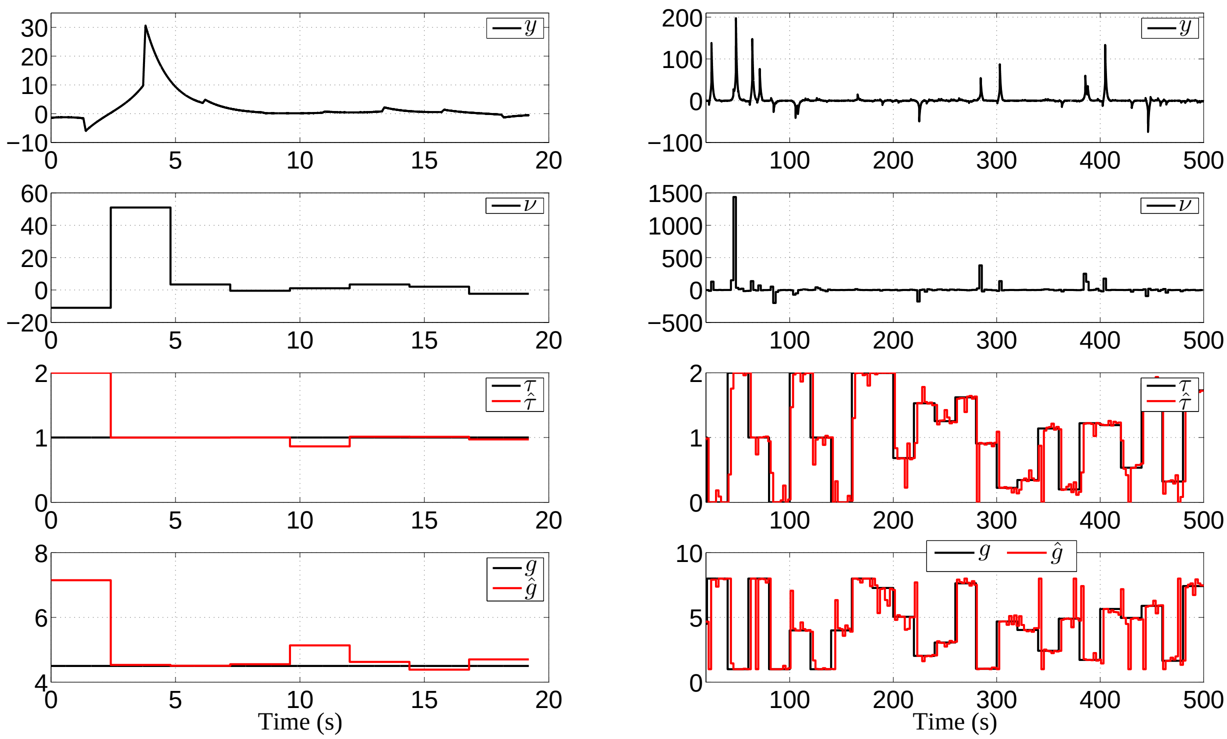

Figure 5.15: Example 5.9: the output, control signal and estimation errors for a simulation at two times the LTI gain and delay margins with jumps every 20 seconds.

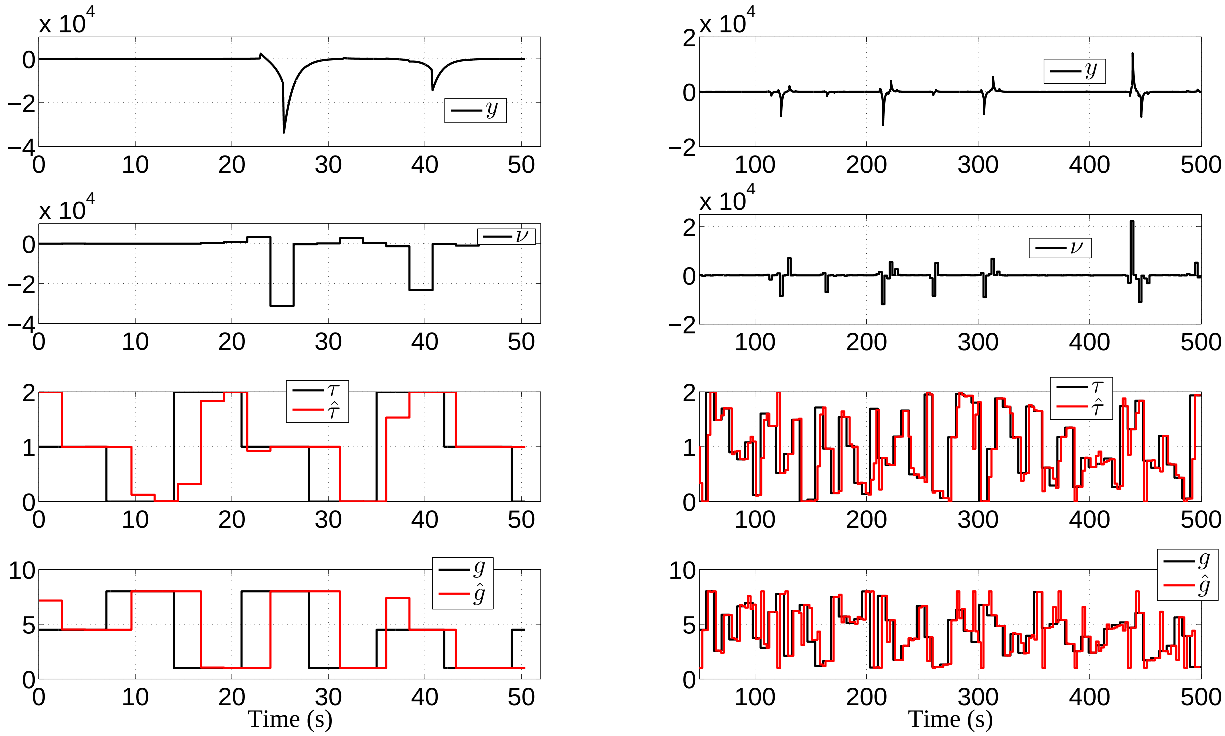

Figure 5.16: Example 5.10: the output, control signal and estimation errors for a simulation at two times the LTI gain and delay margins with jumps every 7 seconds.

and an overall uncertainty set

$$\mathcal{G}_{T_s}^{PC} := \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + g(t)Bu_d(t - \tau(t)) \\ y(t) & = & Cx(t) \end{array} \;\middle|\; (\tau, g) \in \mathcal{T}_{T_s}^{PC} \right\}.$$

To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.095$, $T_3 = 2.005$, resulting in an overall period $T = 2.4$, and we run the simulation with noise starting form time zero and with a maximum magnitude of 10^{-2} as shown in Figure 5.16. While the controller was able to stabilize the system despite switches in the gain and delay every seven seconds, the performance is clearly much worse than Example 5.9 where the switches were every 20 seconds.

Our next example considers the case of plant uncertainty.

Example 5.11. Using the nominal plant (5.120), we perform a simulation at twice the LTI delay and gain margin of the nominal plant with a fixed delay and gain; however, this time we consider plant uncertainty. We start by performing a co-prime factorization on the plant G_0 yielding:

$$N = \frac{s-2}{(s+10)^2} \quad \text{and} \quad M = \frac{(s-1)(s+1)}{(s+10)^2}.$$

We then set

$$\Delta_N = \frac{0.02s}{s+10} \quad \text{and} \quad \Delta_M = \frac{0.005s}{s+10},$$

so

$$\frac{\|\Delta_N\|}{\|N\|} = 0.3922 \quad \text{and} \quad \frac{\|\Delta_M\|}{\|M\|} = 0.005.$$

Using this uncertainty, our actual plant has a transfer function given by

$$G_{act} = \frac{0.0199(s+61.62)(s-1.623)}{(s+1.023)(s-0.9729)} \tag{5.121}$$

and we use the following state space representation for the simulation:

$$A = \begin{bmatrix} -0.0498 & 0.9950 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.5965 & -0.9851 \end{bmatrix}.$$

To better see the difference between the nominal plant given by (5.120) and the actual plant given by (5.121), we show the two Bode plots in Figure 5.17; while both plants are similar at low frequency, we can see a significant difference at higher frequencies, and

$$\sup_{\omega \in \mathbf{R}} \|G_0(j\omega) - G_{act}(j\omega)\| = 0.0798.$$

Furthermore, while our nominal plant has a LTI delay margin of two and a LTI gain margin of four, our actual plant has a LTI delay margin of 0.8234 and a LTI gain margin of 2.7829, so our simulation is at nearly 2.5 times the actual LTI delay margin and almost three times larger than the actual LTI gain margin. To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.095$, $T_3 = 2.005$, resulting in an overall period $T = 2.4$, and we run the simulation with noise starting form time zero and with a maximum magnitude of 10^{-2} as shown in Figure 5.18. Despite the uncertainty, including both of the real plant eigenvalues which are critical for our estimation scheme, we see that our controller stabilizes

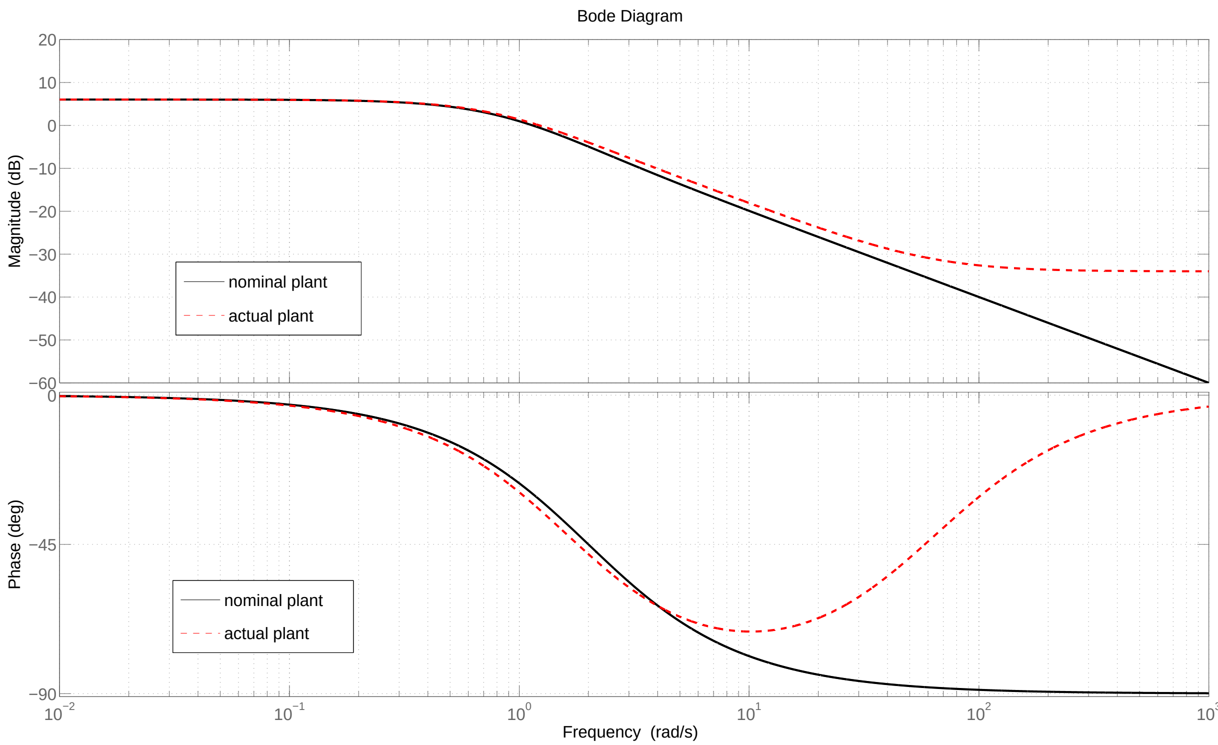

Figure 5.17: The Bode plot of the nominal and actual plants

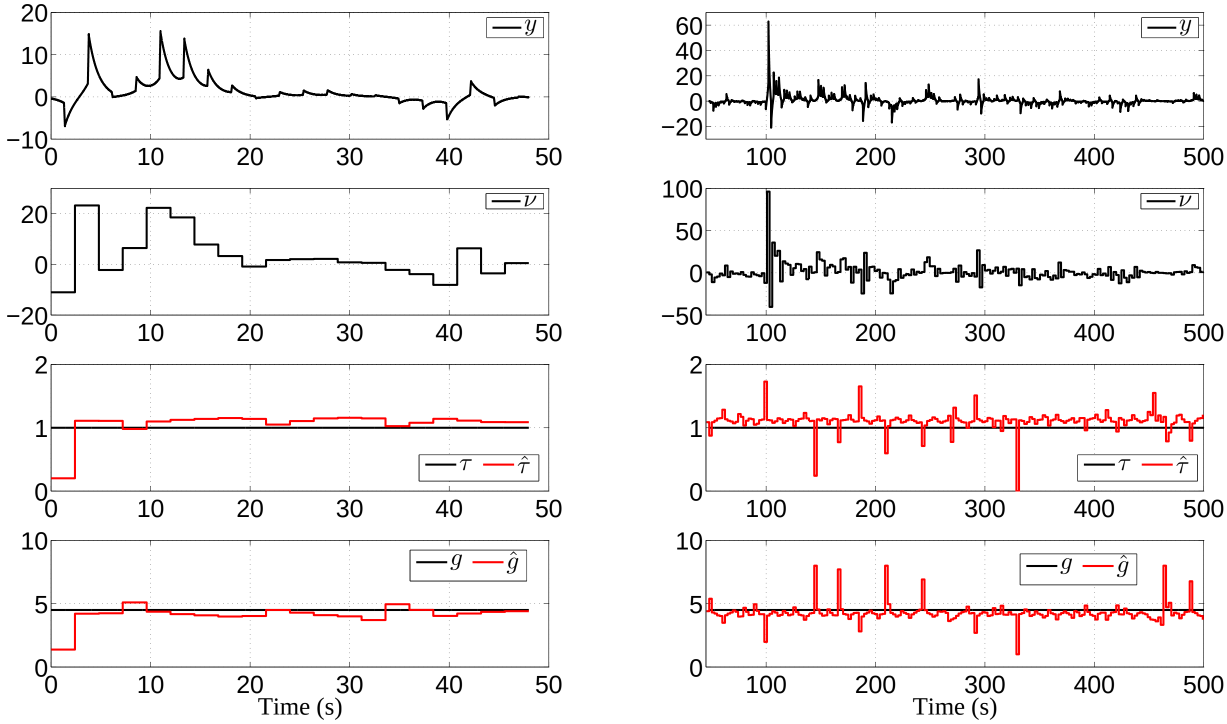

Figure 5.18: Example 5.11: the output, control signal and estimation errors for a simulation at two times the LTI gain and delay margins with plant uncertainty.

the system with reasonable performance when compared to Example 5.5 which considered the same uncertainty in the gain and delay but with a known plant model. Furthermore, while our nominal A matrix was diagonal, as required for our gain and delay estimators, we performed the simulation with a non-diagonal A matrix without difficulty, which highlights the fact our controller only relies on the plant input in order to function.

Example 5.12. Using the nominal plant (5.120), we perform a simulation at the LTI delay and gain margin; however, this time we consider plant uncertainty and switches in the unknown gain and delay. We use the same uncertainty model as the previous example, yielding an actual plant given by (5.121), except this time we allow the unknown gain and delay to switch every 20 seconds. To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.195$, $T_3 = 1.005$, resulting in an overall period $T = 1.5$, and we run the simulation with noise starting form time zero and with a maximum magnitude of 10^{-2} as shown in Figure 5.19. Despite throwing both jumps in the gain and delay and plant uncertainty, we see that the controller stabilizes the system with reasonable performance.

Example 5.13. Using the same plant (5.120), we perform a simulation where we allow the delay to slowly vary, with a maximum delay and gain at twice the respective LTI margins. To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.095$, $T_3 = 2.005$, resulting in an overall period $T = 2.4$, we run the simulation with no noise starting at time zero and with a maximum magnitude of 10^{-2} and with the delay and gain varying as shown in Figure 5.20. While we did not prove that our controller can handle continuous changes in the gain and delay, the simulation results suggest that our controller can handle slowly varying changes in the gain and delay, albeit with some performance degradation in comparison to the fixed delay case of Example 5.5. In the next chapter, we consider the problem of a continuously varying delay for a first order plant.

Example 5.14. For this example, we consider the plant

$$G_0 = \frac{s-2}{(s-4)(s+1)},$$

which has an LTI delay margin less than $\frac{1}{3}$ and a LTI gain margin of four; furthermore, unlike our previous simulations, we now have the unstable pole to the right of the unstable zero, which is a much more difficult problem.. We will perform a simulation with $\overline{\tau} = 0.5$ and $\overline{g} = 4$, yielding the uncertainty set

$$\mathcal{G} = \left\{ g e^{-s\tau} G_0(s) : \ \tau \in [0, 0.5], g \in [1, 4] \right\}.$$

To perform the simulation, we set $h = 0.3$, $T_1 = 0.3$, $T_2 = 0.095$, $T_3 = 0.505$, resulting in an overall period $T = 0.9$, and we run the simulation with no noise starting at time zero

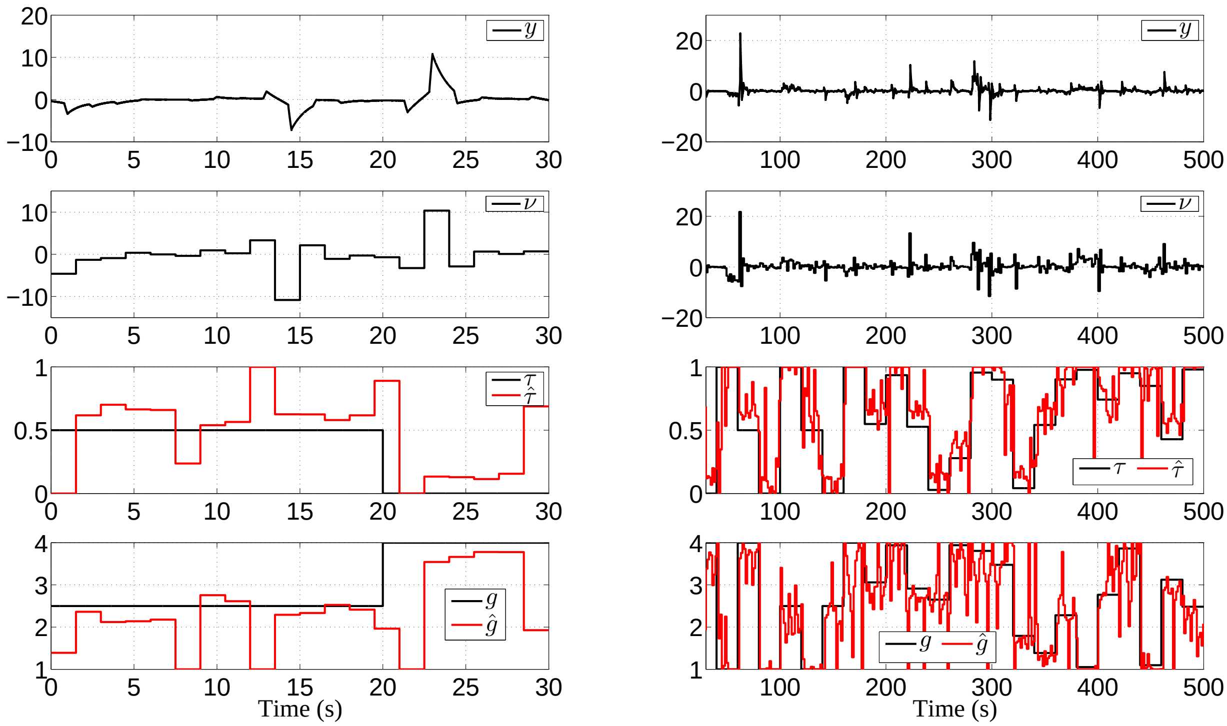

Figure 5.19: Example 5.12: the output, control signal and estimation errors for a simulation at the LTI gain and delay margins with plant uncertainty and jumps in the unknown gain and delay.

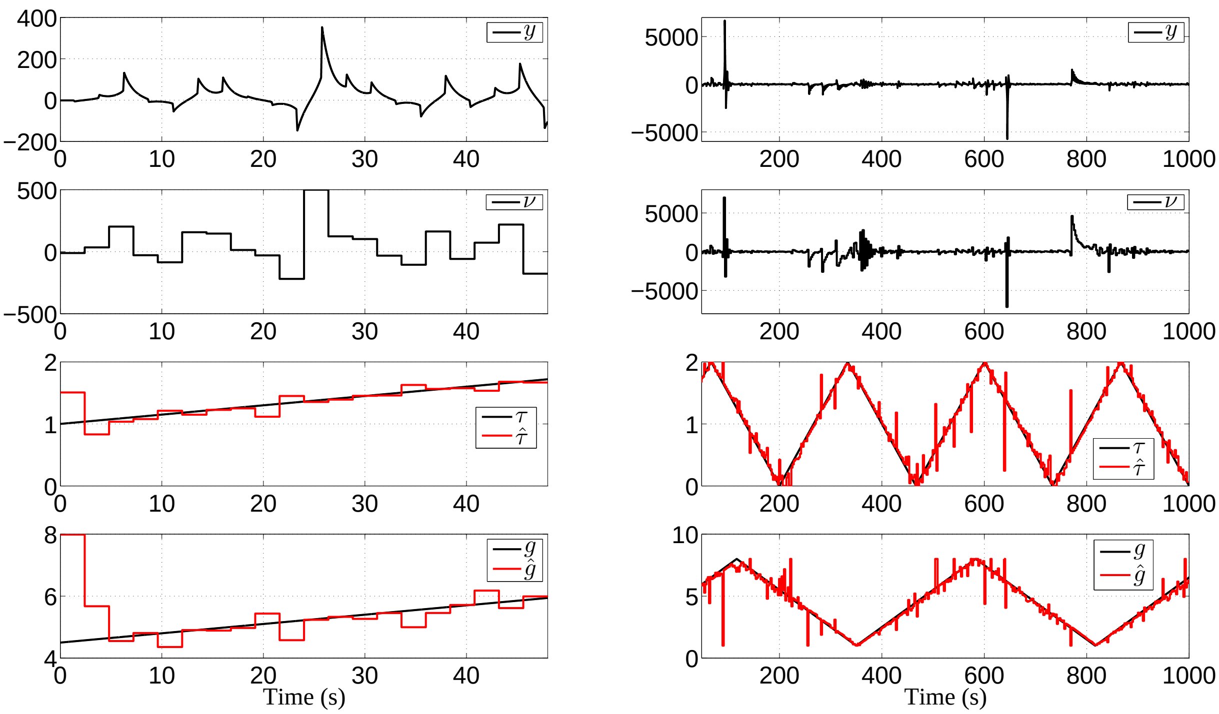

Figure 5.20: Example 5.13: the output, control signal and estimation errors for a simulation at twice LTI gain and delay margins with a slowly varying time delay and gain.

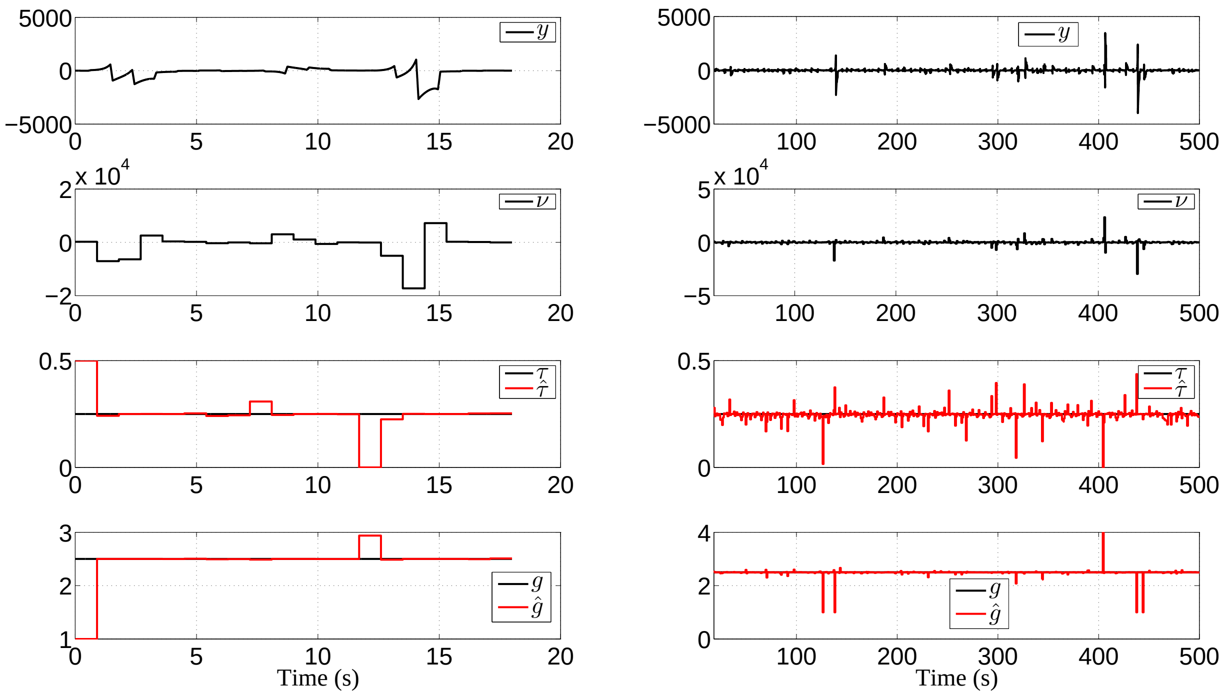

Figure 5.21: Example 5.14: the output, control signal and estimation errors for a simulation at 1.5 times the LTI delay margin and at the LTI gain margin with the unstable pole to the right of the non-minimum phase zero.

and with a maximum magnitude of 10^{-2} as shown in Figure 5.21. As can be seen, the performance is clearly worse than for the previous plant, however despite the difficult to control plant, our controller stabilizes the system and handles the noise reasonably well.

5.6 Conclusions and Future Work

In this chapter, we propose a control scheme that can provide an arbitrarily large gain and delay margin for a given SISO LTI controllable/observable continuous-time plant. The proposed controller uses a generalized hold to produce a discrete-time model which has the delay showing up in a single location. Using this discrete-time model, we are able to estimate the delay and gain at each time step, from which we can calculate a control signal. This controller, while mildly non-linear and periodic, is relatively simple and not only provides BIBO stability, but also guarantees the exponential decay of the initial conditions even when subject to noise; furthermore, the controller is robust to un-modeled dynamics and handles infrequent but possibly persistent jumps in the unknown gain and delay. As far as the authors are aware, this is the only controller that can provide BIBO stability for any SISO LTI controllable/observable plant with an arbitrarily large delay and gain. This work is presented in [16].

For future work, we would like to prove that our controller can maintain stability in the face of slowly varying gains and delays, as suggested by the provided simulation; we do so in the next chapter using a similar controller that only works on a first order plant. We would also like to improve the performance of the controller, which may be possible by exploring the optimal location of the closed loop eigenvalues, or by using previous information on the gain and delay estimates to improve the estimate when the state is small. Finally, we would like to extend this result to multi-input multi-output systems.

Chapter 6

Gain and Delay Margin - Time Varying

In this chapter, we consider a very similar problem to the one considered in the previous chapter. In the previous chapter, we designed a controller which BIBO stabilized an n^{th} order plant with a fixed input delay, although we did allow occasional jumps in the unknown gain and delay. In this chapter, we design a controller which BIBO stabilizes a first order plant with a continuously varying output delay satisfying an explicit bound in terms of the maximum length of the unknown delay and the location of the plant pole.

The controller presented in this chapter is loosely based on our preliminary work in [14] and the results of Chapter 5. In [14], we considered a first order system with a unknown but upper bounded time varying delay, and to solve the problem, we proposed an unimplementable impulse hold to find an explicit bound on the allowable time variation of the delay in terms of the maximum length of the unknown delay and the location of the unstable plant pole; however, in that work, we did not consider noise. In contrast to that work, in this chapter we adopt a similar approach to that of Chapter 5, by replacing the impulse hold with a 'pulse' hold, we are able to prove that our proposed controller BIBO stabilizes the first order plant if the time variation of the delay satisfies an explicit bound in terms of the maximum length of the unknown delay and the location of the plant pole. While the controller that we propose in this chapter is similar to the controller of Chapter 5, it has one key difference; namely, since the system is first order, we no longer require separate estimates of the gain and delay.

This chapter is organized as follows. In Section 6.1 we formally state the problem and define stability, in Section 6.2 we provide our controller, in Section 6.3 we prove that our controller stabilizes the system under suitable constraints on the rate of change of the time delay, in Section 6.4 we simulate our proposed controller, and finally, in Section 6.5 we provide our conclusions and future work.

6.1 Problem Formulation

We are interested in stabilizing a SISO, LTI unstable plant which has an unknown, upper bounded, time varying delay $\tau(t) \in [0, \overline{\tau}]$ and an unknown, upper and lower bounded, fixed gain $g \in [1, \overline{g}]$.[1] In contrast to the previous two chapters where we considered an input delay, in this chapter we consider an output delay in order to simplify the analysis when the delay varies continuously; in particular, this avoids having the time varying delay stretch/compress the input as seen by the plant. As such, with $a > 0$, $b \neq 0$, $c = 1$,[2] our plant G is described by

$$\left.\begin{array}{rcl} \dot{x}(t) & = & ax(t) + gbu(t) \\ y(t) & = & x(t - \tau(t)), \end{array}\right\} \tag{6.1}$$

with $x(t) \in \mathbf{R}$ the plant state, $u(t) \in \mathbf{R}$ the plant input, and $y(t) \in \mathbf{R}$ the plant output. Because of the presence of the delay, the initial condition of the plant is not only on the state at time zero, but also on the state starting at time $-\overline{\tau}$;[3] more specifically, the plant initial condition is given by $x(\theta) = x_0(\theta)$ for $\theta \in [-\overline{\tau}, 0]$; it is natural to require that $x_0 \in AC_\infty([-\overline{\tau}, 0])$. Finally, we impose a constraint on the rate of change of the delay; to this end, we restrict $\tau(t)$ to be absolutely continuous so that the derivative of $\tau(t)$ is defined almost everywhere, and where it is defined, we require it to be at least $\tau_l \leq 0$ and at most $\tau_u \geq 0$. This results in the following constraints on $\tau(t)$:

- $\tau(t)$ is absolutely continuous for all $t \geq 0$.
- $\tau(t) \in [0, \overline{\tau}]$ for all $t \geq 0$.
- $\dot{\tau}(t) \in [\tau_l, \tau_u]$ for almost all $t \geq 0$.

So, we define the following set of admissible time-varying delays:

$$\mathcal{T} := \{\tau \in AC_\infty \mid \tau(t) \in [0, \overline{\tau}] \text{ for all } t \geq 0; \quad \dot{\tau} \in [\tau_l, \tau_u] \text{ for almost all } t \geq 0\},$$

and the corresponding set of admissible plant models:

$$\mathcal{G}(\overline{\tau}, \tau_u, \tau_l, \overline{g}) := \left\{ \begin{array}{rcl} \dot{x}(t) & = & ax(t) + gbu(t) \\ y(t) & = & x(t - \tau(t)) \end{array} \;\middle|\; \tau \in \mathcal{T}, g \in [1, \overline{g}] \right\}.$$

[1] We can always write the unknown gain interval in this form by absorbing any lower gain bound and the sign into b.

[2] Since we can absorb c into b, without loss of generality we let $c = 1$.

[3] This is done so that the same initial condition applies for all possible delays.

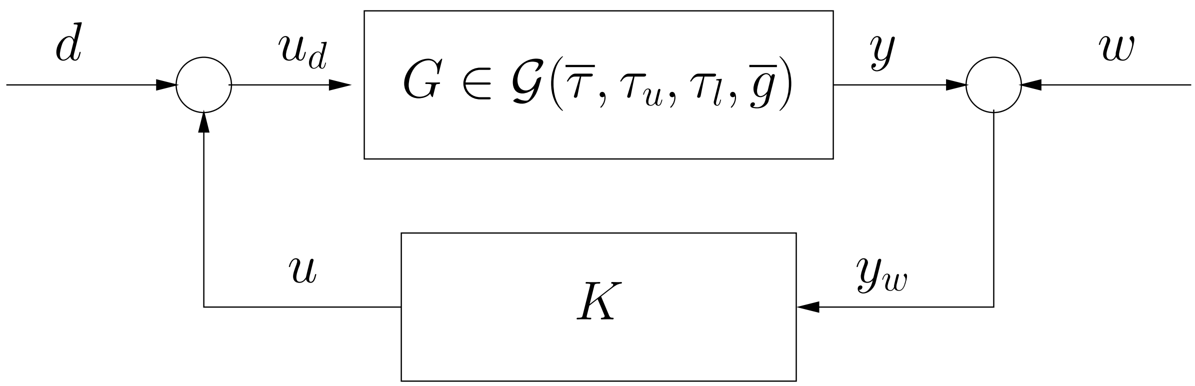

Figure 6.1: The feedback setup considered in this paper.

We consider the standard feedback structure: the controller is input-output of the form

$$u = Ky.$$

The notion of stability in this chapter is very similar to that of Chapter 5, but to formally define stability in this chapter, we introduce noise at the two plant/controller interfaces as shown in Figure 6.1. With the noise added, a plant in $\mathcal{G}(\overline{\tau}, \tau_u, \tau_l, \overline{g})$ is described by:

$$\left.\begin{array}{rcll} \dot{x}(t) & = & ax(t) + gbu(t) + gbd(t) & \\ y_w(t) & = & x(t - \tau(t)) + w(t), & t \geq 0 \end{array}\right\} \tag{6.2}$$

with an initial condition $x_0 \in AC_\infty([-\overline{\tau}, 0])$. To aid in handling the noise, it will be convenient to define the stacked noise vector $\overline{w} := \begin{bmatrix} d \\ w \end{bmatrix}$.

With zero initial conditions on the plant, i.e., $x_0 = 0$, with $\tau \in \mathcal{T}$ and with $g \in [1, \overline{g}]$ we let $\Phi(\tau, g)$ be the closed loop map from $\begin{bmatrix} d \\ w \end{bmatrix} \to \begin{bmatrix} y \\ u \end{bmatrix}$.

Definition 6.1. *We say that K* stabilizes $\mathcal{G}(\overline{\tau}, \tau_u, \tau_l, \overline{g})$ *if $\Phi(\tau, g)$ is uniformly bounded, i.e.*

$$\sup_{\tau \in \mathcal{T}, g \in [1, \overline{g}]} \|\Phi(\tau, g)\| < \infty.$$

The goal of this paper is to design a controller K which stabilizes $\mathcal{G}(\overline{\tau}, \tau_u, \tau_l, \overline{g})$.

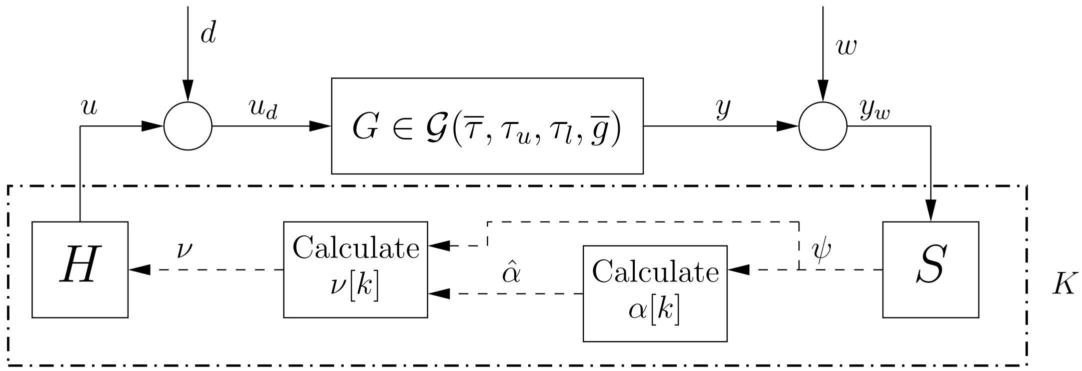

Figure 6.2: The feedback and controller setup considered in this paper.

6.2 The Controller

Our proposed controller K is shown in Figure 6.2; it is non-linear as well as periodic with period T. This controller uses a modified zero-order hold H, along with a regular sampler S as described in Sub-section 6.2.1. In Sub-section 6.2.2, we apply the sampler and hold to the plant, yielding a discretized plant model; using the resulting discretization, in Sub-section 6.2.3 we estimate the delay at the previous sample point and in Sub-section 6.2.4 we use the delay estimate to design a control signal. Finally, in Sub-section 6.2.5 we find a bound on the estimation error.

6.2.1 The Sampler and Hold

The hold H is a partial period pulse parameterized by two quantities of time, T_1 and T_2. The quantity $T_1 > 0$ is the duration of the pulse, and the quantity $T_2 > \overline{\tau}$ is set so that only one pulse arrives at the plant during each period. The resulting period of the hold is given by $T := T_1 + T_2$, and we define the hold $H : \ell(\mathbf{R}) \to PC(\mathbf{R})$ by

$$(H\nu)(t) := \begin{cases} \nu[k] & t \in [kT, kT + T_1] \\ 0 & t \in (kT + T_1, kT + T), \end{cases} \qquad k \geq 0. \tag{6.3}$$

Unlike the hold, the sampler $S : PC(\mathbf{R}) \to \ell(\mathbf{R})$ is completely standard and is given by

$$(Sy_w)[k] := y_w(kT), \quad k \geq 0. \tag{6.4}$$

6.2.2 Discretizing the Plant

We now discretize the plant (6.2) using the hold (6.3) and the sampler (6.4). We sample the output at integer multiples of the period T, yielding

$$\underbrace{y_w(kT)}_{=:\psi[k]} = x(kT - \tau(kT)) + \underbrace{w(kT)}_{=:\omega[k]}, \qquad k \geq 0. \tag{6.5}$$

Since the system is first order, we will combine the state and output equations into a single equation. To do so, we require a solution of the state equation (6.2) starting at time $kT - \tau(kT)$ and an end time of $kT + T - \tau(kT + T)$; due to the initial condition, the discretization for $k = 0$ will be different than for $k \geq 1$.

We start with the $k \geq 1$ case. To do so, we define $\tau[k] := \tau(kT)$ and $\Delta[k] := \tau[k] - \tau[k+1]$, yielding for $k \geq 1$:

$$\begin{aligned}
\underbrace{x((k+1)T - \tau[k+1])}_{=:\chi[k+1]} &= e^{a(T+\Delta[k])} \underbrace{x(kT - \tau[k])}_{=:\chi[k]} + \int_{kT-\tau[k]}^{kT+T-\tau[k+1]} gbe^{a(kT+T-\tau[k+1]-q)} u_d(q)dq \\
&= e^{a(T+\Delta[k])}\chi[k] + \int_0^{T_1} gbe^{a(T-\tau[k+1]-q)}\nu[k]dq + \\
&\quad \underbrace{\int_0^{T+\Delta[k]} gbe^{a(T+\Delta[k]-q)} d(kT - \tau[k] + q)dq}_{=:\zeta[k]} \\
&= e^{a(T+\Delta[k])}\chi[k] + e^{aT}\frac{b}{a}(1 - e^{-aT_1}) \underbrace{ge^{-a\tau[k+1]}}_{=:\alpha[k+1]} \nu[k] + \zeta[k],
\end{aligned} \tag{6.6}$$

Combining (6.5) and (6.6) yields for $k \geq 1$:

$$\psi[k+1] = e^{a(T+\Delta[k])}\psi[k] + e^{aT}\frac{b}{a}(1 - e^{-aT_1})\alpha[k+1]\nu[k] + \underbrace{\zeta[k] - \omega[k+1] - e^{a(T+\Delta[k])}\omega[k]}_{=:\eta[k]}. \tag{6.7}$$

For the $k = 0$ case, we obtain an expression for $\chi[1]$:

$$\begin{aligned}
\underbrace{x(T-\tau[1])}_{=:\chi[1]} &= e^{a(T-\tau[1])}x_0(0) + \int_0^{T-\tau[1]} gbe^{a(T-\tau[1]-q)}u_d(q)dq \\
&= e^{a(T-\tau[1])}x_0(0) + \int_0^{T_1} gbe^{a(T-\tau[1]-q)}\nu[0]dq + \underbrace{\int_0^{T-\tau[1]} gbe^{a(T-\tau[1]-q)}d(q)dq}_{=:\zeta[0]} \\
&= e^{a(T-\tau[1])}x_0(0) + e^{aT}\frac{b}{a}(1-e^{-aT_1})\alpha[1]\nu[0] + \zeta[0]. \qquad (6.8)
\end{aligned}$$

Using (6.8) and (6.5), we obtain the following equations for $\psi[0]$ and $\psi[1]$:

$$\psi[0] = x_0(-\tau(0)) + \omega[0] \qquad (6.9)$$

$$\psi[1] = e^{a(T-\tau[1])}x_0(0) + e^{aT}\frac{b}{a}(1-e^{-aT_1})\alpha[1]\nu[0] + \underbrace{\zeta[0]+\omega[1]}_{:=\eta[0]}. \qquad (6.10)$$

Combining (6.7) and (6.10), and using (6.9) for the initial condition of $\psi[0] = x_0(-\tau(0)) + \omega[0]$ yields the final model:

$$\psi[k+1] = \begin{cases} e^{a(T-\tau[1])}x_0(0) + e^{aT}\frac{b}{a}(1-e^{-aT_1})\alpha[1]\nu[0] + \zeta[0] + \omega[1], & k = 0 \\ e^{a(T+\Delta[k])}\psi[k] + e^{aT}\frac{b}{a}(1-e^{-aT_1})\alpha[k+1]\nu[k] + \eta[k], & k \geq 1 \end{cases} \qquad (6.11)$$

It is also be useful to bound the noise term, η; for every $\tau \in \mathcal{T}$ and $g \in [1, \overline{g}]$ we have that

$$\|\eta\|_\infty \leq \underbrace{\left(e^{a(T+\overline{\tau})} + 1 + \overline{g}\left|\frac{b}{a}\right|(e^{a(T+\overline{\tau})} - 1)\right)}_{=:c_w}\|\overline{w}\|_\infty \qquad (6.12)$$

6.2.3 Estimating the Gain and Delay

To estimate the gain and delay, we adopt a similar approach to our preliminary work [14], namely, at time k, we solve (6.7) (shifted backwards by one time step) for $\alpha[k]$ (recall that $\alpha[k] = ge^{-a\tau[k]}$) under the hypotheses that there is no noise and that the delay is time invariant; since there may be noise and the delay may be time varying, here we will also saturate the estimate to yield $\hat{\alpha}[k]$. To this end, consider (6.7) shifted backwards by one time step under the hypotheses that there is no noise, the time delay is fixed (so $e^{a(T+\Delta[k-1])} = e^{aT}$), and $\nu[k-1] \neq 0$:

$$\begin{aligned}
& & \psi[k] &= e^{aT}\psi[k-1] + \frac{b}{a}e^{aT}(1-e^{-aT_1})\alpha[k]\nu[k-1] \\
\Leftrightarrow & \quad \alpha[k]\frac{b}{a}e^{aT}(1-e^{-aT_1})\nu[k-1] &= \psi[k] - e^{aT}\psi[k-1] & \\
\Leftrightarrow & & \alpha[k] &= \frac{\psi[k]-e^{aT}\psi[k-1]}{\frac{b}{a}e^{aT}(1-e^{-aT_1})\nu[k-1]}.
\end{aligned} \tag{6.13}$$

With initial conditions $\nu[-1] \in \mathbf{R}$ and $\psi[-1] \in \mathbf{R}$, we use (6.13) to define our unsaturated estimate $\check{\alpha}[k]$ for $k \geq 0$ as follows:

$$\check{\alpha}[k] := \begin{cases} \frac{\psi[k]-e^{aT}\psi[k-1]}{\frac{b}{a}e^{aT}(1-e^{-aT_1})\nu[k-1]} & \text{if } \nu[k-1] \neq 0 \\ \overline{g} & \text{if } \nu[k-1] = 0; \end{cases} \tag{6.14}$$

since $ge^{a\tau[k]} \in [e^{-a\overline{\tau}}, \overline{g}]$, we then saturate $\check{\alpha}[k]$ yielding our final estimate:

$$\hat{\alpha}[k] = \begin{cases} e^{-a\overline{\tau}} & \check{\alpha}[k] < e^{-a\overline{\tau}} \\ \check{\alpha}[k] & \check{\alpha}[k] \in [e^{-a\overline{\tau}}, \overline{g}] \\ \overline{g} & \check{\alpha}[k] > \overline{g}. \end{cases} \tag{6.15}$$

A natural question is: how close is our estimate $\hat{\alpha}[k]$ to the actual value $\alpha[k]$? Before we can answer this question we must introduce our control law, which we do in the following section.

6.2.4 The Control Law

The control law will be a time-varying output feedback law of the form $\nu[k] = F[k]\psi[k]$. To derive the control law, we will proceed under the hypothesis that ν has this form and that there is no noise. If this is the case, then (6.7) becomes

$$\begin{aligned}
\psi[k+1] &= e^{aT}e^{-a\tau[k+1]}e^{a\tau[k]}\psi[k] + e^{aT}\frac{b}{a}(1-e^{-aT_1})\alpha[k+1]\nu[k] \\
&= e^{aT}e^{-a\tau[k+1]}\left(e^{a\tau[k]} + \frac{gb}{a}(1-e^{-aT_1})F[k]\right)\psi[k].
\end{aligned}$$

It is clear that $e^{aT}e^{-a\tau[k+1]} \neq 0$ regardless of $\tau \in \mathcal{T}$, so to ensure that the RHS is zero we require that

$$\begin{aligned} 0 &= e^{a\tau[k]} + \frac{gb}{a}(1 - e^{-aT_1})F[k] \\ \Leftrightarrow \quad F[k] &= \frac{-a}{b(1 - e^{-aT_1})\alpha[k]}; \end{aligned}$$

since we do not know $\alpha[k]$, we replace it by our estimate $\hat{\alpha}[k]$, yielding the control law:

$$\nu[k] = \frac{-a\psi[k]}{\hat{\alpha}[k]b(1 - e^{-aT_1})}, \quad k \geq 0. \tag{6.16}$$

The controller K is then given by the sampler (6.4), the estimator (6.14) - (6.15), the control signal (6.16) and the hold (6.3), with initial conditions $\nu[-1] \in \mathbf{R}$ and $\psi[-1] \in \mathbf{R}$.

6.2.5 The Estimator Accuracy

With the control law provided, we can now ascertain how close $\hat{\alpha}[k]$ is to the actual value $\alpha[k]$, which we do in the following Lemma.

Lemma 6.1. *If $\psi[k-1] \neq 0$, and $k \geq 2$ then*

$$|\hat{\alpha}[k] - \alpha[k]| \leq \left|\hat{\alpha}[k-1](1 - e^{a\Delta[k-1]})\right| + \frac{\hat{\alpha}[k-1]c_w\|\overline{w}\|_\infty}{e^{aT}|\psi[k-1]|}.$$

Proof. Let $k \geq 2$ be such that $\psi[k-1] \neq 0$. Next, note that

$$|\hat{\alpha}[k] - \alpha[k]| \leq |\check{\alpha}[k] - \alpha[k]|,$$

so it suffices to prove that $|\check{\alpha}[k] - \alpha[k]|$ is less than the RHS of the Lemma statement.

From the control law (6.16) and the fact that $\hat{\alpha}[k-1] \in [e^{-aT}, \overline{g}]$, it is clear that $\psi[k-1] \neq 0$ implies that $\nu[k-1] \neq 0$. To proceed, we require an expression for $\alpha[k]$ when the system is affected by noise; solving (6.7) for $\alpha[k]$ yields:

$$\alpha[k] = \frac{\psi[k] - e^{aT}e^{a\Delta[k-1]}\psi[k-1] - \eta[k-1]}{\frac{b}{a}e^{aT}(1 - e^{-aT_1})\nu[k-1]}. \tag{6.17}$$

Since $\nu[k-1] \neq 0$, the top line of (6.14) yields an expression for $\check{\alpha}[k]$, so

$$|\check{\alpha}[k] - \alpha[k]| = \left| \frac{\psi[k] - e^{aT}\psi[k-1]}{\frac{b}{a}e^{aT}(1-e^{-aT_1})\nu[k-1]} - \frac{\psi[k] - e^{aT}e^{a\Delta[k-1]}\psi[k-1] - \eta[k-1]}{\frac{b}{a}e^{aT}(1-e^{-aT_1})\nu[k-1]} \right|; \quad (6.18)$$

from (6.16) we have that the denominator simplifies to

$$-e^{aT}\frac{\psi[k-1]}{\hat{\alpha}[k-1]},$$

so (6.18) yields

$$\begin{aligned} |\check{\alpha}[k] - \alpha[k]| &= \left| \frac{\left\{-e^{aT}(e^{a\Delta[k-1]}-1)\psi[k-1] - \eta[k-1]\right\}\hat{\alpha}[k-1]}{e^{aT}\psi[k-1]} \right| \\ &\leq \left|\hat{\alpha}[k-1](1-e^{a\Delta[k-1]})\right| + \frac{\hat{\alpha}[k-1]c_w\|\overline{w}\|_\infty}{e^{aT}|\psi[k-1]|}, \qquad \text{(using (6.12))} \end{aligned}$$

which completes the proof. □

6.3 The Main Result

Theorem 6.1. *If*

$$\tau_l \in \left(\frac{-1}{aT}\ln(1+e^{-aT}), 0\right] \quad \text{and} \quad \tau_u \in \left[0, \frac{-1}{aT}\ln(1-e^{-aT})\right),$$

then

(i) *K stabilizes $\mathcal{G}(\overline{\tau}, \tau_u, \tau_l, \overline{g})$.*

(ii) *There exist positive constants c_1 and γ and a negative constant λ such that for every $\tau \in \mathcal{T}$, $g \in [1, \overline{g}]$, for every initial condition $x_0 \in AC_\infty([-\overline{\tau}, 0])$, and for every $\overline{w} = \begin{bmatrix} w \\ d \end{bmatrix} \in PC_\infty$, we have that when the controller K is applied to the plant (6.2)*

 (a) $\hat{\alpha} \in \ell_\infty$,[4]

 (b) $|\nu[k]| \leq c_1 e^{\lambda k T}\left(\sup_{\theta\in[-\overline{\tau},0]}|x_0(\theta)|\right) + c_2\|\overline{w}\|_\infty$,

 (c) $|y_w(t)| \leq \gamma e^{\lambda t}\left(\sup_{\theta\in[-\overline{\tau},0]}|x_0(\theta)|\right) + c\|\overline{w}\|_\infty$.

Remark 10. *Note that since $\tau_l \leq 0$ and $\tau_u \geq 0$, it follows that if the time delay is fixed (but still unknown) then Theorem 6.1 always holds, i.e., the proposed controller stabilizes $\mathcal{G}(\overline{\tau}, 0, 0, \overline{g})$ for an arbitrarily large fixed time delay and gain.*

Remark 11. *The maximum size of the unknown gain $\overline{g}$ has no impact on the stability result, i.e., we can tolerate the exact same time variation in the delay regardless of $\overline{g}$; however, as $\overline{g}$ increases, we expect that the transient performance will degrade.*

Proof. Fix $\tau_l \in \left(\frac{-1}{aT}\ln(1+e^{-aT}), 0\right]$ and $\tau_u \in \left[0, \frac{-1}{aT}\ln(1-e^{-aT})\right)$. Let $\tau \in \mathcal{T}$, $g \in [1, \overline{g}]$, $d, w \in PC_\infty$, $\psi[-1] \in \mathbf{R}$, $\nu[-1] \in \mathbf{R}$ and $x_0 \in AC_\infty([-\overline{\tau}, 0])$ be arbitrary.

We start by analyzing the startup of the system, i.e., we would like expressions bounding $|\psi[0]|$ and $|\psi[1]|$. Using (6.9) for $|\psi[0]|$, it immediately follows that

$$\begin{aligned} |\psi[0]| &\leq |x_0(-\tau(0))| + |\omega[0]|. \\ &\leq \sup_{\theta \in [-\overline{\tau}, 0]} |x_0(\theta)| + \|\overline{w}\|_\infty. \end{aligned} \tag{6.19}$$

For $|\psi[1]|$ (given by (6.10)), using the control signal (6.16), the bound on $|\psi[0]|$ given by (6.19) and the fact that $\hat{\alpha}[0]$ and $\alpha[1]$ lie in $[e^{-a\overline{\tau}}, \overline{g}]$ regardless of the initial condition, it follows that

$$\begin{aligned} \psi[1] &= e^{a(T-\tau[1])}x_0(0) - e^{aT}\frac{b}{a}(1-e^{-aT_1})\alpha[1]\frac{a(x_0(\tau(0)) + \omega[0])}{b(1-e^{-aT_1})\hat{\alpha}[0]} + \eta[0] \\ |\psi[1]| &\leq e^{aT}\left|e^{-a\tau[1]} - \frac{\alpha[1]}{\hat{\alpha}[0]}\right| \sup_{\theta \in [-\overline{\tau}, 0]} |x_0(\theta)| + e^{aT}\frac{\alpha[1]}{\hat{\alpha}[0]}|\omega[0]| + |\eta[0]| \\ &\leq e^{aT}(1 + \overline{g}e^{a\overline{\tau}}) \sup_{\theta \in [-\overline{\tau}, 0]} |x_0(\theta)| + (e^{aT}\overline{g}e^{a\overline{\tau}} + c_w)\|\overline{w}\|_\infty. \end{aligned} \tag{6.20}$$

With the bound on $|\psi[1]|$, we can now analyze the system update equation for $k \geq 2$. Using the bottom part of (6.11), the definition of $\nu[k]$ given by (6.16), recalling the definitions $\Delta[k] := \tau[k] - \tau[k+1]$ and $\alpha[k] := ge^{-a\tau[k]}$, we can now obtain a bound on $|\psi[k]|$ for $k \geq 2$, namely

$$\psi[k+1] = e^{aT}\alpha[k+1]\left(\frac{1}{\alpha[k]} - \frac{1}{\hat{\alpha}[k]}\right)\psi[k] + \eta[k], \qquad k \geq 1$$

$$\Rightarrow \quad |\psi[k+1]| \leq e^{aT}\alpha[k+1]\left|\frac{1}{\alpha[k]} - \frac{1}{\hat{\alpha}[k]}\right| |\psi[k]| + c_w\|\overline{w}\|_\infty, \qquad k \geq 1 \tag{6.21}$$

$$\leq e^{aT}\frac{\alpha[k+1]}{\alpha[k]}\frac{1}{\hat{\alpha}[k]}\left|\hat{\alpha}[k] - \alpha[k]\right| |\psi[k]| + c_w\|\overline{w}\|_\infty, \quad k \geq 1. \tag{6.22}$$

[4]Note that $\breve{\alpha}$ may not belong to ℓ_∞; however this signal is intermediary in nature and is used in the description of K to enhance clarity.

It will be extremely useful to bound the maximum one time step growth of $|\psi[k]|$:

Claim 4. *There exists a constant c_4 so that*

$$|\psi[k+1]| \leq c_4|\psi[k]| + c_w\|\overline{w}\|_\infty, \quad k \geq 1.$$

Proof. Due to the saturater on $\hat{\alpha}[k]$, it follows that $\left|\frac{1}{\alpha[k]} - \frac{1}{\hat{\alpha}[k]}\right| < e^{a\overline{\tau}}$, and since $\alpha[k+1] \leq \overline{g}$, from (6.21) it follows immediately that

$$|\psi[k+1]| \leq \underbrace{e^{aT}\overline{g}e^{a\overline{\tau}}}_{=:c_4}|\psi[k]| + c_w\|\overline{w}\|_\infty, \quad k \geq 1.$$

□

Before proceeding further, it will be useful to bound $|\psi[2]|$ in terms of the initial condition; using (6.20) and the result of Claim 4, it follows immediately that

$$|\psi[2]| \leq \underbrace{c_4 e^{aT}(1+\overline{g}e^{a\overline{\tau}})}_{=:c_8} \sup_{\theta\in[-\overline{\tau},0]} |x_0(\theta)| + \underbrace{\left[c_4(e^{aT}\overline{g}e^{a\overline{\tau}} + c_w) + c_w\right]}_{=:c_9}\|\overline{w}\|_\infty. \tag{6.23}$$

Using Lemma 6.1 to bound $|\hat{\alpha}[k] - \alpha[k]|$, we can further refine (6.22) so long as $\psi[k-1] \neq 0$:

$$|\psi[k+1]| \leq e^{aT}\frac{\alpha[k+1]}{\alpha[k]}\frac{\hat{\alpha}[k-1]}{\hat{\alpha}[k]}\left(\left|1-e^{a\Delta[k-1]}\right| + \frac{c_w\|\overline{w}\|_\infty}{e^{aT}|\psi[k-1]|}\right)|\psi[k]| + c_w\|\overline{w}\|_\infty, \quad k \geq 2. \tag{6.24}$$

For reasons that will become clear later in the proof, it will be important to determine when

$$e^{aT}\left|1-e^{a\Delta[k-1]}\right| + \frac{c_w\|\overline{w}\|_\infty}{|\psi[k-1]|} < 1,$$

which we show in the following claim.

Claim 5. *There exists constants $\beta > 0$ and $\rho \in [0,1)$ so that if $|\psi[k]| > \beta\|\overline{w}\|_\infty$ then $\psi[k-1] \neq 0$ and*

$$e^{aT}\left|1-e^{a\Delta[k-1]}\right| + \frac{c_w\|\overline{w}\|_\infty}{|\psi[k-1]|} < \rho, \quad k \geq 2.$$

Proof. Let $k \geq 2$ be arbitrary.

From the choice of τ_l and τ_u, it follows that there exists an $\varepsilon \in (0,1)$ so that

$$\tau_l > \frac{-1}{aT} \ln\left(1 + (1-\varepsilon)e^{-aT}\right)$$
$$\tau_u < \frac{-1}{aT} \ln\left(1 - (1-\varepsilon)e^{-aT}\right)$$

We start by obtaining a bound on $e^{aT}\left|1 - e^{a\Delta[k-1]}\right|$ using the bounds on the delay derivatives, τ_u and τ_l; to do so we must analyze two cases:

Case 1: Increasing delay: $\tau[k] \geq \tau[k-1]$

For this case, we have that

$$\begin{aligned}
& \tau[k-1] \leq \quad \tau[k] \quad \leq \tau[k-1] + \tau_u T \\
\Rightarrow \quad & 0 \leq -\Delta[k-1] < \frac{-1}{a} \ln\left(1 - (1-\varepsilon)e^{-aT}\right) \\
\Rightarrow \quad & 0 \geq \ \Delta[k-1] \ > \frac{1}{a} \ln\left(1 - (1-\varepsilon)e^{-aT}\right),
\end{aligned}$$

so

$$\begin{aligned}
e^{aT}\left|1 - e^{a\Delta[k-1]}\right| &= e^{aT}\left(1 - e^{a\Delta[k-1]}\right) \\
&< e^{aT}\left(1 - e^{\ln\left(1-(1-\varepsilon)e^{-aT}\right)}\right) \\
&= e^{aT}\left(1 - 1 + (1-\varepsilon)e^{-aT}\right) \\
&= 1 - \varepsilon.
\end{aligned}$$

Case 2: Decreasing delay: $\tau[k] \leq \tau[k-1]$

For this case, we have that

$$\begin{aligned}
& \tau[k-1] \geq \quad \tau[k] \quad \geq \tau[k-1] + \tau_l T \\
\Rightarrow \quad & 0 \geq -\Delta[k-1] > \frac{-1}{a} \ln\left(1 + (1-\varepsilon)e^{-aT}\right) \\
\Rightarrow \quad & 0 \leq \ \Delta[k-1] \ < \frac{1}{a} \ln\left(1 + (1-\varepsilon)e^{-aT}\right),
\end{aligned}$$

so

$$\begin{aligned}
e^{aT}\left|1 - e^{a\Delta[k-1]}\right| &= e^{aT}\left(e^{a\Delta[k-1]} - 1\right) \\
&< e^{aT}\left(e^{\ln\left(1+(1-\varepsilon)e^{-aT}\right)} - 1\right) \\
&= e^{aT}\left(1 + (1-\varepsilon)e^{-aT} - 1\right) \\
&= 1 - \varepsilon.
\end{aligned}$$

Combining both cases, it is clear that

$$e^{aT}\left|1-e^{a\Delta[k-1]}\right|<1-\varepsilon,$$

so if

$$|\psi[k-1]|>\underbrace{\frac{2}{\varepsilon c_w}}_{=:\beta_1}\|\overline{w}\|_\infty,$$

then

$$e^{aT}\left|1-e^{a\Delta[k-1]}\right|+\frac{c_w\|\overline{w}\|_\infty}{|\psi[k-1]|}<1-\varepsilon+\frac{c_w\varepsilon}{2c_w}<\underbrace{1-\frac{\varepsilon}{2}}_{=:\rho}.$$

All that remains is to convert the bound on $|\psi[k-1]|$ to one on $|\psi[k]|$. To do so, consider Claim 4 (with k replaced with $k-1$): it implies that if $|\psi[k-1]|\le\beta_1\|w\|_\infty$ then

$$|\psi[k]|\le\underbrace{c_4\beta_1+c_w}_{=:\beta}\|\overline{w}\|_\infty,$$

which means that if $|\psi[k]|>\beta\|\overline{w}\|_\infty$, then $|\psi[k-1]|>\beta_1\|\overline{w}\|_\infty$ which completes the proof. □

Claim 6. *There exists constants $c_7>0$ and $\rho\in[0,1)$ such that*

$$|\psi[k]|\le c_7\rho^k\left(\sup_{\theta\in[-\overline{\tau},0]}|x_0(\theta)|\right)+c_7\|\overline{w}\|_\infty,\qquad k\ge0.$$

Proof. We start by bounding the one step growth of the output for $k\ge2$. To do so, we must consider two cases:

Case 1: $|\psi[k]|>\beta\|\overline{w}\|_\infty$.

From Claim 5, we have that $\psi[k-1]\ne0$, so (6.24) holds; applying the rest of of Claim 5 to (6.24) yields

$$|\psi[k+1]|\le\rho\frac{\alpha[k+1]}{\alpha[k]}\frac{\hat{\alpha}[k-1]}{\hat{\alpha}[k]}|\psi[k]|+c_w\|\overline{w}\|_\infty,\quad k\ge2.$$

Case 2: $|\psi[k]|\le\beta\|\overline{w}\|_\infty$.

We apply Claim 4 yielding

$$|\psi[k+1]| \le c_4|\psi[k]| + c_w\|\overline{w}\|_\infty, \quad k \ge 2;$$

since $\psi[k] \le \beta\|\overline{w}\|_\infty$, it follows that

$$|\psi[k+1]| \le (c_4\beta + c_w)\,\|\overline{w}\|_\infty, \quad k \ge 2,$$

and since

$$\rho\frac{\alpha[k+1]}{\alpha[k]}\frac{\hat{\alpha}[k-1]}{\hat{\alpha}[k]}|\psi[k]| \ge 0,$$

it follows that

$$|\psi[k+1]| \le \rho\frac{\alpha[k+1]}{\alpha[k]}\frac{\hat{\alpha}[k-1]}{\hat{\alpha}[k]}|\psi[k]| + (c_4\beta + c_w)\,\|\overline{w}\|_\infty, \quad k \ge 2.$$

We can combine both cases, yielding an expression valid regardless of the size of $|\psi[k]|$:

$$|\psi[k+1]| \le \left(\underbrace{\rho\frac{\alpha[k+1]}{\alpha[k]}\frac{\hat{\alpha}[k-1]}{\hat{\alpha}[k]}}_{=:a_{cl}[k]}\right)|\psi[k]| + (c_4\beta + c_w)\,\|\overline{w}\|_\infty, \quad k \ge 2. \tag{6.25}$$

Defining the state transition matrix

$$\phi_{cl}(k,j) := a_{cl}[k-1] \times a_{cl}[k-2] \times \cdots \times a_{cl}[j], \qquad k \ge j \ge 2, \tag{6.26}$$

it follows that

$$\begin{aligned}\phi_{cl}(k,j) &= \rho\frac{\alpha[k]}{\alpha[k-1]}\frac{\hat{\alpha}[k-2]}{\hat{\alpha}[k-1]}\rho\frac{\alpha[k-1]}{\alpha[k-2]}\frac{\hat{\alpha}[k-3]}{\hat{\alpha}[k-2]} \times \cdots \times \rho\frac{\alpha[j+1]}{\alpha[j]}\frac{\hat{\alpha}[j-1]}{\hat{\alpha}[j]} \\ &= \rho^{k-j}\frac{\alpha[k]}{\alpha[j]}\frac{\hat{\alpha}[j-1]}{\hat{\alpha}[k-1]} \\ &\le \rho^{k-j}\overline{g}^2 e^{2a\overline{\tau}},\end{aligned} \tag{6.27}$$

so it follows that the solution of (6.25) satisfies

$$\begin{aligned}|\psi[k]| &\le |\phi_{cl}(k,2)||\psi[2]| + \sum_{j=2}^{k-1}|\phi_{cl}(k-1,j)|\,(c_4\beta + c_w)\,\|\overline{w}\|_\infty \\ &\le \rho^{k-2}\overline{g}^2 e^{2a\overline{\tau}}|\psi[2]| + \overline{g}^2 e^{2a\overline{\tau}}\,(c_4\beta + c_w)\,\|\overline{w}\|_\infty \sum_{j=2}^{k-1}\rho^{k-1-j} \\ &\le \rho^{k-2}\overline{g}^2 e^{2a\overline{\tau}}|\psi[2]| + \overline{g}^2 e^{2a\overline{\tau}}\,(c_4\beta + c_w)\,\frac{1}{1-\rho}\|\overline{w}\|_\infty, \qquad k \ge 2.\end{aligned} \tag{6.28}$$

To complete the proof, we use (6.23) to replace $|\psi[2]|$ in (6.28) yielding

$$|\psi[k]| \leq \rho^{k-2}\overline{g}^2 e^{2a\overline{\tau}}(c_8 \sup_{\theta\in[-\overline{\tau},0]} |x_0(\theta)| + c_9\|\overline{w}\|_\infty) + \overline{g}^2 e^{2a\overline{\tau}}\,(c_4\beta + c_w)\,\frac{1}{1-\rho}\|\overline{w}\|_\infty$$
$$\leq \underbrace{\overline{g}^2 e^{2a\overline{\tau}} c_8 \rho^{-2}}_{=:\gamma_3} \rho^k \sup_{\theta\in[-\overline{\tau},0]} |x_0(\theta)| + \underbrace{\overline{g}^2 e^{2a\overline{\tau}}\left(c_9 + (c_4\beta + c_w)\,\frac{1}{1-\rho}\right)}_{=:\gamma_4}\|\overline{w}\|_\infty, \quad k \geq 2. \tag{6.29}$$

Defining

$$c_7 := \max\{\gamma_3, \gamma_4, e^{aT}(1 + \overline{g}e^{a\overline{\tau}})\rho^{-1}, e^{aT}\overline{g}e^{a\overline{\tau}} + c_w, 1\},$$

it follows from (6.29) and the bounds on $|\psi[0]|$ and $|\psi[1]|$ given in (6.19) and (6.20) that

$$\psi[k] \leq c_7\rho^k \sup_{\theta\in[-\overline{\tau},0]} |x_0(\theta)| + c_7\|\overline{w}\|_\infty, \qquad k \geq 0,$$

as desired. $\square$

We are now in a position to prove (ii), starting with (ii)-(a). Since, for all $k \geq 0$, we have that $\hat{\alpha}[k] \in [e^{-a\overline{\tau}}, \overline{g}]$, (ii)-(a) clearly holds.

We now consider (ii)-(b). Using (6.16) and noting that $\frac{1}{\hat{\alpha}[k]} \leq e^{a\overline{\tau}} \leq e^{aT}$ we have that

$$|\nu[k]| \leq \underbrace{\left(\frac{ae^{aT}}{b(1-e^{-aT_1})}\right)}_{=:c_{10}}|\psi[k]|, \quad k \geq 0.$$

Using Claim 6, defining $\lambda := \frac{1}{T}\ln(\rho)$ and $c_1 := c_{10}c_7$ it follows that

$$|\nu[k]| \leq c_1 e^{\lambda kT}\left(\sup_{\theta\in[-\overline{\tau},0]} |x_0(\theta)|\right) + c_1\|\overline{w}\|_\infty, \qquad k \geq 0, \tag{6.30}$$

so (ii)-(b) holds.

Now we turn to (ii)-(c). While Claim 6 provides a bound on $|\psi[k]| = |y_w(kT)|$, we need a bound on the inter-sample behaviour; we start with the $k \geq 1$ case. From the second equation of (6.2), we have that

$$y_w(t) = x(t - \tau(t)) + w(t), \quad t \geq 0,$$

and solving the first equation of (6.2) for $t \in [kT, kT+T)$ yields

$$x(t-\tau(t)) = e^{a(t-\tau(t)-kT+\tau(kT))}x(kT-\tau(kT)) + \int_{kT-\tau(kT)}^{t-\tau(t)} gbe^{a(t-\tau(t)-q)}u_d(q)dq; \quad (6.31)$$

recalling that $x(kT-\tau(kT)) = y_w(kT) - w(kT)$, we can re-write (6.31) for $t \in [kT, kT+T)$ as

$$y_w(t) = e^{a(t-\tau(t)-kT+\tau(kT))}(y_w(kT) - w(kT)) + w(t) + \int_{kT-\tau[k]}^{t-\tau(t)} gbe^{a(t-\tau(t)-q)}u_d(q)dq. \quad (6.32)$$

Taking the magnitude of (6.32), using Claim 6 to bound $y_w(kT)$ and using (6.30) to bound $|\nu[k]|$, and extending the integral limits to the maximum possible interval, we obtain:

$$\begin{aligned}
|y_w(t)| &\leq e^{a(T+\overline{\tau})}|y_w(kT)| + (1+e^{a(T+\overline{\tau})})\|\overline{w}\|_\infty + \int_{kT-\overline{\tau}}^{kT+T} gbe^{a(kT+T-q)}(|\nu[k]| + \|\overline{w}\|_\infty)dq \\
&\leq e^{a(T+\overline{\tau})}|y_w(kT)| + (1+e^{a(T+\overline{\tau})})\|\overline{w}\|_\infty + \frac{\overline{g}b}{a}(e^{a(T+\overline{\tau})}-1)|\nu[k]| + \\
&\quad \frac{\overline{g}b}{a}(e^{a(T+\overline{\tau})}-1)\|\overline{w}\|_\infty \\
&\leq \underbrace{\left(e^{a(T+\overline{\tau})}c_7 + \frac{\overline{g}b}{a}(e^{a(T+\overline{\tau})}-1)c_1\right)e^{-\lambda T}}_{=:\gamma_6} e^{\lambda t}\left(\sup_{\theta\in[-\overline{\tau},0]}|x_0(\theta)|\right) + \\
&\quad \underbrace{\left[(1+e^{a(T+\overline{\tau})}) + \frac{\overline{g}b}{a}(e^{a(T+\overline{\tau})}-1)(1+c_1) + c_7^2 e^{a(T+\overline{\tau})}\right]}_{=:\gamma_7}\|\overline{w}\|_\infty, \quad t \in [kT, kT+T).
\end{aligned}$$

We can perform a nearly identical analysis for the $k=0$ case, which we omit for space reasons, so defining $\gamma := \max\{\gamma_6, \gamma_7\}$ we conclude that (ii)-(c) holds.

Finally, part (i) follows immediately from parts (ii)-(b) and (ii)-(c). □

6.4 Simulation

To explore the performance of our proposed controller, we will perform four simulations on the nominal plant

$$\left.\begin{array}{rcl} \dot{x}(t) &=& x(t) + gu_d(t) \\ y_w(t) &=& x(t-\tau(t)) + w(t), \quad t \geq 0 \end{array}\right\} \quad (6.33)$$

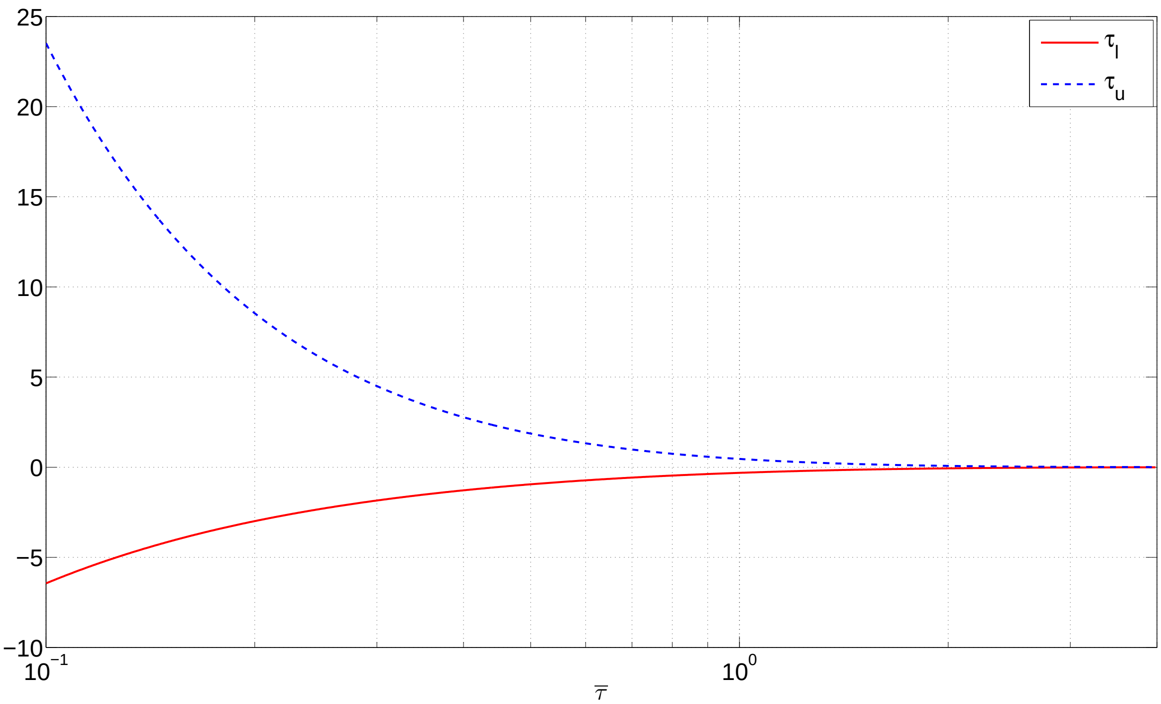

Figure 6.3: The bounds on τ_l and τ_u as a function of the maximum delay for the plant (6.33).

This plant has no gain margin limitation, but Theorem 15 of [35] proves that this plant has an LTI delay margin of two.[5]

Before performing the simulations, we first plot the bounds on τ_l and τ_u given by Theorem 6.1 for the plant (6.33) as a function of the maximum size of the unknown delay; since we can make the period as close as we want to the maximum length of the unknown delay $\overline{\tau}$, we will calculate τ_l and τ_u with $T = \overline{\tau}$, as shown in 6.3. As can be seen, while the allowable time variation is small around the LTI delay margin of two, it rapidly increases as the delay is made smaller.

Example 6.1. For this simulation, we set the maximum delay equal to the LTI delay margin of two and the maximum gain to four. Our uncertainty set for the delay is then

[5]In [35], an LTI controller, parameterized by $\delta > 0$, was constructed which provides a delay margin of at least $2 - \delta$; however, the gain margin provided by this controller was shown to tend to zero as δ tends to zero. Hence, while there is no gain margin limitation when there is no delay, we expect that there will be one when there is a delay, especially when it is unknown.

given by

$$\mathcal{T} := \{\tau \in AC_\infty \mid \tau(t) \in [0,2] \text{ for all } t \geq 0; \dot{\tau} \in [\tau_l, \tau_u] \text{ for almost all } t \geq 0\},$$

and our overall uncertainty model is given by:

$$\mathcal{G}(\overline{\tau}, \tau_u, \tau_l, \overline{g}) := \left\{ \begin{array}{rcl} \dot{x}(t) & = & ax(t) + gbu(t) \\ y(t) & = & x(t-\tau(t)) \end{array} \;\middle|\; \tau \in \mathcal{T}, g \in [1,4] \right\}.$$

To design our controller, we set $T_1 = 0.1$, $T_2 = 2.001$ resulting in an overall period $T = 2.101$. With this period, Theorem 6.1 states that the controller stabilizes $\mathcal{G}$ if $\tau_l \in (-0.05493, 0]$ and $\tau_u \in [0, 0.06211)$, so to perform the simulations, we set $\tau_l = -0.05438$ and $\tau_u = 0.06149$ and run the simulation with random noise with a maximum magnitude of 10^{-2}, with a plant initial condition $x(\theta) = 0.1$ for $\theta \in [-\overline{\tau}, 0]$, with controller initial conditions $\psi[-1] = 0$, $\nu[-1] = 1$, and with the unknown gain set to 2.5, with the results and τ shown in Figure 6.4. Despite changing the delay at 99% of the maximum allowed by Theorem 6.1 for almost every time step, we can see that the controller stabilizes the system and handles the noise.

Example 6.2. For this simulation, we set the maximum delay equal to four, which is twice the LTI delay margin, and the maximum gain to four. Our uncertainty set for the delay is then given by

$$\mathcal{T} := \{\tau \in AC_\infty \mid \tau(t) \in [0,4] \text{ for all } t \geq 0; \dot{\tau} \in [\tau_l, \tau_u] \text{ for almost all } t \geq 0\},$$

and our overall uncertainty model is given by:

$$\mathcal{G}(\overline{\tau}, \tau_u, \tau_l, \overline{g}) := \left\{ \begin{array}{rcl} \dot{x}(t) & = & ax(t) + gbu(t) \\ y(t) & = & x(t-\tau(t)) \end{array} \;\middle|\; \tau \in \mathcal{T}, g \in [1,4] \right\}.$$

To design our controller, we set $T_1 = 0.1$, $T_2 = 4.001$ resulting in an overall period $T = 4.101$. With this period, Theorem 6.1 states that the controller stabilizes $\mathcal{G}$ if $\tau_l \in (-0.004004, 0]$ and $\tau_u \in [0, 0.004071)$, so to perform the simulations, we set them to 90% of the maximum allowed,[6] namely $\tau_l = -0.003604$ and $\tau_u = 0.003664$ and run the simulation with random noise with a maximum magnitude of 10^{-2}, with a plant initial condition $x(\theta) = 0.1$ for $\theta \in [-\overline{\tau}, 0]$, with controller initial conditions $\psi[-1] = 0$, $\nu[-1] = 1$, and with the unknown gain set to 2.5, with the results and τ shown in Figure 6.5. Despite the controller taking a considerable amount of time to handle the initial condition, the controller stabilizes the system and handles the noise with little difficulty.

[6] In the next example, we will see that the output behaves in an unexpected manor when we approach the limits of τ_l and τ_u give by Theorem 6.1.

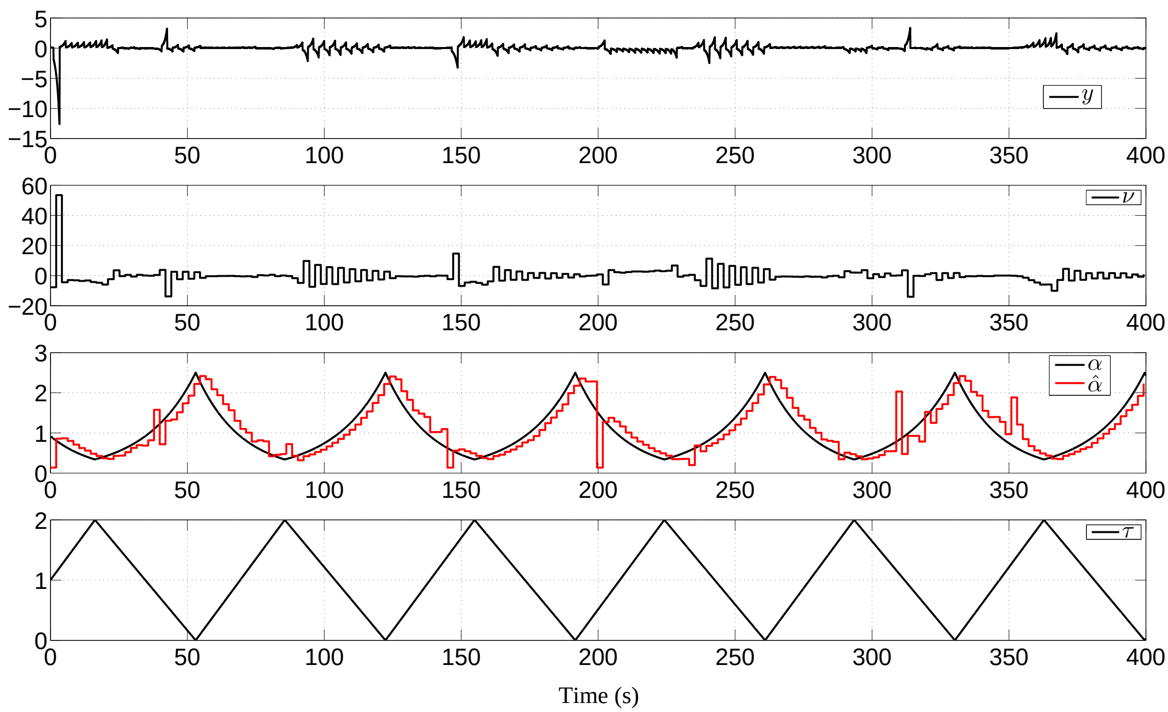

Figure 6.4: Example 6.1: the output, control signal and estimation errors with $\overline{\tau}$ equal to the LTI delay margin.

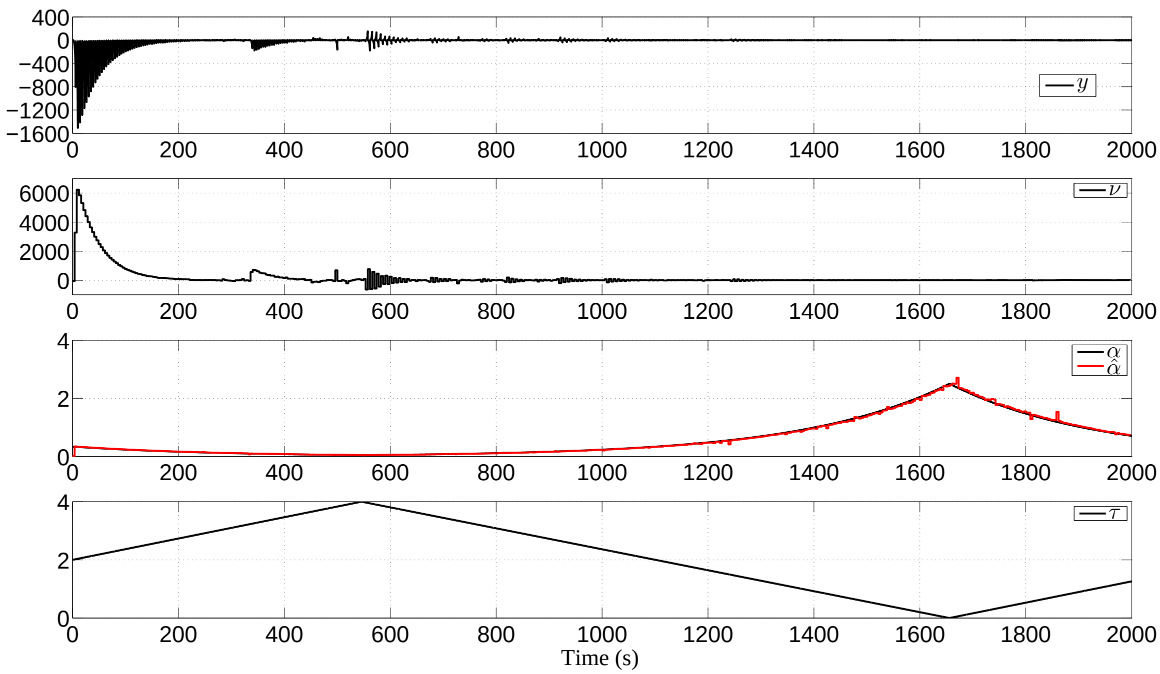

Figure 6.5: Example 6.2: the output, control signal and estimation errors with $\overline{\tau}$ equal to two times the LTI delay margin with τ_u and τ_l at 90% of the maximum allowed by Theorem 6.1.

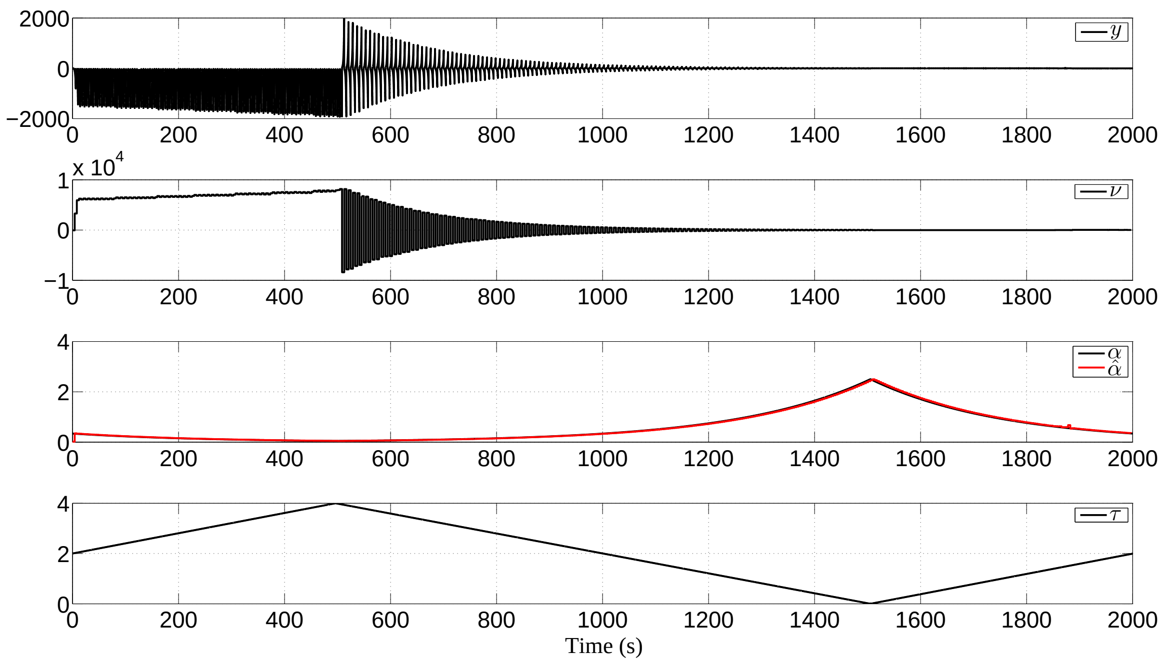

Figure 6.6: Example 6.3: the output, control signal and estimation errors with $\bar{\tau}$ equal to two times the LTI delay margin with τ_u and τ_l at 99% of the maximum allowed by Theorem 6.1.

Example 6.3. For this simulation, we consider the same uncertainty set as the previous example except we will allow the delay to vary at 99% of the maximum allowed by Theorem 6.1 instead of 90%. To design our controller, we set $T_1 = 0.1$, $T_2 = 4.001$ resulting in an overall period $T = 4.101$. With this period, Theorem 6.1 states that the controller stabilizes $\mathcal{G}$ if $\tau_l \in (-0.004004, 0]$ and $\tau_u \in [0, 0.004071)$, so to perform the simulations, we set $\tau_l = -0.003964$ and $\tau_u = 0.004030$ and run the simulation with random noise with a maximum magnitude of 10^{-2}, with a plant initial condition $x(\theta) = 0.1$ for $\theta \in [-\bar{\tau}, 0]$, with controller initial conditions $\psi[-1] = 0$, $\nu[-1] = 1$, and with the unknown gain set to 2.5, with the results and τ shown in Figure 6.6. While the output continues to grow during the first 500 seconds, this type of behaviour is predicted in the proof of Theorem 6.1 via the telescoping product shown in equations (6.26) and (6.27). Despite this initial increase in the output, the controller clearly stabilizes the system and handles the noise with little difficulty.

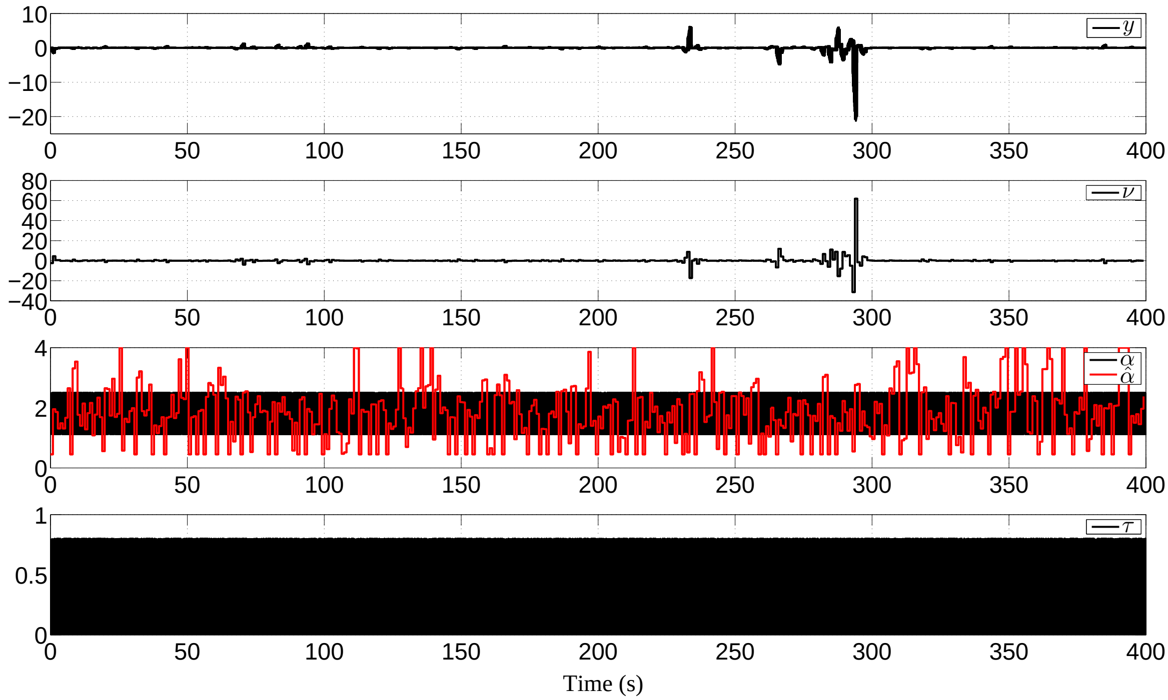

Figure 6.7: Example 6.4: the output, control signal and estimation errors for a randomly varying delay.

Example 6.4. For this simulation, we set the maximum delay to 0.8, and the unknown gain to four. Unlike the previous two examples, this time we allow the delay to vary randomly at each simulation time step of 0.001 seconds. While this scenario was not proven in Theorem 6.1, simulations suggest that stability is maintained for random delays of up to approximately half the LTI delay margin. To design the controller for this simulation, we set $T_1 = 0.1$, $T_2 = 0.801$ resulting in an overall period $T = 0.901$, and we run the simulation with random noise with a maximum magnitude of 10^{-2}, with a plant initial condition $x(\theta) = 0.1$ for $\theta \in [-\overline{\tau}, 0]$, with controller initial conditions $\psi[-1] = 0$, $\nu[-1] = 1$, and with the unknown gain set to 2.5; the results are shown in Figure 6.7, and as can be seen, despite the extreme variation in the delay, the controller stabilizes the system.

6.5 Conclusions and Future Work

In this chapter, we propose a controller which stabilizes a first order unstable LTI continuous-time plant with an uncertain arbitrarily large time varying delay and an arbitrarily large uncertain gain. The proposed controller uses a simple generalized hold which enables a simple update law for estimating the unknown gain and delay at each time step, which is then used for a simple feedback control law. This controller, while mildly nonlinear and periodic, not only provides BIBO stability, but also guarantees the exponential decay of the plant initial conditions so long as the delay varies less than a simple formula relating the maximum delay duration, the location of the unstable pole and the time variation of the delay. The work is reported in [15].

For future work, we would like to consider uncertainty and time variations in the pole location, time variations in the gain, and proving the maximum delay for which the controller can maintain stability when the delay is random.

Chapter 7

Open Problems

In this chapter, we formally state two problems related to the achievable delay margin of discrete-time and sampled data systems. We already considered the discrete-time problem in Chapter 3, where we showed that determining the achievable discrete-time delay margin is a natural simultaneous stabilization problem, and using a classic simultaneous stabilization result, we provided a simple necessary and sufficient condition for when the discrete-time achievable delay margin is non-zero. However, we also showed in Chapter 3 that using existing simultaneous stabilization results provide no new insight into the solution to the general problem, even for the simplest case of a first order plant with a single unstable pole. As for the sampled-data problem, in Chapter 3 we briefly considered the case of a continuous time plant with a discrete-time controller and delay; that problem, so long as the sampling period is non-pathological, would be solved with a solution to the discrete-time problem. So, in this chapter, we consider the other sampled data problem, that of a continuous-time plant with a continuous-time delay and a discrete-time controller. While we do not solve either problem, in both cases, we show how the proof method used to solve the continuous-time problem in [35] breaks down.

This chapter is organized as follows. In Section 7.1, we will explain how [35] was able to solve the continuous-time delay margin problem for the simplest unstable plant, namely $\frac{1}{s-p}$, with $p > 0$. In Section 7.2, we re-state the discrete-time delay margin problem previously introduced in Chapter 3, and show how the proof method used to solve the continuous-time problem in [35] breaks down for the plant $\frac{1}{z-p}$, $p > 1$, in the discrete-time setting. In Section 7.3, we formally state the sampled data delay margin problem, and again, we show how the proof method employed in [35] breaks down and in Section 7.4, we summarize the chapter and discuss some future work.

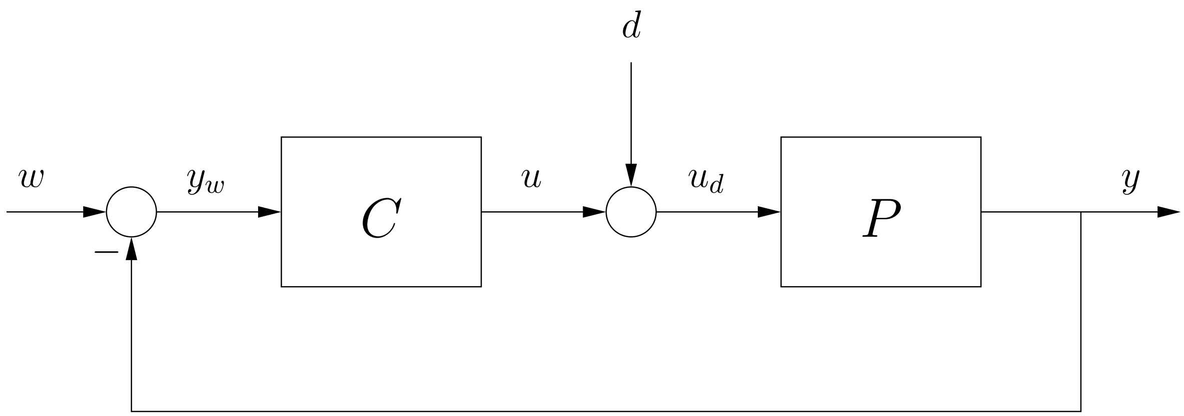

Figure 7.1: The continuous-time delay margin problem setup.

7.1 Continuous-Time Achievable Delay Margin

We first want to quickly summarize how [35] was able to solve certain cases of the continuous-time achievable delay margin problem. To do so, we will first formally state the continuous-time problem, so, with $P_0(s)$ the nominal plant, our set of admissible delayed plants is given by:

$$\mathcal{P}_\tau := \left\{ e^{-s\tau} P_0(s) \ : \ \ \tau \in [0, \overline{\tau}] \right\}.$$

To define stability, we consider the feedback setup shown in Figure 7.1, and we say that a controller $C(s)$ stabilizes a plant $P(s)$ if the transfer function from

$$\begin{bmatrix} d \\ w \end{bmatrix} \to \begin{bmatrix} y \\ u \end{bmatrix}$$

belongs to $H_\infty^{2\times 2}(\mathbf{C}^+)$, i.e.,

$$(1+PC)^{-1}, C(1+PC)^{-1}, P(1+PC)^{-1}, PC(1+PC)^{-1} \in H_\infty(\mathbf{C}^+).$$

With stability defined, the controller dependent delay margin[1] is given by

$$DM(P_0, C) := \sup\{\tau \geq 0 : C \text{ stabilizes } \mathcal{P}_\tau\},$$

and the more fundamental plant property, the achievable delay margin is given by

$$DM(P_0) := \sup\{DM(P_0, C) : C \text{ is FDLTI and stabilizes } P_0\}.$$

[1]See Section 1.2 for the solution to the continuous-time controller dependent delay margin.

In [35], for the nominal plant

$$P_0(s) = \frac{1}{s-p}, \quad p > 0,$$

the authors were able to prove that $DM(P_0) = \frac{2}{p}$,[2] by employing the inner transfer function

$$B_\alpha(s) = \frac{1-\alpha s}{1+\alpha s}, \quad \alpha \geq 0. \tag{7.1}$$

and applying the following logic:

(i) Any controller $C(s)$ that stabilizes $P_0(s)$ also stabilizes $P_0(s)B_\alpha(s)$ for $\alpha = 0$.
(ii) The closed loop poles of $P_0(s)C(s)B_\alpha(s)$ move continuously in α.
(iii) Hence, $C(s)$ stabilizes $P_0(s)B_\alpha(s)$ for small α as well.
(iv) For $\alpha = p^{-1}$, $P_0(s)$ and $B_\alpha(s)$ have an unstable pole/zero cancellation. As a result, there exists an $\alpha^* \in (0, p^{-1})$ and a $\omega^* \in \mathbf{R}$ such that

$$B_\alpha^*(j\omega^*)P_0(j\omega^*)C(j\omega^*) = -1.$$

(v) Therefore, $C(s)$ does not stabilize $P_0(s)B_\alpha^*(s)$.
(vi) Find the smallest value of τ^* such that $e^{-j\omega^*\tau^*} = B_\alpha^*(j\omega^*)$, and note that since both functions have magnitude one on the imaginary axis, that we simply need to equate their phases.
(vii) By equating their phases, we can show that $\tau^* \leq 2\alpha^*$; since $\alpha^* \leq \frac{1}{p}$ it follows that $\tau^* \leq \frac{2}{p}$.
(viii) Observe that

$$e^{-j\omega^*\tau^*}P_0(j\omega^*)C(j\omega^*) = -1,$$

and hence, the system is unstable with a delay of τ^* seconds.
(ix) Since the upper bound on α^* and τ^* is controller independent, $DM(P_0) \leq \frac{2}{p}$.

As will be seen in the next two sections, attempts to extend this logic to the discrete-time and sampled data problems fail.

7.2 Discrete-Time Achievable Delay Margin

We start by re-stating the discrete-time achievable delay margin problem first introduced in Chapter 3; unlike in that chapter, we can consider transfer functions in z instead of

[2] This bound is in fact tight, as also shown in [35] by constructing a controller which provides a delay margin arbitrarily close to $\frac{2}{p}$.

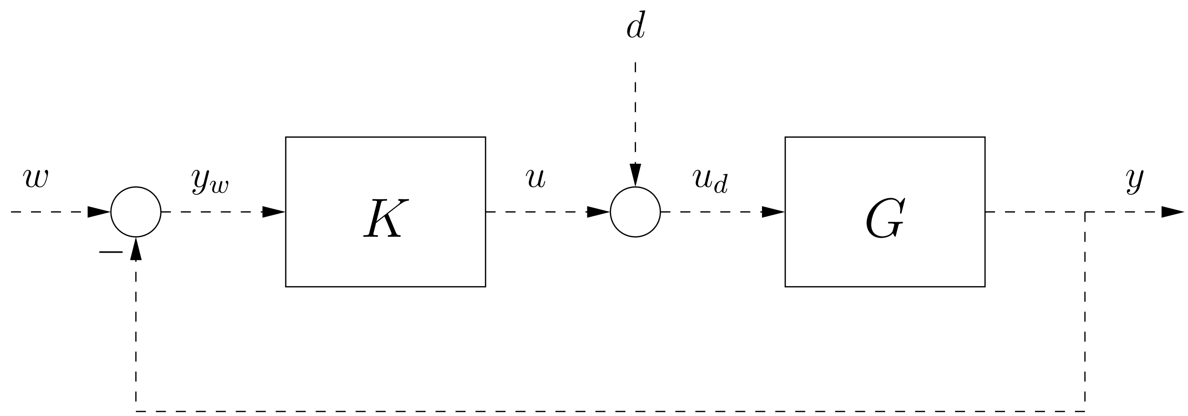

Figure 7.2: The discrete-time delay margin problem setup.

$\lambda = z^{-1}$. Let the real rational and proper transfer function $G_0[z]$ denote the nominal plant, and let

$$G_n[z] := z^{-n} G_0[z], \quad n \in \mathbf{N},$$

denote the delayed versions of the nominal plant.

To define stability in this chapter, we consider the feedback setup shown in Figure 7.2 and say that a controller K *stabilizes* a plant G if the transfer function from

$$\begin{bmatrix} d \\ w \end{bmatrix} \to \begin{bmatrix} y \\ u \end{bmatrix}$$

belongs to $H_\infty^{2\times 2}(\overline{\mathbf{D}})$, i.e.,

$$(1+KG)^{-1}, K(1+KG)^{-1}, G(1+KG)^{-1}, KG(1+KG)^{-1} \in H_\infty(\overline{\mathbf{D}}).$$

To formally define the delay margin, we adopt the notation from [35]. If a controller K stabilizes a plant G_0, then the *delay margin* is

$$DM(G_0, K) := \max\{n \geq 0 : K \text{ stabilizes } G_0, G_1, \cdots, G_n\}.$$

While $DM(G_0, K)$ is a useful quantity to know about a particular plant/controller combination, a more fundamental property of the plant is

$$DM(G_0) := \max\{DM(G_0, K) : K \text{ is FDLTI and stabilizes } G_0\},$$

which is simply the maximum achievable delay margin when using a stabilizing FDLTI controller.

We want to employ the same technique used to solve the continuous-time problem summarized in Section 7.1, to the discrete-time delay margin problem. To that end, define an n sample discrete-time delay

$$F_n[z] = \frac{1}{z^n}, \tag{7.2}$$

and the discrete-time inner transfer function,

$$B_\alpha[z] := \frac{1 - \alpha z}{z - \alpha}, \quad \alpha \in [0, 1) \tag{7.3}$$

which is the discrete-time analogue of (7.1) used for the continuous-time problem summarized in Section 7.1. Applying the logic used for the continuous-time problem yields:

(i) Any controller $K[z]$ that stabilizes $G_0[z]$ also stabilizes $G_0[z]K[z]B_\alpha[z]$ for $\alpha = 0$.
(ii) The closed loop poles of $G_0[z]K[z]B_\alpha[z]$ move continuously in α.
(iii) Hence, $K[z]$ stabilizes $G_0[z]B_\alpha[z]$ for small α as well.
(iv) For $\alpha = p^{-1}$, $G_0[z]$ and $B_\alpha[z]$ have an unstable pole/zero cancellation. As a result, there exists an $\alpha^* \in (0, p^{-1})$ and an $\Omega^* \in \mathbf{R}$ such that

$$B^*_\alpha(e^{j\Omega^*})G_0(e^{j\Omega^*})K(e^{j\Omega^*}) = -1.$$

(v) Therefore, $K[z]$ does not stabilize $G_0[z]B^*_\alpha[z]$.

Unfortunately, step (vi) of Section 7.1 converted to the discrete-time setting requires us to find a permissible delay n^* such that

$$e^{-j\Omega^* n^*} = B^*_\alpha[e^{j\Omega^*}];$$

while this equation has a solution for $n^* \in \mathbf{R}$, it does not (in general) have a solution for $n^* \in \mathbf{N}$. In other words, the proof employed in [35] utilizes the fact that the delay is allowed to vary continuously so that it continuously deforms the resulting Nyquist plot. In discrete-time, since the allowable delays take on integer values, the delay can not vary continuously, and as a result, a change in the delay does not continuously deform the Nyquist plot.

As a result of the difficulties in applying the simultaneous stabilization results to three or more plants and the difficulties in extending the approach adopted in [35], the discrete-time achievable delay margin problem remains open.

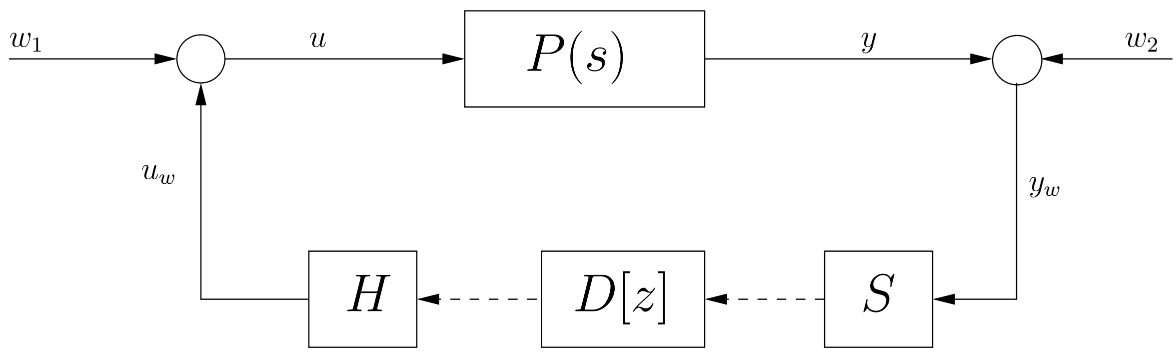

Figure 7.3: A block diagram of the feedback control problem with a time delay.

7.3 Achievable Delay Margin for Sampled Data Systems with a Continuous-Time Delay

In this section we consider the sampled data delay margin problem with a continuous-time delay as shown in Figure 7.3. Like the discrete-time problem described in the previous section, we will show that extending the proof method employed in [35] fails, although the failure occurs for a different reason than the discrete-time case.

Using the feedback setup shown in Figure 7.3, we say that a finite dimensional LTI controller D *stabilizes* a plant P if the map from

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \to \begin{bmatrix} y \\ u \end{bmatrix}$$

has a bounded norm. Since P and D are LTI, this definition is equivalent to the normal notion of bounded input bounded output stability (i.e., if $\|w_1\|_\infty < \infty$ and $\|w_2\|_\infty < \infty$ then $\|y\|_\infty < \infty$ and $\|u\|_\infty < \infty$).

Given a nominal plant $P_0(s)$, our uncertainty set is the same as the continuous-time case, namely,

$$\mathcal{P}_\tau := \left\{ e^{-s\tau} P_0(s) \ : \ \ \tau \in [0, \overline{\tau}] \right\};$$

the controller dependent delay margin is then given by

$$DM(P_0, D, T) := \sup\{\overline{\tau} > 0 : D \text{ stabilizes } P_0(s)e^{-s\tau}, \ \ \tau \in [0, \overline{\tau})\},$$

and the controller independent delay margin is given by

$$DM(P_0, T) := \sup\{DM(P_0, D, T) : D \text{ stabilizes } P_0\}.$$

From [11], we can determine the stability of a sampled data system by analyzing an appropriate discrete-time system. So to that end, consider a continuous-time linear plant P_0 with a state space representation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned}$$

with (A, B) controllable and (C, A) observable; to save space, we will denote $P_0(s)$'s state space representation by (A, B, C, D). We define the discretization of P_0 with a sampling rate of T seconds by G_0, which has a state space matrices given by

$$\begin{aligned} A_d &:= e^{AT} \\ B_d &:= \int_0^T e^{At} B dt \\ C_d &:= C \\ D_d &:= D, \end{aligned} \tag{7.4}$$

and a corresponding transfer function $G_0[z] = D_d + C_d(zI - A_d)^{-1}B_d$. Recalling the definition of pathological sampling given in Section 2.3, it follows from [11] that if T is non-pathological with respect to A, then the map

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \to \begin{bmatrix} y \\ u \end{bmatrix}$$

has bounded norm (and hence the system given by Figure 7.3 is closed loop stable) if and only if the transfer functions

$$(1 + G_0 D)^{-1},\ G_0(1 + G_0 D)^{-1},\ G_0 D(1 + G_0 D)^{-1} \text{ and } D(1 + G_0 D)^{-1}$$

all belong to $H_\infty(\overline{\mathbf{D}})$. From now on, we will assume that T is non-pathological.

We now need a discretization of the delayed version of $P_0(s)$, namely $P_\tau(s) := P_0(s)e^{-s\tau} \in \mathcal{P}_\tau$; to simplify the calculations, we will restrict $P_0(s)$ to be strictly proper, so P_τ has the following state space representation:

$$\begin{aligned} \dot{x} &= Ax + Bu(t - \tau) \\ y &= Cx. \end{aligned} \tag{7.5}$$

We would like to discretize this delayed plant, in order to obtain a discrete-time model $G_\tau[z]$, using the same integration technique that yields (7.4), so that we can use the stability results from [11]. To do this, we use the method outlined in pages 38-42 of [1]; first define

$$d := \left\lceil \frac{\tau}{T} \right\rceil$$

and

$$\tau' := \tau \text{ modulo } T,$$

so that we can write the delay as

$$\tau = (d-1)T + \tau', \quad \tau' \in (0, T], \ d \in \mathbf{N},$$

and then define

$$\begin{aligned} \Phi &= e^{AT} \\ \Gamma_0 &= \int_0^{T-\tau'} e^{As} B ds \\ \Gamma_1 &= e^{A(T-\tau')} \int_0^{\tau'} e^{As} B ds. \end{aligned} \tag{7.6}$$

If $\tau \in (0, T)$, then $d = 1$ and $\tau = \tau'$, and the state space representation for G_τ is

$$\begin{bmatrix} x(k+1) \\ u(k) \end{bmatrix} = \underbrace{\begin{bmatrix} \phi & \Gamma_1 \\ 0 & 0 \end{bmatrix}}_{=:A_\tau} \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix} + \underbrace{\begin{bmatrix} \Gamma_0 \\ I \end{bmatrix}}_{=:B_\tau} u(k); \tag{7.7}$$

if $\tau \geq T$, and hence $d = \lceil \frac{\tau}{T} \rceil > 1$, we get the following state space representation:

$$\begin{bmatrix} x(k+1) \\ u(k-d+1) \\ u(k-d+2) \\ \vdots \\ u(k-1) \\ u(k) \end{bmatrix} = \underbrace{\begin{bmatrix} \phi & \Gamma_1 & \Gamma_0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{:=A_\tau} \begin{bmatrix} x(k) \\ u(k-d) \\ u(k-d+1) \\ \vdots \\ u(k-2) \\ u(k-1) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}}_{:=B_\tau} u(k). \tag{7.8}$$

In both cases, C_τ is given by

$$C_\tau = [C \ \ 0],$$

and

$$G_\tau[z] = C_\tau (zI - A_\tau)^{-1} B_\tau.$$

With this discretization, and using [11], it follows that

$$DM(P_0, D, T) = \sup\{\overline{\tau} > 0 : D \text{ stabilizes } G_\tau,\ \ \tau \in [0, \overline{\tau}]\}.$$

In order to show the difficulties in applying the proof method of [35] to this problem, we will consider the simplest problem, namely, a first order plant with a single unstable pole: with $p > 0$, let

$$P_0(s) = \frac{1}{s - p}, \tag{7.9}$$

which has a state space representation given by

$$\begin{aligned} \dot{x} &= px + u \\ y &= x \end{aligned}$$

Using (7.4), it is easy to show that the discretized version of P_0, denoted by G_0, has a transfer function given by

$$G_0[z] = \frac{e^{pT} - 1}{p(z - e^{pT})}. \tag{7.10}$$

To discretize P_τ, we apply (7.6) yielding

$$\begin{aligned} \Phi &= e^{pT} \\ \Gamma_0 &= \int_0^{T-\tau'} e^{ps} ds \\ &= \frac{e^{pT}(e^{-p\tau'} - e^{-pT})}{p} \\ \Gamma_1 &= e^{p(T-\tau')} \int_0^{\tau'} e^{ps} ds. \\ &= \frac{e^{pT}(1 - e^{-p\tau'})}{p}, \end{aligned} \tag{7.11}$$

with a state space representation given by (7.7) if $\tau < T$ or (7.8) if $\tau \geq T$. However, if we look at the $x(k+1)$ term of (7.7) and (7.8), we get the following expression in both cases:

$$x[k+1] = \Phi x[k] + \Gamma_1 u[k-d] + \Gamma_0 u[k-d+1].$$

Since $y = x$ for this first order problem, we can find the transfer function, denoted $G_\tau[z]$, of the discretized plant, which is given by

$$\begin{aligned} G_\tau[z] &= \frac{Y[z]}{U[z]} \\ &= \frac{\Gamma_1 + z\Gamma_0}{z^d(z-\Phi)} \\ &= \frac{e^{pT}[1 - e^{-p\tau'} + z(e^{-p\tau'} - e^{-pT})]}{p(z - e^{pT})z^d}. \end{aligned} \tag{7.12}$$

Finally, we isolate the effect of the delay in discrete-time by recalling that $d := \lceil \frac{\tau}{T} \rceil$ and $\tau' := \tau$ modulo T and then defining the sampled data delay:

$$\begin{aligned} \Delta_\tau[z] &:= \frac{G_\tau[z]}{G_0[z]} \\ &= e^{pT}\frac{1 - e^{-p\tau'} + z(e^{-p\tau'} - e^{-pT})}{z^d(e^{pT} - 1)}. \end{aligned} \tag{7.13}$$

We would like to use this delay function to find a bound on $DM(P_0, T)$ using the same method employed in [35] and described in Section 7.1. When we tried to apply this method to the discrete-time problem in Section 7.2, the method failed due to the non-continuous nature of a discrete-time delay. Fortunately, it is easy to show that $\Delta_\tau(e^{j\Omega})$ is continuous in the delay for all $\Omega \in [0, \pi)$, so this is no longer the problem. However, there is a new difficulty introduced by Δ_τ, namely, unlike both a continuous-time and discrete-time delay, Δ_τ is not an inner function as seen in the Bode plot of Δ_τ shown in Figure 7.4. As a result, unlike the continuous-time and discrete-time problems, we can not simply equate it with another inner function, i.e., we can not complete step (vi):

$$\Delta_\tau[e^{-j\Omega^*\tau^*}] = B^*_\alpha[e^{j\Omega^*}],$$

since for almost all values of τ^*, we have that

$$|\Delta_\tau[e^{-j\Omega^*\tau^*}]| \neq |B^*_\alpha[e^{j\Omega^*}]|.$$

So, just as for the discrete-time problem, there is no current solution to this problem.

7.4 Conclusions and Future Work

In this chapter, we briefly summarized the proof used in [35] to solve the continuous-time delay margin problem, and then considered the same problem in two different settings,

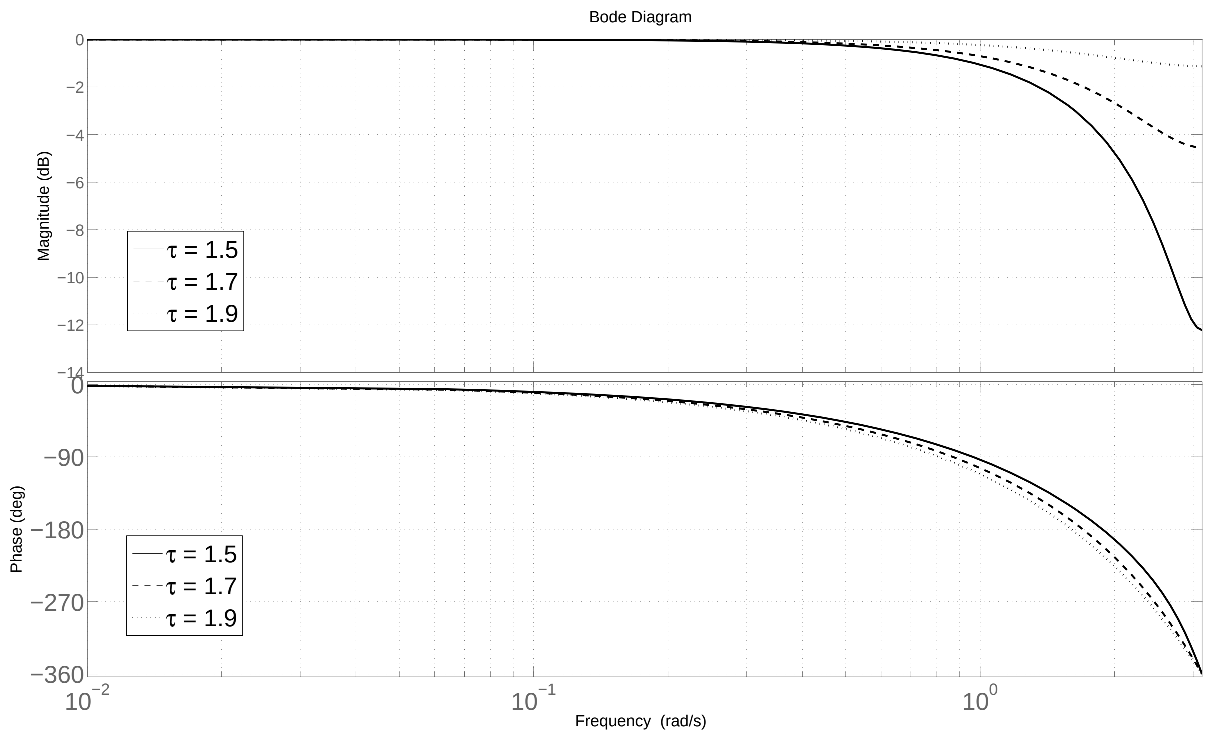

Figure 7.4: Bode plot of Δ_τ with $p = 1$, $T = 1$, for $\tau = 1.5$, $\tau = 1.7$ and $\tau = 1.9$

namely, discrete-time and a sampled data setup. In both cases, we demonstrated how the proof fails to translate to each problem, and as a result, they remain unsolved and are a topic of future research.

Chapter 8

Conclusions and Future Work

8.1 Conclusions

In this thesis, we considered two different problems involving unknown time delays. The first problem was the so-called achievable delay margin problem, which is determining the maximum allowable unknown delay for which a single LTI controller can maintain stability. The second problem was finding a controller design algorithm which stabilizes any continuous-time LTI plant with an arbitrarily large unknown time delay and gain.

For the achievable delay margin problem, we first considered the problem in the discrete-time setting. After setting up the problem, we noted that this is a special case of the classical simultaneous stabilization problem, which has a simple, elegant and necessary and sufficient test for the case of two plants; using this test, we were able to determine that the discrete time achievable delay margin is non-zero if and only if the discrete-time plant has no real, negative unstable poles. Unfortunately, no necessary and sufficient conditions exist for the simultaneous stabilization problem of three or more plants, and we were unable to glean any further insight into the discrete-time achievable delay margin problem from further application of the two plant simultaneous stabilization test. Since the achievable delay margin problem has been solved in the continuous-time setting [35] (though for some plant configurations the bounds are not tight), we also attempted to solve the discrete-time and sampled-data achievable delay margin problems by extending the proof used in [35] to these different settings. Unfortunately, for both the sampled-data and discrete-time settings, we were able to explain why the proof used in [35] does not extend to these new time settings.

For the second problem of stabilizing any continuous-time LTI plant with an arbitrarily large unknown gain and delay, we were able to convert the infinite dimensional continuous-time problem into a finite dimensional discrete-time one through the use of a simple generalized hold. Furthermore, the resulting discrete-time problem was one amenable to a class of classical adaptive controllers, but not to many of the more popular adaptive controllers due to the lack of convexity in the resulting discrete-time uncertainty set. Using an admissible off-the-shelf adaptive controller, we were able to obtain a weak form of stability for any LTI plant with an unknown but upper bounded time delay and gain; unfortunately, the performance was poor in simulation. Hence, we next considered the use of novel estimators for the unknown gain and delay, and using those estimates, we were able to design a purpose built adaptive controller which not only provided BIBO stability, but also guaranteed the exponential decay of the plant initial conditions, tolerated occasional jumps in the unknown gain and delay, and was robust to un-modelled dynamics. Finally, for the first order case, through the use of a similar, but simplified estimator, we were able to prove toleration to continuous variations in the unknown delay, and found an explicit bound on the rate of change of the delay in terms of the maximum allowable delay and the unknown plant pole for which the controller BIBO stabilized the closed loop system.

8.2 Future Work

For the discrete-time achievable delay margin problem, we would like to solve the general problem, as done in [35] for the continuous-time case. To do so, we would likely start with the simple case of a plant with a single unstable pole. While the proof method from [35] breaks down, it does provide insight into the possible solution for this case, and we believe that there does exist a fundamental limit on the achievable delay margin in the discrete-time setting. Furthermore, if we can obtain a solution to the simplest case, we believe that it should be possible to leverage that solution for different plant configurations, and as a result, solve the general problem. For the sampled-data problem, since the proof method of [35] breaks down differently from the discrete-time problem, a clever argument may be able to salvage a solution for the plant $P(s) = \frac{1}{s-p}$, $p > 0$; however it may still be difficult to extend the result to any continuous time plant with a single unstable pole, let alone the general problem. Of course, we would still like to obtain a solution to the general problem in the sampled-data setting.

For the second problem, that of stabilizing any LTI plant with an arbitrarily large unknown gain and delay, we would like to extend the results presented here in numerous ways. For the results of Chapter 4, we would like to consider a larger uncertainty set, namely, linearly parameterized unknown A and B matrices along with the arbitrarily large time delay. We would also like to explore different adaptive controllers with the hope of obtaining a better form of stability, or at least better performance in simulation. For the results of Chapter 5, we would like to show that the proposed controller can tolerate slow variations in the unknown gain and delay; indeed, from our simulations, it appears that the controller can tolerate such variations. We would also like to consider ways to improve the performance of the controller, for example, we would like to consider tracking of certain types of inputs, for example, a step input; we would also like to improve the transient behaviour of the controller by considering factors like the placement of the eigenvalues or by some clever manipulations of the delay and gain estimates to mitigate the effect of a poor estimate. We would also like to consider larger uncertainty sets, for example, dropping the requirement that we know the sign of the unknown gain, or including some uncertainty in both the A and B state space matrices. We would also like to extend this design to multi-input multi-output systems. For the first order case considered in Chapter 6, we would like to consider uncertainty in the pole location, as well as randomly varying delays.

APPENDICES

Appendix A

List of Acronyms

This thesis uses numerous acronyms; for convienence, they are listed below:

LTI	Linear Time Invariant
FDLTI	Finite Dimensional Linear Time Invariant
LTV	Linear Time Varying
NLTV	Non-Linear Time Varying
SISO	Single-Input Single-Output
LMI	Linear Matrix Inequality
BIBO	Bounded Input Bounded Output
RHS	Right Hand Side
LHS	Left Hand Side
PBH	Popov-Belevitch-Hautus

Table A.1: List of Acronyms

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