

Pricing Derivatives by Gram-Charlier Expansions

by

Yin Hei Cheng

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Yin Hei Cheng

Abstract

In this thesis, we provide several applications of Gram-Charlier expansions in derivative pricing. We first give an exposition on how to calculate swaption prices under the CIR2 model. Then we extend this method to CIR2++ model. We also develop a procedure to calculate European call options under Heston's model of stochastic volatility by Gram-Charlier Expansions.

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Chapter 1

Introduction

The Gram-Charlier expansion, discovered by Jørgen Pedersen Gram and Carl Charlier, is an infinite series that approximates a probability distribution in terms of its cumulants (or moments). The idea behind the Gram-Charlier expansions is quite natural. Suppose that the first four moments of a random variable are known. We are able to calculate the mean, variance, skewness and kurtosis of it. Heuristically speaking, the shape of the density function of a random variable can roughly be described using these moments. As a result, the distribution function of a random variable is almost fixed if moments of it are known.

On the other hand, the arbitrage-free price of any European-type derivative of an asset is just an expectation with respect to a equivalent martingale measure. If the moments of the underlying asset with respect to the numeraire measure are known, by the above discussion, we are able to calculate its density (or distribution) function. Hence, the price can be obtained with ease. This is the main idea of this thesis.

This thesis is divided into five chapters. Chapter 1 gives the background knowledge of Gram-Charlier expansions and an important class of ordinary differential equations. The Gram-Charlier expansion will be used throughout this thesis. The solution of the class of ordinary differential equations mentioned above will be heavily used in Chapter 3.

In chapter 2, we review some fundamental concepts on mathematical finance. We

define and discuss the properties of T -forward measures, swap, swaptions and Black's implied volatilities of swaptions.

In the first three sections of chapter 3, we give a detailed survey on [12], which presented a way to calculate the swaption prices under CIR2 model by using Gram-Charlier expansions. Most of the original works, which will be given after section 3.3, are inspired by this paper. In the last section of chapter 3, we present a method to calculate the the swaption prices under CIR2++ model by using Gram-Charlier expansions. This is achieved by a modification of the formula for the bond moment, which is a crucial concept in [12].

We discussed how to apply the Gram-Charlier approach in general diffusion processes in Chapter 4. To make the description simple, we consider the Black-Scholes model and a simplified version of Brennan-Schwarz model on interest rates. Black-Scholes model is chosen as a representation of the class of diffusion processes of which the moments can be easily obtained. In section 4.1, we show how to use Gram-Charlier approach in pricing European call options for this class of models. A simplified version of Brennan-Schwarz model is chosen since the process does not have an easy closed form solution. In section 4.2, we prove that the moments are just solutions of a system of ordinary differential equations. The solutions can be obtained by symbolic calculation software. In general, this can be obtained numerical methods of ordinary differential equations.

In Chapter 5, we study the Gram-Charlier approach in Heston's Model. Since the characteristic functions of the discounted log-price is known, the moments of the log-price are readily obtained by taking derivatives. Then we develop a formula to calculate the truncated moment-generating function in section 5.2. This is the key step of obtaining the approximation formula of the price of European call options. We also suggest a way to simulated the Heston's model, so that negative volatility can always be avoided. Some numerical results and discussions are given at the end of this chapter.

Chapter 6 contains a conclusion of this thesis. We give a summary of the results and methods that we discussed in this thesis. We also discuss some limitations of our

approach.

Chapter 2

Preliminaries

2.1 Hermite polynomials

Let $\phi(x)$ be the density function of the standard normal distribution $N(0, 1)$. Throughout this paper, Hermite polynomials are defined as

$$H_n(x) = (-1)^n \phi(x)^{-1} D^n \phi(x) \text{ with } H_0(x) \equiv 1$$

where

$$n \in \mathbb{N}, D = \frac{d}{dx} \text{ and } \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

The proof of the following lemma is elementary, but not entirely obvious.

Lemma 2.1.1. *We have the following formula:*

$$\int_x^\infty \phi(y) H_n(y) dy = x \phi(x) H_{n-1}(x) + \phi(x) H_{n-2}(x).$$

Proof. Note that $D^n \phi(x) = (-1)^n H_n(x) \phi(x)$. By using integration by parts, we have

$$D((D^{n-1} \phi(x))x) = [D^n(\phi(x))]x + D^{n-1} \phi(x) = (-1)^n x H_n(x) \phi(x) + D^{n-1} \phi(x).$$

Therefore,

$$-(D^{n-1} \phi(x))x = \int_x^\infty (-1)^n y H_n(y) \phi(y) dy - D^{n-2} \phi(x).$$

Hence,

$$\int_x^\infty (-1)^n y H_n(y) \phi(y) dy = -(D^{n-1} \phi(x))x + D^{n-2} \phi(x)$$

$$\text{i.e. } \int_x^\infty (-1)^n y H_n(y) \phi(y) dy = (-1) \cdot (-1)^{n-1} x \phi(x) H_{n-1}(x) + (-1)^{n-2} \phi(x) H_{n-1}(x).$$

□

We now give the Gram-Charlier expansion of a density function and show how to use it to calculate the distribution function and the truncated expectation. The primary reference is [8].

Proposition 2.1.2. *Let Y be a random variable with a continuous density function $f : \mathbb{R} \rightarrow \mathbb{R}$ and finite cumulants $(c_k)_{k \in \mathbb{N}}$. Then the following hold:*

(a) *f is given by the following expansion:*

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right)$$

where $q_0 = 1$, $q_1 = q_2 = 0$,

$$q_n = \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 3, \\ k_1 + \dots + k_m = n}} \frac{c_{k_1} \dots c_{k_m}}{m! k_1! \dots k_m! \sqrt{c_2}^n}, \quad n \geq 3.$$

(b) *for any $a \in \mathbb{R}$,*

$$P(Y > a) = N \left(\frac{c_1 - a}{\sqrt{c_2}} \right) + \sum_{k=3}^{\infty} (-1)^{k-1} q_k H_{k-1} \left(\frac{c_1 - a}{\sqrt{c_2}} \right) \phi \left(\frac{c_1 - a}{\sqrt{c_2}} \right)$$

(c) *for any $a \in \mathbb{R}$,*

$$\begin{aligned} & \mathbb{E}[Y I(Y > a)] \\ &= \sqrt{c_2} \phi \left(\frac{c_1 - a}{\sqrt{c_2}} \right) + c_1 N \left(\frac{c_1 - a}{\sqrt{c_2}} \right) \\ &+ \sum_{n=3}^{\infty} (-1)^{n-1} q_n \phi \left(\frac{c_1 - a}{\sqrt{c_2}} \right) \left[a H_{n-1} \left(\frac{c_1 - a}{\sqrt{c_2}} \right) - \sqrt{c_2} H_{n-2} \left(\frac{c_1 - a}{\sqrt{c_2}} \right) \right]. \end{aligned}$$

Proof. (a)

$$\begin{aligned}
G_Y(t) &:= \mathbb{E}(e^{itY}) \\
&= \int_{-\infty}^{\infty} e^{itx} f(x) dx \\
&= \int_{-\infty}^{\infty} e^{it(c_1 + \sqrt{c_2}x)} f(c_1 + \sqrt{c_2}x) d(c_1 + \sqrt{c_2}x) \\
&= e^{itc_1} \int_{-\infty}^{\infty} e^{i\sqrt{c_2}x} \sqrt{c_2} f(c_1 + \sqrt{c_2}x) dx.
\end{aligned}$$

Since

$$G_Y(t) = e^{\ln G_Y(t)} = e^{\sum_{k=0}^{\infty} \left[\frac{d^k}{dt^k} (\ln G_Y(t)) \right]_{t=0} \frac{t^k}{k!}} = e^{\sum_{k=1}^{\infty} \frac{c_k t^k}{k!}}$$

we have

$$\begin{aligned}
&G_Y(t) \\
&= e^{\sum_{k=1}^{\infty} \frac{c_k (it)^k}{k!}} \\
&= e^{ic_1 t} e^{-\frac{c_2 t^2}{2} + \sum_{k=3}^{\infty} \frac{c_k (it)^k}{k!}} \\
&= e^{ic_1 t} e^{-\frac{c_2 t^2}{2} + \sum_{k=3}^{\infty} \frac{c_k (-1)^k}{k! \sqrt{c_2}^k} (-i\sqrt{c_2} t)^k} \\
&= e^{ic_1 t} \int_{-\infty}^{\infty} e^{i\sqrt{c_2} t x} \left[e^{\sum_{k=3}^{\infty} \frac{c_k (-1)^k}{k! \sqrt{c_2}^k} D^k} \right] (\phi(x)) dx.
\end{aligned}$$

Then

$$\begin{aligned}
& \left[e^{\sum_{k=3}^{\infty} \frac{c_k (-1)^k}{k! \sqrt{c_2}^k} D^k} \right] (\phi(x)) \\
&= \left\{ 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left[\sum_{k=3}^{\infty} \frac{c_k (-1)^k}{k! \sqrt{c_2}^k} D^k \right]^m \right\} (\phi(x)) \\
&= \left\{ 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left[\sum_{k_1, \dots, k_m \geq 3} \frac{c_{k_1} \dots c_{k_m} (-1)^{k_1 + \dots + k_m}}{k_1! \dots k_m! \sqrt{c_2}^{k_1 + \dots + k_m}} D^{k_1 + \dots + k_m} \right] \right\} (\phi(x)) \\
&= \left\{ 1 + \left[\sum_{m=1}^{\infty} \sum_{k_1, \dots, k_m \geq 3} \frac{c_{k_1} \dots c_{k_m} (-1)^{k_1 + \dots + k_m}}{m! k_1! \dots k_m! \sqrt{c_2}^{k_1 + \dots + k_m}} D^{k_1 + \dots + k_m} \right] \right\} (\phi(x)) \\
&= \left[1 + \sum_{n=3}^{\infty} \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 3, \\ k_1 + \dots + k_m = n}} \frac{c_{k_1} \dots c_{k_m}}{m! k_1! \dots k_m! \sqrt{c_2}^n} H_n(x) \right] \phi(x)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& G_Y(t) \\
&= e^{ic_1 t} \int_{-\infty}^{\infty} e^{i\sqrt{c_2} t x} \phi(x) dx \\
&+ e^{ic_1 t} \int_{-\infty}^{\infty} e^{i\sqrt{c_2} t x} \left[\sum_{n=3}^{\infty} \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 3, \\ k_1 + \dots + k_m = n}} \frac{c_{k_1} \dots c_{k_m}}{m! k_1! \dots k_m! \sqrt{c_2}^n} H_n(x) \phi(x) \right] dx
\end{aligned}$$

The rest follows from the inverse Fourier transform and is straightforward.

(b).

$$\begin{aligned}
& \mathbb{E}[I(Y \leq a)] \\
&= \int_{-\infty}^a \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right) dx \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^a \frac{q_n}{\sqrt{c_2}} H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right) dx \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{\frac{a-c_1}{\sqrt{c_2}}} q_n H_n(y) \phi(y) dy \\
&= \int_{-\infty}^{\frac{a-c_1}{\sqrt{c_2}}} \phi(y) dy + \sum_{n=3}^{\infty} q_n \int_{-\infty}^{\frac{a-c_1}{\sqrt{c_2}}} H_n(y) \phi(y) dy \\
&= N \left(\frac{a - c_1}{\sqrt{c_2}} \right) + \sum_{n=3}^{\infty} q_n \int_{-\infty}^{\frac{a-c_1}{\sqrt{c_2}}} (-1)^n D^n \phi(y) dy \\
&= N \left(\frac{a - c_1}{\sqrt{c_2}} \right) + \sum_{n=3}^{\infty} q_n (-1)^n D^{n-1} \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right) \\
&= N \left(\frac{a - c_1}{\sqrt{c_2}} \right) + \sum_{n=3}^{\infty} q_n (-1)^n \cdot (-1)^{n-1} H_{n-1} \left(\frac{a - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right) \\
&= N \left(\frac{a - c_1}{\sqrt{c_2}} \right) - \sum_{n=3}^{\infty} q_n H_{n-1} \left(\frac{a - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \mathbb{E}[I(Y > a)] \\
&= 1 - \left[N \left(\frac{a - c_1}{\sqrt{c_2}} \right) - \sum_{n=3}^{\infty} q_n H_{n-1} \left(\frac{a - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right) \right] \\
&= N \left(\frac{c_1 - a}{\sqrt{c_2}} \right) + \sum_{n=3}^{\infty} q_n H_{n-1} \left(\frac{a - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right) \\
&= N \left(\frac{c_1 - a}{\sqrt{c_2}} \right) + \sum_{n=3}^{\infty} (-1)^{n-1} q_n H_{n-1} \left(\frac{c_1 - a}{\sqrt{c_2}} \right) \phi \left(\frac{c_1 - a}{\sqrt{c_2}} \right).
\end{aligned}$$

(c).

$$\begin{aligned}
& \mathbb{E}[YI(Y \leq a)] \\
&= \int_a^\infty \sum_{n=0}^\infty \frac{q_n}{\sqrt{c_2}} x H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right) dx \\
&= \sum_{n=0}^\infty \int_a^\infty \frac{q_n}{\sqrt{c_2}} x H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right) dx \\
&= \sum_{n=0}^\infty q_n \left[\int_a^\infty \left(\frac{x - c_1}{\sqrt{c_2}} \right) H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right) dx \right. \\
&\quad \left. + \frac{c_1}{\sqrt{c_2}} \int_a^\infty H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right) dx \right] \\
&= \sum_{n=0}^\infty q_n \sqrt{c_2} \left[\int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty y H_n(y) \phi(y) dy + \frac{c_1}{\sqrt{c_2}} \int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty H_n(y) \phi(y) dy \right] \\
&= \sqrt{c_2} \left[\int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty y \phi(y) dy + \frac{c_1}{\sqrt{c_2}} \int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty \phi(y) dy \right] \\
&\quad + \sum_{n=3}^\infty q_n \sqrt{c_2} \left[\int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty y H_n(y) \phi(y) dy + \frac{c_1}{\sqrt{c_2}} \int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty H_n(y) \phi(y) dy \right] \\
&= \sqrt{c_2} \int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty y \phi(y) dy + c_1 N \left(\frac{c_1 - a}{\sqrt{c_2}} \right) \\
&\quad + \sum_{n=3}^\infty q_n \sqrt{c_2} \left[\frac{a - c_1}{\sqrt{c_2}} H_{n-1} \left(\frac{a - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right) + H_{n-2} \left(\frac{a - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right) \right] \\
&\quad + \sum_{n=3}^\infty q_n \sqrt{c_2} \left[\frac{c_1}{\sqrt{c_2}} H_{n-1} \left(\frac{a - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right) \right] \\
&= \sqrt{c_2} \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right) + c_1 N \left(\frac{c_1 - a}{\sqrt{c_2}} \right) \\
&\quad + \sum_{n=3}^\infty q_n \sqrt{c_2} \left[\frac{a}{\sqrt{c_2}} H_{n-1} \left(\frac{a - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right) + H_{n-2} \left(\frac{a - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{a - c_1}{\sqrt{c_2}} \right) \right]
\end{aligned}$$

The second last equality follows from Lemma 2.1.1 □

Remark 2.1.3. In principle, we are able to develop a general formula for $\mathbb{E}[Y^n I(Y > a)]$ for any natural number n .

2.2 An important system of Riccati equations

In this section, we study a system of Riccati equations which is useful in section 3.1.

Consider the following Riccati equation:

$$\frac{dy}{dx} = 1 + ky - \frac{\sigma^2 y}{2}, \quad y(T) = y_0. \quad (2.1)$$

Consider an auxiliary equation

$$\lambda^2 - \frac{2k}{\sigma^2}\lambda - \frac{2}{\sigma^2} = 0$$

The roots of this quadratic equation are given by

$$\lambda_+ = \frac{k + \gamma}{\sigma^2}, \lambda_- = \frac{k - \gamma}{\sigma^2} \text{ where } \gamma = \sqrt{k^2 + 2\sigma^2}.$$

$$\begin{aligned} \frac{dy}{dx} = 1 + ky - \frac{\sigma^2 y}{2} &\Rightarrow \int \frac{dy}{1 + ky - \frac{\sigma^2 y^2}{2}} = \int dt \\ &\Rightarrow \int \frac{dy}{y^2 - \frac{2k}{\sigma^2}y - \frac{2}{\sigma^2}} = -\frac{\sigma^2 t}{2} + C \\ &\Rightarrow \frac{1}{\lambda_+ - \lambda_-} \int \frac{1}{y - \lambda_+} - \frac{1}{y - \lambda_-} dy = -\frac{\sigma^2 t}{2} + C \\ &\Rightarrow \frac{\sigma^2}{2\gamma} \ln \frac{y - \lambda_+}{y - \lambda_-} = -\frac{\sigma^2 t}{2} + C \\ &\Rightarrow \frac{y - \lambda_+}{y - \lambda_-} = D e^{-\gamma t}. \end{aligned}$$

By using the terminal condition, we have

$$D = e^{\gamma T} \frac{y_0 - \lambda_+}{y_0 - \lambda_-}.$$

Therefore,

$$\frac{y - \lambda_+}{y - \lambda_-} = \frac{y_0 - \lambda_+}{y_0 - \lambda_-} e^{\gamma(T-t)}.$$

Let $y_0^* = \frac{y_0 - \lambda_+}{y_0 - \lambda_-}$. Then the solution of the differential equation (2.1) is given by

$$y = \lambda_+ + (\lambda_+ - \lambda_-) \frac{y_0^* e^{\gamma(T-t)}}{1 - y_0^* e^{\gamma(T-t)}}.$$

or

$$y = \frac{1}{\sigma^2} \left[K + \gamma + \frac{2\gamma y_0^* e^{\gamma(T-t)}}{1 - y_0^* e^{\gamma(T-t)}} \right].$$

Next, suppose that we are given the following system of Riccati equations

$$\frac{dx}{dt} = \delta_0 - K_1 \theta_1 y - K_2 \theta_2 z, \quad (2.2)$$

$$\frac{dy}{dt} = 1 + K_1 y - \frac{1}{2} \sigma_1^2 y^2, \quad (2.3)$$

$$\frac{dz}{dt} = 1 + K_2 z - \frac{1}{2} \sigma_2^2 z^2, \quad (2.4)$$

$$x(T) = x_0, y(T) = y_0, z(T) = z_0. \quad (2.5)$$

By the above, we have

$$y = \frac{1}{\sigma_1^2} \left[K_1 + \gamma_1 + \frac{2\gamma_1 y_0^* e^{\gamma_1(T-t)}}{1 - y_0^* e^{\gamma_1(T-t)}} \right].$$

$$z = \frac{1}{\sigma_2^2} \left[K_2 + \gamma_2 + \frac{2\gamma_2 z_0^* e^{\gamma_2(T-t)}}{1 - z_0^* e^{\gamma_2(T-t)}} \right].$$

where $\gamma_j = \sqrt{K_j^2 + \sigma_j^2}$, $j = 1, 2$.

Now,

$$\begin{aligned} x = \delta_0 t - \frac{K_1 \theta_1}{\sigma_1^2} [(K_1 + \gamma_1)t + 2 \ln |1 - y_0^* e^{\gamma_1(T-t)}|] \\ - \frac{K_2 \theta_2}{\sigma_2^2} [(K_2 + \gamma_2)t + 2 \ln |1 - z_0^* e^{\gamma_2(T-t)}|] + C \end{aligned}$$

Therefore,

$$\begin{aligned} C = x_0(T) - \delta_0 T + \frac{K_1 \theta_1}{\sigma_1^2} [(K_1 + \gamma_1)T + 2 \ln |1 - y_0^*|] \\ + \frac{K_2 \theta_2}{\sigma_2^2} [(K_2 + \gamma_2)T + 2 \ln |1 - z_0^*|] \end{aligned}$$

Hence, we have

$$x = x_0 - \delta_0(T - t) - \frac{K_1\theta_1}{\sigma_1^2} \left[(K_1 + \gamma_1)(T - t) + 2 \ln \left| \frac{1 - y_0^* e^{\gamma_1(T-t)}}{1 - y_0^*} \right| \right] \\ - \frac{K_2\theta_2}{\sigma_2^2} \left[(K_2 + \gamma_2)(T - t) + 2 \ln \left| \frac{1 - z_0^* e^{\gamma_2(T-t)}}{1 - z_0^*} \right| \right].$$

Chapter 3

Introduction to Swaptions

3.1 Change of Numeraire - Forward Measures

The primary reference of this section is [2, Chapter 10, 26].

Assumptions:

- The market model consists of asset prices S_0, \dots, S_n , where S_0 is assumed to be strictly positive.
- Under the real-world measure, the S -dynamics are of the following form

$$dS_i = S_i(t)\alpha_i(t)dt + S_i(t)\sigma_i(t)d\bar{W}(t)$$

where α_i, σ_i are adapted processes and \bar{W} is a standard Brownian motion.

Lemma 3.1.1. *Let β be a strictly positive Ito's process and let $Z = \frac{S}{\beta}$. Then a portfolio h is S -self-financing if and only if it is Z -self-financing,*

$$i.e. dV^S(t, h) = h(t) \cdot dS(t) \text{ if and only if } dV^Z(t, h) = h(t) \cdot dS(t)$$

where $V^S(t, h) = h(t) \cdot S(t)$ and $V^Z(t, h) = h(t) \cdot Z(t)$.

Proof.

$$\begin{aligned}
dV^Z(t, h) &= d \left[\frac{V^S(t, h)}{\beta(t)} \right] \\
&= \frac{dV^S(t, h)}{\beta(t)} + V^S(t, h) d \left(\frac{1}{\beta(t)} \right) + dV^S(t, h) \cdot d \left(\frac{1}{\beta(t)} \right) \\
&= \frac{h(t)dS(t)}{\beta(t)} + h(t)S(t)d \left(\frac{1}{\beta(t)} \right) + h(t)dS(t, h) \cdot d \left(\frac{1}{\beta(t)} \right) \\
&= h(t) \left[\frac{dS(t)}{\beta(t)} + S(t)d \left(\frac{1}{\beta(t)} \right) + dS(t, h) \cdot d \left(\frac{1}{\beta(t)} \right) \right] \\
&= h(t)d \left[\frac{S(t, h)}{\beta(t)} \right] \\
&= h(t)dZ(t).
\end{aligned}$$

As a result, the model is S -arbitrage-free if and only if it is Z -arbitrage-free. □

Now, let us recall the Fundamental Theorems of Asset Pricing:

Theorem 3.1.2. *Under the assumption, the following hold:*

- (a) *The market model is free of arbitrage if and only if there exists a probability measure $Q^0 \sim P$ such that*

$$\left(\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, \dots, \frac{S_n(t)}{S_n(t)} \right)$$

are Q_0 -martingales.

- (b) *If the market is arbitrage-free, then any sufficiently integrable T -claim must be priced according to the formula*

$$\Pi(t; X) = S_0(t) \mathbb{E}^{Q^0} \left[\frac{X}{S_0(t)} \middle| \mathcal{F}_t \right]$$

where \mathbb{E}^{Q^0} denotes expectation under Q^0 .

Let S_0 and S_1 be strictly positive assets in an arbitrage-free market. Then there exist probability measures Q^0 and Q^1 , such that for any choice of sufficiently integrable T -claim,

$$\Pi(0; X) = S_0(0) \mathbb{E}^{Q^0} \left[\frac{X}{S_0(T)} \right] = S_1(0) \mathbb{E}^{Q^1} \left[\frac{X}{S_1(T)} \right].$$

Denote by $L_0^1(T)$ the Radon-Nikodym derivative

$$L_0^1(T) = \frac{dQ^1}{dQ^0} \text{ on } \mathcal{F}_T.$$

Then, we have

$$\Pi(0; X) = S_1(0) \mathbb{E}^{Q^0} \left[\frac{X}{S_1(T)} \cdot L_0^1(T) \right].$$

It follows that

$$S_0(0) \mathbb{E}^{Q^0} \left[\frac{X}{S_0(T)} \right] = S_1(0) \mathbb{E}^{Q^0} \left[\frac{X}{S_1(T)} \cdot L_0^1(T) \right].$$

Therefore,

$$\frac{S_0(0)}{S_0(T)} = \frac{S_1(0)}{S_1(T)} \cdot L_0^1(T).$$

As a result, we have

$$L_0^1(T) = \frac{S_0(0)}{S_1(0)} \frac{S_1(T)}{S_0(T)}.$$

Proposition 3.1.3. *Assume that Q^0 is a martingale measure for the numeraire S_0 (on \mathcal{F}_T) and assume that S_1 is a positive asset price process, such that $\frac{S_1}{S_0}$ is a Q^0 -martingale. Define Q^1 on \mathcal{F}_t by the likelihood process*

$$L_0^1(t) = \frac{S_0(0)}{S_1(0)} \frac{S_1(t)}{S_0(t)}, \quad 0 \leq t \leq T.$$

Then Q^1 is a martingale measure for S_1 .

Proof. If Π is an arbitrage-free price process, then $\frac{\Pi}{S_0}$ is also an arbitrage-free price process. Hence,

$$\begin{aligned} \mathbb{E}^{Q^1} \left[\frac{\Pi(t)}{S_1(t)} \middle| \mathcal{F}_s \right] &= \frac{\mathbb{E}^{Q^0} \left[\frac{\Pi(t)}{S_1(t)} \cdot L_0^1(t) \middle| \mathcal{F}_s \right]}{L_0^1(s)} \\ &= \frac{\mathbb{E}^{Q^0} \left[\frac{\Pi(t)}{S_1(t)} \cdot \frac{S_0(0)}{S_1(0)} \frac{S_1(t)}{S_0(t)} \middle| \mathcal{F}_s \right]}{L_0^1(s)} \\ &= \frac{\frac{S_0(0)}{S_1(0)} \cdot \mathbb{E}^{Q^0} \left[\frac{\Pi(t)}{S_0(t)} \middle| \mathcal{F}_s \right]}{L_0^1(s)} \\ &= \frac{\frac{S_0(0)}{S_1(0)} \cdot \frac{\Pi(s)}{S_0(s)}}{L_0^1(s)} = \frac{\Pi(s)}{S_1(s)} \end{aligned}$$

□

We are now ready to define the notion of forward measures.

Definition 3.1.4. Let (r_t) be the short rate process. The money account process is denoted by $B(t) := e^{\int_0^t r_s ds}$. The *risk-neutral measure* Q is defined as the martingale measure for the numeraire process $B(t)$.

Definition 3.1.5. Suppose that we are given a bond market model with a fixed martingale measure Q . Let $P(t, T) := \mathbb{E}^Q[e^{-\int_t^T r_s ds}]$ be the price process of a zero coupon bond issued at time t and maturing at time T . For a fixed T , the *T -forward measure* Q^T is defined as the martingale measure for the numeraire process $P(t, T)$.

Proposition 3.1.6. For any T -claim X , we have

$$\Pi(t; X) = \mathbb{E}^Q[e^{-\int_t^T r_s ds} X | \mathcal{F}_t]$$

where \mathbb{E}^Q denotes expectation under Q .

Proof. By the First Fundamental Theorem and the definition of Q , we have

$$\frac{\Pi(t; X)}{B(t)} = \mathbb{E}^Q \left[\frac{X}{B(T)} \middle| \mathcal{F}_t \right].$$

□

Proposition 3.1.7. For any T -claim X , we have

$$\Pi(t; X) = P(t, T) \mathbb{E}^T[X | \mathcal{F}_t]$$

where \mathbb{E}^T denotes expectation under Q^T .

Proof. By the First Fundamental Theorem of Asset Pricing and the definition of Q^T , we have

$$\Pi(t; X) = P(t, T) \mathbb{E}^T \left[\frac{X}{P(T, T)} \middle| \mathcal{F}_t \right].$$

□

Lemma 3.1.8. *Let Q be the risk-neutral measure. The short r is deterministic if and only if $Q = Q^T$ for any $T > 0$.*

Proof. If $Q = Q^T$, then the Radon-Nikodym derivative

$$\frac{dQ^T}{dQ} = \frac{B(0)P(T, T)}{B(T)P(0, T)} = 1.$$

It is easy to see that $B(T) = P(0, T)^{-1}$ is deterministic. Conversely, if r is deterministic, then

$$P(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T r(s)ds} \right] = e^{-\int_t^T r(s)ds} = \frac{B(t)}{B(T)}.$$

Hence, it follows that $\frac{dQ^T}{dQ} \equiv 1$. □

Let $f(t, T) := -\frac{\partial}{\partial T} \ln P(t, T)$ be the forward rate for the time interval $[t, T]$. The following result tells us that the forward measure is the measure that makes the present forward rate an unbiased estimator of the future short rate.

Lemma 3.1.9. *Assume that, for any $T > 0$, $\frac{r(T)}{B(T)}$ is integrable. Then for all fixed T , $f(t, T)$ is a Q^T -martingale, and we have*

$$f(t, T) = \mathbb{E}^T[r(T)|\mathcal{F}_t].$$

Proof. Let $X = r(T)$. Note that

$$\Pi(t, X) = \mathbb{E}^Q \left[r(T) e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] = P(t, T) \mathbb{E}^T [r(T)|\mathcal{F}_t].$$

It follows that

$$\begin{aligned} \mathbb{E}^T [r(T)|\mathcal{F}_t] &= \frac{1}{p(t, T)} \mathbb{E}^Q \left[r(T) e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] \\ &= -\frac{1}{P(t, T)} \mathbb{E}^Q \left[\frac{\partial}{\partial T} e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] \\ &= -\frac{1}{P(t, T)} \frac{\partial}{\partial T} \mathbb{E}^Q \left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right] \\ &= -\frac{1}{P(t, T)} \frac{\partial}{\partial T} P(t, T) \\ &= -\frac{\partial}{\partial T} \ln P(t, T) \\ &= f(t, T). \end{aligned}$$

□

3.2 Interest rate swaps and swaptions

The interest rate swap is one of the simplest interest rate derivatives. This is basically a scheme where we exchange a payment stream at a fixed interest rate, known as swap rate, for a payment stream at a floating rate (LIBOR rate $L(T_{i-1}, T_i)$). Typically, an interest rate swap is a forward swap settled in arrears, which will be defined clearly as follows:

Let N be the nominal principal and R be the swap rate. By assumption, we have a number of equally spaced dates T_0, T_1, \dots, T_n and payments occur at T_1, \dots, T_n . Let $\delta = T_i - T_{i-1}$, $i = 1, 2, \dots, n$. At time T_i , the swap receiver (or, fixed rate receiver) will receive

$$N\delta \cdot R \text{ (on the fixed rate leg)}$$

and will pay

$$N\delta \cdot L(T_{i-1}, T_i) \text{ (on the floating rate leg).}$$

Hence, the net cash inflow is

$$N\delta \cdot [R - L(T_{i-1}, T_i)].$$

Therefore, at time T , the no-arbitrage price of the $T_0 \times (T_n - T_0)$ receiver's swap is the present value of the cash flow, which is given by

$$\begin{aligned} SV(t; T_0, T_n) &= N \sum_{i=1}^n [\delta R - \delta L(T_{i-1}, T_i)] \times P(t, T_i) \\ &= N \left[\delta R \sum_{i=1}^n P(t, T_i) - \sum_{i=1}^n \delta \times \frac{P(t, T_{i-1}) - P(t, T_i)}{\delta P(t, T_i)} \times P(t, T_i) \right] \\ &= N \left[\delta R \sum_{i=1}^n P(t, T_i) - \sum_{i=1}^n [P(t, T_{i-1}) - P(t, T_i)] \right] \\ &= N \left[-P(t, T_0) + \delta R \sum_{i=1}^n P(t, T_i) + P(t, T_n) \right]. \end{aligned}$$

We write $SV(T_0, T_n) = SV(T_0; T_0, T_n)$.

By definition, the swap rate R is chosen, so that the value of the swap equals zero at the time when the contract is made.

Proposition 3.2.1. *The forward (par) swap rate $R(t; T_0, T_n)$ is given by*

$$R(t; T_0, T_n) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^n P(t, T_i)}.$$

Definition 3.2.2. A $T_0 \times (T_N - T_0)$ receiver swaption with swaption strike K is a contract which at the expiry date T_0 , gives the holder the right but not the obligation to enter into a swap with the fixed swap rates K and payment dates T_1, \dots, T_N . We will call T_0 the *swaption expiry* and $T_N - T_0$ the *tenor* of the swaption.

At time T_0 , the payoff of the swaption is given by $\mathbb{1}_{SV(T_0, T_n) > 0} SV(T_0, T_n)$. As a result, the no-arbitrage price of the swaption $SOV(t)$ at time t is given by

$$SOV(t; T_0, T_n) = P(t, T_0) \mathbb{E}^{T_0}[\mathbb{1}_{SV(T_0, T_n) > 0} SV(T_0, T_n) | \mathcal{F}_t]$$

where the expectation is taken under the T_0 -forward measure.

For more details, the reader is referred to [2, Chapter 27].

3.3 Implied Black's Volatilities for Swaptions

This section is based on part of the material in [2, Chapter 27].

Definition 3.3.1. let $S(t; T_0, T_n)$ be the following process:

$$S(t; T_0, T_n) = \sum_{i=1}^n \delta P(t, T_i).$$

It is referred to as the accrual factor.

The forward swap rate can be expressed by:

$$R(t; T_0, T_n) = \frac{P(t, T_0) - P(t, T_n)}{S(t; T_0, T_n)}.$$

Suppose that the swap rate is K and the nominal principal N is 1. The price of the $T_0 \times (T_N - T_0)$ receiver's swap can be expressed by

$$SV(t; T_0, T_n) = S(t; T_0, T_n)[K - R(t; T_0, T_n)].$$

Therefore, the swaption can be regarded as a put option on $R(t; T_0, T_n)$ with strike price K when expressed in the numeraire $S(t; T_0, T_n)$. The market convention is to compute swaption prices by using the Black-76 formula and to quote prices in terms of the implied Black volatilities.

Definition 3.3.2. The Black-76 formula for a $T_0 \times (T_N - T_0)$ receiver swaption with swaption strike K is defined as

$$SOV(t; T_0, T_n) = S(t; T_0, T_n)[KN(-d_2) - R(t; T_0, T_n)N(-d_1)],$$

where

$$d_1 = \frac{1}{\sigma(T_0, T_n)\sqrt{T_0 - t}} \left[\ln\left(\frac{R(t, T_0, T_n)}{K}\right) + \frac{1}{2}\sigma(T_0, T_n)^2(T_0 - t) \right],$$

$$d_2 = d_1 - \sigma(T_0, T_n)\sqrt{T_0 - t}.$$

The constant $\sigma(T_0, T_n)$ is known as the Black's volatility. Given a market price for the swaption, the Black volatility implied by the Black formula is referred to as the implied Black volatility.

Chapter 4

Pricing swaptions using Gram-Charlier expansions

4.1 Introduction to CIR2 model

The term structure as well as the prices of any interest rate derivatives are completely determined by the short rate dynamics under the risk-neutral measure Q which is assumed to be known. The procedure of specifying the Q -dynamics is called *martingale modeling*.

Definition 4.1.1. If the term structure $\{P(t, T) : 0 \leq t \leq T, T > 0\}$ has the form

$$P(t, T) = F(t, r(t), T),$$

where F has the form

$$F(t, r(t), T) = e^{A(t, T) + B(t, T) \cdot X(t)},$$

and A and B are deterministic functions, then the model is said to possess an affine term structure.

The typical assumption is that r under the Q -measure has dynamics given by

$$r(t) = \delta_0 + \delta_X \cdot X(t)$$

where δ_0 is a constant, δ_X is a vector and $X(t)$ satisfies the following system of SDE:

$$dX(t) = K(\theta - X(t))dt + \Sigma D(X(t))dW(t),$$

$$\alpha_i, \beta_i \in \mathbb{R}, \theta \in \mathbb{R}^n, K \in M_{n \times n}(\mathbb{R}),$$

$$\Sigma \in M_{n \times n}(\mathbb{R}) \text{ such that } \Sigma \Sigma^T \text{ is positive definite,}$$

and

$$D(x) = \text{diag}[\sqrt{\alpha_1 + \beta_1 \cdot x}, \dots, \sqrt{\alpha_n + \beta_n \cdot x}], x \in \mathbb{R}^n.$$

Proposition 4.1.2. *The model of the form assumed above has an affine term structure.*

There are many choices of short rate models. One of the most popular choices is Cox-Ingersoll-Ross(1985) (CIR) model, namely

$$dr = a(b - r) + \sigma\sqrt{r}dW.$$

See [9] for more details. This model ensures mean reversion of the interest rate towards the long run value b , with a speed of adjustment governed by the strictly positive parameter a . One can show that r is always non-negative in this model. Also, r is strictly positive whenever $2ab \leq \sigma^2$.

In order to capture a more complicated shape of yield curves, it is suggested to use two-factor CIR (CIR2) model (See [6, Chapter 4]). The short rate of this model is given by

$$r(t) = X_1(t) + X_2(t) + \delta_0. \tag{4.1}$$

where the Q-dynamics of $X(t) = (X_1(t), X_2(t))$ are given by the following SDEs:

$$dX_1(t) = K_1(\theta_1 - X_1(t))dt + \sigma_1\sqrt{X_1(t)}dW_1(t);$$

$$dX_2(t) = K_2(\theta_2 - X_2(t))dt + \sigma_2\sqrt{X_2(t)}dW_2(t)$$

and the initial conditions

$$X(0) = (X_1(0), X_2(0)) \text{ are given,}$$

where W_1 and W_2 are two independent standard Q -Brownian motions.

Lemma 4.1.3. *Suppose that the discounted price of a derivative is a Q -martingale and admits the affine structure, namely,*

$$V(t, T) = e^{A(t, T) + B(t, T) \cdot X(t)}$$

where A and B are deterministic. Then A and B satisfy the following system of ODEs:

$$\frac{\partial A}{\partial t} = \delta_0 - B_1 K_1 \theta_1 - B_2 K_2 \theta_2, \quad (4.2)$$

$$\frac{\partial B_1}{\partial t} = 1 + B_1 K_1 - \frac{1}{2} B_1^2 \sigma_1^2, \quad (4.3)$$

$$\frac{\partial B_2}{\partial t} = 1 + B_2 K_2 - \frac{1}{2} B_2^2 \sigma_2^2, \quad (4.4)$$

Proof. Assume an affine structure for V , namely

$$V(t, T) = F(t, X_1(t), X_2(t)) = e^{A(t, T) + B_1(t, T) X_1(t, T) + B_2(t, T) X_2(t, T)}.$$

Then

$$F_t = (A_t + B_{1,t} X_1 + B_{2,t} X_2) F, \quad (4.5)$$

$$F_{X_1} = B_1 F, \quad (4.6)$$

$$F_{X_2} = B_2 F, \quad (4.7)$$

$$F_{X_1 X_1} = B_1^2 F, \quad (4.8)$$

$$F_{X_1 X_2} = B_1 B_2 F, \quad (4.9)$$

$$F_{X_2 X_2} = B_2^2 F, \quad (4.10)$$

Let $D(t) = e^{-\int_t^T r(s)ds}$ be the discount factor. Then we have

$$\begin{aligned}
& d(DF) \\
&= -rDFdt + DdF \\
&= -rDFdt + D[F_t dt + \\
&\quad F_{X_1} dX_1 + F_{X_2} dX_2 + \frac{1}{2}F_{X_1 X_1} dX_1 dX_1 + F_{X_1 X_2} dX_1 dX_2 + \frac{1}{2}F_{X_2 X_2} dX_2 dX_2] \\
&= D[-rFdt + F_t dt + F_{X_1}(K_1(\theta_1 - X_1)dt + \sigma_1\sqrt{X_1}dW_1) + \\
&\quad F_{X_2}(K_2(\theta_2 - X_2)dt + \sigma_2\sqrt{X_2}dW_2) + \frac{1}{2}F_{X_1 X_1}\sigma_1^2 X_1 dt + \frac{1}{2}F_{X_2 X_2}\sigma_2^2 X_2 dt] \\
&= D\{[-rF + F_t + F_{X_1}K_1(\theta_1 - X_1) + F_{X_2}K_2(\theta_2 - X_2) + \\
&\quad \frac{1}{2}F_{X_1 X_1}\sigma_1^2 X_1 + \frac{1}{2}F_{X_2 X_2}\sigma_2^2 X_2]dt + \sigma_1\sqrt{X_1}F_{X_1}dW_1 + \sigma_2\sqrt{X_2}F_{X_2}dW_2\}
\end{aligned}$$

Since DF is a Q -martingale, the drift-term is equal to 0. Thus, we have

$$-rF + F_t + F_{X_1}K_1(\theta_1 - X_1) + F_{X_2}K_2(\theta_2 - X_2) + \frac{1}{2}F_{X_1 X_1}\sigma_1^2 X_1 + \frac{1}{2}F_{X_2 X_2}\sigma_2^2 X_2 = 0$$

Therefore,

$$\begin{aligned}
& -(X_1 + X_2 + \delta_0) + (A_t + B_{1,t}X_1 + B_{2,t}X_2) + B_1K_1(\theta_1 - X_1) \\
& \quad + B_2K_2(\theta_2 - X_2) + \frac{1}{2}B_1^2\sigma_1^2 X_1 + \frac{1}{2}B_2^2\sigma_2^2 X_2 = 0
\end{aligned}$$

It follows that

$$\begin{aligned}
& (-\delta_0 + A_t + B_1K_1\theta_1 + B_2K_2\theta_2) + (-1 + B_{1,t} - B_1K_1 + \frac{1}{2}B_1^2\sigma_1^2)X_1 \\
& \quad + (-1 + B_{2,t} - B_2K_2 + \frac{1}{2}B_2^2\sigma_2^2)X_2 = 0.
\end{aligned} \tag{4.11}$$

Since Equation (4.11) holds for any real numbers X_1 and X_2 , we obtain the following system:

$$\frac{\partial A}{\partial t} = \delta_0 - B_1K_1\theta_1 - B_2K_2\theta_2, \tag{4.12}$$

$$\frac{\partial B_1}{\partial t} = 1 + B_1K_1 - \frac{1}{2}B_1^2\sigma_1^2, \tag{4.13}$$

$$\frac{\partial B_2}{\partial t} = 1 + B_2K_2 - \frac{1}{2}B_2^2\sigma_2^2, \tag{4.14}$$

□

Theorem 4.1.4. Assume that the short rate process r follows the CIR2 model given in (4.1). The price at time t of a zero-coupon bond maturing at time T with unit face value is given by

$$P^{CIR}(t, T) = e^{A(t, T) + B(t, T) \cdot X(t)}$$

where

$$\gamma_j = \sqrt{K_j^2 + 2\sigma_j^2}, j = 1, 2$$

$$A(t, T) = -\delta_0(T - t) - \sum_{j=1}^2 K_j \theta_j \left[\frac{2}{\sigma_j^2} \ln \frac{(K_j + \gamma_j)(e^{\gamma_j(T-t)} - 1)}{2\gamma_j} + \frac{2}{K_j - \gamma_j}(T - t) \right],$$

$$B_j(t, T) = \frac{-2(e^{\gamma_j(T-t)} - 1)}{(K_j + \gamma_j)(e^{\gamma_j(T-t)} - 1) + 2\gamma_j}, j = 1, 2.$$

Proof. Consider an affine term structure, namely

$$P^{CIR}(t, T) = F(t, X_1(t), X_2(t)) = e^{A(t, T) + B_1(t, T)X_1(t) + B_2(t, T)X_2(t)}.$$

Then we obtain the following system of ODEs by Lemma 4.1.3:

$$\frac{\partial A}{\partial t} = \delta_0 - B_1 K_1 \theta_1 - B_2 K_2 \theta_2, \quad (4.15)$$

$$\frac{\partial B_1}{\partial t} = 1 + B_1 K_1 - \frac{1}{2} B_1^2 \sigma_1^2, \quad (4.16)$$

$$\frac{\partial B_2}{\partial t} = 1 + B_2 K_2 - \frac{1}{2} B_2^2 \sigma_2^2, \quad (4.17)$$

$$A(T, T) = B_1(T, T) = B_2(T, T) = 0 \quad (4.18)$$

Note that this is a system of Riccati equation described in Section 1.2. Therefore, the solution can be found easily by applying the solution given in Section 1.2. \square

The m^{th} bond moment under the T_0 -forward measure with respect to the time points $\{T_{i_1}, T_{i_2}, \dots, T_{i_m}\}$ is defined by

$$\mu^{T_0}(t, T_0, \{T_{i_1}, T_{i_2}, \dots, T_{i_m}\}) := \mathbb{E}^{T_0} \left[\prod_{k=1}^m P(T_0, T_{i_k}) \middle| \mathcal{F}_t \right].$$

This is the key to the pricing formula given in next section.

The formula given below can be found in [8] and [12]. We provide a detailed proof of it.

Theorem 4.1.5. *The bond moments under the T_0 -forward measure are given by the following formulae:*

$$\mu^{T_0}(t, T_0, \{T_{i_1}, T_{i_2}, \dots, T_{i_m}\}) = \frac{e^{M(t)+N(t) \cdot X(t)}}{P(t, T_0)}.$$

where

$$M(t) = F_0 - \delta_0 \tau - \sum_{j=1}^n K_j \theta_j \left[\frac{2}{\sigma_j^2} \ln \frac{(K_j + \gamma_j - \sigma_j^2 F_j)(e^{\gamma_j \tau} - 1) + 2\gamma_j}{2\gamma_j} + \frac{(K_j + \gamma_j)F_j + 2}{K_j - \gamma_j - \sigma_j^2 F_j} \tau \right],$$

$$N_j(t) = \frac{-[(K_j - \gamma_j)F_j + 2](e^{\gamma_j \tau} - 1) + 2\gamma_j F_j}{(K_j + \gamma_j - \sigma_j^2 F_j)(e^{\gamma_j \tau} - 1) + 2\gamma_j},$$

$\gamma_j = \sqrt{K_j^2 + 2\sigma_j^2}$, $\tau = T_0 - t$, $F_0 = \sum_{i=1}^m A(T_0, T_i) + A(T_0, T)$, $F_j = \sum_{i=1}^m B_j(T_0, T_i) + B_j(T_0, T)$ and the formulas of A and B_j 's are given in Theorem 4.1.4, $j = 1, 2$.

Proof. Since

$$P(T_0, T_{i_k}) = e^{A(T_0, T_{i_k}) + B(T_0, T_{i_k}) \cdot X(T_0)},$$

we have

$$\prod_{k=1}^m P(T_0, T_{i_k}) = e^{F_0(t, T_0, \{T_{i_1}, T_{i_2}, \dots, T_{i_m}\}) + F(t, T_0, \{T_{i_1}, T_{i_2}, \dots, T_{i_m}\}) \cdot X(T_0)}$$

with

$$F_0 := F_0(t, T_0, \{T_{i_1}, T_{i_2}, \dots, T_{i_m}\}) = \sum_{k=1}^m A(T_0, T_{i_k})$$

and

$$F := F(t, T_0, \{T_{i_1}, T_{i_2}, \dots, T_{i_m}\}) = \sum_{k=1}^m B(T_0, T_{i_k}).$$

By the definition of the forward measure, we have

$$\mathbb{E}^{T_0}[e^{F_0 + F \cdot X(T_0)} | \mathcal{F}_t] = \frac{1}{P(t, T_0)} \mathbb{E}[e^{-\int_t^{T_0} r_s ds} e^{F_0 + F \cdot X(T_0)} | \mathcal{F}_t].$$

Let $F = F(t, X)$ be the solution of

$$\begin{cases} \frac{\partial F}{\partial t}(t, X) + \sum_{i=1}^n \mu_i(t, X) \frac{\partial F}{\partial x_i}(t, X) + \frac{1}{2} \sum_{i,j=1}^n C_{ij}(t, X) \frac{\partial^2 F}{\partial x_i \partial x_j}(t, X) - rF(t, X) = 0, \\ F(T_0, X(T_0)) = \Phi(X(T_0)). \end{cases}$$

Assume that F has the form $F(t, X) = e^{M(t)+N(t)\cdot X(t)}$. Then it is easy to see that

$$M(T_0) = \sum_{i=1}^m A(T_0, T_{i_m}), \quad N(T_0) = \sum_{i=1}^m B(T_0, T_{i_m}).$$

Furthermore, the above PDE implies that

$$\begin{aligned} & -(X_1 + X_2 + \delta_0) + (M_t + N_{1,t}X_1 + N_{2,t}X_2) \\ & + N_1K_1(\theta_1 - X_1) + N_2K_2(\theta_2 - X_2) + \frac{1}{2}N_1^2\sigma_1^2X_1 + \frac{1}{2}N_2^2\sigma_2^2X_2 = 0. \end{aligned}$$

It follows that

$$\begin{aligned} & (-\delta_0 + M_t + N_1K_1\theta_1 + N_2K_2\theta_2) + \\ & (-1 + N_{1,t} - N_1K_1 + \frac{1}{2}N_1^2\sigma_1^2)X_1 + (-1 + N_{2,t} - N_2K_2 + \frac{1}{2}N_2^2\sigma_2^2)X_2 = 0. \end{aligned}$$

Since the above equation holds for any real number X_1 and X_2 , we obtain the following system:

$$\frac{\partial M}{\partial t} = \delta_0 - N_1K_1\theta_1 - N_2K_2\theta_2, \quad (4.19)$$

$$\frac{\partial N_1}{\partial t} = 1 + N_1K_1 - \frac{1}{2}N_1^2\sigma_1^2, \quad (4.20)$$

$$\frac{\partial N_2}{\partial t} = 1 + N_2K_2 - \frac{1}{2}N_2^2\sigma_2^2, \quad (4.21)$$

$$M(T_0) = F_0, \quad N_1(T_0) = F_1 \text{ and } N_2(T_0) = F_2. \quad (4.22)$$

Note that this is a system of Riccati equation described in Section 1.2. Therefore, the solution can be found easily by applying the solution given in Section 1.2. \square

4.2 Pricing Swaptions under CIR2 model

The primary reference of this section is [12].

Consider a swaption with the expiry T_0 and the fixed rate K during a period $[T_0, T_N]$. Note that the price of the underlying swap is given by

$$SV(t) = \sum_{i=0}^N a_i P(t, T_i)$$

where

$$a_0 = -1; a_i = \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=1}^N P(t, T_N)} \quad (i = 1, \dots, N-1) \text{ and } a_N = 1 + \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=1}^N P(t, T_N)}.$$

The m^{th} swap moment under the T_0 -forward measure conditioned on \mathcal{F}_t is given by

$$M_m(t) = \mathbb{E}^{T_0} \left[\left(\sum_{i=0}^N a_i P(t, T_i) \right)^m \middle| \mathcal{F}_t \right]$$

Note that

$$\left[\sum_{i=0}^N a_i P(t, T_i) \right]^m = \sum_{0 \leq i_1, \dots, i_m \leq N} a_{i_1} \dots a_{i_m} \left[\prod_{k=1}^m P(T_0, T_{i_k}) \right].$$

So,

$$M_m(t) = \sum_{0 \leq i_1, \dots, i_m \leq N} a_{i_1} \dots a_{i_m} \mathbb{E}^{T_0} \left[\prod_{k=1}^m P(T_0, T_{i_k}) \middle| \mathcal{F}_t \right].$$

Remark 4.2.1. Observe that

$$M_m(t) = \sum_{\substack{0 \leq k_0, \dots, k_N \leq N, \\ k_0 + \dots + k_N = m,}} a_0^{k_0} \dots a_N^{k_N} \mathbb{E}^{T_0} \left[\prod_{j=0}^N P(T_0, T_j)^{k_j} \middle| \mathcal{F}_t \right].$$

By simple combinatorics, we have

$$M_m(t) = \sum_{\substack{0 \leq k_0 \leq \dots \leq k_N \leq N, \\ k_0 + \dots + k_N = m,}} \frac{m!}{k_0! k_1! \dots k_N!} a_0^{k_0} \dots a_N^{k_N} \mathbb{E}^{T_0} \left[\prod_{j=0}^N P(T_0, T_j)^{k_j} \middle| \mathcal{F}_t \right].$$

Therefore, the algorithm for generating the following collection of sets

$$\{ \{k_0, k_1, \dots, k_N\} : 0 \leq k_0, k_1, \dots, k_N \leq N, k_0 + k_1 + \dots + k_N = M \}$$

is crucial in the implementation of our formulas.

Since the bond moments $\mathbb{E}^{T_0} [\prod_{k=1}^m P(T_0, T_{i_k}) | \mathcal{F}_t]$ have closed form formulas (see Theorem 4.1.5), we are able to obtain a closed form formula for the swaptions. To sum up, we have the following theorems.

Theorem 4.2.2. *Suppose that the risk-neutral dynamics of short rates follow a CIR2 model,*

$$\text{i.e. } r(t) = X_1(t) + X_2(t) + \delta_0.$$

where the Q -dynamics of $X(t) = (X_1(t), X_2(t))$ are given by the following SDEs:

$$dX_i(t) = K_i(\theta_i - X_i(t))dt + \sigma_i \sqrt{X_i(t)} dW_i(t), \quad i = 1, 2.$$

Let $c_n(t)$ be the n^{th} cumulant of the swap price $SV(t)$. It can be calculated by the swap moments $\{M_1(t), \dots, M_n(t)\}$ and $C_n(t) = c_n(t)P(t, T_0)^n$ for $n \geq 1$. Let $q_0 = 1$, $q_1 = q_2 = 0$ and

$$q_n = \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 3, \\ k_1 + \dots + k_m = n}} \frac{C_{k_1} \dots C_{k_m}}{m! k_1! \dots k_m! \sqrt{C_2}^n}, \quad n \geq 3.$$

The risk neutral price of the $T_0 \times (T_N - T_0)$ -receiver swaption $SOV(t; T_0, T_n)$ is given by

$$SOV(t; T_0, T_n) = C_1 N \left(\frac{C_1}{\sqrt{C_2}} \right) + \sqrt{C_2} \phi \left(\frac{C_1}{\sqrt{C_2}} \right) \left[1 + \sum_{k=3}^{\infty} (-1)^k q_k H_{k-1} \left(\frac{C_1}{\sqrt{C_2}} \right) \right].$$

4.3 Numerical results

In this section, we provide numerical results of the method of pricing swaptions discussed in the last section. We consider the following parameters in the CIR2 model:

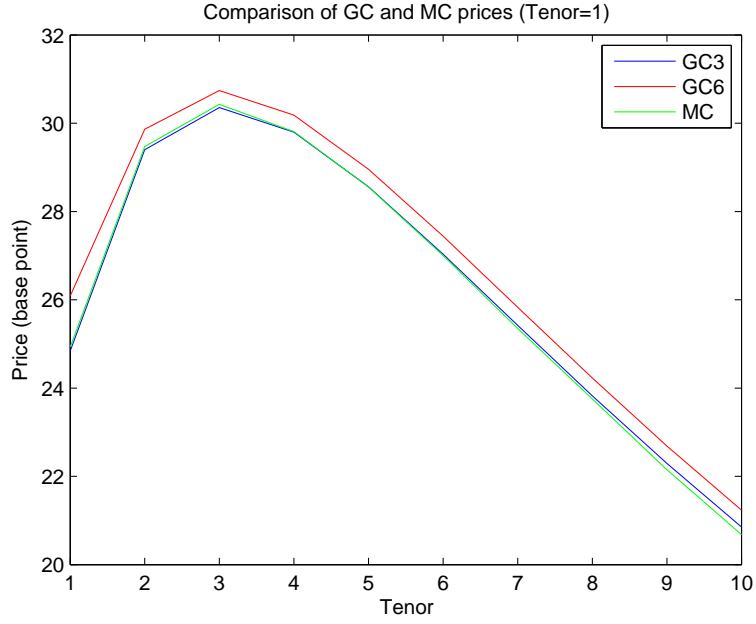


Figure 4.1: Comparison of swaption prices (Tenor = 1)

<i>Parameters</i>	<i>Values</i>
S_0	1
δ_0	0.02
κ_1	0.2
κ_2	0.2
θ_1	0.03
θ_2	0.01
σ_1	0.04
σ_2	0.02
$X_1(0)$	0.04
$X_2(0)$	0.02

We take $N = 5,00,000$ scenarios and 12 time steps per year for the Monte Carlo simulation of swaption prices. We approximate the price of the call option using only first N terms in the Gram-Charlier expansions and denote them by $GC(N)$.

The numerical results are summarized in Figure 3.1 - 3.12.

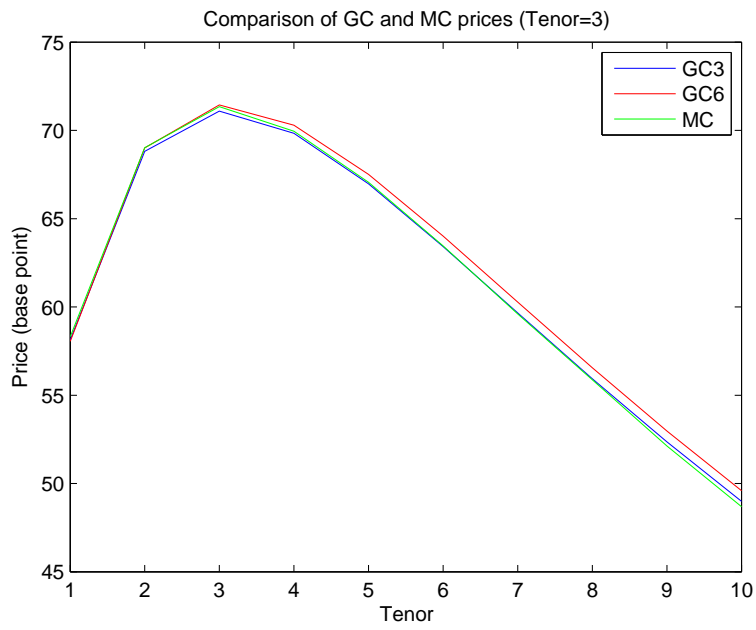


Figure 4.2: Comparison of swaption prices (Tenor = 3)

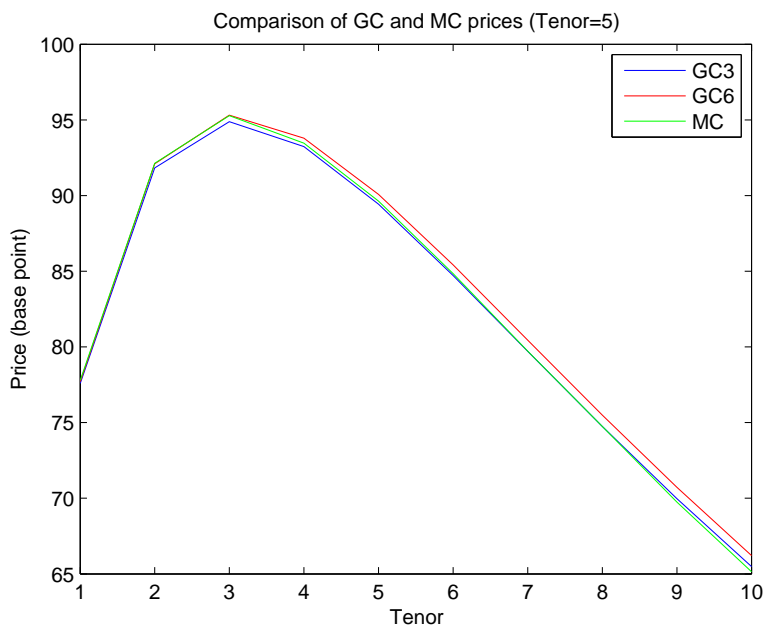


Figure 4.3: Comparison of swaption prices (Tenor = 5)

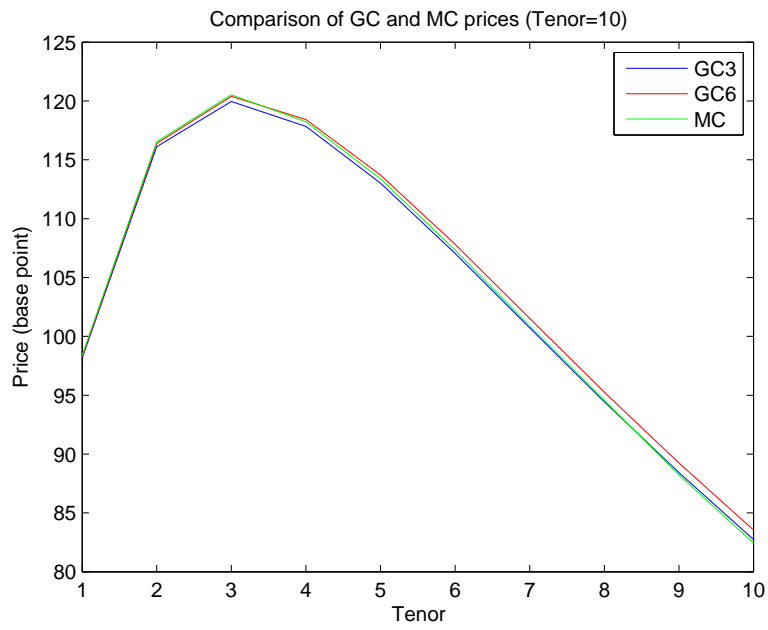


Figure 4.4: Comparison of swaption prices (Tenor = 10)

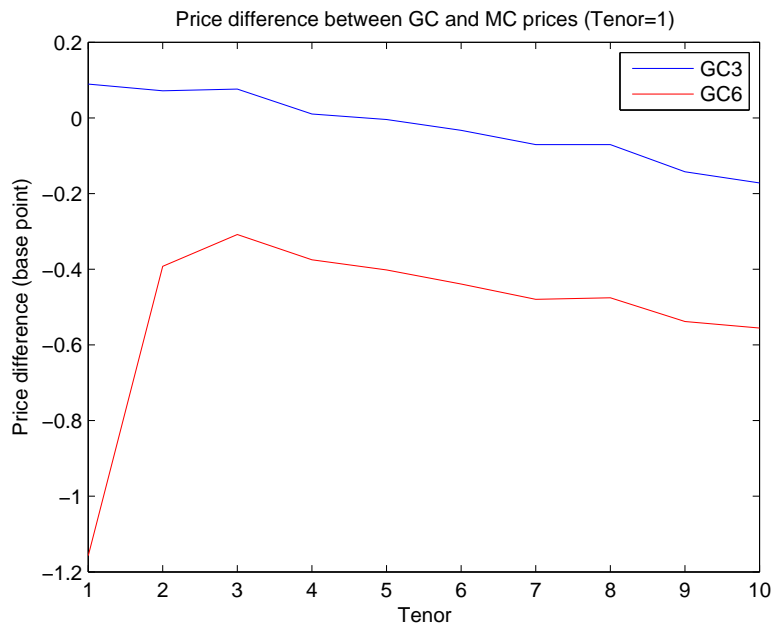


Figure 4.5: Pricing Errors of swaption prices (Tenor = 1)

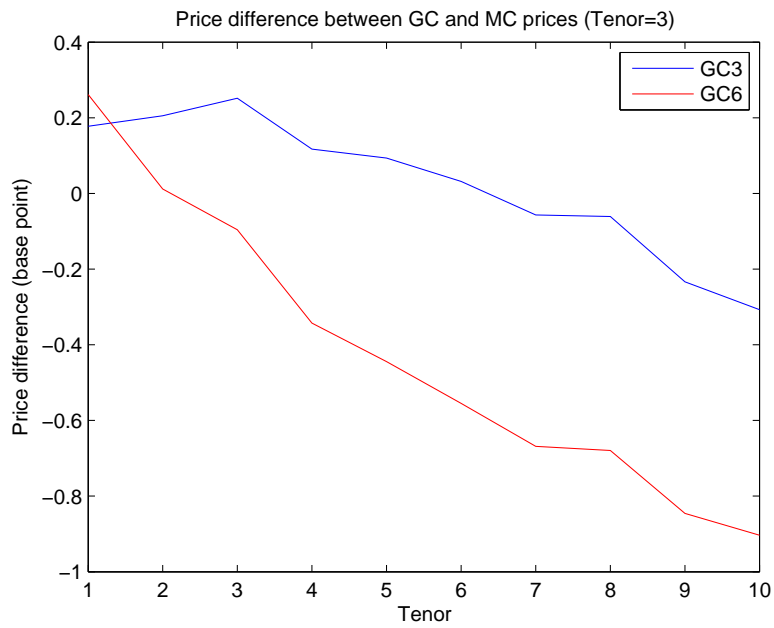


Figure 4.6: Pricing Errors of swaption prices (Tenor = 3)

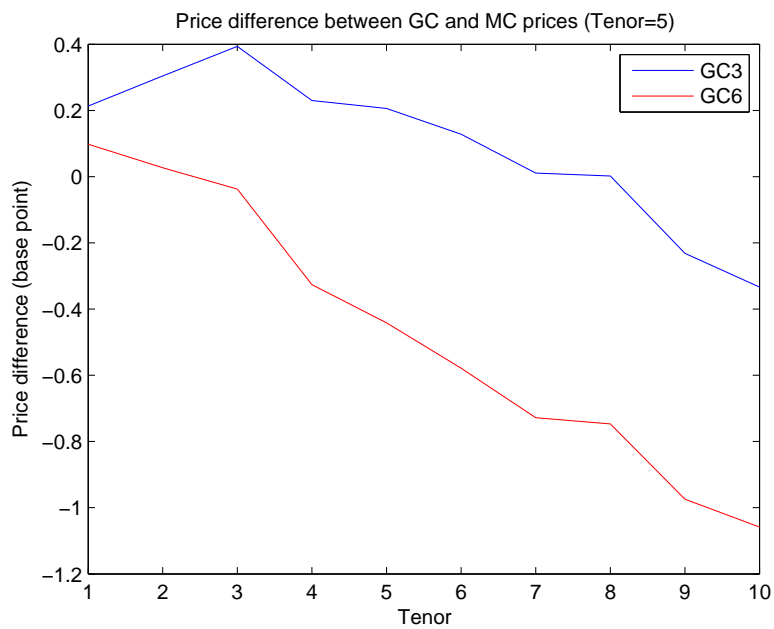


Figure 4.7: Pricing Errors of swaption prices (Tenor = 5)

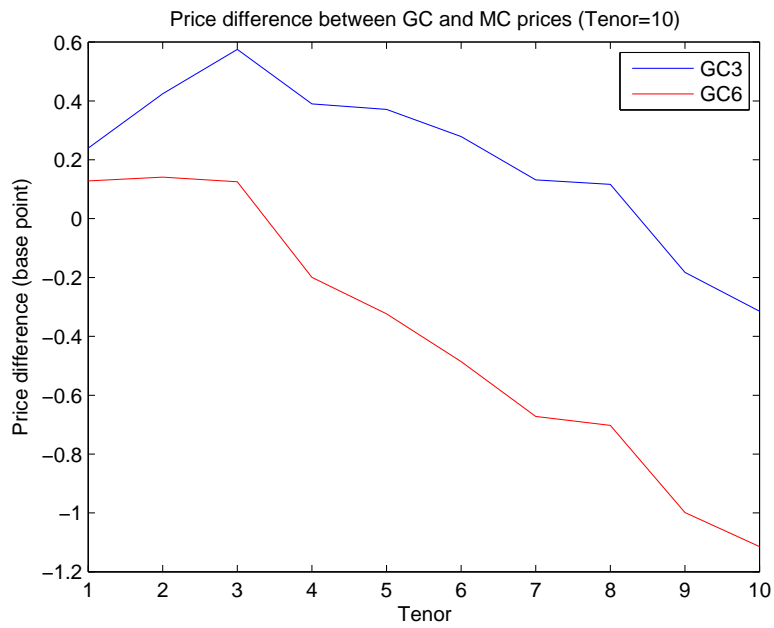


Figure 4.8: Pricing Errors of swaption prices (Tenor = 10)

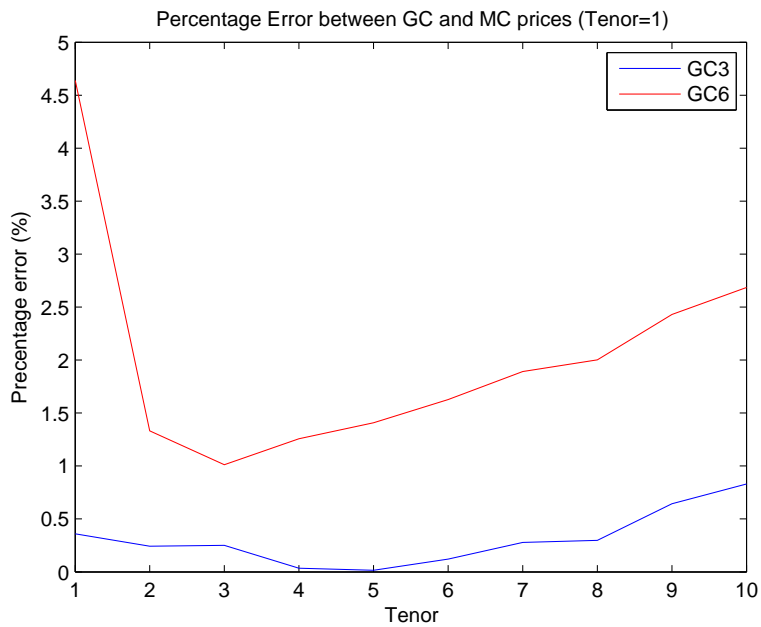


Figure 4.9: Percentage Errors of swaption prices (Tenor = 1)

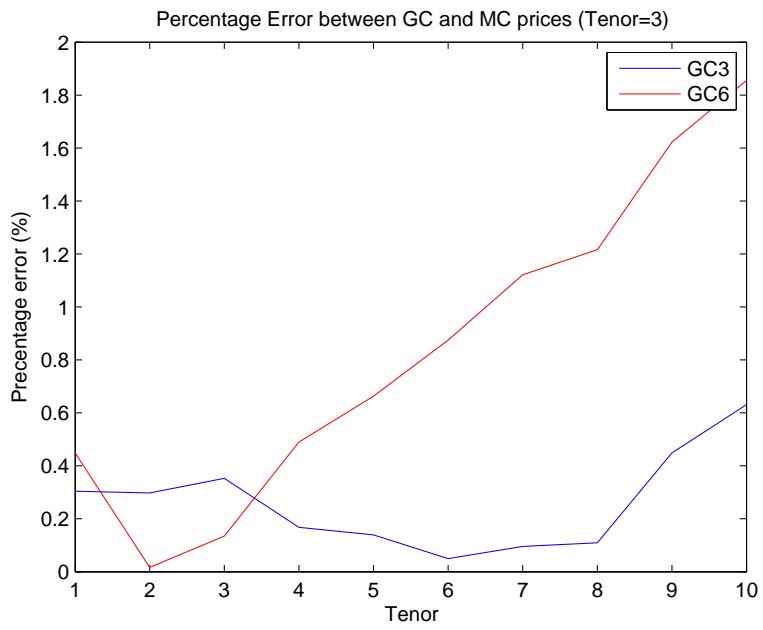


Figure 4.10: Percentage Errors of swaption prices (Tenor = 3)

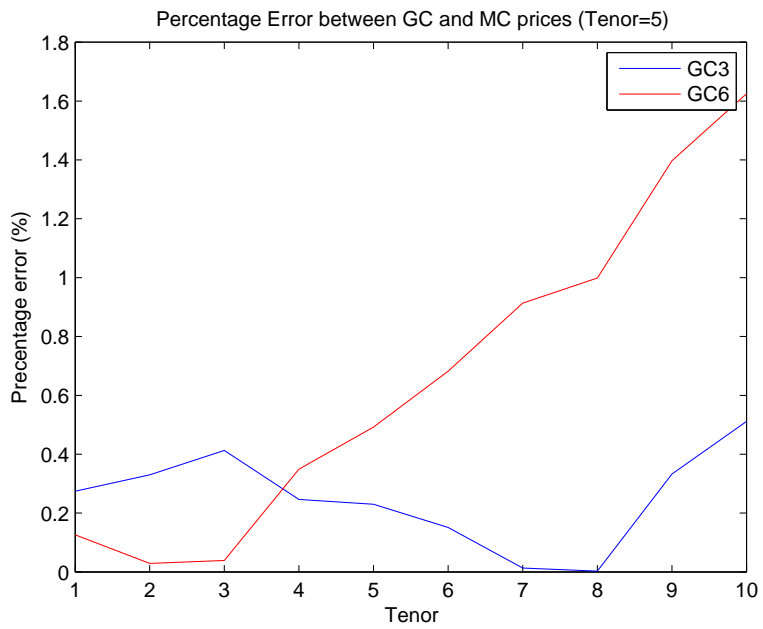


Figure 4.11: Percentage Errors of swaption prices (Tenor = 5)

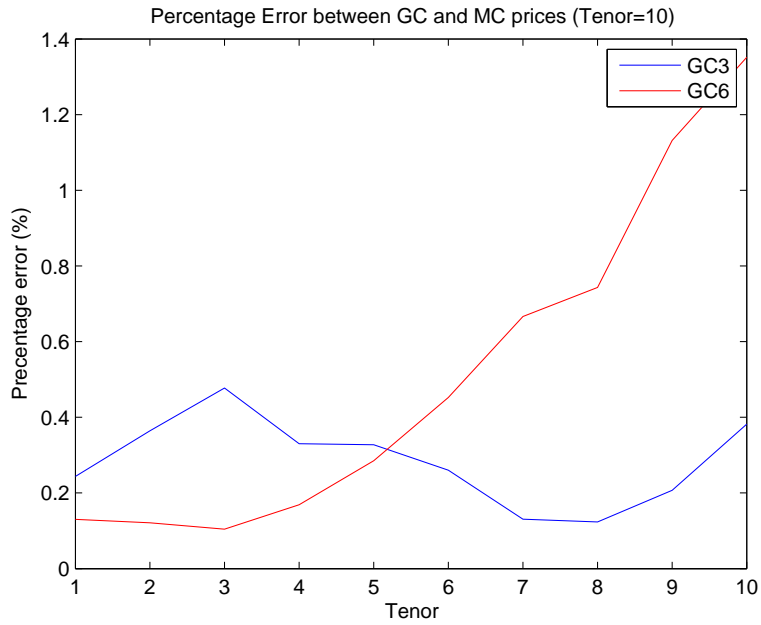


Figure 4.12: Percentage Errors of swaption prices (Tenor = 10)

In Figure 3.1 - 3.12, we see that the GC3 is generally more accurate than GC6. GC6 is slightly more accurate in the short tenor swaptions, but significantly less accurate for long tenor swaptions.

In order to confirm that the result given above is not an artifact, we test another set of parameters:

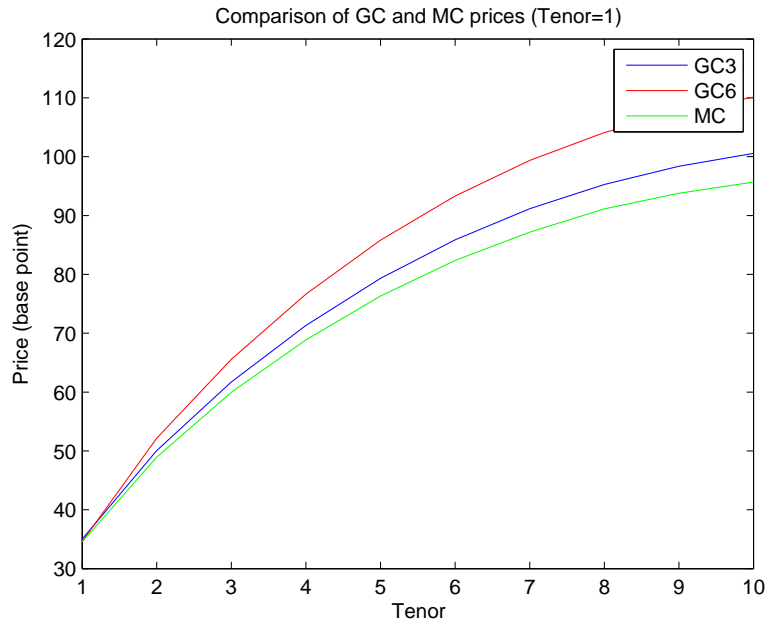


Figure 4.13: Comparison of swaption prices (Tenor = 1)

<i>Parameters</i>	<i>Values</i>
S_0	1
δ_0	-0.02
κ_1	0.05
κ_2	0.5
θ_1	0.085
θ_2	0.01
σ_1	0.08
σ_2	0.05
$X_1(0)$	0.01
$X_2(0)$	0.01

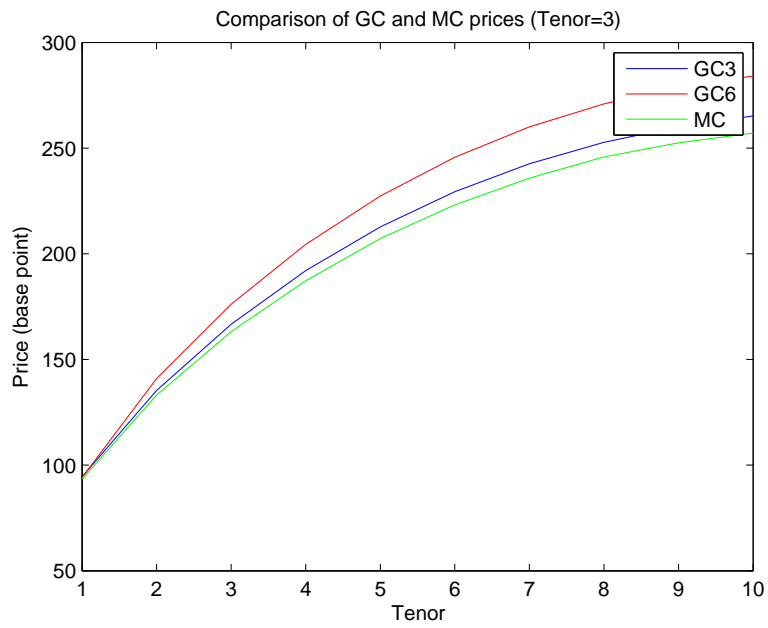


Figure 4.14: Comparison of swaption prices (Tenor = 3)

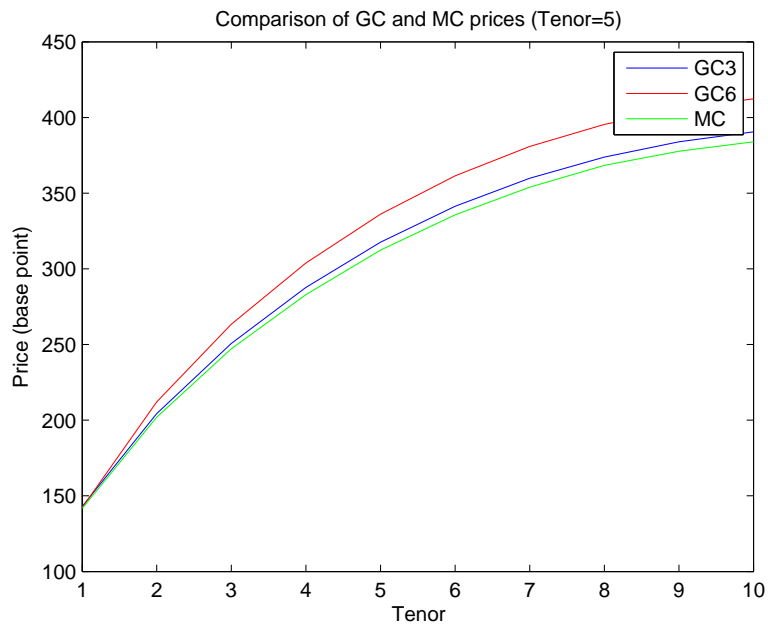


Figure 4.15: Comparison of swaption prices (Tenor = 5)

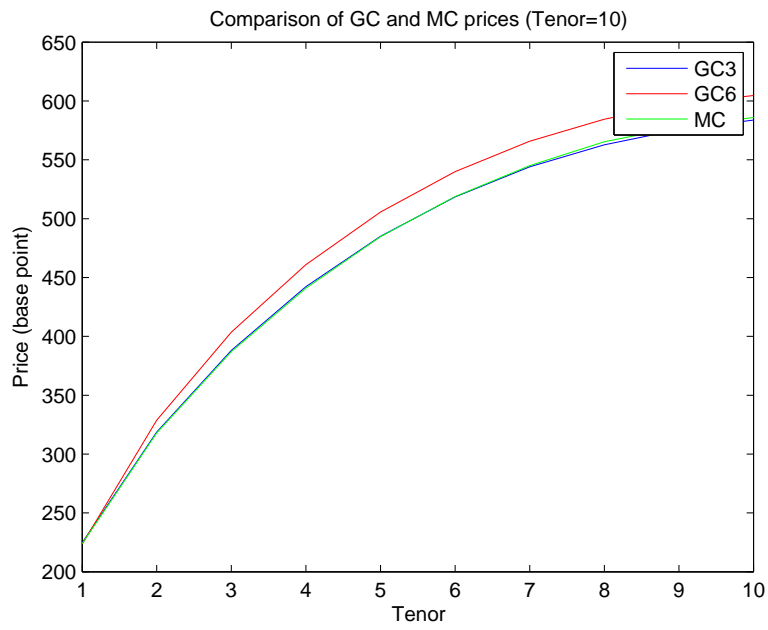


Figure 4.16: Comparison of swaption prices (Tenor = 10)

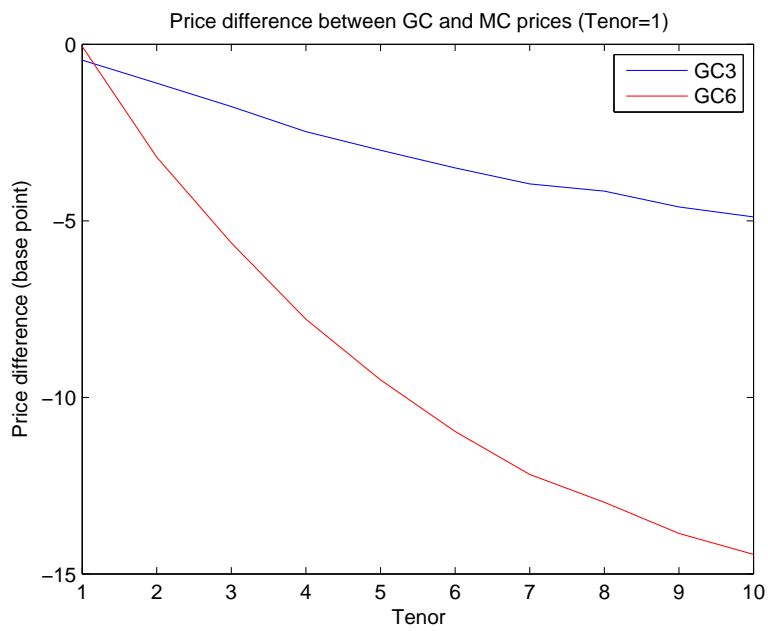


Figure 4.17: Pricing Errors of swaption prices (Tenor = 1)

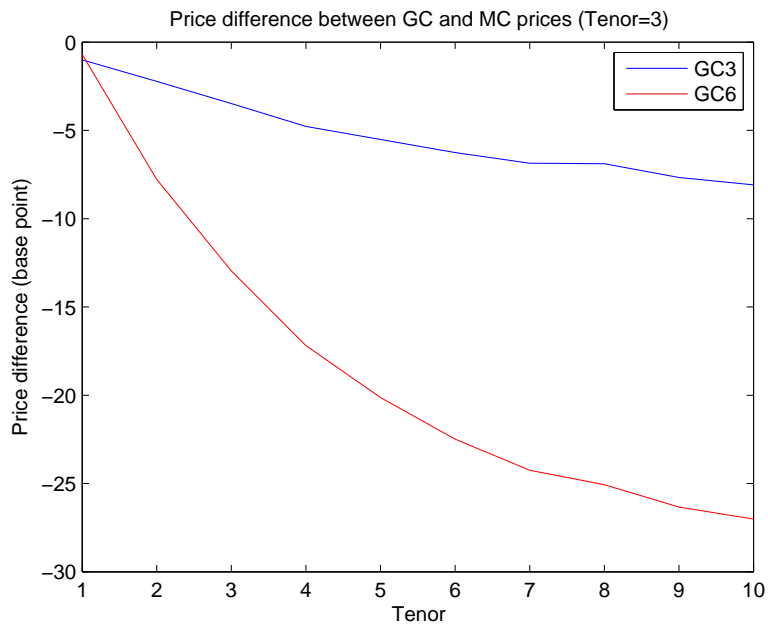


Figure 4.18: Pricing Errors of swaption prices (Tenor = 3)

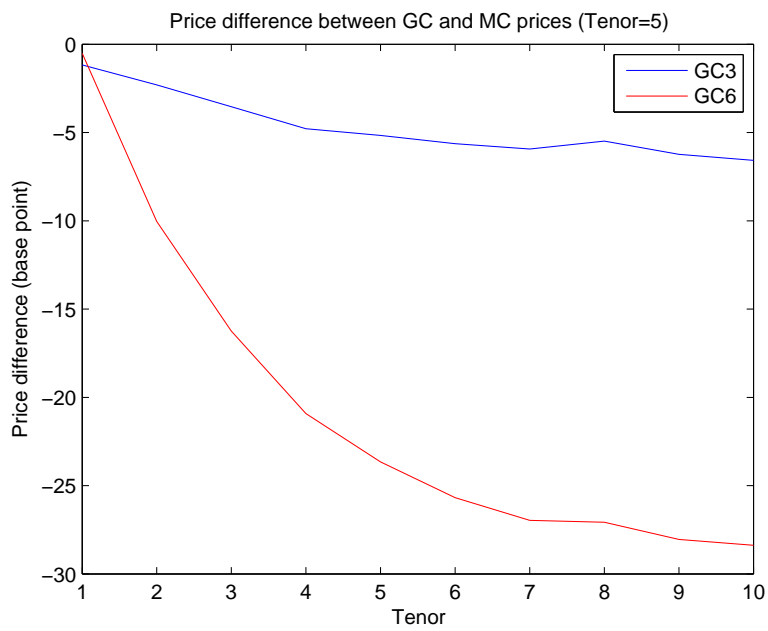


Figure 4.19: Pricing Errors of swaption prices (Tenor = 5)

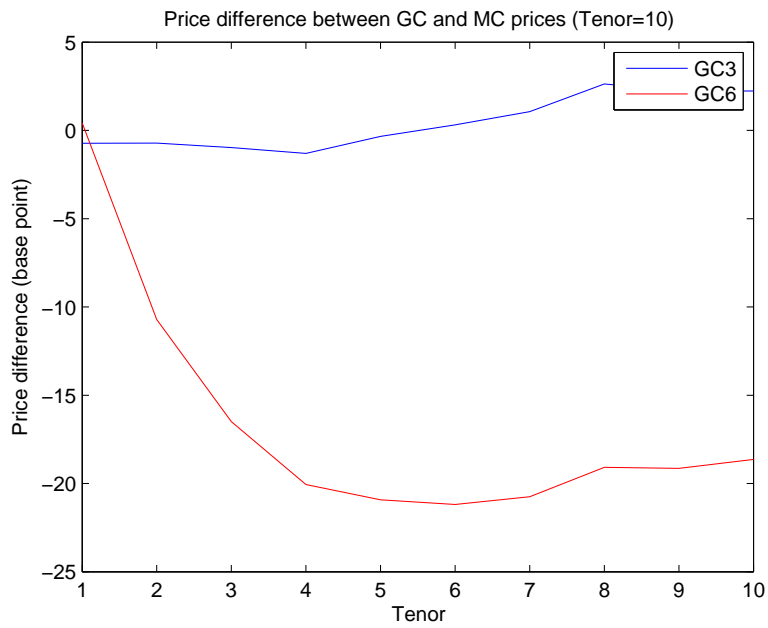


Figure 4.20: Pricing Errors of swaption prices (Tenor = 10)

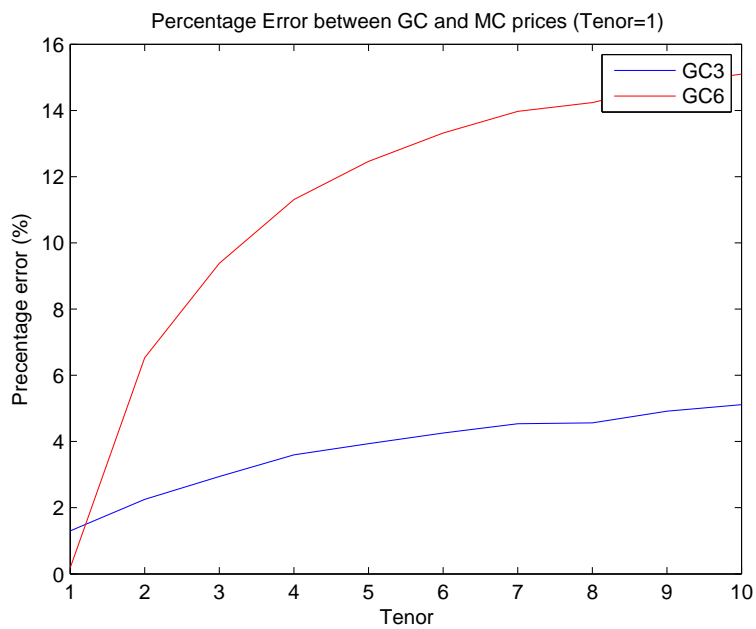


Figure 4.21: Percentage Errors of swaption prices (Tenor = 1)

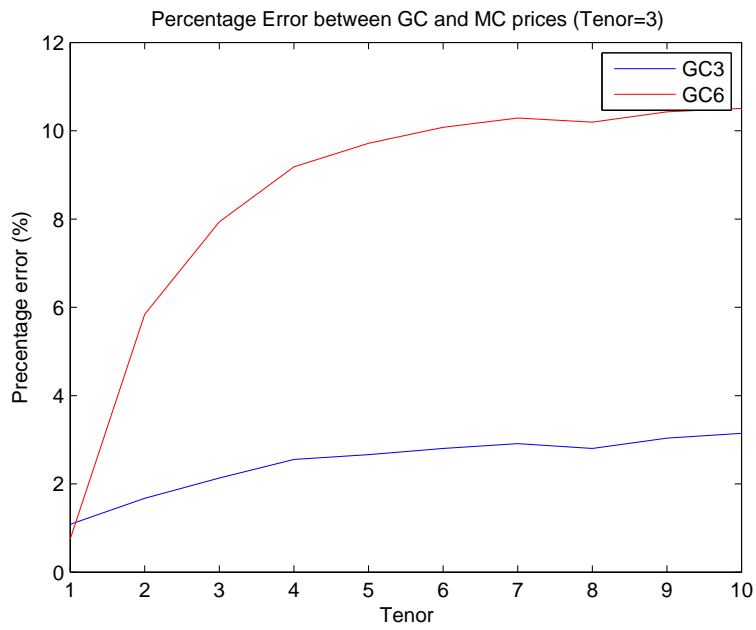


Figure 4.22: Percentage Errors of swaption prices (Tenor = 3)

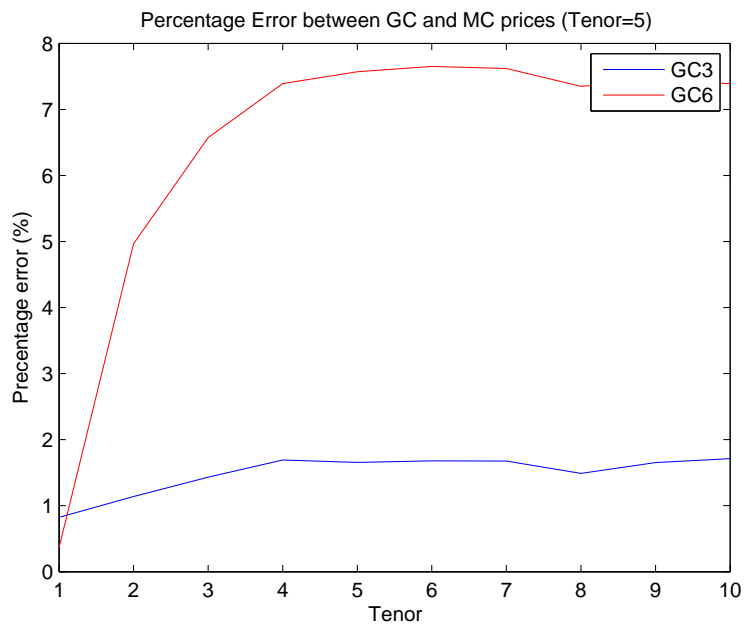


Figure 4.23: Percentage Errors of swaption prices (Tenor = 5)

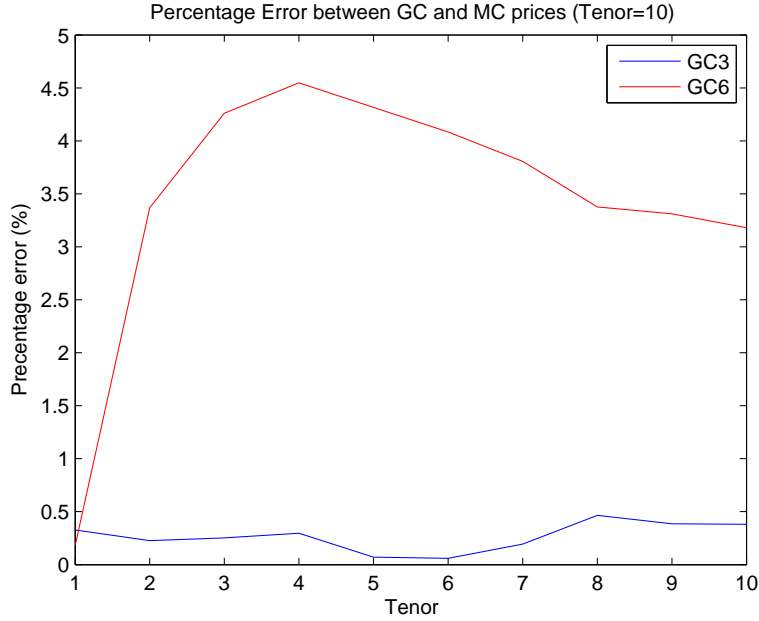


Figure 4.24: Percentage Errors of swaption prices (Tenor = 10)

In Figure 3.13 - 3.24, we again find that the GC3 is generally more accurate than GC6. Adding finitely finite many terms in the Gram-Charlier does not necessarily increase the accuracy of the approximation. This is due to the fact that Gram-Charlier expansions is just an orthogonal series in $L^2(\mu)$ where μ is the Gaussian measure on \mathbb{R} . Also, it is impossible to estimate the error.

4.4 Pricing Swaptions under CIR2++ model

In this section, we will discuss how to use Gram-Charlier expansions to calculate swaption prices under the CIR2++ model.

Consider a two-factor CIR model:

$$\begin{aligned}
 dX_1(t) &= K_1(\theta_1 - X_1(t))dt + \sigma_1\sqrt{X_1(t)}dW_1(t); \\
 dX_2(t) &= K_2(\theta_2 - X_2(t))dt + \sigma_2\sqrt{X_2(t)}dW_2(t)
 \end{aligned}
 \tag{4.23}$$

with the initial conditions $X(0) = (X_1(0), X_2(0))$ and two independent Q -Brownian motions W_1, W_2 .

In the CIR2++ model (see [5]), the short rate is given by

$$r(t) = X_1(t) + X_2(t) + \psi(t).$$

where $\psi(t)$ is chosen so as to fit the initial zero-coupon curve.

Let f_j be the instantaneous forward rate given by the j^{th} SDE in (4.23), $j = 1, 2$, and f_M be the market instantaneous forward rate. Then

$$\psi(t) = f_M(0, t) - f_1(0, t) - f_2(0, t).$$

We define the following:

$$\Phi(u, v) = \frac{P^M(0, v) P^{CIR}(0, u)}{P^M(0, u) P^{CIR}(0, v)}$$

where P_M is the market discount factor.

The price at time t of a zero-coupon bond maturing at time T and with unit face value is given by

$$\bar{P}(t, T) = \Phi(t, T) P^{CIR}(t, T)$$

where

$$P^{CIR}(t, T) = e^{A(t, T) + B(t, T) \cdot X(t)}$$

$$\gamma_j = \sqrt{K_j^2 + 2\sigma_j^2}, j = 1, 2$$

$$A(t, T) = - \sum_{j=1}^2 K_j \theta_j \left[\frac{2}{\sigma_j^2} \ln \frac{(K_j + \gamma_j)(e^{\gamma_j(T-t)} - 1)}{2\gamma_j} + \frac{2}{K_j - \gamma_j} (T - t) \right];$$

$$B_j(t, T) = \frac{-2(e^{\gamma_j(T-t)} - 1)}{(K_j + \gamma_j)(e^{\gamma_j(T-t)} - 1) + 2\gamma_j}, j = 1, 2$$

So, we may write

$$\bar{P}(t, T) = \frac{P^M(0, T)}{P^M(0, t)} \left[\frac{P^{CIR}(0, t)}{P^{CIR}(0, T)} P^{CIR}(t, T) \right].$$

It is obvious to see that

$$\bar{P}(0, T) = \frac{P^M(0, T)}{P^M(0, t)} \left[\frac{P^{CIR}(0, t)}{P^{CIR}(0, T)} P^{CIR}(t, T) \right] = P^M(0, T).$$

Therefore, the discount factors derived from the model match the initial term structure.

Consider a swaption with the expiry T_0 and the fixed rate K during a period $[T_0, T_N]$.

Note that the price of the underlying swap is given by

$$SV(t) = \sum_{i=0}^N a_i \bar{P}(t, T_i)$$

where

$$a_0 = -1; a_i = \frac{\bar{P}(t, T_0) - \bar{P}(t, T_N)}{\sum_{i=1}^N \bar{P}(t, T_N)} \quad (i = 1, \dots, N-1) \text{ and } a_N = 1 + \frac{\bar{P}(t, T_0) - \bar{P}(t, T_N)}{\sum_{i=1}^N \bar{P}(t, T_N)}.$$

In particular, at time $t = 0$, the swap price is given by

$$SV(0) = \sum_{i=0}^N a_i \bar{P}(0, T_i) = \sum_{i=0}^N a_i P^M(0, T_i)$$

where

$$a_0 = -1; a_i = \frac{P^M(0, T_0) - P^M(0, T_N)}{\sum_{i=1}^N P^M(0, T_N)} \quad (i = 1, \dots, N-1) \text{ and } a_N = 1 + \frac{P^M(0, T_0) - P^M(0, T_N)}{\sum_{i=1}^N P^M(0, T_N)}.$$

The m^{th} swap moment under the T_0 -forward measure conditioned on \mathcal{F}_t is given by

$$M_m^*(t) = \mathbb{E}^{T_0} \left[\left\{ \sum_{i=0}^N a_i \bar{P}(t, T_i) \right\}^m \middle| \mathcal{F}_t \right]$$

Note that

$$\left[\sum_{i=0}^N a_i \bar{P}(t, T_i) \right]^m = \sum_{0 \leq i_1, \dots, i_m \leq N} a_{i_1} \dots a_{i_m} \left[\prod_{k=1}^m \bar{P}(T_0, T_{i_k}) \right].$$

So,

$$M_m^*(t) = \sum_{0 \leq i_1, \dots, i_m \leq N} a_{i_1} \dots a_{i_m} \mathbb{E}^{T_0} \left[\prod_{k=1}^m \bar{P}(T_0, T_{i_k}) \middle| \mathcal{F}_t \right].$$

We have to calculate the bond moment under the T_0 -forward measure, which is defined by

$$\mu^{T_0}(t, T_0, \{T_{i_1}, T_{i_2}, \dots, T_{i_m}\}) := \mathbb{E}^{T_0} \left[\prod_{k=1}^m \bar{P}(T_0, T_{i_k}) \middle| \mathcal{F}_t \right].$$

Observe that

$$\bar{P}(T_0, T_{i_k}) = \frac{P^M(0, T_{i_k})}{P^M(0, T_0)} \left[\frac{P^{CIR}(0, T_0)}{P^{CIR}(0, T_{i_k})} P^{CIR}(T_0, T_{i_k}) \right].$$

We have

$$\begin{aligned} M_m^*(t) &= \sum_{0 \leq i_1, \dots, i_m \leq N} a_{i_1} \dots a_{i_m} \mathbb{E}^{T_0} \left[\prod_{k=1}^m \frac{P^M(0, T_{i_k})}{P^M(0, T_0)} \left[\frac{P^{CIR}(0, T_0)}{P^{CIR}(0, T_{i_k})} P^{CIR}(T_0, T_{i_k}) \right] \middle| \mathcal{F}_t \right] \\ &= \left(\frac{P^{CIR}(0, T_0)}{P^M(0, T_0)} \right)^m \sum_{0 \leq i_1, \dots, i_m \leq N} a_{i_1} \dots a_{i_m} \prod_{k=1}^m \frac{P^M(0, T_{i_k})}{P^{CIR}(0, T_{i_k})} \mathbb{E}^{T_0} \left[\prod_{k=1}^m P^{CIR}(T_0, T_{i_k}) \middle| \mathcal{F}_t \right] \\ &= \left(\frac{P^{CIR}(0, T_0)}{P^M(0, T_0)} \right)^m \sum_{0 \leq i_1, \dots, i_m \leq N} a_{i_1}^* \dots a_{i_m}^* \mathbb{E}^{T_0} \left[\prod_{k=1}^m P^{CIR}(T_0, T_{i_k}) \middle| \mathcal{F}_t \right], \end{aligned}$$

where $a_{i_k}^* = a_{i_k} \frac{P^M(0, T_{i_k})}{P^{CIR}(0, T_{i_k})}$.

Therefore, we have a closed form formula for the swaption prices under the CIR2++ model.

Theorem 4.4.1. *Suppose that the risk-neutral dynamics of short rates follow the CIR2++ model.*

$$r(t) = X_1(t) + X_2(t) + \psi(t).$$

where the Q -dynamics of $X(t) = (X_1(t), X_2(t))$ are given by the system of SDEs given in (4.23) and $\psi(t)$ is chosen so as to fit the initial zero-coupon curve.

Let $c_n^*(t)$ be the swap cumulants which can be calculated by the swap moments $\{M_1^*(t), \dots, M_n^*(t)\}$ and $C_n^*(t) = c_n^*(t)P(t, T_0)^n$ for $n \geq 1$. Put $q_0 = 1$, $q_1 = q_2 = 0$ and

$$q_n = \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 3, \\ k_1 + \dots + k_m = n}} \frac{C_{k_1}^* \dots C_{k_m}^*}{m! k_1! \dots k_m! \sqrt{C_2^{*n}}}, \quad n \geq 3.$$

The risk neutral price of the $T_0 \times (T_N - T_0)$ -receiver swaption $SOV(t; T_0, T_n)$ is given by

$$SOV(t; T_0, T_n) = C_1^* N \left(\frac{C_1^*}{\sqrt{C_2^*}} \right) + \sqrt{C_2^*} \phi \left(\frac{C_1^*}{\sqrt{C_2^*}} \right) \left[1 + \sum_{k=3}^{\infty} (-1)^k q_k H_{k-1} \left(\frac{C_1^*}{\sqrt{C_2^*}} \right) \right].$$

Chapter 5

Applications of Gram-Charlier expansions in General Models

In Theorem 2.1.2, we introduced a procedure to calculate the survival function (hence, the distribution function) and the truncated first moment of a random variable when its cumulants (or moments) are known. Theoretically speaking, we are able to calculate the prices of European-type derivatives of any diffusion process if we are able to calculate its moments. In this chapter, we shall discuss the general procedure with worked examples in details.

5.1 A Toy Example: Black-Scholes Model

In this section, we will show how to use Gram-Charlier expansions to calculate the price of an European call option under an ordinary Black-Scholes (1973) model [3]. Since we have a closed form formula for the option prices, this method has limited real-time application. However, the closed form formula gives a benchmark for our approximation method.

Assume that the price process (Q-dynamics) of an asset follows a geometric Brownian motion in the Black-Scholes Model,

$$\text{i.e. } dS_t = S_t(rdt + \sigma dW_t).$$

The solution of the SDE is given by:

$$S_T = S_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma W_T \right).$$

Now, S_t follows a log-normal distribution. We may obtain the moments, and hence cumulants, of S_t easily. In fact, we have

$$M_n(t) := E^Q[S_T^n | S_0] = S_0^n \exp \left(\left(r - \frac{\sigma^2}{2} \right) nT + \frac{n^2 \sigma^2 T}{2} \right) \text{ for } n \geq 1.$$

Moreover, we may decompose the call option price as follows:

$$\begin{aligned} e^{-rt} E^Q[(S_T - K)^+] &= e^{-rt} E^Q[S_T I(S_T > K)] - e^{-rt} E^Q[K I(S_T > K)] \\ &= e^{-rt} E^Q[S_T I(S_T > K)] - e^{-rt} K [Q(S_T > K)]. \end{aligned}$$

By using theorem 2.1.2, we are ready to give a series expansion of the option price:

Proposition 5.1.1. *Suppose that the risk-neutral dynamics of the stock price follow the Black-Scholes Model:*

$$dS_t = S_t(rdt + \sigma dW_t)$$

with initial condition $S_0 = s_0$. Suppose that we have a European call option with strike K . Let c_n be the n^{th} -cumulant of S_t . Let $q_0 = 1$, $q_1 = q_2 = 0$ and

$$q_n = \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 3, \\ k_1 + \dots + k_m = n}} \frac{c_{k_1} \dots c_{k_m}}{m! k_1! \dots k_m! \sqrt{c_2}^n}, \quad n \geq 3.$$

Then the price of the call option price is equal to the following infinite sum:

$$\begin{aligned} &e^{-rt} \sqrt{c_2} \phi \left(\frac{c_1 - K}{\sqrt{c_2}} \right) + e^{-rt} c_1 N \left(\frac{c_1 - K}{\sqrt{c_2}} \right) \\ &+ e^{-rt} \sum_{n=3}^{\infty} (-1)^{n-1} q_n \phi \left(\frac{c_1 - K}{\sqrt{c_2}} \right) \left[K H_{n-1} \left(\frac{c_1 - K}{\sqrt{c_2}} \right) - \sqrt{c_2} H_{n-2} \left(\frac{c_1 - K}{\sqrt{c_2}} \right) \right] \\ &- e^{-rt} K \left[N \left(\frac{c_1 - a}{\sqrt{c_2}} \right) + \sum_{k=3}^{\infty} (-1)^{k-1} q_k H_{k-1} \left(\frac{c_1 - a}{\sqrt{c_2}} \right) \phi \left(\frac{c_1 - a}{\sqrt{c_2}} \right) \right]. \end{aligned}$$

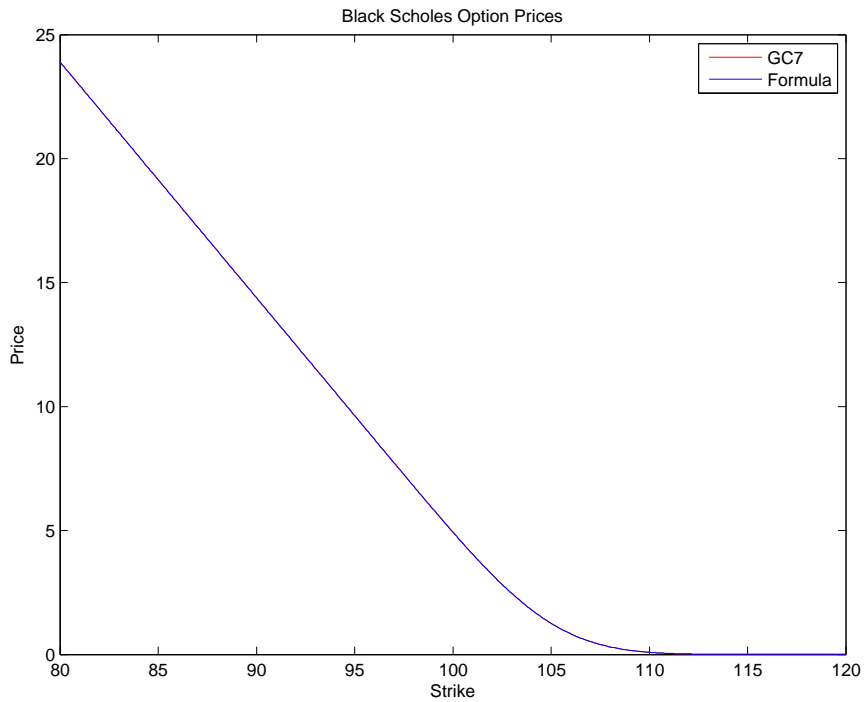


Figure 5.1: Black-Scholes Call Option Prices ($T = 1$)

In the rest of this section, we will present the result of a test on our Gram-Charlier approach. Below is a list of parameters we use for the model.

<i>Parameters</i>	<i>Values</i>
S_0	100
K	{80, 80.1, ..., 119.9, 120.0}
σ	0.03
T	1 or 2
r	0.05

We use 7 terms in the Gram-Charlier expansion. The results are given in Figure 4.1 to Figure 4.6.

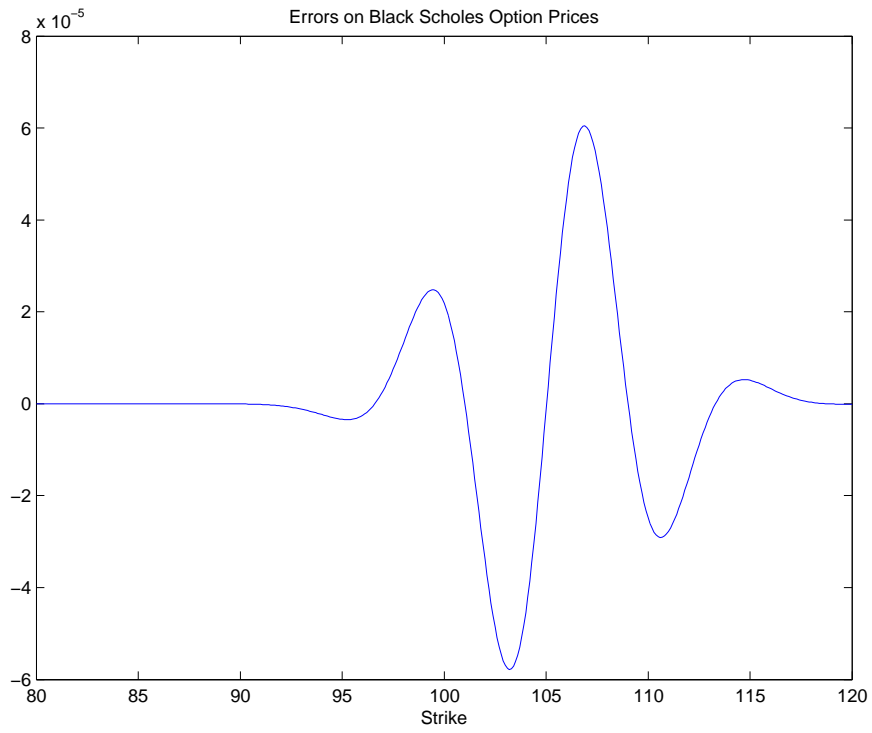


Figure 5.2: Black-Scholes Call Option Prices ($T = 1$)

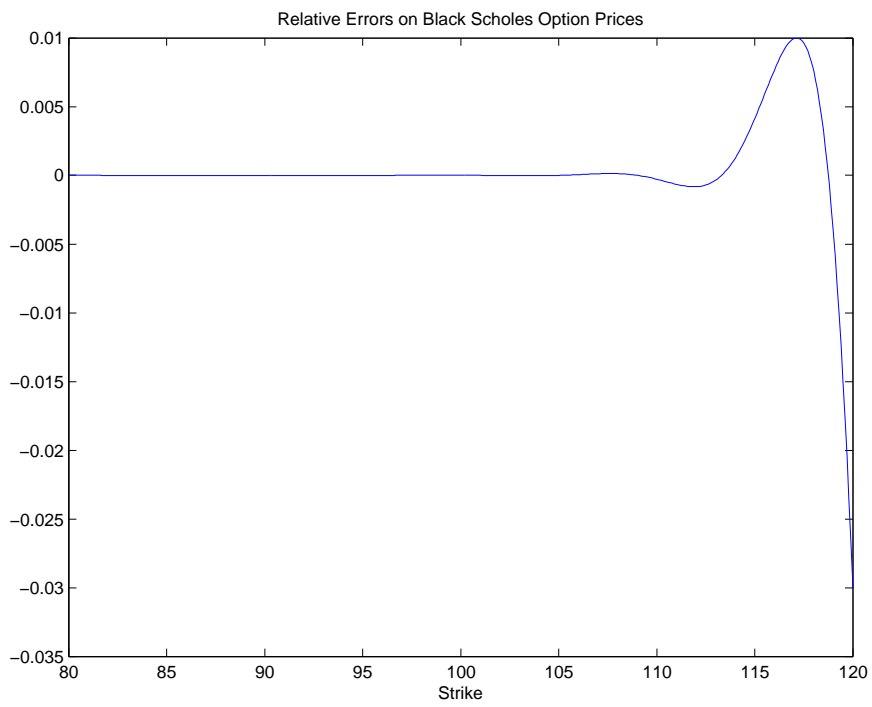


Figure 5.3: Black-Scholes Call Option Prices ($T = 1$)

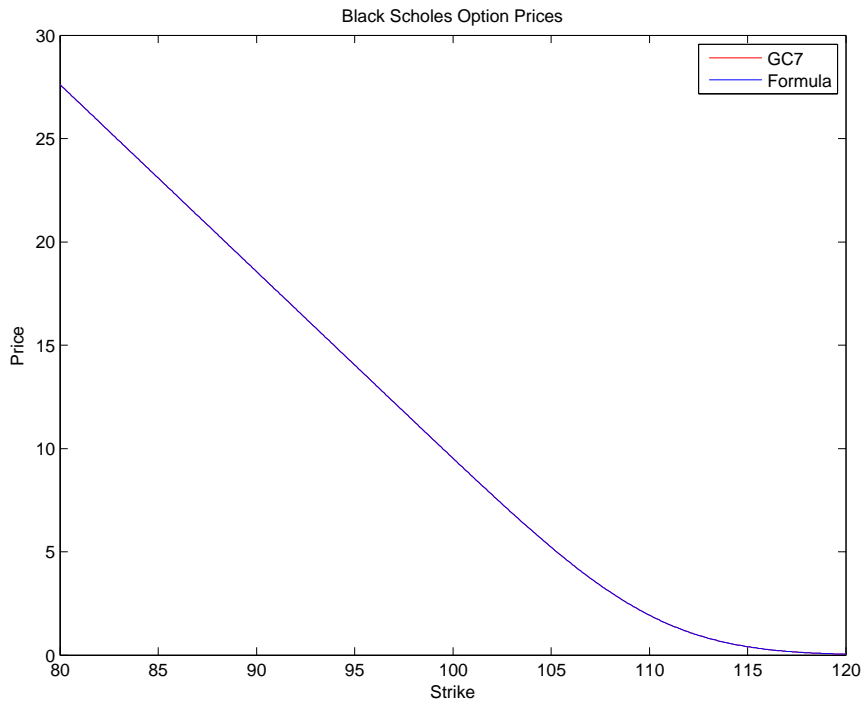


Figure 5.4: Black-Scholes Call Option Prices (T = 2)

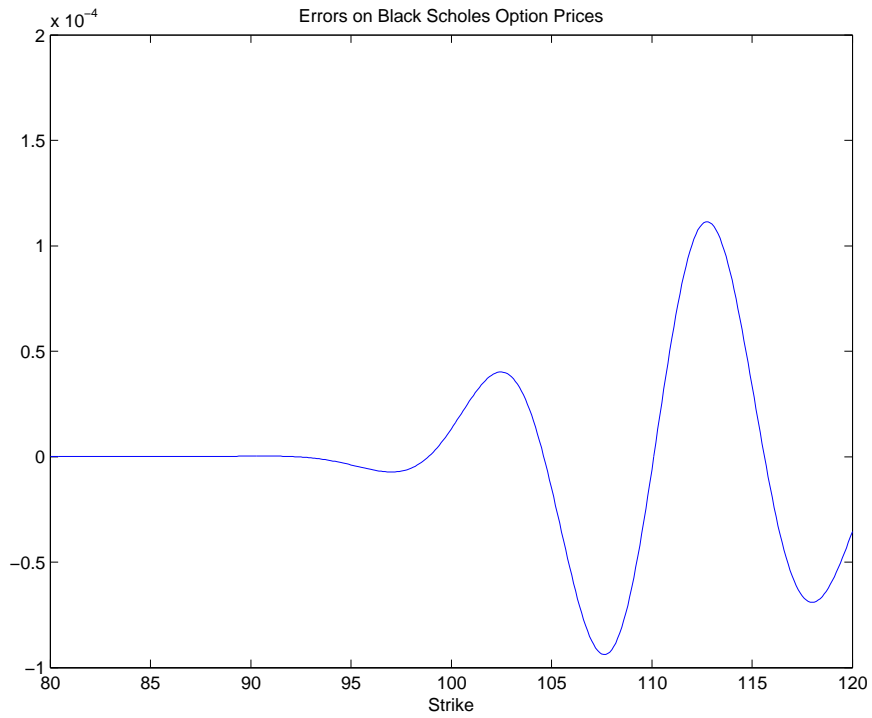


Figure 5.5: Black-Scholes Call Option Prices (T = 2)

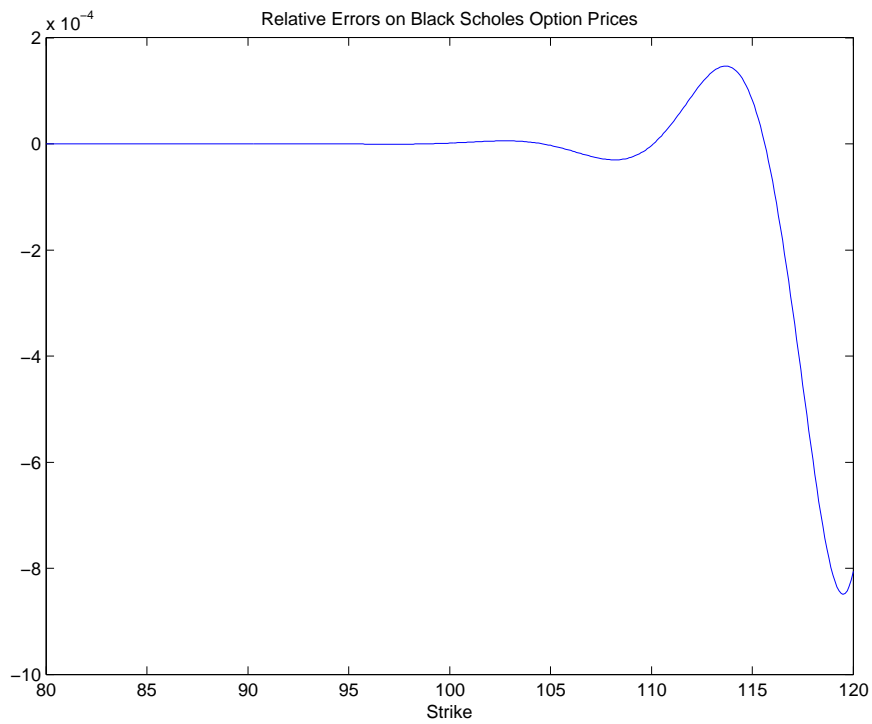


Figure 5.6: Black-Scholes Call Option Prices ($T = 2$)

We see from Figures 4.2, 4.3, 4.5 and 4.6 that the (relative) errors are generally very small. The errors for out-of-the-money options are relatively small. Also, it is more accurate if the time-to-expiry is longer.

5.2 Application to a Simplified Version of Brennan and Schwarz's Model

The Brennan and Schwarz (1983) model (See [4]) is a two-factor model of interest rates which is given by the following:

$$\begin{cases} dr_t = (a_1 + b_1(l_t - r_t))dt + \sigma_1 r_t dW_t^1 \\ dl_t = l_t(a_2 - b_2 r_t + c_2 l_t)dt + \sigma_2 l_t dW_t^2 \end{cases} \quad (5.1)$$

where a_i 's and b_i 's are constants.

To make the demonstration easier, we assume that l_t is a constant process and rewrite the process of r_t as follows:

$$dr_t = \kappa(\theta - r_t)dt + \sigma r_t dW_t \quad (5.2)$$

The first step of our approximation process is to calculate the moments of r_t . We first apply the Itô's Lemma to the process (r_t^n) :

$$\begin{aligned} dr_t^n &= nr_t^{n-1}dr_t + \frac{n(n-1)}{2}r_t^{n-2}(dr_t)^2 \\ &= nr_t^{n-1}[\kappa(\theta - r_t)dt + \sigma r_t dW_t] + \frac{n(n-1)}{2}r_t^{n-2}\sigma^2 r_t^2 dt \\ &= [n\kappa\theta r_t^{n-1} - n\kappa r_t^n]dt + \frac{n(n-1)}{2}r_t^n \sigma^2 dt + \sigma r_t^n dW_t. \end{aligned}$$

In the integral form, we have

$$r_t^n - r_0^n = n\kappa\theta \int_0^t r_s^{n-1} ds + \left[\frac{(n-1)\sigma^2}{2} - \kappa\right] \int_0^t nr_s^n ds + \sigma \int_0^t r_s^n dW_s.$$

Assume that the parameters in (5.2) are nice enough such that r_t is square-integrable.

The last term becomes a martingale. Let $F_n(t) = E[r_t^n]$. By Fubini's theorem, we have

$$F_n(t) = r_0^n + n\kappa\theta \int_0^t F_{n-1}(s) ds + \left[\frac{(n-1)\sigma^2}{2} - \kappa\right] \int_0^t nF_n(s) ds.$$

In other words, $F_n(t)$ can be solved recursively by the following system of ODEs:

$$F'_n(t) = n\kappa\theta F_{n-1}(t) + \left[\frac{(n-1)\sigma^2}{2} - \kappa\right]F_n(t) ; F_n(0) = r_0^n \text{ for } n \geq 1.$$

Note that the ODEs in the system are linear. We may solve the system recursively by using integration factors. However, due to the complexity of the calculation for large n , a symbolic calculation software, called Mathematica, was used for test purpose.

In the rest of this section, we will present the result of a test on our Gram-Charlier approach. Below is a list of parameters we use for the model.

<i>Parameters</i>	<i>Values</i>
r_0	0.06
r	{0.001, 0.002, ...0.1}
T	5
κ	0.2
θ	0.05
σ	0.115

We approximate the distribution of r_t by using Theorem 2.1.2. The moments of r_t are calculated by solving the system of ODEs discussed above with Mathematica. We use 7 terms in the Gram-Charlier expansion. The results are given in Figures 4.7 and 4.8.

We see that the approximation provides a fairly good fit to the model. The error in the middle region is relatively higher, and up to 0.02. This error is acceptable for a risk management purpose.

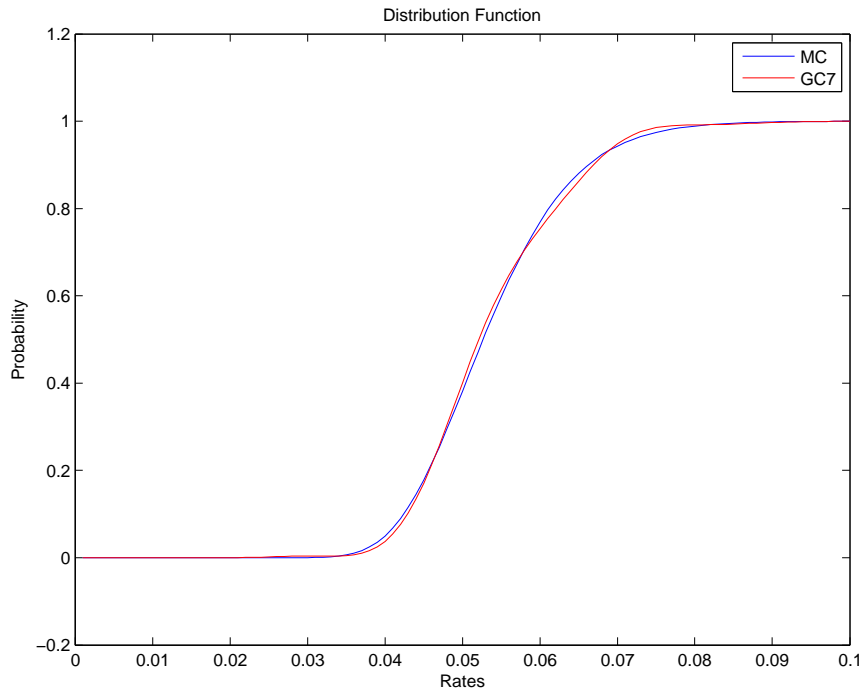


Figure 5.7: Comparison of Distribution Functions

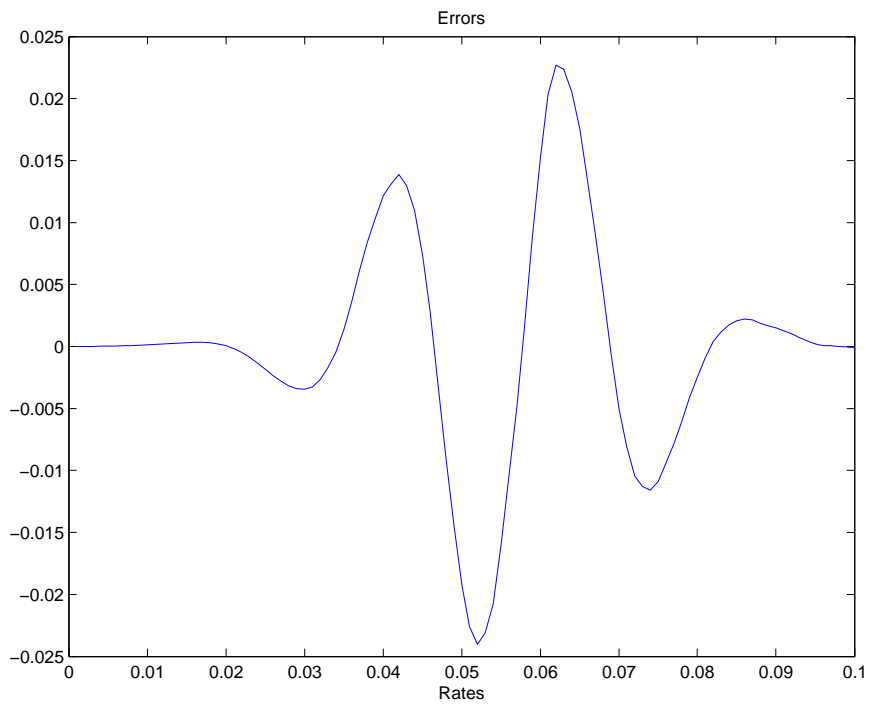


Figure 5.8: Error of Distribution Function

Chapter 6

Pricing Call Options under Heston's Model using Gram-Charlier Expansions

6.1 Introduction to Heston's Model of Stochastic Volatility

We assume that the risk-neutral dynamics of the stock price follows the Heston Model (1993) (See [10]) which is given by the following system of SDEs:

$$\begin{cases} dS_t = S_t(rdt + \sqrt{V_t}dW_t^1) \\ dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^2 \end{cases} \quad (6.1)$$

with initial conditions $S_0 = s_0$ and $V_0 = v_0 \geq 0$ where $\kappa, \theta, \sigma > 0$ and $dW_t^1 dW_t^2 = \rho dt$, $\rho \in [-1, 1]$.

Let $X_t = \ln S_t - rt$ be the logarithm of the discount stock price. By Itô's lemma, we have

$$dX_t = -rdt + \frac{dS_t}{S_t} - \frac{1}{2S_t^2}(dS_t \cdot dS_t) = -\frac{1}{2}V_t dt + \sqrt{V_t}dW_t^1,$$

with initial condition $X_0 = x_0 = \ln S_0$.

Thus, we may transform the system (6.1) into the following system:

$$\begin{cases} dX_t = -\frac{1}{2}V_t dt + \sqrt{V_t} dW_t^1 \\ dV_t = \kappa(\theta - V_t) dt + \sigma\sqrt{V_t} dW_t^2 \end{cases} \quad (6.2)$$

with initial conditions $X_0 = x_0$ and $V_0 = v_0 \geq 0$.

One can show that the moment generating function of X_t (See [11]) is given by

$$\begin{aligned} M_t(u) = \mathbb{E}[e^{uX_t}] &= e^{x_0 u} \left(\frac{e^{(\kappa - \sigma \rho t)/2}}{\cosh(P(u)t/2) + (\kappa - \sigma \rho u) \sinh(P(u)t/2)/P(u)} \right)^{2\kappa\theta/\sigma^2} \\ &\cdot \exp \left(-v_0 \frac{(u - u^2) \sinh(P(u)t/2)/P(u)}{\cosh(P(u)t/2) + (\kappa - \sigma \rho u) \sinh(P(u)t/2)/P(u)} \right) \end{aligned} \quad (6.3)$$

where

$$P(u) = \sqrt{(\kappa - \rho c u)^2 + c^2(u - u^2)}.$$

Hence, the cumulants of X_t can be calculated by

$$c_n = \frac{d^n}{du^n} [\ln M_t(u)] \Big|_{u=0} \quad \text{for } n = 1, 2, \dots$$

In practice, higher derivatives in the expression can be calculated reasonably fast by using any symbolic calculation software.

6.2 Calculating Truncated Moment Generating Function using Gram-Charlier Expansions

Proposition 6.2.1. *Let Y be a random variable with a continuous density function $f(x)$ and finite cumulants $(c_k)_{k \in \mathbb{N}}$. Let $q_0 = 1$, $q_1 = q_2 = 0$ and*

$$q_n = \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 3, \\ k_1 + \dots + k_m = n}} \frac{c_{k_1} \dots c_{k_m}}{m! k_1! \dots k_m! \sqrt{c_2}^n}, \quad n \geq 3.$$

Suppose that e^{aY} is integrable where $a \in \mathbb{R}$. Then the following hold:

(a) The moment generating function of Y truncated below is given by

$$\mathbb{E}[e^{ax}I(Y \leq K)] = e^{aC_1} \sum_{n=0}^{\infty} q_n I_n \left(\frac{K - C_1}{\sqrt{C_2}}, a\sqrt{C_2} \right), \quad (6.4)$$

where $I_n = I_n(x, a)$ satisfies the following recurrence:

$$I_0(x, a) = e^{\frac{b^2}{2}} N(x - a) ; I_n(x, a) = aI_{n-1}(x, a) - H_{n-1}(x)\phi(x)e^{ax}.$$

(b) The moment generating function of Y truncated above is given by

$$\mathbb{E}[e^{ax}I(Y \geq K)] = e^{aC_1} \sum_{n=0}^{\infty} q_n J_n \left(\frac{K - C_1}{\sqrt{C_2}}, a\sqrt{C_2} \right), \quad (6.5)$$

where $J_n = J_n(x, a)$ satisfies the following recurrence:

$$J_0(x, a) = e^{\frac{b^2}{2}} N(a - x) ; J_n(x, a) = aJ_{n-1}(x, a) + H_{n-1}(x)\phi(x)e^{ax}.$$

Proof. We first prove part (a). Recall that

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right).$$

We have

$$\begin{aligned} \mathbb{E}[e^{ax}I(Y \leq K)] &= \int_{-\infty}^K e^{ax} \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right) dx \\ &= \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} \int_{-\infty}^K e^{ax} H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right) dx \\ &= \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} \int_{-\infty}^{\frac{K - C_1}{\sqrt{C_2}}} H_n(y)\phi(y)e^{a\sqrt{C_2}y + aC_1} \sqrt{C_2} dy \\ &= \sum_{n=0}^{\infty} q_n e^{aC_1} \int_{-\infty}^{\frac{K - C_1}{\sqrt{C_2}}} H_n(y)\phi(y)e^{a\sqrt{C_2}y} dy. \end{aligned}$$

Let $I_n(x, a) := \int_{-\infty}^x H_n(y)\phi(y)e^{ay} dy$ and write $I_n := I_n(x, a)$ for convenience. When

$n = 0$, we have

$$\begin{aligned}
I_0 &= \int_{-\infty}^x H_0(y)\phi(y)e^{ay} dy \\
&= \int_{-\infty}^x \phi(y)e^{ay} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} e^{ay} dy \\
&= \frac{e^{\frac{a^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(y-a)^2}{2}} dy \\
&= e^{\frac{a^2}{2}} N(x-a).
\end{aligned}$$

Note that

$$D[(D^{n-1}\phi(x))e^{ax}] = [D^n\phi(x)]e^{ax} + ae^{ax}[D^{n-1}\phi(x)].$$

We have

$$D[(-1)^{n-1}H_{n-1}(x)\phi(x)e^{ax}] = (-1)^n H_n(x)\phi(x)e^{ax} + (-1)^{n-1}aH_{n-1}(x)\phi(x)e^{ax}.$$

It follows that

$$H_{n-1}(x)\phi(x)e^{ax} = - \int_{-\infty}^x H_n(y)\phi(y)e^{ay} dy + a \int_{-\infty}^x H_{n-1}(y)\phi(y)e^{ay} dy.$$

Hence,

$$I_n = aI_{n-1} - H_{n-1}(x)\phi(x)e^{ax}.$$

Therefore, the proof of (a) is completed.

For the proof of part (b), we have

$$\begin{aligned}
\mathbb{E}[e^{ax}I(Y \geq K)] &= \int_K^\infty e^{ax} \sum_{n=0}^\infty \frac{q_n}{\sqrt{c_2}} H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right) dx \\
&= \sum_{n=0}^\infty \frac{q_n}{\sqrt{c_2}} \int_K^\infty e^{ax} H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right) dx \\
&= \sum_{n=0}^\infty \frac{q_n}{\sqrt{c_2}} \int_{\frac{K-c_1}{\sqrt{c_2}}}^\infty H_n(y) \phi(y) e^{a\sqrt{c_2}y+aC_1} \sqrt{c_2} dy \\
&= \sum_{n=0}^\infty q_n e^{aC_1} \int_{\frac{K-c_1}{\sqrt{c_2}}}^\infty H_n(y) \phi(y) e^{a\sqrt{c_2}y} dy
\end{aligned}$$

Let $J_n(x, a) := \int_x^\infty H_n(y) \phi(y) e^{ay} dy$ and write $J_n := J_n(x, a)$ for convenience. When $n = 0$, we have

$$\begin{aligned}
J_0 &= \int_x^\infty H_0(y) \phi(y) e^{ay} dy \\
&= \int_x^\infty \phi(y) e^{ay} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} e^{ay} dy \\
&= \frac{e^{\frac{a^2}{2}}}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{(y-a)^2}{2}} dy \\
&= e^{\frac{a^2}{2}} N(a-x).
\end{aligned}$$

Note that

$$D[(D^{n-1}\phi(x))e^{ax}] = [D^n\phi(x)]e^{ax} + ae^{ax}[D^{n-1}\phi(x)].$$

We have

$$D[(-1)^{n-1}H_{n-1}(x)\phi(x)e^{ax}] = (-1)^n H_n(x)\phi(x)e^{ax} + (-1)^{n-1}aH_{n-1}(x)\phi(x)e^{ax}.$$

It follows that

$$-H_{n-1}(x)\phi(x)e^{ax} = -\int_x^\infty H_n(y)\phi(y)e^{ay} dy + a \int_x^\infty H_{n-1}(y)\phi(y)e^{ay} dy.$$

Hence,

$$J_n = aJ_{n-1} + H_{n-1}(x)\phi(x)e^{ax}.$$

Therefore, the proof of (b) is completed. \square

6.3 Pricing Call Options under Heston's model

Let $X_t = \ln S_t - rt$ be the logarithm of the discounted stock price. We have $e^{X_t} = e^{-rt}S_t$. The price of a European call option with strike K is given by

$$C = \mathbb{E}[e^{-rt}(S_t - K)^+] = \mathbb{E}[(e^{X_t} - e^{-rt}K)^+].$$

Put $k = \ln K - rt$. We may rewrite the price as

$$C = \mathbb{E}[(e^{X_t} - e^k)^+].$$

Hence, the price of the call option can be calculated by the following formula:

$$C = \mathbb{E}[e^{X_t}I(X_t > k)] - e^k\mathbb{E}[I(X_t > k)].$$

The first term in the expression on right-hand-side is just a truncated moment generating function which can be calculated via equation (6.5) and the second term can be calculated by the formula given in Theorem 2.1.2. Therefore, we are able to calculate the price of any European call option whenever the moments (or cumulants) of the log-prices have analytical formulas.

To sum up, we have the following formula:

Theorem 6.3.1. *Suppose that the risk-neutral dynamics of the stock price follow Heston's model.*

$$\begin{cases} dS_t = S_t(rdt + \sqrt{V_t}dW_t^1) \\ dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^2 \end{cases} \quad (6.6)$$

with initial conditions $S_0 = s_0$ and $V_0 = v_0 \geq 0$ where $\kappa, \theta, \sigma > 0$ and $dW_t^1 dW_t^2 = \rho dt$, $\rho \in [-1, 1]$. Suppose that we have a European call option with strike K . Let $X_t = \ln S_t - rt$,

$k = \ln K - rt$ and c_n be the n^{th} -cumulant of X_t . Let $q_0 = 1$, $q_1 = q_2 = 0$ and

$$q_n = \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 3, \\ k_1 + \dots + k_m = n}} \frac{c_{k_1} \dots c_{k_m}}{m! k_1! \dots k_m! \sqrt{c_2}^n}, \quad n \geq 3.$$

Then the price of the call option price is equal to the following infinite sum:

$$e^{c_1} \sum_{n=0}^{\infty} q_n J_n \left(\frac{k - c_1}{\sqrt{c_2}}, \sqrt{c_2} \right) - e^k \left[N \left(\frac{c_1 - k}{\sqrt{c_2}} \right) + \sum_{n=3}^{\infty} (-1)^{n-1} q_n H_{n-1} \left(\frac{c_1 - k}{\sqrt{c_2}} \right) \phi \left(\frac{c_1 - k}{\sqrt{c_2}} \right) \right]$$

where $J_n = J_n(x, a)$ satisfies the following recurrence:

$$J_0(x, a) = e^{\frac{b^2}{2}} N(a - x); \quad J_n(x, a) = a J_{n-1}(x, a) + H_{n-1}(x) \phi(x) e^{ax}.$$

6.4 A Monte Carlo Simulation Method for Heston's Model

In order to investigate the accuracy of our result, we calculate the options prices based on Monte Carlo method. In the second equation of the Heston system, it is a CIR-type mean-reverting process. Thus, it is tempting to use the exact simulation method as the distribution of V_t is known as a non-central chi-square distribution. However, it is hard to include the correlation of the Brownian motions as Cholesky decomposition does not work in this case.

Inspired by Alfonsi's result [1], we use the implicit scheme for $(\sqrt{V_t})$ and an exact simulation for (S_t) . To make it clear, we first obtain the SDE for $(\sqrt{V_t})$ by Itô's lemma:

$$d\sqrt{V_t} = \frac{\kappa\theta - \sigma^2/4}{2\sqrt{V_t}} dt - \frac{\kappa}{2} \sqrt{V_t} dt + \frac{\sigma}{2} dW_t^2.$$

Let the time grid be $\{t_0, \dots, t_n\}$ where $t_0 = 0$, $t_n = T$ and $t_i = \frac{iT}{n}$ for $i = 1, \dots, n$. We obtain the following equation by impliciting the drift term:

$$\sqrt{V_{t_{i+1}}} - \sqrt{V_{t_i}} = \left(\frac{\kappa\theta - \sigma^2/4}{2\sqrt{V_{t_{i+1}}}} - \frac{\kappa}{2} \sqrt{V_{t_{i+1}}} \right) \frac{T}{n} + \frac{\sigma}{2} (W_{t_{i+1}} - W_{t_i}).$$

After simplification, we obtain a quadratic equation in $\sqrt{V_{t_{i+1}}}$,

$$\left(1 + \frac{\kappa T}{2n}\right) (\sqrt{V_{t_{i+1}}})^2 - \left[\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right] \sqrt{V_{t_{i+1}}} - \left(\frac{\kappa\theta - \sigma/4}{2}\right) \frac{T}{n} = 0,$$

which has only one positive root when $\sigma^2 < 4\kappa\theta$, namely:

$$V_{t_{i+1}} = \left[\frac{\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}} + \sqrt{\left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right)^2 + 4\left(1 + \frac{\kappa T}{2n}\right)\left(\frac{\kappa\theta - \sigma^2/4}{2}\right)\frac{T}{n}}}{2\left(1 + \frac{\kappa T}{2n}\right)} \right]^2.$$

Since $\frac{1}{(1+x)^2} \approx 1 - 2x$ for small x , we have

$$V_{t_{i+1}} \approx \frac{1}{4} \left(1 - \frac{\kappa T}{n}\right) \left\{ 2 \left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right)^2 + 4 \left(1 + \frac{\kappa T}{2n}\right) \left(\frac{\kappa\theta - \sigma^2/4}{2}\right) \frac{T}{n} + 2 \left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right) \sqrt{\left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right)^2 + 4 \left(1 + \frac{\kappa T}{2n}\right) \left(\frac{\kappa\theta - \sigma^2/4}{2}\right) \frac{T}{n}} \right\}.$$

Moreover, note that for small $x, y > 0$, we have

$$x\sqrt{x^2 + y} = x^2 \sqrt{1 + \frac{y}{x^2}} \approx x^2 \left(1 + \frac{y}{2x^2}\right) = x^2 + \frac{y}{2}.$$

It follows that

$$2x^2 + y + 2x\sqrt{x^2 + y} \approx 4x^2 + 2y.$$

Thus, we may further approximate $V_{t_{i+1}}$ by

$$V_{t_{i+1}} \approx \frac{1}{4} \left(1 - \frac{\kappa T}{n}\right) \left\{ 4 \left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right)^2 + 8 \left(1 + \frac{\kappa T}{2n}\right) \left(\frac{\kappa\theta - \sigma^2/4}{2}\right) \frac{T}{n} \right\}.$$

We now fix V_{t_i} and conserve the terms in $\frac{T}{n}$, $(W_{t_{i+1}} - W_{t_i})$ and $(W_{t_{i+1}} - W_{t_i})^2$ using a Taylor expansion:

$$\begin{aligned} V_{t_{i+1}} &\approx \left(1 - \frac{\kappa T}{n}\right) \left\{ \left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right)^2 + \left(1 + \frac{\kappa T}{2n}\right) \left(\kappa\theta - \frac{\sigma^2}{4}\right) \frac{T}{n} \right\} \\ &\approx V_{t_i} \left(1 - \frac{\kappa T}{n}\right) + \sigma(W_{t_{i+1}} - W_{t_i})\sqrt{V_{t_i}} + \frac{\sigma^2}{4}(W_{t_{i+1}} - W_{t_i})^2 + \left(\kappa\theta - \frac{\sigma^2}{4}\right) \frac{T}{n} \\ &\approx V_{t_i} \left(1 - \frac{\kappa T}{2n}\right)^2 + \sigma(W_{t_{i+1}} - W_{t_i})\sqrt{V_{t_i}} + \left(\frac{\sigma(W_{t_{i+1}} - W_{t_i})}{2\left(1 - \frac{\kappa T}{2n}\right)}\right)^2 + \left(\kappa\theta - \frac{\sigma^2}{4}\right) \frac{T}{n} \\ &= \left(\sqrt{V_{t_i}} \left(1 - \frac{\kappa T}{2n}\right) + \frac{\sigma(W_{t_{i+1}} - W_{t_i})}{2\left(1 - \frac{\kappa T}{2n}\right)}\right)^2 + \left(\kappa\theta - \frac{\sigma^2}{4}\right) \frac{T}{n} \end{aligned}$$

To sum up, we have the following algorithm for Heston's model:

1. Set $S \leftarrow s_0, V \leftarrow v_0$.
2. Generate a pair of independent $Z_1, Z_2 \sim N(0, 1)$.
3. Let $U_1 = \sqrt{1 - \rho^2}Z_1 + \rho Z_2$ and $U_2 = Z_2$.

4. Generate

$$V \leftarrow \left[\sqrt{V} \left(1 - \frac{kT}{2n} \right) + \frac{\sigma \left(\sqrt{\frac{T}{n}} U_2 \right)}{2 \left(1 - \frac{kT}{2n} \right)} \right]^2 + \left(\kappa \theta - \frac{\sigma^2}{4} \right) \frac{T}{n}.$$

5. Generate

$$S \leftarrow \exp \left(\left(r - \frac{\sigma^2}{2} \right) \frac{T}{n} + \sqrt{V} \sqrt{\frac{T}{n}} U_1 \right).$$

We assume the following parameters in the Heston model to demonstrate the mean behaviors of the scenarios generated by the Alfonsi's scheme.

<i>Parameters</i>	<i>Values</i>
S_0	100
V_0	0.03
κ	0.5, 1, 1.5
θ	0.05
σ	0.30
ρ	-0.45
T	10
r	0.04

We use 250 time steps per year and generate 10,000 scenarios. The results are given in Figures 5.1 and 5.2.

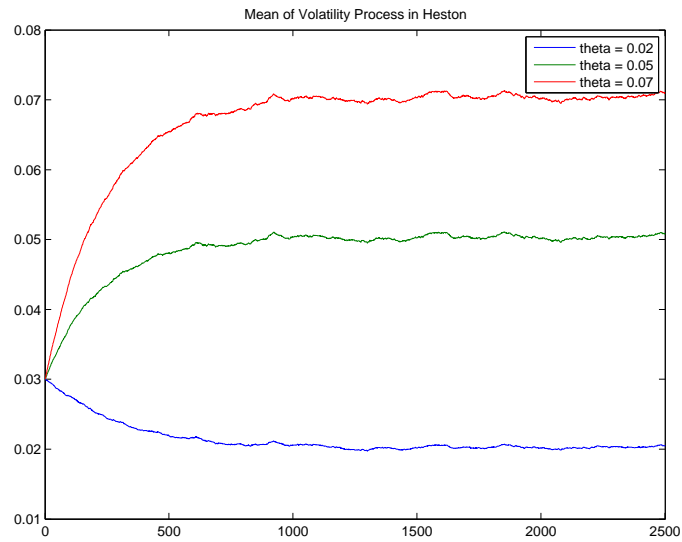


Figure 6.1: Mean reversion levels of Heston's Model using Alfonsi scheme

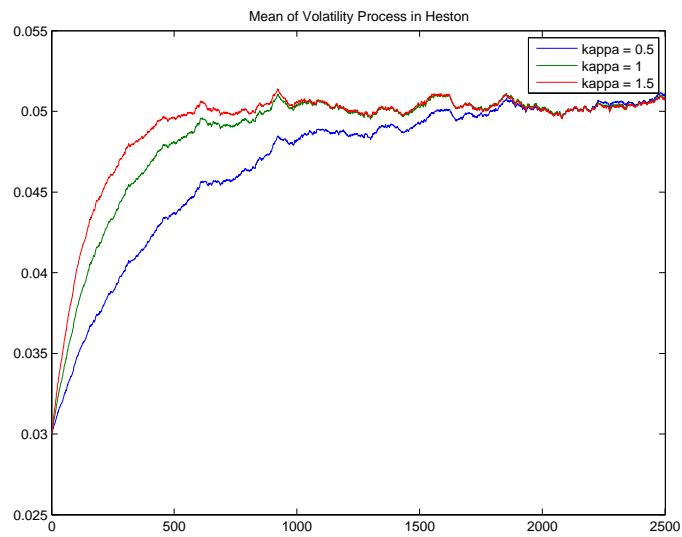


Figure 6.2: Mean reversion speeds of Heston's Model using Alfonsi scheme

6.5 Numerical results

We consider the following parameters in the Heston's model:

<i>Parameters</i>	<i>Values</i>
S_0	100
V_0	0.03
K	{50, 51, ..., 149, 150}
κ	0.15
θ	0.05
σ	0.05
ρ	-0.55
T	1
r	0.04

We take $N = 1,000,000$ Scenarios and 250 time steps per year for the Monte Carlo simulation of call option prices.

We approximate the price of the call option using only first N terms in the Gram-Charlier expansions and denote them by $GC(N)$. We also study $GC(ND)$ where $N = 3, 4, 5$. They are just $GC7$'s with $C_{N+1} = \dots = C_7 = 0$ where $N = 3, 4, 5$.

Since the Fourier Transform (FT) approach is popular in the industry and academic for calculating option price in Heston's model, we also include the FT results in our graph for comparison purpose.

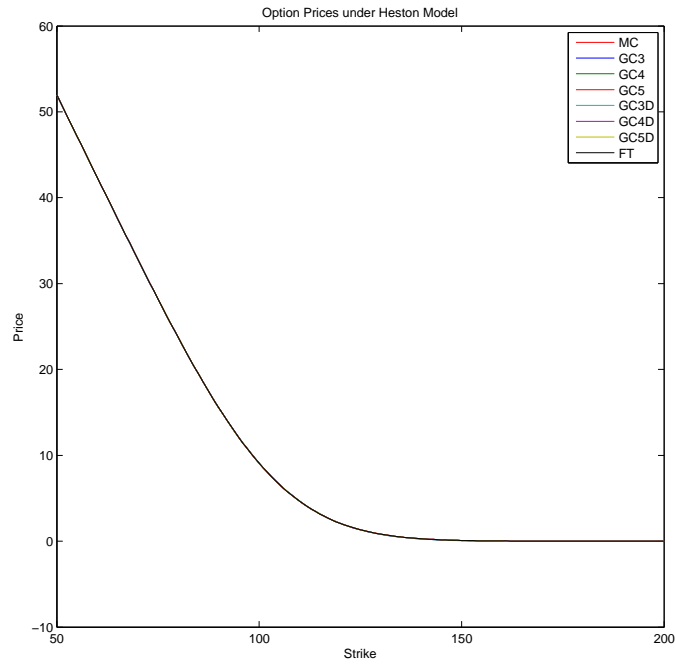


Figure 6.3: Call Option Prices under Heston's Model ($T = 1$)

Selected Numerical results:

Value/ Strike	50	80	90	100	110	120	150
MC	51.9653	23.7206	15.5576	9.0765	4.6458	2.0766	0.0872
FT	51.9612	23.7138	15.5526	9.0761	4.6510	2.0835	0.0880
GC3	51.9603	23.7141	15.5635	9.0932	4.6667	2.0889	0.0694
GC4	51.9608	23.7216	15.5510	9.0620	4.6424	2.0883	0.0893
GC5	51.9608	23.7199	15.5469	9.0612	4.6469	2.0939	0.0871
GC3D	51.9604	23.7076	15.5648	9.1068	4.6755	2.0834	0.0674
GC4D	51.9609	23.7154	15.5523	9.0750	4.6508	2.0831	0.0874
GC5D	51.9609	23.7136	15.5482	9.0742	4.6553	2.0887	0.0852

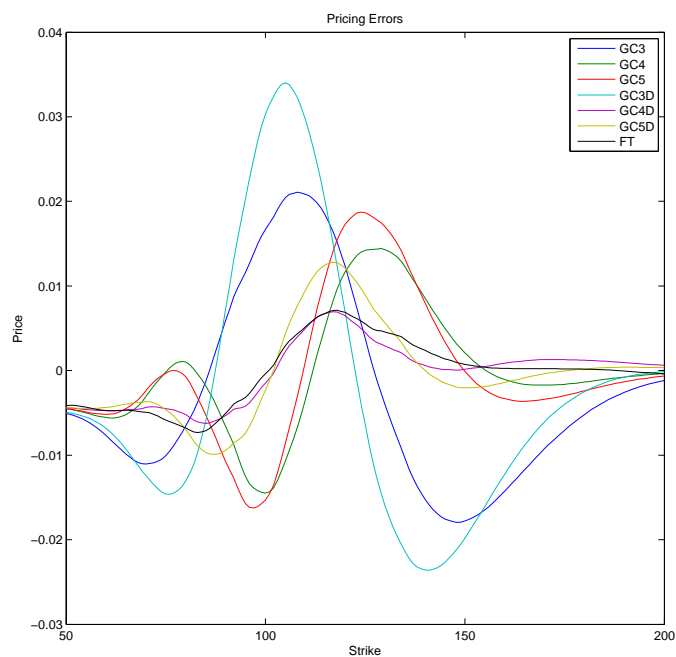


Figure 6.4: Pricing Errors of Call Options under Heston's Model ($T = 1$)

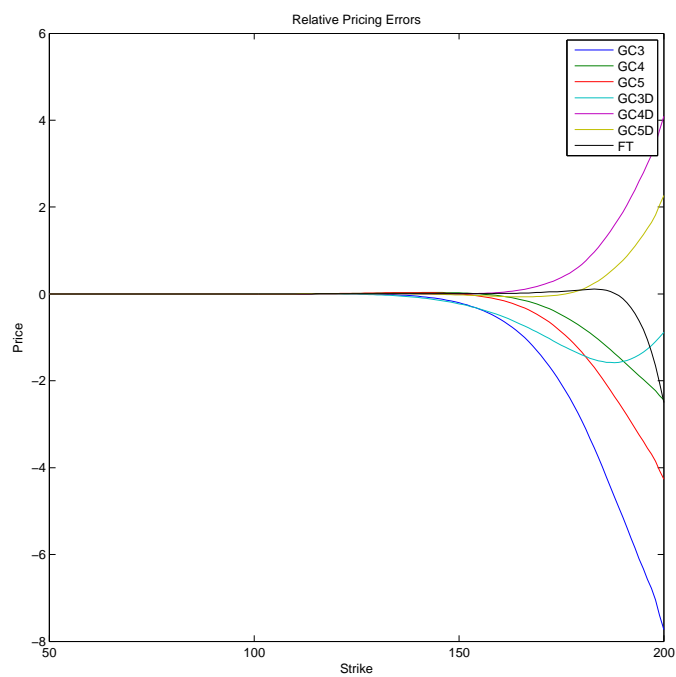


Figure 6.5: Relative Errors of Call Option Prices under Heston's Model ($T = 1$)

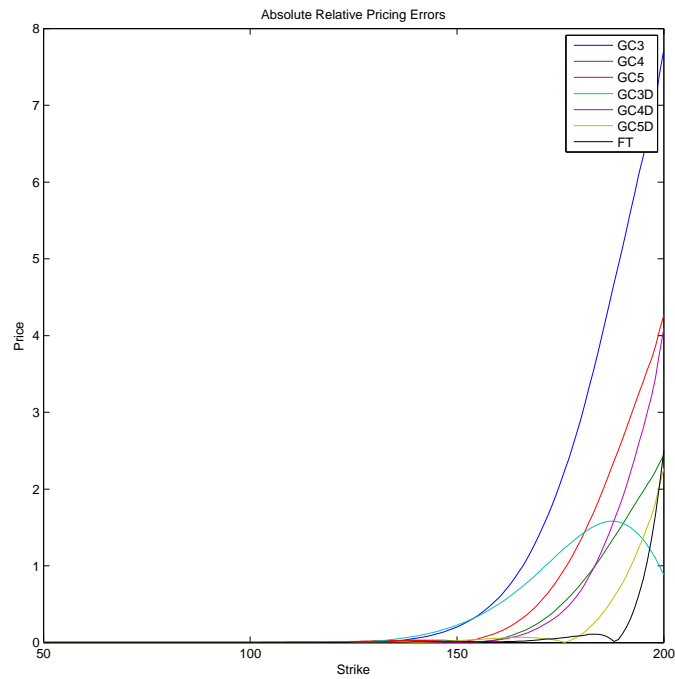


Figure 6.6: Absolute Relative Errors of Call Option Prices under Heston's Model ($T = 1$)

We see from Figures 5.3, 5.4, 5.5 and 6.6 that the (relative) errors are generally very small for out-of-the-money options. The GC4D and GC5D approach are generally better than other approximations. They occasionally outperform the FT approach.

We test the result with other parameters where the time-to-expiry is smaller than the previous one.

<i>Parameters</i>	<i>Values</i>
S_0	100
V_0	0.03
K	{50, 51, ..., 149, 150}
κ	0.15
θ	0.05
σ	0.05
ρ	-0.55
T	4
r	0.04

Selected Numerical results:

Value/ Strike	50	80	90	100	110	120	150
MC	57.5787	34.5536	28.0928	22.4256	17.5883	13.5663	5.6970
FT	57.5982	34.5743	28.1104	22.4405	17.6010	13.5764	5.7030
GC3	57.5589	34.5966	28.1521	22.4947	17.6604	13.6328	5.6958
GC4	57.6126	34.5736	28.0705	22.3704	17.5222	13.5110	5.7302
GC5	57.6134	34.5410	28.0416	22.3582	17.5335	13.5458	5.7918
GC3D	57.5517	34.5983	28.1902	22.5629	17.7389	13.6991	5.6607
GC4D	57.6065	34.5750	28.1026	22.4278	17.5882	13.5668	5.7007
GC5D	57.6073	34.5424	28.0737	22.4155	17.5995	13.6015	5.7623

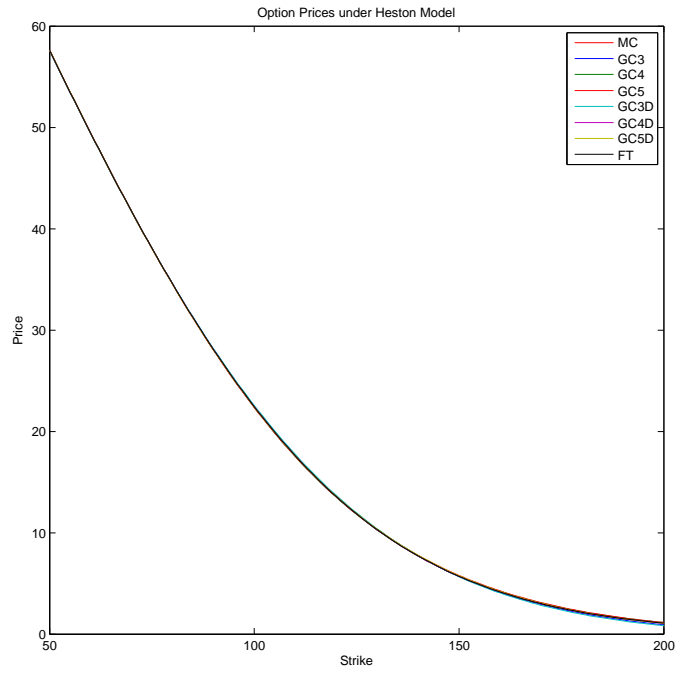


Figure 6.7: Call Option Prices under Heston's Model ($T = 4$)

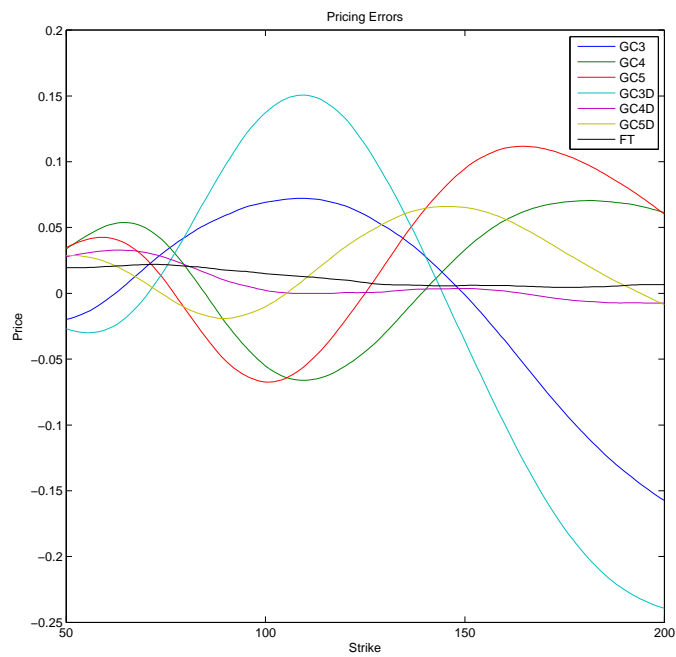


Figure 6.8: Pricing Errors of Call Options under Heston's Model ($T = 4$)

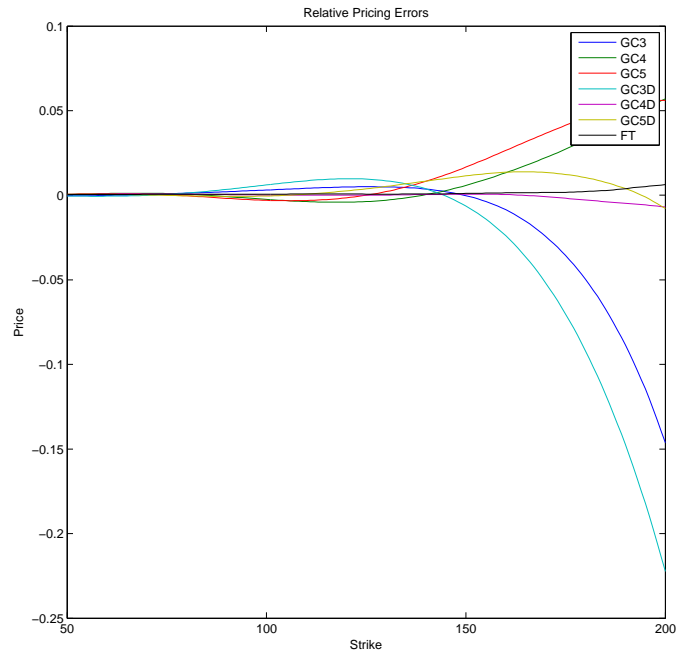


Figure 6.9: Relative Errors of Call Option Prices under Heston's Model ($T = 4$)

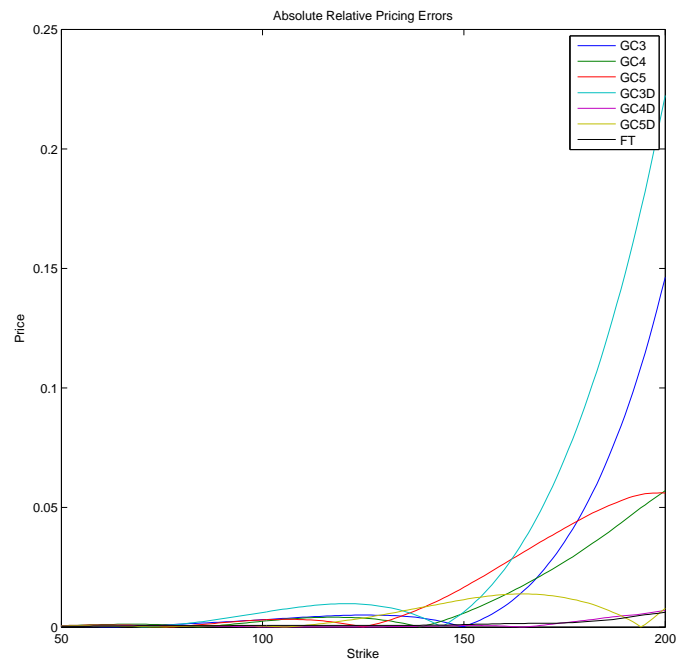


Figure 6.10: Absolute Relative Errors of Call Option Prices under Heston's Model ($T = 4$)

We see from Figures 5.7, 5.8, 5.9 and 5.10 that the (relative) errors are generally very small for out-of-the-money options. The GC4D approach is generally better than other approximations. It outperforms the FT approach when the option is in the at-the-money region.

Increasing the number of terms in the approximation formula does not necessarily increase the accuracy systematically since Gram-Charlier expansions are orthogonal series. Empirical results show that *GC4D* outperforms other methods in general.

Chapter 7

Conclusion

The thesis discussed several applications of Gram-Charlier expansions in pricing swaptions and European call options. It is important to point out that Gram-Charlier expansions can actually be used in any affine-term structure model. Our work on the extension of this method from CIR2 to CIR2++ can actually be generalized to any affine-term structure++ model (i.e. models with the fitting of the initial term structure). Empirical results show that GC3 (Gram-Charlier expansions up to the third cumulants) gives the most efficient and accurate approximation for the swaption prices.

We discussed a procedure to apply the Gram-Charlier approach to general models in Chapter 4. The models are reasonably simple. For example, the drift and diffusion terms are polynomials. Moments can be found by solving a system of ODEs, which is derived by Fubini's Theorem and martingale properties of Itô integrals as shown in section 4.2. This allows us to calculate prices of any European-type derivatives.

For Heston's model, the European call option price is usually obtained by a Fourier Transform; see [7]. This method is proven to be accurate and efficient. For a given set of parameters, we are able to use a fast Fourier transform to calculate the option prices for different strikes. While the logarithm of strike prices is assumed to be equally spaced, the strike prices themselves cannot be equally spaced. However, in our method, cumulants are fixed whenever the parameters are given for the model. Thus, the option prices with

different strikes can be calculated in a parallel manner since Gram-Charlier expansions are easily implemented. For example, we are able to calculate 10,000 option prices with arbitrary strikes within 0.003 seconds. Therefore, our approach is more efficient if the parameters are already calibrated or given in advance.

In principle, we may extend our approach to any stochastic volatility model with a reasonable complexity. To be precise, if the moment generating functions (or moments themselves) can be found in a model, our approach can be readily applied to it.

However, there are a few limitations in our approach. The main assumption in Gram-Charlier expansions is that the cumulants of the random variable are finite. This assumption is stronger than we expected in stochastic modeling. For example, pure jump processes like variance gamma and CGMY do not process this property in general. In these cases, our approach is totally useless. As a result, our approach cannot beat the FT approach in general since they can be applied to this kind of models. Also, the error in the approximations is hard to estimate rigorously since the Gram-Charlier expansions are orthogonal series. There is no guarantee that adding finitely more terms will make the approximation better. A lot of testings are therefore needed, especially for pricing purpose.

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