# Modularity and Structure in Matroids

by

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## AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

# Abstract

This thesis concerns sufficient conditions for a matroid to admit one of two types of structural characterization: a representation over a finite field or a description as a frame matroid.

We call a restriction $N$ of a matroid $M$ modular if, for every flat $F$ of $M$,

$$r_M(F) + r(N) = r_M(F \cap E(N)) + r_M(F \cup E(N)).$$

A consequence of a theorem of Seymour is that any 3-connected matroid with a modular $U_{2,3}$-restriction is binary. We extend this fact to arbitrary finite fields, showing that if $N$ is a modular rank-3 restriction of a vertically 4-connected matroid $M$, then any representation of $N$ over a finite field extends to a representation of $M$.

We also look at a more general notion of modularity that applies to minors of a matroid, and use it to present conditions for a matroid with a large projective geometry minor to be representable over a finite field. In particular, we show that a 3-connected, representable matroid with a sufficiently large projective geometry over a finite field $\mathrm{GF}(q)$ as a minor is either representable over $\mathrm{GF}(q)$ or has a $U_{2,q^2+1}$-minor.

A second result of Seymour is that any vertically 4-connected matroid with a modular $M(K_4)$-restriction is graphic. Geelen, Gerards, and Whittle partially generalized this from $M(K_4)$ to larger frame matroids, showing that any vertically 5-connected, representable matroid with a rank-4 Dowling geometry as a modular restriction is a frame matroid. As with projective geometries, we prove a version of this result for matroids with large Dowling geometries as minors, providing conditions which imply that they are frame matroids.

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# Chapter 1

# Introduction

For many types of combinatorial objects, we can often prove that each object either admits a well-understood structural description or has a small obstruction to it. A famous example is Kuratowski's theorem that any graph is either planar or has one of two specific graphs, $K_5$ and $K_{3,3}$, as a minor [34]. A remarkable extension of this fact is Robertson and Seymour's graph minor theorem, which says that every class of graphs closed under the minor relation has a finite set of excluded minors, or minor-minimal graphs not in the class [43].

Moving to the realm of matroids, Tutte showed that every matroid is either graphic or has one of five particular non-graphic matroids as a minor [53]. Graphicness is one of the two most common types of structural description we encounter for matroids; the other is that of representability over a fixed finite field. Tutte showed that there is a unique excluded minor for the class of binary matroids [51], and there are four and seven excluded minors, respectively, for the matroids representable over the fields of order three and four [4, 48, 14]. For each larger finite field $\mathbb{F}$, Geelen, Gerards, and Whittle have recently announced a proof of Rota's Conjecture [44], which asserts that there are finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.

In this thesis, we consider a variant of these excluded-minor questions for matroids, exhibiting sufficient conditions for a matroid to be representable over a finite field $\mathbb{F}$. When a matroid $M$ has an $\mathbb{F}$-representable minor with particular properties, we show that either $M$ itself is $\mathbb{F}$-representable or that it has a specific small minor obstruction. An example is the following consequence of one of our main theorems.

**Theorem 1.0.1.** *Any vertically* $4$*-connected matroid with* $\mathrm{PG}(2,q)$ *as a restriction is either* $\mathrm{GF}(q)$*-representable or has a* $U_{2,q^2+1}$*-minor.*

In addition to representability, we will also investigate conditions under which a matroid belongs to a certain class of ‘graph-like’ matroids called the frame matroids over a field, which are those that have a representation with at most two non-zero entries per column. A special case of one of our theorems is that if a highly connected representable matroid $M$ has a particular large frame matroid called a Dowling geometry as a minor, then $M$ is either also a frame matroid or has a small minor obstruction to being a frame matroid.

In the next section, we summarize the concepts of matroid theory that we refer to in this chapter. Next, we define varieties of matroids, and then we present two striking theorems of Seymour that give sufficient conditions for a matroid to be binary and to be graphic. We then outline the main results of the thesis, which mainly consist of extensions of these two theorems in various directions. Finally, we discuss two topics to which our results relate: excluded minors for varieties of matroids and growth-rate functions of minor-closed classes.

## 1.1 Matroids

Whitney introduced matroids as a way to capture the linear dependence properties of finite subsets of a finite-dimensional vector space [55]. A comprehensive reference on matroid theory can be found in Oxley [37].

We let $\mathbb{F}$ be a field, $E$ a finite set, and $A \in \mathbb{F}^{k\times E}$ a $k \times |E|$ matrix whose columns are indexed by the elements of $E$. For each $X \subseteq E$, we denote by $A|X$ the restriction of $A$ to the set of columns $X$. The rank of the matrix $A|X$, $\text{rank}(A|X)$, is equal to the dimension of the subspace of $\mathbb{F}^k$ spanned by the columns indexed by $X$. We abbreviate $\text{rank}(A|X)$ as $r(X)$; this function always satisfies three properties:

(R1) $0 \leq r(X) \leq |X|$, for all $X \subseteq E$,

(R2) $r(X) \leq r(Y)$, for all $X \subseteq Y \subseteq E$, and

(R3) $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$, for all $X, Y \subseteq E$.

We take these three properties as axioms to define a more general class of objects. A **matroid** is a pair $M = (E, r)$ consisting of a finite set $E$ and a function $r : 2^E \to \mathbb{Z}$ that satisfies (R1)-(R3). We call $E$ the **ground set** of $M$ and $r(X)$ the **rank** of a set $X \subseteq E(M)$, and to avoid ambiguity we will always write $E(M)$ and $r_M$ for the ground set and rank function of a matroid $M$. The rank of a matroid $M$, denoted $r(M)$, is equal to $r_M(E(M))$.

An **$\mathbb{F}$-representation** of a matroid $M$ is an $\mathbb{F}$-matrix $A$ with columns indexed by $E(M)$ such that for all $X \subseteq E(M)$, $r_M(X)$ is actually the rank of the matrix $A|X$. We call a matroid **representable over** $\mathbb{F}$ or $\mathbb{F}$**-representable** if it has an $\mathbb{F}$-representation, and **representable** if it has a representation over some field. Conversely, when $A$ is an $\mathbb{F}$-matrix, we write $M_{\mathbb{F}}(A)$ for the matroid with $A$ as an $\mathbb{F}$-representation. The three axioms (R1)-(R3) turn out to be quite general, and the class of all matroids includes many examples that do not correspond to the columns of a matrix; these are the non-representable matroids. Moreover, a representable matroid may in general be representable over more than one field, and even for a single field $\mathbb{F}$ there are many different $\mathbb{F}$-representations of the same matroid.

One simple way to get different representations of a matroid is to multiply a column of an $\mathbb{F}$-representation by a non-zero element of $\mathbb{F}$, because the rank of a matrix does not change when we scale its columns. Thus we think of $\mathbb{F}$-representable matroids as lying in the projective space corresponding to a vector space over $\mathbb{F}$, and for this reason we borrow many terms from projective geometry. The **closure** of a set $X$ in a matroid $M$ is the set $\mathrm{cl}_M(X) = \{e \in E(M) : r_M(X \cup \{e\}) = r_M(X)\}$ and $X$ is a **flat**, or **closed** set, if $\mathrm{cl}_M(X) = X$. The unique minimal flat containing $X$ is $\mathrm{cl}_M(X)$. A **point**, **line**, or **plane** is a flat of rank one, two, or three, respectively, and a **hyperplane** is a flat of rank $r(M) - 1$.

A **loop** in $M$ is an element $e$ such that $r_M(\{e\}) = 0$, and two elements $e$ and $f$ are **parallel** if neither are loops and $r_M(\{e, f\}) = 1$. Being parallel is an equivalence relation, so a **parallel class** of $M$ is a maximal set of pairwise parallel elements. A matroid is called **simple** if it has no loops and no pairs of parallel elements, and the simplification, $\mathrm{si}(M)$, of $M$ is the matroid obtained by deleting all but one element from each parallel class of elements. We also refer to matroids of rank two or three as lines and planes, and we denote the simple line with $n$ elements, which is unique up to isomorphism, by $U_{2,n}$.

A set $X \subseteq E(M)$ is called **independent** if $r_M(X) = |X|$, and **dependent** otherwise. A **circuit** of a matroid is a minimal dependent set. A **basis** is an independent set of maximum size, so $X$ is a basis if and only if it is independent and $\mathrm{cl}_M(X) = E(M)$.

### 1.1.1 Minors and duality

When $M$ is a matroid and $D \subseteq E(M)$, $M \backslash D$ is the matroid with ground set $E(M) \setminus D$ and rank function given by $r_{M \backslash D}(X) = r_M(X)$ for each $X \subseteq E(M) \setminus D$. When $C \subseteq E(M)$, $M/C$ is the matroid with ground set $E(M) \setminus C$ and rank function given for each $X \subseteq E(M) \setminus C$ by

$$r_{M/C}(X) = r_M(X \cup C) - r_M(X).$$

We say that $M\backslash D$ is obtained from $M$ by **deleting** $D$ and $M/C$ is obtained from $M$ by **contracting** $C$. When $e \in E(M)$, we abbreviate $M\backslash\{e\}$ and $M/\{e\}$ by $M\backslash e$ and $M/e$, and we do the same for larger sets, for example $M\backslash\{e, f\} = M\backslash e, f$ and $M/\{e, f\} = M/e, f$. Deletion and contraction are commutative operations, so $(M/C)\backslash D = (M\backslash D)/C$ for any disjoint $C, D \subseteq E(M)$. We call any matroid obtained from $M$ by deletion and contraction a **minor** of $M$. If it is obtained only by deletion, we call it a **restriction**, and we write $M|X$ for $M\backslash(E(M) \setminus X)$. An $N$**-minor** of $M$ is a minor of $M$ that is isomorphic to a matroid $N$, and an $N$**-restriction** of $M$ is a restriction of $M$ that is isomorphic to $N$.

The **dual** of a matroid $M$ is the matroid $M^*$ with the same ground set whose rank function is defined for each $X \subseteq E(M)$ by

$$r_{M^*}(X) = |X| - r(M) + r_M(E(M) \setminus X).$$

It is straightforward to check that $r_{M^*}$ satisfies (R1)-(R3). Duality can be an extremely useful property. For example, for any disjoint sets $C, D \subseteq E(M)$, the following identity holds:

$$(M/C\backslash D)^* = M^*\backslash C/D.$$

We define a **cocircuit** in $M$ to be a circuit of $M^*$. A set $X \subseteq E(M)$ is a cocircuit if and only if it is the complement of a hyperplane.

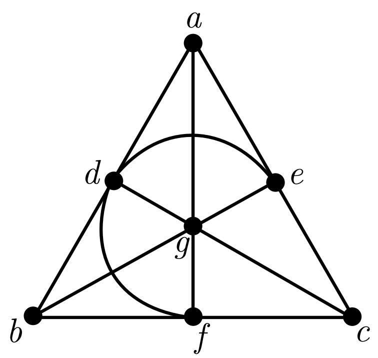
For any field $\mathbb{F}$, the class of $\mathbb{F}$-representable matroids is closed under taking minors. Suppose $A$ is an $\mathbb{F}$-representation of a matroid $M$. Then for any $e \in E(M)$, $A|(E(M) \setminus \{e\})$ is an $\mathbb{F}$-representation of $M\backslash e$. When $e$ is a loop of $M$, then $M/e = M\backslash e$. For contraction of non-loops, we need the following observation: any matrix obtained from $A$ by applying row operations (multiplying on the left by an invertible matrix) is also an $\mathbb{F}$-representation of $M$. So for any non-loop element $e$, we may assume that $A|\{e\}$ is a standard basis vector of $\mathbb{F}^{r(M)}$. It has exactly one non-zero entry, and deleting the row and column of this entry from $A$ results in an $\mathbb{F}$-representation of $M/e$.

### 1.1.2 Projective geometries

For each prime power $q$, we write $\mathrm{GF}(q)$ for the finite field of order $q$. The rank-$n$ **projective geometry** over $\mathrm{GF}(q)$, denoted $\mathrm{PG}(n-1, q)$, is the simple matroid represented over $\mathrm{GF}(q)$ by a matrix whose set of columns consists of a non-zero element from each of the $(q^n-1)/(q-1)$ one-dimensional subspaces of $\mathrm{GF}(q)^n$. Every simple rank-$n$ matroid representable over $\mathrm{GF}(q)$ is isomorphic to a restriction of $\mathrm{PG}(n-1, q)$. We note that the rank-2 projective geometry over

GF($q$) is the line $U_{2,q+1}$, and so the $(q+2)$-point line $U_{2,q+2}$ is not representable over this field.

A rank-3 projective geometry is called a **projective plane**; see Figure 1.1 for the projective plane over GF(2), also known as the Fano plane, and a matrix that represents it over GF(2). A matroid representable over the field GF(2) is called **binary**. A matroid is binary if and only if it has no $U_{2,4}$-minor [51].




$$\begin{array}{c} \begin{matrix} a & b & c & d & e & f & g \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Figure 1.1: The binary projective plane and a GF(2)-representation

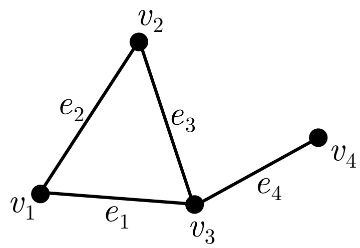
When the order of the field is not relevant, we write PG$(n-1, \mathbb{F})$ for the rank-$n$ projective geometry over a finite field $\mathbb{F}$.

### 1.1.3 Graphic matroids

When $G$ is a graph, we define the rank of a set of edges $X \subseteq E(G)$ to be the size of the largest subset of $X$ that is the edge set of a forest in $G$. The edge set $E(G)$ of $G$ and this rank function form a matroid, which we call the **cycle matroid** of $G$ and denote $M(G)$. The independent sets of $M(G)$ are the edge sets of forests in $G$ and the circuits of $M(G)$ are the edge sets of cycles, the parallel pairs, and the loops of $G$. A matroid that is the cycle matroid of a graph is called **graphic**.

A graphic matroid is representable over any field $\mathbb{F}$. Denote the vertices of a graph $G$ by $v_1, \ldots, v_n$, and let $\chi_{v_1}, \ldots, \chi_{v_n}$ be the standard basis vectors of the vector space $\mathbb{F}^{V(G)}$. We arbitrarily orient each edge of $G$, so that one of its ends is called its **head** and the other its **tail**; when $e$ is a loop, its unique end is both its head and its tail. The signed incidence matrix of $G$ is the matrix $A \in \mathbb{F}^{V(G) \times E(G)}$ such that for each $e \in E(G)$ with head $u$ and tail $v$, $A|\{e\} = \chi_u - \chi_v$. This matrix is a representation of the matroid $M(G)$ over $\mathbb{F}$. See Figure 1.2 for an example of a graph $G$ and a representation of $M(G)$.

The class of graphic matroids is not only minor closed, but minors of graphs correspond to minors of their cycle matroids. When $G$ is a graph and $e \in E(G)$,




$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

Figure 1.2: A graph and a representation of its cycle matroid

we have $M(G\backslash e) = M(G)\backslash e$ and $M(G/e) = M(G)/e$. On the other hand, the class of graphic matroids is not closed under duality. A matroid whose dual is graphic is called **cographic**. Another important fact about graphic matroids is that a matroid and its dual are both graphic if and only if it is the cycle matroid of a planar graph—this follows from Tutte's theorem [53] that a graphic matroid is cographic if and only if it has no $M(K_5)$- or $M(K_{3,3})$-minor, along with Kuratowski's theorem that any non-planar graph has either a $K_5$- or a $K_{3,3}$-minor. In fact, matroid duality corresponds to planar duality of graphs: when $G^*$ is a planar dual of a graph $G$, $M(G)^* = M(G^*)$.

### 1.1.4 Frame and Dowling matroids

For a group $\Gamma$, a **$\Gamma$-labelled graph** $G$ is a pair $(\vec{G}, \gamma_G)$ where $\vec{G}$ is an oriented graph and $\gamma_G \in \Gamma^{E(\vec{G})}$. We use multiplicative notation for $\Gamma$. A $\Gamma$-labelled graph is also known as a *gain graph* [57]. We set $V(G) = V(\vec{G})$ and $E(G) = E(\vec{G})$.

Let $\mathbb{F}$ be a field and $G$ an $\mathbb{F}^\times$-labelled graph with vertices $v_1, \ldots, v_k$. We let $\chi_{v_1}, \ldots, \chi_{v_k}$ be the standard basis vectors of $\mathbb{F}^{V(G)}$ and $A \in \mathbb{F}^{V(G)\times E(G)}$ the matrix such that for each $e \in E(G)$ with head $u$ and tail $v$, $A|\{e\} = \chi_u - \gamma_G(e)\chi_v$. We call the matroid $M_{\mathbb{F}}(A)$ the matroid **represented** by $G$ and write $M(G) = M_{\mathbb{F}}(A)$. A **frame matroid over** $\mathbb{F}$ is any matroid represented by an $\mathbb{F}^\times$-labelled graph. Note that these are precisely the matroids with an $\mathbb{F}$-representation with at most two non-zero entries per column. The frame matroids over a field are part of a more general class of frame matroids that was introduced by Zaslavsky [58].

We define another class of matroids based on $\Gamma$-labelled graphs, where $\Gamma$ is any finite group. For each edge $e$ of a $\Gamma$-labelled graph $G$ that is incident with a vertex $v$, we set $\gamma_G(v, e) = \gamma_G(e)$ when $v$ is the head of $e$, and $\gamma_G(v, e) = \gamma_G(e)^{-1}$ otherwise. Let $C$ be a cycle of $G$ with a vertex $v_1$ and an edge $e_1$ incident with it. If we number the rest of its vertices and edges $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$, appearing in this order as we traverse the cycle, then we define $\gamma_G(C, v_1, e_1) =$

$\gamma_G(v_1, e_1) \cdots \gamma_G(v_k, e_k)$. We call $C$ a **balanced cycle** when $\gamma_G(C, v_1, e_1) = 1$; this is well-defined because $\gamma_G(C, v_1, e_1) = 1$ if and only if $\gamma_G(C, v, e) = 1$ for any choice of vertex $v$ and edge $e$ incident with $v$. A component of the graph $G$ is called balanced if all of its cycles are balanced, and we write $\text{bal}(G)$ for the number of balanced components of $G$.

For any set $X$ of edges of a $\Gamma$-labelled graph $G$, we let $G[X]$ denote the subgraph of $G$ consisting of $X$ and all vertices incident with an element of $X$. If we define the rank of $X$ to be

$$r(X) = |V(G[X])| - \text{bal}(G[X]),$$

then the pair $(E(G), r)$ is a matroid (see [37, Section 6.10]). Such a matroid is called a **Dowling matroid over** $\Gamma$, or a $\Gamma$-Dowling matroid. When $\Gamma$ is a subgroup of the multiplicative group of some field $\mathbb{F}$, then the matroid $(E(G), r)$ coincides with the frame matroid $M(G)$ defined above; therefore, we also write $M(G)$ for this Dowling matroid. We denote by $\mathcal{D}(\Gamma)$ the class of all Dowling matroids over $\Gamma$.

When $\Gamma$ is the trivial group, all the cycles of $G$ are balanced and $M(G)$ is the cycle matroid of the underlying unlabelled graph; hence the class $\mathcal{D}(\Gamma)$ is exactly the class of graphic matroids.

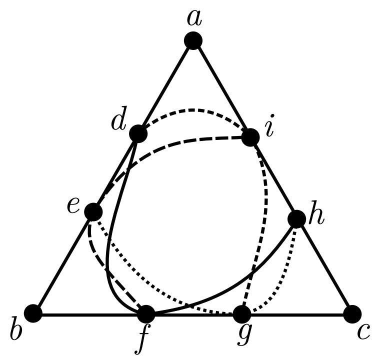
For each $k \geq 1$,the **Dowling geometry over** $\Gamma$, $\text{DG}(k, \Gamma)$, is the simple rank-$k$ matroid in $\mathcal{D}(\Gamma)$ with the maximum number of elements, which is $k + |\Gamma|\binom{k}{2}$. These matroids were introduced by Dowling in 1971 [11, 12]. When $|\Gamma| > 1$, $\text{DG}(k, \Gamma)$ is represented by a $\Gamma$-labelled graph $G$ such that

1. $V(G) = \{v_1, \dots, v_k\}$, and

2. $E(G)$ consists of a loop incident with each $v_i$ labelled by any non-identity element of $\Gamma$, and for each triple $\alpha \in \Gamma$ and $v_i, v_j \in V(G)$ with $i < j$, an edge oriented from $i$ to $j$ with label $\alpha$.

Dowling showed that, for $k \geq 3$, $\text{DG}(k, \Gamma)$ is representable over a field $\mathbb{F}$ if and only if $\Gamma$ is isomorphic to a subgroup of $\mathbb{F}^\times$ [11]. The elements of $\text{DG}(k, \Gamma)$ corresponding to loops in $G$ are called the **joints** of $\text{DG}(k, \Gamma)$. See Figure 1.3 for a diagram and a $\mathbb{C}$-representation of the Dowling geometry $\text{DG}(3, \{1, -1\})$. When $|\Gamma| = 1$, $\text{DG}(k, \Gamma)$ is represented by $K_{k+1}$ (more precisely, by any $\Gamma$-labelled oriented copy of $K_{k+1}$). In this case, we arbitrarily choose a vertex $v$ of $K_{k+1}$ and define the joints of $\text{DG}(k, \Gamma)$ to be the edges of $G$ incident with $v$.

The class of frame matroids over any field is minor closed, as is each class of Dowling matroids. In Chapter 4 we will define minors of group-labelled graphs and show that they coincide with minors of frame matroids they represent.




$$\begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array}$$
$$\begin{pmatrix} 2 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 2 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 1.3: DG$(3, \{-1, 1\})$ and a $\mathbb{C}$-representation

### 1.1.5 Connectivity

The **connectivity function** of a matroid $M$ is defined for each set $X \subseteq E(M)$ by

$$\lambda_M(X) = r_M(X) + r_M(E(M) \setminus X) - r(M).$$

This is equal to $\lambda_M(X) = r_M(X) + r_{M^*}(X) - |X|$.

For any integer $\ell \geq 1$, an **$\ell$-separation** of a matroid $M$ is a partition $(A, B)$ of $E(M)$ such that $|A|, |B| \geq \ell$ and $\lambda_M(A) \leq \ell - 1$ (note that $\lambda_M(B) \leq \ell - 1$ also). A matroid is called **$k$-connected** if it has no $\ell$-separation for any $\ell < k$, and we abbreviate 2-connected by **connected**. A **component** of a matroid $M$ is a minimal non-empty set $A \subseteq E(M)$ with $\lambda_M(A) = 0$.

There are two special types of separations. An **internal $\ell$-separation** is an $\ell$-separation $(A, B)$ such that $|A|, |B| \geq \ell + 1$, and a matroid is **internally $k$-connected** if it is $(k-1)$-connected and has no internal $(k-1)$-separation. A **vertical $\ell$-separation** is an $\ell$-separation $(A, B)$ such that $r_M(A) < r(M)$ and $r_M(B) < r(M)$. A matroid is **vertically $k$-connected** if it has no vertical $\ell$-separation for any $\ell < k$; note that this does not require being $(k-1)$-connected.

Note that for any set $X$ in a matroid $M$, $\lambda_M(X) = \lambda_{M^*}(X)$, so $M$ is $k$-connected if and only if $M^*$ is $k$-connected, and if $M$ is internally $k$-connected, then so is $M^*$. Vertical connectivity, however, is not closed under duality.

For $k \geq 2$, a graph $G$ is $k$-connected (that is, has no vertex cutset of size less than $k$) if and only if its cycle matroid $M(G)$ is vertically $k$-connected [10, 28, 38].

## 1.2 Varieties of matroids

In this thesis we focus on matroids that have as minors two special types of matroids: the projective geometries and the Dowling geometries. These two types of matroids play a central role in matroid theory, arising naturally in several theorems. In this section, we discuss a theorem of Kahn and Kung which partially explains their importance. We will encounter them again in Section 1.8 where they come up in the growth-rate theorem of Geelen, Kabell, Kung, and Whittle.

The **direct sum** $M_1 \oplus M_2$ of two matroids $M_1$ and $M_2$ with disjoint ground sets is the matroid with ground set $E(M_1) \cup E(M_2)$ and rank function given by $r_M(X) = r_{M_1}(X \cap E(M_1)) + r_{M_2}(X \cap E(M_2))$.

A **hereditary class** of matroids is a minor-closed class that is closed under direct sums. Hereditary classes of matroids are analogous to structures called varieties of algebras that come from the field of universal algebra (see, for example, [8, Chapter IV]). These are hereditary classes of algebraic structures that are closed under taking homomorphic images and direct products; for example, the class of all groups is a variety. In a variety of algebras there are *free* objects with a certain *universal property*: for groups these are the free groups, as every group with $n$ generators is a homomorphic image of the free group on $n$ generators. Hereditary classes of matroids, however, do not in general contain members with an analogous property. This motivated Kahn and Kung to make the following definitions [29]. A **sequence of universal models** for a hereditary class of matroids $\mathcal{T}$ is a sequence $\{T_n\}_{n\geq 1}$ of elements of $\mathcal{T}$ such that every simple rank-$n$ matroid in $\mathcal{T}$ is a restriction of $T_n$. A **variety** of matroids is a hereditary class that has a sequence of universal models.

We have already seen two types of varieties. First, there is the class of $\mathbb{F}$-representable matroids for a finite field $\mathbb{F}$; any simple rank-$n$ matroid in this class is a restriction of the projective geometry $\mathrm{PG}(n-1,\mathbb{F})$. The second is the class $\mathcal{D}(\Gamma)$ of Dowling matroids over a finite group $\Gamma$, whose sequence of universal models is the set of Dowling geometries $\{\mathrm{DG}(n,\Gamma)\}_{n\geq 1}$.

Surprisingly, apart from three kinds of varieties whose universal models have low connectivity, all varieties of matroids are of one of these two types.

**Theorem 1.2.1** (Kahn, Kung, [29])**.** *If $\mathcal{T}$ is a variety that contains a 3-connected matroid of rank at least three, then either*

*(i) there is a finite field $\mathbb{F}$ such that $\mathcal{T}$ is the class of $\mathbb{F}$-representable matroids, or*

*(ii) there is a finite group* $\Gamma$ *such that* $\mathcal{T}$ *is the class of* $\Gamma$*-Dowling matroids.*

## 1.3 Three in a circuit and two disjoint rooted paths

A simple graph with no isolated vertices is 2-connected if and only if any pair of edges is contained in a cycle. Three edges $e, f$, and $g$ in a simple 3-connected graph $G$, however, need not be contained in a cycle. There are two ways in which this can happen: when $\{e, f, g\}$ is a 3-edge cutset of $G$ and when $e, f$, and $g$ are incident with a common vertex. Seymour generalized this fact to binary matroids, as follows.

**Theorem 1.3.1** (Seymour, [50])**.** *If* $e, f$*, and* $g$ *are three elements of an internally* 4*-connected binary matroid* $M$ *and there is no circuit of* $M$ *containing* $\{e, f, g\}$*, then either* $\{e, f, g\}$ *is a cocircuit of* $M$ *or there is a graph* $G$ *such that* $M = M(G)$ *and* $e, f$*, and* $g$ *are edges of* $G$ *incident with a common vertex.*

He further conjectured that this is essentially the only way that three elements of a matroid can fail to be contained in a common circuit; specifically, that in a sufficiently highly connected non-graphic matroid, any triple of elements lies in a circuit [50]. This is a weaker version of an earlier conjecture of Robertson that in a 4-connected non-graphic matroid, any triple of elements lies in a circuit [49]. In fact, there are highly connected non-graphic matroids in which this three-in-a-circuit property can fail. However, these conjectures are quite close to being true, at least for representable matroids. Geelen, Gerards, and Whittle have recently shown that if $M$ is a vertically 5-connected, $\mathbb{F}$-representable matroid with a set $X$ of at least four elements no three of which are in a common circuit, then $M$ is a frame matroid over $\mathbb{F}$ [17]. The case when $|X| = 3$, however, is not so well understood.

In the next section we will derive a corollary of Theorem 1.3.1 that provides some motivation for our work in this thesis. First, we look at another application of this theorem, which is to answer the two disjoint rooted paths question: given a graph $G$ and distinct vertices $s_1, s_2, t_1, t_2$, are there disjoint paths $P_1$ and $P_2$ such that each $P_i$ has ends $s_i$ and $t_i$? Algorithms to find such paths are important in the graph minors project of Robertson and Seymour as well as many areas of combinatorial optimization. Surprisingly, characterizing when these paths exist is quite complicated compared to the non-rooted version where we only ask for two disjoint paths joining $\{s_1, s_2\}$ to $\{t_1, t_2\}$ (this version is answered by Menger's Theorem). One case where two disjoint rooted paths do not exist is when $G$ is planar and has a planar embedding in which $s_1, s_2, t_1, t_2$ lie on the boundary of the same face, appearing in that order as we traverse the boundary. It is an easy corollary of Theorem 1.3.1 that this is actually the only obstruction, assuming sufficient connectivity.

**Theorem 1.3.2.** *Let $G$ be a 4-connected graph with distinct vertices $s_1, s_2, t_1$, and $t_2$. If there do not exist disjoint paths $P_1$ and $P_2$ in $G$ such that $P_1$ joins $s_1$ and $t_1$ and $P_2$ joins $s_2$ and $t_2$, then $G$ has a planar embedding in which $s_1, s_2, t_1, t_2$ appear on the boundary of the same face, in that order.*

*Proof.* We may assume that $G$ is simple. We let $H$ be the graph obtained from $G$ by adding three edges: $e$ joining $s_1$ and $s_2$, $f$ joining $s_2$ and $t_1$, and $g$ joining $t_1$ and $t_2$ (unless any such edges already exist, in which case we call them $e, f$, or $g$). The graph $H$ is also 4-connected. Suppose there is a bond (minimal edge cutset) $B$ of $H$ containing $\{e, f, g\}$. Then $H - B$ has exactly two components, one containing $s_1$ and $t_1$, and the other containing $s_2$ and $t_2$. But then there exist disjoint paths joining $s_1$ to $t_1$ and $s_2$ to $t_2$, a contradiction; so there is no bond of $H$ containing $\{e, f, g\}$. This means that $\{e, f, g\}$ is not contained in a cocircuit of the matroid $M(H)$, and so it is not contained in a circuit of the dual matroid $M(H)^*$. The matroid $M(H)$ is vertically 4-connected, and any simple, vertically 4-connected binary matroid is internally 4-connected because it has no four-point lines. Therefore, the dual matroid $M(H)^*$ is also internally 4-connected. Since $\{e, f, g\}$ is not the edge set of a cycle in $H$, it is not a cocircuit in $M(H)^*$, so Theorem 1.3.1 implies that $M(H)^*$ is the cycle matroid of a graph $H'$ in which $e, f$, and $g$ are incident with a common vertex. A matroid and its dual are both graphic if and only if it is the cycle matroid of a planar graph; hence $H$ and $H'$ are planar duals, and $e, f$, and $g$ lie on the boundary of a face in a planar embedding of $H$. This induces a planar embedding of $G$ in which $s_1, s_2, t_1, t_2$ lie on the boundary of a face, in order. □

## 1.4 Modular restrictions

A restriction $N$ of a matroid $M$ is called **modular** if, for every flat $F$ of $M$,

$$r_M(F) + r(N) = r_M(F \cap E(N)) + r_M(F \cup E(N)). \quad (1.1)$$

There is a much more useful equivalent characterization of modularity.

**Proposition 1.4.1.** *A restriction $N$ of a matroid $M$ is modular if and only if $M$ has no minor $N'$ with an element $e$ such that $N' \backslash e = N$ and $e \in \mathrm{cl}_{N'}(E(N))$, but $e$ is not parallel to an element of $E(N)$ in $N'$.*

*Proof.* If such a minor $N' = M/C\backslash D$ exists then the flat $F = \mathrm{cl}_M(C \cup \{e\})$ violates equation (1.1). For the other direction, we choose $M$ to be minimal such that it has a restriction $N$ that is not modular, but no such minor $N'$ of $M$ exists. We choose a flat $F$ that violates equation (1.1). If $F \setminus \mathrm{cl}_M(E(N)) = \emptyset$, then since $r_M(F) > r_M(F \cap E(N))$, we can choose an element $e \in F$ that is not parallel to any element of $E(N)$ and set $N' = M|(E(N) \cup \{e\})$. Otherwise, we pick any element $c \in F \setminus \mathrm{cl}_M(E(N))$. Then $N$ is a restriction of $M/c$. Moreover, $N$ is not modular in $M/c$, for $r_{M/c}(F \setminus \{c\}) = r_M(F) - 1$ and $r_{M/c}(F \setminus \{c\} \cup E(N)) = r_M(F \cup E(N)) - 1$, which contradicts the minimality of $M$. □

We can often show that properties of a modular restriction of a matroid extend to the whole matroid. For example, consider the following elegant consequence of a theorem of Seymour.

**Theorem 1.4.2** (Seymour, [49])**.** *Any* 3*-connected matroid with a modular* $U_{2,3}$*-restriction is binary.*

As a corollary of Theorem 1.3.1, we can prove a similar result for matroids with modular $M(K_4)$-restrictions. We can show that such matroids are not only binary, but graphic.

**Theorem 1.4.3** (Seymour, [49, 50])**.** *If* $M$ *is a vertically* 4*-connected matroid with a modular* $M(K_4)$*-restriction, then* $M$ *is graphic.*

*Proof.* We may assume that $M$ is simple; a simple vertically 4-connected matroid is 3-connected. We let $N$ be the modular $M(K_4)$-restriction of $M$. There is a modular $U_{2,3}$-restriction in $M(K_4)$, and the relation of being a modular restriction is transitive (see [37, Proposition 6.9.7]) so it follows from Theorem 1.4.2 that $M$ is binary. We choose three elements $e, f, g \in E(N)$ that correspond to three edges of $K_4$ incident with a common vertex (see Figure 1.4). Each element of $E(N) \setminus \{e, f, g\}$ is in the closure of some two elements of $\{e, f, g\}$. So if there is a circuit $C$ of $M$ containing $\{e, f, g\}$, then $C \cap E(N) = \{e, f, g\}$. We pick any element $h \in C \setminus \{e, f, g\}$ and consider $M/(C \setminus \{e, f, g, h\})$. In this minor, $\{e, f, g, h\}$ is a circuit; but $N$ has no circuit containing $\{e, f, g\}$, so $h$ is not parallel to any element of $E(N)$. This contradicts the fact that $N$ is modular in $M$ by Proposition 1.4.1, and so $M$ has no circuit containing $\{e, f, g\}$. Unless $M = N$, $\{e, f, g\}$ is not a cocircuit of $M$ because $M$ is vertically 4-connected. Recall that a simple, vertically 4-connected binary matroid is internally 4-connected. We conclude from Theorem 1.3.1 that $M$ is graphic. □

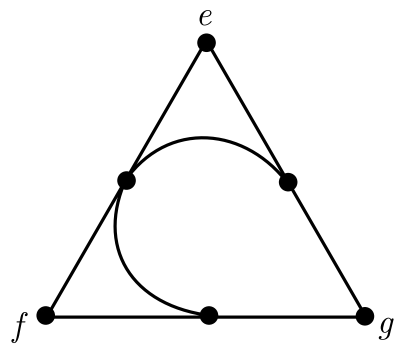



Figure 1.4: $M(K_4)$

One reason to study matroids with modular restrictions is their appearance in structural decomposition theorems for certain minor-closed classes of matroids. An example is Seymour's Decomposition Theorem, which characterizes the **regular** matroids: these are the matroids that can be represented over $\mathbb{Q}$ by a **totally unimodular matrix**, or a matrix whose square submatrices all have determinant $0, 1$, or $-1$. This theorem roughly asserts that any regular matroid $M$ can be built in a tree-like manner from pieces that are graphic or cographic, in the following way.

We start with a set of vertically 4-connected matroids $M_1, \ldots, M_k$ that are each isomorphic to a minor of $M$ and are either graphic, cographic, or a copy of one exceptional matroid. For each $M_j$, there is an $M_i$ with $i < j$ so that the ground sets of $M_j$ and $M_i$ meet in some flat $X_{ij} = E(M_i) \cap E(M_j)$ with $M_j|X_{ij} = M_i|X_{ij}$. In addition, this intersection $N_{ij} = M_j|X_{ij}$ is modular in both $M_i$ and $M_j$. We start with $M_1$ and repeatedly 'glue' each matroid $M_j$, $j > 1$, onto $M_i$ along the flat $X_{ij}$; this is an operation called a *modular sum*, which we will define precisely in the next chapter. Finally, we possibly delete some elements of $X_{ij}$.

It is the case that if $N_{ij}$ has rank three, then it is isomorphic to $M(K_4)$. But then Theorem 1.4.3 implies that $M_i$ and $M_j$ are graphic, and so we can 'merge' $M_i$ and $M_j$ into a single graphic building block before starting our construction. Thus, Theorem 1.4.3 tells us that we may build $M$ by gluing together matroids $M_1, \ldots, M_k$ which meet each other only in sets of rank at most two.

A more complicated but similar example comes from Geelen, Gerards, and Whittle's forthcoming structure theorem for minor-closed classes of matroids representable over a finite field $\mathbb{F}$. Here, we give a brief sketch of how modular restrictions arise, for simplicity only in the case $\mathbb{F} = \mathrm{GF}(4)$. Let $\mathcal{M}$ be a minor-closed proper subclass of the variety of GF(4)-representable matroids.

Then for each matroid $M \in \mathcal{M}$, we can build $M$ by joining together smaller matroids that come from certain basic subclasses of the GF(4)-representable matroids. These basic classes are the variety of binary matroids, the variety of Dowling matroids over $\mathrm{GF}(4)^\times$, and a certain variant of the second class. If we take a highly connected minor $M_0$ of $M$, then $M_0$ will be in one of these basic classes, after applying an operation known as perturbation. In addition, the rest of the matroid is built by 'gluing' matroids $M_1, \dots, M_k$ onto $M_0$ along restrictions $N_i = M_i|(E(M_0) \cap E(M_i))$. Each $N_i$ has the property that it is modular in $M_i$. When $M_0$ belongs to the basic class of binary matroids, then $N_i$ is also binary, and Theorem 1.4.2 tells us that this binary structure extends into $M_i$ if $N_i$ has rank at least two. Hence we may assume that each $M_i$ is glued onto a single point of $M_0$. Similarly, when $M_0$ belongs to the second basic class, $\mathcal{D}(\mathrm{GF}(4)^\times)$, then a generalization of Theorem 1.4.3 tells us about how its Dowling structure extends into $M_i$.

## 1.5 Conditions for representation over a field

Theorems 1.4.2 and 1.4.3 provide the motivation for much of this thesis. We will consider classes of matroids of two types: the variety of matroids representable over a given finite field, and the class of frame matroids over a field (the latter is a variety of Dowling matroids when the field is finite). Suppose that $\mathcal{M}$ is a class of one of these two types. We prove several results like Theorems 1.4.2 and 1.4.3 showing that, under some weak assumptions, if a matroid $M$ has a minor $N \in \mathcal{M}$ with certain properties then $M$ is also in $\mathcal{M}$.

We recall that a graphic matroid is representable over every field, so a corollary of Theorem 1.4.3 is that a vertically 4-connected matroid with a modular $M(K_4)$-restriction is representable over every field. In Chapter 2 we will prove one of our main results, the following generalization of this fact from $M(K_4)$-restrictions to arbitrary restrictions of rank at least three.

**Theorem 1.5.1.** *If $M$ is a vertically 4-connected matroid with a modular restriction $N$ of rank at least three, then for every finite field $\mathbb{F}$, $M$ is $\mathbb{F}$-representable if and only if $N$ is $\mathbb{F}$-representable.*

Theorem 1.5.1 can be thought of as the equivalent to Theorem 1.4.2 for finite fields of order greater than two: unlike the binary case, a rank-2 modular restriction is not sufficient to force representability over a larger finite field, as we point out in Chapter 2.

A natural question would be to ask if there is a version of Theorem 1.5.1 that allows $N$ to be an arbitrary minor of $M$ rather than a restriction. One obstacle to finding such a version is generalizing the property of modularity, which we defined for restrictions, to minors. For the case where $N$ is a projective plane we have the following strengthening of the modularity property that can be generalized to minors.

**Proposition 1.5.2.** *If a matroid $M$ has a $\mathrm{PG}(2,q)$-restriction $N$ that is not modular, then $M$ has a $U_{2,q^2+1}$-minor.*

*Proof.* If $N$ is not modular then by Proposition 1.4.1 $M$ has a minor $N'$ with an element $e$ such that $N'\backslash e = N$, $e \in \mathrm{cl}_{N'}(E(N))$, but $e$ is not parallel to an element of $E(N)$ in $N'$. In the projective plane $N$, there is an element in the intersection of any two distinct lines. Since $e$ is not parallel to any element of $E(N)$ in $N'$, there is at most one line $L$ of $N$ such that $e \in \mathrm{cl}_{N'}(L)$. Therefore, $N'/e$ has at most one parallel class of size more than one. So the simple rank-2 matroid $\mathrm{si}(N'/e)$ contains at least $|\,\mathrm{PG}(2,q)| - q = q^2+1$ elements, and therefore has a restriction isomorphic to $U_{2,q^2+1}$. □

Combining Proposition 1.5.2 with Theorem 1.5.1 we get a theorem characterizing the representability of matroids with a projective plane as a restriction, which we stated earlier as Theorem 1.0.1.

**Corollary 1.5.3.** *Any vertically 4-connected matroid with a $\mathrm{PG}(2,q)$-restriction is either $\mathrm{GF}(q)$-representable or has a $U_{2,q^2+1}$-minor.*

In the $q = 2$ case, the following theorem of Semple and Whittle extends Corollary 1.5.3 from matroids with a $\mathrm{PG}(2,2)$-restriction to matroids with a $\mathrm{PG}(2,2)$-minor, though with some different assumptions.

**Theorem 1.5.4** (Semple, Whittle, [45])**.** *Any 3-connected, representable matroid with a $\mathrm{PG}(2,2)$-minor is either binary or has a $U_{2,5}$-minor.*

We are able to generalize this result to arbitrary finite fields, at the cost of requiring a large projective geometry minor rather than just a projective plane. The main result of Chapter 3 is the following.

**Theorem 1.5.5.** *For each prime power $q$, there is an integer $n$ such that any 3-connected, representable matroid with a $\mathrm{PG}(n-1,q)$-minor is either $\mathrm{GF}(q)$-representable or has a $U_{2,q^2+1}$-minor.*

In both Theorems 1.5.4 and 1.5.5, the assumption of representability is necessary, as we will see in that chapter; it cannot be removed even by requiring vertical 4-connectivity.

## 1.6 Conditions for a frame representation

In Chapter 4 we shift our attention to proving sufficient conditions for being a frame matroid. Geelen, Gerards, and Whittle proved the following version of Theorem 1.4.3 that provides sufficient conditions for being a frame matroid over an arbitrary field, rather than a graphic matroid.

**Theorem 1.6.1** (Geelen, Gerards, Whittle, [17]). *For any field $\mathbb{F}$ and finite subgroup $\Gamma$ of $\mathbb{F}^\times$, if $M$ is a vertically 5-connected $\mathbb{F}$-representable matroid with a modular $\mathrm{DG}(4, \Gamma)$-restriction, then $M$ is a frame matroid over $\mathbb{F}$.*

Note that this theorem requires a modular rank-4 Dowling geometry restriction, $\mathrm{DG}(4, \Gamma)$, whereas in Theorem 1.4.3 the rank-3 clique $M(K_4)$ sufficed to force a matroid to be graphic. Characterizing the structure of matroids with larger rank-3 Dowling geometries as modular restrictions is an open problem that we discuss further at the end of Chapter 4.

As we do for projective geometries, we try to extend this type of result to the case where we have a Dowling geometry as a minor rather than a restriction. The following corollary of Seymour's Decomposition Theorem does this for the binary matroids. It provides conditions that imply that a binary matroid with the rank-4 Dowling geometry $M(K_5)$ as a minor is graphic.

**Theorem 1.6.2** (Seymour, [47]). *Any vertically 4-connected binary matroid with an $M(K_5)$-minor is either graphic or has a $\mathrm{PG}(2, 2)$-minor.*

We note that $\mathrm{PG}(2, 2)$ is the unique binary matroid that is not graphic but from which deleting one element results in the Dowling geometry $M(K_4)$. We extend this theorem from the binary matroids to those representable over other finite fields with the following result.

**Theorem 1.6.3.** *For any finite field $\mathbb{F}$, there is an integer $n$ such that if $M$ is a vertically 5-connected $\mathbb{F}$-representable matroid with a $\mathrm{DG}(n, \mathbb{F}^\times)$-minor, then either*

*(i) $M$ is a frame matroid over $\mathbb{F}$, or*

*(ii) $M$ has a minor $N$ with an element $e$ such that $N\backslash e \cong \mathrm{DG}(3, \mathbb{F}^\times)$ but $N$ is not a frame matroid over $\mathbb{F}$.*

In fact, this generalizes to the following result that holds even for infinite fields.

**Theorem 1.6.4.** *For any field $\mathbb{F}$, finite subgroup $\Gamma$ of $\mathbb{F}^\times$ of order at least two, and $\ell \geq 3$, there is an integer $n$ such that if $M$ is a vertically 5-connected $\mathbb{F}$-representable matroid with a $\mathrm{DG}(n,\Gamma)$-minor, then either*

*(i)* *$M$ is a frame matroid over $\mathbb{F}$,*

*(ii)* *$M$ has a $U_{2,\ell}$-minor, or*

*(iii)* *$M$ has a minor $N$ with an element $e$ such that $N\backslash e \cong \mathrm{DG}(3,\Gamma)$ but $N$ is not a frame matroid over $\mathbb{F}$.*

In Chapter 4 we will prove a more general theorem that applies to representable matroids that have large Dowling geometries over any group as minors. We will be able to avoid any connectivity requirements in this theorem by generalizing from frame matroids to more general structures called patchworks, which we define later.

## 1.7 Excluded minors of varieties

We finish this introduction by discussing two areas in which the main results of this thesis can be applied. The first is characterizing the excluded minors of varieties of matroids. An **excluded minor** for a minor-closed class of matroids $\mathcal{M}$ is a minor-minimal matroid not in $\mathcal{M}$.

For any finite field $\mathbb{F}$ of order at least five, we do not know the set of excluded minors for the variety of $\mathbb{F}$-representable matroids, although Rota's Conjecture states that there are only finitely many. Geelen, Gerards, and Whittle made the following conjecture, which would be a step towards understanding what these excluded minors look like.

**Conjecture 1.7.1** (Geelen, Gerards, Whittle, [16])**.** *For each finite field $\mathbb{F}$, no excluded minor for the variety of $\mathbb{F}$-representable matroids has a $\mathrm{PG}(2,\mathbb{F})$-minor.*

In Chapter 2, we prove that no excluded minor for the $\mathbb{F}$-representable matroids has $\mathrm{PG}(2,\mathbb{F})$ as a restriction. On the other hand, Geelen, Gerards, and Whittle were able to prove a version of the conjecture where the projective plane $\mathrm{PG}(2,\mathbb{F})$ is replaced by some larger projective geometry.

**Theorem 1.7.2** (Geelen, Gerards, Whittle, [16])**.** *For each finite field $\mathbb{F}$, there is an integer $n$ so that no excluded minor for the variety of $\mathbb{F}$-representable matroids has a $\mathrm{PG}(n-1,\mathbb{F})$-minor.*

Theorem 1.5.5 provides an alternate proof of a weak version of this fact: for each prime power $q$, there is an integer $n$ such that no representable excluded minor for the variety of $\mathrm{GF}(q)$-representable matroids has a $\mathrm{PG}(n-1,q)$-minor. Let $M$ be such an excluded minor that is representable. If $M$ has a $\mathrm{PG}(n-1,q)$-minor for large enough $n$, then Theorem 1.5.5 implies that $M$ has a $U_{2,q^2+1}$-minor, a contradiction because $U_{2,q+2}$ is an excluded minor for this variety.

We could thus prove Conjecture 1.7.1 if we could strengthen Theorem 1.5.5 by removing the assumption of representability and requiring only a projective plane minor instead of a large projective geometry. However, we know that this stronger statement is false. For example, let $q$ be a prime power and $p$ a prime that is less than $q$ and does not divide $q$. There is a rank-4 matroid obtained by identifying the $p+1$ points on a line of $\mathrm{PG}(2,p)$ with any $p+1$ collinear points of $\mathrm{PG}(2,q)$, which is 3-connected and has no $U_{2,q^2+1}$-minor but is not $\mathrm{GF}(q)$-representable. On the other hand, if we strengthen the connectivity assumption to vertical 4-connectivity, then we know of only one counterexample and its longest line minor is $U_{2,q+2}$ (we will describe this counterexample in Section 3.1). We therefore have the following conjecture.

**Conjecture 1.7.3.** *For each prime power $q$, any vertically 4-connected matroid with a $\mathrm{PG}(2,q)$-minor is either $\mathrm{GF}(q)$-representable or has a $U_{2,q+2}$-minor.*

Since $U_{2,q+2}$ is not $\mathrm{GF}(q)$-representable, no excluded minor for the variety of $\mathrm{GF}(q)$-representable matroids (other than $U_{2,q+2}$ itself) has it as a minor. Therefore, Conjecture 1.7.3 would go some way towards proving Conjecture 1.7.1; there would be further work because of the vertical 4-connectivity requirement.

Apart from the $\mathbb{F}$-representable matroids, the other interesting type of variety consists of the classes of Dowling matroids. Kahn and Kung [29] asked whether the following strengthening of Rota's Conjecture is true: does every variety have a finite set of excluded minors? In view of this question, perhaps it would be useful to consider this analogue of Conjecture 1.7.1 for the varieties of Dowling matroids.

**Conjecture 1.7.4.** *For any finite group $\Gamma$ of order at least two, no excluded minor for the variety of Dowling matroids over $\Gamma$ has a $\mathrm{DG}(3,\Gamma)$-minor.*

This is not true for the group of order one: the binary projective plane $\mathrm{PG}(2,2)$ is an excluded minor for the variety of graphic matroids and it has $\mathrm{DG}(3,\{1\}) = M(K_4)$ as a restriction. However, this happens because $M(K_4)$ differs by only element from the highly symmetric matroid $\mathrm{PG}(2,2)$, a situation that does not arise with larger Dowling geometries.

## 1.8 Growth rates of minor-closed classes

For a matroid $M$, we define $\varepsilon(M)$ to be the number of points in $M$, or equivalently, $|E(\text{si}(M))|$. For any minor-closed class of matroids $\mathcal{M}$, the **growth-rate function** of $\mathcal{M}$ is

$$g_{\mathcal{M}}(n) = \max\{\varepsilon(M) : M \in \mathcal{M}, r(M) = n\},$$

or we write $g_{\mathcal{M}}(n) = \infty$ when this maximum does not exist.

A well-known example of a growth-rate function is that of the class of planar graphs. It follows from Euler's Formula that for $n \geq 3$, the number of edges in a simple, $n$-vertex planar graph is at most $3n - 6$, with equality achieved by planar triangulations. Therefore, when $\mathcal{P}$ is the class of cycle matroids of planar graphs, we have $g_{\mathcal{P}}(n) = 3n - 3$ for $n \geq 2$ (because an $(n + 1)$-vertex connected graph has a rank-$n$ cycle matroid). In fact, Mader proved in 1967 that every proper minor-closed class of graphs has a growth-rate function bounded by some linear function.

**Theorem 1.8.1** (Mader, [35])**.** *If a simple graph $G$ has no $K_t$-minor, then $|E(G)| \leq 2^{t-3}|V(G)|$.*

On the other hand, the class $\mathcal{G}$ of all graphic matroids is a variety whose sequence of universal models consists of the cycle matroids of the cliques $\{M(K_{n+1})\}_{n\geq 1}$, so its growth-rate function is quadratic: $g_{\mathcal{G}}(n) = \binom{n+1}{2}$. Similarly, for each prime power $q$, the variety of GF($q$)-representable matroids has growth-rate function $\frac{q^n-1}{q-1}$. Perhaps as surprising as the variety theorem of Kahn and Kung is the fact that these three examples essentially characterize all possible growth-rate functions of minor-closed classes of matroids. This is shown by the growth-rate theorem of Geelen, Kabell, Kung, and Whittle, which was conjectured in 1987 by Kung [33].

**Theorem 1.8.2** (Geelen, Kabell, Kung, Whittle, [21, 18, 19])**.** *If $\mathcal{M}$ is a minor-closed class of matroids, then there exists an integer $c_{\mathcal{M}}$ such that either*

*(i)* $g_{\mathcal{M}}(n) \leq c_{\mathcal{M}} n$,

*(ii)* $\binom{n+1}{2} \leq g_{\mathcal{M}}(n) \leq c_{\mathcal{M}} n^2$ *and $\mathcal{M}$ contains all graphic matroids,*

*(iii)* $\frac{q^n-1}{q-1} \leq g_{\mathcal{M}}(n) \leq c_{\mathcal{M}} q^n$ *for some prime power $q$ and $\mathcal{M}$ contains all* GF($q$)*-representable matroids, or*

*(iv)* *$\mathcal{M}$ contains all simple rank-2 matroids.*

For each integer $\ell \geq 2$, we let $\mathcal{U}(\ell)$ be the class of matroids with no $U_{2,\ell+2}$-minor. If $q$ is the largest prime power less than or equal to $\ell$, then $\mathcal{U}(\ell)$ contains all GF$(q)$-representable matroids, but it does not contain all GF$(q')$-representable matroids for any prime power $q' > q$. Thus Theorem 1.8.2 implies that its growth-rate function is $\mathcal{O}(q^n)$. But in fact, its growth rate actually conforms to that of the variety of GF$(q)$-representable matroids.

**Theorem 1.8.3** (Geelen, Nelson, [20]). *For each integer $\ell \geq 2$, when $n$ is sufficiently large, $g_{\mathcal{U}(\ell)}(n) = \frac{q^n-1}{q-1}$ where $q$ is the largest prime power less than or equal to $\ell$, and the only simple rank-$n$ matroid in $\mathcal{U}(\ell)$ with $g_{\mathcal{U}(\ell)}(n)$ points is* $\mathrm{PG}(n-1,q)$.

For sufficiently high rank, Theorem 1.8.3 characterizes the extremal members of the class $\mathcal{U}(\ell)$: they are the projective geometries over GF$(q)$. The next question we might ask is whether it is enough for the number of points in a matroid in $\mathcal{U}(\ell)$ to be 'close' to the value of the growth-rate function to guarantee that it is GF$(q)$-representable. It turns out that this is not true: for $\ell \geq 4$, $\mathcal{U}(\ell)$ contains non-representable matroids with asymptotically the same number of points as the projective geometries, as we see in Chapter 3. However, confining our attention to the representable members of $\mathcal{U}(\ell)$, we will combine Theorem 1.5.5 with a result of Geelen and Kabell to get the following characterization of the sufficiently dense members of $\mathcal{U}(\ell)$.

**Theorem 1.8.4.** *For any positive integer $\ell$, if $M$ is a 3-connected, representable matroid in $\mathcal{U}(\ell)$ of sufficiently large rank and $|E(M)| \geq (2\sqrt{\ell})^{r(M)}$, then $M$ is representable over a field of order at most $\ell$.*

We note that $\mathcal{U}(2)$ is precisely the variety of binary matroids. For $\mathcal{U}(3)$, there is actually a stronger result with almost no requirement on the number of points in the matroid.

**Theorem 1.8.5** (Semple, Whittle, [45]). *Any 3-connected, representable matroid in $\mathcal{U}(3)$ with $|E(M)| \geq r(M) + 3$ is representable over either* GF(2) *or* GF(3).

Another consequence of Theorem 1.8.2 is that when $\mathbb{F}$ is a field of characteristic zero, any minor-closed class of $\mathbb{F}$-representable matroids that does not contain all simple lines has at most a quadratic growth rate (since no projective plane over a finite field is representable over a field of characteristic zero). An example is this theorem of Heller from 1957.

**Theorem 1.8.6** (Heller, [26]). *If $\mathcal{M}$ is a minor-closed class of $\mathbb{C}$-representable matroids that does not contain $U_{2,4}$, then $g_{\mathcal{M}}(n) \leq \binom{n+1}{2}$.*

We define $\mathcal{C}(k)$ to be the class of $\mathbb{C}$-representable matroids with no $U_{2,k+3}$-minor. Heller's Theorem says that $\mathcal{C}(1)$, which contains the variety of graphic matroids, has the same growth-rate function as it. Actually, Heller proved a theorem about the maximum size of a totally unimodular matrix, but it is equivalent to the statement above by Tutte's 1958 result that the matroids represented by these matrices, the regular matroids, are exactly the class $\mathcal{C}(1)$ [51]. Suppose that $\Gamma$ is the subgroup of $\mathbb{C}^\times$ of order $k$. As the Dowling geometries over $\Gamma$ form a sequence of universal models for the variety of Dowling matroids $\mathcal{D}(\Gamma)$, no matroid in $\mathcal{D}(\Gamma)$ has a line with more than $|\,\mathrm{DG}(2,\Gamma)| = k+2$ points as a minor. Hence $\mathcal{C}(k)$ contains $\mathcal{D}(\Gamma)$ and its growth-rate function $g_{\mathcal{C}(k)}(n)$ is at least $n + k\binom{n}{2}$. We saw in Theorem 1.8.3 that the class $\mathcal{U}(\ell)$ matches the growth rate of the 'closest' variety to it (for large enough rank). In a similar way, perhaps we can generalize Heller's Theorem to show that the class $\mathcal{C}(k)$ has the same growth-rate function as the variety of $\Gamma$-Dowling matroids.

**Conjecture 1.8.7.** *If $\mathcal{C}(k)$ is the class of $\mathbb{C}$-representable matroids with no $U_{2,k+3}$-minor, then $g_{\mathcal{C}(k)}(n) = n + k\binom{n}{2}$ for sufficiently large $n$.*

In fact, any frame matroid with no $U_{2,k+3}$-minor and rank $n$ has at most $n + k\binom{n}{2}$ points. So the following easy corollary of our Theorem 1.6.4 makes some progress towards Conjecture 1.8.7 by bounding the number of points in matroids in $\mathcal{C}(k)$ with a large Dowling geometry as a minor.

**Corollary 1.8.8.** *For each $k \geq 2$, if $\Gamma$ is the subgroup of $\mathbb{C}^\times$ of order $k$, there is an integer $n$ such that any vertically 5-connected matroid in $\mathcal{C}(k)$ with a $\mathrm{DG}(n,\Gamma)$-minor is a frame matroid over $\mathbb{C}$.*

*Proof.* By Theorem 1.6.4 it suffices to show that there exists no matroid $N \in \mathcal{C}(k)$ with an element $e$ such that $N\backslash e \cong \mathrm{DG}(3,\Gamma)$ but $N$ is not a frame matroid over $\mathbb{C}$. Suppose such a matroid $N$ exists. We let $B = \{a,b,c\}$ be the joints of $N\backslash e$ and $L_{ab}, L_{bc}$, and $L_{ca}$ the lines $\mathrm{cl}_N(\{a,b\})$, $\mathrm{cl}_N(\{b,c\})$, and $\mathrm{cl}_N(\{c,a\})$. Each of these lines has $k+2$ points, and is thus modular in $N$, which has no $U_{2,k+3}$-minor. We let $A$ be a $\mathbb{C}$-representation of $N$; recall that we can choose $A$ so that $A|B$ is an identity matrix and for each $d \in E(N\backslash e) \setminus B$, the column $A|\{d\}$ has two non-zero entries, 1 and $-\omega$ for some $\omega \in \Gamma$. We index the rows of $A$ by $\{a,b,c\}$ so that $A_{zz} = 1$ for each $z \in B$. Since $N$ is not a frame matroid over $\mathbb{C}$, the column $A|\{e\}$ has three non-zero entries, and by scaling we may assume it has the form $(-x,1,-y)^T \in \mathbb{C}^{\{a,b,c\}}$ for some $x, y \in \mathbb{C}$.

Since $L_{bc}$ is modular in $N$, $\{a,e\}$ spans some point of $L_{bc}$ and thus $(0,1,-y)^T$ is parallel to a column of $A$ indexed by an element of $L_{bc}$, so $y \in \Gamma$. Similarly,

$x \in \Gamma$. We let $f$ be any element of $L_{ab} \setminus B$ other than the one lying on $\mathrm{cl}_N(\{c, e\})$. By the modularity of $L_{bc}$ again, there is an element $g \in L_{bc}$ so that $\{e, f, g\}$ lies on a line of $N$. Since $f \notin \mathrm{cl}_N(\{c, e\})$, $g \in L_{bc} \setminus B$. Up to scaling, for some $v, w \in \Gamma$ the matrix $A|\{e, f, g\}$ is equal to

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} e & f & g \end{array} \\ \begin{pmatrix} -x & 1 & 0 \\ 1 & -v & -w \\ -y & 0 & 1 \end{pmatrix} \end{array}.$$

It has determinant zero, so $xv = 1 - yw$. Note that $\Gamma$ consists of the roots of $x^k - 1$ in $\mathbb{C}$. The number $xv \in \Gamma$ lies on the unit circle centred at 0 in the complex plane, while $1 - yw$ lies on the circle of unit radius with centre 1. These two circles intersect at $e^{i\pi/3}$ and $e^{-i\pi/3}$, so $xv \in \{e^{i\pi/3}, e^{-i\pi/3}\}$. Thus given $x$ there are at most two possible values of $v$, and hence at most two points $f \in L_{ab} \setminus B$ other than the point that lies on the line $\mathrm{cl}_N(\{e, c\})$. This proves that $k = |L_{ab} \setminus B| \leq 3$. However, we also have $e^{i\pi/3} \in \Gamma$ so $k$ is a multiple of six, a contradiction. □

If $\mathcal{M}$ is a minor-closed class with greater than linear growth rate, then for large $n$ we can expect any rank-$n$ matroid $M$ achieving the maximum number of points $g_{\mathcal{M}}(n)$ to have high vertical connectivity. This is because a vertical separation allows us to partition $M$ into two restrictions $M_1$ and $M_2$ of smaller rank, and as long as the growth-rate function $g_{\mathcal{M}}$ is convex and $n$ is large compared to the order of the separation, $|E(M)| \leq g_{\mathcal{M}}(r(M_1)) + g_{\mathcal{M}}(r(M_2))$ will be less than $g_{\mathcal{M}}(n)$. Thus Corollary 1.8.8 reduces the problem of proving Conjecture 1.8.7 to considering the minor-closed subclass of $\mathcal{C}(k)$ that excludes a given Dowling geometry over the cyclic group of order $k$. It is possible that the growth rate of such a class is closer to that of the variety of Dowling matroids over a $(k-1)$-element group.

**Conjecture 1.8.9.** *If $\Gamma$ is the subgroup of $\mathbb{C}^\times$ of order $k$ and $\mathcal{M}$ is a minor-closed subclass of $\mathcal{C}(k)$ that does not contain all Dowling geometries over $\Gamma$, then $g_{\mathcal{M}}(n) \leq cn + (k-1)\binom{n}{2}$ for some constant $c$ and sufficiently large $n$.*

# Chapter 2

# Modular planes

In this chapter, we generalize Seymour's result that a vertically 4-connected matroid with a modular $M(K_4)$-restriction is representable over all fields. The main theorem of this chapter is the following.

**Theorem 2.0.1.** *If $M$ is a vertically 4-connected matroid with a modular restriction $N$ of rank at least three, then every representation of $N$ over a finite field $\mathbb{F}$ extends to an $\mathbb{F}$-representation of $M$.*

Interestingly, we will be able to derive this theorem easily from the following special case.

**Theorem 2.0.2.** *For any finite field $\mathbb{F}$, any vertically 4-connected matroid with a modular $\mathrm{PG}(2, \mathbb{F})$-restriction is $\mathbb{F}$-representable.*

We focus on proving Theorem 2.0.2, and we will see how it implies Theorem 2.0.1 in Section 2.2 after we introduce the concept of 'modular sums'.

The converse of Theorem 2.0.2 is well-known: a restriction isomorphic to a projective geometry over $\mathbb{F}$ is modular in any $\mathbb{F}$-representable matroid. This follows from our equivalent characterization of modularity (Proposition 1.4.1): if a $\mathrm{PG}(n-1, \mathbb{F})$-restriction in a matroid $M$ is not modular, then $M$ has a rank-$n$ minor with more than $|\mathrm{PG}(n-1, \mathbb{F})|$ points, so it is not $\mathbb{F}$-representable.

For the field $\mathrm{GF}(2)$, recall that we have a much stronger result (this was Theorem 1.4.2 — note that $\mathrm{PG}(1, 2)$ is the line $U_{2,3}$).

**Theorem 2.0.3** (Seymour, [49])**.** *Any 3-connected matroid with a modular $\mathrm{PG}(1, 2)$-restriction is binary.*

The direct generalization of Theorem 2.0.3 however is not true: even a 3-connected matroid with a modular $\mathrm{PG}(1, 3)$-restriction need not be $\mathrm{GF}(3)$-representable (see Figure 2.1 for an example that is not $\mathrm{GF}(3)$-representable

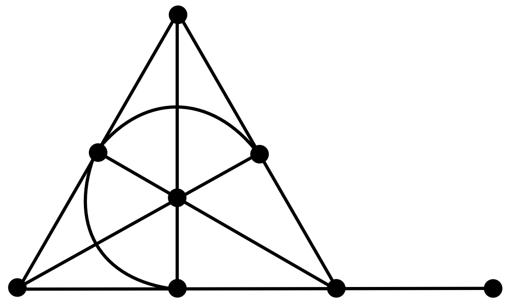


Figure 2.1: A non-GF(3)-representable matroid with a modular PG(1, 3)-restriction

because it has a binary projective plane as a restriction). Therefore, we will have to replace modular lines with modular planes. But even then, to show that a matroid $M$ with a modular $\mathrm{PG}(2,\mathbb{F})$-restriction is $\mathbb{F}$-representable we need $M$ to be vertically 4-connected rather than just 3-connected. For each prime power $q > 2$, there is a prime $p < q$ such that $p$ does not divide $q$. We can construct a rank-4 matroid from $\mathrm{PG}(2,q)$ and $\mathrm{PG}(2,p)$ by identifying the elements of a line in $\mathrm{PG}(2,p)$ with any $p+1$ collinear elements of $\mathrm{PG}(2,q)$. It is 3-connected and has a modular $\mathrm{PG}(2,q)$-restriction, but is not $\mathrm{GF}(q)$-representable.

The main idea of our proof of Theorem 2.0.2 is as follows. Given a certain non-$\mathbb{F}$-representable matroid $M$ with a modular $\mathrm{PG}(2,\mathbb{F})$-restriction, we define an $\mathbb{F}$-representable matroid $M'$ on the same ground set whose rank function equals $r_M$ on every subset of $E(M)$ except those containing a particular two-element subset $\{x,y\}$. We exploit the relation between these two matroids to constrain the structure of $M$. This technique was introduced by Seymour [48] and Kahn and Seymour [30] in their proofs of Rota's Conjecture for GF(3) and arises in many other proofs, such as Geelen, Gerards, and Kapoor's proof of Rota's Conjecture for GF(4) [14].

Although Theorem 2.0.2 does not appear to be related to excluded minors, the majority of our proof is actually an investigation of the minor-minimal 3-connected matroids that have a modular $\mathrm{PG}(2,\mathbb{F})$-restriction but are not $\mathbb{F}$-representable. This consists of proving the following main lemma, which we do in Sections 2.3 to 2.7. It tells us that the only way to construct one of these matroids is as we did in the example above: by 'gluing' a non-$\mathbb{F}$-representable matroid onto a line of $\mathrm{PG}(2,\mathbb{F})$.

**Lemma 2.0.4.** *For any finite field $\mathbb{F}$, if $M_0$ is a 3-connected, non-$\mathbb{F}$-representable matroid with a modular restriction $N_0 \cong \mathrm{PG}(2,\mathbb{F})$, then $M_0$ has a 3-connected, non-$\mathbb{F}$-representable minor $M$ such that $N_0$ is a restriction of $M$ and $\lambda_M(E(N_0)) = 2$.*

In the final section of this chapter after proving our main theorem, we apply this lemma again to show that for every finite field $\mathbb{F}$, no excluded minor for the variety of $\mathbb{F}$-representable matroids has $\mathrm{PG}(2,\mathbb{F})$ as a restriction. This verifies a special case of Conjecture 1.7.1 of Geelen, Gerards, and Whittle.

## 2.1 Modular sums

We define the **local connectivity** of two sets $A, B \subseteq E(M)$ in a matroid $M$ to be

$$\sqcap_M(A,B) = r_M(A) + r_M(B) - r_M(A \cup B).$$

Two sets $A$ and $B$ are called **skew** if $\sqcap_M(A,B) = 0$. Recall that a restriction $N$ of $M$ is called modular if, for every flat $F$ of $M$,

$$r_M(F) + r(N) = r_M(F \cap E(N)) + r_M(F \cup E(N)).$$

Equivalently, $N$ is modular if $\sqcap_M(E(N), F) = r_M(F \cap E(N))$ for every flat $F$ of $M$.

Suppose that the ground sets of two matroids $M_1$ and $M_2$ intersect on a set $T$ such that $M_1|T$ is a modular restriction of $M_1$. We can then define a certain matroid on $E(M_1) \cup E(M_2)$ that generalizes the notions of the direct sum and the 2-sum. The following construction was introduced by Brylawski in 1975.

**Proposition 2.1.1** (Brylawski, [7]). *Let $M_1$ and $M_2$ be matroids and let $T = E(M_1) \cap E(M_2)$. If $M_1|T = M_2|T$ and $M_1|T$ is modular in $M_1$, then there is a unique matroid $M$ such that $E(M) = E(M_1) \cup E(M_2)$, $M|E(M_1) = M_1$, $M|E(M_2) = M_2$, and $r(M) = r(M_1) + r(M_2) - r(M_1|T)$. Moreover, $F \subseteq E(M)$ is a flat of $M$ if and only if $F \cap E(M_1)$ is a flat of $M_1$ and $F \cap E(M_2)$ is a flat of $M_2$, and the rank of a flat $F$ is*

$$r_M(F) = r_{M_1}(F \cap E(M_1)) + r_{M_2}(F \cap E(M_2)) - r_{M_1}(F \cap T).$$

We call the matroid $M$ obtained as in Proposition 2.1.1 the **modular sum** of $M_1$ and $M_2$ and denote it by $M_1 \oplus_m M_2$. We note that the matroid $M_1 \oplus_m M_2$ is often called the *generalized parallel connection* [7].

When $M_1$ and $M_2$ are matroids on disjoint ground sets, $M_1 \oplus_m M_2$ is equal to the direct sum $M_1 \oplus M_2$. When $M_1$ and $M_2$ are matroids whose ground sets intersect in a single element $e$, then we define the **2-sum** of $M_1$ and $M_2$ to be $M_1 \oplus_2 M_2 = (M_1 \oplus_m M_2) \backslash e$.

We remark that the rank function for $M$ given in Proposition 2.1.1 tells us that for any $e \in E(M_2) \setminus T$, $M \backslash e = M_1 \oplus_m (M_2 \backslash e)$, and for any $e \in E(M_2) \setminus \mathrm{cl}_{M_2}(T)$, $M/e = M_1 \oplus_m (M_2/e)$.

We now state four facts about modular sums; the first one was proved by Brylawski and concerns their representability. A matroid $M$ is called **uniquely representable** over a field $\mathbb{F}$ if any $\mathbb{F}$-representation of $M$ can be transformed into any other by row operations, scaling columns, and applying an automorphism of $\mathbb{F}$ to all its entries.

**Proposition 2.1.2** (Brylawski, [7])**.** *If $M = M_1 \oplus_m M_2$ is the modular sum of two matroids $M_1$ and $M_2$ that are representable over a field $\mathbb{F}$ and $M|(E(M_1) \cap E(M_2))$ is uniquely representable over $\mathbb{F}$, then $M$ is representable over $\mathbb{F}$.*

*Proof.* We let $T = E(M_1) \cap E(M_2)$. Consider an $\mathbb{F}$-representation $A_1$ of $M_1$. We can choose $A_1$ to have the form

$$A_1 = \begin{array}{c} \\ \end{array}\overset{\phantom{C_1}\quad T}{\begin{pmatrix} C_1 & 0 \\ C_2 & A \end{pmatrix}}$$

for some matrices $C_1, C_2$, and $A$ with $\mathrm{rank}(A) = r(M_1|T)$. Then $A_1|T$ is an $\mathbb{F}$-representation of $M_2|T$. Let $A_2$ be an $\mathbb{F}$-representation of $M_2$. Since $M|T$ is uniquely $\mathbb{F}$-representable, we may assume by row operations, scaling, and applying an automorphism of $\mathbb{F}$ that $A_2|T$ is obtained by adding zero rows to $A$, and write $A_2$ in the form

$$A_2 = \overset{T\quad\phantom{C_3}}{\begin{pmatrix} A & C_3 \\ 0 & C_4 \end{pmatrix}}.$$

Then the matrix

$$\begin{array}{ccc} E(M_1) \setminus T & T & E(M_2) \setminus T \end{array}$$
$$\begin{pmatrix} C_1 & 0 & 0 \\ C_2 & A & C_3 \\ 0 & 0 & C_4 \end{pmatrix}$$

is an $\mathbb{F}$-representation of $M$. □

The next three facts concern connectivity properties of modular sums. The **corank** of a set $X$ in a matroid $M$ is defined to be the rank of $X$ in its dual $M^*$, and the corank function of $M$ is $r^*_M(X) = r_{M^*}(X)$.

**Proposition 2.1.3.** *If $M = M_1 \oplus_m M_2$ is the modular sum of matroids $M_1$ and $M_2$ with $T = E(M_1) \cap E(M_2)$, then* *(i) for any* $X \subseteq E(M_1) \setminus T$, $r^*_M(X) = r^*_{M_1}(X)$ *and* $\lambda_M(X) = \lambda_{M_1}(X)$, *and*

*(ii) for any* $X \subseteq E(M_2) \setminus T$, $r^*_M(X) = r^*_{M_2}(X)$ *and* $\lambda_M(X) = \lambda_{M_2}(X)$.

*Proof.* We let $X \subseteq E(M_1) \setminus T$ and compute the corank of $X$ in $M$. We see that $r^*_M(X) = |X| - r(M) + r(M \backslash X)$ is equal to

$$|X| - (r(M_1) + r(M_2) - r(M_1|T)) + (r(M_1 \backslash X) + r(M_2) - r(M_1|T)),$$

which is equal to $r^*_{M_1}(X)$. Hence $\lambda_M(X) = r_M(X) + r^*_M(X) - |X| = r_{M_1}(X) + r^*_{M_1}(X) - |X| = \lambda_{M_1}(X)$. The same proof shows that a set $X \subseteq E(M_2) \setminus T$ satisfies $r^*_M(X) = r^*_{M_2}(X)$ and $\lambda_M(X) = \lambda_{M_2}(X)$. □

This next fact is due to Brylawski.

**Proposition 2.1.4** (Brylawski, [7])**.** *If $M$ is a matroid with a modular restriction $N$ and $M/E(N)$ is not connected, then $M$ is a modular sum of two proper restrictions whose ground sets meet in $E(N)$.*

Finally, we have a converse to Proposition 2.1.4.

**Proposition 2.1.5.** *If $M = M_1 \oplus_m M_2$ is the modular sum of matroids $M_1$ and $M_2$ with $T = E(M_1) \cap E(M_2)$, then $(E(M_1) \setminus T, E(M_2) \setminus T)$ is a 1-separation of $M/T$.*

*Proof.* Let $X_1 \subseteq E(M_1) \backslash T$. Then $r_{M/T}(X_1) = r_{M_1}(X_1 \cup T) - r_M(T)$. Similarly, for $X_2 \subseteq E(M_2) \setminus T$, we have $r_{M/T}(X_2) = r_{M_2}(X_2 \cup T) - r_M(T)$. Also, $r_{M/T}(X_1 \cup X_2) = r_M(X_1 \cup X_2 \cup T) - r_M(T)$, and this is equal to $r_{M_1}(X_1 \cup T) + r_{M_2}(X_2 \cup T) - 2r_M(T)$, which is $r_{M/T}(X_1) + r_{M/T}(X_2)$.

Hence for any $X \subseteq E(M/T)$, $r_{M/T}(X) = r_{M/T}(X \cap E(M_1)) + r_{M/T}(X \cap E(M_2))$. □

## 2.2 General modular restrictions

Having defined modular sums, we are ready to show that Theorem 2.0.1 is a corollary of Theorem 2.0.2. We need one more fact, sometimes called the Fundamental Theorem of Projective Geometry (for a proof, see [1, Theorem 5.4.8]) as well as a few definitions about matroid representations.

**Fundamental Theorem of Projective Geometry.** *For each finite field $\mathbb{F}$ and integer $n \geq 3$, the projective geometry $\mathrm{PG}(n-1, \mathbb{F})$ is uniquely representable over $\mathbb{F}$.*

We note that when $B$ is a basis of a matroid $M$, any representation of $M$ can be transformed by row operations into one where the columns indexed by $B$ are an identity matrix. This is called a representation in **standard form with respect to** $B$. When $A$ is a representation of $M$ in standard form with respect to $B$, we index the rows of $A$ by the elements of $B$, so that $A_{bb} = 1$ for each $b \in B$. For each $X \subseteq B$ and $Y \subseteq E(M)$, we write $A[X, Y]$ for the submatrix of $A$ in the rows indexed by $X$ and the columns indexed by $Y$.

A set $D$ in a matroid $M$ is called **coindependent** when it is independent in the dual, $M^*$; a set is coindependent if and only if it is disjoint from a basis of $M$. Whenever $N$ is a minor of a matroid $M$, we can partition $E(M) \setminus E(N)$ into an independent set $C$ and a coindependent set $D$ such that $N = M/C \backslash D$ (see [37, Lemma 3.3.2]). Suppose that $B$ is a basis of $N$ and that $B' = B \cup C$, so $B'$ is a basis of $M$. Let $\mathbb{F}$ be a field and $A'$ an $\mathbb{F}$-representation of $M$ in standard form with respect to the basis $B'$. Then the matrix $A = A'[B, E(N)]$ is an $\mathbb{F}$-representation of $N$ in standard form with respect to the basis $B$. We say that the representation $A'$ of $M$ **extends** the representation $A$ of $N$ and that $A$ **extends to** $A'$. Conversely, $A$ is the representation of $N$ that is **induced** by $A'$. Any representation of $N$ that is row-equivalent to $A$ is also said to extend to $A'$.

We restate Theorem 2.0.1 here and prove it assuming Theorem 2.0.2, towards whose proof we resume working in the next section.

**Theorem 2.0.1.** *If $M$ is a vertically 4-connected matroid with a modular restriction $N$ of rank at least three, then every representation of $N$ over a finite field $\mathbb{F}$ extends to an $\mathbb{F}$-representation of $M$.*

*Proof.* We may assume that $M$ is simple. We let $\mathbb{F}$ be any finite field over which $N$ is representable. We let $N'$ be a copy of the projective geometry $\mathrm{PG}(r(N) - 1, \mathbb{F})$ that has $N$ as a restriction, and whose ground set is disjoint from $E(M) \setminus E(N)$. Since $N$ is modular in $M$, the modular sum $M \oplus_m N'$ exists; we denote it by $M'$.

(1) *$N'$ is a modular restriction of $M'$.*

If not, then by Proposition 1.4.1, there is a set $C \subseteq E(M') \setminus E(N')$ and an element $e \in E(M') \setminus E(N')$ such that $M'/C$ has $N'$ as a restriction and $e \in \mathrm{cl}_{M'/C}(E(N'))$ but $e$ is not parallel to an element of $E(N')$ in $M'/C$. But then $M/C$ has $N$ as a restriction, $e \in \mathrm{cl}_{M/C}(E(N))$, and $e$ is not parallel to an element of $E(N)$, contradicting the modularity of $N$ in $M$.

(2) *$M'$ is vertically 4-connected.*

Suppose $M'$ has a vertical ($\leq 3$)-separation $(A, B)$. Since $M$ is vertically 4-connected, $(A \cap E(M), B \cap E(M))$ is not a vertical ($\leq 3$)-separation of $M$. Hence we may assume that $B \cap E(M) \subseteq \text{cl}_M(A \cap E(M))$. Then as $(A, B)$ is a vertical ($\leq 3$)-separation of $M'$, $B \setminus E(M)$ is not contained in $\text{cl}_{M'}(A)$. But this means that $B \setminus E(M)$ is not in $\text{cl}_{M'}(E(M))$, a contradiction because $E(M') \setminus E(M) = E(N') \setminus E(N)$ which is in $\text{cl}_{N'}(E(N))$ and hence in $\text{cl}_{M'}(E(N))$. This proves (2).

The relation of being a modular restriction is transitive (see [37, Proposition 6.9.7]). Therefore, any $\text{PG}(2, \mathbb{F})$-restriction of $N'$ is modular in $M'$ by (1) and the fact that a $\text{PG}(2, \mathbb{F})$-restriction is modular in any $\mathbb{F}$-representable matroid. So $M'$ is vertically 4-connected and has a modular $\text{PG}(2, \mathbb{F})$-restriction, and Theorem 2.0.2 implies that it is $\mathbb{F}$-representable.

We let $A$ be any $\mathbb{F}$-representation of $N$ and $A'$ an $\mathbb{F}$-representation of $N'$ that extends $A$. By the Fundamental Theorem of Projective Geometry, any $\mathbb{F}$-representation of $M'$ that extends a representation of $N' \cong \text{PG}(2, \mathbb{F})$ can be transformed into one that extends $A'$, by row operations, scaling, and applying an automorphism of $\mathbb{F}$. Thus there is an $\mathbb{F}$-representation of $M'$ that extends $A'$, and its restriction to $E(M)$ is an $\mathbb{F}$-representation of $M$ that extends $A$. □

## 2.3 Duality

A **deletion pair** in a 3-connected matroid $M$ is a pair of elements $\{x, y\}$ such that $M\backslash x$ and $M\backslash y$ are 3-connected and $M\backslash x, y$ is internally 3-connected. A **contraction pair** in $M$ is a deletion pair in the dual, $M^*$.

The proofs in Sections 2.6 and 2.7 will require a counterexample to Lemma 2.0.4 that has a deletion pair. However, we will be able to prove, in Section 2.4, only that it contains either a deletion pair or a contraction pair. We would therefore like a way to show that if a counterexample with a contraction pair exists, then there is another one with a deletion pair. In this section, we describe a useful matroid construction involving modular sums that will let us prove this fact in Section 2.4.

An important fact is that for any field $\mathbb{F}$, the class of $\mathbb{F}$-representable matroids is closed under duality, and further, if we take any $\mathbb{F}$-representation of a matroid $M$ in standard form $(I\ A)$, then $(A^T\ I)$ is an $\mathbb{F}$-representation of its dual, $M^*$ (see [37, Theorem 2.2.8]).

We fix a finite field $\mathbb{F}$ and let $N_0$ and $N_1$ be matroids isomorphic to $\text{PG}(2, \mathbb{F})$ on disjoint ground sets. We let $\varphi : E(N_0) \to E(N_1)$ be an isomorphism between $N_0$ and $N_1$.

We choose some basis $B_0$ of $N_0$ and let $B_1^* = E(N_1) \setminus \varphi(B_0)$; so $B_1^*$ is a basis of $N_1^*$. We choose $A$ to be a matrix such that $(I\ A)$ is an $\mathbb{F}$-representation of $\mathrm{PG}(2, \mathbb{F})$ in standard form with respect to $B_0$; note that $(A^T\ I)$ is an $\mathbb{F}$-representation of $N_1^*$ in standard form with respect to the basis $B_1^*$ . We define the $\mathbb{F}$-matrix $C$ with columns indexed by $E(N_0) \cup E(N_1)$ by

$$C = \begin{array}{c} \begin{array}{cccc} B_0 & E(N_0) \setminus B_0 & E(N_1) \setminus B_1^* & B_1^* \end{array} \\ \left( \begin{array}{cccc} I & A & I & 0 \\ 0 & 0 & A^T & I \end{array} \right) \end{array}.$$

We denote by $R = M_{\mathbb{F}}(C)$ the matroid represented by $C$ over $\mathbb{F}$. We observe that $R \backslash E(N_1) = N_0$ and $R/E(N_0) = N_1^*$. Furthermore, since $R$ is $\mathbb{F}$-representable, $N_0$ is a modular restriction of $R$.

We can now state the main result of this section.

**Proposition 2.3.1.** *If $M_0$ is a 3-connected, non-$\mathbb{F}$-representable matroid with $N_0$ as a restriction and $\lambda_{M_0}(E(N_0)) = 3$, then $M_1 = ((R \oplus_m M_0) \backslash E(N_0))^*$ is internally 3-connected with all parallel pairs containing an element of $E(N_1)$, $M_1$ is non-$\mathbb{F}$-representable, $M_1$ has $N_1$ as a restriction, and $\lambda_{M_1}(E(N_1)) = 3$. Moreover, $N_1$ is modular in $M_1$ if and only if $N_0$ is modular in $M_0$.*

We prove Proposition 2.3.1 through a sequence of lemmas. First, we show that the matroid $M_1$ has $N_1$ as a restriction, then we show that $M_1$ is not $\mathbb{F}$-representable, and finally we prove the required connectivity properties.

For a set $S$ in a matroid $M$ and sets $X \subseteq E(M) \setminus S$, $Y \subseteq S$, we say that $Y$ **subjugates** $X$ **relative to** $S$ **in** $M$ if

$$\sqcap_M(X, S) = \sqcap_M(E(M) \setminus S, Y) = \sqcap_M(X, Y).$$

If for all $X \subseteq E(M) \setminus S$ there is a set $Y \subseteq S$ that subjugates $X$ relative to $S$ in $M$, then we say that $S$ **subjugates** $M$. Whenever $N$ is a modular restriction of a matroid $M$, the set $E(N)$ subjugates $M$. In particular, for any $X \subseteq E(M) \setminus E(N)$, $\mathrm{cl}_M(X) \cap E(N)$ subjugates $X$ relative to $E(N)$. However, unlike modularity, the property of subjugating a matroid is invariant under matroid duality, as we now show.

**Proposition 2.3.2.** *Let $M$ be a matroid and $S \subseteq E(M)$. For any $X \subseteq E(M) \setminus S$ and $Y \subseteq S$, if $Y$ subjugates $(E(M) \setminus S) \setminus X$ relative to $S$ in $M$, then $S \setminus Y$ subjugates $X$ relative to $S$ in $M^*$.*

*Proof.* We start with the following claim:

(1) *If $(A, B, C)$ is a partition of $E(M)$, $\lambda_M(A) = \sqcap_M(A, B) + \sqcap_{M^*}(A, C)$.*

$\sqcap_M(A,B) + \sqcap_{M^*}(A,C)$ is equal to

$$\begin{aligned} &r_M(A) + r_M(B) - r_M(A\cup B) + r_{M^*}(A) + r_{M^*}(C) - r_{M^*}(A\cup C)\\ &= \lambda_M(A) + |A| - r_{M/B\backslash C}(A) - r_{M^*/C\backslash B}(A)\\ &= \lambda_M(A) + |E(M/B\backslash C)| - r(M/B\backslash C) - r((M/B\backslash C)^*)\\ &= \lambda_M(A). \end{aligned}$$

Let $X \subseteq E(M)\setminus S$ and $Y \subseteq S$ such that $Y$ subjugates $(E(M)\setminus S)\setminus X$ relative to $S$ in $M$. Then

$$\sqcap_M((E(M)\setminus S)\setminus X, S) = \sqcap_M(E(M)\setminus S, Y) = \sqcap_M((E(M)\setminus S)\setminus X, Y).$$

By (1) we have $\sqcap_M((E(M)\backslash S)\backslash X, S) = \lambda_M(S) - \sqcap_{M^*}(X,S)$ and $\sqcap_M(E(M)\setminus S, Y) = \lambda_M(E(M)\setminus S) - \sqcap_{M^*}(E(M)\setminus S, S\setminus Y)$, implying that $\sqcap_{M^*}(X,S) = \sqcap_{M^*}(E(M)\setminus S, S\setminus Y)$.

Similarly, from the equality $\sqcap_M(E(M)\setminus S, Y) = \sqcap_M((E(M)\setminus S)\setminus X, Y)$ and (1) we have $\lambda_M(Y) - \sqcap_{M^*}(Y, S\setminus Y) = \lambda_M(Y) - \sqcap_{M^*}((S\setminus Y)\cup X, Y)$. From this we have $-r_{M^*}(S\setminus Y) + r_{M^*}(S) = -r_{M^*}((S\setminus Y)\cup X) + r_{M^*}(S\cup X)$ and hence $\sqcap_{M^*}(X,S) = \sqcap_{M^*}(X, S\setminus Y)$. This proves that $S\setminus Y$ subjugates $X$ relative to $S$ in $M^*$. □

Now we can show that the matroid $M_1$ has $N_1$ as a modular restriction.

**Lemma 2.3.3.** *If $M_0$ is a matroid with $N_0$ as a restriction and $\lambda_{M_0}(E(N_0)) = 3$, then $M_1 = ((R\oplus_m M_0)\backslash E(N_0))^*$ has $N_1$ as a restriction. Moreover, if $N_0$ is modular in $M_0$ then $N_1$ is modular in $M_1$.*

*Proof.* The fact that $\lambda_{M_0}(E(N_0)) = 3$ means that $N_1^* = R/E(N_0) = M_1^*/(E(M_0)\setminus E(N_0))$, so $N_1 = M_1|E(N_1)$ is a restriction of $M_1$. We now assume that $N_0$ is modular in $M_0$.

We observe that $E(N_1)$ subjugates $R^*$, for $R^*$ is $\mathbb{F}$-representable so $R^*|E(N_1) = N_1 \cong \mathrm{PG}(2,\mathbb{F})$ is modular in $R^*$. By Proposition 2.3.2, $E(N_1)$ also subjugates $R$. Then the fact that $E(N_0)$ is coindependent in $R$ implies that for a set $X \subseteq E(M_0)\setminus E(N_0)$, there is a set $Y \subseteq E(N_1)$ with $\mathrm{cl}_R(Y)\cap E(N_0) = \mathrm{cl}_{M_0}(X)\cap E(N_0)$. Let $r = r_{N_0}(\mathrm{cl}_R(Y)\cap E(N_0))$. The modularity of $N_0$ in $M_0$ and in $R$ implies that $\sqcap_{M_0}(X, E(N_0)) = r$ and $\sqcap_R(Y, E(N_0)) = r$. By Proposition 2.1.5, $(E(M_0)\setminus E(N_0), E(N_1))$ is a separation of $(R\oplus_m M_0)/E(N_0)$. This fact, with the coindependence of $E(N_0)$ in $M_0$ and $R$, implies that

$$\sqcap_{(R\oplus_m M_0)\backslash E(N_0)}(X, E(N_1)) = \sqcap_{M_0}(X, E(N_0)) = r, \text{and}$$

$$\sqcap_{(R\oplus_m M_0)\setminus E(N_0)}(Y, E(M_0)\setminus E(N_0)) = \sqcap_R(Y, E(N_0)) = r.$$

Moreover we have $r \leq \sqcap_{M_1^*}(X,Y) \leq \sqcap_{M_1^*}(X, E(N_1)) = r$. This proves that $E(N_1)$ subjugates $M_1^*$, and then Proposition 2.3.2 implies that $E(N_1)$ subjugates $M_1$.

Let $F$ be a flat of $M_1$, let $X = F \setminus E(N_1)$, and let $Y' \subseteq E(N_1)$ be a set that subjugates $X$ relative to $E(N_1)$ in $M_1$. As $E(N_1)$ is independent in $R$, it is coindependent in $M_1$. Therefore, $\sqcap_{M_1}(E(M_1)\setminus E(N_1), Y') = r_{M_1}(Y')$, so $\sqcap_{M_1}(X, Y') = r_{M_1}(Y')$ and $\sqcap_{M_1}(X, E(N_1)) = r_{M_1}(Y')$. The first equation implies that $Y' \subseteq \mathrm{cl}_{M_1}(X)$, which with the second implies that $\sqcap_{M_1}(X, E(N_1)) \leq r_{M_1}(\mathrm{cl}_{M_1}(X)\cap E(N_1))$. Then as $F$ is closed, we have $\sqcap_{M_1}(X, E(N_1)) \leq r_{M_1}(\mathrm{cl}_{M_1}(X)\cap F\cap E(N_1)) \leq \sqcap_{M_1}(X, F\cap E(N_1))$. Thus $\sqcap_{M_1}(X, E(N_1)) = \sqcap_{M_1}(X, F\cap E(N_1))$, implying that $r_{M_1}(E(N_1)) - r_{M_1}(F\cup E(N_1)) = r_{M_1}(F\cap E(N_1)) - r_{M_1}(F)$. This proves that $N_1$ is a modular restriction of $M_1$. □

Next, we show that for a matroid $M_0$ with $N_0$ as a restriction, the operation $M_0 \mapsto ((R\oplus_m M_0)\backslash E(N_0))^*$ is an involution, in the following sense.

**Lemma 2.3.4.** *If $M_0$ is a matroid with $N_0$ as a restriction, $\lambda_{M_0}(E(N_0)) = 3$, and $M_1 = ((R\oplus_m M_0)\backslash E(N_0))^*$, then $M_0 = ((R^*\oplus_m M_1)\backslash E(N_1))^*$.*

*Proof.* By Lemma 2.3.3, $M_1$ has $N_1$ as a restriction, so we can define the matroid $M_2 = ((R^*\oplus_m M_1)\backslash E(N_1))^*$. Let $N_0'$ be a copy of $N_0$ on a disjoint ground set. Let $R'$ and $M_0'$ be the matroids obtained from $R$ and $M_0$, respectively, by relabelling each element in $E(N_0)$ by its copy in $E(N_0')$. The next claim is a straightforward calculation.

(1) $r((R^*\oplus_m (R'\oplus_m M_0')^*)^*) = r((R^*\oplus_m R'^*)^*) + r(M_0') - r(N_0')$.

(2) $(R^*\oplus_m (R'\oplus_m M_0')^*)^* = (R^*\oplus_m R'^*)^*\oplus_m M_0'$.

We observe that $(R^*\oplus_m (R'\oplus_m M_0')^*)^*$, when restricted to the sets $E((R^*\oplus_m R'^*)^*)$ and $E(M_0')$, yields the matroids $(R^*\oplus_m R'^*)^*$ and $M_0'$, respectively. Along with (1), this proves (2) because of the uniqueness of the modular sum as in Proposition 2.1.1.

(3) $M_2 = ((R^*\oplus_m R'^*)^*\oplus_m M_0')\backslash E(N_0')/E(N_1)$.

We have $M_2 = ((R^*\oplus_m ((R'\oplus_m M_0')\backslash E(N_0'))^*)\backslash E(N_1))^* = (R^*\oplus_m ((R'\oplus_m M_0')^*/E(N_0')))^*/E(N_1)$. Since $N_1^* = R'/E(N_0') = (R'\oplus_m M_0')/E(M_0')$, $N_1$ is a restriction of $(R'\oplus_m M_0')^*$. Hence we have $R^*\oplus_m ((R'\oplus_m M_0')^*/E(N_0')) = (R^*\oplus_m (R'\oplus_m M_0')^*)/E(N_0')$. This implies that $M_2 = ((R^*\oplus_m (R'\oplus_m M_0')^*)/E(N_0'))^*/E(N_1)$, and (3) now follows from (2).

(4) $(R^* \oplus_m R'^*)^*/E(N_1)$ *is the matroid obtained from* $N_0'$ *by adding each element of* $E(N_0)$ *in parallel to its copy in* $E(N_0')$.

We observe that $R^*$ is represented over $\mathbb{F}$ by the matrix

$$\begin{array}{c} \quad E(N_0) \quad E(N_1) \\ \begin{pmatrix} A^T & I & 0 & 0 \\ I & 0 & I & A \end{pmatrix}. \end{array}$$

Thus we have the following representation of $R^* \oplus_m R'^*$

$$\begin{array}{c} \quad E(N_0) \quad E(N_1) \quad E(N_0') \\ \begin{pmatrix} A^T & I & 0 & 0 & 0 & 0 \\ I & 0 & I & A & I & 0 \\ 0 & 0 & 0 & 0 & A^T & I \end{pmatrix}, \end{array}$$

hence $(R^* \oplus_m R'^*)^*$ has the representation

$$\begin{array}{c} \quad E(N_0) \quad E(N_1) \quad E(N_0') \\ \begin{pmatrix} I & A & I & 0 & 0 & 0 \\ 0 & 0 & A^T & I & 0 & 0 \\ 0 & 0 & I & 0 & I & A \end{pmatrix}, \end{array}$$

which is row-equivalent to

$$\begin{array}{c} \quad E(N_0) \quad E(N_1) \qquad E(N_0') \\ \begin{pmatrix} I & A & 0 & 0 & -I & -A \\ 0 & 0 & 0 & I & -A^T & -A^TA \\ 0 & 0 & I & 0 & I & A \end{pmatrix}, \end{array}$$

which proves (4).

By (3) we have $M_2 = ((R^* \oplus_m R'^*)^* \oplus_m M_0')\backslash E(N_0')/E(N_1)$. By (4), $(R'^* \oplus_m R^*)^*/E(N_1)$ has $N_0'$ as a restriction, and $M_2 = ((R'^* \oplus_m R^*)^*/E(N_1) \oplus_m M_0')\backslash E(N_0') = M_0$, as required. $\square$

The next lemma shows that our modular sum operation preserves $\mathbb{F}$-representability.

**Lemma 2.3.5.** *If* $M_0$ *is a matroid with* $N_0$ *as a restriction and* $\lambda_{M_0}(E(N_0)) = 3$, *then* $M_1 = ((R \oplus_m M_0)\backslash E(N_0))^*$ *is* $\mathbb{F}$*-representable if and only if* $M_0$ *is.*

*Proof.* First we assume that $M_0$ is $\mathbb{F}$-representable. Then $M_0$ has an $\mathbb{F}$-representation of the form

$$D = \begin{array}{c} \\ \end{array}\begin{pmatrix} \overset{E(M_0)\setminus E(N_0)}{A_1} & \overset{B_0}{0} & \overset{E(N_0)\setminus B_0}{0} \\ A_2 & I & A \end{pmatrix}$$

where $(I\ A)$ is, as before, the matrix representing $N_0 \cong \mathrm{PG}(2,\mathbb{F})$. Then

$$\begin{pmatrix} \overset{E(M_0)\setminus E(N_0)}{A_1} & \overset{B_0}{0} & \overset{E(N_0)\setminus B_0}{0} & \overset{B_1^*}{0} & \overset{E(N_1)\setminus B_1^*}{0} \\ A_2 & I & A & 0 & I \\ 0 & 0 & 0 & I & A^T \end{pmatrix}$$

is an $\mathbb{F}$-representation of $R \oplus_m M_0$ and therefore $M_1 = ((R \oplus_m M_0)\backslash E(N_0))^*$ is $\mathbb{F}$-representable.

Next we assume that $M_1$ is $\mathbb{F}$-representable. By Lemma 2.3.3, $M_1$ has $N_1$ as a restriction, and the matroid $R^* \oplus_m M_1$ exists. By the same argument we applied to $R \oplus_m M_0$, $R^* \oplus_m M_1$ is $\mathbb{F}$-representable. But by Lemma 2.3.4, $M_0 = ((R^* \oplus_m M_1)\backslash E(N_1))^*$, and so $M_0$ is $\mathbb{F}$-representable. □

Next, we show that $M_1$ is internally 3-connected.

**Lemma 2.3.6.** *Let $M_0$ be a matroid with $N_0$ as a restriction, $\lambda_{M_0}(E(N_0)) = 3$, and $M_1 = ((R \oplus_m M_0)\backslash E(N_0))^*$. Then $\lambda_{M_1}(E(N_1)) = 3$ and if $M_0$ is 3-connected then $M_1$ is internally 3-connected and each of its parallel pairs contains an element of $E(N_1)$.*

*Proof.* $M_1$ has $N_1$ as a restriction and $\lambda_{M_1}(E(N_1)) = \lambda_{R\oplus_m M_0}(E(N_1))$, which is three by Proposition 2.1.3, part (i).

We assume that $M_0$ is 3-connected. If $M_1$ is not connected, it has a separation $(W, Z)$. Since $\lambda_{M_0}(E(N_0)) = 3$, $(R \oplus_m M_0)\backslash E(N_0)/(E(M_0) \setminus E(N_0)) = R/E(N_0) = N_1^*$, which is 3-connected. Therefore, we may assume that $E(N_1) \subseteq W$. Then since $E(N_0) \subseteq \mathrm{cl}_R(E(N_1))$, $(W \cup E(N_0), Z)$ is a separation of $R \oplus_m M_0$. This implies that $(W \cup E(N_0) \setminus E(N_1), Z)$ is a separation of $M_0$, a contradiction; so $M_1$ is connected.

If $M_1$ is not internally 3-connected, then $M_1^* = (R \oplus_m M_0)\backslash E(N_0)$ has an internal 2-separation $(U, V)$. Since $M_1^*/(E(M_0) \setminus E(N_0)) = R/E(N_0) = N_1^*$ is 3-connected we may assume that all but at most one element of $E(N_1)$ is contained in $U$. If there is an element $e \in E(N_1) \cap V$ then as $V \setminus \{e\} \subseteq E(M_0)$ we have $r_{M_1^*}(V) = r_{M_1^*}(V \setminus \{e\}) + 1$, implying that $(U \cup \{e\}, V \setminus \{e\})$ is a 2-separation of $M_1^*$. Therefore, $M_1^*$ has a 2-separation $(A, B)$ with $E(N_1) \subseteq A$

(either $A = U$ or $A = U \cup \{e\}$). Since $E(N_0) \subseteq \mathrm{cl}_R(E(N_1))$, $(A \cup E(N_0), B)$ is a 2-separation of $R \oplus_m M_0$, and thus $(A \cup E(N_0) \setminus E(N_1), B)$ is a separation or 2-separation of $M_0$, a contradiction.

Suppose that $M_1$ has a parallel pair $X$ that is disjoint from $E(N_1)$. Then $X$ is a series pair of $(R \oplus_m M_0)\backslash E(N_0)$. Since $E(N_0) \subseteq \mathrm{cl}_{R\oplus_m M_0}(E(N_1))$, $X$ is also a series pair of $R \oplus_m M_0$. But then Proposition 2.1.3, part (ii) implies that $X$ is also a series pair of $M_0$, a contradiction. Therefore, $M_1$ is internally 3-connected and each of its parallel pairs contains an element of $E(N_1)$. □

We conclude this section by remarking that Lemmas 2.3.3, 2.3.4, 2.3.5, and 2.3.6 together prove Proposition 2.3.1.

## 2.4 Finding a deletion pair

We recall that a deletion pair in a 3-connected matroid $M$ is a pair of elements $x, y$ such that $M\backslash x$ and $M\backslash y$ are 3-connected and $M\backslash x, y$ is internally 3-connected. The purpose of this section is to show that if there exists a counterexample to Lemma 2.0.4, then there is one that has a deletion pair. We will start with several useful facts on connectivity. The first, Bixby's Lemma [2, Theorem 1], is one we will use many times throughout this chapter.

**Bixby's Lemma.** *If $M$ is a 3-connected matroid and $e \in E(M)$ then at least one of $M\backslash e$ and $M/e$ is internally 3-connected.*

A **triangle** is a three-element circuit and a **triad** is a three-element cocircuit. A **coloop** in a matroid is a loop in its dual and a **series** pair of elements is a parallel pair in the dual. A **series class** is a parallel class in the dual, and is a maximal set of corank one. Next we state a useful lemma of Tutte [52].

**Tutte's Triangle Lemma.** *If $T = \{a, b, c\}$ is a triangle in a 3-connected matroid $M$ with $|E(M)| \geq 4$ then either $M\backslash a$ is 3-connected, $M\backslash b$ is 3-connected or there is a triad of $M$ that contains $a$ and exactly one of $b$ and $c$.*

The next is a corollary of Tutte's Triangle Lemma and is proved in [16].

**Lemma 2.4.1** (Lemma 2.7, [16])**.** *If $T$ is a triangle in a 3-connected matroid $M$ with $|E(M)| \geq 4$ then there exists $e \in T$ such that $M\backslash e$ is internally 3-connected.*

A **fan** in a matroid is a sequence $(s_1, s_2, \ldots, s_n)$ of distinct elements such that:

- $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle or a triad for each $i = 1, 2, \ldots, n-2$, and
- if $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle then $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad, and if $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad then $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle, for each $i = 1, 2, \ldots, n-3$.

An easy fact about fans is that $\lambda_M(S) \le 2$ for any set $S$ forming a fan in a matroid $M$. Recall that $\lambda_M(S) = r_M(S) + r^*_M(S) - |S|$.

**Lemma 2.4.2.** *If $(s_1, \ldots, s_n)$ is a fan in a matroid $M$, then $\lambda_M(\{s_1, \ldots, s_n\}) \le 2$.*

*Proof.* By duality we may assume that $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad of $M$ for odd $i$ and a triangle for even $i$. Then $r_M(\{s_1, \ldots, s_n\}) \le r_M(\{s_1, s_2, s_3\}) + |\{5 \le i \le n : i \text{ is odd}\}|$. In $M^*$, we have $r_{M^*}(\{s_1, \ldots, s_n\}) \le r_{M^*}(\{s_2, s_3, s_4\}) + |\{6 \le i \le n : i \text{ is even}\}|$. Hence $\lambda_M(\{s_1, \ldots, s_n\}) \le r_M(\{s_1, s_2, s_3\}) + r^*_M(\{s_2, s_3, s_4\}) + n - 4 - n \le 2$. □

An element $e$ of a 3-connected matroid $M$ is called **essential** if neither $M \backslash e$ nor $M/e$ is 3-connected. The next two results of Oxley and Wu are specializations of the statements of Lemma 8.8.6 and Theorem 8.8.8 in [37].

**Lemma 2.4.3** (Oxley, Wu, [39])**.** *If $M$ is a 3-connected matroid containing a projective plane restriction and $S = (s_1, \ldots, s_n)$ is a maximal fan in $M$ with $n \ge 4$, then the set of non-essential elements in $S$ is $\{s_1, s_n\}$.*

**Theorem 2.4.4** (Oxley, Wu, [39])**.** *If $M$ is a 3-connected matroid and $e$ is an essential element of $M$ that is in a four-element fan, then either*

*(i) $e$ is in a unique maximal fan in $M$, or*

*(ii) $e$ is in exactly three maximal fans each of which has exactly five elements, the union $X$ of these three fans has exactly six elements, and one of $M|X$ and $M/(E(M) \setminus X)$ is isomorphic to $M(K_4)$.*

In the following lemma, we find either a deletion pair or a contraction pair in a matroid. It is a generalization of a lemma of Geelen, Gerards and Whittle [16, Lemma 2.8], and we partly follow the outline of their proof. We make extensive use of Tutte's Triangle Lemma, as well as the following consequence of it: when a matroid has no element in a both a triangle and a triad, every triangle contains at least two elements whose deletion leaves the matroid 3-connected.

**Lemma 2.4.5.** *If $M$ is a 3-connected matroid such that* *(a) $M$ has $N_0 \cong \mathrm{PG}(2, \mathbb{F})$ as a modular restriction with $\lambda_M(E(N_0)) = 3$,*

*(b) $M/E(N_0)$ is connected and non-empty, and*

*(c) no element of $M$ is in both a triangle and a triad,*

*then either*

*(i) $M$ has a restriction $K \cong M(K_5)$ with a cocircuit $\{a, b, c, d\}$ such that $M = K \oplus_m (M\backslash a, b, c, d)$,*

*(ii) $M$ has a deletion pair $x, y \in E(M) \setminus E(N_0)$, or*

*(iii) $M$ has a contraction pair $x, y \in E(M) \setminus E(N_0)$.*

*Proof.* We let $M$ be a counterexample; so $M$ is 3-connected and satisfies (a), (b), and (c), but none of the conclusions (i), (ii), or (iii) hold. The fact that $M$ is simple and $N_0$ is modular in $M$ means that $E(N_0)$ is closed in $M$.

We let $\Lambda$ denote the set of elements $e \in E(M) \setminus E(N_0)$ such that $M\backslash e$ is 3-connected and $\Lambda^*$ the set of elements $e \in E(M) \setminus E(N_0)$ such that $M/e$ is 3-connected. The first two claims are straightforward.

(1) *Let $e, f$ be distinct elements of $E(M) \setminus E(N_0)$. If $e \in \Lambda$ then $M\backslash e, f$ is not 3-connected, and if $e \in \Lambda^*$ then $M/e, f$ is not 3-connected.*

(2) *If $N$ is a 3-connected matroid with $|E(N)| \geq 4$ and there are elements $e, f$ such that $N\backslash e/f$ is 3-connected, then either $N/f$ is 3-connected or there is a triangle of $N$ containing $e$ and $f$.*

(3) *Each element of $E(M) \setminus E(N_0)$ is either in $\Lambda \cup \Lambda^*$ or is in a triangle that contains an element of $E(N_0)$ and not in any triangle disjoint from $E(N_0)$.*

Suppose that $e \in E(M) \setminus E(N_0)$ is not in $\Lambda \cup \Lambda^*$ and is either not in a triangle containing an element of $E(N_0)$ or is in a triangle disjoint from $E(N_0)$. By Bixby's Lemma, either $M\backslash e$ or $M/e$ is internally 3-connected. Since neither $M/e$ nor $M\backslash e$ is 3-connected, $e$ is either in a triangle or a triad. If it is contained in a triangle, then it is contained in a triangle disjoint from $E(N_0)$. If it is contained in a triad, then since $N_0$ is 3-connected, the triad is disjoint from $E(N_0)$.

We assume that $e$ is contained in a triangle $T = \{e, a, b\}$ disjoint from $E(N_0)$. A dual argument covers the case where $T$ is a triad disjoint from $E(N_0)$. Neither $a$ nor $b$ is in a triad and $M\backslash e$ is not 3-connected, so by Tutte's Triangle Lemma, both $M\backslash a$ and $M\backslash b$ are 3-connected. We will prove (3) by showing that $M\backslash a, b$ is internally 3-connected so that $M$ satisfies (ii). Let

$(A, B)$ be a 2-separation in $M\backslash e$ with $a \in A$. Then $b \in B$. Since neither $a$ nor $b$ is in a triad, $|A|, |B| \geq 3$. Since $|E(M)| \geq 8$ (from (a) and (b)), by possibly swapping $A$ and $B$ we may assume $|A| \geq 4$. Note that $(A, B \cup \{e\})$ is a 3-separation of $M$, and $a \in \mathrm{cl}_M(B \cup \{e\})$. Thus $(A \setminus \{a\}, B \cup \{e\})$ is a 2-separation of $M/a$ and hence $(A \setminus \{a\}, B \cup \{e\} \setminus \{b\})$ is a 2-separation in $M/a\backslash b$, and it is an internal 2-separation. Thus by Bixby's Lemma, $M\backslash a, b$ is internally 3-connected, contradicting the fact that (ii) does not hold for $M$. This proves (3).

(4) *If $e \in E(M) \setminus E(N_0)$ is in a triad, then $e \in \Lambda^*$, and if $e$ is in a triangle disjoint from $E(N_0)$, then $e \in \Lambda$.*

If $e \in E(M) \setminus E(N_0)$ is in a triad, then $e \notin \Lambda$ and by (c) $e$ is not in a triangle, so (3) implies that $e \in \Lambda^*$. If $e \in E(M) \setminus E(N_0)$ is in a triangle disjoint from $E(N_0)$, then $e \notin \Lambda^*$ and (3) implies that $e \in \Lambda$.

(5) *If $T$ is a triangle of $M$ disjoint from $E(N_0)$, then $\Lambda \subseteq T$, and if $T^*$ is a triad of $M$, then $\Lambda^* \subseteq T^*$.*

Suppose that there is a triangle $T$ disjoint from $E(N_0)$ and an element $e$ of $\Lambda \setminus T$. Then $M\backslash e$ is 3-connected, so by Lemma 2.4.1, there exists $f \in T$ such that $M\backslash e, f$ is internally 3-connected. Also, since $f$ is in a triangle disjoint from $E(N_0)$, it follows from (3) that $M\backslash f$ is 3-connected. Then $e, f$ is a deletion pair of $M$, contradicting the fact that $M$ does not satisfy (ii). This proves the first part of (5) and the dual argument, along with the fact that all triads of $M$ are disjoint from $E(N_0)$, proves the second.

(6) *If $M$ has a triangle $T$ disjoint from $E(N_0)$, it is unique and $\Lambda = T$, and if $M$ has a triad $T^*$, then it is unique and $\Lambda^* = T^*$.*

This follows immediately from (4) and (5).

(7) *If $e \in \Lambda$ and $f \in E(M) \setminus (E(N_0) \cup \{e\})$, then either $M\backslash e, f$ is not internally 3-connected, or $e$ is in a triangle containing exactly one element of $E(N_0)$, or $\Lambda$ is a triangle.*

Suppose that $M\backslash e, f$ is internally 3-connected. Then $M\backslash f$ is not 3-connected, since $M$ does not satisfy (ii). Let $(A, B)$ be a 2-separation in $M\backslash f$ with $e \in A$. If $|A| = 2$ then $A \cup \{f\}$ is a triad of $M$, contradicting the fact that $M\backslash e$ is 3-connected. Thus $|A| \geq 3$, and $(A \setminus \{e\}, B)$ is a 2-separation in $M\backslash e, f$, which is internally 3-connected, so $|A| = 3$. As $(A, B \cup \{f\})$ is a 3-separation of $M$, $A$ is a triangle (not a triad since $e \in \Lambda$). Since $E(N_0)$ is a closed set and $e \notin E(N_0)$, $A$ contains at most one element of $E(N_0)$. Then either $A$ is a triangle containing exactly one element of $E(N_0)$, or (6) implies that $\Lambda = A$.

(8) *If $e \in \Lambda^*$ and $f \in E(M)\backslash(E(N_0)\cup\{e\})$, then either $M/e,f$ is not internally 3-connected, or $\Lambda^*$ is a triad.*

Suppose that $M/e,f$ is internally 3-connected. Then $M/f$ is not 3-connected, since $M$ does not satisfy (iii). Let $(A,B)$ be a 2-separation in $M/f$ with $e \in A$. If $|A| = 2$ then $A \cup \{f\}$ is a triangle of $M$, contradicting the fact that $M/e$ is 3-connected. Thus $|A| \geq 3$, and $(A \setminus \{e\}, B)$ is a 2-separation in $M/e,f$, which is internally 3-connected, so $|A| = 3$. As $(A, B \cup \{f\})$ is a 3-separation of $M$, $A$ is a triad (not a triangle since $e \in \Lambda^*$), and (6) implies that $\Lambda^* = A$.

(9) *There is no cocircuit $\{a,b,c,d\}$ of $M$ such that $\sqcap_M(\{a,b\},E(N_0)) = 1$ and either $\sqcap_M(\{a,c\},E(N_0)) = 1$ or $\sqcap_M(\{c,d\},E(N_0)) = 1$.*

Let $\{a,b,c,d\}$ be a cocircuit of $M$ with $\sqcap_M(E(N_0),\{a,b,c,d\}) \geq 2$. If $\lambda_M(\{a,b,c,d\}) = 2$ then by (b), $E(M) = E(N_0)\cup\{a,b,c,d\}$ and $\lambda_M(E(N_0)) = 2$, a contradiction. So $\lambda_M(\{a,b,c,d\}) = 3$, and $\{a,b,c,d\}$ is independent in $M$.

Suppose that $\sqcap_M(E(N_0),\{a,b,c,d\}) = 3$. Then $r_{M/E(N_0)}(\{a,b,c,d\}) = 1$ and since $\{a,b,c,d\}$ is a cocircuit of $M$, it is a rank-one cocircuit of $M/E(N_0)$, which means it is a component of $M/E(N_0)$. Then since $M/E(N_0)$ is connected, $E(M) = E(N_0) \cup \{a,b,c,d\}$. By the modularity of $N_0$, for each pair of distinct elements $f,g \in \{a,b,c,d\}$, there is an element $e_{fg} \in E(N_0)$ such that $\{f,g,e_{fg}\}$ is a triangle in $M$; let $X$ be the set of these six elements. For each triple of distinct elements $f,g,h \in \{a,b,c,d\}$, $\{e_{fg},e_{gh},e_{hf}\}$ is a triangle. Hence $M|(\{a,b,c,d\}\cup X) \cong M(K_5)$. Moreover, $M = (M|(\{a,b,c,d\}\cup X)) \oplus_m N_0$, so outcome (i) holds. So we may assume that $\sqcap_M(E(N_0),\{a,b,c,d\}) = 2$.

First, we assume that $\sqcap_M(\{a,c\},E(N_0)) = 1$; then also $\sqcap_M(\{b,c\},E(N_0)) = 1$ and there are elements $e_{ab},e_{bc},e_{ca} \in E(N_0)$ such that $\{a,b,e_{ab}\}$, $\{b,c,e_{bc}\}$ and $\{c,a,e_{ca}\}$ are triangles. We claim that $a,b,c \in \Lambda$. If not, by symmetry we may assume that $M\backslash b$ is not 3-connected. Then it has a 2-separation, $(U,V)$, with $e_{ab} \in U$ and $a \in V$. We have $a \notin \mathrm{cl}_M(U)$ and $e_{ab} \notin \mathrm{cl}_M(V)$. Hence since $e_{ca}$ is in triangles of $M\backslash b$ with each of $e_{ab}$ and $a$, $e_{ca} \in \mathrm{cl}_M(U)\cap\mathrm{cl}_M(V)$. Then we have $c \in V$ and $e_{bc} \in U$. Since $|V \setminus \{e_{ca}\}| \geq 2$, we may assume that $e_{ca} \in U$. Now $(U, V\cup\{b\})$ is a 3-separation of $M$ and $U \cap \mathrm{cl}_M(V\cup\{b\}) = \{e_{ab},e_{bc},e_{ca}\}$. So $\lambda_{M/E(N_0)}(U\setminus E(N_0)) = 0$, implying by (b) that $U = E(N_0)$, since $\mathrm{cl}_M(V)\cap E(N_0) = \{e_{ca}\}$. This contradicts the fact that $\lambda_M(E(N_0)) = 3$. Therefore, $a,b,c \in \Lambda$ and by (6), $d$ is not in a triangle of $M$.

We now show that one of $M\backslash a,b$ and $M\backslash b,c$ is internally 3-connected. Suppose that $M\backslash b,c$ is not internally 3-connected. Then $M\backslash b,c$ has an internal 2-separation $(W,Z)$. Since $|E(M)| > 8$ (by (a) and (b)), one of $W$ and $Z$ has size at least four. So since $\{a,d\}$ is a series pair of $M\backslash b,c$, we may assume that

$a, d \in W$. Since $M\backslash b$ is 3-connected, $e_{ca} \in Z$ and $a \notin \mathrm{cl}_M(Z)$. Therefore, since $|W| \geq 3$, $(W \setminus \{a\}, Z)$ is a 2-separation of $M\backslash b, c/a$. Since $\{c, e_{ca}\}$ is a parallel pair in $M\backslash b/a$, $(W \setminus \{a\}, Z \cup \{c\})$ is a 2-separation of $M\backslash b/a$. The unique triangle of $M\backslash b$ containing $a$ is $\{a, c, e_{ca}\}$, so $W \setminus \{a\}$ is not a parallel pair of $M\backslash b/a$; nor is it a series pair as $M\backslash b$ is 3-connected. Hence $|W \setminus \{a\}| > 2$ and $(W \setminus \{a\}, Z \cup \{c\})$ is an internal 2-separation of $M\backslash b/a$. By Bixby's Lemma, $M\backslash a, b$ is internally 3-connected. Therefore, outcome (ii) holds.

We may therefore assume that $\sqcap_M(\{c, d\}, E(N_0)) = 1$. We let $e_{ab}, e_{cd} \in E(N_0)$ be the elements such that $\{a, b, e_{ab}\}$ and $\{c, d, e_{cd}\}$ are triangles. Then by Tutte's Triangle Lemma, we may assume that $b, c \in \Lambda$, and by (6), $a$ and $d$ are not contained in any triangles except these two. If $M\backslash b, c$ is internally 3-connected, then outcome (ii) holds, so there is an internal 2-separation $(W, Z)$ of $M\backslash b, c$. Since $\{a, d\}$ is a series pair of $M\backslash b, c$, we may assume that $a, d \in W$. Then since $M\backslash b$ is 3-connected, $e_{cd} \in Z$. Since $(W, Z)$ is an internal 2-separation, $(W \setminus \{d\}, Z)$ is a 2-separation of $M\backslash b, c/d$. Then as $c$ is parallel to $e_{cd}$ in $M\backslash b/d$, $(W \setminus \{d\}, Z \cup \{c\})$ is a 2-separation of $M\backslash b/d$. But $d$ is in a unique triangle of $M$, so $W \setminus \{d\}$ is not a parallel pair of $M\backslash b/d$. Nor is it a series pair since $M\backslash b$ is 3-connected. So $|W \setminus \{d\}| > 2$ and $(W \setminus \{d\}, Z \cup \{c\})$ is an internal 2-separation of $M\backslash b/d$. By Bixby's Lemma, $M\backslash b, d$ is internally 3-connected. Thus either outcome (ii) holds, or $M\backslash d$ is not 3-connected. We let $(X, Y)$ be a 2-separation of $M\backslash d$ with $c \in X$ and $E(N_0) \in Y$ (this exists as $\{c, d, e_{cd}\}$ is a triangle). Since $M\backslash c$ is 3-connected, $c$ is not in a triad of $M$ so $|X| > 2$. Thus if $\{a, b\} \subseteq Y$ then as $\{a, b, c\}$ is a triad of $M\backslash d$, $(X \setminus \{c\}, Y \cup \{c\})$ is also a 2-separation of $M\backslash d$; but then $(X \setminus \{c\}, Y \cup \{c, d\})$ is a 2-separation of $M$, a contradiction. So at least one of $a$ and $b$ is in $X$. Then as $\{a, b, c\}$ is a triad of $M\backslash d$, $(X \cup \{a, b\}, Y \setminus \{a, b\})$ is also a 2-separation of $M\backslash d$, hence we may assume that $a, b \in X$. But then $(X \cup \{d\}, Y)$ is a 3-separation of $M$. Also, since $e_{cd}$ and $e_{ab}$ are in $Y \cap \mathrm{cl}_M(X \cup \{d\})$, $\sqcap_M(X \cup \{d\}, E(N_0)) = \sqcap_M(X \cup \{d\}, Y) = 2$. Then $X \cup \{d\}$ is a component of $M/E(N_0)$, which is connected, so $Y = E(N_0)$. This contradicts the fact that $\lambda_M(E(N_0)) = 3$. This proves (9).

(10) *$M$ has no $U_{2,4}$-restriction whose ground set is not contained in $E(N_0)$.*

Suppose $\{a, b, c, d\}$ is a set of rank two in $M$ not contained in $E(N_0)$. We may assume that $a, b, c \notin E(N_0)$. By Tutte's Triangle Lemma, we may assume that $M\backslash a$ is 3-connected. Then by Tutte's Triangle Lemma applied to the triangle $\{b, c, d\}$ in $M\backslash a$, we may assume that $M\backslash a, c$ is 3-connected. Then $\{a, c\}$ is a deletion pair and $M$ satisfies (ii), proving (10).

(11) *There is at most one triangle of $M$ containing exactly one element of $N_0$.*

By (10), no two triangles that are not contained in $E(N_0)$ can meet in more than one element. Thus there are three ways in which $M$ could have two triangles that each contain exactly one element of $N_0$: either they meet in an element of $E(M) \setminus E(N_0)$, they are disjoint, or they meet in an element of $E(N_0)$.

First, we assume that there are two triangles $\{a, b, d\}$ and $\{a, c, e\}$ where $a \notin E(N_0)$ and $d, e \in E(N_0)$. By the modularity of $N_0$, there is an element $f \in E(N_0) \cap \mathrm{cl}_M(\{b, c\})$, so $M$ has a triangle containing $\{b, c\}$. By Tutte's Triangle Lemma and the fact that no element of $M$ is in both a triangle and a triad, at least one of $M\backslash b$ and $M\backslash c$ is 3-connected. By symmetry between $b$ and $c$ we may assume that $M\backslash b$ is 3-connected. Since $M$ does not satisfy (ii), neither $M\backslash b, c$ nor $M\backslash b, a$ is 3-connected. So by Tutte's Triangle Lemma applied to the triangle $\{a, c, e\}$, $a$ and $c$ are both in triads of $M\backslash b$. But since $e$ is not in a triad, $a$ and $c$ are contained in the same triad. Since $a$ and $c$ are not in a triad of $M$, this means that $\{a, b, c\}$ is contained in a 4-element cocircuit of $M$, contradicting (9).

Next, we assume there are two triangles $T_1$ and $T_2$ that do not meet in any element of $E(M) \setminus E(N_0)$. Let $\{a, b, e\}$ and $\{c, d, f\}$, where $e, f \in E(N_0)$. So $a, b, c, d$ are distinct, but $e$ and $f$ may not be distinct. By Tutte's Triangle Lemma applied to $\{c, d, f\}$, either $M\backslash c$ or $M\backslash d$ is 3-connected; by symmetry we may assume $M\backslash c$ is. Since $M$ does not satisfy (ii), $M\backslash c, a$ and $M\backslash c, b$ are not 3-connected. Thus by Tutte's Triangle Lemma and the fact that no triad of $M\backslash c$ contains an element of $N_0$, $\{a, b\}$ is contained in a triad of $M\backslash c$. Therefore, as $a$ is not in any triad of $M$, $\{c, a, b\}$ is contained in some 4-element cocircuit of $M$. Since this cocircuit does not contain exactly one element of $\{c, d, e\}$ and does not contain exactly one element of $N_0$, it contains $d$. So $\{a, b, c, d\}$ is a cocircuit of $M$. If $e$ and $f$ are distinct, then $\sqcap_M(\{a, b, c, d\}, E(N_0)) = 2$, contradicting (9). Hence we may assume that $e = f$.

We may assume by Tutte's Triangle Lemma applied to $\{a, b, e\}$ that $M\backslash a$ is 3-connected (recall that we have already applied Tutte's Triangle Lemma to $\{c, d, e\}$ to assume that $M\backslash c$ is 3-connected). Hence $M\backslash a, c$ is not internally 3-connected or $\{a, c\}$ would be a deletion pair. We let $(X, Y)$ be an internal 2-separation of $M\backslash a, c$ with $E(N_0) \subseteq \mathrm{cl}_M(X)$. Since $M\backslash a$ and $M\backslash c$ are 3-connected, we have $b, d \in Y \setminus \mathrm{cl}_M(X)$. Suppose that $|Y| > 3$. Then recall that $\{b, d\}$ is a series pair of $M\backslash a, c$, so $(X \cup \{b, d\}, Y \setminus \{b, d\})$ is also a 2-separation of $M\backslash a, c$. But as $a, c \in \mathrm{cl}_M(X \cup \{b, d\})$, we have a 2-separation $(X \cup \{a, b, c, d\}, Y \setminus \{b, d\})$ of $M$, a contradiction. Hence $|Y| = 3$, and the unique element $w$ of $Y \setminus \{b, d\}$ lies in $\mathrm{cl}_M(X) \cap \mathrm{cl}_M(\{b, d\})$. Applying Tutte's Triangle Lemma to $\{b, d, w\}$, one of $M\backslash b$ and $M\backslash d$ is 3-connected; by symmetry we may assume that $M\backslash b$ is 3-connected. Now since $M\backslash a$ and $M\backslash b$ are both

3-connected, we have restored the symmetry between $a$ and $b$, and so by a symmetric argument $\{a, d\}$ is also contained in a triangle; we let $z$ be its third element. Since $\{a, b, c, d\}$ is a cocircuit of $M$ and $u, w, z \in \mathrm{cl}_M(\{a, b, c, d\})$, it follows that $\{u, w, z\}$ is a triangle of $M$. So by Tutte's Triangle Lemma and the symmetry between $w$ and $z$, we may assume that $M\backslash w$ is 3-connected. Hence $M\backslash w, a$ is not internally 3-connected, or $\{w, a\}$ would be a deletion pair. We let $(A, B)$ be an internal 2-separation of $M\backslash w, a$ with $E(N_0) \subseteq \mathrm{cl}_M(A)$. Since $M\backslash w$ and $M\backslash a$ are 3-connected and $\{u, w, z\}$ and $\{u, a, b\}$ are triangles, we have $z, b \in B \setminus \mathrm{cl}_M(A)$. Also, since $\{b, w, d\}$ and $\{d, u, c\}$ are triangles, we have $d \in A$ and $c \in \mathrm{cl}_M(A)$. Suppose that $|B| = 3$ and let $y$ be the unique element of $B \setminus \{b, z\}$. Then $B$ is a triangle of $M$ and $y \in \mathrm{cl}_M(A)$. A triangle cannot contain just one element of the cocircuit $\{a, b, c, d\}$. But $a, d \in A$, and if $c \in B$ then $c$ and $y$ are both in $\mathrm{cl}_M(A) \cap \mathrm{cl}_M(B)$ and hence parallel, a contradiction. This proves that $|B| > 3$. Now $b$ is in the closure of $B \setminus \{b, z\}$, for if not then $(A \cup \{w, b, z\}, B \setminus \{b, z\}))$ is a 2-separation of $M\backslash a$. This is a contradiction because $\{a, b, c, d\}$ is a cocircuit that is disjoint from $B \setminus \{b, z\}$. This proves (11).

(12) *Either* $\Lambda$ *is not a triangle or* $\Lambda^*$ *is not a triad.*

Suppose that $\Lambda$ is a triangle and $\Lambda^*$ is a triad. Then $\Lambda$ and $\Lambda^*$ are disjoint, and by (3) and (11), $E(M) \setminus E(N_0)$ consists of the union of $\Lambda$, $\Lambda^*$, and at most one triangle, which contains an element of $N_0$.

Suppose that $E(M) \cup E(N_0) = \Lambda \cup \Lambda^*$. Since $\Lambda^*$ has corank two, $\lambda_{M\backslash \Lambda^*}(E(N_0)) \geq \lambda_M(E(N_0)) - 2 = 1$, so $\sqcap_M(\Lambda, E(N_0)) \geq 1$. By the modularity of $N_0$, this means that $\mathrm{cl}_M(\Lambda)$ contains an element of $N_0$, hence $M$ has a four-point line not contained in $E(N_0)$, contradicting (10). Therefore, $E(M) \setminus E(N_0)$ is the union of $\Lambda \cup \Lambda^*$ with a triangle $T$ that contains precisely one element, $u$, of $N_0$.

We write $\Lambda = \{a, b, c\}$ and $\Lambda^* = \{d, e, f\}$. Tutte's Triangle Lemma implies that $T$ cannot be disjoint from $\Lambda$, so we may assume that $c \in T$. We let $w$ denote the element of $T \setminus \{u, c\}$. Since $w \notin \Lambda$, $M\backslash w$ is not 3-connected, and being in a triangle, $w$ is not in a triad, so $M\backslash w$ is not internally 3-connected. Let $(X, Y)$ be an internal 2-separation of $M\backslash w$ with $E(N_0) \subseteq \mathrm{cl}_M(X)$. Then $c \in Y \setminus \mathrm{cl}_M(X)$ and we may assume $a, b \in Y$. Moreover, we may assume that $\Lambda^*$ is contained in either $X$ or $Y$. If $\Lambda^* \subseteq Y$ then $(E(N_0), \Lambda \cup \Lambda^* \cup \{w\})$ is a 3-separation of $M$, contradicting the fact that $\lambda_M(E(N_0)) = 3$. So $\Lambda^* \subseteq X$. Hence $\lambda_M(\{a, b, c, w\}) = 2$ and so $\{a, b, c, w\}$ is a cocircuit of $M$. We let $d, e$, and $f$ denote the elements of $\Lambda^*$. Recall that $M/f$ and $M/d$ are 3-connected; so $M/d, f$ is not internally 3-connected or we would have a contraction pair. So $M/f$ has a 3-separation $(A, B)$ with $E(N_0) \subseteq \mathrm{cl}_{M/f}(A)$,

$d \in \mathrm{cl}_{M/f}(A) \cap \mathrm{cl}_{M/f}(B)$, and $|B \setminus \{d\}| \geq 3$. If $\Lambda \subseteq \mathrm{cl}_{M/f}(A)$ then $e$ would be a coloop of $M/f$; so we may assume that $\Lambda \subseteq B$. We assume that $w$ is in $\mathrm{cl}_{M/f}(A)$. Then so is $c$, and hence $e \in B \setminus \mathrm{cl}_{M/f}(A)$ or else $\{a, b\}$ would be a series pair of $M/f$. Then $\{c, d\}$ is contained in $\mathrm{cl}_{M/f}(A) \cap \mathrm{cl}_{M/f}(B)$, which implies that $E(N_0)$ is disjoint from $\mathrm{cl}_{M/f}(B)$ because otherwise $\{c, d\}$ would be contained in a triangle by the modularity of $N_0$. Thus $\{a, b, c, e\}$ and $E(N_0)$ are skew in $M/f$, and so $\{a, b, c, e, f\}$ and $E(N_0)$ are skew in $M$. But then $\sqcap_M(\{a, b, c, d, e, f, w\}, E(N_0)) \leq 2$, contradicting the fact that $\lambda_M(E(N_0)) = 3$. This proves that $w \in B$. If $e \in A$ then we have $d \in \mathrm{cl}_{M/f}(\{a, b, c, w\})$. But then $\sqcap_M(\{f, d\}, \{a, b, c, w\}) = 1$ so $(\{a, b, c, w, d, f\}, E(N_0))$ is a 2-separation of $M\backslash e$, which means $\lambda_M(E(N_0)) \leq 2$, a contradiction. So $e \in B$. Then since $u$ is the unique element of $N_0$ in $\mathrm{cl}_{M/f}(B)$, $\mathrm{cl}_M(\{a, b, c, e, f\})$ contains no element of $N_0$ except possibly $u$. So $\lambda_{M\backslash w,d}(E(N_0)) \leq 1$, but since $w$ is in the closure of $\{a, b, c, u\}$, we have also $\lambda_{M\backslash d}(E(N_0)) \leq 1$, and then $\lambda_M(E(N_0)) \leq 2$, a contradiction. This proves (12).

(13) *If $\Lambda^*$ is not a triad then for each $e \in \Lambda^*$, $E(M) \setminus (E(N_0) \cup \{e\}) \subseteq \Lambda$.*

We assume that $\Lambda^*$ is not a triad and that there exists $e \in \Lambda^*$. Let $f \in E(M) \setminus (E(N_0) \cup \{e\})$. By (8), $M/e, f$ is not internally 3-connected. Thus by Bixby's Lemma, $M/e\backslash f$ is internally 3-connected. But $M$ has no triads by (6) so $M/e\backslash f$ is actually 3-connected. Also, as $M$ has no triads, the dual of (2) implies that $M\backslash f$ is 3-connected. Thus $f \in \Lambda$, proving (13).

(14) *If $\Lambda$ is not a triangle then for each $e \in \Lambda$ that is not in a triangle of $M$ meeting $E(N_0)$, each $f \in E(M) \setminus (E(N_0) \cup \{e\})$ that is not in a triangle meeting $E(N_0)$ is in $\Lambda^*$.*

We assume that $\Lambda$ is not a triangle and that there exists $e \in \Lambda$ that is not in a triangle of $M$ meeting $E(N_0)$. Let $f \in E(M) \setminus (E(N_0) \cup \{e\})$ such that $f$ is not in a triangle meeting $E(N_0)$. By (7), $M\backslash e, f$ is not internally 3-connected. Thus by Bixby's Lemma, $M\backslash e/f$ is internally 3-connected. But $M$ has no triangles disjoint from $E(N_0)$ by (6) so $M\backslash e/f$ is actually 3-connected. Also, as $M$ has no triangles disjoint from $E(N_0)$, (2) implies that $M/f$ is 3-connected. Thus $f \in \Lambda^*$, proving (14).

(15) *$\Lambda$ is not a triangle and $\Lambda^*$ is not a triad.*

Since $\lambda_M(E(N_0)) = 3$, $|E(M) \setminus E(N_0)| \geq 4$. If this is an equality then $E(M) \setminus E(N_0)$ is a cocircuit, contradicting (9), so $|E(M) \setminus E(N_0)| \geq 5$.

It follows from (3) and (11) that $E(M) \setminus (E(N_0) \cup \Lambda \cup \Lambda^*)$ is contained in a triangle meeting $E(N_0)$. By Tutte's Triangle Lemma, such a triangle contains an element of $\Lambda$, so $|E(M) \setminus (E(N_0) \cup \Lambda \cup \Lambda^*)| \leq 1$. Hence $|\Lambda \cup \Lambda^*| \geq 4$.

Assume that $\Lambda$ is a triangle. Then by (12), $\Lambda^*$ is not a triad. We have $|\Lambda| = 3$, so $\Lambda^*$ is not empty. Thus by (13), for any $e \in \Lambda^*$, $E(M)\setminus(E(N_0)\cup\{e\})$, which has size at least four, is contained in the triangle $\Lambda$. This proves that $\Lambda$ is not a triangle.

Next, assume that $\Lambda^*$ is a triad. We have $|\Lambda^*| = 3$, so $\Lambda$ is not empty. Suppose that there exist two distinct elements $e, f \in E(M) \setminus (E(N_0) \cup \Lambda^*)$ that are not in a triangle meeting $E(N_0)$. Then $e \in \Lambda$ and by (14), $f \in \Lambda^*$, a contradiction. So there is at most one element of $E(M) \setminus (E(N_0) \cup \Lambda^*)$ that is not in a triangle meeting $E(N_0)$. Hence such a triangle, $T$, exists. Denote the elements of $\Lambda^*$ by $\{x, y, z\}$. Suppose that $E(M) \setminus E(N_0) = \Lambda^* \cup T$. Then $r(M) = 5$, and $r(M/x, y) = 3$ and $\mathrm{si}(M/x, y)$ is 3-connected. So either $\{x, y\}$ is a contraction pair, or $M/x, y$ has a parallel class of size at least three, in which case $M/x$ has a line $L$ with at least four points and $y \in L$. If $L$ contains $T\setminus E(N_0)$, then $\sqcap_M(\{x, y\}, T) = 1$, but then $(E(N_0), (T\setminus E(N_0))\cup\{x, y\})$ is a 2-separation of $M\backslash z$, contradicting the fact that $\lambda_{M\backslash z}(E(N_0)) \geq \lambda_M(E(N_0))-1 = 2$. Otherwise, $L$ contains $\{y, z\}$ and exactly one element $a \in T$. But then $r_{M/x}(T \cup \{y, z\}) = 3$ so $r_M(T \cup \Lambda^*) = 4$; but as $r^*_M(T \cup \Lambda^*) < 4$, this means $\lambda_M(E(N_0)) < 3$, a contradiction. Therefore, we may assume that $E(M) \setminus E(N_0)$ contains an element, $w$, that is not in $\Lambda^* \cup T$. If $r(M\backslash\Lambda^*) = 4$, then $\{w\} \cup (T \setminus E(N_0))$ is a triad of $M\backslash\Lambda^*$, and the modularity of $N_0$ implies that $w$ is in two triangles, but we know that $T$ is the unique triangle not contained in $E(N_0)$. So $r(M\backslash\Lambda^*) = 5$ and hence $r(M) = 6$. As $w \notin \Lambda^*$, $M$ has a 3-separation $(A, B)$ with $w \in \mathrm{cl}_M(A) \cap \mathrm{cl}_M(B)$ and $E(N_0) \subseteq \mathrm{cl}_M(A)$. The set $B \setminus \mathrm{cl}_M(A)$ is non-empty because $w$ is not in a triangle. Then $M$ being 3-connected, we must have $|B \setminus \mathrm{cl}_M(A)| \geq 3$. If $B \setminus \mathrm{cl}_M(A)$ contains just one element of $\Lambda^*$, then it is a triad, but the unique triad of $M$ is $\Lambda^*$. So $(A \setminus \Lambda^*, B \cup \Lambda^*)$ is also a 3-separation of $M$ and we may assume that $\Lambda^* \subseteq B$. If any one element of $T \setminus E(N_0)$ is in $B \setminus \mathrm{cl}_M(A)$, then both are; but then $\mathrm{cl}_M(A) = E(N_0)$ and $\lambda_M(E(N_0)) = 2$, a contradiction. So $T \subseteq \mathrm{cl}_M(A)$. Then $w \in \mathrm{cl}_M(A \setminus \mathrm{cl}_M(B))$ because $(A, B)$ is a 3-separation. This means that $r(M\backslash\Lambda^*) = 4$ and $r(M) = 5$, a contradiction. This proves (15).

(16) $\Lambda = E(M) \setminus E(N_0)$ *and* $\Lambda^*$ *contains all elements of* $E(M) \setminus E(N_0)$ *except possibly two, which are in a common triangle.*

Recall that $|E(M) \setminus E(N_0)| \geq 5$ and $M$ has at most one triangle not contained in $E(N_0)$, so the set $X$ of elements of $E(M) \setminus E(N_0)$ not in a triangle meeting $E(N_0)$ has size at least three. Thus either $|\Lambda^*| \geq 2$ or $|X \cap \Lambda| \geq 2$. If $|\Lambda^*| \geq 2$ then it follows from (13) that $\Lambda = E(M) \setminus E(N_0)$. So in either case $|X \cap \Lambda| \geq 2$. Therefore, by (14), $X \subseteq \Lambda^*$. Hence by (13) again we have $\Lambda = E(M) \setminus E(N_0)$, proving (16).

(17) *For any distinct* $e, f \in \Lambda^*$ *and* $g \in \Lambda$, $M\backslash e/f$ *is* 3*-connected and* $g$ *is essential in* $M\backslash e/f$.

We have $e, f \in \Lambda$, so as $\{e, f\}$ is not a deletion pair, $M\backslash e, f$ is not internally 3-connected and Bixby's Lemma says that $M\backslash e/f$ is internally 3-connected. Since $f$ is not in a triangle of $M$, $M\backslash e/f$ is 3-connected. Suppose that $M\backslash e, g/f$ is 3-connected. Then $M\backslash e, g$ is internally 3-connected, and $g \in \Lambda$ so $\{e, g\}$ is a deletion pair. Next, suppose that $M\backslash e/f, g$ is 3-connected. Then $M/f, g$ is internally 3-connected. Also, $g$ is not in a triangle of $M$, for if it were then $M\backslash e/f, g$ would have a parallel pair since $e$ is not in a triangle. Therefore, $g \in \Lambda^*$ and $\{f, g\}$ is a contraction pair. Thus neither $M\backslash e, g/f$ nor $M\backslash e/f, g$ is 3-connected, meaning that $g$ is essential in $M\backslash e/f$. This proves (17).

Let $e, f \in \Lambda^*$. By (17), every $g \in E(M) \setminus (E(N_0) \cup \{e, f\})$ is essential in $M\backslash e/f$. However, Bixby's Lemma implies that one of $M\backslash e, g/f$ and $M\backslash e/f, g$ is internally 3-connected, which means that $g$ is in a triangle or a triad of $M\backslash e/f$. But if $g$ is in a triangle then by Tutte's Triangle Lemma, it is also in a triad, and if $g$ is in a triad, then by the dual of Tutte's Triangle Lemma, it is also in a triangle. A circuit and a cocircuit of a matroid cannot meet in exactly one element, and a 3-connected matroid has no triangle that is also a triad. Hence $g$ is contained in a four-element fan of $M\backslash e/f$.

Let $S = (s_1, \ldots, s_n)$ be a maximal fan of $M\backslash e/f$ containing $g$. By Lemma 2.4.3, the set of non-essential elements of $S$ is $\{s_1, s_n\}$. Note that all elements in $E(N_0)$ are non-essential, because deleting a point from a projective plane leaves a 3-connected matroid. By (17), every element not in $E(N_0)$ is essential. Hence $S \cap E(N_0) = \{s_1, s_n\}$. Moreover, no element of $E(N_0)$ is in a triad, so $\{s_1, s_2, s_3\}$ and $\{s_{n-2}, s_{n-1}, s_n\}$ are triangles and $\{s_2, s_3, s_4\}$ is a triad.

Suppose that $S$ is not the unique maximal fan of $M$ containing $g$. Then we apply Theorem 2.4.4 to $g$ and conclude that $S$ has five elements and there is an element $s \notin S$ such that if $X = \{s, s_1, \ldots, s_5\}$ then $X$ contains another 5-element fan and one of $M|X$ and $M/(E(M) \setminus X)$ is isomorphic to $M(K_4)$. But if we contract all but two or three elements of a projective plane, the remaining two or three elements are loops and hence cannot be part of an $M(K_4)$. So we have $M|X \cong M(K_4)$. Then either $s \in \mathrm{cl}_{M\backslash e/f}(\{s_2, s_5\}) \cap \mathrm{cl}_{M\backslash e/f}(\{s_1, s_4\})$ or $s \in \mathrm{cl}_{N_0}(\{s_1, s_5\}) \cap \mathrm{cl}_{M\backslash e/f}(\{s_2, s_3\}$. But in the former case, the triad $\{s_2, s_3, s_4\}$ meets the triangle $\{s_2, s, s_5\}$ in only one element, a contradiction. So $s \in E(N_0)$ and all maximal fans containing $g$ consist of $S \setminus E(N_0)$ and some two elements of $E(N_0)$.

Hence we can partition the elements of $E(M\backslash e/f) \setminus E(N_0)$ into sets $Y^1, \ldots, Y^k$ such that for each $Y^j$ there is an ordering $(y_2^j, \ldots, y_{n-1}^j)$ of $Y^j$ and elements $y_1^j, y_n^j \in E(N_0)$ so that $(y_1^j, y_2^j, \ldots, y_n^j)$ is a fan of $M\backslash e/f$, and every maximal fan not contained in $E(N_0)$ consists of the union of some $Y^j$ with two elements of $E(N_0)$. Note that every element of $E(M\backslash e/f) \setminus E(N_0)$ is in at most two triangles, and each triangle not contained in $E(N_0)$ has exactly one element in no other triangle. Moreover, this is true for any choice of $e, f \in \Lambda^*$.

(18) *$|Y^j| > 3$ for all $j = 1, \ldots, k$.*

Suppose $|Y^j| = 3$ for some $j$. Then $Y^j$ is a triad of $M\backslash e/f$, and at least one element $y \in Y^j$ is in $\Lambda^*$. In $M\backslash y/f$, at least one of the two elements of $Y^j \setminus \{y\}$ is in a triangle with $e$, otherwise they are contained in a triangle that meets no other triangle outside $E(N_0)$. Let $z$ be the element in a triangle with $e$. If $k > 1$ then there is another part $Y^{j'}$. We choose any element $e' \in Y^{j'} \cap \Lambda^*$. Then $z$ is in three triangles of $M\backslash e'/f$, a contradiction. Hence $k = 1$, and so $j = 1$ and $E(M\backslash e/f) = Y^1$. But then $Y^j \cup \{e\}$ is a four-element cocircuit of $M/f$, and $M/f, e$ is internally 3-connected, a contradiction. This proves (18).

(19) *For all $j = 1, \ldots, k$, $e \in \mathrm{cl}_{M/f}(Y^j)$.*

Let $j \in \{1, \ldots, k\}$. By (18), $|Y^j| > 3$. Hence $Y_j$ contains a triangle $T = \{y_{i-1}^j, y_i^j, y_{i+1}^j\}$ of $M\backslash e/f$ disjoint from $E(N_0)$. We consider $M\backslash y_i^j/f$. Now the triangles $\{y_{i-3}^j, y_{i-2}^j, y_{i-1}^j\}$ and $\{y_{i+1}^j, y_{i+2}^j, y_{i+3}^j\}$ each have two elements in no other triangle. So one of $y_{i-2}^j, y_{i-1}^j$ is in a triangle with $e$, and one of $y_{i+1}^j, y_{i+2}^j$ is in a triangle with $e$. Let $T_1$ and $T_2$ be these triangles. Note that they are disjoint from $Y^m$ for all $m \neq j$: this is because every element of $Y^m$ is either in two other triangles in that fan, or is in one triangle of that fan in which it is the unique element not in any other triangle. Suppose $T_1$ and $T_2$ meet $E(N_0)$. Then whichever of $y_{i-1}^j$ and $y_{i-2}^j$ is in $T_1$ is in three triangles of $M\backslash y_{i+2}^j/f$, a contradiction if $y_{i+2}^j \in \Lambda^*$. But if $y_{i+2}^j \notin \Lambda^*$, then $y_{i-2}^j \in \Lambda^*$, and then whichever of $y_{i+1}^j$ and $y_{i+2}^j$ is in $T_2$ is in three triangles of $M\backslash y_{i-2}^j/f$, a contradiction. This proves that one of $T_1$ or $T_2$ is disjoint from $E(N_0)$, hence contained in $Y^j \cup \{e\}$. Thus $e \in \mathrm{cl}_{M/f}(Y^j)$, proving (19).

If $k = 1$, then (19) implies that $(Y^1 \cup \{e\}, E(N_0))$ is a 3-separation of $M/f$, and so $\lambda_{M/f}(E(N_0)) = 2$. But since $f \notin \mathrm{cl}_M(E(N_0))$, we have $3 = \lambda_M(E(N_0)) = \lambda_{M/f}(E(N_0)) = 2$, a contradiction. Hence $k \geq 2$. But note that $\sqcap_{M/f}(Y^1, E(N_0)) = 2$ and $\sqcap_{M/f}(Y^2, E(N_0)) = 2$. So the fact that $e \notin \mathrm{cl}_{M/f}(E(N_0))$ but $e \in \mathrm{cl}_{M/f}(Y^1) \cap \mathrm{cl}_{M/f}(Y^2)$ means that $\lambda_{M/f}(Y^1) \geq 3$.

But since $e \in \mathrm{cl}_{M/f}(Y^2)$, we then have $\lambda_{M\backslash e/f}(Y^1) = 3$. This contradicts Lemma 2.4.2, which said that a fan always has connectivity at most two. □

We can now prove the main result of this section. It asserts that if there exists a counterexample to Lemma 2.0.4, then there exists one with a deletion pair disjoint from $E(N_0)$, and moreover that we can choose it such that deleting this pair results in a matroid with at most one series pair.

**Lemma 2.4.6.** *Let $M_0$ be a matroid that is 3-connected, non-$\mathbb{F}$-representable and has a modular $\mathrm{PG}(2, \mathbb{F})$-restriction, $N_0$, with $\lambda_{M_0}(E(N_0)) = 3$, such that no proper minor of $M_0$ that has $N_0$ as a restriction is 3-connected and non-$\mathbb{F}$-representable. Then there exists such a matroid $M_0$ with a deletion pair $x, y \in E(M_0) \setminus E(N_0)$ such that $M_0\backslash x, y$ has at most one series pair.*

*Proof.* We use the matroid $R$ defined in Section 2.3. First we show that a contraction pair in one of $M_0$ and $M_1 = ((R \oplus_m M_0)\backslash E(N_0))^*$ is a deletion pair in the other.

(1) *Let $x, y \in E(M_0) \setminus E(N_0)$. If $\{x, y\}$ is a contraction pair in $M_0$, then $\{x, y\}$ is a deletion pair in $M_1$, and any series pair in $M_1\backslash x, y$ is a parallel pair in $M_0/x, y$. If $\{x, y\}$ is a contraction pair in $M_1$, then $\{x, y\}$ is a deletion pair in $M_0$, and any series pair in $M_0\backslash x, y$ is a parallel pair in $M_1/x, y$.*

We prove the first statement, and the second follows by Lemma 2.3.4. By Proposition 2.3.1, $\mathrm{si}(M_1)$ is a 3-connected, non-$\mathbb{F}$-representable matroid with a modular restriction $N_1 \cong \mathrm{PG}(2, \mathbb{F})$ and $\lambda_{M_1}(E(N_1)) = 3$. Let $X$ be the set of elements of $M_1$ parallel to an element of $E(N_1)$. By Lemma 2.3.4, $M_0 = ((R^* \oplus_m M_1)\backslash E(N_1))^*$, so $M_0/X = ((R^* \oplus_m \mathrm{si}(M_1))\backslash E(N_1))^*$. By Proposition 2.3.1 applied to $M_1$, $\mathrm{si}(M_0/X)$ is a 3-connected, non-$\mathbb{F}$-representable matroid with $N_0$ as a modular restriction. Thus the minimality of $M_0$ implies that $\mathrm{si}(M_0/X) = M_0$, and so $X = \emptyset$ and $M_1$ is 3-connected.

We have $M_1\backslash x = ((R \oplus_m (M_0/x))\backslash E(N_0))^*$ so by Lemma 2.3.6 applied to $M_0/x$, $\mathrm{si}(M_1\backslash x)$ is 3-connected; but $M_1$ is 3-connected so this implies that $M_1\backslash x$ is 3-connected. Similarly, $M_1\backslash y$ is 3-connected.

The modularity of $N_0$ in the 3-connected matroid $M_0/x$ implies that $y \notin \mathrm{cl}_{M_0/x}(E(N_0))$ so $M_0/x, y$ has $N_0$ as a restriction. Let $S_1$ be a set consisting of one element from each parallel pair of $M_0/x, y$ that is disjoint from $E(N_0)$, and let $S_2$ be the set of elements of $M_0/x, y$ parallel to an element of $E(N_0)$. Then by Lemma 2.3.6, $(R \oplus_m (M_0/x, y\backslash S_1, S_2))\backslash E(N_0)$ is internally 3-connected with no parallel pairs. But recall that $M_1^* = (R \oplus_m M_0)\backslash E(N_0)$ is 3-connected, so $(R \oplus_m (M_0/x, y\backslash S_1, S_2))\backslash E(N_0)$ has no series pairs either and is thus 3-connected. Therefore, $(R \oplus_m (M_0/x, y\backslash S_1))\backslash E(N_0)$ is also 3-connected as it is obtained by adding non-parallel elements without increasing the rank. So $M_1\backslash x, y/S_1$ is 3-connected. Each parallel pair of $M_0/x, y$ disjoint from $E(N_0)$ is a series pair of $M_1\backslash x, y$, and contracting an element from each of these pairs results in the 3-connected matroid $M_1\backslash x, y/S_1$, so $M_1\backslash x, y$ is internally 3-connected and $\{x, y\}$ is a deletion pair of $M_1$.

Any series pair of $M_1\backslash x, y$ is a parallel pair of $(R \oplus_m M_0)\backslash E(N_0)/x, y$ and hence a parallel pair of $M_0/x, y$. This proves (1).

We see that $M_0/E(N_0)$ is connected, for if not then by Proposition 2.1.4 $M_0$ is a modular sum of two proper restrictions of $M_0$ containing $N_0$. Both of these are 3-connected by Proposition 2.1.3 and at least one of them is not $\mathbb{F}$-representable by Proposition 2.1.2 and the Fundamental Theorem of Projective Geometry, contradicting the minimality of $M_0$. Moreover, $E(M_0) \setminus E(N_0)$ is non-empty as $M_0$ is not $\mathbb{F}$-representable.

For the following claim, we use the fact that binary matroids are uniquely representable over all fields [6].

(2) *No element of $M_0$ is in both a triangle and a triad.*

Suppose $f \in E(M_0)$ is in both a triangle and a triad. Since a circuit and a cocircuit cannot intersect in a single element and a triangle in a 3-connected matroid cannot be a triad, $M_0$ has elements $\{e, f, g, h\}$ such that $\{e, f, g\}$ is a triad and $\{f, g, h\}$ is a triangle. Since $\{e, f, g\}$ is not a triangle, $e$ is not in a four-point line of $M_0$. Suppose that $M_0/e$ has an internal 2-separation, $(A, B)$. Then $(A \cup \{e\}, B)$ is a 3-separation of $M_0$ and $e \in \text{cl}_{M_0}(A) \cap \text{cl}_{M_0}(B)$. Since $|A|, |B| \geq 3$ but $e$ is not in a four-point line in $M_0$, neither $A$ nor $B$ is contained in a parallel class of $M_0/e$. So $(A, B)$ is a vertical separation of $M_0/e$ and hence $(A \cup \{e\}, B)$ is a vertical 3-separation of $M_0$. Therefore, $e \in \text{cl}_{M_0}(A \setminus \text{cl}_{M_0}(B))$ and $e \in \text{cl}_{M_0}(B \setminus \text{cl}_{M_0}(A))$. Then as $\{e, f, g\}$ is a cocircuit, one of $f$ and $g$ is contained in $A \setminus \text{cl}_{M_0}(B)$ and the other in $B \setminus \text{cl}_{M_0}(A)$. But then $\{f, g\}$ cannot be contained in a triangle. So $M_0/e$ is internally 3-connected.

Let $N$ be a copy of $M(K_4)$ such that $E(N) \cap E(M_0) = \{f, g, h\}$ and $\{f, g, h\}$ is a triangle of $N$. Then $M_0$ is isomorphic to $(N \oplus_m (M_0/e))\backslash f, g$. Since $N$, being graphic, is $\mathbb{F}$-representable and $U_{2,3}$ is uniquely $\mathbb{F}$-representable, $M_0/e$ is not $\mathbb{F}$-representable by Proposition 2.1.2.

So $M_0$ has a proper minor $\text{si}(M_0/e)$ that is 3-connected and non-$\mathbb{F}$-representable and has $N_0$ as a restriction, contradicting our minimal choice of $M_0$. This proves (2).

Since $M_0/E(N_0)$ is connected and non-empty and no element of $M_0$ is in both a triangle and a triad, all assumptions of Lemma 2.4.5 hold for $M_0$.

Suppose that $M_0$ has a restriction $K \cong M(K_5)$ with a cocircuit $\{a, b, c, d\}$ such that $M_0 = K \oplus_m (M_0 \backslash a, b, c, d)$. Then $K \backslash a, b, c, d \cong M(K_4)$, which is binary hence uniquely $\mathbb{F}$-representable. Therefore, $M_0 \backslash a, b, c, d$ is non-$\mathbb{F}$-representable by Proposition 2.1.2. Also, it is 3-connected by Proposition 2.1.3, contradicting our minimal choice of $M_0$. This means that outcome (i) of Lemma 2.4.5 does not hold for $M_0$ and so $M_0$ has either a deletion or contraction pair $x, y \in E(M_0) \setminus E(N_0)$.

By (1), $\{x, y\}$ is a deletion pair in a matroid $M \in \{M_0, M_1\}$. We choose such a pair $\{x, y\}$ so that the number of series pairs of $M \backslash x, y$ is minimum. We may therefore assume that for any contraction pair $\{u, v\}$ of $M$ disjoint from $E(N_0)$ and $E(N_1)$, $M/u, v$ has at least as many parallel pairs as $M \backslash x, y$ has series pairs. We now show that $M \backslash x, y$ has at most one series pair.

We denote the series pairs of $M \backslash x, y$ by $S_1 = \{a_1, b_1\}, \ldots, S_k = \{a_k, b_k\}$.

(3) *Let $c \in S_1 \cup \cdots \cup S_k$ such that $c$ is not in a triangle of $M$. If $\{x, y\} \not\subseteq \mathrm{cl}_M(\{c\} \cup S_j)$ for all $j$, then $M/c$ is 3-connected.*

By symmetry we may assume that $c \in S_1$. It suffices to show that $M/c$ is internally 3-connected. Suppose that $M/c$ has an internal 2-separation $(A, B)$ with $y \in B$. If $x \in A$, then as $M/c \backslash x, y$ is internally 3-connected, either $A$ or $B$ is a triad of $M/c$ hence also of $M$, contradicting the fact that $M \backslash x$ and $M \backslash y$ are 3-connected. So $x, y \in B$. Since $\{c, x, y\}$ is not a triangle, $|B| > 2$, and the fact that $M \backslash x$ is 3-connected implies that $B$ is not a triad, so $|B| > 3$. Then since $M/c \backslash x, y$ is internally 3-connected, $B \setminus \{x, y\}$ is a series pair of it, so $B = S_j \cup \{x, y\}$ for some $j > 1$. Since $M/c$ is connected and $\lambda_{M/c}(S_j \cup \{x, y\}) = 1$, either $r_{M/c}(\{x, y\} \cup S_j) = 2$ or $r^*_{M/c}(\{x, y\} \cup S_j) = 2$; but $r^*_{M/c \backslash x,y}(S_j) = 1$ so we have $r_{M/c}(\{x, y\} \cup S_j) = 2$. But this means $\{x, y\} \subseteq \mathrm{cl}_{M/c}(S_j)$ so $\{x, y\} \subseteq \mathrm{cl}_M(\{c\} \cup S_j)$, a contradiction. This proves (3).

(4) *$M \backslash x, y$ does not have exactly two series pairs.*

Suppose that $M \backslash x, y$ has exactly two series pairs. First, we assume that $\{x, y\} \not\subseteq \mathrm{cl}_M(S_1 \cup S_2)$; by symmetry we may assume that $y \notin \mathrm{cl}_M(S_1 \cup S_2)$. Then there is at most one triangle containing an element of $S_1 \cup S_2$; either it contains $\{x, y\}$ or it contains $x$ and an element of each of $S_1$ and $S_2$. So we may assume that $a_1$ and $a_2$ are not contained in any triangles, and if $b_2$ is contained in a triangle then it is $\{x, b_1, b_2\}$. Then by (3), $M/a_1$ and $M/a_2$ are 3-connected. Since $M \backslash x, y/a_1, a_2$ is 3-connected, $\mathrm{si}(M/a_1, a_2)$ is 3-connected and all parallel pairs of $M/a_1, a_2$ contain $x$ or $y$. So if $M/a_1, a_2$ has a parallel class of size greater than two, then it contains $x$ and $y$, and so $\{a_2, x, y\}$ is a triangle of $M/a_1$. But then $\{x, b_1, b_2\}$ is not a triangle of $M$, so by (3), $M/b_2$ is also 3-connected, and $M/a_1, b_2$ has no parallel classes of size greater than two. So by possibly swapping the labels of $a_2$ and $b_2$, we may assume that $M/a_1, a_2$ is internally 3-connected. By our choice of deletion pair $x, y$, there are exactly two parallel pairs in $M/a_1, a_2$. We call them $\{x, u\}$ and $\{y, v\}$.

We consider $M\backslash x, a_2$, which has $\{b_2, y\}$ as a series pair. In $M\backslash x/y$, $S_1 \cup S_2$ is a 4-element cocircuit. Also, $\lambda_{M\backslash x/y}(S_1 \cup S_2) = 3$ because $y \notin \mathrm{cl}_M(S_1 \cup S_2)$. Moreover, $v \in \mathrm{cl}_{M\backslash x/y}(\{a_1, a_2\})$ but $v \notin \mathrm{cl}_{M\backslash x/y}(\{a_1, b_1, b_2\})$, so $\lambda_{M\backslash x, a_2/y}(\{a_1, b_1, b_2, v\}) = 3$ and $M\backslash x, a_2/y$ is 3-connected. Thus $M\backslash x, a_2$ is internally 3-connected with a unique series pair, $\{b_2, y\}$. Then $M\backslash a_2$ is 3-connected because $x \notin \mathrm{cl}_M(E(M) \setminus (S_2 \cup \{x, y\}))$ and $x \notin \mathrm{cl}_M(\{b_2, y\})$. This contradicts our choice of deletion pair $\{x, y\}$. Therefore, we may assume that $\{x, y\} \subseteq \mathrm{cl}_M(S_1 \cup S_2)$.

Suppose $\{x, y\}$ is contained in a triangle; we may assume that either $\{b_1, x, y\}$ is the only such triangle or $\{b_1, x, y\}$ and $\{b_2, x, y\}$ are the only two. Then (3) implies that $M/a_2$ is 3-connected. Also, $\{b_1, b_2, x, y\}$ is a 4-point line of $M/a_2$, so $M/a_2\backslash b_1$ and $M/a_2\backslash b_2$ are 3-connected. Then $M\backslash b_1$ and $M\backslash b_2$ are internally 3-connected and any series pair of each contains $a_2$. But neither $\{a_2, b_1\}$ nor $\{a_2, b_2\}$ is contained in a triad of $M$, so $M\backslash b_1$ and $M\backslash b_2$ are 3-connected. In $M\backslash b_1, b_2$, $\{x, y\}$ is a series pair. In $M\backslash b_1, b_2/y$, $\{a_1, a_2, x\}$ is a triad which is not a triangle, so $\lambda_{M\backslash b_1, b_2/y}(\{a_1, a_2, x\}) = 2$ and $M\backslash b_1, b_2/y$ is 3-connected. Therefore, $M\backslash b_1, b_2$ is internally 3-connected with a single series pair, $\{x, y\}$, contradicting our choice of deletion pair $\{x, y\}$. So $\{x, y\}$ is not contained in a triangle.

At least one of $\{a_1, x\}$ and $\{b_1, x\}$ is skew to $S_2$, for if not then since $\{a_1, b_1, x\}$ is not a triangle we have $\sqcap_M(S_1 \cup \{x\}, S_2) = 2$, contradicting the fact that $r_M(S_1 \cup S_2) = 4$. By symmetry we may assume that $\{a_1, x\}$ is skew to $S_2$. We claim that $M\backslash x, a_1$ is internally 3-connected with one series pair, $\{b_1, y\}$. If not, then $M\backslash x, a_1/b_1$ is not 3-connected. But $M\backslash x/b_1$ is 3-connected and has a triad $\{a_2, b_2, y\}$, so $\{a_2, b_2, y\}$ is also a triad of $M\backslash x, a_1/b_1$. Then we may choose a 2-separation $(A, B)$ of $M\backslash x, a_1/b_1$ with $a_2, b_2, y \in A$. But then $a_1 \in \mathrm{cl}_{M\backslash x/b_1}(\{a_2, b_2, y\}) \subseteq \mathrm{cl}_{M\backslash x/b_1}(A)$, a contradiction. So $M\backslash x, a_1$ is internally 3-connected with a unique series pair. Moreover, $M\backslash a_1$ is 3-connected because $x \notin \mathrm{cl}_M(\{b_1, y\})$ and $x \notin \mathrm{cl}_M(E(M) \setminus (S_1 \cup S_2 \cup \{y\}))$, contradicting our choice of deletion pair $\{x, y\}$. This proves (4).

(5) *There is at most one $c \in S_1 \cup \cdots \cup S_k$ such that $\{c, x, y\}$ is a triangle.*

If not, then we may assume that there are distinct triangles $\{c_1, x, y\}$ and $\{c_2, x, y\}$ with $c_1 \in S_1$ and $c_2 \in S_2$. Then $x, y \in \mathrm{cl}_M(S_1 \cup S_2)$. But (4) implies that there is a third series pair $S_3$ of $M\backslash x, y$, and in this case $S_3$ is also a series pair of $M$, which is 3-connected. This proves (5).

(6) *If $M\backslash x, y$ has more than one series pair, it has exactly three and $\{x, y\}$ is in the closure of their union.*

By (4), we may assume that $M\backslash x, y$ has at least three series pairs, and by (5) we may assume that for any $c \in S_2 \cup S_3$, $\{c, x, y\}$ is not a triangle. Then any triangle of $M$ containing $c$ contains exactly one of $x$ and $y$, say $x$. But if such a triangle exists then for some $r$, $S_r$ is skew to it and is thus a series pair of $M\backslash y$, a contradiction. So $c$ is in no triangles of $M$, and by (3), $M/c$ is 3-connected. For any two such elements $c \in S_2, d \in S_3$, any parallel class of $M/c, d$ contains $x$ or $y$ since $M/c, d\backslash x, y$ has no parallel pairs. If $M/c, d$ is internally 3-connected, this implies that there are at most two parallel pairs in $M/c, d$, which contradicts our choice of $\{x, y\}$. Thus it suffices to show that $M/c, d$ is internally 3-connected.

Assume that $M/c, d$ is not internally 3-connected. Then $M/c, d$ has an internal 2-separation $(A, B)$ with $y \in B$. If $x \in A$ then, as $M/c, d\backslash x, y$ is internally 3-connected, $A$ or $B$ is a triad of $M/c, d$ hence also of $M$, contradicting the fact that $M\backslash x$ and $M\backslash y$ are 3-connected. So $x, y \in B$, and since $M\backslash x, y/c, d$ is internally 3-connected, $|B| \leq 4$.

Let $z \in B \setminus \{x, y\}$. If $B$ is a parallel class, then $\{c, d, z, x\}$ and $\{c, d, z, y\}$ are circuits of $M$. If $z \in S_1$, then (6) holds, while if not then $\{x, y\}$ is in the closure of $E(M) \setminus S_1$ so $S_1$ is a series pair of $M$, a contradiction. Therefore, $r_{M/c,d}(B) \geq 2$. Also, $r^*_{M/c,d}(B) = r^*_M(B) \geq 2$ since $M$ is 3-connected. It follows that if $|B| = 3$, then $B$ is a triad of $M/c, d$ and hence also of $M$, a contradiction. So $|B| = 4$, $r_{M/c,d}(B) = 2$, and $B \setminus \{x, y\}$ is a series pair $S_i$ of $M\backslash x, y$. Therefore, $x, y \in \mathrm{cl}_M(S_2 \cup S_3 \cup S_i)$ and (6) holds.

(7) *$M\backslash x, y$ has at most one series pair.*

If not, then by (6) it has three. By (5) and (3), we may assume that $M/a_1$, $M/b_1$ and $M/a_2$ are 3-connected. $S_3$ is a series pair of $M/a_1, a_2\backslash x, y$ and of $M/b_1, a_2\backslash x, y$, which each have a unique 2-separation. This implies that $M/a_1, a_2\backslash y$ is 3-connected if and only if $x \notin \mathrm{cl}_{M/a_1,a_2}(S_3)$, and $M/b_1, a_2\backslash y$ is 3-connected if and only if $x \notin \mathrm{cl}_{M/b_1,a_2}(S_3)$. If at least one of $M/a_1, a_2\backslash y$ and $M/b_1, a_2\backslash y$ is 3-connected, then one of $M/a_1, a_2$ and $M/b_1, a_2$ is internally 3-connected with at most one parallel pair, contradicting our choice of $\{x, y\}$. So we may assume that $x \in \mathrm{cl}_{M/a_1,a_2}(S_3)$ and $x \in \mathrm{cl}_{M/b_1,a_2}(S_3)$. But then we have $a_1 \in \mathrm{cl}_M(\{x, a_2\} \cup S_3)$ and $b_1 \in \mathrm{cl}_M(\{x, a_2\} \cup S_3)$, so $r_M(S_1 \cup S_3 \cup \{a_2\}) \leq 4$, a contradiction. This proves (7).

If $M = M_0$, then we are done by (7). If $M = M_1$, then it remains to show that $M_1$ has no 3-connected, non-$\mathbb{F}$-representable proper minor with $N_1$ as a restriction. If it does have such a minor $M_1\backslash D/C$, then by Lemma 2.3.4 we have $M_0/D\backslash C = ((R^* \oplus_m (M_1\backslash D/C))\backslash E(N_1))^*$ and then Proposition 2.3.1 implies that $\mathrm{si}(M_0/D\backslash C)$ is a 3-connected, non-$\mathbb{F}$-representable proper minor of $M_0$ with $N_0$ as a restriction, a contradiction. □

## 2.5 Stabilizers

Two representations of a matroid over a field $\mathbb{F}$ are called **equivalent** if one can be obtained from the other by row operations (including adjoining and removing zero rows) and column scaling; they are **inequivalent** otherwise. When $N$ is a minor of a matroid $M$, we say that $N$ **stabilizes** $M$ over $\mathbb{F}$ if no $\mathbb{F}$-representation of $N$ extends to two inequivalent $\mathbb{F}$-representations of $M$. We will use the following fact about stabilizers for matroids over a finite field $\mathbb{F}$.

**Theorem 2.5.1** (Geelen, Whittle, [22])**.** *If $M$ is a 3-connected matroid with $\mathrm{PG}(2,\mathbb{F})$ as a minor, then $\mathrm{PG}(2,\mathbb{F})$ stabilizes $M$ over $\mathbb{F}$.*

A matroid $M$ is called **stable** if it is connected and is not a 2-sum of two non-binary matroids (this definition differs slightly from the original in [14] in that we require that $M$ be connected).

We observe that Theorem 2.5.1 implies that any stable matroid $M$ with $\mathrm{PG}(2,\mathbb{F})$ as a minor is stabilized by it over $\mathbb{F}$; this follows from the fact that binary matroids are uniquely representable over any field [6]. In particular, if $M$ is a direct sum or a 2-sum of a 3-connected matroid $N$ and a binary matroid, then every $\mathbb{F}$-representation of $N$ extends to a unique $\mathbb{F}$-representation of $M$. For a field $\mathbb{F}$, we call a matroid a **stabilizer for** $\mathbb{F}$ if it stabilizes over $\mathbb{F}$ all stable matroids that have it as a minor.

In the next section we will apply the following two lemmas about stabilizers. They were proved by Geelen, Gerards and Whittle in [16] and can also be derived from results in [56].

**Lemma 2.5.2** (Geelen, Gerards, Whittle, [16])**.** *Let $N$ be a uniquely $\mathbb{F}$-representable stabilizer for a finite field $\mathbb{F}$. Let $M$ be a matroid with $x, y \in E(M)$ such that $\{x, y\}$ is coindependent and $M\backslash x, y$ is stable and has an $N$-minor. If $M\backslash x$ and $M\backslash y$ are both $\mathbb{F}$-representable, then there exists an $\mathbb{F}$-representable matroid $M'$ such that $M'\backslash x = M\backslash x$ and $M'\backslash y = M\backslash y$.*

We remark that although their statement of the above lemma [16, Lemma 5.3] requires that $M$ be 3-connected, their proof requires only that $\{x, y\}$ be coindependent in $M$.

**Lemma 2.5.3** (Geelen, Gerards, Whittle, [16]). *Let $\mathbb{F}$ be a finite field and let $M$ and $M'$ be $\mathbb{F}$-representable matroids on the same ground set with elements $x, y \in E(M)$ such that $M\backslash x = M'\backslash x$ and $M\backslash y = M'\backslash y$. If $M\backslash x$ and $M\backslash y$ are both stable, $M\backslash x, y$ is connected, and $M\backslash x, y$ has a minor that is a uniquely $\mathbb{F}$-representable stabilizer for $\mathbb{F}$, then $M = M'$.*

## 2.6 Finding distinguishing sets

In this section, we show that if $M$ is a matroid with a restriction $N_0 \cong \mathrm{PG}(2, \mathbb{F})$ and a deletion pair $x, y \notin E(N_0)$ such that $M\backslash x$ and $M\backslash y$ are $\mathbb{F}$-representable, then there is a unique $\mathbb{F}$-representable matroid $M'$ on $E(M)$ whose rank function can differ from that of $M$ only on sets containing $x$ and $y$. Moreover, we find two such sets with special properties that will be used to prove Lemma 2.0.4.

**Lemma 2.6.1.** *Let $M$ be a 3-connected matroid with a restriction $N_0 \cong \mathrm{PG}(2, \mathbb{F})$ and a deletion pair $x, y \in E(M) \setminus E(N_0)$. If $M\backslash x$ and $M\backslash y$ are $\mathbb{F}$-representable, then there is a unique $\mathbb{F}$-representable matroid $M'$ such that $M'\backslash x = M\backslash x$ and $M'\backslash y = M\backslash y$.*

*Proof.* The definition of a deletion pair implies that $M\backslash x$, $M\backslash y$, and $M\backslash x, y$ are all stable. We recall from Theorem 2.5.1 that $\mathrm{PG}(2, \mathbb{F})$ is a stabilizer for $\mathbb{F}$, and that $\mathrm{PG}(2, \mathbb{F})$ is uniquely $\mathbb{F}$-representable. Therefore, with $N = N_0$ all the hypotheses of Lemma 2.5.2 are satisfied, and there is an $\mathbb{F}$-representable matroid $M'$ such that $M\backslash x = M'\backslash x$ and $M\backslash y = M'\backslash y$. Then by Lemma 2.5.3, $M'$ is the unique such matroid. □

The purpose of the remainder of this section is to find two ways to distinguish the matroids $M$ and $M'$ of Lemma 2.6.1; these are

(a) elements $e \in E(M)$ such that $M\backslash e \neq M'\backslash e$ and $M/e \neq M'/e$, and

(b) sets $S \subseteq E(M) \setminus E(N_0)$ such that $\mathrm{cl}_M(S) \neq \mathrm{cl}_{M'}(S)$.

In a matroid $M$ with a restriction $N$, we call a set $S \subseteq E(M) \setminus E(N)$ a **strand** for $N$ if $\sqcap_M(S, E(N)) = 1$. If $M$ and $M'$ are matroids on the same ground set, both contain a restriction $N$, and $S$ is a strand for $N$ in both $M$ and $M'$, then we say that $S$ **distinguishes** $M$ and $M'$ if $\mathrm{cl}_M(S) \cap E(N) \neq \mathrm{cl}_{M'}(S) \cap E(N)$.

When $B$ is a basis of a matroid $M$ and $e$ is an element not in $B$, then the **fundamental circuit** of $e$ with respect to $B$ is the unique circuit of $M$ contained in $B \cup \{e\}$. The **fundamental matrix** of a matroid $M$ with respect to a basis $B$ is the matrix $A \in \{0,1\}^{B\times(E(M)\setminus B)}$ such that for each $e \in E(M)\setminus B$, the column of $A$ indexed by $e$ is the characteristic vector of the fundamental circuit of $e$ with respect to $B$. If $A'$ is any representation of $M$ in standard form with respect to $B$, then the matrix obtained from $A'|(E(M)\setminus B)$ by replacing each non-zero entry with 1 is the fundamental matrix of $M$ with respect to $B$.

**Lemma 2.6.2.** *Let $M$ and $M'$ be matroids on the same ground set that both have a modular restriction $N_0 \cong \mathrm{PG}(2,\mathbb{F})$, such that $M$ is 3-connected and $M \neq M'$, but for some $x, y \in E(M)\setminus E(N_0)$, $M'\backslash x = M\backslash x$ and $M'\backslash y = M\backslash y$. There are sets $B$ and $B'$ such that*

*(i) $B$ is a basis of both $M$ and $M'$ and contains a basis of $N_0$,*

*(ii) $B'$ is a basis of exactly one of $M$ and $M'$,*

*(iii) $|(B\setminus B')\setminus E(N_0)| \in \{1,2\}$, and*

*(iv) $|B\Delta B'| = 4$.*

*Proof.* Since $M \neq M'$, there exists a set $B'$ that is a basis of exactly one of $M$ and $M'$. We choose $B'$ and a basis $B$ of $M\backslash x, y$ containing a basis of $N_0$ such that $|B\Delta B'|$ is minimum. As $M$ is 3-connected, $B$ is a basis of $M$ and also of $M'$.

(1) *$B'\setminus B \subseteq E(M)\setminus E(N_0)$.*

Suppose there exists an element $u$ of $(B'\setminus B)\cap E(N_0)$. Then by the basis exchange property, there is $v \in B\setminus B'$ such that $B\Delta\{u,v\}$ is a basis in $M$ or $M'$. But since $B\Delta\{u,v\}$ is contained in $E(M)\setminus\{x,y\}$, it is a basis of both $M$ and $M'$. Because $u \in E(N_0)$, $B\Delta\{u,v\}$ contains a basis of $N_0$, contradicting the minimality of $|B\Delta B'|$. This proves (1).

It follows from the fact that $M\backslash x = M'\backslash x$ and $M\backslash y = M'\backslash y$ that the fundamental matrices of $M$ and $M'$ with respect to the basis $B$ are equal; we denote this matrix by $A$.

(2) *$A[(B\setminus B')\setminus E(N_0), (B'\setminus B)\setminus\{x,y\}] = 0$.*

Suppose there exist elements $u \in B'\setminus B\setminus\{x,y\}$ and $v \in (B\setminus B')\setminus E(N_0)$ such that $B\Delta\{u,v\}$ is a basis of $M$ or $M'$. Then since $B\Delta\{u,v\} \subseteq E(M)\setminus\{x,y\}$, it is a basis of both $M$ and $M'$. Furthermore, $B\Delta\{u,v\}$ contains a basis of $N_0$ because $v \notin E(N_0)$, contradicting the minimality of $|B\Delta B'|$. Therefore, for every $u \in (B'\setminus B)\setminus\{x,y\}$ and every $v \in (B\setminus B')\setminus E(N_0)$, the set $B\Delta\{u,v\}$ is dependent in both $M$ and $M'$. This proves (2).

(3) $|(B \setminus B') \setminus E(N_0)| \leq 2$.

Let $e$ and $f$ be two elements of $(B \setminus B') \setminus E(N_0)$. It follows from (2) that $B'$ is a basis of $M'' \in \{M, M'\}$ if and only if $B\Delta\{x, y, e, f\}$ and $B\Delta((B\Delta B') \setminus \{x, y, e, f\})$ are both bases of $M''$. The latter is not a basis of $M''$ in the case when $|(B \setminus B') \setminus E(N_0)| > 2$, as in this case (2) implies that $A[B \setminus (B' \cup \{e, f\}), B' \setminus (B \cup \{x, y\})]$ has a zero row. This contradicts the fact that $B'$ is a basis of exactly one of $M$ and $M'$, proving (3).

(4) $|(B \setminus B') \setminus E(N_0)| \in \{1, 2\}$.

We suppose that $|(B \setminus B') \setminus E(N_0)| = 0$. Then $B \setminus E(N_0) \subseteq B'$. Hence the set $B' \setminus (B \setminus E(N_0))$ is independent in exactly one of the matroids $M/(B \setminus E(N_0))$ and $M'/(B \setminus E(N_0))$. But it follows from the modularity of $N_0$ and the definition of $M'$ that $M/(B \setminus E(N_0)) = M'/(B \setminus E(N_0))$, a contradiction. This proves (4), showing that (iii) holds.

It remains to show that $B$ and $B'$ satisfy (iv). If not, then there is an element $w$ of $B' \setminus B$ other than $x$ and $y$. By the modularity of $N_0$ and (2), there is an element $z \in E(N_0)$ such that $w$ is parallel to $z$ in both $M/(B \cap B' \setminus E(N_0))$ and $M'/(B \cap B' \setminus E(N_0))$. Then $B'\Delta\{w, z\}$ is independent in exactly one of $M$ and $M'$. Also, $|B\Delta(B'\Delta\{w, z\})| = |B\Delta B'|$, so (1) holds with $B'\Delta\{w, z\}$ in place of $B'$, a contradiction. □

We will need the following two facts about fundamental matrices. For matrices $P$ and $Q$ of the same dimensions, we write $P \leq Q$ when $P_{ij} \leq Q_{ij}$ for each row $i$ and column $j$.

**Proposition 2.6.3** (Brualdi, [5])**.** *Let $B$ be a basis of a matroid $M$, $A$ the fundamental matrix of $M$ with respect to $B$, and $X \subseteq B$ and $Y \subseteq E(M) \setminus B$ sets of the same size. If $(B \setminus X) \cup Y$ is a basis of $M$ then there exists a permutation matrix $P$ such that $P \leq A[X, Y]$.*

**Proposition 2.6.4** (Krogdahl, [32])**.** *Let $B$ be a basis of a matroid $M$, $A$ the fundamental matrix of $M$ with respect to $B$, and $X \subseteq B$ and $Y \subseteq E(M) \setminus B$ sets of the same size. If there is a unique permutation matrix $P$ such that $P \leq A[X, Y]$, then $(B \setminus X) \cup Y$ is a basis of $M$.*

Next we find a strand for $N_0$ in one of $M$ and $M'$ that either distinguishes $M$ and $M'$ or is not a strand in the other.

**Lemma 2.6.5.** *Let $M$ be a 3-connected, non-$\mathbb{F}$-representable matroid with a modular restriction $N_0 \cong \mathrm{PG}(2, \mathbb{F})$ and elements $x, y \in E(M) \setminus E(N_0)$. Let*

*$M'$ be an $\mathbb{F}$-representable matroid such that $M'\backslash x = M\backslash x$ and $M'\backslash y = M\backslash y$. Either there exists a strand for $N_0$ that distinguishes $M$ and $M'$, or there is a set $S$ that is a strand for $N_0$ in one of $M$ or $M'$ and skew to $E(N_0)$ in the other.*

*Proof.* We choose $B$ and $B'$ as in the statement of Lemma 2.6.2. We denote the two elements of $B \setminus B'$ by $e$ and $f$ and let $A$ denote the fundamental matrix of $M$ (and also $M'$) with respect to the basis $B$. We observe that all entries of $A[\{e, f\}, \{x, y\}]$ are equal to 1 by Propositions 2.6.3 and 2.6.4.

We let $N = M/((B \cap B') \setminus E(N_0))$ and $N' = M'/((B \cap B') \setminus E(N_0))$. If there is a strand distinguishing $N$ and $N'$, then since $E(N_0)$ is closed, the union of this strand with $B \cap B' \setminus E(N_0)$ is a strand distinguishing $M$ and $M'$. We note that by (iii) of Lemma 2.6.2, $r(N) \leq r(N_0) + 2$ and at most one of $e, f$ is contained in $E(N_0)$.

(1) *If $r(N) = r(N_0) + 2$ then there is a set $S \subseteq E(M) \setminus E(N_0)$ such that one of $\sqcap_M(S, E(N_0)), \sqcap_{M'}(S, E(N_0))$ is $0$ and the other is $1$.*

The set $(B \cap E(N_0)) \cup \{x, y\}$ is independent in exactly one of $N$ and $N'$. So $\{x, y\}$ is a strand for $N_0$ in one of $N$ and $N'$ and skew to $E(N_0)$ in the other. Then $\{x, y\} \cup ((B \cap B') \setminus E(N_0))$ has the same property in $M$ and $M'$, proving (1).

We may assume that $r(N) = r(N_0) + 1$, and by symmetry that $e \notin E(N_0)$ and $f \in E(N_0)$.

(2) *If $r(N) = r(N_0) + 1$ then either $\{x, y\}$ is independent in both $N$ and $N'$ or there is a set $S \subseteq E(M) \setminus E(N_0)$ such that one of $\sqcap_M(S, E(N_0)), \sqcap_{M'}(S, E(N_0))$ is $0$ and the other is $1$.*

We assume that $\{x, y\}$ is a parallel pair in one of $N$ and $N'$. Then they are parallel in exactly one of $N$ and $N'$ because $B' \cap E(N)$, which contains $\{x, y\}$, is a basis of one of $N$ and $N'$. We note that $\{x, e\}$ is not a parallel pair in $N$ or $N'$, otherwise $\{x, y, e\}$ is a parallel class of both matroids. Let $w$ be the element of $N_0$ in $\mathrm{cl}_N(\{x, e\})$ (and hence also in $\mathrm{cl}_{N'}(\{x, e\})$). Then in the matroid in which $x$ and $y$ are parallel, $\{y, e\}$ also spans $w$; this means $\{x, y, e, w\}$ has rank two in both $N$ and $N'$.

In the matroid in which $\{x, y\}$ are independent, $\{x, y\}$ is a strand for $N_0$, while in the other it is skew to $E(N_0)$. Therefore, $S = \{x, y\} \cup (B \cap B' \setminus E(N_0))$ is a strand for $N_0$ in one of $M$ and $M'$ and skew to $E(N_0)$ in the other. This proves (2).

Now we may assume that $\{x, y\}$ is independent in both $N$ and $N'$. Since $A_{ex} = A_{ey} = 1$, neither $x$ nor $y$ are in the closure of $E(N_0)$ in $N$ or $N'$. Then $\{x, y\}$ is a strand in both $N$ and $N'$.

We let $D$ and $D'$ be the fundamental matrices of $N$ and $N'$, respectively, with respect to the basis $B\Delta\{e, y\}$. Since $(B\Delta\{e, y\})\Delta\{f, x\} = B'$ is independent in exactly one of $N$ and $N'$, it follows that exactly one of $D_{fx}$ and $D'_{fx}$ is equal to 1. Thus $D[E(N_0), \{x\}] \neq D'[E(N_0), \{x\}]$. If $w$ is the element of $E(N_0) \cap \mathrm{cl}_N(\{x, y\})$ and $w'$ is the element of $E(N_0) \cap \mathrm{cl}_{N'}(\{x, y\})$, then $D[E(N_0), \{w\}] = D[E(N_0), \{x\}]$ and $D'[E(N_0), \{w'\}] = D'[E(N_0), \{x\}]$. But because $N\backslash x = N'\backslash x$, $D[E(N_0), \{w\}] = D'[E(N_0), \{w\}]$, so $w \neq w'$. This proves that $\{x, y\}$ is a strand distinguishing $N$ and $N'$ and so there is a strand distinguishing $M$ and $M'$. □

For disjoint sets $S, T$ in a matroid $M$, we define

$$\kappa_M(S, T) = \min\{\lambda_M(A) : S \subseteq A \subseteq E(M) \setminus T\}.$$

Let $M$ and $M'$ be two matroids on the same ground set. We write $\Sigma(M, M')$ to denote the set

$$\Sigma(M, M') = \{e \in E(M) : M\backslash e \neq M'\backslash e \text{ and } M/e \neq M'/e\}.$$

We now prove the main lemma of this section.

**Lemma 2.6.6.** *Let $M$ be a 3-connected, non-$\mathbb{F}$-representable matroid with a modular restriction $N_0 \cong \mathrm{PG}(2, \mathbb{F})$ such that no proper minor of $M$ with $N_0$ as a restriction is 3-connected and non-$\mathbb{F}$-representable. If $\lambda_M(E(N_0)) = 3$, $x, y \in E(M) \setminus E(N_0)$ are distinct, and $M'$ is an $\mathbb{F}$-representable matroid with $M'\backslash x = M\backslash x$ and $M'\backslash y = M\backslash y$, then*

*(i) $|\Sigma(M, M')| \geq 2$, and*

*(ii) there are non-nested sets $S, T \subseteq E(M) \setminus E(N_0)$ such that $\mathrm{cl}_M(S) \cap E(N_0) \neq \mathrm{cl}_{M'}(S) \cap E(N_0)$, $\mathrm{cl}_M(T) \cap E(N_0) \neq \mathrm{cl}_{M'}(T) \cap E(N_0)$, and $S\Delta T \subseteq \Sigma(M, M')$.*

*Proof.* We start with some short claims.

(1) *$M/E(N_0)$ and $M'/E(N_0)$ are connected.*

If $M/E(N_0)$ is not connected, then by Proposition 2.1.4, $M$ is a modular sum of two proper restrictions $M_1$ and $M_2$ of $M$ with $E(M_1)\cap E(M_2) = E(N_0)$. Both $M_1$ and $M_2$ are 3-connected by Proposition 2.1.3 and so both are $\mathbb{F}$-representable by the choice of $M$. Then Proposition 2.1.2 implies that $M$ is $\mathbb{F}$-representable, a contradiction. Suppose that $M'/E(N_0)$ is not connected. Then $M'/E(N_0)$ has at least two components. Let $A$ be that containing $x$. We can choose a basis $B$ of $M'$ that contains a basis of $N_0$ and does not contain $x$ or $y$. Then $B \cap (E(N_0) \cup A)$ spans $x$ in $M' \backslash y$ and hence also in $M$. Now $M/E(N_0)\backslash x$ has at least two components, and we can choose one, $A'$, disjoint from $A \setminus \{x\}$. But $B \cap (E(N_0) \cup A)$ spans $x$ and so $A$ and $A'$ are components of $M/E(N_0)$, contradicting the fact that $M/E(N_0)$ is connected and proving (1).

(2) $\lambda_{M'}(E(N_0)) = 3$.

Suppose that $\lambda_{M'}(E(N_0)) < 3$; then $\lambda_{M'}(E(N_0)) = 2$ since $M'\backslash x = M\backslash x$. We let $L$ be the line of $N_0$ that is spanned by $E(M') \setminus E(N_0)$. Since $M'$ is 3-connected, it has a basis $B$ disjoint from $\{x, y\}$; since $\lambda_{M'}(E(N_0)) = 2$, we may further choose $B$ so that it contains at most one element of $E(N_0) \setminus L$. Then the set $B \cap \mathrm{cl}_{M'}(E(M') \setminus E(N_0))$ is a basis for $M'\backslash(E(N_0) \setminus L)$ and hence spans $\{x, y\}$. This means that $x, y \in \mathrm{cl}_M(B \cap \mathrm{cl}_{M'}(E(M') \setminus E(N_0)))$, and then $\lambda_M(E(N_0)) = \lambda_{M\backslash x,y}(E(N_0)) < 3$, a contradiction. This proves (2).

(3) *If $S$ is a strand for $E(N_0)$ in $N \in \{M, M'\}$, then there exist sets $U_1$ and $U_2$ in $E(N) \setminus E(N_0)$ such that $\kappa_{N|(E(N_0)\cup U_1)}(S, E(N_0)) > 1$, $\kappa_{N|(E(N_0)\cup U_2)}(S, E(N_0)) > 1$, and $\mathrm{cl}_N(U_1) \cap E(N_0)$ and $\mathrm{cl}_N(U_2) \cap E(N_0)$ are distinct lines containing $\mathrm{cl}_N(S) \cap E(N_0)$.*

Let $L$ be a line of $N_0$ such that $\sqcap_N(S, L) = 1$, and $A = E(N_0) \setminus L$. Suppose that $\kappa_{N\backslash A}(S, L) < 2$. If $\kappa_{N\backslash A}(S, L) = 0$ then $N/A$ is not connected; so $\kappa_{N\backslash A}(S, L) = 1$. We have a 2-separation $(U, V)$ of $N\backslash A$ with $S \subseteq U$ and $L \subseteq V$. But since $\sqcap_N(S, L) = 1$, we have $\sqcap_N(U, L) = 1$. Then either $N/E(N_0)$ is not connected or $V = L$, which implies that $\lambda_N(E(N_0)) < 3$; this contradicts either (1) or (2).

Therefore, $\kappa_{N\backslash A}(S, L) \geq 2$, and there exists a minimal set $U \subseteq E(N\backslash A)$ such that $\kappa_{N|(L\cup U\cup S)}(S, L) = 2$. Choosing two lines $L$ of $N_0$ with $\sqcap_N(S, L) = 1$, we obtain the two sets $U_1$ and $U_2$ as required, proving (3).

(4) *Let $N''$ be a restriction of $N \in \{M, M'\}$ containing $N_0$ such that $E(N'') \setminus E(N_0)$ is independent. If $X$ and $Y$ are minimal strands for $N_0$ in $N''$ such that $\mathrm{cl}_N(X) \cap E(N_0) = \mathrm{cl}_N(Y) \cap E(N_0)$, then $X = Y$.*

Suppose there are two distinct minimal strands $X$ and $Y$ for $N_0$ in $N''$ such that $\mathrm{cl}_N(X) \cap E(N_0) = \mathrm{cl}_N(Y) \cap E(N_0)$. We denote by $e$ the element of $N_0$ spanned by $X$ and $Y$. By the minimality of $X$ there exists an element $b \in Y \setminus X$. Then $b \in \mathrm{cl}_N(Y \cup \{e\} \setminus \{b\}) \subseteq \mathrm{cl}_N(X \cup Y \setminus \{b\})$, contradicting the fact that $E(N'') \setminus E(N_0)$ is independent. This proves (4).

(5) *If $S$ is a strand for $N_0$ in $N \in \{M, M'\}$ and $U$ is a subset of $E(N) \setminus E(N_0)$ containing $S$ such that $\kappa_{N|(E(N_0)\cup U)}(S, E(N_0)) > 1$, then $U$ contains two strands $T_1$ and $T_2$ such that $\mathrm{cl}_N(S) \cap E(N_0)$, $\mathrm{cl}_N(T_1) \cap E(N_0)$ and $\mathrm{cl}_N(T_2) \cap E(N_0)$ are all distinct.*

We may assume that $U$ is minimal and thus independent, and that $S$ is minimal. We pick any element $z \in S$. Then $\sqcap_N(U \setminus \{z\}, E(N_0)) \geq 1$ so $U \setminus \{z\}$ contains a minimal strand $T_1$. It follows from (4) that $\mathrm{cl}_N(T_1) \cap E(N_0) \neq \mathrm{cl}_N(S) \cap E(N_0)$.

If $S \cap T_1 \neq \emptyset$, then there is an element $s \in S \cap T_1$ and $U \setminus \{s\}$ contains a minimal strand $T_2$ for $N_0$ distinct from $S$ and $T_1$. Similarly, if for some $s \in S$ and $t \in T_1$, $\sqcap_N(U \setminus \{s, t\}, E(N_0)) \geq 1$, then $U \setminus \{s, t\}$ contains a minimal strand $T_2$ for $N_0$ distinct from $S$ and $T_1$. In both cases, (4) implies that $\mathrm{cl}_N(T_2) \cap E(N_0)$ is distinct from $\mathrm{cl}_N(S) \cap E(N_0)$ and $\mathrm{cl}_N(T_1) \cap E(N_0)$.

Therefore, we may assume that $S$ and $T_1$ are disjoint and that for any $s \in S$ and $t \in T_1$, $\sqcap_N(U \setminus \{s, t\}, E(N_0)) = 0$. This means that for any $u \in U$, $\sqcap_N(U \setminus \{u\}, E(N_0)) = 1$, and $S$ and $T_1$ are in the coclosure of $E(N_0)$.

But because $S$ and $T_1$ are disjoint, $\sqcap_N(U \setminus T_1, E(N_0)) = 1$, so $T_1$ is a series class of $N|(E(N_0) \cup U)$. For the same reason, $S$ is a series class of $N|(E(N_0) \cup U)$. This contradicts the fact that $\kappa_{N|(E(N_0)\cup U)}(S, E(N_0)) = 2$, proving (5).

We now let $N$ and $N'$ be matroids such that $\{N, N'\} = \{M, M'\}$.

(6) *If there is an independent strand $S$ for $N_0$ in $N$ such that $\sqcap_{N'}(S, E(N_0)) = 0$, then either*

- *there is a strand for $N_0$ distinguishing $N$ and $N'$, or*
- *there is a strand $T$ for $N_0$ in $N$ such that $\mathrm{cl}_N(T) \cap E(N_0) \neq \mathrm{cl}_N(S) \cap E(N_0)$ and $\sqcap_{N'}(T, E(N_0)) = 0$.*

We let $U$ be a minimal set containing $S$ such that $\kappa_{N|(E(N_0)\cup U)}(S, E(N_0)) > 1$. By (5), $U$ contains two strands $T_1$ and $T_2$ for $N_0$ such that $\mathrm{cl}_N(S) \cap E(N_0), \mathrm{cl}_N(T_1) \cap E(N_0)$, and $\mathrm{cl}_N(T_2) \cap E(N_0)$ are distinct. We may assume that $\sqcap_{N'}(T_1, E(N_0)) > 0$ and $T_1$ is not a strand distinguishing $N$ and $N'$, and that the same holds for $T_2$. Therefore, $\sqcap_{N'}(U, E(N_0)) \geq 2$.

Let $e \in U \setminus S$. By the minimality of $U$, $\sqcap_N(U \setminus \{e\}, E(N_0)) = 1$. If $\mathrm{cl}_{N'}(U \setminus \{e\}) \cap E(N_0) \not\subseteq \mathrm{cl}_N(S) \cap E(N_0)$, then there is a strand $T$ for $N_0$ in $N'$ such that $S \subseteq T \subseteq U \setminus \{e\}$ and $\mathrm{cl}_{N'}(T) \cap E(N_0) \neq \mathrm{cl}_N(S) \cap E(N_0)$, proving (6). So we may assume that $\mathrm{cl}_{N'}(U \setminus \{e\}) \cap E(N_0) = \mathrm{cl}_N(S) \cap E(N_0)$ and $\sqcap_{N'}(U \setminus \{e\}, E(N_0)) = 1$.

The fact that $\sqcap_{N'}(U, E(N_0)) \geq 2$ but $\sqcap_{N'}(U \setminus \{e\}, E(N_0)) = 1$ for all $e \in U \setminus S$ implies that $U \setminus S$ is contained in the coclosure of $E(N_0)$ in $N'$. Then since $\sqcap_{N'}(S, E(N_0)) = 0$, it follows that $r^*_{N'}(U \setminus S) \geq 2$. Furthermore, for any two elements $e, f \in U \setminus S$ that are not in series in $N'$, $\sqcap_{N'}(U \setminus \{e, f\}, E(N_0)) = 0$.

Therefore, for each series class $X$ of $U \setminus S$, there is a circuit containing $\mathrm{cl}_N(S) \cap E(N_0)$ and $(U \setminus S) \setminus X$. This implies that for each series class $X$ of $U \setminus S$, $X \subseteq \mathrm{cl}_{N'}(U \setminus X)$. But then $\sqcap_{N'}(U, E(N_0)) = \sqcap_{N'}(U \setminus X, E(N_0)) = 1$, a contradiction. This proves (6).

(7) *If there exists a strand $S$ for $N_0$ in $N$ such that either $S$ distinguishes $N$ and $N'$ or $\sqcap_{N'}(S, E(N_0)) = 0$, then*

- $|\Sigma(N, N')| \geq 2$, *and*
- *there is another strand $T$ for $N_0$ in $N$ such that* $\mathrm{cl}_N(S) \cap E(N_0) \neq \mathrm{cl}_N(T) \cap E(N_0)$, $\mathrm{cl}_N(T) \cap E(N_0) \neq \mathrm{cl}_{N'}(T) \cap E(N_0)$ *and* $S \Delta T \subseteq \Sigma(N, N')$.

First we suppose that there is no strand that distinguishes $N$ and $N'$; so $\sqcap_{N'}(S, E(N_0)) = 0$. Then by (6), there is a strand $T$ for $N_0$ in $N$ such that $\mathrm{cl}_N(T) \cap E(N_0) \neq \mathrm{cl}_N(S) \cap E(N_0)$ and $\sqcap_{N'}(T, E(N_0)) = 0$. We observe that $S \not\subseteq T$ and $T \not\subseteq S$, and $S \Delta T \subseteq \Sigma(N, N')$.

Therefore, we may assume that $S$ is a strand for $N_0$ that distinguishes $N$ and $N'$. By (3), there are two lines $L_1$ and $L_2$ of $N_0$ containing $\mathrm{cl}_N(S) \cap E(N_0)$, and sets $U_1$ and $U_2$ containing $S$ such that $\mathrm{cl}_N(U_1) \cap E(N_0) = L_1$, $\mathrm{cl}_N(U_2) \cap E(N_0) = L_2$, $\kappa_{N|(E(N_0) \cup U_1)}(S, E(N_0)) > 1$, and $\kappa_{N|(E(N_0) \cup U_2)}(S, E(N_0)) > 1$. Then by (5), each of $U_1$ and $U_2$ contains two more strands that span distinct elements of $E(N_0) \setminus \mathrm{cl}_N(S)$. If all of these four strands are strands for $N_0$ in $N'$ that do not distinguish $N$ and $N'$, then $\mathrm{cl}_{N'}(U_1) = L_1$ and $\mathrm{cl}_{N'}(U_2) = L_2$, so $\mathrm{cl}_{N'}(S) = L_1 \cap L_2 = \mathrm{cl}_N(S)$, contradicting the fact that $S$ distinguishes $N$ and $N'$. Therefore, there exists a strand $S'$ for $N_0$ in $N$ with $\mathrm{cl}_N(S') \cap E(N_0) \neq \mathrm{cl}_N(S) \cap E(N_0)$ such that either $S'$ is a strand for $N_0$ in $N'$ that distinguishes $N$ and $N'$, or $S'$ is not a strand for $N_0$ in $N'$.

Now we observe that $S \Delta S' \subseteq \Sigma(N, N')$, and since $\mathrm{cl}_N(S) \cap E(N_0) \neq \mathrm{cl}_N(S') \cap E(N_0)$, we have $|S \Delta S'| \geq 2$. This proves (7).

According to Lemma 2.6.5, there is a strand for $N_0$ in one of $M$ or $M'$ that is either a strand for $N_0$ in the other matroid and distinguishes $M$ and $M'$, or is skew to $E(N_0)$ in the other matroid. The result now follows from (7). $\square$

## 2.7 Connectivity

In this section we prove Lemma 2.0.4. First, we state two useful results on matroid connectivity.

**Theorem 2.7.1** (Tutte, [52]). *If $M$ is a connected matroid and $e \in E(M)$, then at least one of $M\backslash e$ and $M/e$ is connected.*

The second is another theorem of Tutte [54] that generalizes Menger's Theorem to matroids.

**Tutte's Linking Theorem.** *If $M$ is a matroid and $S, T \subseteq E(M)$ are disjoint then $\kappa_M(S,T) = \max\{\sqcap_{M/Z}(S,T) : Z \subseteq E(M) \setminus (S \cup T)\}$.*

Recall that sets $S$ and $T$ in a matroid $M$ are called skew if $\sqcap_M(S,T) = 0$. We can choose the set $Z$ that attains the maximum in Tutte's Linking Theorem so that it is skew to both $S$ and $T$. We have the following stronger version of the theorem, for which an explicit proof can be found in [37, Theorem 8.5.7] or [15, Theorem 4.2].

**Tutte's Linking Theorem, Version 2.** *If $M$ is a matroid and $S, T \subseteq E(M)$ are disjoint then there is a set $Z \subseteq E(M) \setminus (S \cup T)$ such that $\sqcap_{M/Z}(S,T) = \kappa_M(S,T)$, $(M/Z)|S = M|S$, and $(M/Z)|T = M|T$.*

Before the main result of this section, we prove one last short lemma that is similar to Lemma 2.3 of [14].

**Lemma 2.7.2.** *Let $\mathbb{F}$ be a finite field, $M$ a 3-connected matroid with a restriction $N_0 \cong \mathrm{PG}(2,\mathbb{F})$ and a deletion pair $x, y \in E(M) \setminus E(N_0)$, and $M'$ an $\mathbb{F}$-representable matroid with $M'\backslash x = M\backslash x$ and $M'\backslash y = M\backslash y$. If there are sets $C, D \subset E(M)$ disjoint from $E(N_0) \cup \{x, y\}$ such that*

*(a) $\{x, y\}$ is coindependent in $M\backslash D/C$,*

*(b) $(M\backslash D/C)\backslash x$ and $(M\backslash D/C)\backslash y$ are stable,*

*(c) $(M\backslash D/C)\backslash x, y$ is connected, and*

*(d) $M\backslash D/C \neq M'\backslash D/C$,*

*then* $M\backslash D/C$ *is not* $\mathbb{F}$*-representable.*

*Proof.* Since $M\backslash x$ and $M\backslash y$ are $\mathbb{F}$-representable, so are $(M\backslash D/C)\backslash x$ and $(M\backslash D/C)\backslash y$. By Lemmas 2.5.2 and 2.5.3 applied to $M\backslash D/C$, there is a unique $\mathbb{F}$-representable matroid $N$ such that $N\backslash x = (M\backslash D/C)\backslash x$ and $N\backslash y = (M\backslash D/C)\backslash y$. But $M'\backslash D/C$ satisfies this condition and is $\mathbb{F}$-representable, so $N = M'\backslash D/C$. Then $M\backslash D/C \neq N$, so $M\backslash D/C$ is not $\mathbb{F}$-representable. □

We now prove Lemma 2.0.4, which we restate for convenience. We will use the fact that a matroid $M$ that is not 3-connected is a direct sum or a 2-sum of matroids isomorphic to proper minors of $M$ ([3, 9, 47], see [37, Theorem 8.3.1] for a proof).

**Lemma 2.0.4.** *For any finite field* $\mathbb{F}$*, if* $M_0$ *is a* 3*-connected, non-*$\mathbb{F}$*-representable matroid with a modular restriction* $N_0 \cong \mathrm{PG}(2, \mathbb{F})$*, then* $M_0$ *has a* 3*-connected, non-*$\mathbb{F}$*-representable minor* $M$ *such that* $N_0$ *is a restriction of* $M$ *and* $\lambda_M(E(N_0)) = 2$*.*

*Proof.* First, we need the following easy fact.

(1) *Let* $M = M_1 \oplus_2 M_2$ *with* $B = E(M_2) \setminus E(M_1)$ *such that* $|B| = 3$*. If* $M_2$ *is non-binary, then* $B$ *is a triangle and a triad in* $M$*.*

Since $M_2$ is non-binary and has four elements, it is isomorphic to $U_{2,4}$ and has no series pairs. Thus $B$ is a triangle in $M$. Since $\lambda_M(B) = 1$ and $r_M(B) = 2$, we have $r^*_M(B) = 2$ and $B$ is a triad of $M$. This proves (1).

By choosing $M_0$ minimally, we may assume that it has no proper minor that is 3-connected, non-$\mathbb{F}$-representable, and has $N_0$ as a restriction. We assume that $\lambda_{M_0}(E(N_0)) = 3$ to obtain a contradiction.

By Lemma 2.4.6, there is a 3-connected, non-$\mathbb{F}$-representable matroid $M$ with $N_0$ as a modular restriction and $\lambda_M(E(N_0)) = 3$ such that $M$ has a deletion pair $x, y \in E(M) \setminus E(N_0)$ and no proper minor of $M$ containing $N_0$ is 3-connected and non-$\mathbb{F}$-representable. Furthermore, $M\backslash x, y$ has at most one series pair.

By Lemma 2.6.1 there is an $\mathbb{F}$-representable matroid $M'$ such that $M\backslash x = M'\backslash x$ and $M\backslash y = M'\backslash y$. We recall that $\Sigma(M, M')$ is the set of elements $e \in E(M) \setminus E(N_0)$ such that $M\backslash e \neq M'\backslash e$ and $M/e \neq M'/e$. By Lemma 2.6.6, $|\Sigma(M, M')| \geq 2$.

(2) *If* $e \in E(M) \setminus E(N_0)$*, then* $M\backslash e$ *and* $M/e$ *are* $\mathbb{F}$*-representable.*

We let $P$ be either $M\backslash e$ or $M/e$. Suppose that $P$ is not $\mathbb{F}$-representable. Then the fact that $P$ is a proper minor of $M$ implies that it is not 3-connected. Since $N_0$ is 3-connected, there exists a matroid $P'$ containing $N_0$ such that $\mathrm{si}(P')$ is 3-connected and $P$ is obtained by 2-sums of $P'$ with matroids $M_1, \ldots, M_t$. Since $P'$ is isomorphic to a proper minor of $M$, it is $\mathbb{F}$-representable; this means that for some $i$, $M_i$ is not $\mathbb{F}$-representable. We let $U = E(M_i) \setminus E(P')$ and $V = E(P) \setminus U$. Then $(U, V)$ is a 2-separation of $P$, and $(U \cup \{e\}, V)$ is a 3-separation of $M$. By Tutte's Linking Theorem, there is a minor $N$ of $M$ such that $E(N) = E(N_0) \cup U \cup \{e\}$ and $\lambda_N(E(N_0)) = 2$. Since $\lambda_N(U \cup \{e\}) = \lambda_M(U \cup \{e\})$, we note that $N|(U \cup \{e\}) = M|(U \cup \{e\})$. This means that $N$ is 3-connected, as $M$ is. Furthermore, $\sqcap_N(U, E(N_0)) = 1$ so the modularity of $N_0$ implies that $N$ contains a restriction isomorphic to $M_i$. Therefore, $N$ is not $\mathbb{F}$-representable. But then $N$ is not a proper minor of $M$, so $N = M$, contradicting the fact that $\lambda_M(E(N_0)) \geq 3$. This proves (2).

(3) *If $e \in \Sigma(M, M')$, and $N$ is one of $M\backslash e$ or $M/e$, then either $N\backslash x$ is not stable, $N\backslash y$ is not stable, or $N\backslash x, y$ is not connected.*

Since $M\backslash x, y$ is connected, $\{x, y, e\}$ is not a triad of $M$ and $\{x, y\}$ is coindependent in $M\backslash e$. With $C = \emptyset$ and $D = \{e\}$, hypotheses (a) and (d) of Lemma 2.7.2 are satisfied by $M\backslash D/C$. But $M\backslash e$ is $\mathbb{F}$-representable by (2). Therefore, (b) and (c) of Lemma 2.7.2 do not both hold, and we conclude that either $M\backslash e\backslash x$ is not stable, $M\backslash e\backslash y$ is not stable, or $M\backslash e\backslash x, y$ is not connected.

Similarly, with $C = \{e\}$ and $D = \emptyset$, (a) and (d) of Lemma 2.7.2 are satisfied by $M\backslash D/C$, and $M/e$ is $\mathbb{F}$-representable by (2). Therefore, (b) and (c) of Lemma 2.7.2 do not both hold, and either $M/e\backslash x$ is not stable, $M/e\backslash y$ is not stable, or $M/e\backslash x, y$ is not connected. This proves (3).

(4) *If $e \in \Sigma(M, M')$ then either $M\backslash x\backslash e$ and $M\backslash y/e$ are not stable, or $M\backslash x/e$ and $M\backslash y\backslash e$ are not stable.*

Up to symmetry between $x$ and $y$, (3) implies that one of the following five cases occurs:

*(a)* $M\backslash x, y\backslash e$ is not connected and $M\backslash x, y/e$ is not connected,
*(b)* $M\backslash x, y\backslash e$ is not connected and $M\backslash x/e$ is not stable,
*(c)* $M\backslash x, y/e$ is not connected and $M\backslash x\backslash e$ is not stable,
*(d)* $M\backslash x\backslash e$ is not stable and $M\backslash x/e$ is not stable, or
*(e)* $M\backslash x\backslash e$ is not stable and $M\backslash y/e$ is not stable.

As $M\backslash x, y$ is connected, case (a) contradicts Theorem 2.7.1, and since $M\backslash x$ is 3-connected, case (d) contradicts Bixby's Lemma. We suppose case (b) holds.

Since $M\backslash x/e$ is not stable, Bixby's Lemma implies that $M\backslash x\backslash e$ is internally 3-connected. Then since $M\backslash x,y\backslash e$ is not connected, $y$ is in a series pair of $M\backslash x\backslash e$, so $\{x,y,e\}$ is a triad of $M$. This contradicts the fact that $M\backslash x,y$ is connected. Next, we suppose that case (c) holds. Since $M\backslash x\backslash e$ is not stable, by Bixby's Lemma $M\backslash x/e$ is internally 3-connected. There are no series pairs in $M\backslash x/e$, so $M\backslash x,y/e$ is connected, a contradiction. We conclude that, up to symmety, case (e) holds, which proves (4).

(5) *$M\backslash x,y$ is not 3-connected.*

We say that a 2-separation $(A,B)$ in a matroid $N$ corresponds to a 2-sum of non-binary matroids if $N=N_1\oplus_2 N_2$ for some non-binary matroids $N_1$ and $N_2$ with $A=E(N_1)\setminus E(N_2)$ and $B=E(N_2)\setminus E(N_1)$.

Let $e\in\Sigma(M,M')$. From (4) we may assume that $M\backslash x\backslash e$ and $M\backslash y/e$ are not stable. Suppose $M\backslash x,y$ is 3-connected. Then by Bixby's Lemma, either $M\backslash x,y/e$ or $M\backslash x,y\backslash e$ is internally 3-connected.

First, assume that $M\backslash x,y/e$ is internally 3-connected. Let $(A,B)$ be a 2-separation of $M\backslash y/e$ corresponding to a 2-sum of two non-binary matroids, with $x\in B$. Since $M\backslash y/e\backslash x$ is internally 3-connected, $|B|=3$. Since $M\backslash x,y$ is 3-connected, $M\backslash x,y/e$ has no series pairs, so $B\setminus\{x\}$ is a parallel pair of $M\backslash y/e$, contradicting the fact that $(A,B)$ is a 2-separation that corresponds to a 2-sum of two non-binary matroids.

Therefore, $M\backslash x,y/e$ is not internally 3-connected, and $M\backslash x,y\backslash e$ is. Let $(A,B)$ be a 2-separation of $M\backslash x\backslash e$ corresponding to a 2-sum of two non-binary matroids, with $y\in B$. Then $|B|=3$ since $M\backslash x,y\backslash e$ is internally 3-connected. So by (1), $B$ is a triangle and a triad of $M\backslash x\backslash e$ containing $y$. Denote the other two elements of $B$ by $a$ and $b$. Since $M\backslash x$ is 3-connected, $\{a,b,y,e\}$ is a cocircuit of $M\backslash x$ that contains the triangle $\{a,b,y\}$, and $\{a,b,e\}$ is a triad of $M\backslash x,y$.

Let $(C,D)$ be an internal 2-separation of $M\backslash x,y/e$. Then $e\in \mathrm{cl}_M(C)\cap \mathrm{cl}_M(D)$, so $a$ and $b$ are not both contained in the same one of $C$ or $D$ because $\{a,b,e\}$ is a cocircuit of $M\backslash x,y$. Thus we may assume $a\in C$, $b\in D$. So in $M\backslash x$ we have $\sqcap_{M\backslash x}(\{e,a\},C\setminus\{a\})=1$, $\sqcap_{M\backslash x}(\{e,b\},D\setminus\{b\})=1$, and $\sqcap_{M\backslash x}(\{a,b,y\},C\cup D\setminus\{a,b\})=1$, but $\{a,b,y\}$ and $\{y,e\}$ are each skew to both $C\setminus\{a\}$ and $D\setminus\{b\}$.

We pick a second element $f\in\Sigma(M,M')$. Then either $M\backslash x\backslash f$ and $M\backslash y/e$ are not stable, or $M\backslash x/f$ and $M\backslash y\backslash f$ are not stable. First we assume that $M\backslash x\backslash f$ and $M\backslash y/f$ are not stable. By the same argument that was applied to $e$, $M\backslash x$ has a cocircuit $\{c,d,f,y\}$ with a triangle $\{c,d,y\}$, and there is an internal 2-separation of $M\backslash x,y/f$, $(U,V)$ with $c\in U, d\in V$. So $\{e,a,b\}$ and $\{f,c,d\}$ are both triads in $M\backslash x,y$, and $\{y,a,b\}$ and $\{y,c,d\}$ are both triangles of $M\backslash x$. But the only triangle of $M\backslash x$ containing $y$ is $\{y, a, b\}$, so we may assume that $a = c$ and $b = d$. Now $(V\setminus\{b\})\cup\{f\}$ spans $b$, so $e \in (V\setminus\{b\})\cup\{f\}$. But $e \neq f$ so $e \in V$. Symmetrically, $(U \setminus \{a\}) \cup \{f\}$ spans $a$ so we also have $e \in U$, a contradiction.

Therefore, there are only two elements of $\Sigma(M, M')$, $e$ and $f$, and $M\backslash x/f$ and $M\backslash y\backslash f$ are not stable.

Applying to $f$ the same argument as for $e$ but with $x$ and $y$ swapped, we see that $M\backslash y$ has a cocircuit $\{c, d, x, f\}$ with a triangle $\{c, d, x\}$.

We assume that $e \notin \{c, d\}$ and $f \notin \{a, b\}$. Then $\{a, b, e\}$ and $\{c, d, f\}$ are disjoint because $\{a, b\}$ is a series class of $M\backslash x, y, e$ but $\{c, d\}$ is not. By Lemma 2.6.6, there are two distinct sets $S, T \subseteq E(M) \setminus E(N_0)$ such that $S\Delta T = \{e, f\}$, $\mathrm{cl}_M(S) \cap E(N_0) \neq \mathrm{cl}_{M'}(S) \cap E(N_0)$ and $\mathrm{cl}_M(T) \cap E(N_0) \neq \mathrm{cl}_{M'}(T) \cap E(N_0)$. By symmetry, we may assume $S \setminus T = \{e\}$ and $T \setminus S = \{f\}$. Note that $x, y \in S \cap T$. We may also assume that $S$ and $T$ are minimal, so neither contains $\{a, b\}$ or $\{c, d\}$, both of which are in triangles with $y$. Suppose that $a \in S$. then $\mathrm{cl}_M(S)\cap E(N_0) = \mathrm{cl}_M((S\cup\{b\})\setminus\{y\})\cap E(N_0)$ since $\{a, b, y\}$ is a triangle. But then $\mathrm{cl}_M(S) \cap E(N_0) = \mathrm{cl}_{M'}((S \cup \{b\}) \setminus \{y\}) \cap E(N_0)$, which equals $\mathrm{cl}_{M'}(S)$ because $\{a, b, y\}$ is also a triangle of $M'$. This is a contradiction, so $a \notin S$, and by the symmetric argument, $b \notin S$. Suppose that $a, b, e \notin T$. Then since $\{a, b, e, y\}$ is a cocircuit of $M$, $\mathrm{cl}_M(T) \cap E(N_0) = \mathrm{cl}_M(T\setminus\{y\})\cap E(N_0) = \mathrm{cl}_{M'}(T\setminus\{y\})\cap E(N_0)$. But this equals $\mathrm{cl}_{M'}(T)\cap E(N_0)$, because $\{a, b, y, e\}$ is a union of cocircuits in $M'\backslash x$, a contradiction. Hence $T$ contains at least one of $a$ or $b$, and by symmetry we may assume $a \in T$. Then $a \in S\Delta T = \{e, f\}$, so we have $a = f$, contradicting our assumption that $f \notin \{a, b\}$.

Therefore, we may assume by symmetry that $a = f$. Hence $\{f, b, y\}$ is a triangle, in both $M$ and $M'$. But then $\mathrm{cl}_M(T) = \mathrm{cl}_M((T \setminus \{y\}) \cup \{b\})$ and $\mathrm{cl}_{M'}(T) = \mathrm{cl}_{M'}((T \setminus \{y\}) \cup \{b\})$, so these sets are equal, a contradiction. This proves (5).

Note that, since our deletion pair $\{x, y\}$ was arbitrary up to the assumption that $M\backslash x, y$ has at most one series pair, (5) implies that there is no deletion pair $x', y' \in E(M) \setminus E(N_0)$ such that $M\backslash x', y'$ is 3-connected. Whenever $u, v \in E(M)$ are elements such that $M\backslash u, v$ is 3-connected, then $M\backslash u$ and $M\backslash v$ are internally 3-connected. But they have no parallel pairs so they are actually 3-connected, and $\{u, v\}$ is a deletion pair of $M$. Therefore, there are no two distinct elements $u, v \in E(M) \setminus E(N_0)$ such that $M\backslash u, v$ is 3-connected.

(6) *If $e \in \Sigma(M, M')$, then $e$ is not in a series pair of $M\backslash x, y$.*

From (4) and the symmetry between $x$ and $y$, we may assume that $M\backslash y/e$

is not stable. Suppose $e$ is in a series pair of $M\backslash x, y$. Since $M\backslash x, y$ is internally 3-connected with at most one series pair, $M\backslash x, y/e$ is 3-connected. This contradicts the fact that $M\backslash y/e$ is not internally 3-connected, proving (6).

By Lemma 2.6.6, there is an element $e \in \Sigma(M, M')$ and sets $S, T \subseteq E(M) \setminus E(N_0)$ such that $e \in S \setminus T$, $\mathrm{cl}_M(S) \cap E(N_0) \neq \mathrm{cl}_{M'}(S) \cap E(N_0)$, and $\mathrm{cl}_M(T) \cap E(N_0) \neq \mathrm{cl}_{M'}(T) \cap E(N_0)$. From (4) we may assume that $M\backslash x\backslash e$ and $M\backslash y/e$ are not stable. By (5), $M\backslash x, y$ has exactly one series pair; we denote it by $\{a, b\}$. Then $M\backslash x, y/a$ is 3-connected, and by Bixby's Lemma either $M\backslash x, y/a\backslash e$ or $M\backslash x, y/a/e$ is internally 3-connected.

(7) *For any set $H \subseteq E(M)$, if $a \in H$ or $b \in H$ then $\mathrm{cl}_M(H) = \mathrm{cl}_{M'}(H)$.*

Since $M\backslash x, y/a$ is 3-connected, it follows that $M\backslash x/a$ and $M\backslash y/a$ are both stable and $M\backslash x, y/a$ is connected. Also, $M/a$ is $\mathbb{F}$-representable by (2). Therefore, Lemma 2.7.2 implies that $M/a = M'/a$. For any set $H \subseteq E(M)$ containing $a$, $\mathrm{cl}_{M/a}(H \setminus \{a\}) = \mathrm{cl}_{M'/a}(H \setminus \{a\})$, which means that $\mathrm{cl}_M(H) = \mathrm{cl}_{M'}(H)$. The same argument applies with $b$ in place of $a$, proving (7).

(8) *$M\backslash x, y, e$ is not internally 3-connected.*

We assume that $M\backslash x, y, e$ is internally 3-connected. Since $M\backslash x\backslash e$ is not stable, it has an internal 2-separation $(A, B)$ where $y \in B$, and by (1), $B$ is a triangle and a triad. We let $c$ and $d$ be the other two elements of $B$. Then $B$ is coindependent in $M\backslash x$ and $\{c, d\}$ is coindependent in $M\backslash x, y$. Thus at most one of $a, b$ is in $\{c, d\}$; but $\{y, a, b\}$ is a triad of $M\backslash x$ and $\{y, c, d\}$ is a triangle, so $\{c, d\}$ contains exactly one of $a$ or $b$. We may assume that $c = b$ so $B = \{y, b, d\}$ and $d \neq a$. Then $a \in A$ and $\{a, b\}$ is not a series pair of $M\backslash x, y, e$, because $\{b, d\}$ is and $M\backslash x, y, e$ is internally 3-connected. This contradicts the fact that $\{a, b\}$ is a series pair of $M\backslash x, y$, proving (8).

(9) *$M\backslash x, y/a, e$ is internally 3-connected if and only if $M\backslash x, y/e$ is internally 3-connected.*

Suppose that $M\backslash x, y/e$ is internally 3-connected. Since $\{a, b\}$ is a series pair of $M\backslash x, y$, it is a series pair of $M\backslash x, y/e$, so $M\backslash x, y/a, e$ is also internally 3-connected.

Conversely, suppose that $M\backslash x, y/a, e$ is internally 3-connected and $M\backslash x, y/e$ is not internally 3-connected. Since $\{a, b\}$ is a series pair, $M\backslash x, y/e$ has an internal 2-separation $(W, Z)$ with $a, b \in Z$. But $M\backslash x, y/a, e$ is internally 3-connected, so $|Z| = 3$; let $c$ denote the third element of $Z$. Recall that $M\backslash x, y/a$ is 3-connected, so $M\backslash x, y/a, e$ has no series pairs. Hence $\{b, c\}$ is not a series pair of $M\backslash x,y/a,e$ so it is a parallel pair, and $Z$ is a triangle of $M\backslash x,y/e$. It is not a triad, however, since $\{a,b\}$ is a series pair, so $c \in \mathrm{cl}_{M\backslash x,y/e}(W)$. Note that $(W,\{a,b,c\})$ is the unique (up to ordering parts) 2-separation of $M\backslash x,y/e$. Hence since $M\backslash y/e$ is not stable, $x$ must lie in $\mathrm{cl}_{M\backslash y/e}(\{a,b\})$ but not in $\mathrm{cl}_{M\backslash y/e}(W)$. So $(W,\{a,b,c,x\})$ is a 2-separation of $M\backslash y/e$ and $\{a,b,c,x\}$ is a four-point line in $M\backslash y/e$. Therefore, $(W,\{a,b,c,e,x\})$ is a 3-separation of $M\backslash y$ and $c,e \in \mathrm{cl}_M(W)\cap \mathrm{cl}_M(\{a,b,x\})$. By our remarks after (5), we know that $M\backslash y,c$ is not 3-connected, so it has a 2-separation $(A,B)$ with at least two of $a,b,x$ in $B$. Since $\{a,b,x\}$ is a triad, $\lambda_{M\backslash y,c}(B\cup\{a,b,x\}) = 1$, but $(A\backslash\{a,b,x\}, B\cup\{a,b,x\})$ is not a 2-separation of $M\backslash y,c$ since $c \in \mathrm{cl}_M(\{a,b,x\})$. Therefore, $|A| = 2$, $A$ is a series pair of $M\backslash y,c$, and $A$ contains one element of $\{a,b,x\}$. This implies that there is a triad $L$ of $M\backslash y$ containing $c$ and precisely one element of $\{a,b,x\}$. Since $e \in \mathrm{cl}_M(W)$, $e \notin L$. Therefore, either $c \notin \mathrm{cl}_M(\{b,x,e\})$, $c \notin \mathrm{cl}_M(a,x,e\})$, or $c \notin \mathrm{cl}_M(\{a,b,e\})$. But $r_M(\{a,b,x,e,c\}) = 3$, so one of $\{b,x,e\}$, $\{a,x,e\}$ and $\{a,b,e\}$ is a triangle of $M\backslash y$, contradicting the fact that $\{a,b,c,x\}$ is a four-point line in $M\backslash y/e$. This proves that $M\backslash x,y/e$ is internally 3-connected and proves (9).

(10) *If $M\backslash x,y/e$ is internally 3-connected then $\{a,b,x,e\}$ is a circuit and a flat of $M$ and $M$ has a three- or four-element cocircuit $L$ containing $e$, one of $a$ or $b$, and another element $c \in E(M)\setminus\{a,b,x,y,e\}$, and if $|L| = 4$ then $y \in L$.*

Since $M\backslash y/e$ is not stable, it has an internal 2-separation $(U,V)$ with $x \in V$. Since $(U, V\setminus\{x\})$ is not an internal 2-separation of $M\backslash x,y/e$, $|V| = 3$ and $V$ is closed in $M\backslash y/e$. Hence by (1), $V$ is a triangle and a triad of $M\backslash y/e$. Therefore, $V\setminus\{x\}$ is a series pair of $M\backslash x,y$, so $V = \{a,b,x\}$. Also, $(U, V\cup\{e\})$ is a 3-separation of $M\backslash y$ with $e \in \mathrm{cl}_{M\backslash y}(U)$, so $r_M(\{a,b,x,e\}) = 3$, and since $\{a,b,x\}$ is independent in $M\backslash y/e$, $\{a,b,x,e\}$ is a circuit of $M$. Since $M\backslash x/e$ is internally 3-connected, $(U,\{a,b,y\})$ is not a 2-separation of $M\backslash x/e$, so $y$ is not in $\mathrm{cl}_M(\{a,b,e\})$. Since $V$ is closed in $M\backslash x,y/e$, $V\cup\{e\}$ is a flat of $M$.

By Bixby's Lemma, $M\backslash y,e$ is internally 3-connected. But it is not 3-connected, so it has a series pair and thus $e$ is contained in a triad $L$ of $M\backslash y$. At least one other element of the circuit $\{a,b,x,e\}$ is in $L$, but $x \notin L$ since $e$ is not in a series pair in $M\backslash x,y$. So $L$ contains one of $a$ and $b$. Recall that $\{a,b,x\}$ is a triad of $M\backslash y$, and $e \in \mathrm{cl}_M(\{a,b,x\}$; thus as $M\backslash y$ is 3-connected, $e \in \mathrm{cl}_M(E(M)\backslash\{a,b,x,y\})$ and so $\{a,b,e\}$ is not a triad. Therefore, $L$ contains exactly one of $a$ and $b$, plus another element $c$. Since $L$ is a triad of $M\backslash y$, either $L$ is a triad of $M$ or $L\cup\{y\}$ is a cocircuit of $M$, which proves (10).

(11) *If $M\backslash x,y/a,e$ is internally 3-connected, then there is an element $c \in E(M)\setminus\{x,y,a,b,e\}$ such that $\lambda_M(\{x,y,a,b,c,e\}) = 2$, $\{a,b,x,e\}$ is a circuit, $\{a,b,y,c\}$ is a circuit, and neither $e$ nor $c$ is in $\mathrm{cl}_M(E(M)\setminus\{x,y,a,b,c,e\})$.*

Assume that $M\backslash x, y/a, e$ is internally 3-connected. Then by (9), $M\backslash x, y/e$ is internally 3-connected, and by (10), $\{a, b, x, e\}$ is a circuit and a flat of $M$ and there is an element $c \in E(M) \setminus \{a, b, x, y, e\}$ such that either $\{e, c\}$ or $\{e, c, y\}$ along with one of $a$ or $b$ forms a cocircuit of $M$. By symmetry, we may assume that either $\{e, c, b\}$ or $\{e, c, b, y\}$ is a cocircuit. Note that $y \notin \mathrm{cl}_M(\{a, b, x\})$ because then $\lambda_{M\backslash x/e}(\{a, b, y\}) = 1$ and $M\backslash x/e$ is not internally 3-connected, contradicting Bixby's Lemma and the fact that $M\backslash x, e$ is not stable. Therefore, $\{a, b, x, y\}$ is an independent cocircuit of $M$ and $\lambda_M(\{a, b, x, y\}) = 3$. Let $U = E(M) \setminus \{a, b, x, y, e, c\}$. Since $\{b, c\}$ is a series pair of $M\backslash y, e$, we have $c \notin \mathrm{cl}_M(U)$, hence $\sqcap_M(\{a, b, x, y\}, U) = 2$. If $c \in \mathrm{cl}_M(\{a, b, y\})$, then we have $\lambda_M(\{x, y, a, b, c, e\}) = 2$ and we have proved (11), so we may assume $c \notin \mathrm{cl}_M(\{a, b, y\})$. Also, since $e \notin \mathrm{cl}_M(U)$, $(\{a, b, x, e\}, U)$ is a 2-separation of $M\backslash y, c$. But $M\backslash y$ is 3-connected, so $c \in \mathrm{cl}_M(\{a, b, x\} \cup U)$.

Recall that there are sets $S$ and $T$ such that $e \in S \setminus T$, $\mathrm{cl}_M(S) \cap E(N_0) \neq \mathrm{cl}_{M'}(S) \cap E(N_0)$ and $\mathrm{cl}_M(T) \cap E(N_0) \neq \mathrm{cl}_{M'}(T) \cap E(N_0)$. We note that $x, y \in S \cap T$.

Next, we claim that $\{x, y\}$ and $U$ are skew in both $M$ and $M'$. First, suppose $\sqcap_M(\{x, y\}, U) = 1$ and $\sqcap_{M'}(\{x, y\}, U) = 1$. Note that $\sqcap_M(\{x, y, a\}, U)$ and $\sqcap_{M'}(\{x, y, b\}, U)$ cannot both equal two; we may assume that $\sqcap_M(\{x, y, a\}, U) = 1$. Then $\sqcap_M(\{x, a\}, U) = \sqcap_M(\{y, a\}, U) = 0$ so we have also $\sqcap_{M'}(\{x, y, a\}, U) = 1$. But then $\mathrm{cl}_M(S \cup \{a\}) \cap E(N_0) = \mathrm{cl}_M(S) \cap E(N_0)$ and $\mathrm{cl}_{M'}(S \cup \{a\}) \cap E(N_0) = \mathrm{cl}_{M'}(S) \cap E(N_0)$, contradicting (7). Next, suppose that $\{x, y\}$ and $U$ are skew in one of $M$ and $M'$, which we call $N$, and not in the other, which we call $N'$. By Tutte's Linking Theorem, there is a set $C \subseteq E(N)$ such that $N/C$ has $N|\{x, y, a, b, e, c\}$ and $N_0$ as restrictions, and $\sqcap_{N/C}(\{x, y, a, b, e, c\}, E(N_0)) = 2$. Then $\sqcap_{N/C}(\{x, y\}, E(N_0)) = 1$ and by the modularity of $N_0$, there is an element $z \in \mathrm{cl}_{N/C}(\{x, y\}) \cap E(N_0)$. By (7), $z \in \mathrm{cl}_{N'/C}(\{x, y, a\})$ and $z \in \mathrm{cl}_{N'/C}(\{x, y, b\})$. But this implies that $z \in \mathrm{cl}_{N'/C}(\{x, y\})$, which means that $\sqcap_{N'}(\{x, y\}, U) = 1$, a contradiction. Hence $\{x, y\}$ and $U$ are skew in both $M$ and $M'$.

Since $\mathrm{cl}_M(T) \neq \mathrm{cl}_{M'}(T)$, it follows from (7) that $a, b \notin T$. We also see that $c \in T$; if not then $T \setminus U = \{x, y\}$ and since $\{x, y\}$ is skew to $U$ in both $M$ and $M'$, we would have $\mathrm{cl}_M(T) \cap E(N_0) = \mathrm{cl}_M(T \setminus \{x, y\}) \cap E(N_0)$ and $\mathrm{cl}_{M'}(T) \cap E(N_0) = \mathrm{cl}_{M'}(T \setminus \{x, y\}) \cap E(N_0)$, and these two sets are equal, a contradiction.

Since $c \in T$, $\mathrm{cl}_{M/c}(T \setminus \{c\}) \neq \mathrm{cl}_{M'/c}(T \setminus \{c\})$, so $M/c \neq M'/c$. By Lemma 2.7.2, this implies that either $M\backslash x/c$ is not stable, $M\backslash y/c$ is not stable, or $M\backslash x, y/c$ is not connected. But $M\backslash x, y$ is internally 3-connected and $c$ is not in its unique series pair $\{a, b\}$, so $M\backslash x, y/c$ is connected. We note that $\lambda_{M\backslash y,c}(\{a, b, x, e\}) = 1$ so $M\backslash y, c$ has an internal 2-separation, and by Bixby's

Lemma, $M\backslash y/c$ is internally 3-connected and stable. We conclude that $M\backslash x/c$ is not stable.

Let $(W, Z)$ be a 2-separation of $M\backslash x/c$. Since $\{a, b, y\}$ is a triad of $M\backslash x/c$, we may assume that $a, b, y \in W$. Since $M\backslash x$ is 3-connected, $c \in \mathrm{cl}_M(W) \cap \mathrm{cl}_M(Z)$. But $c \notin \mathrm{cl}_M(U)$, so $Z \not\subseteq U$ and thus we have $e \in Z$. Since $c \notin \mathrm{cl}_M(\{a, b, y\})$, $W \setminus \{a, b, y\}$ is not empty and has an element not in $\mathrm{cl}_M(Z)$. But then $M\backslash y/c$ has a 2-separation $(W \setminus \{a, b, y\}, Z \cup \{a, b, x\})$. But $M\backslash y/c$ is internally 3-connected and $M\backslash y$ is 3-connected, so $M\backslash y/c$ is connected and $W \setminus \{a, b, y\}$ is a parallel pair. But this implies that $c$ is in a triangle with two elements of $W \setminus \{a, b, y\}$, contradicting the fact that $c \notin \mathrm{cl}_M(U)$.

(12) *There is an element $c \in E(M) \setminus \{x, y, a, b, e\}$ such that $\lambda_M(\{x, y, a, b, c, e\}) = 2$, $\{a, b, x, e\}$ is a circuit, $\{a, b, y, c\}$ is a circuit, and neither $e$ nor $c$ is in $\mathrm{cl}_M(E(M) \setminus \{x, y, a, b, c, e\})$.*

By (11), we may assume that $M\backslash x, y/a, e$ is not internally 3-connected; by Bixby's Lemma, $M\backslash x, y/a\backslash e$ is internally 3-connected.

By (8), $M\backslash x, y, e$ has an internal 2-separation $(U, V)$ with $a \in V$. Since $(U, V \setminus \{a\})$ is not an internal 2-separation of $M\backslash x, y, e/a$, $|V| = 3$ and $V$ is a series class of $M\backslash x, y, e$. So $V$ consists of $\{a, b\}$ and another element $c \in E(M) \setminus \{x, y, a, b, e\}$. Moreover, up to ordering the parts, this is the unique internal 2-separation of $M\backslash x, y, e$, so $\{a, b, c\}$ is a flat of $M\backslash x, y, e$. We have $y \notin \mathrm{cl}_M(E(M) \setminus \{a, b\})$ because $M\backslash x$ is 3-connected. Thus if $y \notin \mathrm{cl}_M(\{a, b, c\})$ then $M\backslash x, e$ is internally 3-connected, a contradiction because it is not stable. Therefore, we have $y \in \mathrm{cl}_M(\{a, b, c\})$. Also, $e \notin \mathrm{cl}_M(E(M) \setminus \{x, y, a, b, c, e\})$ because then $\{a, b, c\}$, which is a series class of $M\backslash x, y, e$, would also be a series class of $M\backslash x, y$.

Let $(W, Z)$ be a 2-separation of $M\backslash y/e$; since $\{a, b, x\}$ is a triad of $M\backslash y/e$, we may assume that $a, b, x \in W$. Since $M\backslash y$ is 3-connected, $e \in \mathrm{cl}_M(W) \cap \mathrm{cl}_M(Z)$. But $e \notin \mathrm{cl}_M(E(M) \setminus \{a, b, x, y, c\})$, for then $\lambda_{M\backslash x}(\{a, b, y, c\})$ would equal $\lambda_{M\backslash x, e}(\{a, b, y, c\}) = 1$, but $M\backslash x$ is 3-connected. Thus $Z \cap \{a, b, x, y, c\}$ is non-empty and we have $c \in Z$. If $e \in \mathrm{cl}_M(\{a, b, x\})$ then we are done. If not, then $W \setminus \{a, b, x\}$ is not contained in $\mathrm{cl}_M(\{a, b, x\})$ and $\lambda_{M\backslash x/e}(W \setminus \{a, b, x\}) \leq 1$. Since $M\backslash x/e$ is internally 3-connected, $W \setminus \{a, b, x\}$ is a parallel pair, since if it had a single element that element would be in $\mathrm{cl}_M(Z) \cap \mathrm{cl}_M(\{a, b, x\})$. But this implies that $e$ is in a triangle with two elements of $E(M) \setminus \{x, y, a, b, c\}$, which contradicts the fact that, as we pointed out above, $e \notin \mathrm{cl}_M(E(M) \setminus \{a, b, x, y, c\})$. This proves (12).

See Figure 2.2 for an illustration of $M$. The last step in the proof is to show

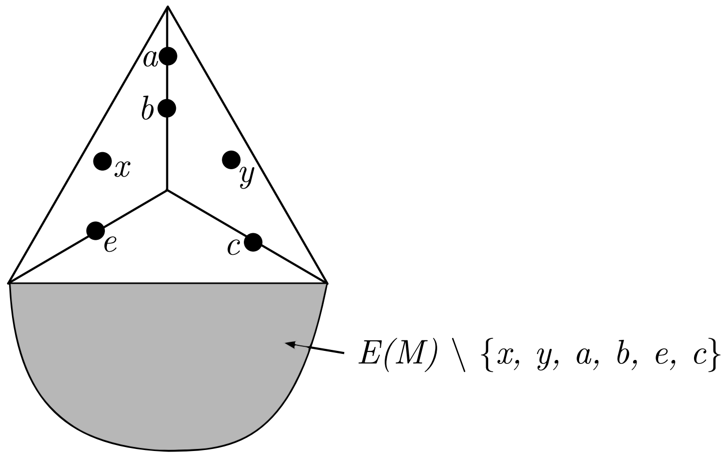



Figure 2.2: The matroid $M$

that $\{x, a\}$ is a deletion pair of $M$ and that $M\backslash x, a$ has one series pair; from this we will get a contradiction.

Let $(U, V)$ be a 2-separation of $M\backslash x, a$; we may assume that $b, y \in V$ since $\{b, y\}$ is a series pair of $M\backslash x, a$. We claim that $V = \{b, y\}$. If not, then $|V| > 2$ and $(U, V \setminus \{b\})$ is a 2-separation of $M\backslash x, a/b$.

Suppose that $|V| = 3$. Then $V \setminus \{b\}$ is a series pair of $M\backslash x, a/b$. But $\{e, c, y\}$ is a triad, so $V \setminus \{b\}$ consists of $y$ and another element $z \notin \{e, c\}$. But then $\{b, y, z\}$ has corank at most two in $M\backslash x, a$, which means that $\{b, z\}$ has corank at most one in $M\backslash x, y, a$. But if it has corank one then $\{a, b, z\}$ is a series class of $M\backslash x, y$, a contradiction; and if it has corank zero then $\{a, b, z\}$ is a series class of $M\backslash x, y$, also a contradiction.

Otherwise, $|V| > 3$. Note that since $\lambda_{M\backslash x,a/b}(\{e, c, y\}) = \lambda_M(\{x, y, a, b, c, e\}) = 2$, there is no 2-separation $(A, B)$ of $M\backslash x, a/b$ with $A$ or $B$ disjoint from $\{y, e, c\}$. If $e, c \in U$, then since $\{y, e, c\}$ is a triad of $M\backslash x, a/b$, $(U \cup \{y\}, V \setminus \{y\})$ is a 2-separation of $M\backslash x, a/b$; a contradiction as $V \setminus \{y\}$ is disjoint from $\{e, c, y\}$. So $e$ and $c$ are not both contained in $U$. But then $(U \setminus \{e, c\}, V \cup \{e, c\})$ is a 2-separation of $M\backslash x, a$, also a contradiction, unless $|U \setminus \{e, c\}| < 2$. This means that either $e$ or $c$ is contained in a series pair of $M\backslash x, a$, and hence $a$ is in a triad of $M\backslash x$ containing $e$ or $c$. But the only triad of $M\backslash x$ containing $a$ is $\{a, b, y\}$, since $\{a, b\}$ is a series class of $M\backslash x, y$. This proves that $V = \{b, y\}$ and so $M\backslash x, a$ is internally 3-connected and its unique series pair is $\{b, y\}$.

Suppose that $M\backslash a$ is not 3-connected, and let $(W, Z)$ be a 2-separation of $M\backslash a$ with $x \in Z$. Since $M\backslash x, a$ is internally 3-connected and $M\backslash a$ has no parallel pairs, $|Z| = 3$ and $Z \setminus \{x\}$ is a series pair of $M\backslash x, a$. Therefore,

$Z = \{x,y,b\}$. This means that $\lambda_M(\{x,y,a,b\}) = 2$, but $\{x,y,a,b\}$ is an independent cocircuit of $M$ and therefore we have $r^*_M(\{x,y,a,b\}) = 2$. Then $r^*_{M\backslash x}(\{y,a,b\}) = 1$, contradicting the fact that $M\backslash x$ is 3-connected. This proves that $M\backslash a$ is 3-connected and that $\{x,a\}$ is a deletion pair of $M$.

Since $\{x,a\}$ is a deletion pair of $M$ and $M\backslash x,a$ has a unique series pair $\{b,y\}$, (12) holds for the deletion pair $\{x,a\}$ in place of $\{x,y\}$, for possibly some new choice of $e$. In particular, there are elements $e',c' \in E(M) \setminus \{x,y,a,b\}$ such that $\lambda_M(\{x,y,a,b,c',e'\}) = 2$, $\{a,b,x,e'\}$ and $\{a,b,y,c'\}$ are circuits of $M$, and neither $e'$ nor $c'$ is in $\mathrm{cl}_M(E(M) \setminus \{x,y,a,b,c',e'\})$. Let $U = E(M) \setminus \{x,y,a,b,e',c'\}$. We have $\sqcap_M(\{x,y,a,b\},U) = 2$.

Since $\{a,b,x,e\}$ is a circuit of $M$ and neither $\{a,b,x,e'\}$ nor $\{a,b,x,c'\}$ are, at least one of $c'$ and $e'$ is distinct from $e$. Then since $e',c' \notin \mathrm{cl}_M(E(M) \setminus \{x,y,a,b,e',c'\})$ and $e',c' \in \mathrm{cl}_M(\{x,y,a,b\})$, we have $\lambda_M(\{x,y,a,b\}) \geq 1 + \sqcap_M(\{x,y,a,b\}, E(M) \setminus \{x,y,a,b,e',c'\}) = 1 + \lambda_M(\{x,y,a,b\})$, a contradiction. □

## 2.8 Vertically 4-connected matroids

We let $M$ be a vertically 4-connected, non-$\mathbb{F}$-representable matroid with $N_0 \cong \mathrm{PG}(2,\mathbb{F})$ as a modular restriction. In this section, we show that $M$ has a minor that is a counterexample to Lemma 2.0.4, and then we finish the proof of the main result of this chapter, Theorem 2.0.2. Before proceeding, we state two useful facts.

**Bixby-Coullard Inequality.** *If $M$ is a matroid, $e \in E(M)$, and $(C_1,C_2)$ and $(D_1,D_2)$ are partitions of $E(M) \setminus \{e\}$, then $\lambda_{M\backslash e}(D_1) + \lambda_{M/e}(C_1) \geq \lambda_M(D_1 \cap C_1) + \lambda_M(D_2 \cap C_2) - 1$.*

The following lemma is proved in a more general form in [16] but we only need a special case. Recall that for disjoint sets $A,B$ in a matroid $M$, $\kappa_M(A,B) = \min\{\lambda_M(U) : A \subseteq U \subseteq E(M) \setminus B\}$.

**Lemma 2.8.1** ([16, Lemma 4.3])**.** *If $(A,B,V)$ is a partition of the elements of a matroid $M$ such that for each $e \in V$, either $\kappa_{M\backslash e}(A,B) < \kappa_M(A,B)$ or $\kappa_{M/e}(A,B) < \kappa_M(A,B)$, then there exists an ordering $v_1,\ldots,v_k$ of $V$ such that $\lambda_M(A) = \kappa_M(A,B)$ and for all $i = 1,\ldots,k$, $\lambda_M(A \cup \{v_1,\ldots,v_i\}) = \kappa_M(A,B)$.*

The main lemma of this section is the following.

**Lemma 2.8.2.** *If $\mathbb{F}$ is a finite field and $M$ is a simple vertically 4-connected matroid that is not $\mathbb{F}$-representable and has a modular restriction $N_0 \cong \mathrm{PG}(2,\mathbb{F})$, then there is a minor $M_0$ of $M$ that is minor-minimal subject to* *(a)* $M_0$ *is* 3*-connected,*

*(b)* $N_0$ *is a restriction of* $M_0$*, and*

*(c)* $M_0$ *is not* $\mathbb{F}$*-representable,*

*such that* $\lambda_{M_0}(E(N_0)) \geq 3$.

*Proof.* First, we make two claims involving connectivity.

(1) *Let* $(A, B, C, D)$ *be a partition of the ground set of a matroid* $Q$ *such that for each* $e \in D$, $\kappa_{Q\backslash e}(A, B) < \kappa_Q(A, B)$ *and for each* $e \in C$, $\kappa_{Q/e}(A, B) < \kappa_Q(A, B)$. *Then for* $e \in D$, $\kappa_{Q/e}(A, B) = \kappa_Q(A, B)$ *and for* $e \in C$, $\kappa_{Q\backslash e}(A, B) = \kappa_Q(A, B)$.

Let $e \in D$ and suppose that $\kappa_{Q/e}(A, B) < \kappa_Q(A, B)$. Since we also have $\kappa_{Q\backslash e}(A, B) < \kappa_Q(A, B)$, there exist partitions $(W_1, W_2)$ and $(U_1, U_2)$ of $E(Q\backslash e)$ such that $B \subseteq W_1 \cap U_1$, $A \subseteq W_2 \cap U_2$, and $\lambda_{Q\backslash e}(W_1) = \kappa_Q(A, B) - 1$ and $\lambda_{Q/e}(U_1) = \kappa_Q(A, B) - 1$. Then the Bixby-Coullard Inequality implies that

$$2\kappa_Q(A, B) - 2 = \lambda_{Q\backslash e}(W_1) + \lambda_{Q/e}(U_1) \geq \lambda_Q(U_1 \cap W_1) + \lambda_Q(U_2 \cap W_2) - 1.$$

But the right hand side of this inequality is at least $2\kappa_Q(A, B) - 1$. This proves the first half of (1) and the second half is the dual argument.

(2) *If* $(A, B, C, D)$ *is a partition of the ground set of a matroid* $Q$ *such that for each* $e \in D$, $\kappa_{Q\backslash e}(A, B) < \kappa_Q(A, B)$ *and for each* $e \in C$, $\kappa_{Q/e}(A, B) < \kappa_Q(A, B)$, *then there exists an ordering* $v_1, \ldots, v_k$ *of* $C \cup D$ *such that for all* $v_i \in D$, $\lambda_{Q\backslash v_i}(A \cup \{v_1, \ldots, v_{i-1}\}) < \kappa_Q(A, B)$ *and for all* $v_i \in C$, $\lambda_{Q/v_i}(A \cup \{v_1, \ldots, v_{i-1}\}) < \kappa_Q(A, B)$.

We apply Lemma 2.8.1 with $V = C \cup D$ to obtain an ordering $v_1, \ldots, v_k$ of the elements of $C \cup D$ such that $\lambda_Q(A) = \kappa_Q(A, B)$ and for all $i = 1, \ldots, k$, $\lambda_Q(A \cup \{v_1, \ldots, v_i\}) = \kappa_Q(A, B)$.

This implies that for each $v_i \in C \cup D$, $v_i$ is either in the closure of $A_i = A \cup \{v_1, \ldots, v_{i-1}\}$ and the closure of $B_i = \{v_{i+1}, \ldots, v_k\} \cup B$, or $v_i$ is in the coclosures of both sets.

If $v_i$ is in the closures of both $A_i$ and $B_i$, then $\lambda_{Q/v_i}(A_i) < \kappa_Q(A, B)$ and by (1), $e \in C$. Similarly, if $v_i$ is in the coclosures of these two sets then $\lambda_{Q\backslash v_i}(A_i) < \kappa_Q(A, B)$ and by (1), $e \in D$. Thus for $v_i \in C \cup D$, $v_i \in C$ if and only if $\lambda_{Q/v_i}(A_i) < \kappa_Q(A, B)$ and $v_i \in D$ if and only if $\lambda_{Q\backslash v_i}(A_i) < \kappa_Q(A, B)$. This proves (2).

We let $N'$ be a minor of $M$ that is minimal subject to (a), (b), and (c). Then we let $M'$ be a minor of $M$ that is minimal such that

- $M'$ has a minor $N$ satisfying (a), (b), and (c) with $|E(N)| = |E(N')|$, and
- $\kappa_{M'}(E(N_0), E(N) \setminus E(N_0)) = 3$.

This exists since $M$ itself satisfies these conditions. We may assume that $N$ is a proper minor of $M'$, otherwise with $M_0 = M'$ we are done.

We let $X = E(N) \setminus E(N_0)$. We may assume that $\lambda_N(E(N_0)) < 3$. Let $(C, D)$ be the partition of $E(M') \setminus E(N)$ such that $N = M'\backslash D/C$. Then by the minimality of $M'$, for each $e \in D$, $\kappa_{M'\backslash e}(E(N_0), X) = 2$ and for each $e \in C$, $\kappa_{M'/e}(E(N_0), X) = 2$.

By (2) applied with $Q = M'$, $A = E(N_0)$ and $B = X$, there exists an ordering $v_1, \dots, v_k$ of $C \cup D$ such that for all $v_i \in D$, $\lambda_{M'\backslash v_i}(E(N_0) \cup \{v_1, \dots, v_{i-1}\}) < \kappa_{M'}(E(N_0), X)$ and for all $v_i \in C$, $\lambda_{M'/v_i}(E(N_0) \cup \{v_1, \dots, v_{i-1}\}) < \kappa_{M'}(E(N_0), X)$.

Since $N_0$ is modular and $M'$ is simple, $v_1 \notin \mathrm{cl}_{M'}(E(N_0))$ so $v_1 \in D$.

(3) *$k \leq 2$ and if $k = 2$ then $v_2 \in C$.*

Suppose there are $v_j, v_{j+1} \in C$. Then $v_j, v_{j+1} \in \mathrm{cl}_{M'}(E(N_0) \cup \{v_1, \dots, v_{j-1}\})$, and since $\lambda_{M'/v_j}(E(N_0) \cup \{v_1, \dots, v_{j-1}\}) = 2$, we have $\lambda_{M'/v_j, v_{j+1}}(E(N_0) \cup \{v_1, \dots, v_{j-1}\}) \leq 1$, a contradiction. Therefore we may assume that there exists $j > 1$ with $v_j \in D$.

Then $(E(N_0) \cup \{v_1, \dots, v_{j-1}\}, X \cup \{v_{j+1}, \dots, v_k\})$ is a 3-separation of $M'\backslash v_j$. Now since $\sqcap_{M'}(E(N_0), X) = 2$, $\sqcap_N(E(N_0), X) = 2$, $M'|E(N_0) = N|E(N_0) = N_0$ and $v_1 \notin \mathrm{cl}_{M'}(E(N_0))$, it follows that $M'\backslash v_j/v_1$ has $N$ as a minor, and hence so does $M'/v_1$. Then with (1), the properties of $M'/v_1$ contradict the minimality of $M'$. This proves (3).

(4) *$k = 2$.*

If not, then (3) and the fact that $M' \neq N$ imply that $k = 1$. We note that $v_1 \notin \mathrm{cl}_{M'}(E(N_0))$ and $v_1 \notin \mathrm{cl}_{M'}(X)$. This implies that $M'/v_1|E(N_0) = N_0$ and $M'/v_1|X = M'|X$. Also, $\mathrm{cl}_{M'}(X) \cap E(N_0) \subseteq \mathrm{cl}_{M'/v_1}(X) \cap E(N_0)$. Since $N$ is not $\mathbb{F}$-representable, $N|X = M'|X$ has no $\mathbb{F}$-representation extending any representation of $\mathrm{cl}_{M'}(X) \cap E(N_0)$ induced by a representation of $N|E(N_0) = N_0$. Therefore, $M'/v_1|X$ also has no $\mathbb{F}$-representation extending any representation of $\mathrm{cl}_{M'/v_1}(X) \cap E(N_0)$ induced by a representation of $N_0$, and $M'/v_1$ is not $\mathbb{F}$-representable.

Therefore, $M'/v_1$ has a minor satisfying (a), (b), and (c), has the same number of elements as $N$, and by (1) satisfies $\kappa_{M'/v_1}(E(N_0), E(M'/v_1) \setminus E(N_0)) = 3$.

This contradicts the minimality of $M'$ and proves (4).

We may now assume from (4) that $k = 2$. Since $\lambda_{M'\backslash v_1}(E(N_0)) < \lambda_{M'}(E(N_0))$, $v_1 \notin \mathrm{cl}_{M'}(X \cup \{v_2\})$, so $M'/v_1|(X \cup \{v_2\}) = M'|(X \cup \{v_2\})$. Also, since $v_1 \notin \mathrm{cl}_{M'}(E(N_0))$, $M'/v_1|E(N_0) = N_0$. Therefore, as no $\mathbb{F}$-representation of $M'|E(N_0)$ extends to $M'|(X \cup \{v_2\})$, $M'/v_1$ is not $\mathbb{F}$-representable. So $M'/v_1$ contains a minor satisfying (a), (b), and (c), and by (1), $\kappa_{M'/v_1}(E(N_0), X) = 3$. But since $N_0$ is modular, and $v_2 \in \mathrm{cl}_{M'}(E(N_0) \cup \{v_1\})$ by (3), $v_2$ is parallel in $M'/v_1$ to an element of $E(N_0)$, so $M'/v_1\backslash v_2$ also contains a minor satisfying (a), (b), and (c); but $|E(M'/v_1\backslash v_2)| = |E(N)|$, so this minor is $M'/v_1\backslash v_2$ itself. Then the fact that $\lambda_{M'/v_1\backslash v_2}(E(N_0)) = \kappa_{M'/v_1\backslash v_2}(E(N_0), X) = 3$ completes the proof with $M_0 = M'/v_1\backslash v_2$. □

Finally, we restate and prove Theorem 2.0.2.

**Theorem 2.0.2.** *For any finite field $\mathbb{F}$, any vertically 4-connected matroid with a modular $\mathrm{PG}(2, \mathbb{F})$-restriction is $\mathbb{F}$-representable.*

*Proof.* We let $M$ be a vertically 4-connected matroid with a modular restriction $N_0 \cong \mathrm{PG}(2, \mathbb{F})$ and assume that $M$ is not $\mathbb{F}$-representable. By Lemma 2.8.2, there is a 3-connected, non-$\mathbb{F}$-representable matroid $M_0$ containing $N_0$ with $\lambda_{M_0}(E(N_0)) \geq 3$, such that $M_0$ has no 3-connected, non-$\mathbb{F}$-representable proper minor $M$ containing $N_0$. But Lemma 2.0.4 implies that $M_0$ has a 3-connected, non-$\mathbb{F}$-representable minor $M$ containing $N_0$ with $\lambda_M(E(N_0)) = 2$, a contradiction. Therefore, $M$ is $\mathbb{F}$-representable. □

## 2.9 Excluded minors

In this section we apply the results of this chapter to prove that no excluded minor for the variety of matroids representable over a finite field $\mathbb{F}$ has a $\mathrm{PG}(2, \mathbb{F})$-restriction.

The rank-$n$ **affine geometry** over a finite field $\mathbb{F}$, denoted $\mathrm{AG}(n-1, \mathbb{F})$, is the matroid obtained from the projective geometry $\mathrm{PG}(n-1, \mathbb{F})$ by deleting a hyperplane. We use the fact that, for each finite field $\mathbb{F}$ and $n \geq 3$, the affine geometry $\mathrm{AG}(n-1, \mathbb{F})$ is uniquely representable over $\mathbb{F}$. This is sometimes called the Fundamental Theorem of Affine Geometry; see [1, Theorem 2.6.3]. Note that this implies that any restriction of $\mathrm{PG}(2, \mathbb{F})$ containing $\mathrm{AG}(2, \mathbb{F})$ is uniquely representable over $\mathbb{F}$: each element of $\mathrm{PG}(2, \mathbb{F})$ lies in the closures of two distinct lines of $\mathrm{AG}(2, \mathbb{F})$ so its column in any representation is uniquely determined, up to scaling, by the representation of these lines. The following corollary verifies a special case of Conjecture 1.7.1.

**Corollary 2.9.1.** *For any finite field* $\mathbb{F}$*, no excluded minor for the variety of* $\mathbb{F}$*-representable matroids has a* $\mathrm{PG}(2, \mathbb{F})$*-restriction.*

*Proof.* We let $M$ be an excluded minor for the variety of $\mathbb{F}$-representable matroids and assume that $M$ has a $\mathrm{PG}(2, \mathbb{F})$-restriction, $N_0$. If $N_0$ is not modular in $M$, then by Proposition 1.5.2 $M$ has a rank-2 minor that is not $\mathbb{F}$-representable, so $M$ is not an excluded minor for $\mathbb{F}$-representability. Hence $N_0$ is a modular restriction of $M$. The class of $\mathbb{F}$-representable matroids is closed under direct sums and 2-sums by Proposition 2.1.2, so $M$ is 3-connected because a matroid that is not 3-connected is a direct sum or 2-sum of its proper minors. It follows from Lemma 2.0.4 that $\lambda_M(E(N_0)) = 2$. Let $L$ be the line of $N_0$ contained in $\mathrm{cl}_M(E(M) \setminus E(N_0))$ and let $e \in E(N_0) \setminus L$. Then $M|L$ is modular in $M \backslash (E(N_0) \setminus L)$ so $N_0 \backslash e$ is modular in $M \backslash e$, and $M$ is equal to the modular sum $(M \backslash e) \oplus_m N_0$. As $M$ is an excluded minor for $\mathbb{F}$-representability, $M \backslash e$ is $\mathbb{F}$-representable. We remarked above that $N_0 \backslash e$ is uniquely representable over $\mathbb{F}$, so it follows from Proposition 2.1.2 that $M$ is $\mathbb{F}$-representable, a contradiction. □

# Chapter 3

# Projective geometries

For each prime power $q$, a line that is representable over $\mathrm{GF}(q)$ can have at most $q+1$ points, so $U_{2,q+2}$ is an excluded minor for the variety of $\mathrm{GF}(q)$-representable matroids. Tutte proved that $U_{2,4}$ is the unique excluded minor for the variety of GF(2)-representable matroids [51]. However, this is the only case where the unique excluded minors are a line and its dual ($U_{2,4}$ is its own dual). The binary projective plane $\mathrm{PG}(2,2)$ is an excluded minor for the $\mathrm{GF}(q)$-representable matroids whenever $q$ is odd, and the non-Fano matroid $F_7^-$ (see Figure 3.1) is an excluded minor whenever $q$ is even and greater than two.

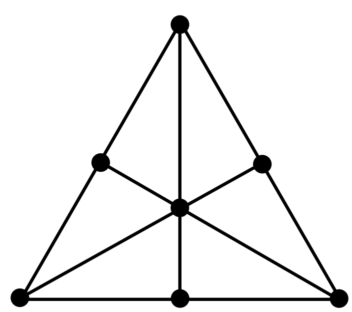


Figure 3.1: The non-Fano matroid, $F_7^-$

In this chapter, we present some conditions under which excluding a line as a minor does suffice to show that a matroid is representable over a given finite field. We look at matroids that have a projective geometry over a field $\mathrm{GF}(q)$ as a minor but do not have a $U_{2,q^2+1}$-minor. This is the longest line representable over $\mathrm{GF}(q^2)$, the smallest field with $\mathrm{GF}(q)$ as a proper subfield. For example, recall the following condition for representation over GF(2).

**Theorem 3.0.1** (Semple, Whittle, [45]). *Any 3-connected, representable matroid with a* PG(2, 2)*-minor is either binary or has a* $U_{2,5}$*-minor.*

It is possible that Theorem 3.0.1 extends to all prime powers $q$, and that any 3-connected, representable matroid with a $\mathrm{PG}(2, q)$-minor is either $\mathrm{GF}(q)$-representable or has a $U_{2,q^2+1}$-minor. We prove a weaker version where the projective plane is replaced by a large projective geometry.

**Theorem 3.0.2.** *For each prime power $q$, there is an integer $n = n_{3.0.2}(q)$ such that any 3-connected, representable matroid with a $\mathrm{PG}(n-1, q)$-minor is either $\mathrm{GF}(q)$-representable or has a $U_{2,q^2+1}$-minor.*

We actually prove a slightly stronger statement, which more directly mirrors our work with modular restrictions in the last chapter.

**Theorem 3.0.3.** *For each prime power $q$ and integers $k, \ell \geq 2$, there is an integer $n$ such that if $M$ is a 3-connected, representable matroid with a $\mathrm{PG}(n-1, q)$-minor, then either*

*(i) $M$ is $\mathrm{GF}(q)$-representable,*

*(ii) $M$ has a $U_{2,\ell}$-minor, or*

*(iii) $M$ has a minor $N$ with an element $e$ such that $N\backslash e \cong \mathrm{PG}(k-1, q)$ but $N$ is not $\mathrm{GF}(q)$-representable.*

A natural way to extend the notion of modularity to minors is to say that a minor $N_0$ of a matroid $M$ is 'modular' if $M$ has no minor $N$ with an element $e$ such that $N\backslash e = N_0$ and $e \in \mathrm{cl}_N(E(N_0))$, but $e$ is not parallel to any element of $E(N_0)$. Thus we can view the three outcomes of Theorem 3.0.3 as saying that either $M$ is $\mathrm{GF}(q)$-representable, $M$ has a given line as a minor, or one of the $\mathrm{PG}(k-1, q)$-minors of $M$ is not modular.

We note that Theorem 3.0.2 is an easy consequence of this stronger theorem: we simply choose $\ell = q^2+1$ and $k = 3$, and as we pointed out in Proposition 1.5.2, the matroid $N$ given by outcome (iii) has a $U_{2,q^2+1}$-minor.

We will conclude this chapter with the application that we promised in the introduction, characterizing the representability of 'dense' matroids in the class $\mathcal{U}(\ell)$ of matroids with no $U_{2,\ell+2}$-minor.

## 3.1 Non-representable matroids

Both the assumptions of 3-connectivity and representability are necessary in Theorem 3.0.3. First, for any prime power $q$, consider a modular sum of copies of $\mathrm{PG}(n-1,q)$ and $U_{2,q+2}$ whose ground sets meet in a single element. This matroid is not 3-connected, but it is representable (over $\mathrm{GF}(q^2)$) yet it is neither $\mathrm{GF}(q)$-representable nor has a $U_{2,q+3}$-minor. Next, we consider the following class of matroids which provide a counterexample to the stronger versions of both Theorems 3.0.3 and 3.0.2 without the assumption of representability. A matroid $M$ in this class is 3-connected and has a $\mathrm{PG}(n-1,q)$-minor, but it is not representable and has no $U_{2,q+3}$-minor.

For $n \geq 3$, we let $H$ be a hyperplane of $\mathrm{PG}(n,q)$, $C$ a circuit of size $n+1$ contained in $H$, and $M(n,q) = \mathrm{PG}(n,q)\backslash(H \setminus C)$. In $M(n,q)$, $C$ is a hyperplane. We let $M'(n,q)$ be the matroid on the same ground set as $M(n,q)$ whose rank function is the same as $r_{M(n,q)}$ except that $r_{M'(n,q)}(C) = |C|$; this is a matroid, referred to as the matroid obtained from $M(n,q)$ by relaxing the circuit-hyperplane $C$ (see [37, Proposition 1.5.14]).

Recall that the affine geometry $\mathrm{AG}(n-1,\mathbb{F})$ over a finite field $\mathbb{F}$ is obtained from $\mathrm{PG}(n-1,\mathbb{F})$ by deleting a hyperplane; we also denote it by $\mathrm{AG}(n-1,q)$ when $\mathbb{F}$ has order $q$. As the complement of a hyperplane in a matroid is a cocircuit, $\mathrm{AG}(n-1,\mathbb{F})$ is equal to the restriction of $\mathrm{PG}(n-1,\mathbb{F})$ to one of its cocircuits. For any element $e \in \mathrm{AG}(n-1,\mathbb{F})$, the minor $\mathrm{si}(\mathrm{AG}(n-1,\mathbb{F})/e)$ is isomorphic to $\mathrm{PG}(n-2,\mathbb{F})$. This is true because $\mathrm{si}(\mathrm{PG}(n-1,\mathbb{F})/e) \cong \mathrm{PG}(n-2,\mathbb{F})$ and each point of this minor is a parallel class corresponding to a line of $\mathrm{PG}(n-1,\mathbb{F})$ containing $e$, and such lines contain at least one other element of $E(\mathrm{AG}(n-1,\mathbb{F}))$.

The matroid $M'(n,q)$ has an $\mathrm{AG}(n,q)$-restriction, so it has a $\mathrm{PG}(n-1,q)$-minor. Suppose that $M'(n,q)$ has a vertical separation $(A,B)$ of order at most three. Neither $A$ nor $B$ is in the closure of the other, so the $\mathrm{AG}(n,q)$-restriction, which has the same rank as $M'(n,q)$, contains an element from each of $A$ and $B$, contradicting the fact that $\mathrm{AG}(n,q)$ is vertically 4-connected. This proves that $M'(n,q)$ is vertically 4-connected, and being simple, it is also 3-connected. Also, this matroid is quite close to being $\mathrm{GF}(q)$-representable: the next fact is an immediate consequence of [37, Proposition 3.3.5].

**Lemma 3.1.1.** *For any element $e$ of $M'(n,q)$, if $e \in C$ then $M'(n,q)\backslash e$ is $\mathrm{GF}(q)$-representable while if $e \notin C$ then $M'(n,q)/e$ is $\mathrm{GF}(q)$-representable.*

It follows from this lemma and the fact that $r_{M'(n,q)}(C) = r(M'(n,q))$ that any minor of $M'(n,q)$ that is not $\mathrm{GF}(q)$-representable has an element of $C$ in its ground set. Thus from any minor of $M'(n,q)$, deleting one element yields a $\mathrm{GF}(q)$-representable matroid. Therefore, $M'(n,q)$ has no $U_{2,q+3}$-minor.

On the other hand, we can show that $M'(n,q)$ is not representable, which means that Theorem 3.0.2 would not hold if we did not require that $M$ be representable. To prove this, we use the following classical result of projective geometry. Figure 3.2 depicts the configuration described in Theorem 3.1.2; see [1, Proposition 5.4.1] for a proof.

**Theorem 3.1.2** (Pappus's Theorem). *Let $L_1$ and $L_2$ be lines in a plane representable over a field with points $a, b, c \in L_1 \setminus L_2$ and $d, e, f \in L_2 \setminus L_1$. If $g, h$, and $i$ are respectively the points on the intersections of the lines spanned by $\{e, a\}$ and $\{d, b\}$, $\{f, a\}$ and $\{d, c\}$, and $\{f, b\}$ and $\{e, c\}$, then $g, h$, and $i$ are collinear.*

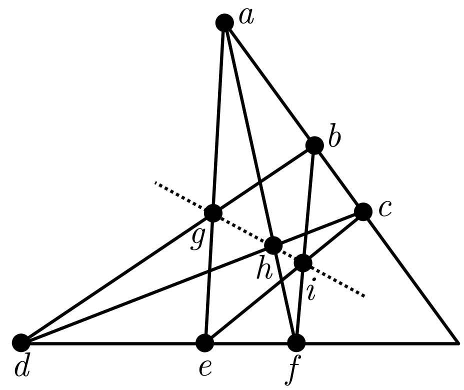



Figure 3.2: A Pappus configuration

**Lemma 3.1.3.** *For each $n \geq 3$, $M'(n, q)$ is not representable.*

*Proof.* We choose a set $X \subseteq C$ with $|X| = |C| - 3$ and let $N = \mathrm{si}(M(n, q)/X)$ and $N' = \mathrm{si}(M'(n, q)/X)$. Then the rank functions of $N'$ and $N$ agree except on the set $C \setminus X$, where we have $r_N(C \setminus X) = 2$ and $r_{N'}(C \setminus X) = 3$.

If $q = 2$, then $N \cong \mathrm{PG}(2, 2)$ and $N'$ is isomorphic to the non-Fano matroid shown in Figure 3.1. The non-Fano matroid is not representable over any field of characteristic two while $\mathrm{PG}(n-1, 2)$ is only representable over fields of characteristic two (see [37, Proposition 6.4.8]). Hence $M'(n, q)$, which has both these minors, is not representable.

The other case is when $q > 2$. We denote by $\{a, b, c\}$ the set $C \setminus X$. Since every line of $N$ has at least four points, we can choose a triangle $\{d, e, f\}$ of $N$ such that $a, b, c \notin \mathrm{cl}_N(\{d, e, f\})$. In addition, we set $g, h$, and $i$ to be the elements of $N$ that respectively lie in $\mathrm{cl}_N(\{e, a\}) \cap \mathrm{cl}_N(\{d, b\})$, $\mathrm{cl}_N(\{f, a\}) \cap \mathrm{cl}_N(\{d, c\})$, and $\mathrm{cl}_N(\{f, b\}) \cap \mathrm{cl}_N(\{e, c\})$. We observe that $r_N(\{g, h, i\}) = 2$ by Pappus's Theorem.

We apply this theorem again to these nine points in $N'$, but in a different order. In $N'$ the two triangles $\{d, e, f\}$ and $\{g, h, i\}$ lie on distinct lines, and $a \in \mathrm{cl}_{N'}(\{f,h\}) \cap \mathrm{cl}_{N'}(\{e,g\})$, $b \in \mathrm{cl}_{N'}(\{f,i\}) \cap \mathrm{cl}_{N'}(\{d,g\})$, and $c \in \mathrm{cl}_{N'}(\{e,i\}) \cap \mathrm{cl}_{N'}(\{d,h\})$. If $N'$ is representable over a field then Pappus's Theorem asserts that $a, b$, and $c$ are collinear. But $r_{N'}(\{a,b,c\}) = 3$, so $N'$, and hence $M'(n,q)$, is not representable. □

## 3.2 Representation over a subfield

In this section we look at the following question: if $\mathbb{F}$ is a field and $\mathbb{F}'$ is a subfield of $\mathbb{F}$, when can we transform an $\mathbb{F}$-representation of a matroid into an $\mathbb{F}'$-representation? First, we consider projective geometries and give a proof of the well-known fact, which we have already mentioned, that the projective plane $\mathrm{PG}(2,\mathbb{F})$ over a finite field $\mathbb{F}$ is representable only over extension fields of $\mathbb{F}$. We also show that any representation of this projective plane over an extension field of $\mathbb{F}$ can be transformed by row operations and column scaling into an $\mathbb{F}$-representation. Next, we present the 'confinement theorem' of Pendavingh and Van Zwam [40]. This theorem gives us convenient conditions under which we can transform a representation of a matroid $M$ over a field $\mathbb{F}$ into a representation over a subfield $\mathbb{F}'$ of $\mathbb{F}$, assuming that $M$ has a certain type of $\mathbb{F}'$-representable minor.

When $\mathbb{F}$ is an extension field of a field $\mathbb{F}'$, we say that an $\mathbb{F}$-matrix $A$ is a **scaled $\mathbb{F}'$-matrix** if there is a $\mathbb{F}'$-matrix obtained from $A$ by scaling rows and columns by elements of $\mathbb{F}^\times$. Recall that a representation $A$ of a matroid $M$ is in standard form with respect to a basis $B$ if $A|B$ is an identity matrix, and that in this case we index the rows of $A$ by the elements of $B$, so that $A_{bb} = 1$ for each $b \in B$. Also, recall that for each $e \in E(M)$, the fundamental circuit of $e$ with respect to $B$ is the unique circuit of $M$ contained in $B \cup \{e\}$. Part (b) of the following result appears in Nelson [36].

**Theorem 3.2.1.** *For any prime power $q$ and $n \geq 3$,*

*(a) $\mathrm{PG}(n-1,q)$ is representable only over fields isomorphic to extension fields of $\mathrm{GF}(q)$, and*

*(b) every standard-form representation of $\mathrm{PG}(n-1,q)$ is a scaled $\mathrm{GF}(q)$-matrix.*

*Proof.* Let $A$ be a representation of $\mathrm{PG}(n-1,q)$ over a field $\mathbb{F}$ in standard form with respect to a basis $B$. For each ordered pair $(x,y)$ of distinct elements of $B$, a third element $e$ in the closure of $\{x,y\}$ can be associated with a field element $-A_{xe}^{-1}A_{ye} \in \mathbb{F}$. We define $H_{xy}$ to be the set of such numbers:

$$H_{xy} = \left\{-A_{xe}^{-1}A_{ye} : e \in \mathrm{cl}_{\mathrm{PG}(n-1,q)}(\{x,y\}) \setminus \{x,y\}\right\}.$$

We note that $|H_{xy}| = |\operatorname{cl}_{\mathrm{PG}(n-1,q)}(\{x,y\})| - 2 = q-1$. Also, for all such pairs $(x,y)$, $H_{xy}$ is equal to $H_{yx}^{-1}$, the set of inverses of elements in $H_{yx}$. We fix some $u \in B$, and for each $z \in B \setminus \{u\}$ we may assume by scaling the row and column of $A$ indexed by $z$ that $1 \in H_{uz}$.

Suppose that $v$ and $w$ are distinct elements of $B \setminus \{u\}$, and $e \in \operatorname{cl}_{\mathrm{PG}(n-1,q)}(\{u,v\}) \setminus \{u,v\}$ and $f \in \operatorname{cl}_{\mathrm{PG}(n-1,q)}(\{u,w\}) \setminus \{u,w\}$. Recall that any two lines in a projective plane intersect in a point; so there exists $g \in \operatorname{cl}_{\mathrm{PG}(n-1,q)}(\{e,f\}) \cap \operatorname{cl}_{\mathrm{PG}(n-1,q)}(\{v,w\})$. Then the matrix

$$\begin{array}{c} \\ u \\ v \\ w \end{array} \begin{array}{c} \begin{array}{ccc} e & f & g \end{array} \\ \begin{pmatrix} A_{ue} & A_{uf} & 0 \\ A_{ve} & 0 & A_{vg} \\ 0 & A_{wf} & A_{wg} \end{pmatrix} \end{array}$$

is singular, so $(A_{ue}^{-1}A_{ve})(A_{wf}^{-1}A_{uf})(A_{vg}^{-1}A_{wg}) = -1$. This proves that

$$H_{uv}H_{wu} = \{\alpha\beta : \alpha \in H_{uv}, \beta \in H_{wu}\} \subseteq H_{wv}.$$

Moreover, since $1 \in H_{uv} \cap H_{wu}$, and the sets $H_{uv}, H_{wu}$, and $H_{wv}$ have the same size, $H_{uv} = H_{wu} = H_{wv}$. We denote this common set by $\Gamma = H_{uv}$. It is closed under multiplication and is finite, so it is a subgroup of $\mathbb{F}^\times$.

(1) $\Gamma \cup \{0\}$ *is a subfield of* $\mathbb{F}$.

Since $\Gamma$ is a finite multiplicative group, it suffices to show that $\Gamma \cup \{0\}$ is closed under addition. Let $\alpha, \beta \in \Gamma$. There exist elements $e \in \operatorname{cl}_{\mathrm{PG}(n-1,q)}(\{u,v\})$ and $f \in \operatorname{cl}_{\mathrm{PG}(n-1,q)}(\{u,w\})$ such that $-A_{ve}^{-1}A_{ue} = \alpha$ and $-A_{wf}^{-1}A_{uf} = \beta$. Since $1 \in \Gamma$, there is also an element $g \in \operatorname{cl}_{\mathrm{PG}(n-1,q)}(\{v,w\})$ such that $-A_{vg}^{-1}A_{wg} = 1$. The columns of $A[\{u,v,w\},\{e,f,g\}]$ can be scaled to

$$\begin{array}{c} \\ u \\ v \\ w \end{array} \begin{array}{c} \begin{array}{ccc} e & f & g \end{array} \\ \begin{pmatrix} -\alpha & -\beta & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \end{array}.$$

Let $z$ be the element in the intersection of the two lines $\operatorname{cl}_{\mathrm{PG}(n-1,q)}(\{e,f\})$ and $\operatorname{cl}_{\mathrm{PG}(n-1,q)}(\{u,g\})$. Then $A[\{u,v,w\},\{z\}]$ is parallel to $(-(\alpha-\beta),1,-1)^T$. Let $z'$ be the element that lies in the two lines spanned by $\{u,v\}$ and $\{w,z\}$. Then $A[\{u,v,w\},\{z'\}]$ is parallel to $(-(\alpha-\beta),1,0)^T$, which means that $\alpha - \beta \in \Gamma \cup \{0\}$. Thus $\Gamma - \Gamma \subseteq \Gamma \cup \{0\}$, and as $\Gamma \cup \{0\}$ is finite it is closed under addition, proving (1).

Since $|\Gamma \cup \{0\}| = q$, it follows from (1) that we may assume that $\mathbb{F}$ is an extension field of $\mathrm{GF}(q)$. Then $\Gamma \cup \{0\} = \mathrm{GF}(q)$ because $\Gamma$ is the unique subgroup of $\mathbb{F}^{\times}$ of order $q-1$ (it consists of the $q-1$ roots of the polynomial $x^{q-1}-1$ in $\mathbb{F}$).

We claim that each column of $A$ is a multiple of a vector with entries in $\mathrm{GF}(q)$. For each $a \in E(\mathrm{PG}(n-1,q)) \setminus B$, we let $C_a$ be the fundamental circuit of $a$ with respect to $B$ in $\mathrm{PG}(n-1,q)$ and $X_a = C_a \setminus \{a\}$. Suppose $|X_a| = 2$ and write $X_a = \{u, v\}$. Then $-A_{ua}^{-1}A_{va} \in \mathrm{GF}(q)$ so $A|\{a\}$ can be scaled to a vector with entries in $\mathrm{GF}(q)$. We proceed by induction on $|X_a|$. If we pick any distinct $u_1, u_2 \in X_a$ then for some $w_1 \in \mathrm{cl}_{\mathrm{PG}(n-1,q)}(X_a \setminus \{u_1\})$ and $w_2 \in \mathrm{cl}_{\mathrm{PG}(n-1,q)}(X_a \setminus \{u_2\})$, $a$ is in both the lines $\mathrm{cl}_{\mathrm{PG}(n-1,q)}(\{u_1, w_1\})$ and $\mathrm{cl}_{\mathrm{PG}(n-1,q)}(\{u_2, w_2\})$. All of $u_1, u_2, w_1$, and $w_2$ are distinct because $|X_a| > 2$. Since $X_{w_1}$ and $X_{w_2}$ are proper subsets of $X_a$, by assumption the four columns of $A|\{u_1, u_2, w_1, w_2\}$ are all scalings of vectors over $\mathrm{GF}(q)$, hence so is $A|\{a\}$. □

Next, we define confinement. Let $N$ be a minor of a matroid $M$ such that $N = M/C\backslash D$ for disjoint sets $C, D \subseteq E(M)$ where $C$ is independent and $D$ is coindependent. We choose a basis $B$ of $N$ and let $B' = B \cup C$, so $B'$ is a basis of $M$. Let $A'$ be an $\mathbb{F}$-representation of $M$ in standard form with respect to the basis $B'$. Recall that if $A = A'[B, E(N)]$ then we say that $A'$ **extends** $A$ and that $A$ is **induced** by $A'$.

Suppose that $N$ is a minor of an $\mathbb{F}$-representable matroid $M$ and $\mathbb{F}'$ is a subfield of $\mathbb{F}$. We say that $N$ **confines** $M$ **to** $\mathbb{F}'$ if whenever $N'$ is a minor of $M$ isomorphic to $N$, every $\mathbb{F}$-representation of $M$ that extends an $\mathbb{F}'$-representation of $N'$ is a scaled $\mathbb{F}'$-matrix. The following theorem of Pendavingh and Van Zwam reduces the problem of proving that a matroid $M$ with a $\mathrm{PG}(n-1,q)$-minor $N$ is $\mathrm{GF}(q)$-representable to checking minors of $M$ with at most $|E(N)|+2$ elements. Although they prove a theorem for representations over a generalization of fields called partial fields [40, Theorem 1.4], we state here only a specialization of it to fields.

**Theorem 3.2.2** (Pendavingh, Van Zwam, [40])**.** *If $\mathbb{F}'$ is a subfield of a field $\mathbb{F}$, $M$ and $N$ are 3-connected matroids, and $N$ is a minor of $M$, then either*

*(i) $N$ confines $M$ to $\mathbb{F}'$, or*

*(ii) $M$ has a 3-connected minor $M'$ such that $N$ does not confine $M'$ to $\mathbb{F}'$ and $N$ is isomorphic to one of $M'/x$, $M'\backslash y$, or $M'/x\backslash y$ for some $x, y \in E(M')$.*

## 3.3 The main theorem

To prove Theorem 3.0.3 we need a result from Ramsey theory. This is a corollary of the Hales-Jewett Theorem [25]; it is also a special case of the Affine Ramsey Theorem of Graham, Leeb, and Rothschild [23], for which a proof can be found in [24, p. 42].

**Theorem 3.3.1.** *For any prime power $q$ and integers $c$ and $k$, there is an integer $n = n_{3.3.1}(q, c, k)$ so that if the elements of $\mathrm{AG}(n-1, q)$ are $c$-coloured, it has a monochromatic restriction isomorphic to $\mathrm{AG}(k-1, q)$.*

We can now prove Theorem 3.0.3, which we restate here.

**Theorem 3.0.3.** *For each prime power $q$ and integers $k, \ell \geq 2$, there is an integer $n$ such that if $M$ is a 3-connected, representable matroid with a $\mathrm{PG}(n-1, q)$-minor, then either*

*(i) $M$ is $\mathrm{GF}(q)$-representable,*

*(ii) $M$ has a $U_{2,\ell}$-minor, or*

*(iii) $M$ has a minor $N$ with an element $e$ such that $N \backslash e \cong \mathrm{PG}(k-1, q)$ but $N$ is not $\mathrm{GF}(q)$-representable.*

*Proof.* We set $n$ to be the integer $n_{3.3.1}(q, \ell, k+1)$ given by Theorem 3.3.1 such that any $\ell$-colouring of the elements of $\mathrm{AG}(n-1, q)$ has a monochromatic restriction isomorphic to $\mathrm{AG}(k, q)$; note that $n \geq k+1 \geq 3$. We let $M$ be a 3-connected, representable matroid with a $\mathrm{PG}(n-1, q)$-minor. Then $M$ is representable over an extension field $\mathbb{F}$ of $\mathrm{GF}(q)$ by part (a) of Theorem 3.2.1. Part (b) of Theorem 3.2.1 implies that $M$ has an $\mathbb{F}$-representation that extends a $\mathrm{GF}(q)$-representation of $\mathrm{PG}(n-1, q)$. Hence if $\mathrm{PG}(n-1, q)$ confines $M$ to $\mathrm{GF}(q)$, then $M$ is $\mathrm{GF}(q)$-representable. Otherwise, we apply Theorem 3.2.2 to $M$ with $N = \mathrm{PG}(n-1, q)$ and $\mathbb{F}' = \mathrm{GF}(q)$, and conclude that there is a 3-connected minor $M'$ of $M$ such that $\mathrm{PG}(n-1, q)$ does not confine $M'$ to $\mathrm{GF}(q)$ and $M'$ has a $\mathrm{PG}(n-1, q)$-minor equal to either $M'/x$, $M' \backslash y$, or $M'/x \backslash y$ for some $x, y \in E(M')$. We rule out the $M' \backslash y$ case with the following claim.

(1) *If $M$ has a minor $P$ with an element $y$ such that $P \backslash y \cong \mathrm{PG}(n-1, q)$ but $\mathrm{PG}(n-1, q)$ does not confine $P$ to $\mathrm{GF}(q)$, then (iii) holds.*

If $y$ were a coloop of $P$ then $\mathrm{PG}(n-1, q)$ would confine $P$ to $\mathrm{GF}(q)$, so $r(P) = n$. Suppose that $y$ is contained in a parallel pair of $P$. Then the only $\mathrm{PG}(n-1,q)$-minors of $P$ are $P\backslash y$ and the one obtained by deleting the element parallel to $y$. Any representation of $P$ that extends a $\mathrm{GF}(q)$-representation of one of these restrictions is a scaled $\mathrm{GF}(q)$-matrix, contradicting the fact that $\mathrm{PG}(n-1,q)$ does not confine $P$. Therefore, $P$ is simple, and hence not $\mathrm{GF}(q)$-representable because it has more points than $\mathrm{PG}(n-1,q)$. We note that any two lines in a projective plane intersect in a point. Thus, since $y$ is not parallel to any element of $P\backslash y$, there is at most one line $L$ of $P\backslash y$ such that $y \in \mathrm{cl}_P(L)$. If we choose any line $L'$ of $P\backslash y$ distinct from $L$, then for any $z \in L'$ not in $L$, $y$ is not in a parallel pair of $P/z$ and $\mathrm{si}(P/z)\backslash y \cong \mathrm{PG}(n-2,q)$. We can repeatedly contract elements of such lines and simplify, $n-k$ times, until we obtain a minor $P'$ of $P$ such that $P'$ is simple and $P'\backslash y \cong \mathrm{PG}(k-1,q)$, so that (iii) holds.

By (1) with $P = M'$ we may assume that $M$ has a $\mathrm{PG}(n-1,q)$-minor $N$ equal to either $M'/x$ or $M'/x\backslash y$ for some $x, y \in E(M')$.

We let $B$ be a basis of $N$ and $A$ an $\mathbb{F}$-representation of $M'$ in standard form with respect to the basis $B \cup \{x\}$ of $M'$. Since $\mathrm{PG}(n-1,q)$ does not confine $M'$ to $\mathrm{GF}(q)$ we may assume that $A$ is not a scaled $\mathrm{GF}(q)$-matrix but it induces a $\mathrm{GF}(q)$-representation $A[B, E(N)]$ of $N$. Moreover, when $N \cong M'/x\backslash y$, applying (1) with $P = M'/x$ lets us assume that $\mathrm{PG}(n-1,q)$ confines $M'/x$ to $\mathrm{GF}(q)$ and that the induced representation $A[B, E(N) \cup \{y\}]$ of $M'/x$ also has all its entries in $\mathrm{GF}(q)$.

(2) *There are two elements $f, g \in E(M'/x)$ such that $A_{xf}, A_{xg} \neq 0$, $A_{xf}^{-1}A_{xg} \notin \mathrm{GF}(q)$, and $\{f, g\}$ is independent in $M'/x$.*

There are elements $f, g \in E(M'/x)$ such that $A_{xf}, A_{xg} \neq 0$ and $A_{xf}^{-1}A_{xg} \notin \mathrm{GF}(q)$, for if not we could scale the row and column of $x$ to get a $\mathrm{GF}(q)$-matrix. If $f, g \in E(N)$ then $\{f, g\}$ is independent in $M'/x$ because $N$ is simple. Otherwise, $N = M'/x\backslash y$ and $y \in \{f, g\}$. Since $M'$ is 3-connected, $M'/x$ has no loops so in $M'/x$, $\{f, g\}$ is either independent or a parallel pair of elements. Suppose $\{f, g\}$ is a parallel pair in $M'/x$ and that there is no other choice of $\{f, g\}$ that is independent in $M'/x$. There is at most one element parallel to $y$ in $M'/x$, so $A_{xh} = 0$ for all $h \in E(M') \setminus \{x, f, g\}$. Then $\{x, f, g\}$ is both a circuit and a cocircuit of $M'$, so $(\{x, f, g\}, E(M') \setminus \{x, f, g\})$ is a 2-separation of $M'$, a contradiction. This proves (2).

We choose a pair of elements $f, g \in E(M'/x)$ as in (2), and by scaling we may assume that $A_{xf} = 1$ and $A_{xg} = \omega$ for some $\omega \notin \mathrm{GF}(q)$. We choose some hyperplane $H$ of $M'/x$ that contains $\{f, g\}$ and we choose an element $z \in E(M'/x) \setminus H$. We let $B'$ be the union of $\{z\}$ with a basis of $H$ in $M'/x$, so

$B' \cup \{x\}$ is a basis of $M'$, and we let $A'$ be a representation of $M'$ in standard form with respect to $B' \cup \{x\}$. We can obtain $A'$ from $A$ by row operations without using the row of $x$, so that $A'[B', E(M')]$ has all its entries in $\mathrm{GF}(q)$. We let $C = E(M'/x) \setminus H$, so $C$ is a cocircuit of $M'/x$ containing $z$. Then the restriction $(M'/x)|C$ is isomorphic to $\mathrm{AG}(n-1, q)$. For each $e \in E(M'/x)$, the entry $A'_{ze}$ is non-zero if and only if $e \in C$, and by scaling columns of $A'$ we may assume that all entries in the row of $z$ are either 0 or 1. The submatrix $A'[\{x, z\}, C]$ represents $(M'/(B' \setminus \{z\}))|C$, which is a rank-2 minor of $M'$. If this matrix contains a set of at least $\ell$ non-parallel columns, then $M'$, and hence $M$, has a $U_{2,\ell}$-minor, and (ii) holds. Otherwise, since $A'_{ze} = 1$ for all $e \in C$, there are fewer than $\ell$ distinct elements of $\mathbb{F}$ that appear in $A'[\{x\}, C]$. We can therefore $\ell$-colour the elements of $(M'/x)|C$ by assigning to each $e \in C$ the colour $A'_{xe}$. Since $(M'/x)|C \cong \mathrm{AG}(n-1, q)$, with our choice of $n = n_{3.3.1}(q, \ell, k+1)$ Theorem 3.3.1 implies that there is a monochromatic restriction of $(M'/x)|C$ isomorphic to $\mathrm{AG}(k, q)$. We denote by $Y$ the ground set of this restriction. The entries $A'_{xe}$ for $e \in Y$ are all equal to some $\beta \in \mathbb{F}$, so $A'[\{x\}, Y]$ is a multiple of $A'[\{z\}, Y]$ and $M'|Y$ is also isomorphic to $\mathrm{AG}(k, q)$. Since $f, g \notin C$, $A'_{zf} = A'_{zg} = 0$, so the row space of $A'$ contains a vector $u \in \mathbb{F}^{E(M')}$ such that $u_e = -\beta$ for all $e \in Y$ and $u_f = u_g = 0$.

As $N$ is 3-connected, $\kappa_N(\{f, g\}, Y) = 2$. Also, when $N = M'/x\backslash y$, $\kappa_{M'/x}(\{f, g\}, Y) = 2$ because $y$ is parallel to an element of $N$ in $M'/x$. By Tutte's Linking Theorem, there is a set $Z \subseteq E(M'/x)$ disjoint from $Y$ and $\{f, g\}$ such that $\sqcap_{(M'/x)/Z}(Y, \{f, g\}) = 2$, and $Z$ and $Y$ are skew. This means that $\{f, g\}$ is independent in $(M'/x)/Z$ and $f, g \in \mathrm{cl}_{(M'/x)/Z}(Y)$. Since $Z$ and $Y$ are skew, there exists a basis $B''$ of $M'/x$ that contains $Z$ and a basis of $Y$. We apply row operations to $A'$ to get a representation $A''$ of $M'$ in standard form with respect to the basis $B'' \cup \{x\}$. The row of $x$ is the same in $A''$ and $A'$ and the vector $u$ is also in the row space of $A''$.

Consider the matrix $D$ obtained from $A''$ by adding the vector $u$ to the row of $x$ then restricting to the submatrix in rows $\{x\} \cup (B'' \cap Y)$ and columns $Y \cup \{f, g\}$. This matrix $D$ represents $M'' = (M'/(B'' \setminus Y))|(Y \cup \{f, g\})$ and it has the form

$$D = \begin{array}{c} \begin{matrix} Y & f & g \end{matrix} \\ \begin{pmatrix} 0 & 1 & \omega \\ D_1 & \alpha & \alpha' \end{pmatrix} \end{array},$$

where $D_1$ is a $\mathrm{GF}(q)$-representation of $\mathrm{AG}(k, q)$ and $\alpha$ and $\alpha'$ are columns with all entries in $\mathrm{GF}(q)$. Since $\{f, g\}$ is independent and contained in the closure of $Y$ in $(M'/x)/Z$, $\alpha$ and $\alpha'$ are both non-zero and are not parallel to each other. The minor $M''/f$ has the following representation

$$\begin{array}{cc} Y & g \\ \left(D_1 \right. & \left. \alpha' - \omega\alpha\right). \end{array}$$

Since $\omega \notin \mathrm{GF}(q)$ and $\alpha$ and $\alpha'$ are both non-zero and are not parallel, the column $\alpha' - \omega\alpha$ is not parallel to a vector over $\mathrm{GF}(q)$. We have $M''/f\backslash g \cong \mathrm{AG}(k, q)$. Suppose there are two distinct lines $L_1$ and $L_2$ of $M''/f\backslash g$ such that $g \in \mathrm{cl}_{M''/f}(L_1) \cap \mathrm{cl}_{M''/f}(L_2)$. Then there is a $\mathrm{GF}(q)$-representation of a matroid isomorphic to $\mathrm{PG}(k, q)$ of the form $(D_1\ D_2)$ for some matrix $D_2$, and as $L_1 \cup L_2$ has rank three, there is a unique element indexing a column of $(D_1\ D_2)$ that is in the closure of both $L_1$ and $L_2$. This column is parallel to $\alpha' - \omega\alpha$, contradicting the fact that it is not parallel to a vector over $\mathrm{GF}(q)$. So there is at most one line $L$ of $M''/f\backslash g$ such that $g \in \mathrm{cl}_{M''/f}(L)$, and there exists an element $e$ of $M''/f\backslash g$ that is not in any such line, so $\mathrm{cl}_{M''/f}(\{e, g\}) = \{e, g\}$. Therefore, $g$ is not in a parallel pair of $M''/f, e$, and $\mathrm{si}(M''/f, e)\backslash g \cong \mathrm{PG}(k-1, q)$, so outcome (iii) holds. □

## 3.4 Growth rates

We can now prove our claim from the introduction about the class $\mathcal{U}(\ell)$ of matroids with no $U_{2,\ell+2}$-minor. Theorem 1.8.4 asserted that any 3-connected, representable matroid $M$ in $\mathcal{U}(\ell)$ with sufficiently large rank and at least $(2\sqrt{\ell})^{r(M)}$ points is representable over a field of order at most $\ell$. This will follow easily from the combination of Theorem 3.0.2 and the following theorem of Geelen and Kabell.

**Theorem 3.4.1** (Geelen, Kabell, [18])**.** *For all integers $\ell, q_0 \geq 2$ and $n$, there exists an integer $c_{3.4.1}(\ell, q_0, n)$ such that if $M$ is a matroid with no $U_{2,\ell+2}$-minor and $\varepsilon(M) \geq c_{3.4.1} q_0^{r(M)}$, then $M$ has a $\mathrm{PG}(n-1, q)$-minor for some prime power $q > q_0$.*

We prove the following more precise result, which implies Theorem 1.8.4 because the smallest prime power greater than or equal to $\sqrt{\ell}$ is at most $2\sqrt{\ell}$ (there is a power of two in this range).

**Theorem 3.4.2.** *Let $\ell \geq 2$ and $q_0$ the smallest prime power greater than or equal to $\sqrt{\ell}$. There is an integer $c$ such that if $M$ is a 3-connected, representable matroid with no $U_{2,\ell+2}$-minor and $|E(M)| \geq c q_0^{r(M)}$, then $M$ is representable over a field of order at most $\ell$.*

*Proof.* We set $n$ to be the maximum of $n_{3.0.2}(q)$ for all prime powers $q \leq \ell$, so that by Theorem 3.0.2, for any such $q$ a 3-connected, representable matroid with a $\mathrm{PG}(n-1,q)$-minor and no $U_{2,q^2+1}$-minor is representable over $\mathrm{GF}(q)$. We set $c = c_{3.4.1}(\ell, q_0, n)$ so that a matroid $M$ with no $U_{2,\ell+2}$-minor and $\varepsilon(M) \geq cq_0^{r(M)}$ has a $\mathrm{PG}(n-1,q)$-minor for some prime power $q > q_0$. Now, we can choose any 3-connected, representable matroid $M$ with no $U_{2,\ell+2}$-minor and $\varepsilon(M) \geq cq_0^{r(M)}$. Then $M$ has a $\mathrm{PG}(n-1,q)$-minor for some prime power $q > q_0$. The fact that $M$ has no $U_{2,\ell+2}$-minor implies that $q \leq \ell$. Also, $q > \sqrt{\ell}$ so $M$ has no $U_{2,q^2+1}$-minor. Therefore, $M$ is $\mathrm{GF}(q)$-representable. □

We point out that, for $\ell > 3$, this theorem is false if we do not assume that $M$ is representable, even if we add the assumption that $M$ is vertically 4-connected. The counterexample is the same set of matroids $\{M'(n,q)\}_{n\geq 3}$ that we defined in Section 3.1, where $q$ is the largest prime power *strictly* less than $\ell$. Since $\ell > 3$, we can easily check that $q$ is strictly larger than $q_0$, the smallest prime power at least $\sqrt{\ell}$. The matroid $M'(n,q)$ has more than $|\mathrm{AG}(n,q)| = q^n = q^{r(M'(n,q))-1}$ points. As we saw, it is 3-connected and vertically 4-connected and it has no $U_{2,q+3}$-minor, hence no $U_{2,\ell+2}$-minor, yet it is not representable.

On the other hand, for $\ell = 2$ we have Tutte's well-known theorem that any matroid with no $U_{2,\ell+2}$-minor is $\mathrm{GF}(2)$-representable. This leaves a gap, the $\ell = 3$ case, in which we know of no counterexample to the version of Theorem 3.4.2 without the requirement of representability. If we set $c = 1$ in the theorem and consider a matroid $M$ with at least $2^{r(M)}$ points (which is too high to be binary) then we have the following open question.

**Problem 3.4.3.** *If $M$ is a 3-connected matroid with no $U_{2,5}$-minor and $|E(M)| \geq 2^{r(M)}$, is $M$ representable over* $\mathrm{GF}(3)$*?*

# Chapter 4

# Dowling geometries

In this chapter we turn away from the varieties of matroids representable over a finite field and look at conditions forcing membership in one of the varieties of representable Dowling matroids, or more generally the frame matroids over a field. In Chapter 1, we saw the following characterization of representable matroids with modular Dowling geometry restrictions.

**Theorem 4.0.1** (Geelen, Gerards, Whittle, [17])**.** *For any field $\mathbb{F}$ and finite subgroup $\Gamma$ of $\mathbb{F}^{\times}$, if $M$ is a vertically 5-connected $\mathbb{F}$-representable matroid with a modular $\mathrm{DG}(4,\Gamma)$-restriction, then $M$ is a frame matroid over $\mathbb{F}$.*

We note that we can apply this theorem not just to representable matroids but to any matroid with a modular $\mathrm{DG}(4,\Gamma)$-restriction for any finite cyclic group $\Gamma$. A matroid with such a restriction is automatically representable over the same set of finite fields, by Theorem 2.0.1. Moreover, $\mathrm{DG}(4,\Gamma)$ is representable over some finite field because it is $\mathbb{C}$-representable, and every representable matroid can be represented over a finite field [37, Corollary 6.8.13].

As we did in the last chapter for projective geometries, we extend this type of result from matroids with Dowling geometries as restrictions to those with them as minors. We will actually prove a theorem involving structures called *patchworks*, which generalize the frame matroids over a field. We postpone the statement of our main theorem until we have defined patchworks precisely in Section 4.2, but the specialization of it to frame matroids is as follows.

**Theorem 4.0.2.** *For any finite group $\Gamma$ and integers $k, \ell \geq 3$, there is an integer $n$ such that if $\mathbb{F}$ is a field and $M$ is a vertically 5-connected $\mathbb{F}$-representable matroid with a $\mathrm{DG}(n,\Gamma)$-minor, then either*

*(i) $M$ is a frame matroid over $\mathbb{F}$,*

*(ii)* *$M$ has a $U_{2,\ell}$-minor, or*

*(iii)* *$M$ has a minor $N$ with a non-coloop element $e$ such that $N\backslash e \cong \mathrm{DG}(k,\Gamma)$ but $e$ is not in the closure of any pair of joints of $N\backslash e$.*

When $|\Gamma| > 1$, outcome (iii) can be replaced with

(iii') $M$ has a minor $N$ with an element $e$ such that $N\backslash e \cong \mathrm{DG}(k,\Gamma)$ but $N$ is not a frame matroid over $\mathbb{F}$.

Thus the obstruction to being a frame matroid is a non-frame matroid that is one element away from a Dowling geometry. The equivalence of (iii') with (iii) holds because when $|\Gamma| > 1$ and $k \geq 3$, every automorphism of $\mathrm{DG}(k,\Gamma)$ sends its set of joints to itself. Hence $e$ is in the closure of two joints of $N\backslash e$ if and only if $N$ is a frame matroid.

However, when $|\Gamma| = 1$ the clique $M(K_{k+1})$, which is isomorphic to $\mathrm{DG}(k,\Gamma)$, can be expressed as a Dowling geometry with respect to many different choices of joints. In particular, for each vertex $v$ of $K_{k+1}$, the set $\delta(v)$ of edges incident with $v$ can be viewed as the set of joints. Consider then the case where $e$ is in the closure of two elements $f, g$ of $N\backslash e \cong \mathrm{DG}(k,\Gamma)$ that are not joints but correspond to edges of $K_{k+1}$ incident with a common vertex. Then we can also view $N\backslash e$ as a Dowling geometry with respect to a different set of joints that contains $f$ and $g$, and $N$ is in fact a frame matroid. See Figure 4.1 for an example where $N\backslash e$ has a set of joints $\{a, b, c\}$, no two of which span $e$, but $N$ is still a frame matroid.

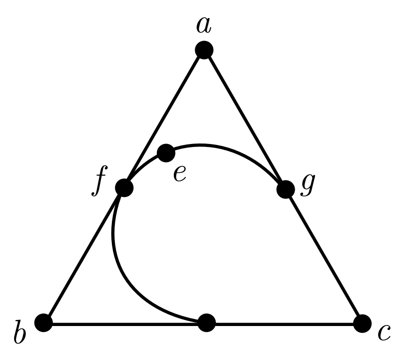



Figure 4.1: A 'non-frame' extension of a clique that is a frame matroid

Our proof has three main parts. First, we have the case where $M$ has a $\mathrm{DG}(n,\Gamma)$-minor of the same corank. This case is equivalent to a problem about minors of certain group-labelled graphs. Second, we have a technical result about patchworks, and finally we proceed by induction on $r^*(M)$.

## 4.1 Represented matroids

Let $\mathbb{F}$ be a field. An **$\mathbb{F}$-represented matroid**, $M$, is a pair $(E, U)$ where $E$ is a finite set and $U$ is a subspace of $\mathbb{F}^E$. A **representation** of an $\mathbb{F}$-represented matroid, $M = (E, U)$, is a matrix $A \in \mathbb{F}^{\dim(U)\times E}$ whose rows are a basis of $U$. Given such a representation $A$, every representation of $M$ has the form $PA$ where $P$ is an invertible $\mathbb{F}$-matrix. For each $\mathbb{F}$-represented matroid $M = (E, U)$ there is a corresponding matroid $M_{\mathbb{F}}(A)$, the matroid represented by the matrix $A$. We note that this matroid is independent of our choice of basis of $U$ used to construct $A$. Its rank is equal to the dimension of $U$.

For $D \subseteq E$, we define $M\backslash D$ to be the pair $(E \setminus D, U')$, where $U'$ is the row space of $A|(E \setminus D)$. This operation corresponds to the deletion of $D$ from the corresponding matroid. We also define $M^* = (E, U^{\perp})$, whose associated matroid is the dual of that of $M$. Lastly, for $C \subseteq E$, we define $M/C = (M^*\backslash C)^*$, which corresponds to contraction of the set $C$.

We therefore treat every $\mathbb{F}$-represented matroid $M = (E, U)$ as a matroid with ground set $E$, and apply the usual minor operations to $M$. We denote the ground set of $M$ by $E(M) = E$.

Let $A$ be a representation of an $\mathbb{F}$-represented matroid $M$, and $\alpha \in \mathbb{F}^{r(M)}$. If $A'$ is constructed from $A$ by adding the column $\alpha$ indexed by a new element $e$, then we denote by $M_{+e}$ the $\mathbb{F}$-represented matroid represented by $A'$ and say that $M_{+e}$ is obtained by **extending** $M$ by $e$. Recall that each representation of $M$ has the form $PA$, where $P \in \mathbb{F}^{r(M)\times r(M)}$ is invertible. Therefore, for each such choice of representation of $M$, adding the column $P\alpha$ indexed by $e$ defines the same $\mathbb{F}$-represented matroid $M_{+e}$, and we do not need to specify a particular representation when extending a represented matroid. Suppose that $C, D \subseteq E(M)$ and $N = M/C\backslash D$ is a minor of $M$ such that $e \notin \mathrm{cl}_{M_{+e}}(C)$. We define $N_{+e}$ to be the corresponding minor of $M_{+e}$, that is, $N_{+e} = M_{+e}/C\backslash D$.

Finally, we write $M_{+W}$ for the matroid obtained by extending $M$ by each element of a finite set $W$ of new elements.

## 4.2 Patchworks

A **bridge** of a set $W$ in a matroid $M$ is a component of $M/W$. When $M$ is a represented matroid and $M_{+W}$ is an extension of $M$ by a set $W$, we define the bridges of the pair $(M, W)$ to be the bridges of $W$ in $M_{+W}$. If $W$ is an independent set of $M_{+W}$, then we can define the **attachments** of a bridge $B$ of $(M, W)$ to be the elements of the minimal subset $A \subseteq W$ such that $\sqcap_M(B, A) = \sqcap_M(B, W)$. A unique such set $A$ exists because whenever $W$

is a basis of a vector space $Y$ and $Y'$ is a subspace of $Y$, there is a unique minimal subset of $W$ that spans $Y'$. The **attachments** of an element $e$ are the attachments of the bridge containing $e$.

Let $M$ be an $\mathbb{F}$-represented matroid and $M_{+V}$ an extension of $M$ by a finite set $V$. The pair $(M, V)$ is called a **patchwork** if $V$ is independent in $M_{+V}$ and for each bridge $B$ of $(M, V)$,

(P1) $B$ has at most three attachments, and

(P2) if $B$ has a set $X$ of three attachments, there is no circuit $C$ of $M_{+V}$ such that $X \subseteq C \subseteq X \cup B$.

If $(M, V)$ is a patchwork then we call the elements of $V$ its **vertices** and the bridges of $(M, V)$ its **patches**. A **$k$-patch** is a patch with $k$ attachments.

A patchwork $(M, V)$ is called a **framework** if every element of $M$ is spanned in $M_{+V}$ by a set of at most two elements of $V$. Equivalently, when $M$ has no coloops, $(M, V)$ is a framework if every patch has a single element. When $(M, V)$ is a framework, it has no 3-patches and $M$ is a frame matroid.

One way to understand patchworks is to think of them as frame matroids onto which we attach patches by modular sums. We can make this precise as follows.

**Theorem 4.2.1.** *If $(M, V)$ is a patchwork, then there is a framework $(N, V)$ and a set of matroids $M_1, \dots, M_k$ such that*

- $M = (M_1 \oplus_m (M_2 \oplus_m \cdots \oplus_m (M_k \oplus_m N)))\backslash E(N)$,
- $E(M_i) \cap E(N)$ *is contained in* $\mathrm{cl}_{N_{+V}}(A)$ *for a set* $A \subseteq V$ *of at most three vertices, and*
- $E(M_i) \cap E(M_j) \subseteq E(N)$ *for all* $i \neq j$.

*Proof.* Let $P$ be a patch of $(M, V)$ with set of attachments $A$. For each set $C \subseteq P$ that is skew to $V$ in $M_{+V}$, we can extend $M$ by a finite number of new elements lying in the closure of $A$ such that every element of $P \setminus C$ is parallel to one of these elements in $M/C$. There are finitely many subsets of $P$, so the set $Z(P)$ of all elements we can add in this way is finite. Since there is no circuit of $M_{+V}$ in $P \cup A$ that contains three elements of $A$, every element of $Z(P)$ is in the closure of at most two elements of $A$. Therefore, if we let $Z$ be the union of the sets $Z(P)$ for all patches $P$ of $(M, V)$, the pair $(M_{+Z}|Z, V)$ is a framework.

Note that for each patch $P$, $M_{+Z(P)}|Z(P)$ is modular in $M_{+Z(P)}|(P \cup Z(P))$ by construction. Therefore, $M$ is obtained from $M_{+Z}|Z$ by taking modular sums with the restrictions $M_{+Z}|(P \cup Z(P))$ for each patch $P$, then deleting $Z$. □

We have defined all the terms necessary to state the main theorem of this chapter, the more general form of Theorem 4.0.2.

**Theorem 4.2.2.** *For any finite group* $\Gamma$ *and integers* $m, \ell \geq 3$*, there is an integer* $n$ *such that if* $\mathbb{F}$ *is a field and* $M$ *is an* $\mathbb{F}$*-represented matroid with* $\mathrm{DG}(n, \Gamma)$ *as a minor, then either*

*(i) there is a patchwork* $(M, V)$ *of which no patch contains a cocircuit of* $\mathrm{DG}(n, \Gamma)$*,*

*(ii)* $M$ *has a* $U_{2,\ell}$*-minor, or*

*(iii)* $M$ *has a minor* $N$ *with a non-coloop element* $e$ *such that* $N \backslash e \cong \mathrm{DG}(m, \Gamma)$ *but* $e$ *is not in the closure of any pair of joints of* $N \backslash e$*.*

We note that this generalizes the version for frame matroids (Theorem 4.0.2) because when $M$ is vertically 5-connected, any patchwork $(M, V)$ with at least four vertices is a framework. To see this, suppose $P$ is a patch of $(M, V)$ that is not spanned in $M_{+V}$ by its attachment set, $A$. Since $|A| \leq 3$, there is some element $v \in V \setminus A$. Then $(P, E(M) \setminus P)$ is a $(\leq 4)$-separation of $M$, and it is a vertical $(\leq 4)$-separation because $v \notin \mathrm{cl}_M(P)$.

The purpose of requiring that no patch contains a cocircuit of $\mathrm{DG}(n, \Gamma)$ in (i) is to ensure that the Dowling geometry minor is not 'hidden inside' a single patch of $(M, V)$, so we can think of the patchwork as extending the graph-like structure of the Dowling geometry.

The **skeleton** of a patchwork $(M, V)$ is the simple graph $G$ with vertex set $V$ in which $u, v \in V$ are adjacent if and only if $(M, V)$ has a patch $P$ with $u$ and $v$ as attachments. We say that a patch $P$ of a patchwork $(M, V)$ is **realizable** if there is a set $C(P) \subseteq P$ that is skew to $V$ in $M_{+V}$ such that for each pair $u, v$ of distinct attachments of $P$, $\mathrm{cl}_{M_{+V}/C(P)}(\{u, v\})$ contains an element of $P$ that is not a loop and is not parallel to $u$ or $v$. We note that 0- and 1-patches are always realizable. The patchwork $(M, V)$ is called **realizable** if each of its patches is realizable. When it is realizable, $M$ has a minor $N$ such that $(N, V)$ is a framework with the same skeleton as $(M, V)$.

We close this section by stating an extension of Seymour's three-in-a-circuit theorem (Theorem 1.3.1) from binary matroids to arbitrary represented matroids, which we will use several times in this chapter.

**Theorem 4.2.3** (Geelen, Gerards, Whittle, [17]). *For any field $\mathbb{F}$, if $M$ is an $\mathbb{F}$-represented matroid and $X \subseteq E(M)$ is independent, then either there is a circuit $C$ of $M$ such that $|C \cap X| \geq 3$, or there is a realizable patchwork $(M, V)$ such that $X \subseteq V$.*

We remark that Theorem 4.2.3 implies Theorem 4.0.1, because no set of three joints of a modular Dowling geometry restriction can be contained in a circuit.

## 4.3 Group-labelled graphs

Recall that when $\Gamma$ is a group, a $\Gamma$-labelled graph $G$ is a pair $(\vec{G}, \gamma_G)$ where $\vec{G}$ is an oriented graph and $\gamma_G \in \Gamma^{E(\vec{G})}$. We write $\widetilde{G}$ for the graph obtained from $\vec{G}$ by disregarding the edge orientations. For each $e \in E(G)$ we call $\gamma_G(e)$ the **label** of $e$.

If $G$ is a $\Gamma$-labelled graph, $v \in V(G)$ and $e \in E(G)$ is an edge incident with $v$, then we define

$$\gamma_G(v, e) = \begin{cases} \gamma_G(e), & \text{if } v \text{ is the head of } e\text{, including when } e \text{ is a loop} \\ \gamma_G(e)^{-1}, & \text{otherwise.} \end{cases}$$

Let $G$ be a $\Gamma$-labelled graph, $\alpha \in \Gamma$, and $v \in V(G)$. We let $G'$ be the $\Gamma$-labelled graph $(\vec{G}, \gamma_{G'})$ where

$$\gamma_{G'}(e) = \begin{cases} \gamma_G(e)\alpha, & \text{if } v \text{ is the head of } e \text{ and } e \text{ is not a loop} \\ \alpha^{-1}\gamma_G(e), & \text{if } v \text{ is the tail of } e \text{ and } e \text{ is not a loop} \\ \alpha^{-1}\gamma_G(e)\alpha, & \text{if } e \text{ is a loop incident with } v \\ \gamma_G(e), & \text{otherwise.} \end{cases}$$

We say that $G'$ is obtained from $G$ by a **shift by $\alpha$ at** $v$. If $e$ is an edge of $G$ and $G''$ is the graph obtained from $G$ by reversing the orientation of $e$ and replacing its label with $\gamma_G(e)^{-1}$, then we say that $G''$ is obtained from $G$ by **flipping** $e$ (when $e$ is a loop, flipping it consists of simply replacing its label with $\gamma_G(e)^{-1}$). We say that two $\Gamma$-labelled graphs are **equivalent** if one can be obtained from the other by a sequence of shifts and by flipping edges.

We can define minors of a $\Gamma$-labelled graph $G$ as follows. Let $e \in E(G)$. We **delete** $e$ to obtain $G \backslash e$ by deleting the edge $e$ from $\widetilde{G}$ and restricting $\gamma_G$ to $E(G) \setminus \{e\}$. We define the contraction of an edge $e$ only when $\gamma_G(e) = 1$. If $e$ is a loop then $G/e = G \backslash e$. If not, then we **contract** $e$ to obtain $G/e$ by contracting $e$ from $\widetilde{G}$ and restricting $\gamma_G$ to $E(G) \setminus \{e\}$. When we contract an edge $e$ with ends $u$ and $v$, we may refer to the new vertex of $G/e$ obtained by identifying $u$ and $v$ by either of the names $u$ or $v$. For $\Gamma$-labelled graphs $H$ and $G$, we say that $H$ is a **minor** of $G$ if it is equivalent to a graph obtained from $G$ by a sequence of contractions and edge- and vertex-deletions.

Let $\Gamma$ be a subgroup of $\mathbb{F}^\times$ for some field $\mathbb{F}$, $G$ a $\Gamma$-labelled graph with vertices $\{v_1, \dots, v_k\}$, and $\chi_{v_1}, \dots, \chi_{v_k}$ the standard basis vectors of $\mathbb{F}^{V(G)}$. Recall that $M(G)$ is represented by the matrix $A \in \mathbb{F}^{V(G)\times E(G)}$ such that $A|\{e\} = \chi_u - \gamma_G(e)\chi_v$ for each edge $e$ with head $u$ and tail $v$. When $\Gamma$ is finite, we call this a **Dowling representation** of $M(G)$, and in general we call it a **frame representation**.

The minor relation on the frame matroids over $\mathbb{F}$ corresponds to the minor relation on the $\mathbb{F}^\times$-labelled graphs that represent them.

**Proposition 4.3.1.** *If $\mathbb{F}$ is a field and $G$ is a $\mathbb{F}^\times$-labelled graph, then for every $e \in E(G)$, $M(G)\backslash e = M(G\backslash e)$ and if $\gamma_G(e) = 1$ then $M(G)/e = M(G/e)$. If $H$ is an $\mathbb{F}^\times$-labelled graph equivalent to $G$, then $M(H) = M(G)$.*

*Proof.* We let $A$ be a frame representation of $M(G)$ over $\mathbb{F}$. For any edge $e \in E(G)$, $M(G)\backslash e$ and $M(G\backslash e)$ are both represented by $A|(E(G) \setminus \{e\})$. If $e$ is a non-loop edge, $\gamma_G(e) = 1$, and $e$ has head $u$ and tail $v$, then $M(G)/e$ is represented by the matrix obtained from $A$ by adding the row of $u$ to the row of $v$ then deleting the row of $u$ and the column of $e$. This matrix is also the frame representation of $M(G)/e$. Shifting at a vertex $v$ by $\alpha \in \mathbb{F}^\times$ corresponds to scaling the row of $v$ by $\alpha^{-1}$ then scaling each column indexed by an edge with head $v$ by $\alpha$. Finally, flipping an edge $e$ corresponds to scaling the column $A|\{e\}$ by $-\gamma_G(e)^{-1}$. □

## 4.4 A non-abelian group

Let $\mathbb{F}$ be a field, $\Gamma$ a finite subgroup of $\mathbb{F}^\times$, and $t \geq 1$ an integer. The set $\{(\alpha, \beta) : \alpha \in \mathbb{F}^t, \beta \in \Gamma\}$ along with the operation defined by

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\beta_2 + \alpha_2, \beta_1\beta_2)$$

forms a group, which we denote by $\Gamma_{\mathbb{F}^t}$. It has identity $(0, 1)$, and the inverse of an element $(\alpha, \beta)$ in $\Gamma_{\mathbb{F}^t}$ is $(\alpha, \beta)^{-1} = (-\alpha\beta^{-1}, \beta^{-1})$.

Let $C$ be an independent set in an $\mathbb{F}$-represented matroid $M$ such that $M/C \in \mathcal{D}(\Gamma)$. There is a $\Gamma$-labelled graph $G'$ that represents $M/C$; we let $A' \in \mathbb{F}^{V(G')\times E(G')}$ be the corresponding Dowling representation of $M/C$. Then

$M$ has an $\mathbb{F}$-representation $A \in \mathbb{F}^{(C\cup V(G'))\times(C\cup E(G'))}$ such that $A[C,C]$ is an identity matrix, $A[V(G'),C] = 0$, and $A[V(G'),E(G')] = A'$. We call such an $\mathbb{F}$-representation $A$ an **extended Dowling representation** that **extends** the representation $A'$ of $M/C$ by $C$.

Let $t = |C|$. We construct a $\Gamma_{\mathbb{F}^t}$-labelled graph $G$ from $G'$ by taking the same oriented graph $\vec{G} = \vec{G}'$ and for each edge $e \in E(G)$, setting $\alpha(e) = A[C,\{e\}]$ and labelling the edge $\gamma_G(e) = (\alpha(e), \gamma_{G'}(e)) \in \Gamma_{\mathbb{F}^t}$. We say that the $\Gamma_{\mathbb{F}^t}$-labelled graph $G$ **represents** the matroid $M$ and we write $M = M_{\mathbb{F},\Gamma}(G)$. Although we are re-using the word 'represents', we use the distinct notation $M_{\mathbb{F},\Gamma}(G)$ and not $M(G)$ because $G$ does not represent $M_{\mathbb{F},\Gamma}(G)$ in the same way that a $\Gamma$-labelled graph $H$ represents the Dowling matroid $M(H)$—in fact, the ground set of $M_{\mathbb{F},\Gamma}(G)$ is not equal to $E(G)$ and $\Gamma_{\mathbb{F}^t}$ is not a subgroup of $\mathbb{F}^\times$.

There is an equivalent to Proposition 4.3.1 for $\Gamma_{\mathbb{F}^t}$-labelled graphs.

**Proposition 4.4.1.** *If $G$ is a $\Gamma_{\mathbb{F}^t}$-labelled graph then for every $e \in E(G)$, $M_{\mathbb{F},\Gamma}(G\backslash e) = M_{\mathbb{F},\Gamma}(G)\backslash e$ and if $\gamma_G(e) = (0,1)$ then $M_{\mathbb{F},\Gamma}(G/e) = M_{\mathbb{F},\Gamma}(G)/e$. If $H$ is a $\Gamma_{\mathbb{F}^t}$-labelled graph equivalent to $G$, then $M_{\mathbb{F},\Gamma}(H) = M_{\mathbb{F},\Gamma}(G)$.*

*Proof.* We let $A$ be the extended Dowling matrix corresponding to $G$; so $M_{\mathbb{F},\Gamma}(G) = M_{\mathbb{F}}(A)$. Let $C$ be the set of size $t$ such that $M_{\mathbb{F}}(A)/C \in \mathcal{D}(\Gamma)$. Then as in the proof of Proposition 4.3.1, deleting or contracting an edge from $G$ corresponds to deleting or contracting it from $M_{\mathbb{F},\Gamma}(G)$ because they correspond to the usual minor operations done on $A$.

Flipping an edge $e$ with label $(\alpha, \beta)$ corresponds to scaling the column $A|\{e\}$ by $-\beta^{-1}$. Finally, we show that shifting at a vertex $v$ by $(\alpha,\beta) \in \Gamma_{\mathbb{F}^t}$ corresponds to the following row operations on $A$. We scale the row of $v$ by $\beta^{-1}$ then add $\alpha$ times this row to each row indexed by an element of $C$. Then we scale each column indexed by an edge with head $v$ by $\beta$ (this includes loops incident with $v$). Note that this affects only columns indexed by elements of $\delta(v)$.

Let $e$ be an edge with head $v$, $f$ an edge with tail $v$, and $g$ a loop incident with $v$, and denote their labels by $(\alpha_e,\beta_e),(\alpha_f,\beta_f),(\alpha_g,\beta_g)$. Then the original submatrix of $A$ indexed by $\{e,f,g\}$ was

$$\begin{array}{c} \\ C \\ v \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} e & f & g \end{array} \\ \begin{pmatrix} \alpha_e & \alpha_f & \alpha_g \\ 1 & -\beta_f & 1-\beta_g \\ -\beta_e & & \\ & 1 & \end{pmatrix} \end{array}.$$

After the row operations, the submatrix becomes

$$\begin{array}{c} \\ C \\ v \\ \\ \\ \end{array}\begin{pmatrix} e & f & g \\ \alpha_e\beta+\alpha & \alpha_f-\alpha\beta^{-1}\beta_f & \alpha_g\beta+\alpha(1-\beta_g) \\ 1 & -\beta^{-1}\beta_f & 1-\beta_g \\ -\beta_e\beta & & \\ & 1 & \end{pmatrix}.$$

and the entries here are precisely the result of the multiplications that we do to shift at $v$ by $(\alpha,\beta)$ in $G$: $(\alpha_e,\beta_e)\cdot(\alpha,\beta) = (\alpha_e\beta+\alpha, \beta_e\beta)$, $(\alpha,\beta)^{-1}\cdot(\alpha_f,\beta_f) = (-\alpha\beta^{-1}\beta_f+\alpha_f, \beta^{-1}\beta_f)$, and $(\alpha,\beta)^{-1}\cdot(\alpha_g,\beta_g)\cdot(\alpha,\beta) = (\alpha_g\beta+\alpha(1-\beta_g), \beta_g)$. □

We define one more group-labelled graph operation specific to $\Gamma_{\mathbb{F}^t}$-labelled graphs. If $G$ is a $\Gamma_{\mathbb{F}^t}$-labelled graph, then for any $x \in \mathbb{F}^\times$ and $1 \leq i \leq t$, we let $G'$ be the graph obtaind by changing each edge label $(\alpha,\beta)$ by multiplying the $i$th coordinate of $\alpha$ by $x$. This corresponds to scaling rows in the matrix corresponding to $G$, so $M_{\mathbb{F},\Gamma}(G') = M_{\mathbb{F},\Gamma}(G)$. We say that $G'$ is obtained from $G$ by **scaling**.

## 4.5 Coextensions of Dowling matroids

In this section we prove the special case of our main theorem for coextensions of Dowling matroids. We fix the following notation: in any $\Gamma_{\mathbb{F}^t}$-labelled graph $G$, for each edge $e \in E(G)$ with label $\gamma_G(e) = (\alpha,\beta)$, we define $\alpha_G(e) = \alpha$ and $\beta_G(e) = \beta$. If $e$ has a vertex $v$ as an end and $\gamma_G(v,e) = (\alpha,\beta)$, we define $\alpha_G(v,e) = \alpha$ and $\beta_G(v,e) = \beta$ (recall that $\gamma_G(v,e)$ equals $\gamma_G(e)$ when $e$ has head $v$ and $\gamma_G(e)^{-1}$ otherwise).

First we state Ramsey's Theorem [41] and a version of it for bipartite graphs, both of which we shall use in the following lemma.

**Ramsey's Theorem.** *For all integers $c, s > 0$ there is an integer $R(c,s)$ so that any clique with $R(c,s)$ vertices whose edges are $c$-coloured has a monochromatic copy of $K_s$ as a subgraph.*

This bipartite version has a short direct proof but also follows easily from the Erdős-Stone Theorem [13].

**Theorem 4.5.1.** *For all integers $c, s > 0$ there is an integer $n = B(c,s)$ so that if the edges of $K_{n,n}$ are $c$-coloured then it has a monochromatic copy of $K_{s,s}$ as a subgraph.*

We can view the labels on a group-labelled graph as colours, and thus obtain Ramsey-type results for group-labelled graphs if we can bound the number of distinct labels appearing in the graph. In the next lemma, we show that if a certain $\Gamma_{\mathbb{F}}$-labelled clique or complete bipartite graph represents a matroid without some fixed line minor, then a graph equivalent to it has a large monochromatic clique (or complete bipartite graph) as a subgraph. Moreover, such a subgraph represents a Dowling matroid in $\mathcal{D}(\Gamma)$.

**Lemma 4.5.2.** *Let $\ell \geq 2$ and $s \geq 1$ be integers, $\mathbb{F}$ a field, $\Gamma$ a subgroup of $\mathbb{F}^{\times}$, and $\beta \in \Gamma$. There is an integer $n = n_{4.5.2}(\ell, s)$ such that if $G$ is a $\Gamma_{\mathbb{F}}$-labelled graph with*

$$\widetilde{G} \cong \begin{cases} K_n, & \beta \neq 1 \\ K_{n,n}, & \beta = 1 \end{cases},$$

*$\beta_G(e) = \beta$ for all $e \in E(G)$, and $M_{\mathbb{F},\Gamma}(G)$ has no $U_{2,\ell}$-minor, then there is a graph $G'$ equivalent to $G$ with a subgraph $H$ such that*

$$\widetilde{H} \cong \begin{cases} K_s, & \beta \neq 1 \\ K_{s,s}, & \beta = 1 \end{cases},$$

*$\beta_{G'}(e) = \beta$ for all $e \in G'$, and $\gamma_{G'}(e) = (0, \beta)$ for all $e \in E(H)$.*

*Proof.* First, we suppose $\beta \neq 1$. We set $n = 2R(\ell - 1, s) - 1$ and let $G$ be a $\Gamma_{\mathbb{F}}$-labelled copy of $K_n$ such that $\beta_G(e) = \beta$ for all $e \in E(G)$ and $M_{\mathbb{F},\Gamma}(G)$ has no $U_{2,\ell}$-minor.

We choose a vertex $v \in V(G)$ and let $G'$ be the graph obtained from $G$ by shifting at each $u \in V(G - v)$ by $(-\alpha_G(u, uv), 1)$. The result is that for all edges $e$ incident with $v$, $\gamma_{G'}(e) = (0, \beta)$ and for all $e \in E(G')$, $\beta_{G'}(e) = \beta$.

We partition $V(G - v)$ into two sets $X, Y$ so that each $x \in X$ is incident with an edge with tail $v$ and each $y \in Y$ is incident with an edge with head $v$. We let $G''$ be the graph obtained by shifting at each $u \in V(G - v)$ by $(0, \beta_G(u, uv)^{-1})$. Then for each edge $e$ incident with $v$, $\gamma_{G''}(e) = (0, 1)$ and for each edge $e \in E(G''[X]) \cup E(G''[Y])$, $\gamma_{G''}(e) = \gamma_{G'}(e)$.

The minor of $G''$ obtained by deleting all edges joining $X$ and $Y$ and contracting every edge incident with $v$ consists of the vertex $v$ and a set of loops with labels $\{\gamma_{G'}(e) : e \in E(G')\}$. It represents the matroid with the matrix representation

$$\begin{array}{cccccccc} & e_1 & \cdots & e_k & f_1 & \cdots & f_m \end{array}$$
$$\begin{pmatrix} 1 & \alpha_{G'}(e_1) & \cdots & \alpha_{G'}(e_k) & \alpha_{G'}(f_1) & \cdots & \alpha_{G'}(f_m) \\ & 1-\beta & \cdots & 1-\beta & 1-\beta & \cdots & 1-\beta \end{pmatrix},$$

where $e_1, \ldots, e_k$ are the elements of $E(G'[X])$ and $f_1, \ldots, f_m$ are the elements of $E(G'[Y])$. Since $\beta \neq 1$ and this matroid has no $U_{2,\ell}$-minor we conclude that $|\{\alpha_{G'}(e) : e \in E(G'[X]) \cup E(G'[Y])\}| < \ell$.

We consider the edges of $G'[X] \cup G'[Y]$ to be coloured by $\alpha_{G'}$. One of $X$ and $Y$ has at least $R(\ell-1, s)$ elements, so by Ramsey's Theorem there is a monochromatic subgraph $H \cong K_s$ of either $G'[X]$ or $G'[Y]$. Hence, there is an element $\alpha \in \mathbb{F}$ such that every edge in $H$ has the same label, $(\alpha, \beta)$. If $\alpha \neq 0$ we shift at every $z \in V(H)$ by $(-\alpha(1-\beta)^{-1}, 1)$ so that $\alpha_{G'}(e) = 0$ for all $e \in E(H)$ and $\beta_{G'}(e) = \beta$ for all $e \in E(G')$, as required.

We consider next the case where $\beta = 1$. We set $n = B(\ell-1, s)$ and let $G$ be a $\Gamma_{\mathbb{F}}$-labelled copy of $K_{n,n}$ such that $M_{\mathbb{F},\Gamma}(G)$ has no $U_{2,\ell}$-minor and $\beta_G(e) = 1$ for all $e \in E(G)$. We denote by $X$ and $Y$ the two independent sets that partition $V(G)$.

We choose vertices $x \in X$ and $y \in Y$, and let $G'$ be the graph obtained from $G$ by first shifting at each $u \in X \setminus \{x\}$ by $(-\alpha_G(u, uy), 1)$ and at each $u \in (Y \setminus \{y\})$ by $(-\alpha_G(u, ux), 1)$, then flipping edges so that every edge has its head in $Y$ and tail in $X$. Then each edge incident with precisely one of $x$ or $y$ has label $(0, 1)$ while every edge $e$ of $G'$ has $\beta_{G'}(e) = 1$.

We let $G''$ be the minor of $G'$ obtained by contracting all edges incident with precisely one of $x$ or $y$. This graph consists of a single parallel class and $M_{\mathbb{F},\Gamma}(G'')$ has the following matrix representation

$$\begin{array}{c} \\ \\ x \\ y \end{array} \begin{array}{c} \begin{array}{ccccc} \phantom{1} & xy & e_1 & \cdots & e_k \end{array} \\ \begin{pmatrix} 1 & 0 & \alpha_{G'}(e_1) & \cdots & \alpha_{G'}(e_k) \\ & 1 & 1 & \cdots & 1 \\ & -1 & -1 & \cdots & -1 \end{pmatrix} \end{array},$$

where $e_1, \ldots, e_k$ are the edges of $G''$ other than $xy$. Since this matroid has no $U_{2,\ell}$-minor, we conclude that $|\{\alpha_{G'}(e) : e \in E(G'')\}| < \ell$. But all edges of $G'$ not in $E(G'')$ have the same label as $xy$, so in fact $|\{\alpha_{G'}(e) : e \in E(G')\}| < \ell$.

We consider the edges of $G'$ to be coloured by $\alpha_{G'}$. Since $n \geq B(\ell-1, s)$, by Theorem 4.5.1, there is a monochromatic subgraph $H \cong K_{s,s}$ of $G'$. There is an element $\alpha \in \mathbb{F}$ such that every edge in $H$ has the same label, $(\alpha, 1)$. If $\alpha \neq 0$ we shift at every $z \in V(H) \cap Y$ by $(-\alpha, 1)$ so that $\alpha_{G'}(e) = 0$ for all $e \in E(H)$ and $\beta_{G'}(e) = 1$ for all $e \in E(G')$, as required. □

*Remark.* Although Lemma 4.5.2 is stated for $\Gamma_{\mathbb{F}}$-labelled graphs, for each fixed integer $t \geq 1$ the corresponding statement holds as well for $\Gamma_{\mathbb{F}^t}$-labelled graphs. This follows by induction on $t$: we find a large clique (or complete bipartite subgraph) in which all edge labels are the same in all but one coordinate of $\alpha_G$, then apply the argument to this subgraph for the final coordinate.

Next we show that a large clique whose edges are all labelled with a generator of a finite group $\Gamma$ has a minor representing a Dowling geometry over $\Gamma$. Recall that when $|\Gamma| > 1$, $\mathrm{DG}(k, \Gamma)$ is represented by a $\Gamma$-labelled graph $G$ such that

1. $V(G) = \{v_1, \ldots, v_k\}$, and
2. $E(G)$ consists of a loop incident with each $v_i$ labelled by any non-identity element of $\Gamma$, and for each triple $\alpha \in \Gamma$ and $v_i, v_j \in V(G)$ with $i < j$, an edge oriented from $i$ to $j$ with label $\alpha$,

and that when $|\Gamma| = 1$, $\mathrm{DG}(k, \Gamma) \cong M(K_{k+1})$.

**Lemma 4.5.3.** *Let $k > 0$ be an integer and $\Gamma$ a finite cyclic group with generator $\beta$. There is an integer $n = n_{4.5.3}(k, \Gamma)$ such that if $H$ is a $\Gamma$-labelled graph with $\gamma_H(e) = \beta$ for all $e \in E(H)$ and*

$$\widetilde{H} \cong \begin{cases} K_n, & |\Gamma| > 1 \\ K_{n,n}, & |\Gamma| = 1 \end{cases},$$

*then for any set $X \subseteq V(H)$ of size $k$, there is a minor $G$ of $H$ on vertex set $X$ such that $M(G) \cong \mathrm{DG}(k, \Gamma)$ when $|\Gamma| > 1$ and $M(G) \cong \mathrm{DG}(k-1, \Gamma)$ when $|\Gamma| = 1$. Moreover, $G$ is obtained from $H$ without shifting at any vertex of $X$.*

*Proof.* When $|\Gamma| = 1$, we set $n = n_{4.5.3}(k, \Gamma) = \binom{k}{2}$ and the result is equivalent to the fact that $K_{n,n}$ has a $K_k$-minor on any set of $k$ vertices. We therefore assume that $|\Gamma| > 1$, and so $\widetilde{H} \cong K_n$.

Suppose that there is a set $X' \subseteq V(H) \setminus X$ of size $k$ such that $H - X$ has a minor $G$ on vertex set $X'$ with $M(G) \cong \mathrm{DG}(k, \Gamma)$. Then we can pick a matching $P$ joining $X$ to $X'$, shift at each vertex of $X'$ so that the label of each edge in $P$ is 1, perform the contractions necessary to get the minor $G$ on vertex set $X'$, and contract the edges of $P$. A graph representing a Dowling geometry has an edge with every possible label in each parallel class, so the shifts at vertices in $X'$ simply permute the labels within each parallel class. The result is that there is a minor on vertex set $X$ that represents $\mathrm{DG}(k, \Gamma)$. We thus need only show that we can find a minor $G$ with $M(G) \cong \mathrm{DG}(k, \Gamma)$ on some set of $k$ vertices of $H$.

A straightforward Ramsey-type argument shows that for every integer $n$, if $m \geq 2^{n-1}$ is large enough then every oriented copy of $K_m$ has a set of vertices $v_1, \ldots, v_n$ so that whenever $i < j$ the edge $v_i v_j$ has tail $v_i$ and head $v_j$. It therefore suffices to show that there is an integer $n$ so that if $H$ is a $\Gamma$-labelled clique on vertices $\{v_1, \ldots, v_n\}$, every edge has label $\beta$, and $v_i v_j$ has head $v_j$ for all $i < j$, then $H$ has a minor representing $\mathrm{DG}(k, \Gamma)$.

We choose any set $Y$ of $k$ vertices of $H$. For each pair of distinct vertices $v_i, v_j \in Y$ and each $t \in \{1, \dots, |\Gamma|\}$, we choose a directed path $P_0$ in $H - Y$ with tail $u$ and head $w$ of length $t+2$. Consider four consecutive vertices $a, b, c, d$ on $P_0$ and let $P_1$ be the path obtained from $P_0$ by deleting $b$ and adding the edge $ac$ and $P_2$ the path obtained from $P$ by deleting $b$ and $c$ and adding the edge $ad$. For $P \in \{P_0, P_1, P_2\}$, if we shift at vertices of $P$, contract $\{v_i u, v_j w\}$, and contract $P$ to a single edge $e$, then $e$ has its label in $\{\beta^{|E(P)|}, \beta^{|E(P)|-2}, \beta^{|E(P)|+2}\}$ depending on the orientations of the edges $v_i u$ and $v_j w$. Since $\{|E(P_0)|, |E(P_1)|, |E(P_2)|\} = \{t+2, t, t-2\}$, this edge label is equal to $\beta^t$ for some choice of $P$. We therefore obtain an edge $e$ joining $v_i$ and $v_j$ with head $v_j$ and label $\beta^t$.

Next, for each $v_i \in Y$ we choose a single directed path $P_0$ in $H - Y$ of length three; by the same argument we can use it to contract a loop onto $v_i$ with label $\beta$.

As long as $n$ is large enough to choose all such paths $P_0$ that we used to be disjoint, we obtain in this manner a minor with vertex set $Y$ such that every $v_i, v_j \in Y$ with $i < j$ is joined by a parallel class directed from $i$ to $j$ with every distinct label in $\Gamma$, and every $v_i \in Y$ is incident with a loop with a non-identity label. This is a minor representing $\mathrm{DG}(k, \Gamma)$. □

When $v$ is a vertex in a graph, we denote by $\delta(v)$ the set of edges incident with $v$. A subgraph of a graph $G$ is called a **star** if it is a tree whose edges are contained in $\delta(v)$ for some $v \in V(G)$, called its **centre**. A matroid is called **cosimple** when its dual is simple; equivalently, when it has no coloops or series pairs. We can now prove the case of Theorem 4.2.2 for matroids with a Dowling geometry minor of the same corank. We observe that every finite subgroup of the multiplicative group of a field is cyclic (see [46, Section 1.2]).

**Lemma 4.5.4.** *Let $\mathbb{F}$ be a field, $\Gamma$ a finite subgroup of $\mathbb{F}^\times$ with generator $\beta$, and $\ell$ and $k \geq 5$ integers. There is an integer $n = n_{4.5.4}(\ell, k, |\Gamma|)$ such that if $H$ is a $\Gamma$-labelled graph,*

$$\widetilde{H} \cong \begin{cases} K_n, & \text{when } |\Gamma| > 1 \\ K_{n,n}, & \text{when } |\Gamma| = 1 \end{cases},$$

*and $\gamma_H(e) = \beta$ for all $e \in E(H)$, and $M$ is a matroid with an extended Dowling representation over $\mathbb{F}$ that extends the Dowling representation of $M(H)$, then*

*(i) there is a patchwork $(M, V)$ of which no patch contains a cocircuit of $M(H)$,*

*(ii) $M$ has a $U_{2,\ell}$-minor, or*

*(iii)* *$M$ has a minor $N$ with a non-coloop element $e$ such that $N\backslash e \cong \mathrm{DG}(k,\Gamma)$ but $e$ is not in the closure of any pair of joints of $N\backslash e$.*

Before the proof, we outline its main ideas. Using $M(H)$ instead of $\mathrm{DG}(n,\Gamma)$ simplifies our proof and is equivalent for our purposes because when $n$ is large enough, $M(H)$ is both a restriction of a Dowling geometry and, by Lemma 4.5.3, has a given Dowling geometry as a minor.

First, we look at a represented matroid $M$ with an element $c$ such that $M/c$ is $M(H)$. Then $M$ can be represented by a $\Gamma_{\mathbb{F}}$-labelled graph $G$. We need to either express $M$ as a patchwork, or find a non-frame minor $N$ with an element $e$ such that $N\backslash e$ is a large Dowling matroid. We can get a large Dowling matroid as a restriction of $M$ by applying Lemma 4.5.2 to find a clique in $G$ whose edges $e$ all satisfy $\alpha_G(e) = 0$. We then try to build our non-frame minor $N$ from this using the edges $e \in E(G)$ with $\alpha_G(e) \neq 0$. There are three cases: either there are two such edges that form a matching, or the set of such edges is contained in a star, or it is contained in a triangle. In each case, we either find the desired minor $N$, or we constrain the possible values of the labels $\alpha_G(e)$ enough that we can express $M$ as a patchwork.

More generally, we have a matroid $M$ with a set $C$ such that $M/C$ is $M(H)$. We apply the above argument to the matroids $M/(C \setminus \{c\})$ for each $c \in C$, and then we consider the interaction between the group-labels in the graphs representing each of these matroids. We either find the desired minor $N$ or we are able to express $M$ as a patchwork.

*Proof of Lemma 4.5.4.* We may assume that $M$ has no $U_{2,\ell}$-minor. We set $n = n_{4.5.2}(\ell, n_{4.5.3}(k,\Gamma))$; we recall that if $H$ has this many vertices then (up to equivalence) it has a $n_{4.5.3}(k,\Gamma)$-vertex clique subgraph with all labels $(0,\beta)$, and such a clique has a minor representing $\mathrm{DG}(k,\Gamma)$.

First, we observe that if any element $e \in E(M) \setminus E(H)$ is a coloop or is contained in a non-trivial series class, then $M$ satisfies whichever of outcomes (i) or (iii) that $M/e$ does. Therefore, we may assume that $M$ is cosimple.

We let $C$ be the independent set in $M$ such that $M/C = M(H)$ and $M$ has an extended Dowling representation, $A$, that extends the Dowling representation of $M(H)$ by $C$. We let $G$ be the corresponding $\Gamma_{\mathbb{F}^t}$-labelled graph that represents $M$. We observe that for each $e \in E(G)$ with label $(\alpha_G(e), \beta_G(e))$, $\alpha_G(e)$ is a vector in $\mathbb{F}^C$ and $\beta_G(e) = \beta$. For each $c \in C$, the submatrix of $A$ that represents $M/(C \setminus \{c\})$ corresponds to a $\Gamma_{\mathbb{F}}$-labelled graph $G_c$. Each of the labelled graphs $G_c$ has the same underlying oriented graph as $G$, and its edge labels $(\alpha_{G_c}, \beta_{G_c})$ are obtained from the labels $(\alpha_G, \beta_G)$ by restricting $\alpha_G$ to the coordinate indexed by $c$.

In the next four claims, we choose an arbitrary $c \in C$ and determine the properties that $G_c$ must satisfy if outcome (iii) does not hold. We remark that if a matrix $D$ is a Dowling representation of $\mathrm{DG}(k, \Gamma)$ and we add to it a column with at least three non-zero entries, then we get a representation of a matroid $N$ satisfying outcome (iii).

(1) *If* $\mathrm{supp}(\alpha_{G_c})$ *is the edge set of a triangle* $T$ *of* $G_c$*, then either outcome (iii) holds, or* $\beta = -1$ *and* $\alpha_{G_c}(e) = \alpha_{G_c}(f)$ *for all* $e, f \in E(T)$

Suppose there is a vertex $v \in V(T)$ and edges $e, f \in E(T)$ incident with $v$ such that $\alpha_{G_c}(v, e) \neq \alpha_{G_c}(v, f)$. We choose a set $Y$ of $n_{4.5.3}(k, \Gamma)$ vertices of $G_c - V(T)$ and a subset $X \subset Y$ of size $k$. By Lemma 4.5.3, $G_c[Y]$ has a minor on vertex set $X$ that represents a minor of $M/(C \setminus \{c\})$ isomorphic to $\mathrm{DG}(k, \Gamma)$. We pick a matching $P$ in $G_c$ joining any three elements of $X$ to $V(T)$. We let $G'$ be the minor of $G_c$ obtained by doing the appropriate operations on $G_c[Y]$ to get a $\mathrm{DG}(k, \Gamma)$-minor on vertex set $X$, then shifting on the ends of $P$ in $X$ so that the edges in $P$ have label $(0, 1)$, contracting the elements of $P$, and deleting all vertices other than $X$ and deleting the edge of $T$ other than $e$ and $f$. We note that the shifts on the ends of $P$ do not change the fact that there is a subgraph of $G'$ representing a copy of $\mathrm{DG}(k, \Gamma)$, because each pair of vertices is joined by a parallel class containing an edge with every label in the set $\{(0, \beta^t) : \beta^t \in \Gamma\}$ and these simply get permuted by the shifts. Then the minor $M_{\mathbb{F},\Gamma}(G')$ of $M/(C \setminus \{c\})$ has a restriction $M' \cong \mathrm{DG}(k, \Gamma)$. Moreover, if $D$ is the Dowling representation of $\mathrm{DG}(k, \Gamma)$, then $M_{\mathbb{F},\Gamma}(G')$ has the following matrix representation

$$\begin{array}{c} \\ c \\ v \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} c & e & f & E(M') \end{array} \\ \begin{pmatrix} 1 & \alpha_{G_c}(v, e) & \alpha_{G_c}(v, f) & 0 \\ 0 & 1 & 1 & \\ 0 & -\beta_{G_c}(v, e) & 0 & D \\ 0 & 0 & -\beta_{G_c}(v, f) & \\ 0 & 0 & 0 & \end{pmatrix} \end{array},$$

and contracting $f$ and deleting $c$ yields a minor represented by

$$\begin{array}{c} \\ v \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cc} e & E(M') \end{array} \\ \begin{pmatrix} 1 - \alpha_{G_c}(v, e)\alpha_{G_c}(v, f)^{-1} & \\ -\beta_{G_c}(v, e) & D \\ \beta_{G_c}(v, f)\alpha_{G_c}(v, e)\alpha_{G_c}(v, f)^{-1} & \\ 0 & \end{pmatrix} \end{array},$$

which, since $\alpha_{G_c}(v, e) \neq \alpha_{G_c}(v, f)$, satisfies (iii). We may therefore assume that for each $v \in V(T)$, if $e, f \in E(T)$ are incident with $v$ then $\alpha_{G_c}(v, e) = \alpha_{G_c}(v, f)$.

A straightforward calculation shows that if this holds for all three vertices $v$ of $T$ then $\beta = -1$ and $\alpha_{G_c}(e) = \alpha_{G_c}(f)$ for all $e, f \in E(T)$, proving (1).

(2) *If* $\mathrm{supp}(\alpha_{G_c})$ *is contained in a star of* $G_c$ *with centre* $v$ *then either outcome (iii) holds or* $|\{\alpha_{G_c}(v,e) : e \in \delta(v)\}| \leq 2$.

We suppose that $\mathrm{supp}(\alpha_{G_c})$ is contained in a star of $G_c$ with centre $v$ and that $|\{\alpha_{G_c}(v,e) : e \in \delta(v)\}| \geq 3$. We let $e, f$, and $g$ be three edges incident with $v$ such that $\alpha_{G_c}(v,e)$, $\alpha_{G_c}(v,f)$ and $\alpha_{G_c}(v,g)$ are all distinct.

We let $x, y$ and $z$ respectively be ends of $e, f$, and $g$ other than $v$. We pick a set $Y$ of $n_{4.5.3}(k,\Gamma)$ vertices of $G_c$ containing $\{x,y,z\}$ and disjoint from $v$. By Lemma 4.5.3, $G_c[Y]$ has a minor on vertex set $\{x,y,z\}$ that represents a minor of $M/(C \setminus \{c\})$ isomorphic to $\mathrm{DG}(k,\Gamma)$. We let $G'$ be the minor of $G_c$ obtained by deleting all edges incident with $v$ except $e, f$, and $g$, doing the appropriate operations on $G_c[Y]$ to obtain the $\mathrm{DG}(k,\Gamma)$-minor on $\{x,y,z\}$, shifting at $v$ by $(\alpha_{G_c}(v,g),\beta)^{-1}$, contracting $g$, and deleting all vertices except $x, y$, and $z$.

The minor of $M_{\mathbb{F},\Gamma}(G')$ of $M/(C \setminus \{c\})$ has the following matrix representation, where $D$ is the Dowling representation of $\mathrm{DG}(k,\Gamma)$.

$$\begin{array}{cc} & \begin{array}{cccc} c & e & f & E(G') \setminus \{e,f\} \end{array} \\ \begin{array}{c} c \\ v \\ \\ \\ \\ \end{array} & \left(\begin{array}{cccc} 1 & \alpha_{G_c}(v,e) & \alpha_{G_c}(v,f) & 0 \\ 0 & 1 & 1 & \\ 0 & -\beta_{G_c}(v,e) & 0 & D \\ 0 & 0 & -\beta_{G_c}(v,f) & \\ 0 & 0 & 0 & \end{array}\right). \end{array}$$

The minor $N$ of $M_{\mathbb{F},\Gamma}(G')$ obtained by contracting $f$ and deleting $c$ is represented by the matrix

$$\begin{array}{cc} & \begin{array}{cc} e & E(G') \setminus \{e,f\} \end{array} \\ \begin{array}{c} v \\ \\ \\ \\ \end{array} & \left(\begin{array}{cc} 1 - \alpha_{G_c}(v,e)\alpha_{G_c}(v,f)^{-1} & \\ -\beta_{G_c}(e) & D \\ \alpha_{G_c}(v,e)\beta_{G_c}(v,f)\alpha_{G_c}(v,f)^{-1} & \\ 0 & \end{array}\right), \end{array}$$

and since $\alpha_{G_c}(v,e) \neq \alpha_{G_c}(v,f)$, the column indexed by $e$ has three non-zero entries, so $N$ satisfies outcome (iii), proving (2).

(3) *If* $G_c$ *has a minor* $G'$ *with two non-incident edges* $f, g$ *such that* $\alpha_{G'}(f), \alpha_{G'}(g) \neq 0$, $\gamma_{G'}(e) = (0,\beta)$ *for all* $e \neq f, g$, *and*

$$\widetilde{G'} \setminus f, g \cong \begin{cases} K_{n_{4.5.3}(k,\Gamma)}, & \beta \neq 1 \\ K_{n_{4.5.3}(k,\Gamma),n_{4.5.3}(k,\Gamma)}, & \beta = 1 \end{cases}$$

*then outcome (iii) holds.*

We let $u_f, v_f$ denote the ends of $f$ and $u_g, v_g$ denote the ends of $g$. By Lemma 4.5.3, $G'\backslash f, g$ has a minor whose vertex set contains $\{u_f, v_f, u_g, v_g\}$ that represents a minor of $M/(C \setminus \{c\})$ isomorphic to $\mathrm{DG}(k, \Gamma)$. We let $G''$ be the minor of $G'$ obtained by performing the same operations that give the $\mathrm{DG}(k, \Gamma)$-minor of $G'\backslash f, g$.

The minor of $M_{\mathbb{F},\Gamma}(G'')$ of $M/(C \setminus \{c\})$ has the following matrix representation, where $D$ is the Dowling representation of $\mathrm{DG}(k, \Gamma)$.

$$\begin{array}{c} \\ c \\ u_f \\ v_f \\ u_g \\ v_g \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} c & f & g & E(G'') \setminus \{f,g\} \end{array} \\ \begin{pmatrix} 1 & \alpha_{G'}(u_f, f) & \alpha_{G'}(u_g, g) & 0 \\ 0 & 1 & 0 & \\ 0 & -\beta_{G'}(u_f, f) & 0 & D \\ 0 & 0 & 1 & \\ 0 & 0 & -\beta_{G'}(u_g, g) & \\ 0 & 0 & 0 & \end{pmatrix} \end{array}.$$

The minor $N$ of $M_{\mathbb{F},\Gamma}(G'')$ obtained by contracting $f$ and deleting $c$ is represented by the matrix

$$\begin{array}{c} \\ u_f \\ v_f \\ u_g \\ v_g \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} g & E(G'') \setminus \{f,g\} \end{array} \\ \begin{pmatrix} -\alpha_{G'}(u_g, g)\alpha_{G'}(u_f, f)^{-1} & \\ \alpha_{G'}(u_g, g)\alpha_{G'}(u_f, f)^{-1}\beta_{G'}(u_f, f) & D \\ 1 & \\ -\beta_{G'}(u_g, g) & \\ 0 & \end{pmatrix} \end{array}$$

and since $\alpha_{G'}(f), \alpha_{G'}(g) \neq 0$, there are four non-zero entries in the column indexed by $g$. Therefore, outcome (iii) holds, proving (3).

(4) *Either outcome (iii) holds, or there is a graph $G'$ equivalent to $G_c$ for which $|\{\alpha_{G'}(e) : e \in E(G')\}| = 2$ and either*

*(a)* $\mathrm{supp}(\alpha_{G'})$ *is contained in a star of $G'$, or*

*(b)* $\mathrm{supp}(\alpha_{G'})$ *is the edge set of a triangle of $G'$ and $\beta = -1$.*

We apply Lemma 4.5.2 to find a graph $G'$ equivalent to $G_c$ with a subgraph $J$ of at least $s \geq n_{4.5.3}(k, \Gamma)$ vertices such that $\alpha_{G'}(e) = (0, \beta)$ for all $e \in E(J)$, and such that

$$\widetilde{J} \cong \begin{cases} K_s, & \beta \neq 1 \\ K_s \text{ or } K_{s,s}, & \beta = 1 \end{cases}.$$

We choose $J$ so that $\widetilde{J}$ is a clique rather than a bipartite graph if possible. Subject to this restriction, we choose $G'$ and $J$ so that $|V(J)|$ is maximal; thus shifting at vertices of $G'$ does not allow us to find a larger such subgraph $J$. We may further assume that $G'$ does not satisfy (a) or (b).

We first consider the case that $\widetilde{J} \cong K_{s,s}$; in this case, $\beta = 1$. Since we could not choose $J$ so that $\widetilde{J}$ is a clique, there is at least one edge $e$ of $G'[V(J)]$ with $\alpha_{G'}(e) \neq 0$. If two such edges exist and they are non-incident, then outcome (iii) holds by (3). Otherwise, there is a set $X \subset V(J)$ of size at most three such that every edge $e$ of $G[V(J)]$ with $\alpha_{G'}(e) \neq 0$ has an end in $X$. But then we could have chosen $J - X$ in place of $J$, since $\widetilde{J} - X$ is a clique with at least $s$ vertices, a contradiction. We may therefore assume that $\widetilde{J} \cong K_s$.

By (2), we may assume that there are at least two vertices of $G'$ not contained in $V(J)$. If $v \notin V(J)$ and the set $\{\alpha_{G'}(v, e) : e \text{ joins } v \text{ to } X\}$ has size one, then by shifting at $v$ we may assume that $\alpha_{G'}(e) = 0$ for all $e$ joining $v$ to $V(J)$, contradicting our maximal choice of $J$. Therefore, for each vertex $v \notin V(J)$ there are two edges $e_v, f_v$ joining $v$ to $V(J)$ with $\alpha_{G'}(v, e_v) \neq \alpha_{G'}(v, f_v)$.

We claim that there exist two vertices $u, v \notin V(J)$ and four edges $e_u, f_u, e_v, f_v$ such that $e_u$ and $f_u$ join $u$ to $V(J)$, $e_v$ and $f_v$ join $v$ to $V(J)$, the ends of $e_u, f_u, e_v$ and $f_v$ in $V(J)$ are distinct, $\alpha_{G'}(u, e_u) \neq \alpha_{G'}(u, f_u)$ and $\alpha_{G'}(v, e_v) \neq \alpha_{G'}(v, f_v)$. If this holds, then we obtain a minor $G''$ as follows; we shift at $u$ and $v$ so that $e_u$ and $e_v$ both have label $(0, 1)$, then we delete all vertices other than $V(J) \cup \{u, v\}$ and all edges other than $E(J) \cup \{e_u, f_u, e_v, f_v\}$ and contract $e_u$ and $e_v$. Since $\alpha_{G'}(u, e_u) \neq \alpha_{G'}(u, f_u)$, we have $\alpha_{G''}(f_u) \neq 0$, and similarly $\alpha_{G''}(f_v) \neq 0$. But $G'' \backslash f_u, f_v = J$, and (3) implies that outcome (iii) holds. So we may assume that such vertices $u$ and $v$ and edges $e_u, f_u, e_v, f_v$ do not exist. This implies that there is a vertex $x \in V(J)$ such that for any $v \notin V(J)$, the set $\{\alpha_{G'}(v, e) : e \text{ is incident with } v \text{ and an element of } V(J) \setminus \{x\}\}$ has size one. But then shifting at each vertex not in $J$, we may assume that $\alpha_{G'}(e) = 0$ for all $e$ with an end in $V(J) \setminus \{x\}$.

If $|V(G') \setminus V(J)| = 2$ then (iii), (a), or (b) holds by (1), and if $\alpha_{G'}(e) = 0$ for all $e$ with no end in $J$ then (iii) or (a) holds by (2). Therefore, we may assume that $|V(G') \setminus V(J)| \geq 3$, and so there is an edge $f$ joining $x$ to a vertex $v \notin V(J)$ and an edge $g$ with no ends in common with $f$ such that both $\alpha_{G'}(f)$ and $\alpha_{G'}(g)$ are non-zero. We let $e_1, e_2, e_3$ be three edges that join the ends of $f$ and $g$ not in $J$ to three distinct vertices of $J - \{x\}$. We recall that $\alpha_{G'}(e_1), \alpha_{G'}(e_2), \alpha_{G'}(e_3) = 0$. We let $G''$ be the minor obtained as follows. We shift at $v$ and the ends of $g$ so that $e_1, e_2$ and $e_3$ have label $(0, 1)$, then we delete all edges except $E(J) \cup \{f, g, e_1, e_2, e_3\}$ and all resulting isolated vertices and then contract $e_1, e_2$, and $e_3$. Since $\alpha_{G'}(e_1), \alpha_{G'}(e_2), \alpha_{G'}(e_3) = 0$ but

$\alpha_{G'}(f), \alpha_{G'}(g) \neq 0$, we have $\alpha_{G''}(f), \alpha_{G''}(g) \neq 0$. We note that $G''\backslash f, g = J$ so outcome (iii) holds by (3). This proves (4).

We recall that $G$ is a $\Gamma_{\mathbb{F}^t}$-labelled graph representing $M$ and that each $G_c$ is obtained from $G$ by restricting the vectors $\alpha_G(e) \in \mathbb{F}^C$ to the coordinate indexed by $c$, so $G$ and all $G_c$ have a common underlying graph $\widetilde{G}$.

We may assume by (4) and by shifting and scaling labels that for each $c \in C$, either

(A) $\mathrm{supp}(\alpha_{G_c})$ is contained in a star of $\widetilde{G}$ centred at a vertex $v$ and $\{\alpha_{G_c}(v, e) : e \in \delta(v)\} \subseteq \{0, 1\}$, or

(B) $\mathrm{supp}(\alpha_{G_c})$ is the edge set of a triangle $T$ of $\widetilde{G}$, $\alpha_{G_c}(e) = 1$ for all $e \in E(T)$, and $\beta = -1$.

We now consider the interaction between the various graphs $G_c$. In the next claim, we show that the labellings of two graphs $G_{c_1}$ and $G_{c_2}$ cannot 'cross' each other too much without outcome (iii) occuring.

(5) *If there are distinct elements $c_1, c_2 \in C$, $v \in V(\widetilde{G})$, and four edges $e_1, e_2, e_3, e_4 \in \delta(v)$ such that $e_1 \in \mathrm{supp}(\alpha_{G_{c_1}}) \setminus \mathrm{supp}(\alpha_{G_{c_2}})$, $e_2 \in \mathrm{supp}(\alpha_{G_{c_1}}) \cap \mathrm{supp}(\alpha_{G_{c_2}})$, $e_3 \in \mathrm{supp}(\alpha_{G_{c_2}}) \setminus \mathrm{supp}(\alpha_{G_{c_1}})$, and $e_4 \notin \mathrm{supp}(\alpha_{G_{c_1}}) \cup \mathrm{supp}(\alpha_{G_{c_2}})$, then outcome (iii) holds*

We let $G'$ be the $\Gamma_{\mathbb{F}^2}$-labelled graph representing $M/(C \setminus \{c_1, c_2\})$ that is obtained from $G$ by restricting the vectors $\alpha_G(e)$ to the coordinates indexed by $c_1$ and $c_2$. We let $X$ be the set of ends of $e_1, e_2$, and $e_3$ and let $w$ be the end of $e_4$ other than $v$. We note that by (A) and (B), $\mathrm{supp}(\alpha_{G'})$ consists only of edges incident with $v$ and possibly the two edges joining the ends of $e_1$ and $e_2$ and the ends of $e_2$ and $e_3$. We can therefore choose a set $Y$ of $n_{4.5.3}(k, \Gamma)$ vertices of $G - X$ that contains $w$ and a set $X'$ of four elements of $Y$ that contains $w$. By Lemma 4.5.3, $G'[Y]$ has a minor of vertex set $X'$ that represents a minor of $M$ isomorphic to $\mathrm{DG}(k, \Gamma)$. We pick a matching $P$ in $G$ containing $e_4$ that joins $X'$ to $X$; all edges of $P$ have label $(0, \beta)$. We let $G''$ be the minor of $G'$ obtained by doing the appropriate operations on $G'[Y]$ to get the $\mathrm{DG}(k, \Gamma)$-minor on vertex set $X'$, then shifting on the ends of $P$ in $X'$ so that the edges in $P$ have label $(0, 1)$ and contracting the elements of $P$. Then the minor $M_{\mathbb{F},\Gamma}(G'')$ of $M/(C \setminus \{c_1, c_2\})$ has a restriction $M' \cong \mathrm{DG}(k, \Gamma)$. Moreover, $M$ has a minor with the following matrix representation, where $D$ is the Dowling representation of $\mathrm{DG}(k,\Gamma)$.

$$\begin{array}{c} \\ c_1 \\ c_2 \\ v \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} c_1 & c_2 & e_1 & e_2 & e_3 & E(M') \end{array} \\ \left(\begin{array}{cccccc} 1 & & 1 & 1 & & 0 \\ & 1 & & 1 & 1 & 0 \\ & & 1 & 1 & 1 & \\ & & -\beta & & & D \\ & & & -\beta & & \\ & & & & -\beta & \end{array}\right) \end{array}.$$

The minor obtained by contracting $\{e_1, e_2\}$ has the following representation

$$\begin{array}{c} \\ v \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} c_1 & c_2 & e_3 & E(M') \end{array} \\ \left(\begin{array}{cccc} -1 & & 1 & \\ \beta & -\beta & -\beta & D \\ & \beta & \beta & \\ & & -\beta & \end{array}\right) \end{array}.$$

All four entries in the column indexed by $e_3$ are non-zero, so outcome (iii) holds. This proves (5).

We have identified all cases in which outcome (iii) holds. We can now assume that (iii) does not hold and complete the proof after the following technical claim about set systems. We recall that a collection of sets is called **laminar** if each pair is either disjoint or one contains the other.

(6) *Let $S_1, \ldots, S_k$ be subsets of a set $S$ such that for any two $S_i, S_j$, at least one of $S_i \setminus S_j$, $S_i \cap S_j$, $S_j \setminus S_i$, and $S \setminus (S_i \cup S_j)$ is empty. Then there is a set $T_i \in \{S_i, S \setminus S_i\}$ for each $i = 1, \ldots, k$ such that $\{T_1, \ldots, T_k\}$ is laminar.*

We pick any $x \in S$ and for each $i$, set $T_i = S_i$ if $x \notin S_i$ and $T_i = S \setminus S_i$ if $x \in S_i$. Then $\{T_1, \ldots, T_k\}$ is laminar (this proof is from [31, p. 22]).

We recall that $V(G) = \{v_1, \ldots, v_n\}$. Since each $c \in C$ satisfies either (A) or (B), there is a set of triangles $T_1, \ldots, T_m$ of $G$ such that for each $c \in C$, $\mathrm{supp}(\alpha_{G_c})$ is either contained in some $\delta(v_i)$ or is equal to some $E(T_i)$. We denote by $C_i$ the set of $c \in C$ with $\mathrm{supp}(\alpha_{G_c}) \subseteq \delta(v_i)$. We may assume by (5) and (6) that for each $v_i$, the collection of sets $\{\mathrm{supp}(\alpha_{G_c}) : \mathrm{supp}(\alpha_{G_c}) \subseteq \delta(v_i)\}$ is laminar, and that every pair $T_i, T_j$ of distinct triangles is disjoint. It also follows from (5) and the fact that $M$ is cosimple that, for any $v_i$ and $T_j$, $\{\mathrm{supp}(\alpha_{G_{c_1}}), \mathrm{supp}(\alpha_{G_{c_2}}) \cap \delta(v_i)\}$ is laminar whenever $\mathrm{supp}(\alpha_{G_{c_1}}) \subseteq \delta(v_i)$ and $\mathrm{supp}(\alpha_{G_{c_2}}) = E(T_j)$.

We recall that $A$ that is the extended Dowling matrix representing $M$ that corresponds to $G$. The rows of $A$ are indexed by $C \cup V(G)$. We also index the standard basis vectors of $\mathbb{F}^{V(G)}$ by the elements of $V(G)$, so that $(M/C, V(G))$ is a framework with vertex set $V(G)$.

Since $M$ is cosimple, the sets $\mathrm{supp}(\alpha_{G_c})$ for $c \in C_i$ are distinct. For each $v_i$ and each $c \in C_i$, we add a column $w_c$ to the matrix $A$ so that $w_c$ has a 1 in the row of $v_i$ and in the row of any $c' \in C$ such that $\mathrm{supp}(\alpha_{G_c}) \subseteq \mathrm{supp}(\alpha_{G_{c'}})$, and 0 elsewhere. We let $W_i$ be the set of all these new elements for each $v_i$.

We let $C'$ be the union of all sets $C_i$. We claim that $M/(C \setminus C')$ forms a framework with vertices $V(G) \cup \{W_1, \ldots, W_h\}$. Let $e \in E(M) \setminus C$ have ends $v_i$ and $v_j$ in $G$. If $e$ is not in $\mathrm{supp}(\alpha_{G_c})$ for any $c \in C_i$, then we let $x = v_i$; otherwise we choose $c \in C_i$ with $\mathrm{supp}(\alpha_{G_c})$ minimal such that it contains $e$, and we let $x = w_c$. Similarly, if $e$ is not in $\mathrm{supp}(\alpha_{G_c})$ for any $c \in C_j$, then we let $y = v_j$; otherwise we choose $c \in C_j$ with $\mathrm{supp}(\alpha_{G_c})$ minimal such that it contains $e$, and we let $y = w_c$. Then $e$ is in the span of $\{x, y\}$ in $M/(C \setminus C')$, showing that $(M/(C \setminus C'), V(G) \cup \{W_1, \ldots, W_h\})$ is a framework.

We let $M_{+V}$ be the $\mathbb{F}$-represented matroid obtained by extending $M$ by the set $V = V(G) \cup \{W_1, \ldots, W_h\}$. For each triangle $T_i$ there is a unique $c \in C$ so that $\mathrm{supp}(\alpha_{G_c}) = E(T_i)$; we let $U_i = \{c\} \cup E(T_i)$. Then $U_i$ is a bridge of $(M, V)$. It has three attachments; for each $v_j \in V(T_i)$, if $E(T_i) \cap \delta(v_j)$ is not contained in $\mathrm{supp}(\alpha_{G_c})$ for any $c \in C_j$ then $v_j$ is an attachment of $U_i$, and otherwise we choose $c \in C_j$ so that $\mathrm{supp}(\alpha_{G_c})$ is minimal such that it contains $E(T_i) \cap \delta(v_j)$, and $w_c$ is an attachment of $U_i$. We denote the attachments of $U_i$ by $\{x, y, z\}$; then the representation of $M/(C \setminus \{c\})$, restricted to the set $U_i \cup \{x, y, z\}$ is (recall that $\beta = -1$ whenever a triangle $T_i$ exists)

$$\begin{array}{c} \\ c \\ \\ V(T_i) \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccccc} c & & E(T_i) & & x & y & z \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & & & \\
 & 1 & 1 & & 1 & & \\
 & & 1 & 1 & & 1 & \\
 & 1 & & 1 & & & 1
\end{pmatrix}
\end{array}.$$

From this matrix it is apparent that there is no circuit of $M_{+V}$ contained in $U_i \cup \{x, y, z\}$ that contains all three of $x, y$, and $z$. Hence $(M, V)$ is a patchwork of which each $U_i$ is a patch. $\square$

## 4.6 Vertex- and edge-labelled graphs

We generalize group-labelled graphs to allow labels on vertices as well as edges. Let $\mathbb{F}$ be a field and $\Gamma$ a subgroup of $\mathbb{F}^{\times}$. An $(\mathbb{F}, \Gamma)$-**labelled graph** $G$ is a triple $(\vec{G}, \gamma_G, x_G)$, where $(\vec{G}, \gamma_G)$ is a $\Gamma$-labelled graph and $x_G \in \mathbb{F}^{V(G)}$. For each vertex $v \in V(G)$, we call $x_G(v)$ the **label** of $v$. The $(\mathrm{GF}(2), \mathrm{GF}(2)^\times)$-labelled graphs are often called *grafts*.

Let $G$ be an $(\mathbb{F}, \Gamma)$-labelled graph, $\alpha \in \Gamma$, and $v \in V(G)$. A **shift by** $\alpha$ **at** $v$ consists of shifting by $\alpha$ at $v$ in the underlying $\Gamma$-labelled graph $(\vec{G}, \gamma_G)$ and multiplying the label $x_G(v)$ of $v$ by $\alpha^{-1}$. An $(\mathbb{F}, \Gamma)$-labelled graph is **equivalent** to $G$ if it can be obtained from $G$ by a sequence of shifts, flipping edges of the underlying $\Gamma$-labelled graph, and scaling the vector $x_G$ by an element of $\mathbb{F}^\times$.

We extend the definition of minors of $\Gamma$-labelled graphs to $(\mathbb{F}, \Gamma)$-labelled graphs. We **delete** any edge of $G$ by deleting it from the underlying $\Gamma$-labelled graph and restricting the domain of $x_G$. If $e \in E(G)$ is a non-loop edge with $\gamma_G(e) = 1$, head $u$, and tail $v$, we **contract** $e$ to obtain $G/e$ by contracting $e$ in the underlying $\Gamma$-labelled graph and assigning the label $x_G(u) + x_G(v)$ to the vertex obtained by identifying $u$ and $v$.

For $(\mathbb{F}, \Gamma)$-labelled graphs $H$ and $G$, we say that $H$ is a **minor** of $G$ if it is equivalent to a graph obtained from $G$ by a sequence of contractions, edge-deletions, and deletions of vertices $v$ with $x_G(v) = 0$. The reason we restrict the allowed vertex-deletions is that, as we see below, $(\mathbb{F}, \Gamma)$-labelled graphs represent certain matroids and deleting a vertex with non-zero label does not correspond to a minor operation on these matroids. In this thesis, however, we never delete vertices from $(\mathbb{F}, \Gamma)$-labelled graphs.

When $H$ is a $\Gamma$-labelled graph and $G$ is an $(\mathbb{F}, \Gamma)$-labelled graph, we say that $H$ is a minor of $G$ if it is a minor of the $\Gamma$-labelled graph obtained from $G$ by discarding the vertex labels $x_G$.

Let $G$ be a $(\mathbb{F}, \Gamma)$-labelled graph and let $A'$ be a frame representation of $(\vec{G}, \gamma_G)$ over $\mathbb{F}$, so the rows are indexed by $V(G)$ and the column indexed by each edge $e$ has non-zero entries only in the rows of its ends. We construct the matrix $A$ from $A'$ by adding the vector $x_G$ as a new column indexed by an element $x$. We say that $G$ **represents** $M_{\mathbb{F}}(A)$ and write $M(G) = M_{\mathbb{F}}(A)$. We note that shifting and flipping in $G$ correspond to scaling in $A$, so whenever $G'$ is a graph equivalent to $G$, $M(G') = M(G)$. Also, minor operations on the $(\mathbb{F}, \Gamma)$-labelled graph $G$ correspond to minor operations on the matrix $A$, so we have the following generalization of Proposition 4.3.1.

**Proposition 4.6.1.** *If $\mathbb{F}$ is a field, $\Gamma$ is a subgroup of $\mathbb{F}^\times$, and $G$ is a $(\mathbb{F}, \Gamma)$-labelled graph, then for every $e \in E(G)$, $M(G)\backslash e = M(G\backslash e)$ and $M(G)/e = M(G/e)$.*

An **extended framework** is a triple $(M, V, x)$ where $x$ is a non-coloop element of $M$ and $(M\backslash x, V)$ is a framework. Let $A$ be a representation of

$M_{+V}$ in standard form with respect to the basis $V$. There is an $\mathbb{F}^\times$-labelled graph $G'$ corresponding to $A$ with vertex set $V$ so that $M(G') = M_{+V} \backslash x$. We can extend $G'$ to an $(\mathbb{F}, \mathbb{F}^\times)$-labelled graph $G$ by setting the label $x_G(v)$ of each vertex $v$ to be $A_{vx}$. Then $M(G) = M$. We say that $G$ **represents** the extended framework $(M, V, x)$.

## 4.7 Unique maximal skeletons

In this section, we prove that if an $M$ is an $\mathbb{F}$-represented matroid that can be expressed as a patchwork containing a set $X$ of three vertices, then there is (almost) a unique maximal way to express $M$ as a patchwork with $X$ as vertices. Some of the general ideas of our proof are inspired by Geelen, Gerards, and Whittle's proof of Theorem 4.2.3.

We start with a fact about 2-patches and then a result about $(\mathbb{F}, \mathbb{F}^\times)$-labelled graphs.

**Lemma 4.7.1.** *Every 2-patch in a patchwork is realizable.*

*Proof.* Let $P$ be a 2-patch in a patchwork $(M, V)$ with attachments $a$ and $b$. First, we assume that $M_{+V}|(P \cup \{a, b\})$ has a separation $(A, B)$. Since $P$ is connected in $M_{+V}/\{a, b\}$ (by the definition of a patch), we may assume that $P \subseteq A$, and so we may also assume that $b \in B$. If $a \in B$ also, then $\sqcap_{M_{+V}}(P, V) = \sqcap_{M_{+V}}(P, \{a, b\}) = 0$ and $P$ is a 0-patch. So $a \in A$, and thus $b \notin \mathrm{cl}_{M_{+V}}(P \cup \{a\})$, which means $\sqcap_{M_{+V}}(P, V) = 1$ and that $a \in \mathrm{cl}_{M_{+V}}(P)$. But then $a$ is the unique attachment of $P$, a contradiction. Therefore, $M_{+V}|(P \cup \{a, b\})$ is connected. It follows from the Bixby-Coullard Inequality that in any connected matroid, each element can be either contracted or deleted to preserve connectivity. So we repeatedly remove elements $e \in P \setminus \mathrm{cl}_{M_{+V}}(\{a, b\})$ by either contracting them or deleting them, until we have a connected minor with basis $\{a, b\}$. Being connected, it necessarily has a third element that is not parallel to either $a$ or $b$. The set of elements we contracted is skew to $\{a, b\}$ in $M_{+V}$, so $P$ is realizable. □

Next is our last lemma before the main theorem of this section. A **separation** of a graph $G$ is an ordered pair of subgraphs $(G_1, G_2)$ such that $E(G_1) \cup E(G_2) = E(G)$ and $E(G_1) \cap E(G_2) = \emptyset$. Its **vertex boundary** is the set $V(G_1) \cap V(G_2)$ and its **order** is $\mathrm{ord}(G_1, G_2) = |V(G_1) \cap V(G_2)|$. A separation of order $k$ is called a $k$**-separation**.

**Lemma 4.7.2.** *If $\mathbb{F}$ is a field, $G$ is an $(\mathbb{F}, \mathbb{F}^\times)$-labelled graph and $X \subseteq V(G)$ such that* *(i)* $\widetilde{G}$ *is* 2*-connected,*

*(ii)* $\widetilde{G}[X]$ *is a clique and* $|X| \geq 3$*,*

*(iii)* $|\operatorname{supp}(x_G)| \geq 2$*, and*

*(iv) there is no* 2*-separation* $(A, B)$ *of* $\widetilde{G}$ *with* $|V(A)| \geq 3$*,* $\operatorname{supp}(x_G) \subseteq V(A)$ *and* $X \subseteq V(B)$*,*

*then there is a minor* $H$ *of* $G$ *on vertex set* $X$ *with* $|\operatorname{supp}(x_H)| \geq 2$*.*

*Proof.* We choose $G$ to be a minimum counterexample. First, we assume that $\widetilde{G}$ is 3-connected. Suppose that every edge of $G$ has an end in $X$. Let $v$ be a vertex of $G - X$ and let $e$ and $f$ be two edges joining $v$ to $X$. Either $G/e$ or $G/f$ is a smaller counterexample, or $|\operatorname{supp}(x_{G/e})| < 2$ and $|\operatorname{supp}(x_{G/f})| < 2$. But then $\operatorname{supp}(x_G)$ consists of $v$ and the other ends of $e$ and $f$, and we can contract any third edge incident with $v$ to get a smaller counterexample. Thus there exists an edge $e$ with no end in $X$. Then $G\backslash e$ is a smaller counterexample unless it has a 2-separation $(A, B)$ with $\operatorname{supp}(x_G) \subseteq V(A)$ and $X \subseteq V(B)$. Then $e$ has an end $z \in V(B) \setminus V(A)$ and since there are three paths joining $\{z\} \cup (V(A) \cap V(B))$ to $X$, the graph obtained from $G[V(A) \cup \{z\}]$ by putting a clique on $\{z\} \cup (V(A) \cap V(B))$ is a smaller counterexample, with this set in place of $X$.

This proves that $\widetilde{G}$ is not 3-connected. There is a 2-separation $(A, B)$ with $X \subseteq V(B)$ and $|V(A)| \geq 3$. Note that $\operatorname{supp}(x_G)$ contains a vertex in $V(B) \setminus V(A)$. If $\operatorname{supp}(x_G) \cap V(A)$ is empty, then we can contract edges in $A$ to get a minor on vertex set $V(B)$ that has an edge joining $u$ and $v$, and this is a smaller counterexample; so $\operatorname{supp}(x_G) \cap V(A) \neq \emptyset$. Since every edge in a 2-connected graph can either be deleted or contracted while maintaining 2-connectivity, and contracting an edge decreases $|\operatorname{supp}(x_G)|$ by at most two, by contracting edges in $A$ we may assume that $|\operatorname{supp}(x_G) \cap V(A)| \in \{1, 2\}$. Then we can contract edges in $A$ to get a minor $G'$ on vertex set $V(B)$ with at least one vertex of $V(A) \cap V(B)$ in $\operatorname{supp}(x_{G'})$. This graph is a smaller counterexample. □

We would like to prove that if $(M, V)$ is a patchwork and $X \subseteq V$ induces a clique in its skeleton, then there is a unique maximal choice of $V$ containing $X$ such that $(M, V)$ is a realizable patchwork with a 3-connected skeleton. This is essentially true, but there is one type of exception.

Let $(M, V)$ be a realizable patchwork with skeleton $G$ such that $M$ is 3-connected. Suppose that $G$ has a triangle $T$ with a vertex $v_1$ that is adjacent to exactly one vertex $v_2$ outside $V(T)$, and $v_2$ has exactly one other neighbour,

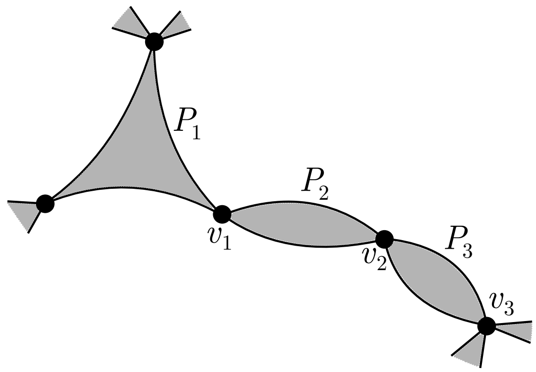



Figure 4.2: Shrinking a 3-patch

$v_3$ (see Figure 4.2). Suppose that $G$ is 2-connected and has no 2-vertex-cutsets besides $\{v_1, v_3\}$. We let $P_1$ be the union of patches with all attachments in $V(T)$, $P_2$ the union of patches with attachment set $\{v_1, v_2\}$, and $P_3$ the union of patches with attachment set $\{v_2, v_3\}$ or $\{v_2\}$. There are two ways to choose a maximal set $V'$ so that $(M, V')$ is realizable and has a 3-connected skeleton: $V_1 = V \setminus \{v_2\}$ or $V_2 = V \setminus \{v_1\}$. In the first case, $P_2 \cup P_3$ is a patch of $(M, V_1)$ with attachments $\{v_1, v_3\}$, and its skeleton contains the triangle $T$ and $v_1$ is adjacent to $v_3$. In the second case, $P_1 \cup P_2$ is a patch of $(M, V_2)$ with attachments $(V(T) \setminus \{v_1\}) \cup \{v_2\}$, and $v_2$ is adjacent to $v_3$ in its skeleton. We say that $(M, V_1)$ is obtained from $(M, V_2)$ by **shrinking a 3-patch**.

We can identify a canonical maximal skeleton by always choosing the first of these two options. If $(M, V)$ is a patchwork and $X \subseteq V$, we say that $(M, V)$ is **$X$-strong** if it is realizable and has a 2-connected skeleton of which there is no 2-separation $(A, B)$ with $X \subseteq V(A)$, and there is no other such patchwork obtained from it by shrinking a 3-patch. If, in addition, the skeleton of $(M, V)$ is actually 3-connected, then we simply call it a **strong** patchwork; this is necessarily the case when it is $X$-strong and $X$ induces a clique in its skeleton.

Whenever $(A, B)$ is a separation of the skeleton of a patchwork $(M, V)$, we let $(A_M, B_M)$ be the partition of $E(M)$ in which $A_M$ is the union of all patches with attachments in $V(A)$, and $B_M = E(M) \setminus A_M$.

**Theorem 4.7.3.** *If $M$ is a 3-connected $\mathbb{F}$-represented matroid, $X \subseteq E(M)$, $|X| \geq 3$, and no triple of elements of $X$ is contained in a circuit of $M$, then there is a unique maximal set $V \subset \mathbb{F}^{r(M)}$ containing $X$ such that $(M, V)$ is an $X$-strong patchwork.*

*Proof.* We choose $M$ to be a counterexample minimizing $r(M)$. Consider an extension $M_{+x}$ of $M$ by a new point in the span of exactly two elements of $X$.

Whenever $(M_{+x}, V)$ is a patchwork with $X \subseteq V$, $(M, V)$ is also a patchwork. Moreover, they have the same skeleton except for possibly one new edge with ends in $X$, and $(M, V)$ is realizable and $X$-strong if and only if $(M_{+x}, V)$ is. Therefore, by adding such points $x$, we may assume that every pair of elements of $X$ is contained in a triangle of $M$. Then $X$ induces a clique in the skeleton of any patchwork whose vertices contain $X$, so any patchwork is $X$-strong if and only if it is strong.

(1) *$M$ is vertically 4-connected.*

Suppose that $M$ has a vertical 3-separation $(A, B)$. We recall that every pair of elements of $X$ is contained in a triangle, so $X$ is in the closure of either $A$ or $B$; we may thus assume that $X \subseteq B$. We let $M'$ be the matroid obtained by extending $M$ by every possible point $z$ for which there is a set $C_z \subseteq A$ with $\sqcap_M(C_z, B) = 1$ and $z \in \mathrm{cl}_{M_{+z}}(C_z) \cap \mathrm{cl}_{M_{+z}}(B)$, then deleting $A$; the set $Z$ of new points is finite so $M'$ is a represented matroid. Note that $|Z| \geq 2$ because $M$ is 3-connected. Moreover, if we pick two such elements $a, b \in Z$, then $(M, \{a, b\})$ is a patchwork in which $A$ is a union of patches with attachments $\{a, b\}$; by Lemma 4.7.1 any such patch is realizable so in fact, $|Z| \geq 3$. No triple of elements of $X$ is in a circuit of $M'$, for if such a circuit $C$ contained an element $z \in Z$ then $(C \setminus Z) \cup C_z$ would be a circuit of $M$ (if we chose $C_z$ to be minimal). Also, $r(M') < r(M)$, so by our minimal choice of $M$ there is a strong patchwork $(M', V)$ for some set $V$. Then, because $Z$ forms a line in $M'$ of size at least three, there is a single patch $P$ of $(M', V)$ containing all elements of $Z$ except those that form one-element patches. This implies that $(M, V)$ is also a strong patchwork, in which $(P \setminus Z) \cup A$ is a patch, a contradiction. This proves (1).

It follows from Theorem 4.2.3 that there exists a realizable patchwork $(M, V)$ with $X \subseteq V$. By (1), it has no 0-patches, it has no 1- or 2-patches with more than one element, and its skeleton is 3-connected. Therefore, by shrinking 3-patches as necessary we may assume that $(M, V)$ is strong.

We assume that there are two distinct maximal sets $V, W$ containing $X$ such that $(M, V)$ and $(M, W)$ are strong patchworks. We choose an element $v \in W \setminus V$. Since $(M, W)$ is a patchwork, there is no circuit of $M_{+v}$ containing $v$ and at least two elements of $X$. Our goal is to exploit the properties of the patchwork $(M, V)$ to find such a circuit, thereby obtaining a contradiction.

As $(M, V)$ is realizable, each patch $P$ contains a set $C(P)$ such that $(M/C(P), V)$ has the same skeleton as $(M, V)$ and every element of $P \setminus C(P)$ is in $\mathrm{cl}_{M_{+V}/C(P)}(V)$. We let $C$ be the union of all such sets $C(P)$. Then $(M/C, V)$ is a framework with the same skeleton as $(M, V)$.

(2) *The fundamental circuit of $v$ with respect to the basis $V$ of $M_{+V\cup\{v\}}/C$ contains at most one element of $V$.*

We assume that $v$ is in a circuit with at least two elements of $V$ in $M_{+V\cup\{v\}}/C$. Then $(M_{+v}/C, V, v)$ is an extended framework represented by an $(\mathbb{F}, \mathbb{F}^{\times})$-labelled graph $G$ in which $|\operatorname{supp}(x_G)| \geq 2$. We claim that there is a circuit of $M_{+v}$ containing $v$ and at least two elements of $X$. This is true for $M_{+v}$ if it is true in $M_{+v}/C$ or any minor thereof, so it suffices to show that $G$ has a minor $G'$ on vertex set $X$ with $|\operatorname{supp}(x_{G'})| \geq 2$. Since $(M, V)$ is a strong patchwork and $(M/C, V)$ has the same skeleton, $\widetilde{G}$ is 3-connected. Therefore, the required minor exists by Lemma 4.7.2, which proves (2).

(3) *If $P_1$ is a patch of $(M, V)$, $C' = C \setminus C(P_1)$, and $v$ is not parallel to an element of $V$ in $M_{+v}/C'$, then $v \notin \operatorname{cl}_{M_{+v}/C'}(P_1)$.*

Suppose that $v \in \operatorname{cl}_{M_{+v}/C'}(P_1)$ and $v$ is not parallel to an attachment of $P_1$. If $P_1$ is a 2-patch, then by (1), $|P_1| = 1$, so $C' = C$ and (3) follows from (2). Thus $P_1$ is a 3-patch. We denote by $Z$ the set of attachments of $P_1$. We extend $M$ by an element $x \in \operatorname{cl}_{M_{+x}}(P_1)$ so that $v$ is parallel to $x$ in $M_{+\{v,x\}}/C'$.

First, we assume that there is no circuit of $M_{+V\cup\{x\}}|(P_1 \cup Z \cup \{x\})$ containing at least three elements of $Z \cup \{x\}$. By Theorem 4.2.3, there is some $V'$ containing $Z \cup \{x\}$ such that $(M|P_1, V')$ is a realizable patchwork. By shrinking 3-patches we may assume that it is $(Z \cup \{x\})$-strong. Then $(M, V \cup V')$ is a realizable patchwork; if it is strong then this contradicts the maximality of $V$. Thus either it is possible to shrink a 3-patch or its skeleton is not 3-connected. But $(M, V)$ is strong so any such 3-patch is contained in $P_1$, and $(M|P_1, V')$ has no 3-patches that we can shrink. If the skeleton of $(M, V \cup V')$ is not 3-connected, then it has a 2-separation separating some element of $Z$ from the others; but then the 3-patch $P_1$ could have been shrunk in $(M, V)$.

We may now assume that there is a circuit $Y'$ of $M_{+V\cup\{x\}}|(P_1 \cup Z \cup \{x\})$ containing three elements of $Z \cup \{x\}$. As the elements of $Z$ are vertices of the patchwork $(M, V)$, $Y'$ necessarily contains $x$. We note that $Y'$ is still a circuit in $M_{+V\cup\{x\}}/C'$, so $Y = (Y' \setminus \{x\}) \cup \{v\}$ is a circuit in $M_{+V\cup\{v\}}/C'$. We will find a circuit of $M_{+v}$ containing $v$ and at least two elements of $X$ to obtain a contradiction.

We let $N$ be the minor of $M_{+v}$ obtained by contracting $Y \cap P_1$ then deleting any elements of $P_1$ that are not in $\operatorname{cl}_{M_{+V}/(C'\cup(Y\cap P_1))}(V)$, and $N_{+V}$ the corresponding minor of $M_{+V\cup\{v\}}$. In $N_{+V}$, there is a circuit, $Y \setminus P_1$, consisting of $v$ and two or three elements of $Z$. Then $(N\backslash v, V)$ is a framework, and its skeleton is obtained from the skeleton of $(M, V)$ by possibly deleting some edges with ends in $Z$. Since $v$ is contained in the circuit $Y \setminus P_1$ of $N_{+V}$

along with two or three elements of $V$, $(N, V, v)$ is an extended framework represented by an $(\mathbb{F}, \mathbb{F}^\times)$-labelled graph $G$ such that $\mathrm{supp}(x_G)$ consists of two or three elements of $Z$. Since $(M, V)$ has a 3-connected skeleton, there are three disjoint paths joining $Z$ to $X$ in its skeleton and hence also in that of $(N \backslash v, V)$ as well as in $G$. We get a minor $G'$ of $G$ by shifting so that all edges of these paths have label 1 and contracting them, then repeatedly contracting edges with an end not in $X$ until there are none. Then $\mathrm{supp}(x_{G'}) \subseteq X$, $|\mathrm{supp}(x_{G'})| \geq 2$, and the matroid $M(G')$ is a minor of $N$. Hence there is a minor of $N$ in which some two or three vertices of $X$ (those in $\mathrm{supp}(x_{G'})$) form a circuit with $v$, a contradiction. This proves (3).

We let $P_1, \ldots, P_k$ be a minimal collection of patches of $(M, V)$ such that $v \in \mathrm{cl}_{M_{+V\cup\{v\}}}(V \cup P_1 \cup \cdots \cup P_k)$, and we let $Z_i$ denote the set of attachments of each patch $P_i$. Since $(M, V)$ has no 2-patches that are disjoint from $\mathrm{cl}_{M_{+V}}(V)$, all the patches $P_1, \ldots, P_k$ are 3-patches. Since $M$ is vertically 4-connected, $Z_i \subset \mathrm{cl}_{M_{+V}}(P_i)$ for each $i$, so $v \in \mathrm{cl}_{M_{+V\cup\{v\}}}(P_1 \cup \cdots \cup P_k \cup V \setminus (Z_1 \cup \cdots \cup Z_k))$.

If $v \in \mathrm{cl}_{M_{+V\cup\{v\}}}(V)$, then there is a circuit of $M_{+V\cup\{v\}}$ containing $v$ and at least two elements of $V$, and this is also a circuit of $M_{+V\cup\{v\}}/C$, contradicting (2); hence $k \geq 1$. We can extend $M_{+V\cup\{v\}}$ by new points $y_1, \ldots, y_k$ such that each $y_i$ lies in the closure of $P_i$ and $v$ lies in the closure of $\{y_1, \ldots, y_k\} \cup V \setminus (Z_1 \cup \cdots Z_k)$. No point $y_i$ is in the closure of $Z_i$ in $M_{+V\cup\{y_i\}}$ because of our minimal choice of the set of patches $\{P_1, \ldots, P_k\}$.

For each $i = 1, \ldots, k$, we claim that there is a circuit $Y_i$ of $M_{+V\cup\{y_i\}}$ contained in $P_i \cup Z_i \cup \{y_i\}$ that contains $y_i$ and at least two elements of $Z_i$. If not, then by Theorem 4.2.3, there is a patchwork $(M|P_i, V')$ for some $V'$ containing $Z_i \cup \{y_i\}$, and we choose $V'$ so that it is $(Z_i \cup \{y_i\})$-strong; then $(M, V \cup V')$ is a strong patchwork, and since $y_i$ is not parallel to any element of $Z_i$, this contradicts the maximality of $V$.

We let $C' = C \backslash P_1$. In $M_{+V\cup\{v,y_1\}}/C'$, the point $v$ is spanned by $\{y_1\} \cup V \setminus Z_1$, and the circuit $F$ contained in $\{v, y_1\} \cup V \setminus Z_1$ contains $y_1$. However, by (3) $v$ is not parallel to $y_1$ in this matroid, so $|F \cap V \setminus Z_1| \geq 1$. But if $|F \cap V \setminus Z_1| > 1$, then $\{v\} \cup (F \cap V \setminus Z_1)$ is a circuit of size at least three in $M_{+V\cup\{v\}}/C$, contradicting (2). Hence $|F \cap V \setminus Z_1| = 1$, and we denote by $w$ its unique element, so $\{v, y_1, w\}$ is a circuit of $M_{+V\cup\{v\}}/C'$.

If $y_1$ is not spanned by $C(P_1)$ in $M_{+y_1}$, then there is a circuit of $M_{+y_1}/C$ consisting of $y_1$ and a non-empty subset $Z$ of $Z_1$; then $\{v, w\} \cup Z$ is a circuit of $M_{+v}/C$, contradicting (2). So we may assume that $y_1$ is spanned by $C(P_1)$ in $M_{+y_1}$. We choose any element $y$ in the minimal subset of $C(P_1)$ that spans $y_1$. We define $N = M_{+v}/(C \setminus \{y\})$, and $N_{+y_1}$ and $N_{+V}$ the corresponding minors of $M_{+\{v,y_1\}}$ and $M_{+V\cup\{v\}}$. Then $y_1$ is parallel to $y$ in $N_{+y_1}$, so $\{w, v, y\}$ is a circuit of $N_{+V}$. Since $(N/y, V)$ is a framework, $r(N) = |V| + 1$ and for any element $y'$ of $P_1 \setminus C(P_1)$ that is not in $\mathrm{cl}_{N_{+V}}(V)$, $(N/y', V)$ is a framework.

We let $a, b, c$ be the three vertices in $Z_1$. We say that an edge $st$ in the skeleton of a patchwork is **realized** if there is a one-element 2-patch with attachments $\{s, t\}$. We have four cases, depending on how many of the three edges $ab, bc$, and $ca$ are realized in $(N, V)$.

(4) *At most two of $ab, bc$, and $ca$ are realized in* $(N, V)$*.*

If all three of these edges are realized in $(N, V)$, we contract any one element of $P_1 \setminus (\mathrm{cl}_{N_{+V}}(V) \cup \{y\})$ in $N$ to get a minor $N'$, and then $y \in \mathrm{cl}_{N'_{+V}}(Z_1)$, so there is a circuit of $N'_{+V}$ consisting of $\{v, w\}$ and at least one element of $Z_1$. Then $(N', V, v)$ is an extended framework with the same skeleton as $(M, V)$, and is represented by a 3-connected $(\mathbb{F}, \mathbb{F}^{\times})$-labelled graph $G$ with $|\operatorname{supp}(x_G)| \geq 2$. Then by Lemma 4.7.2 there is a minor $G'$ of $G$ on vertex set $X$ with $|\operatorname{supp}(x_{G'})| \geq 2$, so $M(G')$ has a circuit containing $v$ and at least two elements of $X$. Then $N = M(G)$ also has such a circuit, as does $M_{+v}$, a contradiction.

(5) *At least one of $ab, bc$, and $ca$ is realized in* $(N, V)$*.*

Suppose none of the three edges $ab, bc, ca$ are realized in $(N, V)$. Then there are three elements $t_1, t_2, t_3$ of $N$ such that $\{y, t_1, a, b\}$, $\{y, t_2, b, c\}$ and $\{y, t_3, c, a\}$ are circuits of $N_{+V}$. We let $N' = N/t_1$ and $N'_{+V} = N_{+V}/t_1$. Then $\{y, a, b\}$ is a circuit of $N'_{+V}$. Also, $\{y, t_2, b, c\}$ is a circuit, and thus so is $\{t_2, c, a\}$ because the definition of a patchwork means that $\{t_2, a, b, c\}$ is not a circuit. Simiarly, $\{t_3, b, c\}$ is a circuit of $N'_{+V}$. Therefore, all three edges $ab, bc$ and $ca$ are realized in $(N', V)$, so it has the same skeleton as $(M, V)$ and the extended framework $(N', V, v)$ is represented by an $(\mathbb{F}, \mathbb{F}^{\times})$-labelled graph $G$ with $\operatorname{supp}(x_G) = \{a, b, w\}$. As before, Lemma 4.7.2 implies that $N' = M(G)$ has a circuit containing $v$ and at least two elements of $X$, hence $M_{+v}$ does too, a contradiction.

(6) *Exactly one of $ab, bc$, and $ca$ is realized in* $(N, V)$*.*

If not, then by (4) and (5), exactly one of $ab, bc$, and $ca$ is not realized in $(N, V)$; say $ab$ is not. Then there is a four-element circuit of $N_{+V}$ containing $\{a, b, y\}$; call its fourth element $t$. We let $N' = N/t$ and $N'_{+V} = N_{+V}/t$, so $\{a, b, y\}$ is a circuit of $N'_{+V}$ and $\{w, v, a, b\}$ is a also circuit. Then $(N', V, v)$ is an extended framework with the same skeleton as $(M, V)$ and it is represented by an $(\mathbb{F}, \mathbb{F}^{\times})$-labelled graph $G$ with $\operatorname{supp}(x_G) = \{w, a, b\}$. As before, Lemma 4.7.2 implies that $N' = M(G)$ has a circuit containing $v$ and at least two elements of $X$, hence so does $M_{+v}$, a contradiction. This proves (6).

We may assume that $ca$ is the one of $\{ab, bc, ca\}$ that is realized in $(N, V)$. There are two elements $t_1$ and $t_2$ of $N$ such that $\{y, t_1, a, b\}$ and $\{y, t_2, b, c\}$ are circuits of $N_{+V}$. We let $N' = N/t_1$ and $N'_{+V} = N_{+V}/t_1$, so $\{y, a, b\}$ is a circuit in $N'_{+V}$ and $ab$ is realized in $(N', V)$. The skeleton of $(N', V)$ is obtained from the skeleton of $(M, V)$ by possibly deleting $bc$, so it is 2-connected. The extended framework $(N', V, v)$ is represented by an $(\mathbb{F}, \mathbb{F}^\times)$-labelled graph $G$ with $\mathrm{supp}(x_G) = \{w, a, b\}$. If $G$ has no 2-separation $(A, B)$ with $\{w, a, b\} \subseteq V(A)$ and $X \subseteq V(B)$, then it follows from Lemma 4.7.2 that $N' = M(G)$ and hence $M_{+v}$ has a circuit containing $v$ and at least two elements of $X$, a contradiction. Otherwise, such a separation $(A, B)$ exists. Then $b \in V(A) \setminus V(B)$ and $c \in V(B) \setminus V(A)$, so $a \in V(A) \cap V(B)$. We denote by $d$ the other element of $V(A) \cap V(B)$. Since $\widetilde{G}$ is obtained from a 3-connected graph by deleting $bc$, there is a path $R$ in $A$ joining $w$ and $d$ that is disjoint from $\{a, b, c\}$. Either $w = d$, or $E(R) \cup \{w, d\}$ is a circuit of $N_{+V}$, and since the path $R$ is disjoint from $Z_1 = \{a, b, c\}$, $E(R) \cup \{w, d\}$ is skew to $P_1 \cup Z_1$ in $N_{+V}$. Also, $\{d, a, c\}$ is a 3-vertex cutset in the skeleton of $(M, V)$, so it has a 3-separation $(A', B')$ with $\{w, d, a, b\} \subseteq V(A')$ and $\{c\} \cup X \subseteq V(B')$. This corresponds to a 4-separation $(A'_{M/C'}, B'_{M/C'})$ in the matroid $M/C'$ (where $A'_{M/C'}$ is the union of all patches with all attachments in $V(A')$).

Recall that $y_1$ is a point in the span of $P_1$ in $M_{+V}/C'$ such that $\{y_1, v, w\}$ is a circuit of $M_{+V \cup \{y_1\}}/C'$. By Tutte's Linking Theorem there is a set $Y \subseteq P_1$ skew to both $\{a, c\}$ and $y_1$ such that $y_1 \in \mathrm{cl}_{M_{+V \cup \{y_1\}}/(C' \cup Y)}(\{a, c\})$. We let $N'$ be the minor of $M_{+v}/C'$ obtained by contracting $E(R)$ and $Y$, then deleting all elements of $B'_{M/C'}$ that are not in $\mathrm{cl}_{M/(C' \cup E(R) \cup Y)}(\{d, a, c\})$, and let $N'_{+V \cup \{y_1\}}$ the corresponding minor of $M_{+V \cup \{v, y_1\}}/C'$. Then $y_1 \in \mathrm{cl}_{N'_{+V \cup \{y_1\}}}(\{a, c\})$ and $\{v, y_1, d\}$ is a circuit of $N'_{+V \cup \{y_1\}}$, so there is a circuit of $N'_{+V}$ that contains $\{v, d\}$ and at least one of $a$ and $c$. Therefore $(N', V(B), v)$ is an extended framework represented by an $(\mathbb{F}, \mathbb{F}^\times)$-labelled graph $G'$ obtained from $B$ by possibly adding edges between $d, a$, and $c$, and $\mathrm{supp}(x_{G'})$ consists of $d$ and one or both of $a$ and $c$. Since there are three disjoint paths joining $\{d, a, c\}$ to $X$ in the skeleton of $(M, V)$, there are also such paths in $B$ and hence also in $G'$. As before, it follows from Lemma 4.7.2 that $N' = M(G')$ has a circuit containing $v$ and at least two elements of $X$, and hence so does $M_{+v}$. $\square$

## 4.8 Tangles

When $H$ and $G$ are graphs, an $H$**-model** in $G$ is a collection of disjoint trees $\{T_v : v \in V(H)\}$ and distinct edges $\{im(e) : e \in E(H)\}$ such that for each $e \in E(H)$ with ends $u$ and $v$, $im(e)$ has ends in $T_u$ and $T_v$ in $G$ (if $e$ is a loop incident with $u$, $im(e)$ has ends only in $T_u$). The trees $T_v$ are called **vertex-images** of the model. We observe that $G$ has an $H$-model if and only if $G$ has an $H$-minor — the trees of the model contain the edges we contract to get the minor. When $H$ itself is a minor of $G$, we will always choose each $im(e)$ to be equal to $e$.

Recall that a separation of a graph $G$ is an ordered pair of subgraphs $(G_1, G_2)$ such that $E(G_1) \cup E(G_2) = E(G)$ and $E(G_1) \cap E(G_2) = \emptyset$, and that its order $\mathrm{ord}(G_1, G_2)$ is the size of its vertex boundary $V(G_1) \cap V(G_2)$.

The order function of separations is *submodular*: if $(A, B)$ and $(C, D)$ are separations of a graph $G$, then $\mathrm{ord}(A \cap B, C \cup D) + \mathrm{ord}(A \cup B, C \cap D) \leq \mathrm{ord}(A, B) + \mathrm{ord}(C, D)$.

A **tangle** in $G$ of order $\theta$ is a collection $\mathcal{T}$ of separations of $G$ of order less than $\theta$, such that

(T1) for each separation $(A, B)$ of order less than $\theta$, exactly one of $(A, B)$ and $(B, A)$ is in $\mathcal{T}$,

(T2) if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$, and

(T3) if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Tangles were introduced by Robertson and Seymour [42]. For a tangle $\mathcal{T}$ of order $\theta$, we define the $\mathcal{T}$**-rank** of a set $X \subseteq V(G)$ to be

$$r_{\mathcal{T}}(X) = \min\{\mathrm{ord}(A, B) : (A, B) \in \mathcal{T}, X \subseteq V(A)\},$$

when $X \subseteq V(A)$ for some $(A, B) \in \mathcal{T}$, and $r_{\mathcal{T}}(X) = \theta$ otherwise. We say that $X$ is $\mathcal{T}$**-independent** when its $\mathcal{T}$-rank is $|X|$. The submodularity of the order function implies that whenever $(A, B) \in \mathcal{T}$, $V(A) \cap V(B)$ is $\mathcal{T}$-independent if and only if there is no separation $(A', B') \in \mathcal{T}$ of order less than $\mathrm{ord}(A, B)$ with $A$ contained in $A'$. If there is no $(A', B') \in \mathcal{T}$ of order at most $\mathrm{ord}(A, B)$ with $A$ contained in $A'$, then we call $(A, B) \in \mathcal{T}$ $\mathcal{T}$**-closed**. These definitions are motivated by the fact that $(V(G), r_{\mathcal{T}})$ is a matroid [42].

Robertson and Seymour showed that the clique $K_n$ has a tangle of order $\lceil 2n/3 \rceil$ [42]. Whenever $H$ is a clique we let $\mathcal{T}_H$ denote this tangle. A clique has no vertex-cutsets, so for any $(A, B) \in \mathcal{T}_H$, $V(B) = V(H)$ by (T3).

Let $H$ be a minor of a graph $G$ and $\mathcal{T}$ a tangle of $H$ of order $\theta$. Let $\mathcal{T}'$ be the set of separations $(A, B)$ of $G$ of order less than $\theta$ such that $E(A) \cap E(H) =$

$E(A')$ for some $(A', B') \in \mathcal{T}$. Then $\mathcal{T}'$ is a tangle of $G$ of order $\theta$ and we call it the tangle **induced** by $\mathcal{T}$. The **truncation** of a tangle $\mathcal{T}''$ to order $\theta$ is the subset of $\mathcal{T}''$ of separations of order less than $\theta$. Whenever $\mathcal{T}''$ is a tangle of $G$ of which $\mathcal{T}'$ is a truncation, we say that $\mathcal{T}''$ **controls** $\mathcal{T}$. When $H$ is a clique and $\mathcal{T}''$ controls the tangle $\mathcal{T}_H$, then it **controls** $H$.

When $H$ is a $K_n$-minor of $G$, $\mathcal{T}$ is a tangle of $G$ that controls $\mathcal{T}_H$, and $(A, B) \in \mathcal{T}$, then there is no vertex-image $T_v$ of the $H$-model in $G$ with $V(T_v) \subseteq V(A) \setminus V(B)$. If there were, then every edge of $H$ incident with $v$ would be in $E(A)$ and hence contained in $E(A')$ for some $(A', B') \in \mathcal{T}_H$. But then $V(H) = V(A')$, contradicting (T3).

When $G$ is a group-labelled graph, we define a tangle in $G$ to be a tangle in its underlying graph $\widetilde{G}$. The next fact is that when a tangle controls a large clique minor, then for any separation $(A, B)$ in the tangle we can find a clique minor of $B$ with $V(A) \cap V(B)$ in its vertex set. The version of Proposition 4.8.1 without group-labels is a standard fact about tangles, and this extension to group-labelled graphs introduces almost no changes to the proof. We present a proof using the argument of [27, Lemma 4.5.3].

**Proposition 4.8.1.** *Let $\Gamma$ be a group, $\beta \in \Gamma$, $G$ a $\Gamma$-labelled graph, and $H$ a minor of $G$ such that $\widetilde{H} \cong K_n$ and $\gamma_H(e) = \beta$ for all $e \in E(H)$. Let $t$ be the order of $\mathcal{T}_H$, $\mathcal{T}$ the tangle of of $G$ induced by $\mathcal{T}_H$, and $(A, B) \in \mathcal{T}$ such that $V(A) \cap V(B)$ is $\mathcal{T}$-independent. Then $B$ has a minor whose vertex set contains $V(A) \cap V(B)$ that is an oriented copy of $K_t$ with all edges labelled $\beta$.*

*Proof.* We choose a counterexample with $|V(G)|$ minimum. We let $X = V(A) \cap V(B)$. We choose $(A', B') \in \mathcal{T}$ such that $\text{ord}(A', B') = t$ and $A$ is contained in $A'$. Since $X$ is $\mathcal{T}$-independent, there is a collection $\mathcal{P}$ of $\text{ord}(A, B)$ disjoint paths joining $X$ to $V(A') \cap V(B')$. Hence if $B'$ has a minor on vertex set $V(A') \cap V(B')$ that is a copy of $K_t$ with all edges labelled $\beta$, then there is such a minor of $B$ whose vertex set contains $X$: we obtain it by contracting the edges of each path in $\mathcal{P}$, after shifting as necessary (note that we can make all edges in these paths have label 1 without shifting at the ends of the paths in $V(A') \cap V(B')$). Therefore, we may assume that $(A, B)$ has order $t$, and that $(A, B)$ is $\mathcal{T}$-closed.

There is a sequence of shiftings, contractions and deletions on $G$ that results in the minor $H$; by doing the shiftings first we may assume that $G$ has an $H$-model with vertex-images $\{T_v : v \in V(H)\}$ such that for each edge $e$ of $H$, $\gamma_G(e) = \gamma_H(e) = \beta$.

(1) *There is no $v \in V(H)$ such that $|V(T_v)| > 1$ and $V(T_v) \not\subseteq X$.*

Suppose there is $v \in V(H)$ so that $|V(T_v)| > 1$ and $V(T_v) \not\subseteq X$. There exists $e \in E(T_v)$ with an end outside $X$. The graph $G/e$ has $H$ as a minor; we let $\mathcal{T}'$ be the tangle of $G/e$ induced by $H$. Then, depending on whether $e \in E(A)$ or $e \in E(B)$, $(A/e, B)$ or $(A, B/e)$ is a separation of $G/e$ in $\mathcal{T}'$ that has vertex-boundary $X$. If $X$ is $\mathcal{T}'$-independent, then as $G$ is a minimum counterexample the required minor exists in $G/e$ and hence also in $G$. So we may assume that $X$ is not $\mathcal{T}'$-independent and there is a separation $(C', D') \in \mathcal{T}'$ of order less than $t = |X|$ with $X \subseteq V(C')$. Since $X$ is $\mathcal{T}$-independent, there is no such separation in $G$, so by uncontracting $e$ we get a separation $(C, D) \in \mathcal{T}$ of order exactly $t$ with $X \subseteq V(C)$. Since $e \in E(C) \setminus E(A)$, $A$ does not contain $C$. Hence there is a separation $(A \cup C, B \cap D)$ distinct from $(A, B)$. Since $(A, B)$ is $\mathcal{T}$-closed, it follows that $(A \cup C, B \cap D) \notin \mathcal{T}$. But nor is $(B \cap D, A \cup C)$ in $\mathcal{T}$, for then the fact that $(B \cap D) \cup A \cup C = G$ contradicts (T2). Therefore, $\mathrm{ord}(A \cup C, B \cap D) > t$. But then $\mathrm{ord}(A \cap C, B \cup D) < t$ by submodularity, contradicting the fact that $X$ is $\mathcal{T}$-independent and proving (1).

We may therefore assume that for all $v \in V(H)$, either $V(T_v) \subseteq X$ or $|V(T_v)| = 1$. This implies that there are at most $|X|/2 \leq n/3$ vertex-images $T_v$ with $|V(T_v)| > 1$, so there are at least $2n/3 \geq |X|$ vertex-images $T_1, \ldots, T_k$ that consist of a single vertex, and these are all contained in $B$ since no vertex-image can be disjoint from $B$. We let $Y = V(T_1) \cup \cdots \cup V(T_k)$. Then $G[Y]$, after possibly deleting parallel edges, is a clique whose edges all have label $\beta$. There exists a collection $\mathcal{P}$ of $|X|$ disjoint paths joining $Y$ to $X$, for if not then by Menger's Theorem there is a separation $(C, D)$ of $G$ of order less than $|X|$ with $X \subseteq V(C)$ and a vertex-image of the $H$-model contained in $D$. This means $(C, D) \in \mathcal{T}$, contradicting the fact that $X$ is $\mathcal{T}$-independent. For each path $P \in \mathcal{P}$, we can shift at the vertices of $V(P) \setminus Y$ so that all edges of $P$ have label 1. This does not affect the labels in the graph $G[Y]$. Then we contract all edges in each $P \in \mathcal{P}$, thereby obtaining the required minor with vertex set $X$. □

Recall that a separation $(A, B)$ in the skeleton of a patchwork $(M, V)$ corresponds to a partition $(A_M, B_M)$ of $E(M)$ where $A_M$ is the union of all patches whose attachments are contained in $V(A)$. The next proposition concerns 'gluing together' certain patchwork representations of $M|A_M$ and $M|B_M$.

**Proposition 4.8.2.** *Let $M$ be a represented matroid and $e \in E(M)$. Let $(M \backslash e, V)$ be a realizable patchwork whose skeleton $H$ is 3-connected and has a tangle $\mathcal{T}$. Let $(A, B) \in \mathcal{T}$ be a $\mathcal{T}$-closed separation such that $V(A)$ contains the* *attachments of $e$ in $(M,V)$. If $(M|(A_{M\backslash e}\cup\{e\}),V')$ is a realizable patchwork where $V'$ contains $V(A)\cap V(B)$, then there is a set $V''$ containing $V(B)$ so that $(M,V'')$ is a realizable patchwork with a 3-connected skeleton.*

*Proof.* We recall that $(A_{M\backslash e},B_{M\backslash e})$ is the separation of $M\backslash e$ corresponding to the separation $(A,B)$. Since the attachments of $e$ in $(M,V)$ are in $V(A)$, $(M|B_{M\backslash e},V(B))$ is a realizable patchwork.

We set $V''=V(B)\cup V'$, so $(M,V'')$ is a realizable patchwork. Let $G$ be its skeleton. Note that two distinct components $U_1,U_2$ of $G-V(B)$ correspond to distinct bridges of $(M|C,V(B))$, so $V(U_1)$ and $V(U_2)$ cannot be joined by an edge of $H$, the skeleton of $(M\backslash e,V)$. Suppose that $G$ is not 3-connected and let $X$ be a vertex-cutset of size at most two. Since no edge of $H$ joins vertices in different components of $G-V(B)$, $X$ cannot be contained in $V(B)$ or it would also be a vertex-cutset of $H$. If $X$ is contained in $V'$, then there is a separation $(A',B')$ of $G$ with vertex-boundary $X$ such that $B'$ contains $B$, and we can replace $V''$ with $V(B')$ to get a realizable patchwork $(M,V(B'))$. Hence we may assume that $X$ consists of one vertex in $V'\setminus V(B)$ and another in $V(B)\setminus V'$.

Let $a$ be the vertex of $X$ in $V'\setminus V(B)$ and $b$ the one in $V(B)\setminus V'$. Then $G$ has a separation $(A',B')$ of order two with vertex boundary $\{a,b\}$.

Note that neither $V(A')$ nor $V(B')$ can contain $V(A)\cap V(B)$, otherwise $a$ or $b$ would be a cut-vertex. But now $\{b\}\cup(V(A)\cap V(B)\cap V(B'))$ and $\{b\}\cup(V(A)\cap V(B)\cap V(A'))$ are both vertex-boundaries of separations of $H$. Let $(U_1,W_1)$ and $(U_2,W_2)$ be separations of $H$ with these two sets as vertex boundaries such that $V(A)\subseteq V(U_1)$, $V(A)\subseteq V(U_2)$, and $W_1$ and $W_2$ are edge-maximal. They both have order at most $\mathrm{ord}(A,B)$ because neither $V(B')$ nor $V(A')$ contains $V(A)\cap V(B)$. Then $W_1\cup W_2\cup A=H$, so by (T2), $(W_1,U_1)$ and $(W_2,U_2)$ are not both in $\mathcal{T}$, hence one of $(U_1,W_1)$ and $(U_2,W_2)$ is in $\mathcal{T}$. But $U_1$ and $U_2$ both contain $A$ as well as $b$, contradicting the fact that $(A,B)$ is $\mathcal{T}$-closed. $\square$

## 4.9 Matroids with a Dowling geometry minor

In this section we complete the proof of Theorem 4.2.2. First, we briefly outline the proof. Given a represented matroid $M$ that has a large Dowling geometry minor and excludes a given line minor, we want to either express $M$ as a patchwork $(M,V)$ of which no patch contains a cocircuit of the Dowling geometry, or find a minor $N$ that extends a smaller Dowling geometry by a single element not in the span of any pair of joints. We take an element $e\in E(M)$

such that $M\backslash e$ also has the Dowling geometry minor (the case where no such $e$ exists was done in Section 4.5). By induction we may assume that $(M\backslash e, V)$ is a patchwork for some set $V$, but $(M, V)$ is not a patchwork. The skeleton of $(M\backslash e, V)$ has a tangle induced by the Dowling geometry minor. There are two cases, depending on the set $X$ of attachments of $e$ in $(M, V)$. The first is that this tangle contains a separation $(A, B)$ with $X \subseteq V(A)$. Theorem 4.7.3 implies that $V$ is essentially the only possible vertex set containing $V(B)$ with which $M\backslash e$ forms a patchwork. Therefore, we cannot replace $V(A)$ with another set of vertices $V'$ such that $(M, V(B) \cup V')$ is a patchwork. Then we use Theorem 4.2.3 to get a circuit containing at least three elements of $V(A) \cap V(B)$, and we use this to get the required minor $N$. Otherwise, $|X|$ is large. We repeatedly contract sets skew to $V$ in $M_{+V}$ to reduce the number of patches of $(M\backslash e, V)$ that are not spanned by $V$. Either we eventually get a separation $(A, B)$ in the tangle of the patchwork's skeleton with $X \subseteq V(A)$, and we apply the earlier argument, or we lose all such patches and obtain a framework, which reduces to a problem on vertex- and edge-labelled graphs.

Before the main proof we have two short results.

**Proposition 4.9.1.** *For any field $\mathbb{F}$, let $G_n$ be an $\mathbb{F}^\times$-labelled graph such that $\widetilde{G_n} = K_n$ and $\mathcal{T}_{G_n}$ is a tangle in $\widetilde{G_n}$. Let $G$ be a 3-connected $(\mathbb{F}, \mathbb{F}^\times)$-labelled graph with $G_n$ as a minor such that $\operatorname{supp}(x_G)$ has rank at least 7 in the tangle $\mathcal{T}_G$ of $\widetilde{G}$ induced by $\mathcal{T}_{G_n}$. Then $G$ has a 3-connected minor $G'$ which has $G_n$ as a minor, and either*

- *(i) $|V(G')| = n$ and $|\operatorname{supp}(x_{G'})| \geq 3$, or*
- *(ii) the rank of $\operatorname{supp}(x_{G'})$ in the tangle of $G'$ induced by $T_{G_n}$ is in $\{7, 8, 9, 10\}$.*

*Proof.* If $|V(G)| = n$ then (i) holds with $G' = G$. Otherwise we proceed by induction on $|V(G)|$. We may assume that $\operatorname{supp}(x_G)$ has rank at least 11 in $\mathcal{T}_G$, otherwise (ii) holds with $G' = G$. Note that any minor of a connected labelled graph with fewer vertices can be obtained with at least one edge contraction: otherwise some edge deletion creates an isolated vertex, and this edge can be contracted instead, after shifting at that vertex if necessary. Therefore, since $|V(G)| > |V(G_n)|$, there is an edge $e \in E(G)$ such that (after shifting so that $\gamma_G(e) = 1$) $G/e$ has $G_n$ as a minor.

Let $u$ and $v$ be the ends of $e$ and let $X = \operatorname{supp}(x_G) \setminus \{u, v\}$. Then $r_{\mathcal{T}_G}(X) \geq r_{\mathcal{T}_G}(\operatorname{supp}(x_G)) - 2 \geq 9$. Let $\mathcal{T}_{G/e}$ be the tangle of $G/e$ induced by $\mathcal{T}_{G_n}$. Then $r_{\mathcal{T}_{G/e}}(X) \geq r_{\mathcal{T}_G}(X) - 1$ so $r_{\mathcal{T}_{G/e}}(\operatorname{supp}(x_G)) \geq r_{\mathcal{T}_{G/e}}(X) \geq 8$. If $G/e$ is 3-connected, then we are done by induction, so we may assume that $G/e$ is not 3-connected.

We let $u$ denote the vertex of $G/e$ formed by identifying $u$ and $v$ in contracting $e$. Let $(A, B) \in \mathcal{T}_{G/e}$ be a separation of order two and $z$ the vertex of $V(A) \cap V(B)$ other than $u$. If $V(A) \setminus V(B)$ is disjoint from $\text{supp}(x_{G/e})$ then we repeatedly contract edges with an end in $V(A) \setminus V(B)$ until there are none. This does not change the rank of the set of labelled vertices. Otherwise, there is a minimal path of $A$ joining $z$ to a vertex in $\text{supp}(x_{G/e})$. We contract the edges of this path, after shifting if required at vertices other than $z$. We then repeatedly contract edges joining $u$ to vertices of $V(A) \setminus V(B)$ as long as there are any. Let $G'$ be the graph obtained from $G/e$ by doing these contractions for every such separation $(A, B) \in \mathcal{T}_{G/e}$, and $T_{G'}$ the tangle of $G'$ induced by $\mathcal{T}_{G_n}$. Now for every separation $(A, B) \in \mathcal{T}_{G/e}$ of order two, if $V(A) \setminus V(B)$ contains a vertex of $\text{supp}(x_{G/e})$ then the vertex of $V(A) \cap V(B)$ other than $u$ is in $\text{supp}(x_{G'})$. Therefore, we have $r_{\mathcal{T}_{G'}}(\text{supp}(x_{G'})) \geq r_{\mathcal{T}_{G/e}}(\text{supp}(x_{G/e})) - 1 \geq 7$. Moreover, $G'$ is 3-connected, so we are done by induction. □

Let $\mathbb{F}$ be a field, $M$ an $\mathbb{F}$-represented matroid, and $N$ a minor of $M$ that is a frame matroid over $\mathbb{F}$. When $(M, V)$ is a patchwork, we say that $(M, V)$ **respects** $N$ if there is a minor $N'$ of $M$ such that $(N', V)$ is a framework and $N$ is a minor of $N'$. In this case there are $\mathbb{F}^{\times}$-labelled graphs $G$ and $H$ such that $M(G) = N'$ and $M(H) = N$ and $H$ is a minor of $G$. The following is a useful equivalent characterization.

**Proposition 4.9.2.** *If $(M, V)$ is a patchwork and $N$ is a minor of $M$ of rank at least four with no non-spanning cocircuits, then $(M, V)$ respects $N$ if and only if no patch of $(M, V)$ contains a cocircuit of $N$.*

*Proof.* Suppose that $(M, V)$ respects $N$. Then if a patch $P$ of $(M, V)$ contains a cocircuit $X$ of $N$, $r_N(X) \leq 3$, and $X$ does not span $N$, a contradiction.

Suppose that no patch of $(M, V)$ contains a cocircuit of $N$. We let $C$ be a maximal subset of $E(M)$ such that $M/C$ has $N$ as a restriction, and we let $C'$ be a maximal subset of $C$ skew to $V$ in $M_{+V}$. Then $M/C'$ has $N$ as a minor, and $(M/C', V)$ is a patchwork. No patch of $(M/C', V)$ contains a cocircuit of $N$, as every such patch is a subset of a patch of $(M, V)$. Let $P$ be a patch of $(M/C', V)$ that is not spanned by $V$, and let $D = P \setminus E(N)$. Then $M/C' \backslash D$ has $N$ as a minor, since each element of $D$ is skew to $V$ yet is not contained in $C'$. If $P \cap E(N)$ is non-empty, then it contains a cocircuit of $N$ because no element of $P$ is spanned by $E(M/C') \setminus P$. Therefore, $P \cap E(N)$ is empty for every patch $P$ that is not spanned by $V$. Hence, we can delete all such patches from $M/C'$ to obtain a minor $M'$ of $M$ such that $(M', V)$ is a framework and $M'$ has $N$ as a minor. □

Recall that when $(M, V)$ is a patchwork and $(A, B)$ is a separation of its skeleton, there is a corresponding partition of $E(M)$ that we denote $(A_M, B_M)$: $A_M$ consists of the union of all patches whose attachments are contained in $V(A)$ and $B_M$ consists of the union of all other patches (those with all attachments in $V(B)$ but not all in $V(A) \cap V(B)$).

We now prove our main theorem, Theorem 4.2.2, which is restated below.

**Theorem 4.2.2.** *For any finite group $\Gamma$ and integers $m, \ell \geq 3$, there is an integer $n$ such that if $\mathbb{F}$ is a field and $M$ is an $\mathbb{F}$-represented matroid with* $\mathrm{DG}(n, \Gamma)$ *as a minor, then either*

*(i) there is a patchwork $(M, V)$ of which no patch contains a cocircuit of* $\mathrm{DG}(n, \Gamma)$*,*

*(ii) $M$ has a $U_{2,\ell}$-minor, or*

*(iii) $M$ has a minor $N$ with a non-coloop element $e$ such that $N \backslash e \cong \mathrm{DG}(m, \Gamma)$ but $e$ is not in the closure of any pair of joints of $N \backslash e$.*

*Proof.* Recall that if $\mathrm{DG}(n, \Gamma)$ is $\mathbb{F}$-representable then $\Gamma$ is isomorphic to a subgroup of $\mathbb{F}^{\times}$. All finite subgroups of the multiplicative group of a field are cyclic (see [46, Section 1.2]), so we can choose a generator $\beta$ of $\Gamma$. For each integer $n$, we define $\mathcal{G}_n$ to be the set of $\Gamma$-labelled graphs $G_n$ such that $\widetilde{G_n} \cong K_n$ and $\gamma_{G_n}(e) = \beta$ for all $e \in E(G_n)$ (these are the graphs $H$ of Lemmas 4.5.3 and 4.5.4). We will prove that there is an integer $n$ such that if $M$ is an $\mathbb{F}$-represented matroid with no $U_{2,\ell}$-minor and for some $G_n \in \mathcal{G}$,

- if $|\Gamma| > 1$, $M$ has $M(G_n)$ as a minor, and
- if $|\Gamma| = 1$, $M$ has $M(K_{n,n})$ as a minor (which in turn has $M(G_n)$ as a minor),

then either

(a) there is a patchwork $(M, V)$ that respects $M(G_n)$, or

(b) for some $G_k \in \mathcal{G}_k$, $M$ has a minor $N$ with an element $e$ such that $N \backslash e \cong M(G_k)$ and $N$ is not represented by a group-labelled graph with $G_k$ as a subgraph.

It is sufficient to prove this by the following claim along with the fact that $\mathrm{DG}(n, \Gamma)$ has an $M(G_n)$-restriction and for large enough $n$, $M(G_n)$ has any $\mathrm{DG}(n', \Gamma)$-minor, by Lemma 4.5.3.

(1) *If (b) holds with* $k = 3 + n_{4.5.3(m+1,\Gamma)}$ *then (iii) holds.*

We may assume that $m \geq 6$; this is because if (iii) holds with any $m > 3$, then for any joint $f$ of the Dowling geometry $N \backslash e$, $\mathrm{si}(N/f \backslash e) \cong \mathrm{DG}(m-1, \Gamma)$, and $e$ is not in the closure of any two joints of $\mathrm{si}(N/f)$ unless $e$ is in the closure in $N$ of a triple of joints of $N \backslash e$ that includes $f$. But when $m > 3$, we can choose $f$ so that this does not occur.

As $N \backslash e$ is represented by $G_k$, $N$ is represented by a $(\mathbb{F}, \Gamma)$-labelled graph $G$ obtained from $G_k$ by adding vertex-labels $x_G$, and $|\operatorname{supp}(x_G)| \geq 3$. It suffices to show that $G$ has a minor $G'$ with $|\operatorname{supp}(x_{G'})| \geq 3$ whose underlying $\Gamma$-labelled graph represents $\mathrm{DG}(m, \Gamma)$. We let $Z \subseteq \operatorname{supp}(x_G)$ be a set of three vertices, and for each $z \in Z$, we pick a set $N_z$ of two neighbours of $z$ such that the three sets $N_z$ are disjoint. We let $X \subseteq V(G) \setminus Z$ be a set of $m$ vertices (or $m + 1$ if $|\Gamma| = 1$) that contains each of the three sets $N_z$. By Lemma 4.5.3, $G_k - Z$ has a minor representing $\mathrm{DG}(m, \Gamma)$ on vertex set $X$. We note that this minor can be obtained without deleting vertices, as follows. First, we do all the contractions, so we have a connected minor $J$ with a subgraph representing $\mathrm{DG}(m, \Gamma)$. If this minor $J$ has a vertex $v$ not in that subgraph, we pick any edge $e$ incident with $v$, shift at $v$ so that $e$ has label 1, then contract $e$. We repeat as long as there are such vertices, then do any required edge-deletions. We do the contractions and deletions necessary to obtain this minor of $G_k - Z$ on the $(\mathbb{F}, \Gamma)$-labelled graph $G$ itself; let $H$ be the resulting minor. We have $Z \subseteq \operatorname{supp}(x_H)$. For each $z \in Z$, we pick one of the edges $f$ joining $z$ to $N_z$, and if possible we do so such that the end of $f$ in $N_z$ is not in $\operatorname{supp}(x_H)$. Then we shift at $z$ so that $f$ has label 1 and contract $f$. We obtain a minor $G'$ representing $\mathrm{DG}(m, \Gamma)$ in which $|\operatorname{supp}(x_{G'})| \geq 3$, as required. This proves (1).

We set $n = \max\{\frac{3}{2}k + 26, n_{4.5.3}(k, \Gamma), n_{4.5.4}(\ell, k, \Gamma)\}$. Our proof is by induction on $r^*(M)$. First, suppose that $|\Gamma| > 1$ and $r^*(M) = r^*(M(G_n))$ or $|\Gamma| = 1$ and $r^*(M) = r^*(M(K_{n,n}))$. As $n \geq n_{4.5.4}(\ell, k, \Gamma)$, we have already done this case in Lemma 4.5.4. We may therefore assume that, if $|\Gamma| > 1$, $r^*(M) > r^*(M(G_n))$, and if $|\Gamma| = 1$, $r^*(M) > r^*(M(K_{n,n}))$. But $M(K_{n,n})$ has $M(G_n)$ as a minor, so in both cases we have $r^*(M) > r^*(M(G_n))$ and there is an element $e \in E(M)$ such that $M \backslash e$ has $M(G_n)$ as a minor.

In addition, we can assume that $M$ is 3-connected: if not, then it has a proper 3-connected minor $M'$ with $M(G_n)$ as a minor such that if $(M', V')$ is a patchwork for some $V'$ then $(M, V')$ is also a patchwork with the same skeleton.

If $M \backslash e$ satisfies (b) then so does $M$, so by induction we may assume that there is a patchwork $(M \backslash e, V)$ that respects the minor $M(G_n)$.

(2) *We can choose $V$ so that $(M\backslash e, V)$ is realizable and has a 3-connected skeleton.*

Since there exists some $V$ such that $(M\backslash e, V)$ is a patchwork, we can apply Theorem 4.2.3 with $X = V$ and conclude that we can extend $V$ to a set $V'$ so that $(M\backslash e, V')$ is realizable. Thus we may assume that $(M\backslash e, V)$ is realizable. Suppose that the skeleton of $(M\backslash e, V)$ has a separation $(A, B)$ of order $t$, $t \leq 2$, with neither $V(A)$ nor $V(B)$ containing the other. We have the corresponding $(\leq t+1)$-separation $(A_M, B_M)$ of $M$; since $M$ is 3-connected, $t = 2$. Since $M(G_n)$ has no vertical 3-separations, at most one of $A_M$ and $B_M$ contains a cocircuit of $M(G_n)$, so we may assume that $A_M$ does not. Then $(M\backslash e, V(B))$ is a patchwork in which $A_M$ is the union of all patches with attachments in $V(A) \cap V(B)$, and all other patches are also patches of $(M\backslash e, V)$. Thus no patch contains a cocircuit of $M(G_n)$ so by Proposition 4.9.2, $(M\backslash e, V(B))$ respects the minor $M(G_n)$. Also, it has no 3-patches that were not already patches of $(M\backslash e, V)$, which was realizable, so it is also realizable. We can repeat this process of shrinking the set $V$ as long as $(M\backslash e, V)$ has a 2-separation, until we obtain a 3-connected skeleton. This proves (2).

We choose $V$ to be maximal such that $(M\backslash e, V)$ is strong and respects $M(G_n)$. If $(M, V)$ is a patchwork then we are done; we will assume that $(M, V)$ is not a patchwork.

We let $X$ be the set of attachments of the bridge of $(M, V)$ containing $e$. If $|X| \leq 2$ then $(M, V)$ is a patchwork; so we may assume that $|X| \geq 3$. Since the patchwork $(M\backslash e, V)$ respects $M(G_n)$, its skeleton has $\widetilde{G_n}$ as a minor. We let $\mathcal{T}_{G_n}$ be the tangle of order $\lceil 2n/3 \rceil$ in $\widetilde{G_n}$ and $\mathcal{T}$ the tangle in the skeleton of $(M\backslash e, V)$ induced by $\mathcal{T}_{G_n}$. Whenever $M'$ is a minor of $M\backslash e$ with $M(G_n)$ as a minor and, for some $V'$, $(M', V')$ is a patchwork that respects $M(G_n)$, we denote by $\mathcal{T}_{(M',V')}$ the tangle of the skeleton of $(M', V')$ induced by $\mathcal{T}_{G_n}$.

(3) *If $r_{\mathcal{T}}(X) \leq 13$ then (a) or (b) holds.*

We choose a $\mathcal{T}$-closed separation $(A, B) \in \mathcal{T}$ with $X \subseteq V(A)$, so $V(A) \cap V(B)$ is independent in $\mathcal{T}$. We distinguish two cases: either there is no circuit $Y$ of $M_{+V}$ such that $Y \subseteq A_{M\backslash e} \cup \{e\} \cup V$ and $Y$ contains at least three elements of $V(A) \cap V(B)$, or there is such a circuit.

If there is no such circuit $Y$, then by Theorem 4.2.3, $(M|(A_{M\backslash e} \cup \{e\}), V')$ is a patchwork for some set $V'$ that contains $V(A) \cap V(B)$. This means that $(M, V(B) \cup V')$ is a patchwork. Any patch of it is either also a patch of $(M\backslash e, V)$, or is contained in $A_{M\backslash e} \cup \{e\}$. As $(A_{M\backslash e} \cup \{e\}, B_M)$ is a $(\leq 14)$-separation of

$M$, $A_{M\backslash e}$ contains no cocircuit of $M(G_n)$. Therefore, by Proposition 4.9.2, the patchwork $(M, V(B) \cup V')$ respects $M(G_n)$ so (a) holds.

Otherwise, such a circuit $Y$ exists. We recall that each patch $P$ of $(M\backslash e, V)$ contains a set $C(P)$ such that, when $C$ is the union of the sets $C(P)$, $(M\backslash e/C, V)$ is a framework and $M\backslash e/C$ has $M(G_n)$ as a minor. When we contract a set $C(P)$ in $M\backslash e$ for a patch $P$ with all attachments in $V(B)$, the skeleton of the resulting patchwork $(M\backslash e/C(P), V)$ contains $A$ as a subgraph, so the rank of $X$ in $\mathcal{T}_{(M\backslash e/C(P),V)}$ is at most its rank in $\mathcal{T}$. We let $M'$ be the minor of $M\backslash e$ obtained by contracting as many sets $C(P)$ as possible for patches $P$ with all attachments in $V(B)$, so that $X$ has the same rank in $\mathcal{T}_{(M',V)}$ as it does in $\mathcal{T}$.

Let $(A', B') \in \mathcal{T}_{(M',V)}$ be a $\mathcal{T}_{(M',V)}$-closed separation such that $X \subseteq V(A')$; it has order $\mathrm{ord}(A, B)$. Then $A$ is contained in $A'$ since $A$ remains a subgraph of the skeleton of $(M', V)$. Suppose that $(M', V)$ has a 3-patch $P$ with all attachments in $V(B')$. Then $X$ has rank less than $\mathrm{ord}(A', B')$ in $(M'/C(P), V)$, so there is a separation $(S, T) \in \mathcal{T}_{(M'/C(P),V)}$ of order less than $\mathrm{ord}(A', B')$ with $A'$ contained in $S$. But the skeleton of $(M'/C(P), V)$ contains all edges of the skeleton of $(M', V)$ except possibly those between attachments of $P$. So $P$ has an attachment in $V(S)\backslash V(T)$ and an attachment in $V(T)\backslash V(S)$. Moreover, one of these sets contains only one attachment of $P$, so $\mathrm{ord}(S, T) = \mathrm{ord}(A', B') - 1$. If $P$ has a unique attachment $w$ in $V(T) \setminus V(S)$, we let $S'$ be the graph obtained from $S$ by adding the vertex $w$ and the edges between attachments of $P$. We have a separation $(S', T) \in \mathcal{T}_{(M',V)}$ of order $\mathrm{ord}(A', B')$, and since $(A', B')$ is $\mathcal{T}_{(M',V)}$-closed, $S' = A'$, which implies that $w \in V(A')$ and the other attachments of $P$ are in $V(A') \setminus V(B')$, a contradiction. Otherwise, $P$ has a unique attachment $z$ in $V(S) \setminus V(T)$, and we let $T'$ be the graph obtained from $T$ by adding the vertex $z$ and the edges between attachments of $P$. We have a separation $(S, T') \in \mathcal{T}_{(M',V)}$ of order $\mathrm{ord}(A', B')$, and since $(A', B')$ is $\mathcal{T}_{(M',V)}$-closed, $S = A'$, which implies that $z \in V(A')$ and that $z$ has just two neighbours in $V(B') \setminus V(A')$. Therefore, every 3-patch of $(M', V)$ with all attachments in $V(B')$ has an attachment in $V(A') \cap V(B')$. Moreover, if there is such a 3-patch $P$ with an attachment $z \in V(A') \cap V(B')$, then the other two attachments of $P$ are the only neighbours of $z$ in $V(B') \setminus V(A')$. Thus there is a separation $(A'', B'') \in \mathcal{T}_{(M',V)}$ of order at most $2\,\mathrm{ord}(A', B') \leq 26$ so that $A''$ contains $A'$ and there are no 3-patches of $(M', V)$ with all attachments in $V(B'')$.

Recall that $M\backslash e/C$ is represented by a graph $G$ with $G_n$ as a minor, and the simplification of $\widetilde{G}$ is a subgraph of the skeleton of $(M', V)$. There exist at most $\mathrm{ord}(A'', B'') \leq 26$ vertex-images of the $G_n$-model of $G$ that contain any vertex of $A''$. Thus there is an induced subgraph $G'$ of $G_n$ with at least $n-26$ vertices that is a minor of $G[V(B'')]$. The tangle $\mathcal{T}_{(M',V)}$ controls $\widetilde{G'}$. Since $V(A') \cap V(B')$ is independent in this tangle and $n \geq 3/2k+26$, by Proposition 4.8.1 there is a graph $G_k \in \mathcal{G}_k$ such that $G[V(B'')]$ has a $G_k$-minor with vertex set $V(A'') \cap V(B'')$. This represents a minor of $M'|B''_{M'}$ isomorphic to $M(G_k)$. Recall that $(A'', B'')$ induces a separation $(A''_{M\backslash e}, B''_{M\backslash e})$ of $M\backslash e$ and so $(A''_{M\backslash e} \cup \{e\}, B''_{M\backslash e})$ is a separation of $M$ of the same order. Thus to show that (b) holds, it suffices to show that $M_{+V}|(A''_{M\backslash e} \cup \{e\} \cup V(A'' \cap B''))$ has a circuit containing at least three elements of $V(A'' \cap B'')$. Since every patch of $(M\backslash e, V)$ is realizable, there is a set $C'$ contained in the union of the 3-patches of $(M', V)$ with attachments in $V(B) \cap V(A'')$ such that $(M'/C', V)$ has the same skeleton as $(M', V)$ and has no 3-patches with attachments in $V(B)$. Then the skeleton of $(M'/C', V)$ has a set $\mathcal{P}$ of disjoint paths joining each vertex of $V(A \cap B)$ to a vertex of $V(A'' \cap B'')$. Recall that $Y$ is a circuit of $M_{+V}$ such that $Y \subseteq A_{M\backslash e} \cup \{e\} \cup V$ and $Y$ contains at least three elements of $V(A \cap B)$. We let $\mathcal{P}'$ be the subset of $\mathcal{P}$ consisting of the paths with an end in $Y$. Then the union of $Y \setminus V(A \cap B)$ with the edges of paths in $\mathcal{P}'$ and their other ends in $V(A'' \cap B'')$ is a circuit of $(M_{+V}/C')|((A''_{M\backslash e} \setminus C') \cup \{e\} \cup V(A'' \cap B''))$ that contains at least three elements of $V(A'' \cap B'')$, as required. This proves (3).

We recall that for each patch $P$ of $(M\backslash e, V)$, there is a set $C(P)$ such that $(M\backslash e/C(P), V)$ is a patchwork and $M\backslash e/C(P)$ has $M(G_n)$ as a minor. We claim that for each $P$, the skeleton of $(M\backslash e/C(P), V)$ is obtained from the skeleton of $(M\backslash e, V)$ by deleting at most one edge, whose ends are attachments of $P$. This is clear if $P$ is a 2-patch. If it is a 3-patch, then suppose the skeleton of $(M\backslash e/C(P), V)$ does not contain some two edges of the skeleton of $(M\backslash e, V)$: these will be edges joining an attachment $v$ of $P$ to its two other attachments, $u$ and $w$. Then in $M_{+V}/C(P)$ every element of $P \setminus C(P)$ lies either in the closure of $\{u, w\}$ or is parallel to $v$. But since $P$ is not a 2-patch, there is at least one such element, $f$, parallel to $v$. So $\sqcap_{M_{+V}}(\{v, f\}, C(P)) = 1$ and we can extend the represented matroid $M$ by an element $x$ that lies in the closures of $\{v, f\}$ and $C(P)$. Then $(M\backslash e, V \cup \{x\})$ is a patchwork, in which $P \setminus \mathrm{cl}_{M_{+V}}(\{v, f\})$ is a patch with attachments $\{x, u, w\}$, and this patch is realizable because $P$ is realizable in $(M\backslash e, V)$. If this patchwork has a 3-connected skeleton, that contradicts our maximal choice of $V$. If not, then its unique 2-vertex-cutset consists of $x$ and another neighbour of $v$, but then $(M\backslash e, (V \setminus \{v\}) \cup \{x\})$ is a patchwork obtained by shrinking the 3-patch $P$, contradicting the fact that $(M\backslash e, V)$ is strong. Thus the skeleton of $(M\backslash e/C(P), V)$ is obtained from that of $(M\backslash e, V)$ by deleting at most one edge. Moreover, the skeleton of

$(M\backslash e/C(P), V)$ is 2-connected and any 2-vertex cutset in it is contained in a 3-vertex-cutset of the skeleton of $(M\backslash e, V)$.

We recall that $X$ is the set of attachments of the bridge of $(M, V)$ containing $e$, and by (3), we may assume that $|X| > 13$ and that the skeleton of $(M\backslash e, V)$ has no separation $(A, B) \in \mathcal{T}$ of order at most thirteen with $X \subseteq V(A)$.

When $M' = M/C(P)$ for some patch $P$, the set of attachments $X'$ of $e$ in $(M', V)$ is contained in $X$, and contains all attachments of $e$ in $(M, V)$ that are not attachments of $P$. Thus $10 < |X| - 3 \leq |X'| \leq |X|$. The set $X'$ has rank at least 9 in $\mathcal{T}_{(M'\backslash e,V)}$, otherwise it would have rank less than ten in $\mathcal{T}_{(M\backslash e,V)}$ and so $X$ would have rank less than 13 in this tangle, a contradiction. If the skeleton of $(M'\backslash e, V)$ is not 3-connected, then it has a $\mathcal{T}_{(M'\backslash e,V)}$-closed 2-separation $(A, B) \in \mathcal{T}_{(M'\backslash e,V)}$. So $(M'\backslash e, V(B))$ is a patchwork with a 3-connected skeleton. If $X''$ is the set of attachments of $e$ in $(M', V(B))$, then $X' \setminus V(A) \subseteq X''$ so $|X''| \geq |X' \setminus V(A)|$. We claim that $|X''| \geq 7$. If not, then $|X' \setminus V(A)| < 7$. But recall that if $(A, B)$ is a 2-separation of the skeleton of $(M'\backslash e, V)$ then $V(A) \cap V(B)$ is contained in a 3-vertex cutset in the skeleton of $(M\backslash e, V)$. So there is a separation $(A', B')$ of order three in the skeleton of $(M\backslash e, V)$ with $V(A) \subseteq V(A')$. If $A''$ is the subgraph of the skeleton obtained from $A'$ by adding the vertices in $X' \backslash V(A')$, then $(A'', B') \in \mathcal{T}$ is a separation of the skeleton of $(M\backslash e, V)$ of order at most nine and $X \subseteq V(A'')$, a contradiction. Also, $X''$ has rank at least 7 in $\mathcal{T}_{(M'\backslash e,V(B))}$ for if not then $X'$ would have rank less than nine in $\mathcal{T}_{(M'\backslash e,V)}$.

We therefore have a minor $M'$ of $M$ obtained by contracting some set $C(P)$, and a subset $V'$ of $V$ (either $V' = V$ or $V' = V(B)$ when there is a 2-separation $(A, B) \in \mathcal{T}_{(M'\backslash e,V)}$) such that $(M'\backslash e, V')$ is a realizable patchwork with a 3-connected skeleton respecting $M(G_n)$ and the bridge of $(M', V')$ containing $e$ has a set of attachments with rank at least 7 in $\mathcal{T}_{(M'\backslash e,V')}$.

If the set of attachments of $e$ in $(M', V')$ has rank at least 13 in $\mathcal{T}_{(M'\backslash e,V')}$ and there is another patch $P'$ with $C(P') \neq \emptyset$, then we can repeat the argument and contract $C(P')$. We do this as many times as possible, and the result is a minor $M'$ of $M$ and a subset $V'$ of $V$ such that either

1. $(M'\backslash e, V')$ is a framework, or
2. the set of attachments of $e$ in $(M', V')$ has rank less than 13 in $\mathcal{T}_{(M'\backslash e,V')}$,

but in both cases, the set $X'$ of attachments of $e$ in $(M', V')$ has rank at least 7 in $\mathcal{T}_{(M'\backslash e,V')}$, and $(M'\backslash e, V')$ is realizable, has a 3-connected skeleton, and respects $M(G_n)$.

We claim that we can reduce the first case to the second. We assume that $(M'\backslash e, V')$ is a framework. In this case, $(M', V', e)$ is an extended framework, and there is an $(\mathbb{F}, \mathbb{F}^\times)$-labelled graph $G$ with vertex set $V'$ such that $M(G) = M'$. The attachments of the bridge of $e$ in $(M', V')$ are $\mathrm{supp}(x_G)$. The graph $G$ is 3-connected and has $G_n$ as a minor. If $|V'| = n$, then $M'\backslash e$ has $M(G_n)$ as a spanning restriction, and the fact that $|\mathrm{supp}(x_G)| > 3$ means (b) holds.

Otherwise, $|V'| > n$. By Proposition 4.9.1, $G$ has a 3-connected minor $G'$ with $G_n$ as a minor such that either $V(G') = n$ and $|\mathrm{supp}(x_{G'})| \geq 3$, in which case (b) holds, or the rank of $\mathrm{supp}(x_{G'})$ in the tangle of $G'$ induced by $T_{G_n}$ is between 7 and 10. We let $M''$ be the corresponding minor of $M$, that is $M'' = M(G')$, so $(M''\backslash e, V(G'))$ is a framework with a 3-connected skeleton and the bridge $\{e\}$ of $(M'', V')$ has a set of attachments with rank in $\{7, 8, 9, 10\}$ in $\mathcal{T}_{(M'', V(G'))}$. Thus we may take $M'$ to be equal to $M''$ and $V'$ to be equal to $V(G')$, and the second case holds.

Therefore, we may assume that the bridge of $(M', V')$ containing $e$ has a set of attachments $X'$ with rank at least 7 and less than 13 in $\mathcal{T}_{(M'\backslash e, V')}$. We choose $(A, B) \in \mathcal{T}_{(M'\backslash e, V')}$ of minimum order such that $X' \subseteq V(A)$ and $(A, B)$ is $\mathcal{T}_{(M'\backslash e, V')}$-closed. As we did in the proof of (3), we distinguish two cases, depending on whether or not there is a circuit $Y$ of $M'_{+V'}$ such that $Y \subseteq A_{M'\backslash e} \cup \{e\} \cup V(A \cap B)$ and $Y$ contains at least three elements of $V(A) \cap V(B)$.

Suppose that there is such a circuit $Y$. Since $(M'\backslash e, V')$ respects the minor $M(G_n)$, there is a framework $(N, V')$ such that $N$ is a minor of $M'\backslash e$ and has $M(G_n)$ as a minor. Then $N$ is represented by an $\mathbb{F}^\times$-labelled graph $G$ with $G_n$ as a minor. As $(A, B)$ has order at most 13, $G[V(B)]$ has a minor $G_{n-13}$ for some $G_{n-13} \in \mathcal{G}_{n-13}$. Then since $n \geq \frac{3}{2}k + 26$, Proposition 4.8.1 implies that for some $G_k \in \mathcal{G}_k$, $G[V(B)]$ has a $G_k$-minor with vertex set $V(A) \cap V(B)$. Then the existence of the circuit $Y$ means that (b) holds.

Thus we may assume that no such circuit $Y$ exists, and by Theorem 4.2.3, $(M'|(A_{M'\backslash e} \cup \{e\}), V_A)$ is a patchwork for some set $V_A$ that contains $V(A) \cap V(B)$. This means that $(M', V(B) \cup V_A)$ is a patchwork. By Proposition 4.8.2 we can choose $V_A$ so that it is realizable and has a 3-connected skeleton; we can further choose it so that no 3-patch can be shrunk and so that $V_A$ is maximal.

Recall Theorem 4.7.3 which said that there is a unique maximal way in which we can express any represented matroid as a realizable patchwork with a 3-connected, strong skeleton that contains some three given vertices. So if the skeleton of $(M'\backslash e, V(B) \cup V_A)$ is also 3-connected and realizable, then by Theorem 4.7.3, $V(B) \cup V_A \subseteq V'$. Since $(M', V(B) \cup V_A)$ is a patchwork, the bridge of $e$ has a set $Z$ of at most 3 attachments; but then the bridge of $e$ in $(M', V')$ also has at most three attachments, a contradiction because its attachment set has rank at least 7 in $\mathcal{T}_{(M'\backslash e, V')}$. Therefore, the skeleton of the patchwork $(M'\backslash e, V(B) \cup V_A)$ is either not 3-connected or not realizable.

But we claim that this skeleton is either 3-connected or realizable. For if it is not realizable, it has a patch $P$ that is not realizable, and thus $P$ has three attachments and $P \cup \{e\}$ is a realizable patch of $(M', V(B) \cup V_A)$ with the same attachments, so the skeleton of $(M' \backslash e, V(B) \cup V_A)$ is the same as the skeleton of $(M', V(B) \cup V_A)$, which is 3-connected.

So we first consider the case that the skeleton of $(M' \backslash e, V(B) \cup V_A)$ is not 3-connected but is realizable. Let $(U, W)$ be a 1- or 2-separation of this skeleton that is in $\mathcal{T}_{(M' \backslash e, V(B) \cup V_A)}$ and is closed in this tangle. Then we note that $V(U) \subseteq V_A$, by Proposition 4.8.2. There are vertices $u \in V(U) \setminus V(W)$ and $w \in V(W) \setminus V(U)$ that are both attachments of some patch $P$ of $(M', V(B) \cup V_A)$ but are not both attachments of any patch of $(M' \backslash e, V(B) \cup V_A)$. Thus $e \in P$.

We claim that any other attachment of $P$ is in $V(U) \cap V(W)$. If not, by symmetry suppose $P$ has another attachment $v \in V(W)$. Then $P \setminus \{e\}$ is a patch of $(M' \backslash e, V(B) \cup V_A)$ with two attachments, $v$ and $w$. We have $\sqcap_{M'_{V(B) \cup V_A}}(\{u, e\}, P \setminus \{e\}) = 1$ so we can extend the matroid $M'$ by a new point $x$ that is spanned by both $\{u, e\}$ and $P \setminus \{e\}$. Then $(M', V(B) \cup V_A \cup \{x\})$ is a realizable patchwork (because $P$ is a realizable patch of $(M', V(B) \cup V_A)$). If its skeleton is also 3-connected, this contradicts our maximal choice of $V_A$. The skeleton of $(M', V(B) \cup V_A)$ is obtained from the skeleton of $(M', V(B) \cup V_A \cup \{x\})$ by contracting the edge $ux$. So if the skeleton of $(M', V(B) \cup V_A \cup \{x\})$ is not 3-connected, it is because $u$ has degree two. In this case removing it from the vertex set yields a realizable patchwork with a 3-connected skeleton. This is a patchwork obtained from $(M', V(B) \cup V_A)$ by shrinking the 3-patch $P$, which again contradicts our choice of $V_A$. This proves that $u$ and $w$ are the only attachments of $P$ not in $V(U) \cap V(W)$.

We note that either there is a unique such $(U, W) \in \mathcal{T}_{(M' \backslash e, V(B) \cup V_A)}$ that is closed in the tangle, or there are exactly two, $(U_1, W_1)$ and $(U_2, W_2)$ with $u \in V(U_1) \setminus V(W_1)$ and $w \in V(U_2) \setminus V(W_2)$. In the first case, $(M' \backslash e, (V(B) \cup V_A) \cap V(W))$ is a patchwork in which $U_{M' \backslash e}$ is a patch with attachments $V(U) \cap V(W)$. It is realizable and has a 3-connected skeleton, so by Theorem 4.7.3, $V_A \cap V(W) \subseteq V(A)$. But also $U_{M' \backslash e} \cup \{e\}$ is a patch with attachments $\{w\} \cup (V(U) \cap V(W))$ (a set of size at most three), which contradicts the fact that the attachment set $X'$ of the bridge of $e$ in $(M' \backslash e, V')$ has rank at least 7 in $\mathcal{T}_{(M' \backslash e, V')}$. Similarly, if there are two separations $(U_1, W_1)$ and $(U_2, W_2)$, then $(M' \backslash e, (V(B) \cup V_A) \cap V(W_1) \cap V(W_2))$ is a patchwork in which $(U_1)_{M' \backslash e}$ and $(U_2)_{M' \backslash e}$ are patches with attachment sets $V(U_1) \cap V(W_1)$ and $V(U_2) \cap V(W_2)$, respectively. It is realizable and has a 3-connected skeleton, so by Theorem 4.7.3, $V_A \cap V(W_1) \cap V(W_2) \subseteq V(A)$. But then the attachment set of the bridge of $e$ in $(M', (V(B) \cup V_A) \cap V(W_1) \cap V(W_2))$, and hence also the attachment set $X'$ of the bridge of $e$ in $(M', V')$, has rank at most six in the tangle $\mathcal{T}_{(M'\backslash e, V')}$, a contradiction.

This concludes the case that the skeleton of $(M'\backslash e, V(B) \cup V_A)$ is not 3-connected, so we may assume that is it 3-connected but is not realizable. So it has a unique unrealizable patch $P$ with three attachments and $P \cup \{e\}$ is a patch of $(M', V(B) \cup V_A)$ with the same attachments. We let $Z$ be the set of attachments of $P$ in $(M'\backslash e, V(B) \cup V_A)$ and $Z'$ the set obtained from $Z$ by removing any vertex with only one neighbour in the skeleton of $(M'\backslash e, V(B) \cup V_A)$ that is not in $Z$ and replacing it with that neighbour. Then by Theorem 4.2.3, $(P \setminus \{e\}, Z'')$ is a patchwork for some set $Z''$ containing $Z'$, and we can choose $Z''$ so that it is realizable and $Z'$-strong. Then $(M'\backslash e, (V(B) \cup V_A \setminus Z) \cup Z'')$ is a realizable patchwork with a 3-connected skeleton. By Theorem 4.7.3, $(V_A \setminus Z) \cup Z'' \subseteq V(A)$. Also the attachments of the bridge of $e$ in $(M', (V(B) \cup V_A \setminus Z) \cup Z'')$ are contained in $Z''$, so the attachments of the bridge of $e$ in $(M', V')$ are also contained in $Z''$, which means there is a separation $(S, T)$ of its skeleton with $V(S) = Z''$ and all attachments of the bridge of $e$ are in $V(S)$. But $V(S) \cap V(T) = Z'$ so $(S, T)$ has order three and $(S, T) \in \mathcal{T}_{(M'\backslash e, V')}$, a contradiction because the attachments of $e$ are contained in $V(S)$ and have rank at least seven in this tangle. □

## 4.10 Three elements in a circuit

We conclude this chapter with a discussion of an open question related to both patchworks and modular restrictions. We recall that Seymour proved that if a highly connected binary matroid has a triple of elements not contained in any circuit, then it is a graphic matroid. On the other hand, we still do not have a precise characterization of the non-binary matroids with a triple of elements not contained in a circuit.

**Problem 4.10.1.** *Characterize the non-binary matroids with a set of three elements that is not contained in any circuit.*

Frame matroids over fields larger than GF(2) fall into this class, but they are not the only examples, as we see below.

When a patchwork $(M, V)$ has a 3-patch $P$, its attachment set $X$ is not contained in any circuit of $M_{+X}|(P \cup X)$. Therefore, the problem of characterizing represented matroids with a triple of elements not in a circuit is exactly the problem of describing the structure of 3-patches in a patchwork.

This is also closely related to the problem of describing matroids with a rank-3 frame matroid as a modular restriction, since we saw in Theorem 4.2.1

that a patchwork can be constructed from a frame matroid by modular sums on planes spanned by up to three vertices.

We have seen that a modular $M(K_4)$-restriction forces any vertically 4-connected matroid to be graphic. We might therefore be tempted to conjecture for any finite field $\mathbb{F}$ that, up to connectivity, a matroid with a modular $\mathrm{DG}(3, \mathbb{F}^\times)$-restriction is a Dowling matroid over $\mathbb{F}^\times$. Although Geelen, Gerards, and Whittle showed that the corresponding result is true when the matroid has a modular $\mathrm{DG}(4, \mathbb{F}^\times)$-restriction, they also found the following example showing that a modular $\mathrm{DG}(3, \mathbb{F}^\times)$-restriction does not suffice, even when $\mathbb{F} = \mathrm{GF}(3)$.

We observe that the non-Fano matroid $F_7^-$ is in $\mathcal{D}(\mathrm{GF}(3)^\times)$; see Figure 4.3 for an illustration. We can actually construct highly connected GF(3)-

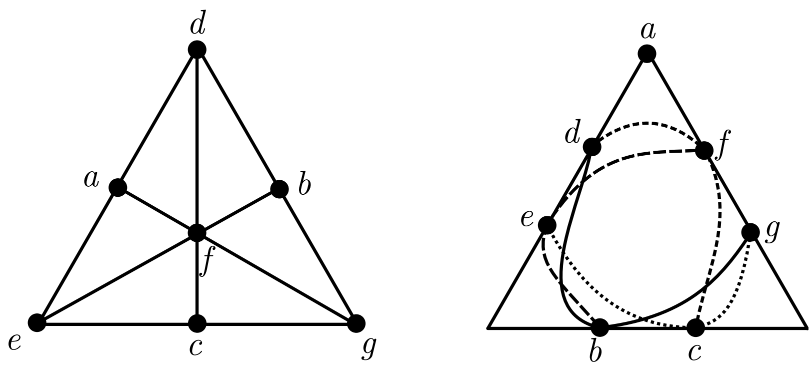



Figure 4.3: Two drawings of the non-Fano matroid, $F_7^-$

representable matroids with modular $F_7^-$-restrictions that are not in $\mathcal{D}(\mathrm{GF}(3)^\times)$. The matroid $F_7^-$ is represented by a $(\mathrm{GF}(3), \{1\})$-labelled copy of $K_4$ where two vertices $\{a, b\}$ have label 1 and two vertices $\{c, d\}$ have label $-1$. Equivalently, we add the column $(1, -1, 1, -1)^T \in \mathrm{GF}(3)^{\{a,b,c,d\}}$ to a graphic representation of $M(K_4)$ — note that, although $F_7^- \in \mathcal{D}(\mathrm{GF}(3)^\times)$, this is a non-Dowling representation of $F_7^-$.

We draw this copy of $K_4$ in the plane by drawing the edges $ab, bc, cd$, and $da$ bounding a disc and then drawing the edges $ac$ and $bd$ inside this disc with a crossing (see Figure 4.4). Then we draw an arbitrary planar graph around this drawing of $K_4$ with an even cycle $C$ bounding the infinite face, and we assign the label 0 to every vertex except $a, b, c$, and $d$. This gives us a graph $G$ drawn in the plane with one crossing. Finally, we add a second set of vertex-labels $y \in \mathrm{GF}(3)^{V(G)}$ such that every vertex of $G - C$ has label 0 and the vertices of $C$ are labelled 1 and $-1$ alternately as we traverse the cycle. This second set of labels corresponds to adding a new element to the matroid by adjoining the column $y$ to the representation of $M(G)$. This matroid is not in $\mathcal{D}(\Gamma)$, as this last column has too many non-zero entries, regardless of which set of joints we choose to get a Dowling representation of $M(\widetilde{G})$.

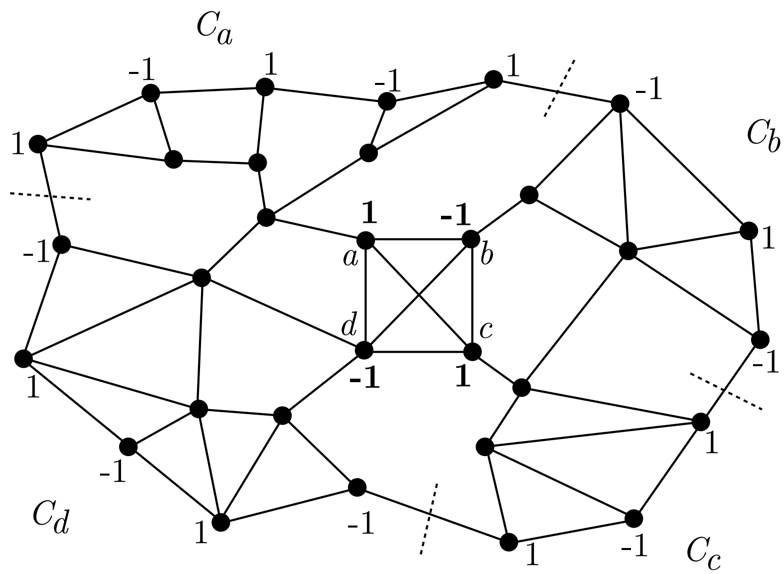



Figure 4.4: A graph representing a matroid with a modular $F_7^-$-restriction

Since cliques are modular in graphs, the only way for the $F_7^-$-restriction to fail to be modular is if we can find a minor where the final element $y$ forms a new point in the closure of the non-Fano restriction. But this is impossible thanks to the planarity of the graph — we can partition $V(C)$ into four consecutive sequences $C_a, C_b, C_c$, and $C_d$, which each get 'contracted onto' the four vertices $a, b, c$, and $d$ of the $K_4$. For any such partition, the resulting vertex-labels on $a, b, c$, and $d$ are the sums of the labels in $C_a, C_b, C_c$, and $C_d$, and either at most two of these are non-zero or they alternate between 1 and $-1$.

We can also extend the non-Fano restriction by two new elements to get a modular $\mathrm{DG}(3, \mathrm{GF}(3)^\times)$-restriction. Its set of three joints is not contained in a circuit, providing an example of a highly connected non-frame matroid that has three elements not in a common circuit.

This raises the problem of characterizing the matroids with modular Dowling restrictions like $F_7^-$, with which we end this thesis.

**Problem 4.10.2.** *Characterize the vertically 4-connected matroids with a modular $F_7^-$-restriction.*

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# Glossary of Notation

| | |
|---|---|
| $\oplus$ | direct sum. 9 |
| $\oplus_2$ | 2-sum. 25 |
| $\oplus_m$ | modular sum. 25 |
| $A[X, Y]$ | the submatrix of $A$ in the rows indexed by $X$ and the columns indexed by $Y$. 28 |
| $\mathrm{AG}(n-1, \mathbb{F})$ | the rank-$n$ affine geometry over $\mathbb{F}$. 74 |
| $(A_M, B_M)$ | the partition of $E(M)$ corresponding to a separation $(A, B)$ of the skeleton of a patchwork $(M, V)$. 113 |
| $\mathcal{C}(k)$ | the class of $\mathbb{C}$-representable matroids with no $U_{2,k+3}$-minor. 21 |
| $\mathrm{cl}_M$ | the closure function of a matroid $M$. 3 |
| $\mathrm{DG}(n, \Gamma)$ | the rank-$n$ Dowling geometry over $\Gamma$. 7 |
| $\mathcal{D}(\Gamma)$ | the variety of Dowling matroids over $\Gamma$. 7 |
| $\delta(v)$ | the set of edges incident with a vertex $v$ in a graph. |
| $E(M)$ | the ground set of a matroid $M$. 2 |
| $\varepsilon(M)$ | $\lvert E(\mathrm{si}(M))\rvert$, the number of points in $M$. 19 |
| $E(G)$ | the edge set of a graph $G$. |
| $\mathbb{F}^\times$ | the group of units of a field $\mathbb{F}$. |
| $F_7^-$ | the non-Fano matroid. 77 |
| $\widetilde{G}$ | the graph corresponding to a labelled graph $G$. 94 |
| $\vec{G}$ | the oriented graph corresponding to a labelled graph $G$. 6 |
| $\Gamma_{\mathbb{F}^t}$ | the group $\mathbb{F}^t \times \Gamma$, where $\Gamma$ is a finite subgroup of $\mathbb{F}^\times$, with operation $(\alpha_1, \gamma_1) \cdot (\alpha_2, \gamma_2) = (\alpha_1\gamma_2 + \alpha_2, \gamma_1\gamma_2)$ and identity $(0, 1)$. 95 |
| $\mathrm{GF}(q)$ | the finite field of order $q$. |
| $\gamma_G$ | the edge labels in a group-labelled graph $G$. 6 |
| $g_{\mathcal{M}}(n)$ | the growth-rate function of a class $\mathcal{M}$; $\max\{\varepsilon(M) : M \in \mathcal{M}, r(M) = n\}$. 19 |

| | |
|---|---|
| $G[X]$ | the subgraph of a graph $G$ induced by a set of vertices $X$, or the subgraph consisting of a set of edges $X$ and the ends of its elements. |
| $\kappa_M(S,T)$ | $\min\{\lambda_M(A) : S \subseteq A \subseteq E(M) \setminus T\}$ for disjoint $S,T \subseteq E(M)$. 57 |
| $K_n$ | the $n$-vertex complete graph or clique. |
| $K_{n,n}$ | the complete bipartite graph with two independent sets of size $n$. |
| $\lambda_M(X)$ | the connectivity function of a matroid $M$, equal to $r_M(X) + r_M(E(M) \setminus X) - r(M)$ and $r_M(X) + r^*_M(X) - \lvert X \rvert$. 8 |
| $M^*$ | the dual of a matroid $M$. 4 |
| $M_{+e}, M_{+V}$ | the extension of a represented matroid $M$ by an element $e$ or a finite set of elements $V$. 91 |
| $M_{\mathbb{F}}(A)$ | the matroid represented by a matrix $A$ over the field $\mathbb{F}$. 3 |
| $M_{\mathbb{F},\Gamma}(G)$ | the matroid represented by a $\Gamma_{\mathbb{F}^t}$-labelled graph $G$. 96 |
| $M(G)$ | the cycle matroid of a graph $G$ or the matroid represented by a group-labelled graph $G$. 5 |
| $\mathrm{ord}(G_1, G_2)$ | the order of a separation $(G_1, G_2)$. 111 |
| $\mathrm{PG}(n-1, \mathbb{F})$ | the rank-$n$ projective geometry over a finite field $\mathbb{F}$. 5 |
| $\mathrm{PG}(n-1, q)$ | the rank-$n$ projective geometry over a field of order $q$. 4 |
| $r_M$ | the rank function of a matroid $M$. 2 |
| $r^*_M$ | the corank function of a matroid $M$, equal to $r_{M^*}$. 26 |
| $r_{\mathcal{T}}(X)$ | the rank of a set of vertices $X$ in the tangle $\mathcal{T}$. 119 |
| $\mathrm{si}(M)$ | the simplification of a matroid $M$. 3 |
| $\Sigma(M, M')$ | $\{e : M \backslash e \neq M' \backslash e \text{ and } M/e \neq M'/e\}$. 57 |
| $\sqcap_M(S,T)$ | local connectivity, $r_M(S) + r_M(T) - r_M(S \cup T)$. 25 |
| $U_{2,n}$ | the simple rank-2 matroid with $n$ elements. 3 |
| $\mathcal{U}(\ell)$ | the class of matroids with no $U_{2,\ell+2}$-minor. 20 |
| $V(G)$ | the vertex set of a graph $G$. |
| $x_G$ | the vertex labels in a vertex- and edge-labelled graph $G$. 110 |

# Index