Tight Orthogonal Main Effect Plans

by

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Abstract

In this thesis, we study orthogonal main effect plans (OMEPs), also known as orthogonal resolution III fractional designs. OMEPs are are a generalization of orthogonal arrays, and play a role in experimental design, in particular in screening experiments. We show that for any OMEP parameter set P, there is a special parameter set P', called a tight parameter set, so that if an OMEP with parameters P' exists then it can be used to obtain an OMEP with parameters P easily. Tight OMEPs are more structured than general OMEPs and therefore are easier to analyze. We find all tight OMEPs on three, four, and five rows, and use this to answer the existence question for four and five-factor OMEPs. The same procedure can be used to help answer the existence question for OMEPs on any number of rows. We also show that, asymptotically, for any tight parameter set there is a corresponding OMEP (with one small class of exceptions). We use this information to gain insight into Jacroux's lower bound on the number of runs in an OMEP. We demonstrate that any OMEP (not just every OMEP parameter set) having three rows can be uncollapsed to a tight OMEP, so in the case of OMEPs having three rows all the structural information about OMEPs is contained in the subset of three-row tight OMEPs. We also develop recursive constructions for equally replicated OMEPs. Often these constructions produce OMEPs having more rows than a direct product construction could achieve. Sometimes the OMEPs produced are tight. One of these constructions produces resolvable orthogonal arrays. Other miscellaneous results concerning OMEPs are also proven.

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Chapter 1

Introduction

In recent years, combinatorial designs have found increasing use in various fields. In particular, combinatorial designs are being used not only in traditional fields such as statistics, but also in computer science and engineering.

The field of experimental design uses many combinatorial designs, such as transversal designs and balanced incomplete block designs, as well as similar structures such as fractional factorial designs and orthogonal main effect plans. In this setting, these structures are often used to define a sequence of experiments so that the collection of experiments as a whole has desirable properties for statistical analysis.

In this thesis we investigate orthogonal main effect plans, or OMEPs. (OMEPs are sometimes called orthogonal resolution III fractional designs.) This investigation begins by considering the existence question. This question naturally leads to the concept of tight OMEPs, which are a particularly nice subclass of OMEPs. In later chapters, we prove existence results about tight OMEPs and show why tight OMEPs are useful in answering questions about general OMEPs.

1.1 Definitions and Examples from Combinatorial Design

In this section we define some of the structures that occur in this thesis. We also give examples and describe some well known construction methods and existence results.

The most well known type of design is probably a *balanced incomplete block design*, or BIBD. A BIBD is a pair $(V, \mathcal{B})$, where V is a set of points and $\mathcal{B}$ is a collection of k-subsets of V, called *blocks*, with the property that any pair of points in V is contained in exactly λ blocks in $\mathcal{B}$. Since each pair of points is contained in a constant number of blocks these designs are also called 2-designs. For example, by taking $V = Z_7$ and $B = \{0,1,3\} + i, i \in Z_7$, we obtain a BIBD with $k = 3$ and $\lambda = 1$. Often the parameters of a BIBD are included by calling the design a (v, k, λ)-BIBD. One can easily calculate $b = |\mathcal{B}|$ using v, k, λ. For instance, each of the $v(v-1)$ ordered pairs of distinct points in V must occur λ times in the blocks, and each block contains $k(k-1)$ such ordered pairs. Thus we find

$$\lambda v(v-1) = bk(k-1). \tag{1.1}$$

In a BIBD, any point occurs in the same number of blocks, as the following argument shows. For a fixed point p, consider the $v-1$ ordered pairs (p, x) where x is a point of the design other than p. Each such ordered pair must occur λ times in total, and any block containing p contributes $k-1$ such ordered pairs. Hence if r_p denotes the number of blocks containing point p, we find

$$\lambda(v-1) = r_p(k-1), \tag{1.2}$$

and thus $r_p = \lambda(v-1)/(k-1)$, which is independent of the point p. Thus each point occurs in $\lambda(v-1)/(k-1)$ blocks. We denote this common value by r.

These equations give necessary conditions for the existence of BIBDs with specified parameters. In some literature, a (v, k, λ)-BIBD is also called a (b, v, r, k, λ)-BIBD.

BIBDs are known to exist for many possible parameter sets. In particular, Hanani ([14], [13]) has shown that the necessary conditions 1.1 and 1.2 are sufficient when $k = 3, 4$. Furthermore, Wilson [28] has shown that for fixed k, v, there is an N depending on k, v for which these necessary conditions are sufficient for $\lambda \geq N$. Wilson [26] has also shown that for fixed k and λ, these necessary conditions are sufficient for all but finitely many values of v.

A well known family of BIBDs consists of the finite projective geometries. For example, all cosets of each 1-dimensional subspace in a vector space of dimension two over GF(q) yield an *affine plane*, and these subsets are also the block set of a $(q^2, q, 1)$-BIBD. Furthermore, if V is a vector space of dimension three over GF(q), and we take each 1-dimensional subspace of V as a point and each 2-dimensional subspace of V as a block, we obtain a $(q^2 + q + 1, q + 1, 1)$-BIBD. A BIBD with these parameters is also called a *projective plane*.

Another important type of design is a *transversal design*. Such designs are used in constructing other combinatorial designs, in experimental statistics, and in the study of error correcting codes. A transversal design is an ordered triple $(V, \mathcal{G}, \mathcal{B})$, where again V is a set of points, $\mathcal{G}$ is a partition of V into parts of equal size (the parts are called *groups*), and $\mathcal{B}$ is a collection of subsets of V, called *blocks*, where every block contains exactly one point from each group (hence the name transversal). The defining property is that any pair of points in V in *distinct* groups occurs in exactly λ blocks in $\mathcal{B}$, and any pair of points from the same group occurs in no blocks in $\mathcal{B}$. If g is the size of the groups, then the block size is $k = |V|/g$ and we call the design a TD$_\lambda(k, g)$. When $\lambda = 1$ the convention is to just write TD(k, g).

As with BIBDs, counting arguments can be used to find relationships among the parameters of a transversal design. For example, in a $\text{TD}_\lambda(k, g)$ there are λg^2 blocks and each point lies in λg blocks. Necessary conditions for the existence of transversal designs are not as easy to derive as with BIBDs, because if k is small enough then the transversal design always exists. For example, a $\text{TD}(3, g)$ exists for all $g \geq 1$, and hence for all $\lambda \geq 1$. However, it is possible to place an upper bound on k given the other parameters.

In the above $(7, 3, 1)$-BIBD, the blocks containing 0 are $\{0, 1, 3\}$, $\{0, 4, 5\}$, $\{0, 2, 6\}$. We can form a $\text{TD}(3, 2)$ by taking $V = Z_7 \setminus \{0\}$, $G = \{\{1, 3\}, \{4, 5\}, \{2, 6\}\}$, and $B = \{$those blocks of our $(7, 3, 1)$-BIBD that do not contain $0\}$. More generally, removing a point in a $(n^2+n+1, n+1, 1)$-BIBD and considering the blocks through that point as groups yields a $\text{TD}(n + 1, n)$.

Orthogonal arrays are closely related to transversal designs. An *orthogonal array* $\text{OA}_\lambda(k, g)$ is a $k \times \lambda g^2$ matrix, having symbols from the g-set S in each row (usually $S = \{1, 2, \ldots, g\}$), and with the property that for any pair of rows the $2 \times \lambda g^2$ submatrix induced by these rows consists of every possible column of symbols from S with each such column occuring λ times. Given a $\text{TD}_\lambda(k, g)$, we can write each of its blocks as a column vector, where the symbol in row i is the point of the block occurring in the ith group of the transversal design. If we then rename the symbols in each row (there are g distinct symbols in each row) to coincide with S, then we obtain an $\text{OA}_\lambda(k, g)$. This process is reversible, and so transversal designs and orthogonal arrays really represent the same concept. Table 1.1 is an orthogonal array obtained from the above $\text{TD}(3, 2)$ using this construction.

Many objects in design theory are examples of incidence structures. An *incidence structure* is a triple $(V, B, \mathcal{I})$, where V and B are two sets (called the point set and the block set) and $\mathcal{I}$ is a binary relation between V and B. We say that

Table 1.1: An OA(3, 2)

0	0	1	1
0	1	0	1
0	1	1	0

point $p \in V$ is *incident* with a block (or line) $\ell \in B$ if $p\mathcal{I}\ell$. In the case of transversal designs and BIBDs, the incidence relation $\mathcal{I}$ is just $p\mathcal{I}\ell$ if $p \in \ell$. In any object considered in this thesis, this is the case and so any block is always considered as a collection of points. We refer to incidence structures mainly to describe concepts which apply to more than one kind of design.

One such concept is resolvability. An incidence structure is *resolvable* if its block set can be partitioned into classes so that the blocks in each class form a partition of the point set. Each such class is called a parallel class. Resolvability is useful in extending designs — adding points and blocks to an existing design to obtain a new design. For example, a resolvable TD(k, g) has g parallel classes. By adding a new point ∞_i to each block in the ith parallel class, adding the group $\{\infty_1, \infty_2, \ldots, \infty_g\}$ to the group set, and adding the new points to the point set results in a TD$(k+1, g)$. Adding points to a resolvable design in this manner is often called "extending parallel classes" or "adding points at infinity". In fact one sees that a TD$(k+1, g)$ gives a resolvable TD(k, g) by essentially reversing this procedure. Resolvability of transversal designs with $\lambda > 1$ also allows one to obtain another transversal design with a larger block size, but in this case it is no longer necessarily true that a TD$_\lambda(k+1, g)$ can be used to obtain a resolvable TD$_\lambda(k, g)$.

Resolvability of an orthogonal array OA$_\lambda(k, g)$ implies that we can partition the columns into λg classes so that each row of each class contains each symbol exactly once. In this case, we can group the λg classes into g classes $P_1, P_2, \ldots, P_g$, each having λg columns, and add a new row having symbol i below each column in the P_i'th class. In this case we obtain an $\text{OA}_\lambda(k+1, g)$. This operation of extending a resolvable orthogonal array by a row is also called extending the parallel classes.

Another very common incidence structure is a *pairwise balanced design*, or PBD. A PBD is a pair $(V, \mathcal{B})$ where as usual V is a set of points and $\mathcal{B}$ is a collection of subsets of V, called blocks, so that any pair of points in V is contained in exactly λ blocks in $\mathcal{B}$. There are many results about PBDs; perhaps the most well known is Fisher's inequality, namely $|\mathcal{B}| \geq |V|$ in any PBD. Many recursive constructions for designs involve PBDs.

Many known incidence structures admit an automorphism. An *automorphism* of an incidence structure is a bijection π mapping points to points and blocks to blocks so that for any block B of the incidence structure, $\pi(B)$ is also a block of the incidence structure. The set of automorphisms of an incidence structure form a group called the *automorphism group* of the structure. When an incidence structure has a automorphism group, the group action partitions the block set into orbits. Using any block in any particular orbit and the group action, one can obtain all the blocks in the orbit. For this reason, we sometimes call a block representing an orbit a *base block*.

In the case of orthogonal arrays, a "common" automorphism is the permutation that permutes the points of each row in a cycle. (It is common in the sense that many *known* orthogonal arrays have such an automorphism.) Such a permutation partitions the column set into λg orbits of size g each, and thus we can generate the columns in one orbit using any column in the orbit and applying π to this column g times. This operation of generating a collection of columns from one particular column using a group action is called *developing* the column, or developing the

Table 1.2: A $(3,3,1)$-Difference Matrix

0	0	0
0	1	2
0	2	1

Table 1.3: An OA$(3,3)$

0	1	2	0	1	2	0	1	2
0	1	2	1	2	0	2	0	1
0	1	2	2	0	1	1	2	0

block in the more general case of incidence structures. For example, an OA$(3,3)$ can be obtained by developing each column in the matrix in Table 1.2 using the permutation (0 1 2). Since many orthogonal arrays can be described in this way, it is common to give the "generator matrix" a special name. A (g,k,λ)-*difference matrix* over a group G is a $k \times g\lambda$ matrix $D = (d_{ij})$ with entries from G with the property that for any $i \neq j$, the list $(d_{il} - d_{jl}), l = 1 \dots g\lambda$ contains each element of G precisely λ times. The subtraction is in the group G. If D is a (g,k,λ)-difference matrix, then the set of columns $\{D + g | g \in G\}$ is a OA$_\lambda(k,g)$ with a set of automorphisms $\{x \to x + g | g \in G\}$. Difference matrices, or some variant of them, are often used in conjunction with other automorphisms to find specific orthogonal arrays with a computer search.

To illustrate another concept, we develop the matrix in Table 1.2 using the permutation (0 1 2), giving an OA$(3,3)$. The group action partitions the nine columns into three orbits, each of size three. For example, the first orbit consists of the first three columns of Table 1.3. If we pick any orbit, and choose any row, then each of the symbols in $\{0, 1, 2\}$ occurs in that row in the columns of the orbit. Thus the columns of the OA(3, 3) can be partitioned into λg classes of size g, so that in any class, each row is a permutation of the elements $\{0, 1, 2, \ldots, g-1\}$. Thus the orthogonal array is resolvable. As with general incidence structures each class is called a parallel class. The development of any difference matrix always gives a resolvable orthogonal array. When an OA(k, g) is resolvable, we can add a new row to the orthogonal array, and for each column in it we place an i in the new row if that column lies in the i'th parallel class. This results in an OA($k+1, g$). (This is analogous to extending parallel classes in a resolvable transversal design). As with transversal designs, if an OA($k+1, g$) exists then a resolvable OA(k, g) exists. For $\lambda > 1$, it remains true that if a resolvable $\mathrm{OA}_\lambda(k, g)$ exists, then a $\mathrm{OA}_\lambda(k+1, g)$ exists; however, the converse no longer holds.

Various results are known about the existence of $\mathrm{OA}_\lambda(k, g)$ for various g, k, λ. One of the most important of these is the existence of an OA($q+1, q$) whenever q is a prime power. In particular, if c is any $q \times 1$ column vector consisting of distinct elements of GF(q), then the matrix with columns $\{\alpha c | \alpha \in \mathrm{GF}(q)\}$ is a $(q, q, 1)$-difference matrix over GF(q). Thus its development gives a resolvable OA(q, q), and by extending the parallel classes we obtain an OA($q+1, q$).

Another useful fact is that a resolvable $\mathrm{OA}_{p^j}(p^{i+j}, p^i)$ exists for any prime p and any nonnegative integers i, j. For if D is a (v, k, λ)-difference matrix over G, and H is a normal subgroup of G, then D can be considered as a $(g/|H|, k, \lambda|H|)$-difference matrix over the factor group G/H. There is a $(p^{i+j}, p^{i+j}, 1)$-difference matrix over $(\mathrm{GF}(p^{i+j}), +)$, and furthermore this group has a normal subgroup of order p^j. Thus this matrix, as viewed from the factor group, is a (p^i, p^{i+j}, p^j)-difference matrix whose development (in the factor group!) gives the desired resolvable orthogonal array.

Further orthogonal arrays can be constructed by using various recursive constructions. For example, it is well known [17] that the direct product of an $\text{OA}_{\lambda_1}(k, g_1)$ and a $\text{OA}_{\lambda_2}(k, g_2)$ is an $\text{OA}_{\lambda_1\lambda_2}(k, g_1g_2)$. Using this construction, and the above facts about difference matrices, if $n = q_1q_2 \dots q_n$ is the prime power factorization of n, with $q_i < q_{i+1}$, then an $\text{OA}(q_1 + 1, n)$ exists. However, better results than this are known. For example, an $\text{OA}(5, n)$ exists for all $n \neq 2, 3, 6, 10$. As an illustration of the difficulty in constructing and proving non-existence of orthogonal arrays, we mention that it is still not known whether an $\text{OA}(5, 10)$ exists, despite the relatively small values of the parameters.

Another useful recursive construction is the Kronecker product of two difference matrices. The *Kronecker product* of two matrices $A = \{a_{ij}\}$ (an m by n matrix) and $B = \{b_{ij}\}$ is the matrix

$$\begin{matrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{matrix}$$

The Kronecker product of a (g, k_1, λ_1)-difference matrix over G and a (g, k_2, λ_2)-difference matrix over G is a $(g, k_1k_2, \lambda_1\lambda_2g)$-difference matrix over G.

In fact, if A is an $\text{OA}_\lambda(k, g)$, and B is a (g, k', λ')-difference matrix over a group G, then the Kronecker product of A and B is an $\text{OA}_{g\lambda\lambda'}(kk', g)$. See [20] for details. This result has been generalized by Wang and Wu [25].

Various results exist bounding the block size k in an $\text{TD}_\lambda(k, g)$. If the transversal design is resolvable, then $k \leq g\lambda$. One method of proving this is to use linear algebra. However, in [15], Hine and Mavron, proved this result using a nice counting argument, as follows. Let $\mathcal{D}$ be a $\mathrm{TD}_\lambda(k,g)$ having two disjoint blocks B and C. (If the $\mathrm{TD}_\lambda(k,g)$ is resolvable, then certainly there are two disjoint blocks.) Define $b_X = |B \cap X|, c_X = |C \cap X|$, for a block $X \neq B, C$. By counting blocks (other than B) through a point in B, we find $\sum b_X = k(\lambda g - 1)$. Similarly, $\sum c_X = k(\lambda g - 1)$. By considering pairs of points in B, we find $\sum b_X(b_X - 1) = k(k-1)(\lambda - 1)$, which is also the value of $\sum c_X(c_X - 1)$. Finally, by counting triples (p, q, X) with $p \in B, X$, $q \in C, X$, and $X \neq B, C$, we find $\sum b_X c_X = k(k-1)\lambda$. Now, using the fact that $0 \leq \sum(b_X - c_X)^2 = \sum b_X(b_X - 1) + \sum c_X(c_X - 1) + \sum b_X + \sum c_X - 2\sum b_X c_X$ and substituting in the values above, we find $0 \leq 2k(\lambda g - k)$ which gives the result.

1.2 Definitions and Examples from Experimental Design

Definition 1.2.1 *An* orthogonal main effect plan, *or OMEP, is an array having k rows (or factors), n columns (or runs), s_i symbols in row i, for $1 \leq i \leq k$, and which satisfies the property: if $1 \leq i < j \leq k$, and if x is any symbol in row i, and y is any symbol in row j, then the number of columns with an x in row i and a y in row j equals the number of times x appears in row i, multiplied by the number of times y appears in row j, divided by n. We call the array an $s_1 \times s_2 \times \ldots s_k//n$ OMEP.*

We denote the number of times symbol x occurs in row i by r_{ix}. These numbers are called the *replication numbers* of the OMEP. For example, the matrix of Table 1.4 is a $2 \times 3 \times 4//16$ OMEP.

Orthogonal main effect plans are used in the design of statistical experiments. In [16], Jacroux writes

Table 1.4: A $2 \times 3 \times 4//16$ OMEP.

1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
1	1	2	3	1	1	2	3	1	1	2	3	1	1	2	3
1	1	3	4	2	2	4	3	3	4	2	2	3	4	1	1

In many industrial situations, investigators will often begin an experimental study by employing a screening design to help them identify key factors for further investigation. Orthogonal main-effect plans are often the screening designs of choice used in such situations. OMEPs allow for the estimation of all main effects of a factorial arrangement without correlation when the interactions are all assumed negligible.

A representive application is as follows. Suppose we have a machine that creates ground coffee from fresh beans. There are three dials on the machine, and for each dial there are a number of settings. The first dial controls the length of time for which we roast the beans, and has two settings. The second dial controls how quickly the beans are cooled, and has three settings. The third dial controls the amount of grinding done on the beans, and has four settings. We wish to determine the effect of the settings of the dials on the quality of the final product (which we measure quantitatively in some manner.) One way to do this is to perform a set of experiments, varying the settings on the dials in each and recording the quality of the resulting coffee for each setting. If we perform sixteen experiments, and for experiment i we set dial j to the k'th setting, where k is the symbol in position (j, i) of the matrix in Table 1.4, then the special structure of the OMEP allows us to say something about how the settings of the dials affect the coffee produced.

Often we consider OMEPs with many rows having the same number of symbols in them. In such a situation the following notation is useful.

Definition 1.2.2 *If an OMEP on n rows has m_1 rows each containing s_1 symbols, m_2 rows each containing s_2 symbols, and so on up to m_t rows each containing s_t symbols, then we call it an*

$$s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_t^{m_t} // n \text{ OMEP}.$$

For example, a $2 \times 2 \times 2 \times 3 \times 3//9$ OMEP is compactly described using this notation as a $2^3 \times 3^2//9$ OMEP. Furthermore, a $5 \times 5 \times 5 \times 5 \times 5//25$ OMEP is also described as a $5^5//25$ OMEP. To avoid confusion, we avoid using exponents when writing the number of symbols in a row of an OMEP. Thus a $25 \times 1//25$ OMEP would never be written as a $5^2 \times 1//25$ OMEP, since we reserve this to mean a $5 \times 5 \times 1//25$ OMEP. We might write it as a $25 \times 1//5^2$ OMEP, however, or even as a $(5^2) \times 1//5^2$ OMEP.

OMEPs have close ties with orthogonal arrays, and thus with transversal designs. For example, an orthogonal array $\mathrm{OA}_\lambda(k, g)$ is also a $g^k//\lambda g^2$ OMEP. A similar relation between OMEPs and transversal designs also holds since any transversal design corresponds to an orthogonal array.

In fact, OMEPs can be used to describe resolvable orthogonal arrays also. Suppose we have a resolvable orthogonal array $\mathrm{OA}_\lambda(k, g)$ where each row contains the symbols 1 through g. Since the array is resolvable the columns can be partitioned into λg parallel classes so that for any class, and for any row in that class, each of the symbols in $\{1, 2, \ldots, g\}$ occurs exactly once. Add a new row to the matrix and for each column, put the symbol i in the new row if the column is in the i'th class. (This completely specifies the new row.) The resulting matrix is a $g^k \times \lambda g//\lambda g^2$

Table 1.5: An equally replicated OMEP

0	0	1	1	2	2	3	3
0	1	0	1	0	1	0	1
0	1	1	0	0	1	1	0

OMEP, having $k+1$ rows. Clearly we can obtain the original orthogonal array by removing the new row. Thus there is a mapping from resolvable orthogonal arrays to OMEPs and vice-versa.

Orthogonal main effect plans were apparently introduced by Addelman [1] in 1962, although similar structures have been considered earlier. Addelman also introduced a useful way of constructing other OMEPs from a given OMEP. Given an $s_1 \times s_2 \times \ldots s_k//n$ OMEP, pick a pair of symbols in a given row (say the ith row). Replace every occurence of these symbols in this row by some new symbol. The resulting matrix is a $s_1 \times s_2 \times \ldots \times s_{i-1} \times (s_i - 1) \times s_{i+1} \ldots \times s_k//n$ OMEP. Thus we have reduced the number of symbols in row i by one. This construction, or repeated applications of it, is called *collapsing levels* in the OMEP.

There are certain classes of OMEPs that are of particular interest to statisticians. An OMEP in which every symbol in every row occurs the same number of times as each other symbol in that row is said to be *equally replicated.* For example, Table 1.5 is an equally replicated OMEP. In particular, this does not mean that the symbols in different rows must occur the same number of times.

Recall that the number of times symbol x occurs in row i is denoted r_{ix}, the replication number for symbol x in row i. Thus, equally replicated OMEPs have $r_{ix} = r_{iy}$ for each row i and for any choice of symbols x, y in row i. An equally replicated OMEP is also called an *orthogonal array on a variable number of symbols,* or simply an OAVS. We see later that tight OMEPs are equally replicated OMEPs.

Another feature of certain OMEPs of use to statisticians is the presence of repeated columns. In a practical setting, having repeated columns means repeating an experiment but holding the parameters of interest constant. An article which discusses the usefulness of this is [5].

Various results are known about OMEPs. Since orthogonal arrays are special cases of OMEPs, many known results about orthogonal arrays apply. We attempt to give some idea of the known results specifically concerning the existence of OMEPs. Much of this material is in the survey by Street [23].

In [19], a construction is given for $s_1^{t_1} \times \ldots \times s_m^{t_m}//s^n$ OMEPs, where $\sum t_i \leq n, s_i \leq s$ for $1 \leq i \leq m$ and where s is a prime or prime power.

A $n \times n$ matrix H with entries in $\{1, -1\}$ so that $HH^T = nI_n$ is called a *Hadamard matrix.* If a Hadamard matrix of order n exists, then n is called a *Hadamard number.* It is known that, with the exception $n = 2$, all Hadamard numbers are divisible by 4. Cheng [7] shows that if t and n are Hadamard numbers with $t \geq n \geq 4$, then a $4^{t-1} \times 2^{nt-3t+2}//nt$ OMEP exists. He also shows that if t,n and $t/2$ are Hadamard numbers, with $n, t \geq 4$, then there is a $n \times 4^{h-1} \times 2^{n(t-1)-3(h-1)}//nt$ OMEP, where $h = \min(n, t)$. Special cases of this result have been given by various authors.

By extending the parallel classes, a resolvable $\mathrm{OA}_\lambda(k, g)$ can be used to a construct a $\lambda g \times g^k//\lambda g^2$ OMEP [12].

If $m \leq 2k$ and q is a prime power, then for $j = 0, 1, \ldots, (k - m/2)$ we can construct a $(2^m q^{k-j}) \times q^{2^m(q^k+q^{k-1}+\ldots+q^{k-j})}//2^m q^{k+1}$ OMEP [24]. In addition, if $2 \leq m \leq 2k$, where again q is a prime power, and $2^m q^{k-j}$ is a Hadamard number, then for $j = 0, 1, \ldots, (k-m/2)$ we can construct a $(2^{2^m q^{k-j}-1}) \times q^{2^m(q^k+q^{k-1}+\ldots+q^{k-j})}//2^m q^{k+1}$

OMEP.

Another useful construction due to Addelman, called the "method of replacement", is to replace each occurrence of the j'th distinct symbol in the i'th row of an $s_1 \times s_2 \times \ldots \times s_{i-1} \times s_i \times s_{i+1} \times \ldots \times s_k//n$ OMEP by the j'th column of a $t_1 \times t_2 \times \ldots \times t_m//s_i$ OMEP. An easy calculation shows this gives a $s_1 \times s_2 \times \ldots \times s_{i-1} \times t_1 \times t_2 \times \ldots \times t_m \times s_{i+1} \times \ldots \times s_k//n$ OMEP. Using this method, a $2 \times 2 \times 2 \times 4 \times 4 \times 4 \times 4//16$ OMEP is obtained using a $4 \times 4 \times 4 \times 4 \times 4//16$ OMEP and a $2 \times 2 \times 2//4$ OMEP.

Various authors have proved direct product constructions for variants of orthogonal arrays. For example, see [18], [2], or [17].

Orthogonal arrays and transversal designs can be used to construct OMEPs, but have been studied by many authors simply as combinatorial objects. Fundamental results were proven by Bose, Shrikhande, Parker [4] and Wilson [27], although many authors have since contributed to the theory. A standard reference for results on orthogonal arrays (and most other objects in design theory) is [3].

For industrial applications of OMEPs, it may be desirable to have as few columns as possible since this means fewer experiments and therefore less experimental effort. Given $s_1, s_2, \ldots, s_k$, if n is the smallest number so that a $s_1 \times s_2 \times \ldots s_k//n$ OMEP exists, then the corresponding OMEP is called *minimal*.

This thesis concentrates mainly on the existence question for OMEPs, and in particular, on the concept and existence of a special class of OMEPs called tight OMEPs. More precisely, given $s_1, s_2, \ldots, s_k$, and n, can a $s_1 \times s_2 \times \ldots s_k//n$ OMEP exist? This relates to the minimality question, for if we characterize those n for which a $s_1 \times s_2 \times \ldots s_k//n$ OMEP exists, then we can find the minimal such n.

In answering the existence question, the concept of a tight OMEP arises. This concept, and its implications, are discussed in the second chapter of this thesis. We show that every OMEP parameter P set has a corresponding tight parameter set P'. Furthermore, if an OMEP with parameters P' exists, then one can collapse levels in this OMEP to obtain an OMEP with parameters P. We determine all tight OMEPs having three, four, and five rows. Finally we use the knowledge of four-factor tight OMEPs to answer the existence question for general four-factor OMEPs.

In the third chapter of this thesis, we consider uncollapsing levels in an OMEP. We show that any three-factor OMEP can be uncollapsed to a tight OMEP. Therefore, any three-factor OMEP can be obtained by collapsing levels in a tight three-factor OMEP. We give examples of OMEPs on four and more factors that cannot be obtained by collapsing tight OMEPs. Also in this chapter, we consider the question of unconcatenating OMEPs. We prove a finite basis type result for the concatenation of tight OMEPs.

In the fourth chapter, we show asymptotic existence of tight OMEPs. More specifically, we show that for any fixed number of rows k, and with the exception of parameters of the form $2^{k-1} \times s//2s$ for s odd, then there are only a finite number of tight OMEP parameter sets for which the tight OMEP does not exist. This information is used to gain further insight on Jacroux's lower bound on the number of columns needed in an OMEP with a specified number of symbols in each row.

In the fifth chapter of this thesis we develop some recursive constructions for equally replicated OMEPs. The basic theme is that by using a resolvable PBD and some smaller designs we can unite blocks from the smaller designs to obtain OMEPs with more rows than a direct product construction could obtain. Some of the constructions produce tight OMEPs.

Chapter 2

Tight Orthogonal Main Effect Plans

In the last chapter we introduced some common design theoretic structures and the general definition of an orthogonal main effect plan. In this chapter we motivate and define an important subclass of OMEPs, called tight OMEPs. We also give an application of tight OMEPs in the determination of those parameters for which four factor OMEPs exist. We take the view that the structure of OMEPs having one or two rows is somewhat trivial, and so in what follows we assume that the number of rows k is three or more.

2.1 Definition and Motivation

Suppose $\mathcal{D}$ is an $s_1 \times s_2 \times \ldots \times s_k//n$ OMEP, and that $n = p_1^{m_1} p_2^{m_2} \ldots p_d^{m_d}$ is the prime power factorization of n. Let

$$g_i = \gcd\{r_{ix} |\ x \text{ a symbol in row } i\ \}.$$

Since $\mathcal{D}$ is an OMEP

$$n | r_{ix} r_{jy} \text{ for } i \neq j,\ x \text{ in row } i,\ y \text{ in row } j\ ,$$

and so

$$n | g_i g_j \text{ for all } 1 \leq i < j \leq k\ . \tag{2.1}$$

For each prime p_t dividing n, let l_t be the greatest integer such that $p_t^{l_t} | g_j$ for each j, and choose c_t so that $p_t^{l_t}$ exactly divides g_{c_t}. (By exactly divides we mean $p_t^{l_t} | g_{c_t}$, but $p_t^{l_t+1} \not| g_{c_t}$. Note that c_t is not necessarily uniquely determined.) Then, by (2.1), we have $p_t^{m_t - l_t}$ divides g_j for $j \neq c_t$. If $p_t^{m_t - l_t}$ exactly divides g_j for $j \neq c_t$, and furthermore if $s_j = n/g_j$ for each $j \in \{1, 2, \ldots k\}$, then we call the OMEP *tight*. In this case, since $p_t^{l_t} | g_j$ for each j, and $p_t^{m_t - l_t} | g_j$ exactly for $j \neq c_t$, we have $l_t \leq m_t - l_t$ and so $l_t \leq m_t/2$. Also observe that a tight OMEP is equally replicated, since $s_j = n/g_j$ for such OMEPs and this forces $r_{jx} = g_j$ for each x..

If $\mathcal{D}$ itself is not tight, then the l_t's and the c_t's still exist, and these determine the parameter set of a tight OMEP, $s'_1 \times s'_2 \times \ldots \times s'_k // n$, where

$$g'_i = r'_{ix} = \prod_{t:c_t \neq i} p_t^{m_t - l_t} \prod_{t:c_t = i} p_t^{l_t}, \text{ and } s'_i = n/g'_i. \tag{2.2}$$

Note that $s_i \leq s'_i$ for each i, since $g_i \geq g'_i$ for each i. Hence, if this tight OMEP exists, then an OMEP with the same parameters as $\mathcal{D}$ can be obtained by collapsing levels in the tight OMEP. We state this formally.

Theorem 2.1.1 *Given OMEP parameters $s_1 \times s_2 \times \ldots \times s_k // n$ with associated replication numbers r_{ix}, there exists a tight parameter set $s'_1 \times s'_2 \times \ldots \times s'_k // n$ with associated replication numbers r'_{ix}, so that if an OMEP with parameters $s'_1 \times s'_2 \times \ldots \times s'_k // n$ and replication numbers r'_{ix} exists, then it can have levels collapsed to obtain an OMEP with the original parameters and replication numbers.*

In fact the above analysis holds whether or not any of the OMEPs actually exist, so in fact any set of OMEP parameters and replication numbers must come from a tight set of parameters and replication numbers by "collapsing the tight parameter set".

For example, consider the $2 \times 3 \times 4//16$ OMEP in Table 1.4. In this case we have $n = 16 = 2^4$, so $r_{11} = r_{12} = 8$, and thus $g_1 = \gcd(8,8) = 8$. Similarly we find $g_2 = 4$, and $g_3 = 4$. In this case, the only prime dividing n is 2, and 2^2 divides each g_i, and it is the largest power of 2 to do so. Also, 2^2 exactly divides g_2. Thus we find $p_1 = 2, l_1 = 2$, and we can take $c_1 = 2$. Now, for the OMEP to be tight we require 2^{4-2} to exactly divide g_1 and g_3. Since $g_1 = 8$ this is not the case. However, from equations 2.2 we find $g_1' = g_2' = g_3' = 4$, and so if we can find an OMEP with $g_1' = g_2' = g_3' = 4$ and $s_1 = s_2 = s_3 = 16/4 = 4$ then it would be a tight OMEP and furthermore we could collapse levels in it top obtain a $2 \times 3 \times 4//16$ OMEP. Of course such an OMEP exists; it is an OA(3, 4)! As promised, this OA(3, 4) can indeed be collapsed to obtain a $2 \times 3 \times 4//16$ OMEP (i.e. an OMEP with the same parameters as our original).

Equations 2.2 provide a method of simply describing a tight parameter set. Suppose $s_1' \times s_2' \times \ldots \times s_k'//n$ is a tight parameter set. Since $s_i' = n/g_i'$ for such OMEPs, $p_t^{l_t}$ divides s_i' for $i \neq c_t$, and $p_t^{m_t - l_t}$ divides s_i' for $i = c_t$. Since $m_t - l_t \geq l_t$ for each t, if we set $q_t = p_t^{l_t}$, then q_t divides each s_i', and $p_t^{m_t - 2l_t} q_t$ divides exactly one s_i'. This is true for each prime divisor of n, and hence for each possible divisor of any s_i'. It follows that a tight OMEP parameter set can be written as

$$\lambda_1 g \times \lambda_2 g \ldots \times \lambda_k g // \lambda_1 \ldots \lambda_k g^2, \tag{2.3}$$

where g is the product of the q_t's, $\lambda_i = \prod_{t:c_t=i} p_t^{m_t - 2l_t}$. The λ_i's are pairwise relatively prime.

However, not every OMEP with these parameters is a tight OMEP. (We refer the reader ahead to Theorem 2.2.1 and Theorem 2.1.2, which we use here.) For example, take $g = 1, k = 3$ and $\lambda_i = p_i^3$, where $p_1 = 2, p_2 = 3, p_3 = 5$. Then for each $i = 1, 2, 3$ an equally replicated $p_i^2 \times p_i \times p_i // p_i^3$ OMEP exists. By Theorem 2.2.1 and Theorem 2.1.2, a tight $p_1^2p_2p_3 \times p_1p_2^2p_3 \times p_1p_2p_3^2 // p_1^3p_2^3p_3^3$ OMEP exists. Now, since the product of any two of these primes is larger than the third prime, we can collapse levels in this OMEP to obtain a $p_1^3 \times p_2^3 \times p_3^3 // p_1^3p_2^3p_3^3$ OMEP, with $g_1 = p_1p_2^2p_3^2, g_2 = p_1^2p_2p_3^2, g_3 = p_1^2p_2^2p_3$. This OMEP has parameters as in (2.3), but it is not tight, since $g_i \neq n/s_i$ (we do not have equal replication).

If an OMEP has parameters as in 2.3, and in addition the OMEP has equal replication, then indeed the OMEP is tight.

Theorem 2.1.2 *An equally replicated OMEP with parameters*

$$\lambda_1 g \times \lambda_2 g \times \ldots \times \lambda_k g // \lambda_1 \lambda_2 \ldots \lambda_k g^2$$

and with the λ_i pairwise relatively prime is a tight OMEP.

Proof: To conform with the notation used in the definition of tight OMEPs, let $s_i =$ the number of symbols in row $i = \lambda_i g$, and let $n =$ the number of columns $= \lambda_1\lambda_2 \ldots \lambda_k g^2$. Since the OMEP is equally replicated, we have

$$r_{ix} = n/s_i = \frac{\lambda_1\lambda_2 \ldots \lambda_k g}{\lambda_i},$$

and so

$$g_i = \gcd\{r_{ix} | x \text{ in row } i\} = \frac{\lambda_1\lambda_2 \ldots \lambda_k g}{\lambda_i}$$

also. Suppose $p_1^{m_1} p_2^{m_2} \ldots p_d^{m_d}$ is the prime power factorization n. Equal replication ensures that $s_i = n/g_i$ for each i. All that remains is to show for each t, $p^{m_t - l_t}$

exactly divides g_j for $j \neq c_t$, where as before l_t is the greatest integer such that $p_t^{l_t} | g_j$ for each j, and c_t is such that $p_t^{l_t}$ exactly divides g_{c_t}. Suppose p_t^a is the largest power of p_t dividing g, and that p_t^b is the largest power of p_t dividing any λ_i, say λ_{i^*}. ($b = 0$ is allowed, but if $b > 0$ then since the λ_i's are pairwise relatively prime, λ_{i^*} is the only λ_i divided by p_t.) Since $n = \lambda_1 \lambda_2 \ldots \lambda_k g^2$, we have $m_t = 2a + b$. Since $\gcd\{g_i | 1 \leq i \leq k\} = g$, we see that $l_t = a$. Further, p_t^a exactly divides g_{i^*}, so we can take $c_t = i^*$. Thus for $i \neq c_t$, g_i has a factor $\lambda_{c_t} g$, and no other factors of g_i are divisible by p_t. Therefore $p_t^{m_t - l_t} = p_t^{a+b}$ exactly divides g_i for $i \neq c_t$, and so the OMEP is tight.

□

The idea that should be emphasized here is that tight OMEP parameters are as "uncollapsed" as possible. In other words, there is no OMEP in which we could collapse levels to obtain a tight OMEP.

Sometimes it is desirable that an OMEP have equal replication, and so $r_{ix} = r_{iy}$ for every pair of symbols x and y in each row i. Tight OMEPs have this property since $r_{ix} = g_i$ for such OMEPs. But in fact tight OMEPs have a connection with equally replicated OMEPs similar to their connection with the usual notion of OMEPs. Consider any $s_1 \times s_2 \times \ldots \times s_k // n$ OMEP with equal replication. Then $r_{ix} = g_i$ for each i. As above, the replication numbers of this OMEP determine a tight parameter set, say $s'_1 \times s'_2 \times \ldots \times s'_k // n$. Since $g'_i | g_i$ for each i, and we have $n = s_i g_i = s'_i g'_i$, it must be that $s_i | s'_i$ for each i. Hence, if the corresponding tight OMEP exists, we can collapse levels in it to obtain an equally replicated OMEP with the same parameters as the original OMEP. Again we state this result formally.

Theorem 2.1.3 *Given parameters $s_1 \times s_2 \times \ldots \times s_k // n$ with associated replication numbers r_{ix} of an equally replicated OMEP, there exists a tight parameter set $s'_1 \times$*

$s'_2 \times \ldots \times s'_k//n$ with associated replication numbers r'_{ix}, so that if an OMEP with parameters $s'_1 \times s'_2 \times \ldots \times s'_k//n$ and replication numbers r'_{ix} exists, then it can have levels collapsed to obtain an equally replicated OMEP with the original parameters and replication numbers.

Given $n = p_1^{m_1} p_2^{m_2} \ldots p_t^{m_t}$, we can determine the parameters $s_1 \times s_2 \times \ldots \times s_k$ of all possible tight OMEPs on n columns, since for tight OMEPs these parameters are determined by the l_t's and the c_t's, and there are only finitely many possibilities for each of these. In particular, there are only $\lceil m_t/2 \rceil$ possibilities for l_t, and k possibilities for c_t. Some of these choices may give rise to the same parameters $s_1 \times s_2 \times \ldots \times s_k$, but with the s_i's possibly reordered.

For example, let us compute the possible tight parameter sets for a three-factor OMEP with $n = 24 = 2^3 3^1$. Now l_1 can be 0 or 1, and l_2 must be 0. Both c_1 and c_2 can be any value in $\{1, 2, 3\}$. If $l_1 = 1$, $c_1 = 1$, and $c_2 = 2$, say, then $g_1 = 2 \cdot 3$, $g_2 = 2^2$, and $g_3 = 2^2 \cdot 3$. Hence the corresponding tight parameter set is $4 \times 6 \times 2//24$. In this way, we find the possible parameter sets (reordered so that $s_1 \geq s_2 \geq s_3$) are: $24 \times 1 \times 1//24$, $8 \times 3 \times 1//24$, $12 \times 2 \times 2//24$, and $6 \times 4 \times 2//24$.

In the next section we give an application of tight OMEPs.

2.2 Existence of Tight OMEPs on Four Factors

In this section, we determine the tight parameter sets for which there corresponds a tight OMEP with four rows. By using this information, we then give a method for determining the minimal n for which an $s_1 \times s_2 \times s_3 \times s_4//n$ OMEP exists. This is the first explicit method for finding the minimal n in the case of four factor OMEPs, and so it further suggests that tight OMEPs are a useful concept. Furthermore, the method is easily generalizable to OMEPs on more factors, and it can also be used to determine the minimal n for which an $s_1 \times s_2 \times s_3 \times s_4//n$ OMEP with equal replication exists.

We first introduce some recursive constructions for OMEPs, and we prove some existence results about tight OMEPs on four factors.

As mentioned in the introduction, direct product type constructions for combinatorial designs have been studied by many authors. Here is a direct product construction for OMEPs, which is credited to Adhikary and Das [2].

Theorem 2.2.1 (**Direct Product Construction**) *If an $s_1 \times s_2 \times \ldots \times s_k//n$ OMEP exists, and an $s'_1 \times s'_2 \times \ldots \times s'_k//n'$ OMEP exists, then an $s_1s'_1 \times s_2s'_2 \times \ldots \times s_ks'_k//nn'$ OMEP exists.*

Proof: The direct product of the first two OMEPs gives the third OMEP. For a fixed row, and a symbol x in this row in the first OMEP and a symbol x' in this row in the second OMEP, there is a symbol (x, x') in the resultant OMEP. Furthermore, if the replication number of x is r_{ix}, and the replication number of x' is $r'_{ix'}$, then the replication number of (x, x') is $r_{ix}r'_{ix'}$. For distinct rows i, j in the resultant OMEP, and a symbol (x, x') in row i and a symbol (y, y') in row j, the number of columns in the resultant OMEP containing the symbol (x, x') in row i and symbol (y, y') in row j is the number of columns in the first OMEP containing symbol x in row i and symbol y in row j multiplied by the number of columns in the second OMEP containing symbol x' in row i and symbol y' in row j. Since the first two arrays are OMEPs, then by definition this product equals

$$(r_{ix}r_{jy}/n)(r'_{ix'}r'_{jy'}/n') = (r_{ix}r'_{ix'})(r_{iy}r'_{iy'})/(nn') = r_{i(x,x')}r_{j(y,y')}/(nn').$$

This last expression is precisely the product of the replication numbers of symbol (x, x') in row i and a symbol (y, y') divided by the number of columns of the resultant array. Thus by definition the resultant array is an OMEP. □

Notice that the direct product construction preserves equal replication, since the replication number of a symbol in the resulting OMEP is the product of the replication numbers of the associated symbols in the ingredient OMEPs.

Theorem 2.2.2 (**Concatenation Construction**) *Suppose $\mathcal{D}$ is an $s_1 \times s_2 \times \ldots \times s_k//n$ OMEP, and $\mathcal{D}'$ is an $s_1 \times s_2 \times \ldots \times s_{k-1} \times s'_k//n'$ OMEP, with replication numbers r_{ix} and r'_{jy} respectively. Further suppose that these OMEPs have the same symbol sets in the first $k-1$ rows, $r_{ix}/n = r'_{ix}/n'$ when $1 \leq i \leq k-1$, and for the remaining row, the symbols in the first OMEP are all different from the symbols of the second OMEP. Then the concatenation of these matrices is an $s_1 \times s_2 \times \ldots \times s_{k-1} \times (s_k + s'_k)//(n + n')$ OMEP.*

Proof: Let $\mathcal{M}$ be the k by $n + n'$ array obtained by juxtaposing the two OMEPs. Consider $1 \leq i \neq j \leq k$, and a symbol x in row i and a symbol y in row j. The number of columns in $\mathcal{M}$ that have an x in row i and a y in row j is

$$n\frac{r_{ix}}{n}\frac{r_{jy}}{n} + n'\frac{r'_{ix}}{n'}\frac{r'_{jy}}{n'}.$$

For $\mathcal{M}$ to be an OMEP, this must equal

$$(n+n')\left[\frac{(r_{ix}+r'_{ix})}{(n+n')}\frac{(r_{jy}+r'_{jy})}{(n+n')}\right].$$

Since $i \neq j$, we may assume without loss of generality that $r_{ix}/n = r'_{ix}/n'$. Then $(r_{ix} + r_{ix})/(n + n') = r_{ix}/n$ and so the above equation reduces to the first. □

Corollary 2.2.3 *If there is an equally replicated $s_1 \times s_2 \times \ldots \times s_{k-1} \times s_k//n_1$ OMEP and an equally replicated $s_1 \times s_2 \times \ldots \times s_{k-1} \times s'_k//n_2$ OMEP, then there exists an equally replicated $s_1 \times s_2 \times \ldots \times s_{k-1} \times (s_k + s'_k)//n_1 n_2$ OMEP.*

Proof: Equally replicated OMEPs have $r_{ix}/n = 1/s_i$ so if they have the same number of symbols in the first $k-1$ rows the conditions of Theorem 2.2.2 are satisfied. □

Since tight OMEPs are equally replicated, the corollary applies to tight OMEPs also. We use this corollary extensively later.

We know that every tight OMEP on four rows has parameters of the form $\lambda_1 g \times \lambda_2 g \times \lambda_3 g \times \lambda_4 g // \lambda_1\lambda_2\lambda_3\lambda_4 g^2$. Thus it is natural to consider cases based on the value of g. Given that we are breaking the cases up in this way, there are some simple observations that considerably eases our work.

Lemma 2.2.4 *If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are pairwise relatively prime, then a tight $\lambda_1 \times \lambda_2 \times \ldots \times \lambda_k // \lambda_1\lambda_2 \ldots \lambda_k$ OMEP exists.*

Proof: For each i we have a $1^{i-1} \times \lambda_i \times 1^{k-i} // \lambda_i$ OMEP since it is simply a $k \times \lambda_i$ array consisting of $k-1$ rows containing a single symbol and another row containing λ_i distinct symbols. We now apply the direct product construction to k such OMEPs (one for each λ_i) to obtain an OMEP $\mathcal{M}$. $\mathcal{M}$ is equally replicated, since it is the direct product of equally replicated OMEPs, and it is tight since it is equally replicated and has the parameters of a tight OMEP. □

Lemma 2.2.5 *If an $OA(k, g)$ exists, then all tight OMEPs with k rows and with parameters $\lambda_1 g \times \lambda_2 g \ldots \lambda_k g // \lambda_1 \ldots \lambda_k g^2$ exist.*

Proof: An OA(k, g) is an equally replicated $g \times g \times \ldots \times g // g^2$ tight OMEP (with k rows), and an equally replicated $\lambda_1 \times \lambda_2 \times \ldots \times \lambda_k // \lambda_1 \ldots \lambda_k$ tight OMEP exists by the above lemma. The direct product of these two OMEPs is an equally replicated

OMEP having the desired parameters, which are the parameters of a tight OMEP, and so this OMEP is the desired OMEP. □

Armed with this lemma, we see that for four-factor OMEPs, the only cases we must further consider are when $g = 2, 6$, since, for all other values of g, an OA$(4, g)$ exists (see [3] for example), and so Lemma 2.2.5 applies. Let us consider the case $g = 2$ first.

Any tight OMEP parameter set with $g = 2$ has the form $2\lambda_1 \times 2\lambda_2 \times 2\lambda_3 \times 2\lambda_4 // 4\lambda_1\lambda_2\lambda_3\lambda_4$, with the λ_i's pairwise relatively prime. If all λ_i's are 1, then the OMEP would correspond to an OA$(4, 2)$ which does not exist for trivial reasons. Before considering the other cases, we first prove some lemmas.

Lemma 2.2.6 *An OMEP with parameters* $2 \times 2 \times 2 \times 2s//4s$ *does not exist if* s *is odd.*

Proof: Suppose to the contrary that such an OMEP exists. Since the only tight parameter set, $s_1 \times s_2 \times s_3 \times s_4//4s$, with $s_1, s_2, s_3 \geq 2$ and $s_4 \geq 2s$ is in fact $2 \times 2 \times 2 \times 2s//4s$, we see that if this OMEP exists it must be tight, and so the replication numbers of the symbols in the first three rows is $2s$. We may assume the symbols in each of the first three rows are 0 and 1. Let a_{ijk} be the number of columns with an i in row 1, a j in row 2, and a k in row 3, and as usual let r_{ij} be the number of times symbol j occurs in row i. By definition $r_{10}r_{20}/4s$ equals the number of columns containing a 0 in rows 1 and 2, which is also the value of $a_{000} + a_{001}$. Since $r_{10} = r_{20} = 2s$ we see this value is also s. Proceeding in this way we find that the sum

$$\frac{r_{10}r_{20}}{4s} - \frac{r_{10}r_{31}}{4s} + \frac{r_{21}r_{31}}{4s}$$

evaluates to

$$(a_{000} + a_{001}) - (a_{001} + a_{011}) + (a_{011} + a_{111}) = s,$$

and so

$$a_{000} + a_{111} = s.$$

Every symbol in the fourth row appears twice, and the columns above it are complementary (their components sum to 1 mod 2). Thus if we restrict the OMEP to the first three rows, we can pair up the columns into pairs which are complementary. However the above formula reduces to $a_{000} + a_{111} = s$, and since s is odd, this means we cannot pair up the corresponding columns. This is a contradiction. □

The reader familiar with transversal designs should observe that this last lemma is simply a statement that a $\mathrm{RTD}_\lambda(3,2)$ cannot exist for odd λ.

Lemma 2.2.7 *An equally replicated* $2s_1 \times 2s_2 \times 2 \times 2//4s_1s_2$ *OMEP exists for all odd* $s_1, s_2 \geq 3$.

Proof: We can collapse levels in a $6 \times 6 \times 6 \times 5//36$ OMEP to obtain a equally replicated $6 \times 6 \times 2 \times 2//36$ OMEP, such that for each i, $r_{ix}/36 = 1/s_i$ for each symbol x in row i. Using this OMEP, and an equally replicated $4 \times 6 \times 2 \times 2//24$ OMEP, and Theorem 2.2.2, it follows inductively that we can construct an equally replicated $2s_1 \times 6 \times 2 \times 2//12s_1$ OMEP for odd $s_1 \geq 3$. But now using this OMEP, an equally replicated $2s_1 \times 4 \times 2 \times 2//8s_1$ OMEP, and Theorem 2.2.2, the result follows inductively. □

Lemma 2.2.8 *A* $2 \times 2 \times 2 \times 4//8$ *tight OMEP exists.*

Proof: The following array is such an OMEP.

$$\begin{array}{cccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \end{array}$$

□

The tight parameter sets with $g = 2$ are handled by these last three lemmas, as the following argument shows. Again recall the general form of the parameters in this case is $2\lambda_1 \times 2\lambda_2 \times 2\lambda_3 \times 2\lambda_4 // 4\lambda_1\lambda_2\lambda_3\lambda_4$, and furthermore the λ_i's are pairwise relatively prime. Since when all $\lambda_i = 1$ the OMEP cannot exist, we assume that at least one λ_i, say λ_1, is greater than 1. Suppose some λ_i is even, say $\lambda_4 = 2\lambda_4'$. Then a $2 \times 2 \times 2 \times 4//8$ tight OMEP exists (by Lemma 2.2.8), and an equally replicated $\lambda_1 \times \lambda_2 \times \lambda_3 \times \lambda_4' // \lambda_1\lambda_2\lambda_3\lambda_4'$ tight OMEP exists by Lemma 2.2.4, and the direct product of these OMEPs gives an OMEP with the desired parameters. This OMEP is tight by Theorem 2.1.2. Next suppose all the λ_i's are odd. If three of the λ_i's are 1, then this case is handled by Lemma 2.2.6 and the tight OMEP cannot exist. Otherwise, at least two of the λ_i's are greater than or equal to 3, say λ_1 and λ_2, and the corresponding OMEP exists, since it can be obtained by using Theorem 2.2.1 with an equally replicated $2\lambda_1 \times 2\lambda_2 \times 2 \times 2//4\lambda_1\lambda_2$ OMEP (exists by Lemma 2.2.7), and a $1 \times 1 \times \lambda_3 \times \lambda_4 // \lambda_3\lambda_4$ tight OMEP, and by applying Theorem 2.1.2.

We now turn to the case $g = 6$, where the analysis is similar to the case $g = 2$. In this case the general form of the parameters of a tight OMEP are $6\lambda_1 \times 6\lambda_2 \times 6\lambda_3 \times 6\lambda_4 // 36\lambda_1\lambda_2\lambda_3\lambda_4$, with the λ_i's pairwise relatively prime. Again, if all the λ_i's are 1, then the tight OMEP does not exist as it corresponds to an $OA(4, 6)$ which is known not to exist, though this fact is not obvious (see [3], for example). Next suppose some λ_i is even, say $\lambda_1 = 2\lambda_1'$. Since a $12 \times 6 \times 6 \times 6//72$ tight OMEP exists (take direct product of $4\times2\times2\times2//8$ and $3\times3\times3\times3//9$ tight OMEPs and apply Theorem 2.1.2), and a $\lambda_1' \times \lambda_2 \times \lambda_3 \times \lambda_4//\lambda_1'\lambda_2\lambda_3\lambda_4$ tight OMEP exists (Lemma 2.2.4), we see their direct product is an equally replicated $6\lambda_1 \times 6\lambda_2 \times 6\lambda_3 \times 6\lambda_4//36\lambda_1\lambda_2\lambda_3\lambda_4$ OMEP, which is tight by Theorem 2.1.2. Thus the only remaining case is with all λ_i's odd, and at least one $\lambda_i \geq 3$. To finish this case we need the following fact.

Proposition 2.2.9 *An* $18 \times 6 \times 6 \times 6//108$ *tight OMEP exists.*

Proof: We give a completely resolvable $OA_3(3,6)$, (an orthogonal array with 6 symbols, 3 rows, and $108 = 3 \cdot 6^3$ columns), from which you just extend the 18 parallel classes to get the desired OMEP. The solution is cyclic modulo 5, with one fixed point x in each row. The first five parallel classes are obtained by developing the following parallel class modulo 5, where x is fixed.

0	1	x	3	4	2
0	2	4	1	x	3
0	x	1	4	2	3

The next five parallel classes are obtained by developing the following parallel class modulo 5.

0	1	2	x	4	3
x	2	4	1	3	0
0	3	1	4	x	2

The next five parallel classes are obtained by developing the following parallel class modulo 5.

0	1	2	x	4	3
0	x	4	1	3	2
0	3	x	4	2	1

The sixteenth, seventeenth, and eighteenth parallel classes are

0	1	2	3	4	x	0	1	2	3	4	x	0	1	2	3	4	x
0	1	2	3	4	x	3	4	0	1	2	x	3	4	0	1	2	x
1	2	3	4	0	x	4	0	1	2	3	x	2	3	4	0	1	x

Verification that this gives the desired OMEP is routine. □

Lemma 2.2.10 *An equally replicated* $6\lambda_1 \times 6 \times 6 \times 6//36\lambda_1$ *OMEP exists for all odd* $\lambda_1 \geq 3$.

Proof: Using the above $18 \times 6 \times 6 \times 6//108$ tight OMEP, a $12 \times 6 \times 6 \times 6//72$ tight OMEP, and Theorem 2.2.2, we can construct an equally replicated $6(s+2) \times 6 \times 6 \times 6//36(s+2)$ OMEP from an equally replicated $6s \times 6 \times 6 \times 6//36s$ OMEP. The result now follows inductively. □

It now follows that any tight $6\lambda_1 \times 6\lambda_2 \times 6\lambda_3 \times 6\lambda_4//36\lambda_1\lambda_2\lambda_3\lambda_4$ OMEP with all λ_i's odd and at least one $\lambda_i \geq 3$ (say λ_1) exists since it can be obtained by applying Theorem 2.1.2 to the direct product of an equally replicated $6\lambda_1 \times 6 \times 6 \times 6//36\lambda_1$ OMEP and a equally replicated $1 \times \lambda_2 \times \lambda_3 \times \lambda_4//\lambda_2\lambda_3\lambda_4$ OMEP.

Thus we have shown the following result.

Result 2.2.11 *The only tight four-factor OMEPs that do not exist have parameters* $6 \times 6 \times 6 \times 6//36$ *or* $2 \times 2 \times 2 \times 2s//4s$ *for* s *odd.*

We now wish to apply these results about tight OMEPs to the general existence question for four-factor OMEPs. The general theme is to show existence of an OMEP with a given parameter set by collapsing some tight OMEP. However we must be careful in those cases where the tight OMEP we would "want" to collapse does not exist. The following results help in these cases.

Corollary 2.2.12 *The minimal n for which a $2 \times 2 \times 2 \times s//n$ OMEP exists, with $s \geq 2$, is $n = 2(s+1)$ for $s \equiv 1, 3 \pmod 4$, $n = 2(s+2)$ for $s \equiv 2 \pmod 4$, and $n = 2s$ for $s \equiv 0 \pmod 4$.*

Proof: If $s \equiv 0 \pmod 4$, then $s = 4s'$, and we can apply the direct product construction using a $2 \times 2 \times 2 \times 4//8$ OMEP and a $1 \times 1 \times 1 \times s'//s'$ OMEP as ingredient OMEPs to obtain a $2 \times 2 \times 2 \times s//2s$ OMEP.

If $s \equiv 1 \pmod 4$, then every tight OMEP on $2s$ columns exists by Result 2.2.11, but by inspection none of them can have levels collapsed to obtain a $2 \times 2 \times 2 \times s//2s$ OMEP. Hence, $n \geq 2(s+1)$. However, for each $t \geq 1$, we can concatenate an equally replicated $2 \times 2 \times 2 \times 1//4$ OMEP and t copies of an equally replicated $2 \times 2 \times 2 \times 4//8$ OMEP to obtain a $2 \times 2 \times 2 \times (4t+1)//2(4t+2)$ OMEP. Hence, a $2 \times 2 \times 2 \times s//2(s+1)$ OMEP exists and so $n = 2(s+1)$ is minimal in this case.

If $s \equiv 3 \pmod 4$, then every tight OMEP on $2s$ columns exists by Result 2.2.11, but by inspection none of them can have levels collapsed to obtain a $2 \times 2 \times 2 \times s//2s$ OMEP. Hence, $n \geq 2(s+1)$. But then as $s+1 \equiv 0 \pmod 4$, a $2 \times 2 \times 2 \times (s+1)//2(s+1)$ OMEP exists, and so we can collapse levels to obtain the desired OMEP.

Finally, if $s \equiv 2 \pmod 4$, then Theorem 2.2.6 implies $n > 2s$. As $n = (2+i)(s+j)$ for some nonnegative integers i, j, the next possible value is $n = 2(s+1)$. However all tight OMEPs on $2(s+1)$ columns exist and none can have levels collapsed to obtain the desired OMEP. So in fact we have $n \geq 2(s+2)$. As $s+2 \equiv 0 \pmod 4$, there exists a $2 \times 2 \times 2 \times (s+2)//2(s+2)$ OMEP, in which we can collapse levels to obtain a $2 \times 2 \times 2 \times s//2(s+2)$ OMEP. □

Proposition 2.2.13 *The minimal n for which a $6 \times 6 \times 6 \times 6//n$ OMEP exists is $n = 49$.*

Proof: We know a $6 \times 6 \times 6 \times 6//36$ OMEP does not exist. By considering the tight parameter sets with $37 \leq n \leq 48$ we find no possible OMEP on this number of columns could be collapsed to obtain a $6 \times 6 \times 6 \times 6//36$ OMEP. The next possible value for n is $7^2 = 49$, and since a $7 \times 7 \times 7 \times 7//49$ OMEP exists we can collapse levels in it to obtain the desired OMEP. □

Proposition 2.2.14 *A* $6 \times 6 \times 6 \times 5//36$ *OMEP exists.*

Proof: Euler found a pair of latin squares L_1, L_2 of order 6, having a common 2×2 subsquare, but otherwise orthogonal. Suppose the symbols in the subsquare are x and y. Then by identifying x and y in L_2, and constructing the corresponding matrix (with columns $(i, j, L_1[i,j], L_2[i,j])$), we get a $6 \times 6 \times 6 \times 5//36$ OMEP. □

We also want to apply knowledge about tight OMEPs to answer the existence question about equally replicated OMEPs, and so we also need the next propositions.

Corollary 2.2.15 *The minimal n for which an equally replicated* $2 \times 2 \times 2 \times s//n$ *OMEP exists, with* $s \geq 2$, *is* $n = 4s$ *for* $s \equiv 1,3 \pmod 4$, $n = 4s$ *for* $s \equiv 2 \pmod 4$, *and* $n = 2s$ *for* $s \equiv 0 \pmod 4$.

Proof: Suppose s is odd. Since we desire equal replication, we have $2|n$ and $s|n$ and so $2s|n$. From Corollary 2.2.12, $n \neq 2s$. However, using the direct product construction, a tight (and so equally replicated) $2 \times 2 \times 2 \times 1//4$ OMEP, and a (equally replicated) $1 \times 1 \times 1 \times s//s$ OMEP gives a $2 \times 2 \times 2 \times s//4s$ OMEP, so $n = 4s$ is minimal in this case. If $s \equiv 0 \pmod 4$, then a tight $2 \times 2 \times 2 \times s//2s$ exists and so the result follows in this case. Consider the case $s \equiv 2 \pmod 4$. Since we want an equally replicated OMEP, and $s_1 = s_2 = 2$, we must have $4|n$.

Hence $2s|n$. We know $n \neq 2s$, by Lemma 2.2.6, and so $n \geq 2(2s) = 4s$. Let $s = 2s'$. Using a tight $2 \times 2 \times 2 \times 4//8$ OMEP, a tight $1 \times 1 \times 1 \times s'//s'$ OMEP, and the direct product construction yields a $2 \times 2 \times 2 \times 2s//4s$, and so $n = 4s$ is minimal in this case. □

Proposition 2.2.16 *The minimal n for which a $6 \times 6 \times 6 \times 6//n$ OMEP with equal replication exists is $n = 72$.*

Proof: Any equally replicated $s_1 \times s_2 \times s_3 \times s_4//n$ with $6|s_i$ for each i has $9|n$ and $4|n$ and so $36|n$. We know $n > 36$, since a $6 \times 6 \times 6 \times 6//36$ OMEP does not exist, but using Theorem 2.2.1, an equally replicated $2 \times 2 \times 2 \times 4//8$ OMEP, and a equally replicated $3 \times 3 \times 3 \times 3//9$ OMEP (both of which exist) we obtain an equally replicated $6 \times 6 \times 6 \times 12//72$ OMEP. This can have levels collapsed to obtain the desired OMEP. □

Proposition 2.2.17 *The minimal n for which an equally replicated $6 \times 6 \times 6 \times s//n$ exists, $s = 2, 3, 4, 5$, is (respectively) $n = 36, 36, 72, 180$.*

Proof: An equally replicated $6 \times 6 \times 6 \times 2//36$ OMEP and an equally replicated $6 \times 6 \times 6 \times 3//36$ OMEP can be obtained by collapsing levels in a $6 \times 6 \times 6 \times 5//36$ OMEP. In these cases we need at least $6 \cdot 6 = 36$ columns, so $n = 36$ is minimal in these cases. For an equally replicated $6 \times 6 \times 6 \times 4//n$ OMEP, we need $n|(n/6)(n/6)$ and $n|(n/6)(n/4)$. Hence $36|n$, and $24|n$, and so $\text{lcm}(24, 36)|n$ which implies $72|n$. However, $n = 72$ suffices, since a $6 \times 6 \times 6 \times 4//72$ OMEP can be obtained as the direct product of an equally replicated $2 \times 2 \times 2 \times 4//8$ OMEP (Lemma 2.2.8) and an equally replicated $3 \times 3 \times 3 \times 3//9$ OMEP (an OA(4, 3)). Similarly, an equally replicated $6 \times 6 \times 6 \times 5//n$ OMEP must have $\text{lcm}(36, 30)|n$, and so $180|n$. Again,

$n = 180$ suffices, since the desired OMEP can be obtained as the direct product of an equally replicated $6 \times 6 \times 6 \times 1//36$ OMEP (an OA(3,6) with a row of all 1's added) and an equally replicated $1 \times 1 \times 1 \times 5//5$ OMEP (Lemma 2.2.4). □

2.3 Existence for general four-factor OMEPs

We now show how tight OMEPs go a long way in answering the existence question for general OMEPs. We give algorithms for determining the minimal n for which four-factor OMEPS and four-factor OMEPs with equal replication exist.

In the introduction we saw that any $s_1 \times s_2 \times s_3 \times s_4//n$ OMEP gives rise to a tight parameter set $s'_1 \times s'_2 \times s'_3 \times s'_4//n$, where $s_i \leq s'_i$ for each i. Furthermore, we saw that if the original OMEP had equal replication, then we in fact have $s_i | s'_i$. Hence in both cases, if the corresponding tight OMEP exists, then it can have levels collapsed to obtain the original OMEP. Since we know exactly when tight OMEPs on four factors exist, it is not surprising that two rather trivial algorithms work.

Suppose we are given s_1, s_2, s_3, s_4, and we do not require an OMEP with equal replication. Since we know how to generate all the tight parameter sets for a given n, we can find the smallest n so that there is a tight parameter set, $\mathcal{P}'$: $s'_1 \times s'_2 \times s'_3 \times s'_4//n$, with $s_i \leq s'_i$. Now, if the corresponding OMEP exists then n is minimal and we are done. Otherwise, there are only two possibilities for $\mathcal{P}'$: $2 \times 2 \times 2 \times 2s//4s$, with s odd, or $6 \times 6 \times 6 \times 6//36$. In the first case, it must be that s_1, s_2, s_3, s_4 actually equals $2, 2, 2, 2s$ in some order, in which case Theorem 2.2.12 applies. In the second case, either each $s_i = 6$, in which case we know $n = 7^2$ is minimal, otherwise $n = 36$ is minimal since an OMEP with the desired parameters can be obtained by collapsing levels in a $6 \times 6 \times 6 \times 5//36$ OMEP.

Suppose we are given s_1, s_2, s_3, s_4, and we do require an OMEP with equal replication. Since we know how to generate all the tight parameter sets for a given n, we can find the smallest n so that there is a tight parameter set, $\mathcal{P}'$: $s'_1 \times s'_2 \times s'_3 \times s'_4//n$, with $s_i|s'_i$. Now, if the corresponding OMEP exists then n is minimal and we are done. Otherwise, there are only two possibilities for $\mathcal{P}'$: $2 \times 2 \times 2 \times 2s//4s$, with s odd, or $6 \times 6 \times 6 \times 6//36$. In the first case, it must be that $s_1 = s_2 = s_3 = 2$ in which case Theorem 2.2.15 applies. In the second case, either each $s_i = 6$, in which case we know $n = 72$ is minimal, otherwise $n = 36$ is minimal since an equally replicated OMEP with the desired parameters can be obtained by collapsing levels in a $6 \times 6 \times 6 \times 5//36$ OMEP.

2.4 Five-Factor Tight OMEPs

In this brief section we determine all the tight parameter sets on five factors for which a tight OMEP exists.

As in the last section, we assume the tight parameter set has the form

$$\lambda_1 g \times \lambda_2 g \times \lambda_3 g \times \lambda_4 g \times \lambda_5 g//\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 g^2. \tag{2.4}$$

Also, we break up the cases depending on the value of g. A TD$(5, g)$ exists for any $g \notin \{2, 3, 6, 10\}$, so for any such g and any choice of the λ_i's a tight OMEP exists having these parameters, since it can be obtained as the direct product of a $g \times g \times g \times g \times g//g^2$ OMEP and a $\lambda_1 \times \lambda_2 \times \lambda_3 \times \lambda_4 \times \lambda_5//\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5$ OMEP.

Consider the case $g = 10$. It is unknown whether a $10^5//100$ OMEP exists, and we make no attempt to prove or disprove its existence here. In the Appendix, we give a resolvable TD$_3(4, 10)$, so by extending parallel classes we obtain an equally replicated $10 \times 10 \times 10 \times 10 \times 30//300$ OMEP. The direct product of an equally replicated $2^4 \times 4//8$ OMEP and an equally replicated $5^5//25$ OMEP is an equally replicated $10 \times 10 \times 10 \times 10 \times 20//200$ OMEP. By extending parallel classes and applying the concatenation construction we obtain an equally replicated $10 \times 10 \times 10 \times 10 \times 10\lambda//100\lambda$ OMEP for all $\lambda > 1$. Any tight OMEP with parameters as in (2.4) and $g = 10$ and having at least one $\lambda_i > 1$, say λ_5, can now be obtained using Theorem 2.1.2 applied to the direct product of an equally replicated $10 \times 10 \times 10 \times 10 \times 10\lambda_5//100\lambda_5$ OMEP and an equally replicated $\lambda_1 \times \lambda_2 \times \lambda_3 \times \lambda_4 \times 1//\lambda_1\lambda_2\lambda_3\lambda_4$ OMEP. It follows that any tight parameter set having the form in (2.4) and having $g = 10$ has a corresponding tight OMEP, with the possible exception of a $10^5//100$ OMEP.

Next consider the case $g = 6$. It is known that a $6^4//36$ OMEP does not exist, and thus a $6^5//36$ OMEP does not exist either. In the Appendix, we give a resolvable $TD_2(4,6)$ and a resolvable $TD_3(4,6)$, and so by extending parallel classes and applying the concatenation construction we obtain an equally replicated $6 \times 6 \times 6 \times 6 \times 6\lambda//36\lambda$ OMEP for all $\lambda > 1$. It follows then, as with the case $g = 10$, that any tight parameter set having the form in (2.4) and having $g = 6$ has a corresponding tight OMEP, with the exception of a $6^5//36$ OMEP.

The case $g = 3$ is similar to the last two cases. For trivial reasons, a $3^5//9$ OMEP does not exist. In the Appendix, we give a resolvable $TD_2(4,3)$, and so by extending parallel classes we obtain an equally replicated $3 \times 3 \times 3 \times 3 \times 6//18$ OMEP. An equally replicated $3 \times 3 \times 3 \times 3 \times 9//27$ OMEP exists since it can be obtained by extending parallel classes in a resolvable $TD_3(4,3)$, which was shown to exist in the introduction. By applying the concatenation construction we obtain an equally replicated $3 \times 3 \times 3 \times 3 \times 3\lambda//9\lambda$ OMEP for all $\lambda > 1$. As in the previous two cases, it follows that any tight parameter set having the form in 2.4 and having $g = 3$ has a corresponding tight OMEP, with the exception of a $3^5//9$ OMEP.

Finally, consider the case $g = 2$. For trivial reasons, a $2^5//4$ OMEP does not exist. First consider the case where some λ_i is even, say λ_5. We have seen that an equally replicated $2^4 \times 4//8$ OMEP exists, and so an equally replicated $2\lambda_1 \times 2\lambda_2 \times 2\lambda_3 \times 2\lambda_4 \times 2\lambda_5//4\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5$ OMEP can be obtained as the direct product of an equally replicated $2^4 \times 4//8$ OMEP and an equally replicated $\lambda_1 \times \lambda_2 \times \lambda_3 \times \lambda_4 \times (\lambda_5/2)//\lambda_1\lambda_2\lambda_3\lambda_4(\lambda_5/2)$ OMEP. Next suppose all λ_i are odd. If just one λ_i is greater than 1, say $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$ and $\lambda_5 > 1$, then Lemma 2.2.6 applies and the tight OMEP cannot exist. On the other hand, suppose at least two λ_i are greater than 1, say λ_4 and λ_5. In the Appendix, an equally replicated $2 \times 2 \times 2 \times 6 \times 6//36$ OMEP is given. Further, a tight $2^3 \times 6 \times 4//24$ OMEP exists, and so by the concatenation construction we can construct an equally replicated $2^3 \times 6 \times 2\lambda_5//12\lambda_5$ OMEP for all odd $\lambda_5 \geq 3$. For such λ_5, an equally replicated $2^3 \times 4 \times 2\lambda_5//8\lambda_5$ OMEP exists, and so again by the concatenation construction we obtain an equally replicated $2^3 \times 2\lambda_4 \times 2\lambda_5//4\lambda_4\lambda_5$ OMEP for all odd $\lambda_4, \lambda_5 \geq 3$. Finally, the direct product of this OMEP with a $\lambda_1 \times \lambda_2 \times \lambda_3 \times 1 \times 1//\lambda_1\lambda_2\lambda_3$ OMEP gives an equally replicated $2\lambda_1 \times 2\lambda_2 \times 2\lambda_3 \times 2\lambda_4 \times 2\lambda_5//4\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5$ OMEP. If the λ_i are pairwise relatively prime, then Theorem 2.1.2 gives the result. Thus, we have shown that in the case $g = 2$, the only tight parameter sets for which the tight OMEP does not exist is $2^5//4$ and $2^4 \times 2s//4s$ for s odd.

In summary then, the only tight five-factor parameter sets for which the tight OMEP does not exist are $2^5//4$, $2^4 \times 2s//4s$ for s odd, $3^5//9$, $6^5//36$, and possibly $10^5//100$.

2.5 Summary

There are some obvious applications of the techniques in this chapter. First of all, the results of this chapter show all tight OMEPs with three rows exist, and so we can say that an $s_1 \times s_2 \times s_3//n$ OMEP exists if and only if there exists a tight OMEP on n columns with at least as many different symbols in each row. Similarly, an $s_1 \times s_2 \times s_3//n$ OMEP with equal replication exists if and only if there exists a tight $s_1' \times s_2' \times s_3'//n$ OMEP such that $s_i|s_i'$ for each i. The same techniques apply in determining minimal OMEPs with more rows. The only problem is that fewer of the "ingredient" OMEPs exist. For example, in the case of six rows, there is no $2 \times 2 \times 2 \times 2 \times 2 \times 4//8$ OMEP so some other "base" ingredients must be used.

Chapter 3

Collapsing and Uncollapsing

In this chapter we give new results concerning the collapsing and uncollapsing of three row OMEPs. As seen in the last chapter, for almost any OMEP parameter set with three or four rows, there is a tight OMEP which we can collapse to obtain an OMEP with these parameters. For example, It is possible to obtain a $7 \times 7 \times 7 \times 6$ OMEP by collapsing a tight OMEP. However, there may exist a *particular* $7 \times 7 \times 7 \times 6$ OMEP that cannot be obtained by collapsing a tight OMEP.

We show that there are indeed OMEPs on four factors that cannot be obtained by collapsing a tight OMEP. More importantly, however, we show that *any* OMEP on three factors can be obtained by collapsing some tight OMEP. Put another way, any three factor OMEP can be uncollapsed to obtain a tight OMEP. We also prove some results concerning the concatenation and unconcatenation of tight OMEPs.

3.1 Uncollapsing Three-Factor OMEPs

In this section we show that *any* OMEP on three rows can be obtained by collapsing some tight three-factor OMEP. In other words, for any three-factor OMEP it is possible to "uncollapse" levels in it to obtain a tight three-factor OMEP.

Let $\mathcal{D}$ be an $s_1 \times s_2 \times s_3//n$ OMEP, having symbol set $\{1, 2, \dots s_i\}$ in row i. If $\mathcal{D}$ is already tight then we are done. Otherwise, as we prove in the second chapter, there is an associated tight OMEP parameter set $s'_1 \times s'_2 \times s'_3//n$, which we can alternatively write as $\lambda_1 g \times \lambda_2 g \times \lambda_3 g // \lambda_1 \lambda_2 \lambda_3 g^2$, where the λ_i's are pairwise relatively prime. Also $s'_i \geq s_i$ and some $s'_X > s_X$ (so X indexes a row). If the replication numbers of the symbols in row i of $\mathcal{D}$ are $r_{i1}, r_{i2}, \dots r_{is_i}$, then $\gcd(r_{i1}, r_{i2}, \dots, r_{is_i})$ is divisible by $g'_i = n/s'_i$. Since $s_X < s'_X$, at least one r_{Xj} is not equal to g'_X, so $r_{XY} = dg'_X$, for some $d > 1$ and some symbol Y. We now explain how to "uncollapse" symbol Y in row X into d distinct symbols. In what follows, we assume without loss of generality that $X = 1$ and $Y = 1$ for ease of description. Let C be those columns of $\mathcal{D}$ containing a 1 (i.e. Y) in row 1 (i.e. X). We construct a bipartite graph T with bipartition classes $A = \{1, 2, \dots, s_2\}$ and $B = \{1, 2, \dots, s_3\}$ by joining the vertices α in A and β in B for each column $(1, \alpha, \beta)^T$ in C. With this construction, there is a one to one correspondence between the edges of T and the columns in C. Further, the degree of the vertex corresponding to the symbol j in row i, $i \neq 1$, is $r_{11} r_{ij}/n$, which is divisible by $dg'_1 g'_i/n$ which in turn is divisible by d. Thus, every vertex in T has degree divisible by d, and each bipartition class has a total of $|C|$ edges leaving it. Therefore, there is a d-regular bipartite graph T', having two bipartition classes of size $|C|/d$ from which we can obtain T by identifying sets of vertices in T''s bipartition classes. (For example, to get such a graph T', pick a vertex in T of degree sd and arbitrarily "split" this vertex into s

vertices of degree d. Repeat this for each vertex in T to get T'.) The edge set of T' can be decomposed into d one-factors of T', since T' is a regular bipartite graph. (This is an easy corollary of Hall's Theorem or König's Theorem; see any good book on graph theory.) These one-factors correspond to a partition of the edge set of T into d (spanning) subgraphs of T, say $T_1, T_2, \ldots, T_d$, where each vertex v in any T_i has degree

$$\frac{\text{degree of } v \text{ in } T}{d}.$$

Also, each edge in each subgraph corresponds to a column in C, and thus corresponds to a column in $\mathcal{D}$. Finally, for each edge in each subgraph T_i, find the associated column in $\mathcal{D}$ and replace the symbol 1 in row 1 by a 1^i. This has the effect of replacing the symbol 1 in row 1 of $\mathcal{D}$ by d new symbols $1^1, 1^2, \ldots, 1^d$. Why is the resultant array an OMEP? A little thought shows that we only need to check the number of occurences of one of the new symbols with a symbol from another row. Consider symbol 1^z in row 1, $1 \le z \le d$, and symbol y in row j, $2 \le j \le 3$. Symbol 1 in row 1 occurs r_{11} times in the original array, and symbol y in row j occurs r_{jy} times. In the new array, symbol y in row j still occurs r_{jy} times, but symbol 1^z only occurs r_{11}/d times. Furthermore, any vertex in the subgraph T_z (as defined above) has degree $1/d$ of its degree in T, that is, $r_{11}r_{jy}/nd$. Thus in the new array, symbol y in row j occurs $r_{11}r_{jy}/nd$ times in a column with symbol 1^z in row 1. As

$$r_{11^z}r_{jy}/n = (r_{11}/d)(r_{jy})/n,$$

the two symbols occur the exact number of times required for the resultant array to be an OMEP. Thus the new array is an $(s_1 + d - 1) \times s_2 \times s_3//n$ OMEP.

If this new OMEP is not tight, we can repeat this construction. Eventually, we obtain a tight OMEP (which cannot be further uncollapsed). This tight OMEP can be collapsed to obtain $\mathcal{D}$, by just reversing the uncollapsing operations.

The above argument proves the following theorem.

Theorem 3.1.1 *Any three-factor OMEP can be obtained by collapsing a tight three-factor OMEP. Equivalently, any three factor OMEP can be uncollapsed to obtain a tight three-factor OMEP.*

Therefore, to enumerate all possible three-factor OMEPs (not just the possible parameters, but the actual OMEPs), it suffices to generate every possible tight three-factor OMEP, and then record the OMEPs that can be obtained by collapsing levels in these tight OMEPs. Generating all possible tight three-factor OMEPs, even with a given set of parameters, is nontrivial. However it seems clear that this two step approach is computationally easier than directly generating all three-factor OMEPs, especially if we are using backtracking to generate the OMEPs.

As an example, we do one "iteration" of this procedure to partially uncollapse the $2 \times 3 \times 4//16$ OMEP of Table 1.4. We uncollapse the symbol 1 in row 2 into two distinct symbols. In this case the set C of columns containing symbol 1 in row 2 is

1	1	1	1	2	2	2	2
1	1	1	1	1	1	1	1
1	1	2	2	3	4	3	4

Now we form the bipartite graph having bipartition classes $\{1,2\}, \{1,2,3,4\}$. This graph is shown in Figure 3.1. Now, we must partition this graph into two (spanning) subgraphs so that the degree of each vertex in each subgraph is exactly half what it was in the original bipartite graph. In this case, the desired partition is easily found. Furthermore we mention that there are "nonisomorphic" partitions, corresponding to different ways to uncollapse. In any case, one such partition is shown in Figure 3.2. From this, we replace the symbol 1 in row 2 by the symbols

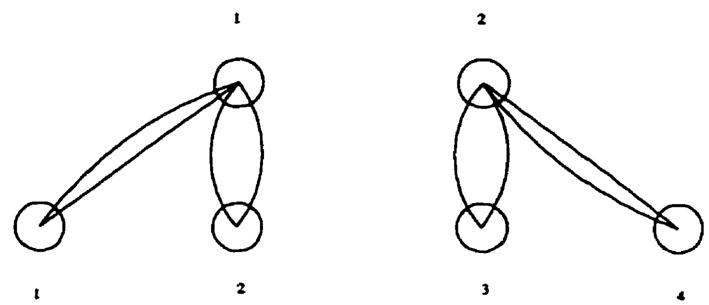

Figure 3.1: The Bipartite Graph T

$1^1, 1^2$, as follows.

1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
1^1	1^2	2	3	1^1	1^2	2	3	1^1	1^1	2	3	1^2	1^2	2	3
1	1	3	4	2	2	4	3	3	4	2	2	3	4	1	1

This array is a $2 \times 4 \times 4//16$ OMEP.

3.2 Four-Factor OMEPs

Although every three-factor OMEP can be uncollapsed to a tight OMEP, this is not true of four-factor OMEPs. For example, a $6 \times 6 \times 6 \times 5//36$ OMEP exists (see Lemma 2.2.14, for example), but it cannot be uncollapsed to obtain a tight OMEP, since the tight OMEP would have parameters $6 \times 6 \times 6 \times 6//36$ and such an OMEP does not exist, as it would correspond to two MOLS of order six. However, we mention that the method of the last section can be generalized to OMEPs having four or more factors. For example, given a four-factor OMEP and a symbol x in row 1 (say) that we wish to uncollapse into d distinct symbols. Form a tripartite graph analogous to the graph of the last section, except that every column containing an

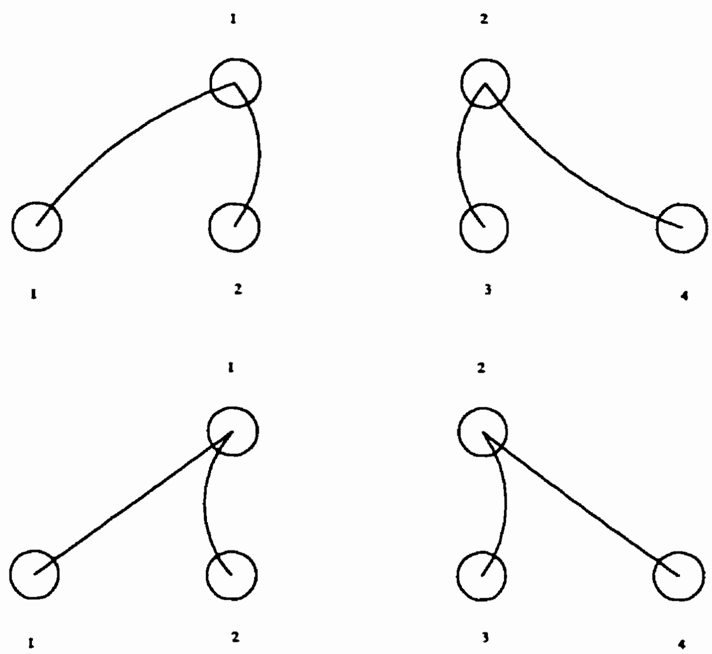

Figure 3.2: A Partition of the Graph T

x in row 1 now contributes a 3-clique to the tripartite graph, instead of a 2-clique (edge). To uncollapse, one would have to partition these 3-cliques into d classes, forming spanning subgraphs $T_1, T_2, \ldots, T_d$ so that the degree of any vertex in any T_i is $1/d$ times its degree in the original tripartite graph. As (implicitly) shown above, this is not always possible. In fact, deciding whether a tripartite graph has a partition into 3-cliques is NP-complete [8].

It is tempting to wonder if the only thing that prevents uncollapsing is the non-existence of the tight OMEP (as in the above example). This is not the case, as can be seen as follows. Consider a matrix M with 4 rows and 45 columns, with the symbols $1, 2, 3, 4, 5, 6, 7$ occurring in each row, so that for each pair of rows, each ordered pair of symbols except $(1,1)^T, (1,2)^T, (2,1)^T, (2,2)^T$ occurs exactly once. Thus if a TD(4,2) actually existed, we could add its corresponding columns to obtain a TD(4,7). For this reason such a matrix is sometimes called a TD(4,7) $-$ TD(4,2). Replace each symbol 2 in row 1 of M by the symbol 1. Now add the columns of a $1 \times 2 \times 2 \times 2//4$ OMEP, with symbol set $\{1\}, \{1,2\}, \{1,2\}, \{1,2\}$ in each row, respectively. This results in a $6 \times 7 \times 7 \times 7//49$ OMEP. Furthermore, this OMEP cannot be uncollapsed to form a $7 \times 7 \times 7 \times 7//49$ OMEP, since the process of uncollapsing would uncollapse the $1 \times 2 \times 2 \times 2//4$ sub-OMEP into a $2 \times 2 \times 2 \times 2//4$ OMEP. Thus there is a $6 \times 7 \times 7 \times 7//49$ OMEP that cannot be uncollapsed to form a $7 \times 7 \times 7 \times 7//49$ OMEP, although there does exist a $7 \times 7 \times 7 \times 7//49$ OMEP. Furthermore, it is known that a TD(4,g)$-$TD(4,2) exists for all $g \geq 6$, and so the above construction provides many examples of such OMEPs.

3.3 Unconcatenating

As seen in the concatenation construction, Theorem 2.2.2, the juxtaposition of two OMEPs sometimes gives a new OMEP. In the last section, we considered the question of uncollapsing a given OMEP. One may ask a similar question about unconcatenating a given OMEP. In the context of tight OMEPs, this question becomes: When is a tight $\lambda_1 g \times \lambda_2 g \times \ldots \times \lambda_k g // \lambda_1 \lambda_2 \ldots \lambda_k g^2$ OMEP the concatenation of a $\lambda_1 g \times \lambda_2 g \times \ldots \times \lambda_{k-1} g \times \mu_k g // \lambda_1 \lambda_2 \ldots \lambda_{k-1} \mu_k g^2$ OMEP and a $\lambda_1 g \times \lambda_2 g \times \ldots \times \lambda_{k-1} g \times \mu'_k g // \lambda_1 \lambda_2 \ldots \lambda_{k-1} \mu'_k g^2$ OMEP?

We mention in passing that a tight $3 \times 3 \times 3s//9s$ OMEP is always the concatenation of s $3 \times 3 \times 3//9$ OMEPs, as the following argument shows. A tight $3 \times 3 \times 3s//9s$ OMEP is equivalent to a completely resolvable $OA_s(2,3)$, so it is enough to show that a completely resolvable $OA_s(2,3)$ is the union of s completely resolvable OA(2, 3)s. A completely resolvable $OA_s(2,3)$ is by definition a collection of parallel classes. Let a_{ijk} denote the number of times the parallel class

$$\begin{matrix} 0 & 1 & 2 \\ i & j & k \end{matrix}$$

occurs. (Here we are assuming the symbol set in each row is $\{0,1,2\}$.) If $a_{012} = \mu$, then since $a_{012} + a_{021} = s$, we must have $a_{021} = s - \mu$. Similarly since $a_{012} + a_{102} = s$, we find $a_{102} = s - \mu$. Continuing in this manner, we find $a_{012} = a_{120} = a_{201} = \mu$ and $a_{021} = a_{102} = a_{210} = s - \mu$. Notice that the collection of parallel classes corresponding to the variables $a_{012}, a_{120}, a_{201}$ form a completely resolvable OA(2, 3), as do the collection of parallel classes corresponding to the other three variables. Thus the completely resolvable $OA_s(2,3)$ is the union of s completely resolvable OA(2, 3)'s : μ of one kind and $s - \mu$ of the other kind.

Call a tight OMEP *decomposable* if it is the concatenation of two other OMEPs.

Theorem 3.3.1 *For fixed k and fixed $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, g$, there is only a finite number of tight $\lambda_1 g \times \lambda_2 g \times \ldots \times \lambda_k g // \lambda_1\lambda_2 \ldots \lambda_k g^2$ OMEPs that cannot be expressed as the concatenation of a $\lambda_1 g \times \lambda_2 g \times \ldots \times \lambda_{k-1} g \times \mu g // \lambda_1\lambda_2 \ldots \lambda_{k-1}\mu g^2$ OMEP and a $\lambda_1 g \times \lambda_2 g \times \ldots \times \lambda_{k-1} g \times \mu' g // \lambda_1\lambda_2 \ldots \lambda_{k-1}\mu' g^2$ OMEP.*

The proof is similar to the finite basis result for PBDs, see [21] for example.

Proof: For convenience define $\lambda = \lambda_1\lambda_2 \ldots \lambda_k$. We first remark that the columns containing any fixed symbol in the kth row of a tight $\lambda_1 g \times \lambda_2 g \ldots \times \lambda_k g // \lambda g^2$ OMEP form a set of $\lambda g/\lambda_k$ columns so that any symbol in the ith row, $1 \leq i \leq k-1$ occurs $\lambda/(\lambda_i\lambda_k)$ times in this set of columns. Let S be that set of all $k-1 \times \lambda g/\lambda_k$ matrices so that any symbol in the ith row, $1 \leq i \leq k-1$ occurs $\lambda/(\lambda_i\lambda_k)$ times in this set of columns. Therefore, up to the naming of the symbols in row k, any $\lambda_1 g \times \lambda_2 g \ldots \times \lambda_k g // \lambda g^2$ tight OMEP D is specified by a $|S|$-tuple T_D, where the number in coordinate position j of this tuple indicates the number of times the jth matrix in S occurs above some symbol in row k of our OMEP. Now, if D is decomposable, then there are $|S|$-tuples T_{D_1}, T_{D_2} so that $T_{D_1} + T_{D_2} = T_D$. Such tuples can be partially ordered by $\leq$, where $T_1 \leq T_2$ if every coordinate of T_1 is less than or equal to its corresponding coordinate in T_2. Tuples T_1, T_2 are said to be incomparable if neither $T_1 \leq T_2$ nor $T_2 \leq T_1$. The tuples representing two indecomposable tight OMEPs are incomparable, for if $T_{D_a} \leq T_{D_b}$, for two distinct indecomposable tight OMEPs D_a, D_b, then $T_{D_b} - T_{D_a}$ is a vector representing an OMEP D_c such that D_b is the concatenation of D_a and D_c, and therefore D_b is not indecomposable. Hence, to show that there are a finite number of indecomposable tight OMEPs (for fixed $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, g$), it suffices to show that there are no infinite antichains in the partial order. However this is exactly what is guaranteed by Lemma 1 of [21]. We review the proof here. Suppose $K = \{T_{D_1}, T_{D_2}, \ldots\}$ is an infinite antichain in the partial order. Consider the first coordinate of elements in K. Since an infinite sequence of non-negative integers has an nondecreasing subsequence, K contains an infinite subsequence K_1 so that the first coordinate of elements of K_1 is nondecreasing. Now the same argument shows K_1 contains an infinite subsequence K_2 so that the second coordinate of elements of K_2 is also nondecreasing. The number, $|S|$, of coordinates is finite, so eventually we produce a subsequence $K_{|S|}$ that contains two comparable elements. However this is a contradiction since K is an antichain. □

3.4 Summary

The main result of this chapter is that any three-factor OMEP can be uncollapsed to obtain a tight three-factor OMEP. For four or more factors, there are examples of OMEPs that cannot be uncollapsed to a tight OMEP. Also in this chapter we have shown a finite basis type result for concatenation of OMEPs, which says, roughly, that tight OMEPs with n columns and having a fixed number of symbols in the first $k-1$ rows are the concatenation of two smaller OMEPs if n is large enough.

Chapter 4

Asymptotic Existence of Tight OMEPs

Asymptotic existence of tight OMEPs is established in this chapter. This is accomplished mainly through the use of a recursive construction, and simple arithmetic facts. We also give an application of the asymptotic result to Jacroux's lower bound for OMEPs. In the second chapter, it was shown that every tight OMEP parameter set on three or fewer rows has a corresponding tight OMEP, so we make the implicit assumption that $k \geq 4$.

4.1 The Main Result

The following incidence structure is useful.

Definition 4.1.1 *Let $\mathcal{S} = \{v_{ij} | 1 \leq i \leq k, 1 \leq j \leq g\}$. Let $\mathcal{B}$ be a set of subsets (blocks) of $\mathcal{S}$. The pair $(\mathcal{S}, \mathcal{B})$ is called a $R(g, k, \mu, \lambda)$-design if the block set can* *be partitioned into parallel classes and if pairs of points v_{ix}, v_{jy} are in no blocks if $i = j$, in λ blocks if $i \neq j$ and $x \neq y$, and in μ blocks if $i \neq j$ and $x = y$.*

The main idea in this section is to introduce a recursive construction creating these R-designs, so that the recursion allows for increasing the block size without necessarily making μ a multiple of g. This results in the construction of resolvable transversal designs $\text{RTD}_\lambda(k, g)$ with k as large as desired, and with λ not necessarily a multiple of g (as in the Kronocker Product construction).

Lemma 4.1.2 *Let g and k be fixed. Then an $R(g, k, g^{k-2} - 1, g^{k-2})$-design exists.*

Proof: Let the point set be $\{(i, j) | 1 \leq i \leq g, 1 \leq j \leq k\}$. The set of blocks

$$\{\{(p_1, 1), (p_2, 2), \ldots, (p_k, k)\} |\ 1 \leq p_i \leq g,\ p_i\text{'s not all equal}\}$$

is a $\text{R}(g, k, g^{k-2} - 1, g^{k-2})$-design. □

If we do not exclude blocks with all p_i equal, we get a $\text{RTD}_{g^{k-2}}(k, g)$.

Remark 1 *For any fixed g and k, a $RTD_{g^{k-2}}(k, g)$ exists.*

The union of the block sets of two RTD's on the same point set gives a third RTD. We use the following consequence of this fact repeatedly.

Lemma 4.1.3 *If a $RTD_{\lambda_1}(k, g)$ and a $RTD_{\lambda_2}(k, g)$ exist, with $\gcd(\lambda_1, \lambda_2) = 1$, then a $RTD_\lambda(k, g)$ exists for all $\lambda \geq \lambda_1\lambda_2$. Hence if a $RTD_\mu(k, g)$ with $\gcd(\mu, g) = 1$ exists, then a $RTD_\lambda(k, g)$ exists for all λ sufficiently large.*

Proof: The first statement holds since $\lambda = s\lambda_1 + t\lambda_2$ has a nonnegative integral solution in s, t for all $\lambda \geq \lambda_1\lambda_2$, and since the union of the block sets of s

$\mathrm{RTD}_{\lambda_1}(k,g)$'s and t $\mathrm{RTD}_{\lambda_2}(k,g)$'s gives a $\mathrm{RTD}_{s\lambda_1+t\lambda_2}(k,g)$. The second follows by using Remark 1. □

The advantage of the following construction is that it allows for increasing the block size without necessarily making the "index" a multiple of g. We see similar constructions in the next chapter.

Theorem 4.1.4 *If an RBIBD*(v,k,λ) *and a RTD*$_\mu(v/k,g)$ *exist, then a* $R(g,v,\lambda\mu(g+(v-k)/(k-1)),\lambda\mu(v-k)/(k-1))$*-design exists.*

Proof: We construct blocks on the point set

$$S=\{(i,j)|1\leq i\leq g, 1\leq j\leq v\}.$$

Assume that the $\mathrm{RTD}_\mu(v/k,g)$ is on the points $\{(i,j)|1\leq i\leq g,1\leq j\leq v/k\}$, and the groups are $G_j=\{(i,j)|1\leq i\leq g\}$. Assume the RBIBD is on the point set $\{1,2,\ldots v\}$. For each parallel class of the RTD, say $\{B_1,B_2,\ldots,B_g\}$, and each parallel class of the RBIBD, say $\{B_1',B_2',\ldots,B_{\frac{v}{k}}'\}$, we construct a parallel class on S as follows. If

$$B_i=\{(\delta_{i,1},1),(\delta_{i,2},2),\ldots,(\delta_{i,\frac{v}{k}},\frac{v}{k})\},$$

then our parallel class on S has blocks $\{\beta_j|1\leq j\leq g\}$ defined by

$$\beta_j=(\{\delta_{j,1}\}\times B_1')\cup(\{\delta_{j,2}\}\times B_2')\cup\ldots\cup(\{\delta_{j,\frac{v}{k}}\}\times B_{\frac{v}{k}}'). \tag{4.1}$$

We now count the blocks of the resulting design containing a given pair of distinct points. In what follows, let $r=\lambda(v-1)/(k-1)$, and let $\mathcal{R}$ denote the set of blocks of the final structure. First, any pair of points in S with a common second index never occur together in a block $\mathcal{R}$, by virtue of 4.1 and the fact that the B_i' form a parallel class. Next consider a pair of points $(i_1,j_1),(i_2,j_2)$, where

$i_1 \neq i_2$ and $j_1 \neq j_2$. Again by 4.1, the only blocks in $\mathcal{R}$ in which these points occur together come from parallel classes of the RBIBD where j_1 and j_2 lie in distinct blocks. There are $r - \lambda$ such parallel classes, and for each one there are μ blocks of the resolvable transversal design that combine with this parallel class to make a block of $\mathcal{R}$ containing the points $(i_1, j_1), (i_2, j_2)$. This makes for a total of

$$\mu(r - \lambda) = \lambda\mu(v - k)/(k - 1)$$

blocks of $\mathcal{R}$ containing the given pair of points. Finally consider points where only the second index differs, say $(i, j_1), (i, j_2)$. In this case every parallel class of the RBIBD combines with some block of the resolvable transversal design to give a block of $\mathcal{R}$ containing the given pair of points. There are λ blocks of the RBIBD containing the pair of points j_1, j_2, and for each such block B, there are μg blocks of the RTD that combine with it to give a block of $\mathcal{R}$ containing the pair $(i, j_1), (i, j_2)$. (These μg blocks are all blocks through a certain point of the RTD; that certain point depending on the value of i and the parallel class of the RBIBD.) There are $r - \lambda$ parallel classes of the RBIBD in which the points j_1, j_2 lie in different blocks, and for each such parallel class there are μ blocks of the RTD that combine with it to give a block of $\mathcal{R}$ containing the pair $(i, j_1), (i, j_2)$. (These μ blocks correspond to the blocks through a fixed pair of points of the RTD.) This makes for a total of

$$\lambda\mu g + \mu(r - \lambda) = \lambda\mu(g + (v - k)/(k - 1))$$

blocks of $\mathcal{R}$ containing the given pair of points. □

The above theorem allows us to prove, with minimal work, the asymptotic existence of resolvable transversal designs. This is the first step in proving the asymptotic existence of tight OMEPs.

Corollary 4.1.5 *Let $g \geq 4$ be a fixed number not divisible by 3, and let k be fixed. Then for all λ large enough, a $RTD_\lambda(k,g)$ exists.*

Proof: Choose i such that $3^{i+1} \geq k$. Apply Theorem 4.1.4 using an RBIBD$(3^{i+1}, 3^i, (3^i - 1)/2)$ and a RTD$(3,g)$ to obtain an R$(g, 3^{i+1}, g(3^i-1)/2 + 3^i, 3^i)$-design which we truncate to a R$(g,k,g(3^i-1)/2+3^i,3^i)$-design. Now take $g(3^i-1)/2$ copies of the blocks of a R$(g,k,g^{k-2}-1,g^{k-2})$ and one copy of the blocks of our R$(g,k,g(3^i - 1)/2 + 3^i, 3^i)$-design, to give a RTD$_\mu(k,g)$, where $\mu = g^{k-1}(3^i-1)/2+3^i$. Since μ is relatively prime to g, Lemma 4.1.3 gives the result. □

Lemma 4.1.6 *For any k, any m, and any λ sufficiently large, a $RTD_\lambda(k,3^m)$ exists.*

Proof: First suppose $m = 1$. Choose i such that $4^{i+1} \geq k$. Apply Theorem 4.1.4 using an RBIBD$(4^{i+1}, 4^i, (4^i-1)/3)$ and a RTD$_2(4,3)$ to obtain an R$(3, 4^{i+1}, 6(4^i - 1)/3 + 2\cdot 4^i, 2\cdot 4^i)$-design which we truncate to a R$(3,k,6(4^i-1)/3+2\cdot 4^i, 2\cdot 4^i)$-design. Now take $6(4^i-1)/3$ copies of the blocks of a R$(3,k,3^{k-2}-1,3^{k-2})$ and one copy of the blocks of our R$(3,k,6(4^i-1)/3+2\cdot 4^i, 2\cdot 4^i)$-design, to give a RTD$_\mu(k,g)$, where $\mu = 2\cdot 3^{k-1}(4^i-1)/3+2\cdot 4^i$. Since μ is relatively prime to 3, Lemma 4.1.3 gives the result in this case.

Next suppose $m > 1$. Set $g = 3^m$. Choose i such that $4^{i+1} \geq k$. Apply Theorem 4.1.4 using an RBIBD$(4^{i+1}, 4^i, (4^i-1)/3)$ and a RTD$(4,g)$ to obtain an R$(g, 4^{i+1}, g(4^i-1)/3+4^i, 4^i)$-design which we truncate to a R$(g,k,g(4^i-1)/3+4^i,4^i)$-design. Now take $g(4^i-1)/3$ copies of the blocks of a R$(g,k,g^{k-2}-1,g^{k-2})$ and one copy of the blocks of our R$(g,k,g(4^i-1)/3+4^i,4^i)$-design, to give a RTD$_\mu(k,g)$, where $\mu = g^{k-1}(4^i-1)/3+4^i$. Since μ is relatively prime to g, Lemma 4.1.3 again gives the result. □

Corollary 4.1.7 *For any k and any g with $3|g$, and all λ sufficiently large, a $RTD_\lambda(k, g)$ exists.*

Proof: We first consider the case $g = 6$. In this case, choose i such that $5^{i+1} \geq k$. Applying Theorem 4.1.4 using an $\mathrm{RTD}_5(5,6)$ and an $\mathrm{RBIBD}(5^{i+1}, 5^i, (5^i-1)/4)$ gives an $R(6, 5^{i+1}, 6\cdot 5(5^i-1)/4+5^{i+1}, 5^{i+1})$-design which we truncate to a $R(6, k, 6\cdot 5(5^i-1)/4 + 5^{i+1}, 5^{i+1})$-design. Adding $6\cdot 5(5^i-1)/4$ copies of the blocks of an $R(6, k, 6^{k-2}-1, 6^{k-2})$-design gives an $\mathrm{RTD}_\lambda(k, 6)$, where $\lambda = 5\frac{5^i-1}{4}\cdot 6^{k-1} + 5^{i+1}$, which is relatively prime to 6. Thus Lemma 4.1.3 now gives the result.

For $g \neq 6$, write $g = 3^m g'$, with $3 \not| g'$. Since $g \neq 6$, we have $g' \neq 2$. From Lemma 4.1.6, there exists a $\mathrm{RTD}_{\lambda_1}(k, 3^m)$ with $\gcd(\lambda_1, g) = 1$, and by Corollary 4.1.5 there is a $\mathrm{RTD}_{\lambda_2}(k, g')$ with $\gcd(\lambda_2, g) = 1$. The direct product of these is a $\mathrm{RTD}_{\lambda_1\lambda_2}(k, g)$. Since $\gcd(\lambda_1\lambda_2, g) = 1$, Lemma 4.1.3 now gives the result. □

These last few observations show that for fixed k and $g \geq 3$, a $\mathrm{RTD}_\lambda(k-1, g)$ exists for all λ large enough, say all $\lambda \geq M(g, k)$. Hence a tight $\lambda g \times g^{k-1}//\lambda g^2$ OMEP exists for all $\lambda \geq M(g, k)$. The product theorem implies that for any set of λ_i's pairwise relatively prime with at least one $\lambda_i \geq M(g, k)$ a tight

$$\lambda_1 g \times \lambda_2 g \times \ldots \times \lambda_k g // \lambda_1\lambda_2 \ldots \lambda_k g^2 \tag{4.2}$$

OMEP exists. Since for fixed k, g there are only a finite number of parameters of the form in (4.2) with the λ_i's all less than λ, there are at most a finite number of such tight OMEP parameters for which the tight OMEP does not exist. Furthermore, for all sufficiently large g a $\mathrm{TD}(k, g)$ exists, and for such g and any choice of pairwise relatively prime λ_i's a tight $\lambda_1 g \times \lambda_2 g \times \ldots \times \lambda_k g // \lambda_1\lambda_2 \ldots \lambda_k g^2$ OMEP exists. Thus, there are only a finite number of tight parameter sets of the form in (4.2) with $g \neq 2$ for which the tight OMEP does not exist. It remains to show that there are only a finite number that do not exist when $g = 2$.

Since a TD$(k, 2\alpha)$ exists for some α odd (depending on k), by collapsing levels in it we obtain an equally replicated $2\alpha \times 2\alpha \times 2^{k-2}//4\alpha^2$ OMEP. Also, there is a tight $2\alpha' \times 2^{k-1}//4\alpha'$ OMEP for α' a sufficiently large power of 2, and hence a tight $2\alpha' \times 2\alpha \times 2^{k-2}//4\alpha'\alpha$ OMEP. By using the concatenation construction (Theorem 2.2.2) we obtain a tight $2\mu \times 2\alpha \times 2^{k-2}//4\mu\alpha$ OMEP for all $\mu \geq \alpha\alpha'$. For such μ, there is also a tight $2\mu \times 2\alpha' \times 2^{k-2}//4\mu\alpha'$ OMEP. Again using concatenation we obtain a tight $2\mu \times 2\mu' \times 2^{k-2}//4\mu\mu'$ OMEP for all $\mu' \geq \alpha\alpha$. Thus, for any choice of μ_i's pairwise relatively prime with at least two of the μ_i's at least $\alpha\alpha'$, there is a tight

$$2\mu_1 \times 2\mu_2 \times \ldots \times 2\mu_k//4\mu_1\mu_2\ldots\mu_k \tag{4.3}$$

OMEP. We must now consider OMEPs of the form in (4.3) but where all but one of the μ_i's are less than $\alpha\alpha'$. We need some lemmas first.

Lemma 4.1.8 *For any k, there is an odd λ such that a tight $2\lambda \times 4 \times 2^{k-2}//8\lambda$ OMEP exists.*

Proof: Choose i such that $3^{i+1} \geq k-2$ and i is even. For neatness define $v = 3^{i+1}$. Let $\mathcal{D}_1$ be a $2\times4\times2\times2\times2//8$ OMEP, and let $\mathcal{D}_2$ be an RBIBD$(3^{i+1}, 3^i, (3^i-1)/2)$. Let

$$\mathcal{T} = \{\text{all } v\text{-tuples using 0,1 except } (0,0,\ldots,0) \text{ and } (1,1\ldots,1)\ \}\,.$$

Let the j'th parallel class of $\mathcal{D}_2$ be $\{B_{j1}, B_{j2}, B_{j3}\}$, $1 \leq j \leq (3^{i+1}-1)/2$. We construct an OMEP on $v+2$ rows, with rows labelled $\infty, 0, 1, 2, \ldots, v$. We construct the OMEP so that the symbols in row ∞ are $\mathcal{T} \times \{1, 2, \ldots, (3^i-1)/2\} \cup \{1, 2, \ldots v-1\}$, the symbols in row 0 are $\{0, 1, 2, 3\}$, and the symbols in each other row are $\{0, 1\}$. Assume the symbols in the rows of $\mathcal{D}_1$ are $\{0, 1\}$, $\{0, 1, 2, 3\}$, $\{0, 1\}$, $\{0, 1\}$, and $\{0, 1\}$, respectively. Assume the point set of $\mathcal{D}_2$ is $\{1, 2, \ldots v\}$. For each column

$(p_\infty, p_0, p_1, p_2, p_3)^T$ of $\mathcal{D}_1$ and each parallel class $\{B_{j1}, B_{j2}, B_{j3}\}$ of $\mathcal{D}_2$ we construct a column with $2j - p_\infty$ in the row ∞, p_0 in row 0, p_1 in each row indexed in B_{j1}, p_2 in each row indexed in B_{j2}, and p_3 in each row indexed in B_{j3}. (Since $\{B_{j1}, B_{j2}, B_{j3}\}$ is a parallel class this defines the entire column.) Further, for each $\alpha \in \{1, 2, \ldots, (3^i - 1)/2\}$, each v-tuple $T = (t_1, t_2, \ldots, t_v)$ in $\mathcal{T}$, and each $s \in \{0, 1, 2, 3\}$ we construct a column with (T, α) in row ∞, s in row 0, and $t_l + s$ in row l for each row l, $1 \leq l \leq v$ (where addition is done modulo 2). These columns together form an OMEP where symbols from row ∞ and row 0 occur together once, symbols from row ∞ and row l, $1 \leq l \leq v$, occur together twice, symbols from row 0 and row l, $1 \leq l \leq v$, occur together λ times, where $\lambda = (3^{i+1} - 1)/2 + (2^{v-1} - 1)(3^i - 1)/2$, and symbols from any pair of distinct rows with labels between 1 and v occur together 2λ times. (Since i is even, $(3^i - 1)/2$ is even, and $(3^{i+1} - 1)/2$ is odd, so λ is odd.) Thus this is a $2\lambda \times 4 \times 2^v // 8\lambda$ OMEP on $v + 2$ rows, which gives the desired OMEP by possibly removing some rows. □

Lemma 4.1.9 *For any k, there is a λ that is a power of 2 such that a tight $2\lambda \times 4 \times 2^{k-2} // 8\lambda$ OMEP exists.*

Proof: Choose i so that $4^i \geq k - 1$. A $4^i \times 4 \times 4^{4^i - 1} // 4^{i+1}$ OMEP exists, since a $\mathrm{RTD}_{4^i}(4^{i+1}, 4)$ exists. By collapsing levels we obtain a $4^i \times 4 \times 2^{4^i - 1} // 4^{i+1}$ OMEP. Taking $\lambda = 2^{2i-1}$, we find this is a tight $2\lambda \times 4 \times 2^{4^i - 1} // 8\lambda$ OMEP, which can be truncated to give the desired OMEP. □

Corollary 4.1.10 *For any k, and for all sufficiently large λ, a tight $2\lambda \times 4 \times 2^{k-2} // 8\lambda$ OMEP exists.*

Proof: This follows from Lemma 4.1.8, Lemma 4.1.9, and Lemma 4.1.3. □

We now show asymptotic existence of OMEPs with parameters as in (4.3), but where all but one of the μ_i's are less than $\alpha\alpha'$. Again recall that k is some fixed number of rows.

In the first case at least one μ_i is even, say $\mu_2 = 2\mu_2'$. Then by (4.1.9), a tight $2\lambda \times 4 \times 2 \times 2 \ldots \times 2//8\lambda$ OMEP on k rows exists for all λ large, say $\lambda \geq M'(k)$. Using the product construction we find a tight $2\lambda \times 2\mu_2 \times 2 \times 2 \ldots \times 2//4\lambda\mu_2$ OMEP exists for $\lambda \geq M'(k)$, and so again using the product construction, a tight $2\lambda \times 2\mu_2 \times 2\mu_3 \times 2 \ldots \times 2\mu_k//4\lambda\mu_2\mu_3 \ldots \mu_k$ OMEP exists for such λ. Thus if some $\mu_i \geq M'(k)$, and some μ_j is even, then a tight $2\mu_1 \times 2\mu_2 \times \ldots \times 2\mu_k//4\mu_1\mu_2 \ldots \mu_k$ OMEP exists. Hence there are at most a finite number of OMEP parameters in the first case for which the OMEP does not exist.

In the second case, no μ_i is even. We know if $k \geq 4$ and at most one μ_i is greater than one then the OMEP cannot exist, and in this case the parameters have the form $2 \times 2 \times \ldots \times 2 \times 2s//4s$ for s odd. (See Lemma 2.2.6.) Otherwise at least two μ_i's are greater than one. Suppose $\mu_1 \geq \mu_2 > 1$. By the earlier results a $\text{RTD}_{\lambda'}(k-1, 2\mu_2)$ exists for some odd λ', so an equally replicated $2\mu_2\lambda' \times 2\mu_2^{k-1}//4\lambda'\mu_2^2$ OMEP exists, and so by collapsing levels we obtain an equally replicated $2\lambda'\mu_2 \times 2\mu_2 \times 2^{k-2}//4(\lambda'\mu_2)\mu_2$ OMEP. Also a $2^i \times 2\mu_2 \times 2^{k-2}//2^{i+1}\mu_2$ OMEP exists for $2^i \geq k-1$, since a tight $2^i \times 2^{2^i}//2^{i+1}$ OMEP exists. Thus again by an argument similar to the proof of Lemma 4.1.3, an equally replicated $2\lambda \times 2\mu_2 \times 2^{k-2}//4\lambda\mu_2$ OMEP exists for all large λ, and so an equally replicated $2\lambda \times 2\mu_2 \times 2\mu_3 \times 2\mu_4 \times \ldots \times 2\mu_k//4\lambda\mu_2\mu_3 \ldots \mu_k$ OMEP exists for all large λ. Thus if μ_1 is sufficiently large the OMEP exists, and hence there are at most a finite number of OMEP parameters in the second case for which the OMEP does not exist.

These are the only possible cases and so there are at most a finite number of OMEP parameters with the form in (4.3) for which the OMEP does not exist, with the one exception of parameters of the type $2s \times 2 \times 2 \ldots \times 2//4s$ with s odd and with four or more rows.

Combining all these results we find, for any fixed k, and with the exception of parameters of the form $2 \times 2 \times 2 \times \ldots \times 2 \times 2s//4s$ with s odd and having 4 or more rows, there are a finite number of tight OMEP parameters on k rows for which the OMEP does not exist.

4.2 An Application to Jacroux's Bound

With these results we can show that the Jacroux's lower bound on the number of runs n needed to construct an $s_1 \times s_2 \times \ldots \times s_k//n$ OMEP is "almost asymptotically tight". To explain what we mean here we need to make some observations.

Jacroux's [16] lower bound on the number of columns in a $s_1 \times s_2 \times \ldots \times s_k//n$ OMEP is as follows.

Theorem 4.2.1 *Suppose that an OMEP $\mathcal{D}$ has $k \geq 3$ factors in which factor i has s_i levels, $i = 1, 2, \ldots, k$, with $s_i \geq s_{i+1}$, and n experimental runs. If $n = s_1' s_2'$ for s_1', s_2' satisfying*

$$s_1' s_2' = \min\{xy | x \geq s_1, y \geq s_2, xy < 2s_1 s_2, s_3 \leq \gcd(x, y)\}$$

then $\mathcal{D}$ is a minimal OMEP.

Essentially we are bounding the number of runs required by bounding the number of runs required for the truncated $s_1 \times s_2 \times s_3//n$ OMEP.

Street [22] has extended Jacroux's result when $k = 3$:

Theorem 4.2.2 *A minimal $s_1 \times s_2 \times s_3//n$ OMEP, with $s_1 \leq s_2 \leq s_3$, exists if and only if*

- $n = (s_2 + x)(s_3 + y)$ *for some nonnegative integers* x, y,
- $s_1 \leq \gcd(s_2 + x, s_3 + y)$.

The concept of a tight OMEP quickly leads to the above results, as follows. In the second chapter, it is shown that the minimal n for which an $s_1 \times s_2 \times s_3//n$ OMEP exists is the minimal n for which a tight $s_1' \times s_2' \times s_3'//n$ OMEP exists with $s_i' \geq s_i$ for $i = 1, 2, 3$. Let $d = \gcd(s_1', s_2', s_3')$, and let $u_i = n/s_i'$ for $i = 1, 2, 3$. Now since we are dealing with three row OMEPs, $s_1' \times s_2' \times s_3'//n$ is the parameter set of a tight OMEP if and only if u_1, u_2, u_3 are pairwise relatively prime, and $n = d^2 u_1 u_2 u_3$. All tight three-factor OMEPs exist, so the minimal n for which a tight $s_1' \times s_2' \times s_3'//n$ OMEP exists is given by

$$\min d^2 u_1 u_2 u_3$$

$$\text{subject to}$$

$$u_i d \geq s_i,$$

$$\gcd(u_i, u_j) = 1 \text{ for } i \neq j.$$

$$u_i, d \text{ positive integers.}$$

We claim this system has an optimal solution with $u_3 = 1$. Since $s_1 \geq s_2 \geq s_3$, any solution (d, u_1, u_2, u_3) can be assumed to have $u_1 \geq u_2 \geq u_3$. Furthermore, the constraint $\gcd(u_i, u_j) = 1$ for $i \neq j$ does not change the minimum value achieved, for if (d, u_1, u_2, u_3) is a solution with u_1, u_2 having a common factor f, say, then $(df, u_1/f, u_2/f, u_3)$ is a solution with the same objective value. So in what follows we forget about this constraint. Now assume (d, u_1, u_2, u_3) is an optimal solution with $u_3 > 1$. Then $(du_3, \lceil\frac{u_1}{u_3}\rceil, \lceil\frac{u_1}{u_3}\rceil, 1)$ is a solution with an objective value no larger than the first solution. Proving this amounts to showing that

$$u_3\lceil\frac{u_1}{u_3}\rceil\lceil\frac{u_2}{u_3}\rceil \le u_1u_2. \tag{4.4}$$

To verify the inequality, first notice it is trivial if $u_1 = u_3$ or $u_2 = u_3$. So assume $u_1 > u_3$ and $u_2 > u_3$. Then as

$$u_3\lceil\frac{u_1}{u_3}\rceil\lceil\frac{u_2}{u_3}\rceil \le u_3\frac{(u_1+u_3-1)}{u_3}\frac{(u_2+u_3-1)}{u_3}, \tag{4.5}$$

it is enough to show that

$$(u_1+u_3-1)(u_2+u_3-1) \le u_1u_2u_3. \tag{4.6}$$

After expanding and cancelling a common factor $(u_3 - 1)$ we find the equivalent inequality

$$u_3 \le (u_1-1)(u_2-1) \tag{4.7}$$

which is trivially true as $u_1, u_2 > u_3$ and all are integral.

Thus the system has an optimal solution with $u_3 = 1$. Taking $s_i' = u_id$, we find there is an optimal solution with $n = s_1's_2'$, and $s_3' = \gcd(s_1', s_2')$, which should be compared with Theorem 4.2.2 and Theorem 4.2.1.

Thus Jacroux's lower bound is actually telling us the smallest n for which there is a tight OMEP parameter set $s_1' \times s_2' \times s_3'//n$ with $s_i' \ge s_i$. Furthermore, since the above integral system has an optimal solution with $u_3 = 1$, the smallest n for which there is a tight OMEP parameter set $s_1' \times s_2' \times \ldots \times s_k'//n$ with $s_i' \ge s_i$ can be assumed to have the form $n = \mu_1\mu_2g^2$, and the tight parameter set can be assumed to have the form

$$\mu_1g \times \mu_2g \times g \times \ldots \times g//\mu_1\mu_2g^2 \tag{4.8}$$

Now if $g \geq 3$ then there are at most a finite number of parameters with the form (4.8) for which the tight OMEP does not exist. Thus if $s_1 \geq s_2 \ldots \geq s_k$, and $s_3 \geq 3$, then there are at most a finite number of choices for the other s_i for which Jacroux's bound is not tight. Even if $s_3 = 2$ and both s_1, s_2 are greater than 2 then there are still at most a finite number of cases where Jacroux's bound is not tight. This is what we mean by "Jacroux's bound is almost asymptotically tight".

4.3 Summary

In this chapter we have proven the asymptotic existence of tight orthogonal main effect plans, in the sense that for a fixed number k of rows, and with the exception of tight OMEPs with parameters $2^{k-1} \times 2s//4s$ for s odd, there is an N depending on k such that all tight OMEPs having N or more rows exist. We have applied this result to show that Jacroux's lower bound for OMEPs is often met with equality. We also found that OMEPs with parameters of the form

$$\mu_1 g \times \mu_2 g \times g \times \ldots \times g//\mu_1\mu_2 g^2 \tag{4.9}$$

are important when we are looking for minimal OMEPs. We give constructions which can produce OMEPs with these parameters in the next chapter.

Chapter 5

Recursive Constructions

Although direct product type constructions for OMEPs are useful, they give OMEPs having no more rows than the ingredient designs. Normally, one wishes to have as many rows as possible (for a fixed number of columns) since this means that more factors can be analyzed. There are constructions for producing OMEPs having large numbers of rows, though these constructions usually involve Hadamard matrices (see [10] for example). Furthermore, these constructions typically produce OMEPs having parameters of the form $t \times 4^{m_1} \times 2^{m_2}//n$, which can be restrictive if we have two factors having many levels, or if all factors have more than two levels. In the next section we give constructions which give equally replicated OMEPs having large numbers of rows which have neither of these restrictions. Similar methods have been applied to construct difference matrices in [9].

In the last section, we saw that OMEPs with parameters of the form

$$\mu_1 g \times \mu_2 g \times g \times \ldots \times g//\mu_1\mu_2 g^2 \tag{5.1}$$

are important when we are looking for minimal OMEPs. The constructions in this chapter can be used to construct OMEPs with these parameters.

5.1 Constructions using PBDs

For the first construction, we need the following structure.

Definition 5.1.1 *A $(\mu g - 1) \times g^{k-1}//\mu g^2 - g$ Modified OMEP (or MOMEP) is a $k \times (\mu g^2 - g)$ array having $\mu g - 1$ distinct symbols in row 1, having the symbol set $\{1, 2, \ldots g\}$ in each of the other rows, and with the property that any symbol from the first row occurs in a column exactly once with any symbol from row i, $1 < i \leq k$, any pair of* distinct *symbols from rows i and j, $1 < i < j \leq k$ occur together in a column exactly μ times, and identical symbols from rows i and j, $1 < i < j \leq k$, occur together in a column exactly $\mu - 1$ times.*

A resolvable orthogonal array $\mathrm{ROA}_\mu(k, g)$ with a parallel class $\{(i, i, i, \ldots, i)^T | 1 \leq i \leq g\}$ can be used to construct a $\mu g - 1 \times g^{k-1}//\mu g^2 - g$ MOMEP by removing this parallel class and by extending each other parallel class.

Recall that a PBD(v, λ) is a pair $(V, \mathcal{B})$, where $\mathcal{B}$ is a collection of subsets (called blocks) of the v-set V such that any pair of elements from V is contained in exactly λ of the blocks of $\mathcal{B}$. In some discussions of PBDs, blocks of size one are forbidden, but we do not require such a restriction in this chapter. If we can partition $\mathcal{B}$ into l classes each of size at most w so that each class is a partition of V then the PBD is said to be resolvable and we call it a $l \times w$ PBD(v, λ). As usual, each such class is called a parallel class.

Here is the main construction.

Theorem 5.1.2 *Suppose a $l \times w$ PBD(v, λ) exists, and a $\mu g - 1 \times g^{w+1}//\mu g^2 - g$ MOMEP $\mathcal{M}$ with $\lambda \mu g = l$ exists. Suppose $\lambda \mu = \alpha\beta$, for some positive integers α, β. Then an equally replicated $\alpha(\mu g - 1)g \times \beta g \times g^v//\alpha(\mu g - 1)\beta g^2$ OMEP exists.*

Proof: We construct the desired OMEP by concatenating together l smaller arrays; one smaller array A_i for each parallel class of the PBD(v, λ). The resulting array has symbol set $\{(x,a) | 1 \leq x \leq \mu g - 1, 1 \leq a \leq \alpha g\}$ in the first row, symbol set $\{(y,b) | 1 \leq y \leq g, 1 \leq b \leq \beta\}$ in the second row, and symbol set $\{z | 1 \leq z \leq g\}$ in each other row.

Let $\mathcal{N}$ be a $\alpha g \times \beta // g\alpha\beta$ OMEP, whose i'th column is $(n_{1i}, n_{2i})^T$. Let $L = [l_{ij}]$ be a latin square of side g. Let the PBD(v, λ) have point set $\{3, 4, \ldots, v+2\}$, and parallel classes $\{B_{i,1}, B_{i,2}, \ldots, B_{i,w_i}\}$, $1 \leq i \leq l$. Let the (i,j) entry of $\mathcal{M}$ be m_{ij}. The array A_i, $1 \leq i \leq l$ has $v+2$ rows and $\mu g^2 - g$ columns, and the symbol in row r and column c is

- (m_{1c}, n_{1i}) , if $r = 1$,
- $(l_{qm_{2c}}, n_{2i})$ where $q = (n_{1i} \bmod g) + 1$, if $r = 2$, and,
- m_{tc} where $B_{i,t-2}$ is the block in the i'th parallel class containing the point r, if $r > 2$.

We now verify that the array $\mathcal{D}$ obtained by concatenating these subarrays gives the desired OMEP. Any two symbols from the same row of $\mathcal{D}$ occur the same number of times in that row so if $\mathcal{D}$ is an OMEP it is equally replicated. Let (x,a) be a symbol from row 1 of $\mathcal{D}$, and let (y,b) be a symbol from row 2 of $\mathcal{D}$. Exactly one column of $\mathcal{N}$ is $(a,b)^T$; suppose it is the p'th column. Then the subarray corresponding to the p'th parallel class of the PBD has exactly one column with a (x,a) in row 1 and a (y,b) in row 2, and no other subarray has such a column. So this pair of symbols occurs in these rows in exactly one column of $\mathcal{D}$.

Let (x,a) be a symbol from row 1 of $\mathcal{D}$, and let z be a symbol from row i of $\mathcal{D}$, $i > 2$. The symbol (x,a) occurs in row 1 in β of the subarrays, and in each such subarray there is exactly one column containing (x, a) in row 1 and z in row i. So the pair of symbols occurs in the given rows a total of β times.

Let (y, b) be a symbol from the second row of $\mathcal{D}$, and let i index a row, $2 < i \leq v+2$. The symbol y also occurs in row i of $\mathcal{D}$. The symbol (y, b) occurs in row 2 in αg of the subarrays, and in $\alpha(g-1)$ of the subarrays there are μ columns containing the given symbols in the given rows, and in α of the subarrays there are $\mu - 1$ columns containing the given symbols in the given rows. Thus in total there are $\alpha(\mu g - 1)$ columns of $\mathcal{D}$ containing (y, b) in row 2 and y in row i.

Let x be a symbol from row i of $\mathcal{D}$, and y be a symbol from row j of $\mathcal{D}$, with $i \neq j$ and $x \neq y$, $2 < i < j \leq v + 2$. For those subarrays corresponding to the λ parallel classes where i and j are contained in a block of the parallel class, there are no columns containing these symbols in these rows. For the subarrays corresponding to the $l - \lambda$ other parallel classes, these symbols occur in these rows in μ columns, for a total of $\mu(l-\lambda) = \lambda\mu(\mu g - 1) = \alpha\beta(\mu g - 1)$. (Here we use the fact that $l = \lambda\mu g$, and $\lambda\mu = \alpha\beta$.)

Let x be a symbol from row i of $\mathcal{D}$, and y be a symbol from row j of $\mathcal{D}$, with $i \neq j$ and with $x = y$, $2 < i < j \leq v + 2$. For those subarrays corresponding to the λ parallel classes where i and j are contained in a block of the parallel class, there are $\mu g - 1$ columns containing these symbols in these rows. For the subarrays corresponding to the $l-\lambda$ other parallel classes, these symbols occur in these rows in $\mu-1$ columns, for a total of $\lambda(\mu g-1)+(l-\lambda)(\mu-1) = \lambda(\mu g-1)+\lambda(\mu g-1)(\mu-1) = \lambda\mu(\mu g - 1) = \alpha\beta(\mu g - 1)$. (Again using the fact that $l = \lambda\mu g$, and $\lambda\mu = \alpha\beta$.)

Thus we see that symbols from different rows occur in the correct number of columns and so the resulting array is an equally replicated $\alpha(\mu g - 1)g \times \beta g \times g^v // (\mu g - 1)\mu g^2$ OMEP. □

A nice corollary of this theorem is the following result, which produces tight OMEPs having a large number of rows.

Corollary 5.1.3 *Suppose a RBIBD*$(v, k, 1)$ *exists, and a resolvable orthogonal array ROA*$_\mu(v/k+1, g)$ *with* $\mu g = (v-1)/(k-1)$ *exists. Then a tight* $(\mu g - 1)g \times \mu g \times g^v//(\mu g-1)\mu g^2$ *OMEP exists.*

The fundamental idea in Theorem 5.1.2 is that we can obtain a design by judiciously taking unions of the blocks of smaller designs. The following construction helps illustrate the idea further.

Theorem 5.1.4 *If an equally replicated* $\lambda_1 g \times \lambda_2 g \times \ldots \times \lambda_t g \times g^w//\lambda_1\lambda_2\ldots\lambda_t g^2$ *OMEP* $\mathcal{M}$*, a ROA*$_\mu(k, g)$ $\mathcal{T}$*, a* $l \times w$ *PBD*(v, λ) $\mathcal{P}$ *having maximum block size at most* k*, and an equally replicated* $\mu_1 \times \mu_2 \times \ldots \mu_t//l\mu g$ *OMEP* $\mathcal{N}$ *all exist then an equally replicated* $\mu_1\lambda_1 g \times \mu_2\lambda_2 g \times \ldots \mu_t\lambda_t g \times g^v//l\mu\lambda_1\lambda_2\ldots\lambda_t g^3$ *OMEP exists.*

Proof: We construct the OMEP by concatenating together $l\mu g$ subarrays. The resulting OMEP has symbol set $\{(i, x)|1 \le i \le \lambda_r g, 1 \le x \le \mu_r\}$ in the rth row, $1 \le r \le t$, and symbol set $\{z|1 \le z \le g\}$ in each other row.

Let the (i, j)th entry of $\mathcal{M}$ be m_{ij}, and let the (i, j)th entry of $\mathcal{N}$ be n_{ij} Let the PBD(v, λ) have point set $\{t+1, t+2, \ldots, t+v\}$, and parallel classes $\{B_{i,1}, B_{i,2}, \ldots, B_{i,w_i}\}$, $1 \le i \le l$.

There are $\mu g l$ pairs (P, Q) where P is a parallel class of the PBD(v, λ) and Q is a parallel class of the ROA(k, g). For each block B of each parallel class $\{B_{i,1}, B_{i,2}, \ldots, B_{i,w_i}\}$ of the PBD(v, λ) we fix a ROA$_\mu(|B|, g)$ $\mathcal{T}$ where the rows are indexed by the points in B. Fix an ordering of the parallel classes of each ROA$_\mu(|B|, g)$ and fix an ordering of the blocks (columns) in each parallel class.

For the jth such pair (P, Q), we construct a subarray $A_{P,Q}$, having $t + v$ rows and $\lambda_1\lambda_2 \ldots \lambda_t g^2$ columns, and the symbol in row r and column c is

- (m_{rc}, n_{rj}) , if $r \leq t$,
- z, where z is the point in the group indexed by r in the m_{xc}th block of Q, where $B_{p,x-t}$ is the block in P containing the point r, if $r > t$.

We now verify that the array $\mathcal{D}$ obtained by concatenating together these subarrays gives the desired OMEP. It is clear that the result is equally replicated, so let (x, a) be a symbol from row i of $\mathcal{D}$, and let (y, b) be a symbol from row j of $\mathcal{D}$, where $1 \leq i < j \leq t$. There are $l\mu g/(\mu_i\mu_j)$ columns of $\mathcal{N}$ with an a in row i and a b in row j. The subarrays corresponding to these columns each contain $\lambda_1\lambda_2 \ldots \lambda_t/(\lambda_i\lambda_j)$ columns with (x, a) in row i and (y, b) in row j, for a total of $(l\mu g\lambda_1\lambda_2 \ldots \lambda_t)/(\lambda_i\mu_i\lambda_j\mu_j)$ columns alltogether (which is the number required for the OMEP property).

Next let (x, a) be a symbol from row i and let z be a symbol from row j of $\mathcal{D}$ with $1 \leq i \leq t < j \leq v$. There are $l\mu g/\mu_i$ columns of $\mathcal{N}$ with symbol a in row i. The subarrays corresponding to these columns each have $\lambda_1\lambda_2 \ldots \lambda_t/\lambda_i$ columns containing symbol (x, a) in row i and symbol z in row j, giving a total of $(l\mu g\lambda_1\lambda_2 \ldots \lambda_t)/(\mu_i\lambda_i)$ such columns, which again is the desired number of columns.

Finally, let z_1 be a symbol from row i and let z_2 be a symbol from row j of $\mathcal{D}$, with $t < i < j \leq v$. The λ subarrays corresponding to the parallel classes of the PBD in which points i, j are in a block of the parallel class each contain $\mu(\lambda_1\lambda_2 \ldots \lambda_t g)$ columns with a z_1 in row i and a z_2 in row j. Each of the $l - \lambda$ other subarrays contain $(\mu g)\lambda_1\lambda_2 \ldots \lambda_t$ columns with these symbols in these rows. In total this makes for $\lambda_1\lambda_2 \ldots \lambda_t\mu g l$ such columns, which is the correct number of columns.

□

To describe another construction, we need another definition.

Definition 5.1.5 *An (s, t, g, w, λ)-MOMEP is an $s^2 \times (t+w)$ array, having symbol set $\{1, 2, \ldots, s\}$ in rows 1 through t, and symbol set $\{1, 2, \ldots, g\}$ in rows $t+1$ through $t+w$, such that*

- *each symbol in row i occurs in a column with each symbol from row j exactly once, $1 \leq i < j \leq t$,*
- *each symbol from row i occurs in a column with symbol x from row j exactly $\mu_{i,x}$ times, $1 \leq i \leq t < j \leq t+w$,*
- *each symbol x from row i occurs in a column with symbol x from row j exactly λ_x times, $t+1 \leq i < j \leq t+w$,*
- *each symbol x from row i occurs in a column with symbol y from row j exactly λ times, $t+1 \leq i < j \leq t+w$, $x \neq y$.*

Table 5.1 gives an example of such a structure. Although the parameters μ_{ix}, λ_x may be of interest in actually constructing such objects, only the values of s, t, g, w, λ will be of interest in the next construction, which explains why we list only these parameters when describing the object. Some relations hold among the parameters. In particular, $\sum_{x=1}^{g} \mu_{i,x} = s$.

Theorem 5.1.6 *Suppose we have an $l \times w$ PBD$(v, 1)$ and a (s, t, g, w, λ)-MOMEP $\mathcal{M}$ so that*

- $\lambda = (ls^2)/(g^2(l-1))$,
- *$l = \alpha_1\alpha_2 \ldots \alpha_t g^2$, for some positive integers α_i,*

1	2	3	1	2	3	1	2	3
1	2	3	2	3	1	3	1	2
1	1	1	2	2	2	2	2	2
2	2	2	1	1	1	2	2	2
2	2	2	2	2	2	1	1	1

Table 5.1: A $(3,2,2,3,3)$-MOMEP

- *an equally replicated* $\alpha_1 g \times \alpha_2 g \times \ldots \times \alpha_t g \times g//l$ *OMEP exists.*

Then an equally replicated $\alpha_1 sg \times \alpha_2 sg \times \ldots \times \alpha_t sg \times g^v//ls^2$ *OMEP exists.*

Proof: We construct the OMEP by concatenating together l subarrays. The resulting OMEP has symbol set $\{(x,a)|1 \le x \le s, 1 \le a \le \alpha_i g\}$ in the row i, $1 \le i \le t$, and symbol set $\{x|1 \le x \le g\}$ in each other row.

Let $\mathcal{N}$ be an equally replicated $\alpha_1 g \times \alpha_2 g \times \ldots \times \alpha_t g \times g//l$ OMEP. Let the (r,c)th entry of $\mathcal{M}$ be $m_{r,c}$, and let the (r,c)th entry of $\mathcal{N}$ be $n_{r,c}$. Let $L = [l(r,c)]$ be a latin square of side g. Let the PBD$(v,1)$ have point set $\{t+1, t+2, \ldots, t+v\}$, and parallel classes $\{B_{i,1}, B_{i,2}, \ldots, B_{i,w_i}\}$, $1 \le i \le l$.

The pth subarray, $1 \le p \le l$ has $t+v$ rows and s^2 columns, and the symbol in row r and column c is

- (m_{rc}, n_{rp}) , if $r \le t$,
- $l(n_{t+1,p}, m_{z,c})$, where z is such that the block $B_{p,z-t}$ contains the point r, if $r > t$.

We now verify that $\mathcal{D}$, the concatenation of these subarrays, is the desired OMEP. Any two symbols in the same row of $\mathcal{D}$ occur the same number of times so if $\mathcal{D}$ is an OMEP it is equally replicated.

Let (x_1, a_1) be a symbol from row i of $\mathcal{D}$, and let (x_2, a_2) be a symbol from row j of $\mathcal{D}$, with $1 \le i < j \le t$. Exactly $l/(\alpha_i \alpha_j g^2)$ columns of $\mathcal{N}$ have an a_1 in row i and a a_2 in row j. Suppose the pth column of $\mathcal{N}$ is one such column. Then the subarray corresponding to the pth parallel class of the PBD has exactly one column with an (x_1, a_1) in row i and a (x_2, a_2) in row j. Only these $l/(\alpha_i \alpha_j g^2)$ subarrays have such a column, and so this pair of symbols occurs in these rows in exactly $l/(\alpha_i \alpha_j g^2)$ columns of $\mathcal{D}$.

Let (x, a) be a symbol from row i of $\mathcal{D}$, and let y be a symbol from row j of $\mathcal{D}$, $1 \le i \le t < j \le t + w$. The symbol (x, a) occurs in row i in $l/(\alpha_i g)$ of the subarrays, and for each z, $1 \le z \le g$ there are $l/(\alpha_i g^2)$ of these subarrays that have μ_{iz} columns containing (x, a) in row i and y in row j. So the pair of symbols occurs in the given rows a total of $\frac{l}{(\alpha_i g^2)}(\sum_{z=1}^{g} \mu_{iz}) = (ls)/(\alpha_i g^2)$ times.

Let x be a symbol from row i of $\mathcal{D}$, and y be a symbol from row j of $\mathcal{D}$, with $t < i < j \le v + t$ and $x \neq y$. For that subarray corresponding to the parallel class where i and j are contained in a block of the parallel class, there are no columns containing these symbols in these rows. For the subarrays corresponding to the $l - 1$ other parallel classes, these symbols occur in these rows in λ columns, for a total of $\lambda(l-1) = ls^2/g^2$ columns. (Here we use the fact that $\lambda = (ls^2)/(g^2(l-1))$.)

Let x be a symbol from row i of $\mathcal{D}$, and let j index another row of $\mathcal{D}$, with $t < i < j \le v + t$. Exactly one parallel class of the PBD has a block containing the points i, j, suppose it is the pth. The number of columns of $\mathcal{D}$ containing x in row i and row j is $\frac{l}{g}(\lambda_1 + \lambda_2 + \ldots + \lambda_g) + b_q - \lambda_q$, where q is the column of L containing symbol x in row $n_{t+1,p}$, and b_q is the number of columns of M containing a q in row $t + 1$. (Actually, b_q is the same for all rows except the first t rows.) As

$b_q - \lambda_q = (g-1)\lambda$, and $\sum_{z=1}^{g} \lambda_z = s^2 - g(g-1)\lambda$, a little arithmetic shows this reduces to ls^2/g^2.

Thus we see that symbols from different rows occur in the correct number of columns and so $\mathcal{D}$ is an equally replicated $\alpha_1 sg \times \alpha_2 sg \times \ldots \times \alpha_t sg \times g^v$ OMEP. □

For example, applying this construction using a RBIBD(9, 3, 1) and the (3, 2, 2, 3, 3)-MOMEP in Table 5.1, one obtains an equally replicated $6 \times 6 \times 2^9//36$ OMEP.

It is possible to generalize the form of the MOMEP given in Table 5.1.

Lemma 5.1.7 *Suppose we have ROA(k, s) and a OA$_\mu(w, g)$ with $\mu g^2 - 1 = s$. Then a $(s, k, g, w, \mu s)$-MOMEP exists.*

Proof: Without loss of generality, the OA$_\mu(w, g)$ has symbol set $\{1, 2, \ldots, g\}$, and has a column consisting entirely of 1's. Let $c_1, c_2, \ldots, c_{\mu g^2 - 1}$ be the remaining columns. Let $P_1, P_2, \ldots, P_s$ be the parallel classes of the ROA(k, s). (So each P_i is a set of s columns.) Then a $(s, k, g, w, \mu s)$-MOMEP is given by appending the column c_i to column j of the ROA(k, s) whenever column j is in P_i. Verification that this is the desired MOMEP is routine. □

Finally, we give a new construction for resolvable orthogonal arrays. Such designs are useful to us since by extending parallel classes we obtain tight OMEPs.

Theorem 5.1.8 *Suppose there exists a ROA$_{\mu_1}(k, g_1)$, a ROA$_{\tau_2}(w, g_2)$, ROA$_{\tau_1}(w, g_1)$, and a $l \times w$ PBD(v, λ) with maximum block size at most k, so that $\mu_1 g_1 = \tau_2 g_2 - 1$ and $(l - \lambda) = \lambda \mu_1 g_1$. Then a ROA$_{(l-\lambda)\tau_2\tau_1}(v, g_1 g_2)$ exists.*

Proof: The points in each row of the resulting orthogonal array will be from the set $\{(x, y) | 1 \le x \le g_1, 1 \le y \le g_2\}$. Let the ith parallel class of the PBD be

$B_{i,1}, B_{i,2}, \ldots, B_{i,w_i}$. For each i, $1 \le i \le l$, we will use a $\mathrm{ROA}_{\tau_2}(w_i, g_2)$ missing a parallel class $\{(i, i, \ldots, i)^T | 1 \le i \le g_2\}$. (Any $\mathrm{ROA}_{\tau_2}(w_i, g_2)$ can be assumed to have such a parallel class, by possibly permuting the symbols set in each row.) Denote the column set of this structure by C_i

For each block B of the PBD, fix a $\mathrm{ROA}_{\mu_1}(|B|, g_1)$ with rows indexed by the points in B. For each i, $1 \le i \le l$, and each symbol x in each row α of the $\mathrm{ROA}_{\tau_2}(w_i, g_2)$, fix a bijection $f_{i,x,\alpha}$ between the $\mu_1 g_1$ parallel classes of the $\mathrm{ROA}_{\mu_1}(|B_{i,\alpha}|, g_1)$ associated with block $B_{i,\alpha}$ and the $\tau_2 g_2 - 1$ columns in C_i containing symbol x in row α.

Next, for each i, $1 \le i \le l$, and each column $(a_1, a_2, \ldots, a_{w_i})^T$ in C_i, and each column $(b_1, b_2, \ldots, b_{w_i})^T$ in a $\mathrm{ROA}_{\tau_1}(w_i, g_1)$, we construct a column

$$((x_1, y_1), (x_2, y_2), \ldots, (x_v, y_v))^T$$

in the final array, where x_r, y_r are defined as follows. Let B_{ij} be the unique block in the ith parallel class of the PBD which contains the point r, and let D be the $\mathrm{ROA}_{\mu_1}(|B|, g_1)$ associated with this block. Then y_r is defined to be a_j, and x_r is defined to be the symbol in the row indexed by r and in the b_rth column of the parallel class of D whose image under $f_{i,a_j,j}$ is $(a_1, a_2, \ldots, a_w)^T$. We claim that the resulting collection of columns is the desired $\mathrm{ROA}_{(l-\lambda)\tau_2\tau_1}(v, g_1 g_2)$

To verify that the resulting array is an $\mathrm{ROA}_{(l-\lambda)\tau_2\tau_1}(v, g_1 g_2)$, choose a pair of symbols $(x_1, y_1), (x_2, y_2)$ from the set $\{(x, y) | 1 \le x \le g_1, 1 \le y \le g_2\}$. and a distinct pair of rows r_1, r_2 from $\{1, 2, \ldots, v\}$. We count the number of columns in the final array which contain these symbols in these rows (respectively).

First suppose $y_1 \ne y_2$. For each of the $l - \lambda$ parallel classes of the PBD in which r_1 and r_2 do not occur together in a block, there are a total of $\tau_2\tau_1$ columns in the final array containing the given symbols in the given rows. The λ parallel classes of the PBD in which r_1, r_2 occur in a block contribute no such columns to the final array. Thus there are a total of $(l-\lambda)\tau_2\tau_1$ columns of the resulting array containing the given symbols in the given rows.

Next suppose $y_1 = y_2$. For each of the $l-\lambda$ parallel classes of the PBD in which r_1 and r_2 do not occur together in a block, there are a total of $(\tau_2 - 1)\tau_1$ columns in the final array containing the given symbols in the given rows. Further, for the λ parallel classes of the PBD in which r_1, r_2 occur in a block, there are $\mu_1\tau_1 g_1$ columns of the final array in which these symbols occur in these rows. This makes for a total of $(l-\lambda)(\tau_2-1)\tau_1 + \lambda\mu_1\tau_1 g_1$ which simplifies to $(l-\lambda)\tau_2\tau_1$ since $(l-\lambda) = \lambda\mu_1 g_1$. Thus in this case the given symbols occur in the given rows in $(l-\lambda)\tau_2\tau_1$ columns of the final array.

It is easy to see that the resulting array is resolvable, since the columns of the final array arising from a parallel class of the PBD, a parallel class in some C_i and a parallel class of the $\mathrm{ROA}_{\tau_1}(w_i, g_1)$s are a parallel class of the final orthogonal array.

Thus we see that the resulting array is a $\mathrm{ROA}_{(l-\lambda)\tau_2\tau_1}(v, g_1g_2)$. □

We mention that if each of the ingredient orthogonal arrays come from difference matrices (over G_1, G_2), then the resulting orthogonal array has an automorphism in the group $G_1 \times G_2$, and thus an associated difference matrix exists. Furthermore we mention that if our ingredient designs are instead a $\mathrm{ROA}_{\mu_1}(k, g_1)$, an $\mathrm{OA}_{\tau_2}(w, g_2)$ (with a parallel class), an $\mathrm{OA}_{\tau_1}(w, g_1)$, and a $l \times w$ $\mathrm{PBD}(v, \lambda)$ with maximum block size at most k, with $\mu_1 g_1 = \tau_2 g_2 - 1$ and $(l-\lambda) = \lambda\mu_1 g_1$, then a $\mathrm{OA}_{(l-\lambda)\tau_2\tau_1}(v, g_1g_2)$ exists.

This theorem appears to give the best known results for certain values of g and λ. Colbourn and Kreher [9] contains a table giving the best lower bounds on k in a (g, k, λ)-difference matrix given g, λ. The above construction provides better bounds in a number of cases. For example, using a $\mathrm{RTD}_2(4,2)$, a $\mathrm{RTD}(4,5)$, a 5×4 PBD(16, 4), and a $\mathrm{RTD}_2(4,2)$ in Theorem 5.1.8 gives a $\mathrm{RTD}_8(16,10)$. If the ingredients come from difference matrices then the resulting RTD has an associated difference matrix, and so a $(10,16,8)$-difference matrix exists, whereas the best known k for $g = 10$, $\lambda = 8$ in [9] is 10. Using a $\mathrm{RTD}(3,3)$, a $\mathrm{RTD}(3,4)$, a 4×3 PBD(9, 3), and a $\mathrm{RTD}(3,3)$ in Theorem 5.1.8 gives a $\mathrm{RTD}_3(9,12)$. As above one can construct a $(12,9,3)$-difference matrix if the ingredient RTD's come from difference matrices. The lower bound on k for $g = 12$, $\lambda = 3$ in [9] is 6. Similarly we can obtain a $(6,9,6)$-difference matrix and a $(15,25,10)$-difference matrix using Theorem 5.1.8. The best lower bounds on k for these parameters in [9] are 6 and 7 respectively.

5.2 Summary

In this chapter we have given new recursive constructions for orthogonal main effect plans. These constructions have the advantage that the resultant OMEPs have more rows than the ingredient designs. Further, the number of levels for each factor is not particularly restricted, as is the case with some constructions based on Hadamard matrices, for example. The constructions have the additional advantage that the OMEPs constructed are equally replicated.

Chapter 6

Conclusions

In the previous chapters we see several ideas involving tight OMEPs. The second chapter introduces the concept of a tight OMEP, and shows that answering the existence question for tight OMEPs helps in answering the existence question for general OMEPs and equally replicated OMEPs. In the third chapter it is shown that any three-factor OMEP can be uncollapsed to a tight OMEP. Hence, modulo the collapsing of levels, all structural information about three-factor OMEPs is contained in the class of three-factor tight OMEPs. In the fourth chapter it is shown that practically all tight OMEPs exist, in the sense that for a fixed number of rows and with the exception of one small infinite class, there are only a finite number of parameters for which the corresponding tight OMEP does not exist. This result allows for a better understanding of Jacroux's lower bound on the number of runs in an OMEP. Even the constructions given in the fifth chapter were found by considering OMEPs of the form

$$\lambda_1 g \times \lambda_2 g \ldots \times \lambda_k g // \lambda_1 \ldots \lambda_k g^2, \tag{6.1}$$

which is a form suggested by tight OMEPs. Thus there is considerable evidence that the concept of tight OMEPs is a useful one.

What further research is suggested by the results in this thesis? An obvious problem is in answering the existence question for tight OMEPs having six or more factors. As with OMEPs having fewer factors, such research will help in answering the general existence question.

More information about the uncollapsing of OMEPs having four or more factors would be helpful. Although we have shown that a general result like the three factor case is not possible, perhaps a large class of OMEPs can be uncollapsed to tight OMEPs. Even results specifically concerning four-factor OMEPs would be useful. It may be possible to apply the uncollapsing result about three-factor OMEPs to obtain structural information about such OMEPs. For example, a result concerning the existence of repeated columns in tight three-factor OMEPs might give a result about repeated columns in general three-factor OMEPs.

It would be interesting to obtain non-existence results for OMEPs also. For example, it is well known that an equally replicated $g^k//g^2$ OMEP cannot exist if $k > g + 1$. This is a non-existence result for tight OMEPs having parameters as in (6.1) and with all λ_i's equal to one. Perhaps there are more general results available if we allow the λ_i's to vary a small amount. Such a result would give a better idea of just how plentiful tight OMEPs are.

Of course, more constructions for OMEPs, and tight OMEPs in particular, would be most helpful. The recursive constructions in chapter five are powerful, but as with many recursive constructions, they often produce designs having a large number of runs. Thus more direct constructions for tight OMEPs would be useful, as such constructions might give OMEPs which can be used both for practical use and for use in recursive constructions.

Chapter 7

Appendix

An equally replicated $2 \times 2 \times 2 \times 6 \times 6//36$ OMEP.

0	0	1	0	1	1	0	1	0	1	0	1	0	0	0	1	1	1	0	0	1	0	1	1	1	0	1	1	0	0	0	0	0	1	1	1
0	0	0	1	1	1	0	0	1	0	1	1	0	1	0	1	0	1	0	0	0	1	1	1	0	0	1	0	1	1	1	0	1	1	0	0
0	1	0	1	0	1	0	0	0	1	1	1	0	0	1	0	1	1	1	0	1	1	0	0	0	0	0	1	1	1	0	0	1	0	1	1
0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4	5	5	5	5	5	5
0	1	2	3	4	5	0	1	2	3	5	4	4	1	2	3	5	0	3	4	1	2	0	5	3	5	1	2	0	4	3	5	1	2	0	4

There is a completely resolvable $OA_2(5, 6)$, from which we obtain a $6 \times 6 \times 6 \times 6 \times 6 \times 12//72$ OMEP. We provide a difference matrix over Z_6 whose development gives the desired orthogonal array.

0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	2	2	3	3	4	4	5	5
5	4	5	3	2	1	4	3	1	0	2	0
5	5	2	4	2	3	0	1	0	4	1	3
5	5	3	2	4	0	1	3	4	1	0	2

An equally replicated $6 \times 6 \times 6 \times 6 \times 6 \times 18//108$ OMEP can be obtained from the following completely resolvable $OA_3(5,6)$. Develop the following parallel class over Z_5 to obtain the first five parallel classes. In all of what follows, X is a fixed point (so that $X + \text{anything} = X$).

X	0	1	2	3	4
4	X	3	2	1	0
4	3	X	2	0	1
4	2	1	X	3	0
4	2	1	0	X	3

Develop the following parallel class over Z_5 to get the next five parallel classes.

X	0	1	2	3	4
4	X	3	2	1	0
3	4	X	0	2	1
0	2	4	X	1	3
3	1	0	4	X	2

Develop the following parallel class over Z_5 to get the next five parallel classes.

X	0	1	2	3	4
4	X	3	2	1	0
1	0	X	2	4	3
3	1	0	X	2	4
4	0	2	3	X	1

Finally, here are the sixteenth, seventeenth, and eighteenth parallel classes.

0	1	2	3	4	X	0	1	2	3	4	X	0	1	2	3	4	X
4	0	1	2	3	X	4	0	1	2	3	X	4	0	1	2	3	X
3	4	0	1	2	X	1	2	3	4	0	X	1	2	3	4	0	X
3	4	0	1	2	X	2	3	4	0	1	X	1	2	3	4	0	X
4	0	1	2	3	X	0	1	2	3	4	X	3	4	0	1	2	X

An equally replicated $10 \times 10 \times 10 \times 10 \times 30//300$ OMEP. Develop the following parallel class over Z_9 to obtain the first 9 parallel classes.

X	0	1	2	3	4	5	6	7	8
8	X	7	6	5	4	3	2	1	0
8	7	X	6	5	3	4	1	0	2
8	7	6	X	5	1	2	0	4	3

Develop the following parallel class over Z_9 to obtain the next 9 parallel classes.

X	0	1	2	3	4	5	6	7	8
8	X	6	7	4	5	1	3	0	2
7	8	X	6	3	4	2	0	1	5
3	4	1	X	0	2	8	6	5	7

Develop the following parallel class over Z_9 to obtain the next 9 parallel classes.

X	0	1	2	3	4	5	6	7	8
8	X	5	4	3	1	0	2	7	6
2	0	X	8	4	7	6	5	1	3
7	6	8	X	3	0	1	4	5	2

Finally, the 28th, 29th, and 30th parallel classes

0	1	2	3	4	5	6	7	8	X	0	1	2	3	4	5	6	7	8	X
5	6	7	8	0	1	2	3	4	X	5	6	7	8	0	1	2	3	4	X
7	8	0	1	2	3	4	5	6	X	6	7	8	0	1	2	3	4	5	X
8	0	1	2	3	4	5	6	7	X	7	8	0	1	2	3	4	5	6	X

0	1	2	3	4	5	6	7	8	X
5	6	7	8	0	1	2	3	4	X
6	7	8	0	1	2	3	4	5	X
3	4	5	6	7	8	0	1	2	X

There is a completely resolvable $OA_2(6,3)$, from which we obtain a $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 6//18$ OMEP. We provide a difference matrix over Z_3 whose development gives the desired orthogonal array.

0	0	0	0	0	0
0	0	1	1	2	2
0	1	0	2	1	2
0	1	2	0	2	1
0	2	1	2	0	1
0	2	2	1	1	0

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