Complexity of Classes of Structures

by

Carolyn Knoll

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2013

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The main theme of this thesis is studying classes of structures with respect to various measurements of complexity. We will briefly discuss the notion of computable dimension, while the breadth of the paper will focus on calculating the Turing ordinal and the back-and-forth ordinal of various classes, along with an exploration of how these two ordinals are related in general.

Computable structure theorists study which computable dimensions can be realized by structures from a given class. Using a structural characterization of the computably categorical equivalence structures due to Calvert, Cenzer, Harizanov and Morozov, we prove that the only possible computable dimension of an equivalence structure is 1 or ω.

In 1994, Jockusch and Soare introduced the notion of the Turing ordinal of a class of structures. It was unknown whether every computable ordinal was the Turing ordinal of some class. Following the work of Ash, Jocksuch and Knight, we show that the answer is yes, but, as one might expect, the axiomatizations of these classes are complex. In 2009, Montalbán defined the back-and-forth ordinal of a class using the back-and-forth relations. Montalbán, following a result of Knight, showed that if the back-and-forth ordinal is $n+1$, then the Turing ordinal is at least n. We will prove a theorem stated by Knight that extends the previous result to all computable ordinals and show that if the back-and-forth ordinal is α (infinite) then the Turing ordinal is at least α.

It is conjectured at present that if a class of structures is relatively nice then the Turing ordinal and the back-and-forth ordinal of the class differ by at most 1. We will present many examples of classes having axiomatizations of varying complexities that support this conjecture; however, we will show that this result does not hold for arbitrary Borel classes. In particular, we will prove that there is a Borel class with infinite Turing ordinal but finite back-and-forth ordinal and show that, for each positive integer d, there exists a Borel class of structures such that the Turing ordinal and the back-and-forth ordinal of the class are both finite and differ by at least d.

Acknowledgements

First, I would sincerely like to thank the Department of Pure Mathematics at the University of Waterloo for creating such an enjoyable environment over the years. I truly believe that the kindness and support provided by the members of this department cannot be matched, including their commitment to mentor and encourage those students with a desire to teach. I would like to say a special thank you to the exceptional staff, Lis D'Alessio, Nancy Maloney, Pavlina Penk and Shonn Martin, on whom I have relied on countless occasions.

I wish also to recognize all of those in the department who participated in the computability learning seminar for a year, a term or even just one or two lectures. It was a pleasure discussing the wonders of computability theory with you. And a particular thank you to the small but mighty algebra and logic community at the University of Waterloo for providing me with enriching seminars outside my area during the terms when I needed a substitute mathematical family.

I would like to thank the readers of this thesis, Joseph Miller, Rahim Moosa, Ross Willard and Ming Li, for all of their comments and advice. In addition, I would like to thank Antonio Montalbán and Stanley Burris for various helpful conversations during the production of this work.

Finally, I wish to express my gratitude to my advisor, Barbara Csima. Nearly six years of support, instruction, entertainment and weekly meetings amount to a debt that I cannot quantify. The fact that your commitment did not diminish while attempting many months of long distance supervision and during the addition of two new members to your family is simply astounding.

Dedication

To my parents, Jo Wearing and Sandor Knoll, and to my sister, Jenny. This is what I have been doing in Ontario for the past decade.

Table of Contents

Chapter 1

Introduction

The main theme of this thesis is assigning complexity to classes of structures using various measurements. Chapter 2 will discuss the notion of computable dimension, while Chapters 3-7 will focus on the Turing ordinal and the back-and-forth ordinal.

Computable structure theorists study which computable dimensions can be realized by structures from a given class. When there is a structural characterization of the computably categorical structures of a class, we generally expect the only possible computable dimensions to be 1 and ω. In Chapter 2, using such a characterization of Calvert, Cenzer, Harizanov and Morozov of the computably categorical equivalence structures, we prove that the only possible computable dimension of an equivalence structure is 1 or ω.

In Chapter 3, we will discuss a measurement called the back-and-forth ordinal, defined by Montalbán in 2009. This is defined to be the first ordinal α such that there are uncountably many infinitary Σ_α types realized by tuples from structures in the class. Montalbán, following Knight, discovered the relationship between the back-and-forth ordinal of a class and the ease of coding non-trivial information into structures in the class. In the last section, we will explain why the result of Montalbán cannot be improved.

The results from Chapter 3 suggest a connection between the back-and-forth ordinal and a computability-theoretic measurement, introduced by Jockusch and Soare in 1994, called the Turing ordinal. Roughly speaking, this ordinal measures how difficult it is to code information into jumps of structures in a given class. Montalbán shows that if the back-and-forth ordinal is $n+1$, then the Turing ordinal is at least n. A result of Knight's can be generalized to extend this bound to transfinite levels. For completeness, we present a proof of this generalization that uses forcing. It will follow that if the back-and-forth ordinal is α and infinite, then the Turing ordinal is at least α. At the time the back-and-forth ordinal was introduced, all classes where both ordinals were known actually had the back-and-forth ordinal *equal* to the successor of the Turing ordinal (in the finite case) or *equal* to the Turing ordinal (in the infinite case). In this Chapter, we will show that this is not the case in general.

In Chapter 5, we calculate the back-and-forth ordinal of a family of classes of linear orderings, defined by Downey and Jockusch, and observe that they follow the same pattern as discussed above. In 2009, it was unknown whether or not every computable ordinal was the Turing ordinal of some class of structures. In Chapter 6, following the work of Ash, Jockusch and Knight, we show that the answer is yes. We also calculate the back-and-forth ordinals of the classes for all successor ordinals $\alpha \leq \omega + 2$ and show that the two ordinals are off by at most 1 in each case.

In the final chapter, we will complete the analysis for the class corresponding to $\alpha = \omega$ and show that there exist Borel classes where the ordinals are finite and arbitrarily far apart. More precisely, for each positive integer d, we will define a Borel class of structures such that the Turing ordinal and the back-and-forth ordinal of the class are both finite and differ by at least d.

Before we begin, we will introduce the relevant notation that will appear throughout the paper, and review the main concepts needed from computability theory and computable structure theory.

1.1 Notation and Background

For computability theory, we will follow the notational conventions from [29].

Definition 1.1.1. (i) Let $\{\varphi_e\}_{e\in\omega}$ be an effective listing of all *partial computable functions*, and hence $W_e := dom(\varphi_e)$ denotes the e^{th} *computably enumerable (c.e.) set*.

(ii) Let $\{\Phi_e^X\}_{e\in\omega}$ be an effective listing of all *Turing functionals* and hence $\{W_e^A\}_{e\in\omega}$ is an effective listing of all the sets the are *computably enumerable relative to (c.e. in) A*.

Definition 1.1.2. Fix any computable bijection from $\omega \times \omega$ to ω which we will denote by $\langle \cdot, \cdot \rangle$. This is called a pairing function. Let $\langle x, y, z\rangle$ denote $\langle\langle x, y\rangle, z\rangle$, and in general, the n^{th} pairing function is denoted by $\langle x_1, x_2, \ldots, x_n\rangle = \langle \ldots \langle\langle x_1, x_2\rangle, x_3\rangle, \ldots, x_n\rangle$ where $\langle \cdot, \cdot \rangle$ is the fixed pairing function.

For computable structure theory, we will follow [2].

Definition 1.1.3. A formal *structure*, $\mathcal{A}$, consists of a domain along with some basic functions and relations on that domain. Structures will always have domain ω or, if finite, an initial segment of ω.

Taking the domain of a structure to be a subset of the natural numbers allows us to study the complexity of the structure as follows: Let $\mathcal{A}$ be a structure, in a language $\mathcal{L}$, and let A denote the domain of $\mathcal{A}$. An $(\mathcal{L} \cup A)$-sentence is a sentence in the language $\mathcal{L}$ where we allow parameters from A. Every language will be countable (and effectively presented) and hence we can fix an enumeration of all $(\mathcal{L} \cup A)$-sentences, say $\{\psi_i\}_{i\in\omega}$. We will associate the sentence ψ_i with its index $i \in \omega$.

Definition 1.1.4. The *atomic diagram* of $\mathcal{A}$, denoted by $D(\mathcal{A})$, is the set of atomic and negated atomic $(\mathcal{L} \cup A)$-sentences true in $\mathcal{A}$.

We say that a structure is *computable* if its atomic diagram is computable. Observe that this definition is equivalent to all of the basic functions and relations of the structure being computable. One natural way to assign a degree to a structure that is not necessarily computable is to let the degree of $\mathcal{A}$ be the Turing degree of the set $D(\mathcal{A})$. It not hard to see that different isomorphic copies of a structure can have different degrees in this sense. In fact, there need not be any relationship between the degrees of $D(\mathcal{A})$ and $D(\mathcal{B})$ even if $\mathcal{A} \cong \mathcal{B}$. For example, if we let $\mathcal{A}$ be the natural numbers as a linear ordering, then for any Turing degree $\mathbf{d}$, there is a structure $\mathcal{B} \cong \mathcal{A}$ such that $D(\mathcal{B})$ is of degree $\mathbf{d}$. Therefore, in order to assign a degree to an isomorphism class of a structure, we consider the following invariant notion of degree.

Definition 1.1.5. Let $\mathcal{A}$ be a structure.

(1) We define the *degree spectrum of* $\mathcal{A}$ to be $\mathrm{Spec}(\mathcal{A}) = \{\deg(D(\mathcal{B})) : \mathcal{B} \cong \mathcal{A}\}$, the collection of Turing degrees of all presentations of $\mathcal{A}$.

(2) We say that $\mathcal{A}$ has *degree* $\mathbf{d}$ if $\mathbf{d}$ is the least member of $\mathrm{Spec}(\mathcal{A})$.

We say that a structure $\mathcal{A}$ is *trivial* if there is a finite tuple $\vec{a}$ in $\mathcal{A}$ such that any permutation of the domain of $\mathcal{A}$ that fixes the tuple $\vec{a}$ is an automorphism of $\mathcal{A}$. A result of Knight shows that the degree spectrum of a trivial structure is a singleton, while the degree spectrum of a non-trivial structure is upward closed [21]. Therefore if a (non-trivial) structure has degree $\mathbf{d}$, then its spectrum forms an upper cone in the Turing degrees and hence the degree $\mathbf{d}$ reflects the information that is contained in this isomorphism class.

Even if the spectrum is not an upper cone, then we can still attempt to assign a complexity to the structure. Perhaps there is information contained in the jump of the structure, or even some iterated jump. Recall that by iterating the jump operator we can define the n^{th} jump of an arbitrary set X, denoted by $X^{(n)}$. Jockusch suggested the following family of measures which generalizes Definition 1.1.5 to n^{th} jump degrees for all $n \geq 0$.

Definition 1.1.6 (Jockusch)**.** Let $\mathcal{A}$ be a structure.

(1) The n^{th} *jump degree spectrum of* $\mathcal{A}$ is defined as $\mathrm{Spec}^{(n)}(\mathcal{A}) = \{\deg(D(\mathcal{B}))^{(n)} : \mathcal{B} \cong \mathcal{A}\}$.

(2) We say that $\mathcal{A}$ has n^{th} *jump degree* $\mathbf{d}$ if $\mathbf{d}$ is the least member of $\mathrm{Spec}^{(n)}(\mathcal{A})$.

We can define corresponding spectra for all computable ordinals as well. For a full treatment of computable ordinals and the hyperarithmetical hierarchy, see [2]. We will provide a brief summary here, for our purposes.

A *computable ordinal* is an ordinal which is isomorphic to some computable well-ordering. Clearly, all computable ordinals are countable and it can be shown that the computable ordinals form an initial segment of the ordinals. The first noncomputable ordinal is denoted by ω_1^{CK}. For an effective treatment of ordinals, we need to attach "names" (more precisely natural numbers) to as many ordinals as possible, in some "useful" way. As there are uncountably many ordinals, we cannot assign names to the entire collection.

Definition 1.1.7. Kleene's $\mathcal{O}$ is a system of *notations* for ordinals consisting of a set of natural numbers, $\mathcal{O}$, a function, $|\cdot|_{\mathcal{O}}$, that maps each number $a \in \mathcal{O}$ to an ordinal and a strict partial order, $<_{\mathcal{O}}$, on $\mathcal{O}$ with the following properties:

(1) $1 \in \mathcal{O}$ and $|1|_{\mathcal{O}} = 0$.

(2) If $a \in \mathcal{O}$ is a notation for an ordinal α then $2^a \in \mathcal{O}$ and $|2^a|_{\mathcal{O}} = \alpha + 1$. In the partial ordering we define $b <_{\mathcal{O}} 2^a$ if $b <_{\mathcal{O}} a$ or $b = a$.

(3) Given a limit ordinal γ, the notations for γ in $\mathcal{O}$ are numbers of the form $3 \cdot 5^e$ such that φ_e is a total computable function with range contained in $\mathcal{O}$ and such that

$$\varphi_e(0) <_{\mathcal{O}} \varphi_e(1) <_{\mathcal{O}} \varphi_e(2) <_{\mathcal{O}} \dots$$

and γ is the least upper bound of the sequence of ordinals $|\varphi_e(n)|$. In the partial ordering, we define $b <_{\mathcal{O}} 3 \cdot 5^e$ if there exists an n such that $b <_{\mathcal{O}} \varphi_e(n)$.

It is known that the computable ordinals are precisely the ordinals having at least one notation in Kleene's $\mathcal{O}$. As one might expect, the set $\mathcal{O}$ is extremely complicated and the notations are not unique in general. Every finite ordinal has a unique notation in $\mathcal{O}$ while, for any infinite ordinal, if it has one notation in $\mathcal{O}$ then it has infinitely many. Now we can iterate the jump operator through all the computable ordinals.

Definition 1.1.8. For $a \in \mathcal{O}$, define the set $H(a)$ by transfinite recursion as follows:

(1) $H(1) = \emptyset$,

(2) $H(2^a) = H(a)'$,

(3) $H(3 \cdot 5^e) = \{\langle u, v\rangle : u <_{\mathcal{O}} 3 \cdot 5^e \ \&\ v \in H(u)\} = \{\langle u, v\rangle : \exists n\ (u \leq_{\mathcal{O}} \varphi_e(n) \ \&\ v \in H(u))\}$.

Spector showed that, for every computable ordinal α, the Turing degree of $H(a)$, where a is any notation for α, is independent of the choice of a [2]. The Turing degree of $H(a)$ is denoted by $\mathbf{0}^{(\alpha)}$. For $n \in \omega$, a set $X \subseteq \omega$ is Σ^0_n if and only if X is c.e. relative to $0^{(n-1)}$, and for a computable ordinal $\alpha \geq \omega$, X is Σ^0_α if and only if X is c.e. in $H(a)$ for some $a \in \mathcal{O}$ such that $|a|_{\mathcal{O}} = \alpha$, and hence c.e. in $H(a)$ for all such notations a.

We can also relativize the hierarchy to any set $X \subseteq \omega$ as follows:

Definition 1.1.9. For $a \in \mathcal{O}$, let

(1) $H(1)(X) = X$,

(2) $H(2^a)(X) = (H(a)(X))'$,

(3) $H(3 \cdot 5^e)(X) = \{\langle u, v\rangle : \exists n\ (u \leq_{\mathcal{O}} \varphi_e(n) \ \&\ v \in H(u)(X))\}$.

In this paper we will be working with $\mathcal{L}_{\omega_1,\omega}$ formulas over various finite languages $\mathcal{L}$. The "ω_1" parameter indicates that we are permitted to take conjunctions and disjunctions over countable sets of formulas; the "ω" parameter indicates that formulas are only allowed finitely many nested quantifiers. All formulas are first order. We will call $\mathcal{L}_{\omega_1,\omega}$ formulas *infinitary formulas*, and call a formula *finitary* when we are restricting ourselves to finite conjunctions and disjunctions. We do not have a prenex normal form for infinitary formulas, but we say that such a formula is in normal form if it is Σ_α or Π_α for some countable ordinal α as defined below by transfinite induction.

Definition 1.1.10. The Σ_0 *and* Π_0 *formulas* are the finitary quantifier free formulas. For any ordinal $\alpha > 0$, a Σ_α *formula* is a countable disjunction of formulas of the form $\exists\vec{u}\varphi$ where φ is Π_β for some $\beta < \alpha$. Similarly a Π_α *formula* is a countable conjunction of formulas of the form $\forall\vec{u}\varphi$ where φ is Σ_β for some $\beta < \alpha$. Every $\mathcal{L}_{\omega_1,\omega}$ formula is equivalent to a formula of this form.

On many occasions, we will be restricting ourselves to a subcollection of the above formulas, called the *computable infinitary formulas.* Informally, these are $\mathcal{L}_{\omega_1,\omega}$ formulas in which all disjunctions and conjunctions are taken over c.e. sets of formulas. In a formal treatment, the computable infinitary formulas are defined in terms of ordinal notations. We will write Σ^c_α and Π^c_α to denote the computable Σ_α and Π_α formulas respectively.

Chapter 2

Computable Dimension

2.1 Introduction

In mathematics, we often classify structures up to isomorphism; however, two isomorphic structures can exhibit vastly different computability theoretic behaviour. As such, when studying the effective properties of structures, we identify two structures if and only if they are computably isomorphic, that is isomorphic via a computable function. It is of interest to study how many "different" computable presentations there are of a given computable structure. The *computable dimension* of a computable structure $\mathcal{A}$ is defined to be the number of computable presentations of $\mathcal{A}$ up to computable isomorphism. It is clear that the computable dimension of a computable structure is at least 1 and, as there are at most countably many computable presentations of any structure, the computable dimension of any structure is bounded above by ω. Structures of computable dimension 1 are called *computably categorical*.

There are many familiar examples of classes of structures that admit only computably categorical structures or structures of full computable dimension, ω. In other words, if we can find two distinct computable copies of a structure, up to computable isomorphism, then we are guaranteed to find infinitely many. The following theorem is a collection of known results due to Goncharov ([12]); Goncharov and Dzgoev ([13]); Goncharov, Lempp and Solomon ([15]); Laroche ([23]); Metakides and Nerode ([26]); Nurtazin ([30]); and Remmel ([32] and [31]).

Theorem 2.1.1. *Computable structures in the following classes have computable dimension* 1 *or* ω*: Linear orders, Boolean algebras, abelian groups, algebraically closed fields, real closed fields.*

It is known that, for each $0 < n \leq \omega$, there exists a structure with computable dimension n. Moreover, there are classes of structures that admit all possible computable dimensions. The next theorem, compiling results of Goncharov ([11]); Goncharov, Molokov and Romanovskii ([14]); Hirschfeldt, Khoussainov, Shore and Slinko ([17]); and Kudinov ([22]), presents the known classes with this property.

Theorem 2.1.2. *For each $0 < n \leq \omega$, there are computable structures in the following classes with computable dimension n: graphs, lattices, partial orders, nilpotent groups, integral domains.*

The results in this Chapter will show that the class of equivalence structures fall into the former category, in other words, the only possible computable dimension of an equivalence structure is 1 or ω.

2.2 Computably categorical equivalence structures

Definition 2.2.1. An equivalence structure, $\mathcal{A}$, consists of a domain A and a binary equivalence relation E on that domain, i.e. a relation that is reflexive, symmetric, and transitive.

Calvert, Cenzer, Harizanov and Morozov (in [3]) give a complete structural characterization of the computably categorical equivalence structures. We will use this description to prove that any equivalence structure that is not computably categorical has computable dimension ω.

Theorem 2.2.2 ([3])**.** *A computable equivalence structure $\mathcal{A}$ is computably categorical if and only if one of the following holds:*

(i) *$\mathcal{A}$ has finitely many finite equivalence classes, or*

(ii) *$\mathcal{A}$ has finitely many infinite classes, there is a uniform bound on the size of all finite classes, and at most one $k < \omega$ such that $\mathcal{A}$ has infinitely many classes of size k.*

From the above theorem, if a computable equivalence structure is not computably categorical then it satisfies at least one of the following properties:

(1) There is no bound on the sizes of finite classes.

(2) There exist $k_1 < k_2 \leq \omega$ such that there are infinitely many classes of size k_1 and infinitely many classes of size k_2.

By generalizing Corollary 3.15 from [3], one can show that if a computable equivalence structure satisfies property (1) then it cannot have finite computable dimension. The general argument from [3] is as follows: For every total function $\varphi_e : \omega \to \{0, 1\}$, let $(\mathcal{C}_e, \equiv_{C_e})$ be the structure with $m \equiv_{C_e} n$ if and only if $\varphi_e(\langle m, n\rangle) = 1$. It is not hard to see that the set $\{e : \mathcal{C}_e$ is an equivalence structure$\}$ is a Π^0_2 set. The authors show that, if $\mathcal{A}$ is a computably categorical equivalence structure, then the set $\{e : \mathcal{C}_e \cong \mathcal{A}\}$ is a Σ^0_3 set. We can generalize this easily to show that the same is true if $\mathcal{A}$ has any finite computable dimension. Suppose that $\mathcal{A}$ has computable dimension $1 \leq n < \omega$ and suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are n computable copies of $\mathcal{A}$ that are pairwise non-computably isomorphic. Then we have $\mathcal{C}_e \cong \mathcal{A}$ if and only if $\mathcal{C}_e$ is computably isomorphic to $\mathcal{A}_i$ for some $1 \leq i \leq n$. More precisely, we have the Σ^0_3 defnition

$$\mathcal{C}_e \cong \mathcal{A} \Longleftrightarrow \bigvee_{i=1}^{n} (\exists a)\Big[a \in Tot \text{ and } (\forall m)(\forall n)\big(m \equiv_{c_e} n \leftrightarrow \varphi_a(m) \equiv_{\mathcal{A}_i} \varphi_a(n)\big)\Big]$$

where Tot is the Π^0_2-complete set consisting of the indices of all total computable functions. The authors show (in Theorems 3.9, 3.11 and 3.13) that if $\mathcal{A}$ satisfies property (1) then the set $\{e : \mathcal{C}_e \cong \mathcal{A}\}$ is either Π^0_3-complete, D^0_3-complete (where D^0_3 is the difference of Σ^0_3 sets), or Π^0_4-complete. As this would contradict the Σ^0_3 definition above, such a structure $\mathcal{A}$ cannot have finite computable dimension.

The proof that structures satisfying property (2) are not computably categorical does not generalize. Given $\mathcal{A}$ satisfying (2), the authors in [3] built two copies of $\mathcal{A}$, say $\mathcal{B}$ and $\mathcal{C}$, such that the set of elements which are members of equivalence classes of size k_1 is computable in $\mathcal{B}$ and not computable in $\mathcal{C}$. In the next section, we will provide a new proof that the computable dimension of an equivalence structure satisfying property (2) is at least 2, and this new method will generalize to show that, in fact, any structure of this kind must have computable dimension ω.

2.3 Computable dimension of equivalence structures

For this section, let $\mathcal{A}$ be a computable equivalence structure consisting of infinitely many equivalence classes of size k_1 and infinitely many classes of size k_2 where $k_1 < k_2 \leq \omega$.

From now on, for any $0 \le n \le \omega$, we will call an equivalence class of size n an *n-class*. We will begin by showing that the computable dimension of $\mathcal{A}$ is at least 2, and then extend this construction to build an infinite family of computable copies of $\mathcal{A}$ that each lie in a distinct computable isomorphism class.

Proposition 2.3.1. *Let $\mathcal{A}$ be a computable equivalence structure with infinitely many k_1-classes and k_2-classes for $1 \le k_1 < k_2 \le \omega$. Then the computable dimension of $\mathcal{A}$ is at least* 2.

We will build two computable copies, $\mathcal{B}$ and $\mathcal{C}$, of $\mathcal{A}$ that are not computably isomorphic. We will build the structures $\mathcal{B}$ and $\mathcal{C}$ in stages such that $\mathcal{B} = \cup_s \mathcal{B}_s$ and $\mathcal{C} = \cup_s \mathcal{C}_s$. The structure $\mathcal{B}$ will be computable as if n enters $\mathcal{B}$ at stage s, we will determine (once and for all) whether or not $n \equiv_{\mathcal{B}} m$ for each $m \in \mathcal{B}_s$ (and similarly for $\mathcal{C}$). To ensure that $\mathcal{B}$ and $\mathcal{C}$ are not computably isomorphic, we will meet the following requirements for each $e \in \omega$:

R_e: If φ_e is a bijection between $\mathcal{B}$ and $\mathcal{C}$ then φ_e sends a member of a k_1-class in $\mathcal{B}$ to a k_2-class in $\mathcal{C}$.

Basic module: In each of $\mathcal{B}$ and $\mathcal{C}$, we will copy the structure $\mathcal{A}$ in stages (using the odd numbers) and, in addition, introduce extra classes of size k_1 (using the even numbers) in order to diagonalize against any possible computable isomorphism. We will attach a k_1-class in $\mathcal{B}$ to φ_e, say $\vec{b}_{\varphi_e}$, that will witness R_e being met. If φ_e is an isomorphism between $\mathcal{B}$ and $\mathcal{C}$ then it must send $\vec{b}_{\varphi_e}$ to a k_1-class in $\mathcal{C}$. As soon as φ_e selects such a k_1-class, say $\vec{c}_e$, we will grow $\vec{c}_e$ to a k_2-class. As we will have infinitely many k_1-classes in the portion of $\mathcal{C}$ that is copying $\mathcal{A}$ along with infinitely many auxiliary k_1-classes, φ_e can select a k_1-class from either category. We will deal with each type separately.

All requirements: We will label all auxiliary k_1-classes that appear in $\mathcal{C}$. At stage s we will ensure that there are s auxiliary k_1-classes, all labeled. We will achieve this by adding a new k_1-class each time we grow a labeled k_1-class to be of size k_2.

If φ_e selects a (labeled) auxiliary k_1-class, then we will act following our priority argument (that will be explained in the construction). If φ_e selects a k_1-class $\vec{c}$ in the portion of $\mathcal{C}$ that is copying the structure $\mathcal{A}$ (let $\vec{a}$ be the pre-image of $\vec{c}$ in $\mathcal{A}$), then we grow $\vec{c}$ into a k_2-class to meet R_e, and introduce k_1 new odd numbers into $\mathcal{C}$ to (re)copy $\vec{a}$. Note that if $\vec{a}$ truly is a k_1-class, then we may infinitely often change the image of $\vec{a}$ in $\mathcal{C}$, but $\mathcal{A}$ and $\mathcal{C}$ will still be isomorphic. If $\vec{a}$ is actually part of a class of size $> k_1$ then we will only change the isomorphism on $\vec{a}$ finitely often.

Now we proceed to the construction. In this construction we will assume that we have $k_1 < k_2 < \omega$ such that there are infinitely many classes both of size k_1 and of size k_2. Following this construction we will explain how to amend the procedure in the case where $k_2 = \omega$.

Construction 1:

As $\mathcal{A}$ is a computable structure, we can fix a computable sequence of finite structures $\{\mathcal{A}_s\}_{s\in\omega}$ such that $\mathcal{A}_s \subseteq \mathcal{A}_{s+1}$ and $\lim_s \mathcal{A}_s = \mathcal{A}$. Let $\equiv_{\mathcal{B}}$ and $\equiv_{\mathcal{C}}$ denote the equivalence relations, in the structures $\mathcal{B}$ and $\mathcal{C}$ respectively, that we are building. When defining the relation on $\mathcal{B}_s$, we will write $\equiv_{\mathcal{B}}$ instead of $\equiv_{\mathcal{B}_s}$ as we will never redefine any equivalence made at any stage s. We will define two auxiliary functions, $f^{\mathcal{B}}$ and $f^{\mathcal{C}}$, that will be copying $\mathcal{A}$ into $\mathcal{B}$ and $\mathcal{C}$ in stages. At every stage s, the function $f^{\mathcal{B}}_s$ (respectively $f^{\mathcal{C}}_s$) will be an embedding of $\mathcal{A}_s$ into $\mathcal{B}_s$ (respectively $\mathcal{C}_s$).

$\boxed{\textit{Stage 0:}}$ Let $\mathcal{A}_0 = \emptyset$. Define $\mathcal{B}_0$ and $\mathcal{C}_0$ as follows: Define the first k_1 even numbers to be a k_1-class in both $\mathcal{B}_0$ and $\mathcal{C}_0$. Label the k_1-class in $\mathcal{B}_0$ by $\vec{b}_{\varphi_{0,0}}$, attaching the class to requirement R_0 and label the k_1-class in $\mathcal{C}_0$ by $\vec{c}_{0,0}$. (In general, we can grow $\vec{c}_{e,s}, \vec{c}_{e+1,s}, \ldots$ on behalf of R_e.) Let $f^{\mathcal{B}}_0 : \mathcal{A}_0 \hookrightarrow \mathcal{B}_0$ and $f^{\mathcal{C}}_0 : \mathcal{A}_0 \hookrightarrow \mathcal{C}_0$ be empty maps.

$\boxed{\textit{Stage s+1:}}$ At the end of stage s we have defined $\vec{b}_{\varphi_{i,s}}$ and $\vec{c}_{j,s}$ for all $i, j \leq s$.

1. Check whether the following holds for any $e \leq s$:
   - $\varphi_{e,s+1}$ is 1-1.
   - For all $x, y \in \mathcal{B}_s$ if $\varphi_{e,s+1}(x)\downarrow \in \mathcal{C}_s$ and $\varphi_{e,s+1}(y)\downarrow \in \mathcal{C}_s$ then $x \equiv_{\mathcal{B}} y \Leftrightarrow \varphi_e(x) \equiv_{\mathcal{C}} \varphi_e(y)$.
   - $\varphi_{e,s+1}(\vec{b}_{\varphi_{e,s}})\downarrow = \vec{c}$ and $\vec{c}$ is a k_1-class in $\mathcal{C}_s$.
2. If so, find the least such e. There are three cases:
   (A) $\vec{c} = \vec{c}_{i_0,s}$ for some $i_0 < e$: In this case, due to our prioriy argument, we need to redefine the k_1-class in $\mathcal{B}$ that will witness R_e being met. Find the least k_1 even numbers not yet in $\mathcal{B}_s$, declare them to be a new k_1-class in $\mathcal{B}$, and label this class as $\vec{b}_{\varphi_{e,s+1}}$. Finally let $\vec{b}_{\varphi_{i,s+1}} = \vec{b}_{\varphi_{i,s}}$ for all $i \neq e$ and $\vec{c}_{i,s+1} = \vec{c}_{i,s}$ for all i.
   (B) $\vec{c} = \vec{c}_{i_0,s}$ for some $e \leq i_0 \leq s$: In this case, we are permitted to grow the k_1-class, $\vec{c}_{i_0,s}$, to a k_2-class in order to meet R_e. Find the least $(k_2 - k_1)$ even numbers not

yet in $\mathcal{C}_s$ and add them to the equivalence class, $\vec{c}$. Now $\vec{c}$ is part of a k_2-class in $\mathcal{C}$. Next, we introduce a new k_1-class to replace the one that we grew. Use the first k_1 even numbers not yet in $\mathcal{C}_s$, and declare them to be a new k_1-class in $\mathcal{C}$ labeled $\vec{c}_{i_0,s+1}$. Finally let $\vec{b}_{\varphi_{i,s+1}} = \vec{b}_{\varphi_{i,s}}$ for all i and $\vec{c}_{i,s+1} = \vec{c}_{i,s}$ for all $i \neq i_0$.

(C) $\vec{c} = f_s^{\mathcal{C}}(\vec{a})$ for some $\vec{a} \in \mathcal{A}_s$: If we are not in case (A) or (B), then the k_1-class $\vec{c}$ is necessarily copying $\mathcal{A}$. Here again, we are allowed to grow the k_1-class to meet R_e. Find the least $(k_2 - k_1)$ even numbers not yet in $\mathcal{C}_s$ and add them to the current equivalence class $\vec{c}$. Now $\vec{c}$ is included in a k_2-class in $\mathcal{C}$. Finally, we need to redefine the map $f^{\mathcal{C}}$ on the preimage of $\vec{c}$. Let $\vec{d}$ be the least k_1 odd numbers not yet in $\mathcal{C}_s$ and declare $\vec{d}$ to be a k_1-class in $\mathcal{C}$. Define $f_{s+1}^{\mathcal{C}}(\vec{a}) = \vec{d}$. Finally let $\vec{b}_{\varphi_{i,s+1}} = \vec{b}_{\varphi_{i,s}}$ for all i and $\vec{c}_{i,s+1} = \vec{c}_{i,s}$ for all i.

If no such e exists, let $\vec{b}_{\varphi_{i,s+1}} = \vec{b}_{\varphi_{i,s}}$ and $\vec{c}_{i,s+1} = \vec{c}_{i,s}$ for all i.

For all $a \in \mathcal{A}_s$, except members of the tuple $\vec{a}$ from Step 2(C), let $f_{s+1}^{\mathcal{C}}(a) = f_s^{\mathcal{C}}(a)$.

3. For any $x \in \mathcal{A}_{s+1} - \mathcal{A}_s$, let b be the least odd number not yet in $\mathcal{B}$, declare b to be in $\mathcal{B}_{s+1}$, and define $f_{s+1}^{\mathcal{B}}(x) = b$. Define all equivalences and non-equivalences necessary involving b so that
$$f_{s+1}^{\mathcal{B}} : \mathcal{A}_{s+1} \hookrightarrow \mathcal{B}_{s+1}.$$
Follow the same procedure in $\mathcal{C}$.

4. Define a new k_1-class, labeled $\vec{b}_{\varphi_{s+1,s+1}}$, in $\mathcal{B}_{s+1}$ for requirement R_{s+1}, and a new k_1-class, labeled $\vec{c}_{s+1,s+1}$, in $\mathcal{C}_{s+1}$ (using the first available even numbers).

End Construction

Verification:

It is clear from the construction that both $\mathcal{B}$ and $\mathcal{C}$ are computable, and that each requirement grows at most one tuple.

Lemma 2.3.2. *Each R_e is met.*

Proof. First, the k_1-class with priority i, namely $\vec{c}_i$, is redefined at most finitely many times. For each $i \in \omega$, $\vec{c}_i$ is only redefined if some requirement R_e with $e < i$ wants to grow the class. Each of these finitely many requirements grows at most one k_1-class and hence $\vec{c}_i$ is redefined finitely often.

Suppose for a contradiction that R_e is not met. Then in particular, φ_e is a bijection. From the construction, the witness $\vec{b}_{\varphi_e}$ is redefined if and only if φ_e "selects" $\vec{c_i}$ for some $i < e$. As each $\vec{c_i}$ is changed finitely many times and φ_e is a bijection, there must be some first stage s after which φ_e no longer selects a k_1-class that R_e is not permitted to grow. At this stage, the final witness, say $\vec{b}$, is chosen. As φ_e is an isomorphism, it must send $\vec{b}$ to a k_1-tuple, say $\vec{c}$, in $\mathcal{C}$. By assumption, we are permitted to grow $\vec{c}$ on behalf of R_e, and so we act in either Step 2(B) or 2(C) by growing $\vec{c}$ to a k_2-tuple. This is a contradiction and so R_e is met. □

Lemma 2.3.3. *$\mathcal{C} \cong \mathcal{A} \cong \mathcal{B}$.*

Proof. Justifying that $\mathcal{B} \cong \mathcal{A}$ is easier so we will start with this case of $\mathcal{C}$. Let

$$S_{\mathcal{A}} := \{n : n \text{ is not in a class of size } k_1 \text{ or } k_2 \text{ in } \mathcal{A}\}$$

and

$$S_{\mathcal{C}} := \{n : n \text{ is not in a class of size } k_1 \text{ or } k_2 \text{ in } \mathcal{C}\}.$$

We claim $\lim_s f_s^{\mathcal{C}}$ is an isomorphism from $\mathcal{A}\restriction_{S_{\mathcal{A}}}$ to $\mathcal{C}\restriction_{S_{\mathcal{C}}}$. We can see this as follows: Let $\vec{n}$ be a class consisting of elements from $S_{\mathcal{A}}$ and let s be the first stage where all members of $\vec{n}$ appear in $\mathcal{A}_s$. Then we will define $f_s^{\mathcal{C}}(n_i) := m_i$, where $\vec{m}$ is a class of size $|\vec{n}|$ in $\mathcal{C}_s$, by the end of stage s. We only grow classes of size k_1 in $\mathcal{C}$ on behalf of requirements, so the class $\vec{m}$ will never be selected by a requirement, nor will it grow at any later stage. So $\vec{m} \in S_{\mathcal{C}}$. Moreover, we have $f_t^{\mathcal{C}}(n_i) := m_i$ for all $t \geq s$.

Let $m \in S_{\mathcal{C}}$. If m is even, then m must lie in a class of size k_1 or k_2 in $\mathcal{C}$, so we must have m odd. Therefore, m was introduced in step $2(C)$ or step 3 of the construction. Suppose that m first appears at stage s.

Case 1: m was introduced in step 2(C).

We define $f_s^{\mathcal{C}}(a) := m$ where $a \in \vec{a}$, a k_1-class in $\mathcal{A}_s$, and $m \in \vec{m}$, a k_1-class in $\mathcal{C}_s$. As m cannot be a member of a k_1-class in $\mathcal{C}$, we know that the class containing m must gain an element at a later stage. If $\vec{a}$ truly is a k_1-class in $\mathcal{A}$ then the only way m can gain an element is if some requirement selects $\vec{m}$ and we grow $\vec{m}$ to a k_2-class that remains a k_2-class in the final structure $\mathcal{C}$. This cannot happen as $m \in S_{\mathcal{C}}$. So we can conclude that $f_t^{\mathcal{C}}(a) := m$ for all $t \geq s$ and that $\vec{a}$ is part of a class of size strictly larger than k_1. As $\vec{a}$ grows, we will introduce elements in $\mathcal{C}$ to match and the function f will send the equivalence class of $\vec{a}$ to the equivalence of $\vec{m}$ which will be of equal size.

Case 2: m was introduced in step 3.

This time, we define $f_s^{\mathcal{C}}(a) := m$ because a enters $\mathcal{A}$ at stage s. The class of m grows (if necessary) as the class of a grows and the function f is defined accordingly. If a is a member of a class of size $< k_1$ then the argument is easy. If the class containing a ever reaches k_1 elements (and hence the class containing m reaches k_1 elements) then the argument proceeds as in Case 1.

So we have an isomorphism between the $\mathcal{A} \upharpoonright S_{\mathcal{A}}$ and $\mathcal{C} \upharpoonright S_{\mathcal{C}}$. Observe that there are infinitely many k_1-classes and k_2-classes in $\mathcal{C}$. In the construction, we replace every k_1-class that happens to be grown with a new k_1-class and, as $\mathcal{A}$ has infinitely many k_2-classes, we will have infinitely many k_2-classes appearing in the portion of $\mathcal{C}$ that copies $\mathcal{A}$. It follows that $\mathcal{A} \cong \mathcal{C}$.

Proving that $\mathcal{A} \cong \mathcal{B}$ is much easier. The function $f^{\mathcal{B}}$ is never redefined on any $a \in \mathcal{A}$, and at the end of each stage s, the map $f_s^{\mathcal{B}}$ is an embedding of $\mathcal{A}$ into $\mathcal{B}$. The map $f^{\mathcal{B}} = \cup_s f_s^{\mathcal{B}}$, when restricted to $S_{\mathcal{A}}$, is an isomorpism between $\mathcal{A}\upharpoonright_{S_{\mathcal{A}}}$ and $\mathcal{A}\upharpoonright_{S_{\mathcal{B}}}$. It is clear that there are infinitely many k_1-classes and k_2-classes in $\mathcal{B}$ and hence we have $\mathcal{A} \cong \mathcal{B}$. □

Note 2.3.4. In the case where $k_2 = \omega$: Instead of growing a certain k_1-class, $\vec{c}$, to size k_2 we will "declare" this class to be of size $k_2 = \omega$ and, at the end of each later stage s, add an element to each "declared k_2-class" to ensure that in the limit, these classes are infinite. The rest of the construction remains unchanged.

Now we will use the idea in Construction 1 to tackle our main theorem.

Theorem 2.3.5. *Let $\mathcal{A}$ be a computable equivalence structure with infinitely many k_1-classes and k_2-classes for $1 \leq k_1 < k_2 \leq \omega$. Then the computable dimension of $\mathcal{A}$ is ω.*

Again, we will assume in our construction that we have $k_1 < k_2 < \omega$, but we can amend the construction as in Note 2.3.4 to deal with the case where $k_2 = \omega$.

We will build an infinite sequence $\{\mathcal{A}_l\}_{l\in\omega}$ of computable copies of $\mathcal{A}$ that are not computably isomorphic. This time we will meet the following requirements:

$$R_e = R_{\langle i,l,m\rangle} : \varphi_i \text{ is not an isomorphism between } \mathcal{A}_l \text{ and } \mathcal{A}_m$$

for all triples $\langle i, l, m\rangle$ with $l < m$.

Construction 2

We will reveal the computable structure $\mathcal{A}$ in stages and build each copy $\mathcal{A}_l$ in stages. We will denote the approximations of $\mathcal{A}$ and $\mathcal{A}_l$ at stage s by $\mathcal{A}^s$ and $\mathcal{A}_l^s$ respectively. We will also build a sequence of auxiliary functions $\{f_l\}_{l\in\omega}$, in stages, such that, at each stage s, f_l^s is an embedding of $\mathcal{A}^s$ into $\mathcal{A}_l^s$.

Remark 2.3.6. During the construction, every k_1-class, say $\vec{c}$, in some $\mathcal{A}_l$ at stage s will be labeled in one of three ways. Either $\vec{c}$ is currently attached to a requirement $R_{\langle i,l,m\rangle}$ for some $i, m \in \omega$ and is labeled $\vec{b}_{\langle i,l,m\rangle,s}$, or $\vec{c}$ was previously attached to some requirement $R_{\langle i,l,m\rangle}$ (and has since been abandoned) and is labeled $\vec{b}_{\langle i,l,m\rangle,t}$ for some $t < s$, or $\vec{c}$ was introduced to copy part of $\mathcal{A}$ and hence is in the image of the function f_l^s.

$\boxed{\textit{Stage 0:}}$ Let $\mathcal{A}^0 = \emptyset$. Define the first k_1 even numbers to be a k_1-class in $\mathcal{A}_l^0$ for each $l \in \omega$. Label the k_1-class in $\mathcal{A}_l^0$ as $\vec{b}_{\langle 0,l,l+1\rangle,0}$. Note that every k_1-class is labeled as in the above remark. The maps $f_l^0 : \mathcal{A}^0 \hookrightarrow \mathcal{A}_l^0$ for all l are all empty.

$\boxed{\textit{Stage s+1:}}$ At the end of stage s, for each $l \in \omega$, we have defined tuples $\vec{b}_{\langle i,l,m\rangle,s}$ in $\mathcal{A}_l^s$ for all $\langle i, l, l+k\rangle$ satisfying $i, k \leq s$. The tuple $\vec{b}_{\langle i,l,m\rangle,s}$ is the k_1-class in $\mathcal{A}_l^s$ currently attached to requirement $\langle i, l, m\rangle$.

1. Check whether the following holds for any $e = \langle i, l, m\rangle$:
   - $\varphi_{i,s+1}$ is 1-1.
   - For all $x, y \in \mathcal{A}_l^s$, if $\varphi_{i,s+1}(x)\!\downarrow = z \in \mathcal{A}_m^s$ and $\varphi_{i,s+1}(y)\!\downarrow = w \in \mathcal{A}_m^s$ then
   $$x \equiv_{\mathcal{A}_l} y \Leftrightarrow z \equiv_{\mathcal{A}_m} w.$$
   - $\varphi_{i,s+1}(\vec{b}_{\langle i,l,m\rangle,s})\!\downarrow = \vec{c}$ and $\vec{c}$ is a k_1-class in $\mathcal{A}_m^s$.
2. If so, find the least such $e = \langle i, l, m\rangle$. There are three cases:

   (A) $\vec{c} = \vec{b}_{\langle j,m,n\rangle,s}$ where $\langle j, m, n\rangle < \langle i, l, m\rangle$: In this case, we need to redefine the witness $\vec{b}_{\langle i,l,m\rangle}$ as we are not permitted to grow the k_1-class $\vec{b}_{\langle j,m,n\rangle}$. Find the least k_1 even numbers not yet in $\mathcal{A}_l^s$ and declare them to form a k_1-class in $\mathcal{A}_l$ labeled $\vec{b}_{\langle i,l,m\rangle,s+1}$.
   Let $\vec{b}_{e,s+1} = \vec{b}_{e,s}$ for all $e \neq \langle i, l, m\rangle$.

(B) $\vec{c} = \vec{b}_{\langle j,m,n\rangle,s}$ where $\langle i,l,m\rangle \leq \langle j,m,n\rangle$: In this case we can go ahead and grow the k_1-class $\vec{b}_{\langle j,m,n\rangle}$ on behalf of $R_{\langle i,l,m\rangle}$. Find the least $k_2 - k_1$ even numbers not yet in $\mathcal{A}_m^s$ and add them to the current equivalence class $\vec{b}_{\langle j,m,n\rangle,s}$. Now the current witness $\vec{b}_{\langle j,m,n\rangle,s}$ is part of a k_2-class in $\mathcal{A}_m$. Next, we introduce a new k_1-class as a new witness for $R_{\langle j,m,n\rangle}$. Find the least k_1 even numbers not yet in $\mathcal{A}_m^s$ and declare them to be a new k_1-class in $\mathcal{A}_m$ labeled $\vec{b}_{\langle j,m,n\rangle,s+1}$.

Let $\vec{b}_{e,s+1} = \vec{b}_{e,s}$ for all $e \neq \langle j,m,n\rangle$.

(C) $\vec{c}$ is not equal to $\vec{b}_{\langle j,m,n\rangle,s}$ for any $j,n \in \omega$: In this case, we are allowed to grow the k_1-class $\vec{c}$ on behalf of $R_{\langle i,l,m\rangle}$. First, find the least $k_2 - k_1$ even numbers not yet in $\mathcal{A}_m^s$ and add them to the current equivalence class $\vec{c}$. Since $\vec{c}$ is not currently a witness, it is either in the portion of $\mathcal{A}_m^s$ that is copying $\mathcal{A}^s$, or it was a witness for $R_{\langle j,m,n\rangle}$ for some j,m at some earlier stage and has since been abandoned (i.e. labeled $\vec{b}_{\langle j,m,n\rangle,t}$ for some j,n and some $t < s$). If we are in the latter case, then no further action is necessary. If we are in the former case, then we need to redefine the function f_m on the preimage of $\vec{c}$. Suppose that $f_m^s(\vec{a}) = \vec{c}$ for $\vec{a} \in \mathcal{A}^s$. Let $\vec{d}$ be the first k_1 odd numbers not yet in $\mathcal{A}_m^s$. Delcare $\vec{d}$ to be a new k_1-class in $\mathcal{A}_m$ and define $f_m^{s+1}(\vec{a}) = \vec{d}$.

Finally, let $\vec{b}_{e,s+1} = \vec{b}_{e,s}$ for all $e \in \omega$.

If no such e exists, then do not act for any requirement and let $\vec{b}_{e,s+1} = \vec{b}_{e,s}$ for all e.

3. For any $x \in \mathcal{A}^{s+1} - \mathcal{A}^s$ do the following: For each m, introduce the least odd number not yet in $\mathcal{A}_m$, say d, into $\mathcal{A}_m^{s+1}$ and define $f_m^{s+1}(x) = d$. Make any new $\equiv_{\mathcal{A}_m}$ definitions, involving d, as necessary to ensure that that $f_m^{s+1} : \mathcal{A}^s \hookrightarrow A_m^s$.

4. For each m, do the following: For each $j = 0, 1, \ldots, s+1$, introduce a new k_1-class in $\mathcal{A}_m^{s+1}$ labeled $\vec{b}_{\langle j,m,m+s+1\rangle}$, and for each $k = 1, 2 \ldots, s$, introduce a new k_1-class labeled $\vec{b}_{\langle s+1,m,l+k\rangle}$, all using the first available even numbers.

End Construction 2

Verification

The fact that each structure $\mathcal{A}_m$ is computable is clear from the construction. The domain of each structure is ω as we use all even numbers and odd numbers in turn, and $|\mathcal{A}_m^{s+1}| > |\mathcal{A}_m^s|$ for all stages s. To compute whether $n \equiv_{\mathcal{A}_m} m$, we simply wait for both n and m to appear in $\mathcal{A}_m$ and, at that stage, the answer is determined.

We may injure each requirement finitely many times by forcing R_e to change its witness $\vec{b}_e$. Note that, on a fixed witness, each requirement will act by growing a tuple at most once. If we act in Step 2 on behalf of R_e by growing a k_1-class, $\vec{c}$, to size k_2, then this computation will stand for the remainder of the construction. Requirement R_e may be required to change its witness at a later stage, if the current witness is grown to size k_2 by a higher priority requirement. We will show that every requirement will eventually settle on a true witness, and that we will act to meet R_e in Step 2 if necessary.

Lemma 2.3.7. *For each $e \in \omega$, the witness $\vec{b}_e$ is redefined finitely often.*

Proof. We will prove this by induction on e with base case $e = \langle 0, 0, 1\rangle$. By the construction, $\vec{b}_{\langle 0,0,1\rangle}$ is never redefined. Assume that s is a stage after which the true witnesses b_e for all $e < e_0 = \langle i_0, l_0, m_0\rangle$ have been chosen. As each requirement grows a k_1-class for its true witness, b_e, at most once during the construction, let $t > s$ be a stage where every R_e for $e < e_0$ has acted by growing a class (if it ever does). Then, by assumption, $\vec{b}_{e_0}$ will not be grown on behalf of any requirement after stage t. If $\vec{b}_{e_0}$ is redefined at some later stage, then it must be because φ_{i_0} satisfied all of the conditions from Step 1 of the construction, and "selected" one of the k_1-classes labeled $\vec{b}_e$ for $e < e_0$. If φ_{i_0} "selects" the same class $\vec{b}_e$ more than once (with different witnesses), then φ_{i_0} is not 1-1 and hence will never again satisfy the conditions in Step 1 of the construction. So, in the worst case, φ_{i_0} "selects" each tuple $\vec{b}_e$ exactly once for each $e < e_0$ and the witness $\vec{b}_{e_0}$ is redefined those finitely many times. Therefore $\vec{b}_{e_0}$ is redefined only finitely often. □

Lemma 2.3.8. *For each $e \in \omega$, the requirement R_e is met.*

Proof. Fix $e = \langle i, l, m\rangle \in \omega$. Let s be the first stage after which the true witness, $\vec{b}_e = \vec{b}$, has been chosen and every requirement with higher priority has finished growing k_1-classes (if it ever does). If φ_i does not converge on all of $\vec{b}$, then R_e is met. So suppose that at some stage $t > s$ we see $\varphi_{i,t}(\vec{b})\downarrow = \vec{c}$. If $\vec{c}$ is part of an equivalence class in $\mathcal{A}_m$ of size larger than k_1, then R_e is met. If not, then, by assumption, R_e is the least requirement needing attention and hence we act in Step 2(B) or 2(C)and grow $\vec{c}$ to a k_2-class to meet R_e. □

Lemma 2.3.9. *$\mathcal{A}_l \cong \mathcal{A}$ for all $l \in \omega$.*

Proof. First we will prove that the function $f_l = \lim_s f_l^s$ is an isomorphism between the structures $\mathcal{A}^*$ and $\mathcal{A}_l^*$ where $\mathcal{A}^*$ is obtained from $\mathcal{A}$ by deleting all classes of size k_1 and k_2, and $\mathcal{A}_l^*$ is defined similarly.

Suppose that $\vec{n} = (n_1, \dots, n_k)$ is an equivalence class in $\mathcal{A}^*$. Of course we are assuming that all n_i's are distinct. We need to show that $\lim_s f_l^s(n_i)$ exists and that f_l sends the class $\vec{n}$ to a class of the same size in $\mathcal{A}_l^*$. Let s be the first stage where all of $\vec{n}$ appears in $\mathcal{A}^s$. By the end of stage s we will have defined $f_l^s(n_i) := m_i$ where $m_i \neq m_j \in \mathcal{A}_l^s$ and $\vec{m}$ forms an equivalence class in $\mathcal{A}_l^s$. As $\vec{n} \subseteq \mathcal{A}^*$, the class $\vec{m}$ is cannot be of size k_1, nor will it ever grow to be one. As such, $\vec{m}$ will never be selected by any requirement. So the class $\vec{m}$ is in $\mathcal{A}_l^*$ and is of the correct size, and $f_l^t(n_i) := m_i$ for all $t \geq s$.

Now suppose that $\vec{m} = (m_1, \dots, m_k)$ is an equivalence class in $\mathcal{A}_l^*$. We need to show that $\lim(f_l^s)^{-1}(m_i)$ exists and that $(f_l)^{-1}$ sends the class $\vec{m}$ to a class of the same size in $\mathcal{A}^*$. If any member of $\vec{m}$ is even then, by our construction, $\vec{m}$ must be a k_1-class or a k_2-class. So we can assume that $\vec{m}$ consists of odd numbers. Let m be the least member of $\vec{m}$. This means that, in our construction, m appears first in $\mathcal{A}_l$, let's say at stage s. We have two cases:

Case 1: m is introduced in Step 2(C). Then we introduce m as a part of a new k_1-class to recopy part of $\mathcal{A}$. Say we define $f_l^{s+1}(a) = m$ and the k_1-class of a in $\mathcal{A}^s$, say $\vec{a}$, is mapped to some k_1-class in $\mathcal{A}_l^s$, say $\vec{v}$, containing m. Since $\vec{v} \subseteq \vec{m}$, the class of m must be of size at least k_1. As we know m cannot be a member of a k_1-class, the current k_1-class, $\vec{v}$, must gain another element at a later stage. If $\vec{a}$ truly is a k_1-class, then the only way the equivalence class of m can grow is if some requirement selects $\vec{v}$ and we grow $\vec{v}$ to a k_2-class (that stays the same size for all later stages). Since m cannot be a member of a k_2-class either, this cannot be the case. So $\vec{a}$ must be part of a larger class in $\mathcal{A}$. As $\vec{a}$ grows, we will introduce new odd numbers into the class $\vec{v}$ to match. Once $\vec{a}$ has grown to $\tilde{a}$ and stopped, we will have grown $\vec{v}$ to $\vec{m}$ and from that stage onward we will have $f_l^t(\tilde{a}) = \vec{m}$. So $(f_l)^{-1}$ is defined on all of $\vec{m}$ and maps $\vec{m}$ to a class of the same size in $\mathcal{A}^*$.

Case 2: m is introduced in Step 3. This time we define $f_l^s(a) = m$ for some a that enters $\mathcal{A}_l$ at stage s. The class of m grows (if necessary) as the class of a grows and the function f_l is defined accordingly. If a is a member of a class of size at most $k_1 - 1$ then the argument is easy. If the class containing a ever reaches k_1 members (and hence the class containing m reaches k_1 elements as well) then the argument proceeds as in Case 1.

So f_l is an isomorphism between $\mathcal{A}^*$ and $\mathcal{A}_l^*$.

In our construction, we ensure that each structure $\mathcal{A}_l$ has infinitely many classes of size k_1 and k_2, and so there is also an isomorphism between the structures $\mathcal{A} - \mathcal{A}^*$ and $\mathcal{A}_l - \mathcal{A}_l^*$. Therefore we can conclude that $\mathcal{A} \cong \mathcal{A}_l$ as desired. □

Corollary 2.3.10. *The computable dimension of a computable equivalence structure is either* 1 *or* ω.

Chapter 3

The Back-and-forth ordinal

In this Chapter we investigate a model-theoretic method of comparing classes of structures, introduced by Montalbán in [28], called the back-and-forth ordinal. This measurement assigns an ordinal to a class based on the number of types realized by finite tuples of elements from structures in this class. In the following sections, we will see that while the back-and-forth ordinal is defined in terms of model theory, it relates to the ease or difficulty of coding non-trivial information into structures from the given class.

3.1 Definitions and background

Consider a class of structures, $\mathbb{K}$. Given two structures $\mathcal{A}$ and $\mathcal{B}$ from $\mathbb{K}$, not necessarily distinct, and two fixed finite tuples $\vec{a}$ and $\vec{b}$ from the respective structures, we can ask how difficult it is to distinguish the tuple $\vec{a}$ in $\mathcal{A}$ from the tuple $\vec{b}$ in $\mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, with an isomorphism mapping $\vec{a}$ to $\vec{b}$, then the tuples are indistinguishable. If not then, from a complexity point of view, we can ask how difficult it is to separate the two tuples. More precisely, what is the minimal complexity of a formula φ witnessing this distinction. This idea is represented in the notion of the back-and-forth relations.

Definition 3.1.1 (Back-and-forth relations [2]). Let $\mathcal{A}$ be a countable structure in a finite language and let $\vec{a}$ be an n-tuple from $\mathcal{A}$. Let $\mathcal{B}$ be another structure in the same language as $\mathcal{A}$ and let $\vec{b}$ be an n-tuple from $\mathcal{B}$. For all ordinals α, define the back-and-forth relations, $\leq_\alpha$, inductively as follows:

1. $(\mathcal{A}, \vec{a}) \leq_0 (\mathcal{B}, \vec{b})$ if and only if $\vec{a}$ and $\vec{b}$ satisfy the same atomic formulas in $\mathcal{A}$ and $\mathcal{B}$ respectively, and

2. for $\gamma \geq 1$, $(\mathcal{A}, \vec{a}) \leq_\gamma (\mathcal{B}, \vec{b})$ if and only if for each $\vec{d} \in \mathcal{B}$ and each $0 \leq \beta < \gamma$ there exists $\vec{c} \in \mathcal{A}$ such that $(\mathcal{B}, \vec{b}, \vec{d}) \leq_\beta (\mathcal{A}, \vec{a}, \vec{c})$, where $\vec{c}$ and $\vec{d}$ are of the same length.

Note that this definition includes the case where $\vec{a}$ and $\vec{b}$ are both the empty tuple. We will denote $(\mathcal{A}, \emptyset) \leq_\gamma (\mathcal{B}, \emptyset)$ simply by $\mathcal{A} \leq_\gamma \mathcal{B}$.

Note 3.1.2. To deal with infinite languages, it is helpful to define $\leq_0$ using a standard enumeration function for the atomic and negated atomic formulas (also called literals). We define $(\mathcal{A}, \vec{a}) \leq_0 (\mathcal{B}, \vec{b})$ if every literal with Gödel number less than $|\vec{a}|$ that is true of $\vec{a}$ is also true of $\vec{b}$. We will only deal with finite languages here so we will use Definition 3.1.1 as it is sufficient for our purposes.

There is a known correspondence between the back-and-forth relations above and the infinitary formulas in the language of $\mathcal{A}$. For information about infinitary formulas see [2].

Theorem 3.1.3 (Ash and Knight [2])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be structures in the same language and let $\vec{a}$ and $\vec{b}$ be tuples, from $\mathcal{A}$ and $\mathcal{B}$ respectively, with $|\vec{a}| = |\vec{b}|$. Then, for all ordinals α, the following are equivalent.*

(i) $(\mathcal{A}, \vec{a}) \leq_\alpha (\mathcal{B}, \vec{b})$

(ii) Every Σ_α formula true of $\vec{b}$ in $\mathcal{B}$ is also true of $\vec{a}$ in $\mathcal{A}$.

(iii) Every Π_α formula true of $\vec{a}$ in $\mathcal{A}$ is also true of $\vec{b}$ in $\mathcal{B}$.

Note: The formulas are arbitrary $\mathcal{L}_{\omega_1,\omega}$ formulas, not necessarily computable.

We define $(\mathcal{A}, \vec{a}) \equiv_\gamma (\mathcal{B}, \vec{b})$ if both $(\mathcal{A}, \vec{a}) \leq_\gamma (\mathcal{B}, \vec{b})$ and $(\mathcal{B}, \vec{b}) \leq_\gamma (\mathcal{A}, \vec{a})$ and get the following back-and-forth structures defined in [28]:

Definition 3.1.4 (Montalbán)**.** Let $\mathbb{K}$ be a class of structures. Let $\mathbf{bf}_\gamma(\mathbb{K}) = \frac{\{(\mathcal{A},\vec{a}):\mathcal{A}\in\mathbb{K},\vec{a}\in\mathcal{A}\}}{\equiv_\gamma}$ where $\mathbf{bf}_\gamma(\mathbb{K})$ is partially ordered by $\leq_\gamma$ in the obvious way.

It is not hard to see that $(\mathcal{A}, \vec{a}) \leq_\alpha (\mathcal{B}, \vec{b})$ implies $(\mathcal{A}, \vec{a}) \leq_\beta (\mathcal{B}, \vec{b})$ for all $\beta \leq \alpha$. To measure the complexity of a class of structures, we are interested in the number of back-and-forth equivalence classes and, in particular, the first ordinal α where there are a large number of different tuples up to α-equivalence.

Definition 3.1.5 (Montalbán)**.** The *back-and-forth ordinal* of a class $\mathbb{K}$ is the least ordinal α such that $\mathbf{bf}_\alpha(\mathbb{K})$ is uncountable, if such an α exists.

By Theorem 3.1.3, the $\equiv_\alpha$-equivalence classes correspond to Σ_α-types. It is easy to see that there are uncountably many existential types realized by tuples from graphs, and it follows that the back-and-forth ordinal of the class of graphs is 1. Montalbán analyzes the back-and-forth classes of equivalence structures and linear orderings in [28] and shows that the back-and-forth ordinal of these classes are 2 and 3 respectively.

In the next section, we will see how the back-and-forth ordinal can provide computability-theoretic information about the given class of structures. In particular, it will help to describe the collection of sets that can be *coded* into structures in the class. We now present the necessary background for this analysis.

Definition 3.1.6 (Montalbán [28])**.** We say that a set $X \subseteq \omega$ is *coded by a structure* $\mathcal{A}$ if X is c.e. in $D(\mathcal{B})$ for all $\mathcal{B} \cong \mathcal{A}$. More generally, $X \subseteq \omega$ is *coded by the* n^{th} *jump of a structure* $\mathcal{A}$ if X is c.e. in $D(\mathcal{B})^{(n)}$ for all $\mathcal{B} \cong \mathcal{A}$.

Montalbán also defined a slightly weaker notion of coding requiring only that the set be left-c.e. rather than c.e. in each copy.

Definition 3.1.7. Let $X \subseteq \omega$.

(1) For $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \leq_L \tau$ if $\sigma \subseteq \tau$ or for the least n such that $\sigma(n)$ and $\tau(n)$ are both defined and $\sigma(n) \neq \tau(n)$, we have $\tau(n) = 1$.

Note that $\leq_L$ *is total order on* $2^{<\omega}$*.*

(2) Let $\sigma \in 2^{<\omega}$ and $X, Y \in 2^\omega$. We write $\sigma \leq_L X$ if $\sigma \subseteq X$ or there exists a least n such that $\sigma(n)$ is defined and $\sigma(n) \neq X(n) = 1$. If there is a least n such that $\sigma(n)$ is defined and $1 = \sigma(n) \neq X(n)$ then we write $X \leq_L \sigma$. Finally, we write $X \leq_L Y$ if for the least n such that $X(n) \neq Y(n)$ we have $Y(n) = 1$.

Note that $\leq_L$ *is total order on* $2^{\leq\omega}$*.*

(3) We will write $<_L$ if we have $\leq_L$ but not equality. Observe that for any $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$ we have either $\sigma <_L X$ or $X <_L \sigma$. Let $X_L := \{\sigma \in 2^{<\omega} : \sigma <_L X\}$. We say that X is *left-c.e.* if the set X_L is c.e.

Definition 3.1.8 (Montalbán in [28])**.** We say that a set $X \subseteq \omega$ is *weakly coded by a structure* $\mathcal{A}$ if X is left-c.e. in $D(\mathcal{B})$ for all $\mathcal{B} \cong \mathcal{A}$. More generally, $X \subseteq \omega$ is *weakly coded by the* n^{th} *jump of* $\mathcal{A}$ if X is left-c.e. in $D(\mathcal{B})^{(n)}$ for all $\mathcal{B} \cong \mathcal{A}$.

Remark 3.1.9. The notion of a set being left-c.e. is slightly weaker than c.e. Instead of requiring that we can enumerate the members of X, we only require that we can enumerate all the finite strings that are "to the left" of X in the binary tree. It is not hard to see that for every $X \subseteq \omega$, $X_L \equiv_T X$ and hence every left-c.e. set must have c.e. degree. The fact that $X_L \leq_T X$ is obvious. For $X \leq_T X_L$, we have $0 \in X$ if and only if $\sigma := 1 \in X_L$ and, in general, $n \in X$ if and only if $\sigma := X(0)X(1)\dots X(n-1)\,1 \in X_L$. However, there are left-c.e. sets that are not c.e. For example, let $A = \lim_s A_s$ where $A_0 := \{2n : n \in \omega\}$ and if, at stage s, $2n+2 \in W_{n,s} - W_{n,s-1}$ then enumerate $2n+1 \in A_s$ and $2n+2 \in \overline{A_s}$ (i.e. add $2n+1$ to A and remove $2n+2$ from A).

We will also be using the notion of enumeration reducibility. Informally, we want A to be enumeration reducible to B if we can computably enumerate A from an enumeration of B, where the enumeration of A does not depend on the order in which the set B is enumerated. For a formal treatment, we need a coding of pairs n, D where n is a natural number and D is a finite set of natural numbers. Fix an effective list of all finite sets of natural numbers, say $D_0, D_1, D_2, \dots$, and let $\langle n, D_j \rangle = \langle n, j \rangle$.

Definition 3.1.10 ([6])**.** We say that a set A is *enumeration reducible* to a set B, denoted $A \leq_e B$, if for some c.e. set W_i,

$$n \in A \iff (\exists \text{ finite } D \subseteq B)[\langle n, D \rangle \in W_i].$$

If we have $A \leq_e B$ via the set W_i then we write $A = \Psi_i^B$.

Recall the following equivalent definition of enumeration reducibility due to Selman [35]:

$$A \leq_e B \Leftrightarrow (\forall X)[B \text{ is c.e. in } X \to A \text{ is c.e. in } X].$$

A result of Knight's relates the two previous definitions:

Theorem 3.1.11 (Knight [2])**.** *Let $\mathcal{A}$ be a structure. A set $X \subseteq \omega$ is coded by the n^{th} jump of $\mathcal{A}$ if and only if X is enumeration reducible to the Σ^c_{n+1}-type of some tuple $\vec{a} \in \mathcal{A}$.*

Note that the Σ^c_{n+1}-type of $\vec{a}$ in $\mathcal{A}$ is the set of Σ^c_{n+1} formulas true of $\vec{a}$ in $\mathcal{A}$. The proof of the $n = 0$ case can be found in [2] and this proof can be generalized to obtain the above result for all $n \geq 0$.

3.2 Size of the n-back-and-forth structure

It follows from Theorem 3.1.11 that if there are only countably many $\equiv_{n+1}$-classes of tuples from $\mathbb{K}$, then only countably many sets can be coded by n^{th} jumps of structures in $\mathbb{K}$. It follows from a result of Silver's in [36] that, if $\mathbb{K}$ is Borel class — i.e. a class axiomatizable via countably many $\mathcal{L}_{\omega_1,\omega}$ formulas — then $\mathbf{bf}_n(\mathbb{K})$ is either countable or has size continuum. The following results from [28] characterize exactly when each of these two sizes occur, relative to the difficulty of coding into structures of the given Borel class.

Theorem 3.2.1 (Montalbán)**.** *Let $\mathbb{K}$ be a Borel class of structures. Then the following are equivalent:*

- *(i)* $|\boldsymbol{bf}_n(\mathbb{K})| = \aleph_0$
- *(ii) There exists an oracle relative to which the only sets of numbers that can be coded by the $(n-1)^{st}$ jump of a structure in $\mathbb{K}$ are the sets computable in the oracle.*

Theorem 3.2.2 (Montalbán)**.** *Let $\mathbb{K}$ be a Borel class of structures. Then the following are equivalent:*

- *(i)* $|\boldsymbol{bf}_n(\mathbb{K})| = 2^{\aleph_0}$
- *(ii) Relative to some fixed oracle, every set can be weakly coded into the $(n-1)^{st}$ jump of some structure in $\mathbb{K}$.*

To have a proper dichotomy in the above theorems we would need to replace *weak coding* in Theorem 3.2.2 with *coding*, but unfortunately, this cannot be done. It is clear that the direction $(ii) \Rightarrow (i)$ remains true if we replace the statement with coding, but the direction $(i) \Rightarrow (ii)$ is false. This is not as obvious. A class of structures defined by Montalbán (Example 2.17 in [27]) exhibits a class with uncountable 1-back-and-forth structure, but where arbitrary coding is not possible, even relative to any fixed oracle. Montalbán presented this example and explained why we have arbitrary weak coding; we will provide verification of the other desired properties.

Definition 3.2.3 (Montalbán)**.** Let $\mathcal{L} = \{U, V, f, \{c_\sigma : \sigma \in 2^{<\omega}\}\}$ where U and V are unary relations, f is a unary function and each c_σ is a constant. Let $\mathbb{K}_W$ be the class of countable $\mathcal{L}$ structures, $\mathcal{A}$, that satisfy the following properties:

(i) U and V partition $|\mathcal{A}|$

(ii) x is named by a constant iff $x \in V$

(iii) If $\sigma \neq \tau$ then $c_\sigma \neq c_\tau$

(iv) $rng(f) \subseteq V$

(v) $f\restriction_U$ is 1-1

(vi) $f\restriction_V = id$, and

(vii) If $\sigma <_L \tau$ and $(\exists x \in U)[f(x) = c_\tau]$ then $(\exists x \in U)[f(x) = c_\sigma]$.

For each $\mathcal{A} \in \mathbb{K}_W$, consider the set $R_\mathcal{A} := \{\sigma : \mathcal{A} \models (\exists x \in U)[f(x) = c_\sigma]\}$. Recall that $R_\mathcal{A}$ is coded in $\mathcal{A}$ if and only if $\text{Spec}(\mathcal{A}) \subseteq \{X : R_\mathcal{A} \text{ is c.e. in } X\}$.

Proposition 3.2.4. *For every $\mathcal{A} \in \mathbb{K}_W$, Spec$(\mathcal{A}) = \{X : R_\mathcal{A}$ is c.e. in $X\}$.*

Proof. Clearly, $R_\mathcal{A}$ is c.e. in $\mathcal{A}$. We claim that, for any $\mathcal{B} \cong \mathcal{A}$, we have

$$R_\mathcal{A} = \{\sigma : \mathcal{B} \models (\exists x \in U)[f(x) = c_\sigma]\} = R_\mathcal{B}.$$

Let $\pi : \mathcal{A} \cong \mathcal{B}$. By the properties above, we must have $\pi(U) = U$ and $\pi(V) = V$. Let $\sigma \in R_\mathcal{A}$. Then, for some $a \in U^\mathcal{A}$ and some $b \in V^\mathcal{A}$, we have

$$f^\mathcal{A}(a) = c_\sigma^\mathcal{A} = b.$$

Then $f^\mathcal{B}(\pi(a)) = \pi(c_\sigma^\mathcal{A}) = c_\sigma^\mathcal{B}$. As $\pi(a) \in U^\mathcal{B}$ we have $\sigma \in R_\mathcal{B}$. The other direction is symmetric. As $R_\mathcal{B}$ is c.e. in $\mathcal{B}$, so is $R_\mathcal{A}$, and hence $R_\mathcal{A}$ is coded in $\mathcal{A}$.

It remains to show that $\text{Spec}(\mathcal{A}) \supseteq \{X : R_\mathcal{A} \text{ is c.e. in X}\}$. Suppose that $R_\mathcal{A}$ is c.e. in X. We want to build an X-computable copy $\mathcal{B}$ of $\mathcal{A}$. Let $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ be a computable listing of all strings in $2^{<\omega}$. By properties (ii) and (iii), the set V must be infinite. First, let $Y = \{b_0, b_1, b_2, \ldots\}$ be a (coinfinite) computable subset of ω, declare $b_i \in V^\mathcal{B}$ for all $i \in \omega$ and let $c_{\sigma_i}^\mathcal{B} = b_i$. Let $\{R_\mathcal{A}^s\}_{s\in\omega}$ be an X-computable enumeration of $R_\mathcal{A}$. At stage s, use X to compute $R_\mathcal{A}^s$ and let

$$R_\mathcal{A}^s - R_\mathcal{A}^{s-1} = \{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_k}\}.$$

Take the first k numbers that are not in Y and not yet in the domain of $f^{\mathcal{B}}$, say $a_1, a_2, \ldots, a_k$. Declare $a_j \in U^{\mathcal{B}}$ for all $j = 1, \ldots, k$, and let $f^{\mathcal{B}}(a_j) = b_{i_j}$ and $f^{\mathcal{B}}(b_{i_j}) = b_{i_j}$.

By construction, the structure $\mathcal{B}$ is computable from X and satisfies properties (i)-(vii). Let's define a map, π, between $\mathcal{A}$ and $\mathcal{B}$ as follows: For each $v \in V^{\mathcal{A}}$, we have $v = c_\sigma^{\mathcal{A}}$ for some σ. Let $\pi(v) = \pi(c_\sigma^{\mathcal{A}}) = c_\sigma^{\mathcal{B}}$. For each $u \in U^{\mathcal{A}}$, we must have $f^{\mathcal{A}}(u) = v = c_\sigma^{\mathcal{A}}$ for some $v \in V^{\mathcal{A}}$ and some $\sigma \in 2^{<\omega}$. There must exist exactly one $\tilde{u} \in U^{\mathcal{B}}$ such that $f^{\mathcal{B}}(\tilde{u}) = c_\sigma^{\mathcal{B}}$ and so we let $\pi(u) = \tilde{u}$. This map, π, is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. □

Proposition 3.2.5 (Montalbán)**.** *Every set $D \subseteq \omega$ is weakly coded in some $\mathcal{A} \in \mathbb{K}_W$.*

Proof. Let $D \subseteq \omega$. Consider the set $E = \{\sigma : \sigma <_L D\} = \{\sigma_0, \sigma_1, \sigma_2, \ldots\}$. We will define the structure $\mathcal{A}$ as follows: Let U consist of the even numbers, and V the odd numbers. Let $c_{\sigma_i}^{\mathcal{A}} = 2i+1$ and let $f^{\mathcal{A}}(2i+1) = 2i+1$ and $f^{\mathcal{A}}(2i) = 2i+1$. Then $R_{\mathcal{A}} = E$. As $R_{\mathcal{A}} = E$ is coded in $\mathcal{A}$, the set D is weakly coded in $\mathcal{A}$. □

By Therorem 3.2.2, we must have $|\mathbf{bf}_1(\mathbb{K}_W)| = 2^{\aleph_0}$. It remains to prove the following result:

Theorem 3.2.6. *There is a set $D \subset \omega$ such that D is not coded in any structure $\mathcal{A} \in \mathbb{K}_W$.*

First note that, for any set D and any $\mathcal{A} \in \mathbb{K}_W$, we have

$$\begin{aligned} D \text{ is coded in } \mathcal{A} \quad &\Leftrightarrow \quad \mathrm{Spec}(\mathcal{A}) \subseteq \{X : D \text{ is c.e. in } X\} \\ &\Leftrightarrow \quad \{X : R_{\mathcal{A}} \text{ is c.e. in } X\} \subseteq \{X : D \text{ is c.e. in } X\} \\ &\Leftrightarrow \quad (\forall X)[R_{\mathcal{A}} \text{ is c.e. in } X \to D \text{ is c.e. in } X] \\ &\Leftrightarrow \quad D \leq_e R_{\mathcal{A}} \end{aligned}$$

Therefore, to prove Theorem 3.2.6, we need to show that

$$\bigcup_{\mathcal{A} \in \mathbb{K}} \{D : D \leq_e R_{\mathcal{A}}\} \neq 2^{\omega}.$$

In the proof of Proposition 3.2.5, we show that for every $X \subseteq \omega$ there is a structure $\mathcal{A} \in \mathbb{K}_W$ such that $R_{\mathcal{A}} = X_L$. Conversely, for every structure $\mathcal{A} \in \mathbb{K}_W$, we have $R_{\mathcal{A}} = X_L$ for some $X \subseteq \omega$. It follows from this observation that

$$\bigcup_{\mathcal{A} \in \mathbb{K}} \{D : D \leq_e R_{\mathcal{A}}\} = \bigcup_{X \subseteq \omega} \{D : D \leq_e X_L\}.$$

We will prove Theorem 3.2.6 by showing that $\bigcup_{X\subseteq\omega}\{D : D \leq_e X_L\} \neq 2^\omega$.

We wish to build a set D such that $D \not\leq_e X_L$ for all $X \subseteq \omega$. We will build D satisfying the following requirements, for all $e \in \omega$:

$$R_e : \qquad D \neq \Psi_e^{X_L} \text{ for all } X \subseteq \omega.$$

Given a set $X \subseteq \omega$, finite subsets of X_L will be finite sets of strings $\{\sigma_1, \dots, \sigma_k\}$ such that $\sigma_i <_L X$ for all $0 \leq i \leq k$. As such, $\vec{\sigma} := \{\sigma_1 \dots, \sigma_k\}$ is a subset of X_L if and only if the "rightmost" string in $\vec{\sigma}$ is in X_L. Let $R(\vec{\sigma}) := \{\sigma \in \vec{\sigma} : \tau \leq_L \sigma \text{ for all } \tau \in \vec{\sigma}\}$ denote the rightmost string of $\vec{\sigma}$.

Recall that we write $D = \Psi_e^{X_L}$ if for all $n \in \omega$,

$$n \in D \iff (\exists \text{ finite } \vec{\sigma} \subseteq X_L)[\langle n, \vec{\sigma}\rangle \in W_e].$$

<u>To meet Requirement R_e</u>: We will use the numbers $\{\langle i, e\rangle\}_{i\in\omega}$ to meet the requirement R_e.

Let $S_0^e := \{\vec{\sigma} \in 2^{<\omega} : \langle\langle 0, e\rangle, \vec{\sigma}\rangle \in W_e\}$ and for each $n > 0$ let

$$S_n^e := \{\vec{\sigma} <_L S_{n-1}^e : \langle\langle n, e\rangle, \vec{\sigma}\rangle \in W_e\}$$

where we write "$\vec{\sigma} <_L S$" for some set $S \subseteq 2^{<\omega}$ if $R(\vec{\sigma}) <_L R(\vec{\tau})$ for all $\vec{\tau} \in S$.

Definition 3.2.7. We define the "e -slice" of D as follows, depending on the collection of sets S_n^e:

(1) If $S_0^e = \emptyset$ then set $D(\langle 0, e\rangle) = 1$. The rest of this slice can be defined arbitrarily, so let $D(\langle n, e\rangle) = 1$ for all $n \in \omega$.

(2) If $N > 0$ is the least index n such that $S_n^e = \emptyset$ then set $D(\langle N-1, e\rangle) = 0$ and $D(\langle N, e\rangle) = 1$. The rest of the slice can be defined arbitrarily, so let $D(\langle n, e\rangle) = 1$ for all $n \neq N-1, N$.

(3) If $S_n^e \neq \emptyset$ for all n then set $D(\langle 0, e\rangle) = 1$ and $D(\langle n, e\rangle) = 0$ for all $n > 0$.

How we defined the "e -slice" of D will ensure that requirement R_e is met.

Lemma 3.2.8. *The set D defined above satisfies $D \not\leq_e X_L$ for all $X \subseteq \omega$.*

Proof. We will show that R_e is met for each $e \in \omega$ by cases:

Case 1 ($S_0^e = \emptyset$): If S_0^e is empty then, by definition of S_0^e, we have $\Psi_e^{X_L}(\langle 0, e\rangle) = 0$ for all $X \subseteq \omega$. So since $D(\langle 0, e\rangle) = 1$ we satisfy R_e.

Case 2 ($N > 0$ is least index such that $S_N^e = \emptyset$): By assumption $S_{N-1}^e \neq \emptyset$, and $S_N^e = \emptyset$. We have two subcases:

(1) There is some $\vec{\tau} \in S_{N-1}^e$ satisfying $R(\vec{\tau}) <_L X$:

In this case we have $\vec{\tau} \subset X_L$ with $\langle\langle N-1, e\rangle, \vec{\tau}\rangle \in W_e$ and hence

$$\Psi_e^{X_L}(\langle N-1, e\rangle) = 1 \neq 0 = D(\langle N-1, e\rangle).$$

(2) $X <_L R(\vec{\tau})$ for all $\vec{\tau} \in S_{N-1}^e$:

In this case, for every $\vec{\rho} \in 2^{<\omega}$, we must have either $\vec{\rho} \not\subseteq X_L$ or $\langle\langle N, e\rangle, \vec{\rho}\rangle \notin W_e$. Suppose for a contradiction that we have both $\vec{\rho} \subset X_L$ and $\langle\langle N, e\rangle, \vec{\rho}\rangle \in W_e$. Then we have $R(\vec{\rho}) <_L X <_L R(\vec{\tau})$ for all $\vec{\tau} \in S_{N-1}^e$, or in other words, $\vec{\rho} <_L S_{N-1}^e$. As $\langle\langle N, e\rangle, \vec{\rho}\rangle \in W_e$, it follows that $\vec{\rho} \in S_N^e = \emptyset$ which is a contradiction. Therefore we have $\langle\langle N, e\rangle, \vec{\rho}\rangle \notin W_e$ for all $\vec{\rho} \subseteq X_L$ and hence $\Psi_e^{X_L}(\langle N, e\rangle) = 0 \neq 1 = D(\langle N, e\rangle)$.

Case 3 ($S_n^e \neq \emptyset$ for all n): Let $S^e = \bigcup_n S_n^e$. Again we have two subcases:

(1) There is some $\vec{\sigma} \in S^e$ such that $R(\vec{\sigma}) <_L X$:

As $\vec{\sigma} \in S^e$, there is some N such that $\vec{\sigma} \in S_N^e$. If $N > 0$ then we are done. If not, as $S_n^e \neq \emptyset$ for all n, we can choose a string $\vec{\tau} \in S_n^e$ for some $n > 0$ such that $\vec{\tau} <_L \vec{\sigma} <_L X$ and hence $\vec{\tau} \subset X_L$ and $\langle\langle n, e\rangle, \vec{\tau}\rangle \in W_e$. As $n > 0$, we have $D(\langle n, e\rangle) = 0 \neq 1 = \Psi_e^{X_L}(\langle n, e\rangle)$.

(2) $X <_L R(\vec{\sigma})$ for all $\vec{\sigma} \in S^e$:

We will show in this case that $\langle\langle 0, e\rangle, \vec{\tau}\rangle \notin W_e$ for all $\vec{\tau} \subset X_L$. Suppose that $\vec{\tau} \subset X_L$. Then we have $R(\vec{\tau}) <_L X$ and hence $R(\vec{\tau}) <_L X <_L R(\vec{\sigma})$ for all $\vec{\sigma} \in S^e$. In particular, we have $R(\vec{\tau}) <_L R(\vec{\sigma})$ for all $\vec{\sigma} \in S_0^e$ and so $\vec{\tau} \notin S_0^e$. The only way we could have $\vec{\tau} \notin S_0^e$ is if $\langle\langle 0, e\rangle, \vec{\tau}\rangle \notin W_e$. So we have $\Psi_e^{X_L}(\langle 0, e\rangle) = 0 \neq 1 = D(\langle 0, e\rangle)$.

In all cases, R_e is met. □

Remark 3.2.9. It should be noted there that the proof of Theorem 3.2.6 can be relativized to include an arbitrary fixed oracle. In other words, if we fix an oracle Y, then we can build a set D such that D is not coded in any structure in $\mathbb{K}_W$, even relative to the oracle Y. We amend the previous construction as follows: We write $A \leq_e^Y B$ if there is some e such that for all $n \in \omega$,

$$n \in A \Longleftrightarrow (\exists \text{ finite } D \subseteq B)\left[\langle n, D\rangle \in W_e^Y\right].$$

Then for any structure $\mathcal{A} \in \mathbb{K}_W$, we have

$$D \text{ is coded in } \mathcal{A} \text{ relative to } Y \Leftrightarrow (\forall X)[R_{\mathcal{A}} \text{ is c.e. in } X \to D \text{ is c.e. in } X \oplus Y] \Leftrightarrow D \leq_e^Y R_{\mathcal{A}}.$$

The first equivalence follows immediately from previous work, and the second equivalence is a relativization of Selman's theorem. Now can prove (the relativized version of) Theorem 3.2.6 by fixing any oracle Y, and building a set D such that $D \not\leq_e^Y X_L$ for all $X \subseteq \omega$. The construction and verification are the same, except that every occurrence of the set W_e must be replaced by the set W_e^Y.

Corollary 3.2.10. *There is a class of structures $\mathbb{K}$ such that $|\boldsymbol{bf}_1(\mathbb{K})| = 2^{\aleph_0}$ but such that there is no fixed oracle relative to which every set can be coded in some $\mathcal{A} \in \mathbb{K}$.*

Proof. Let $\mathbb{K}_W$ be the previously defined class. As every set can be weakly coded into some $\mathcal{A} \in \mathbb{K}_W$ then, by Theorem 3.2.2, we must have $|\mathbf{bf}_1(\mathbb{K}_{\mathbb{W}})| = 2^{\aleph_0}$.

The set D from Definition 3.2.7 is not coded in any $\mathcal{A} \in \mathbb{K}_W$ (even relative to a fixed oracle) by Lemma 3.2.8, Remark 3.2.9 and earlier observations. □

Chapter 4

An upper bound on the back-and-forth ordinal

In 1994, Jockusch and Soare introduced the notion of the Turing ordinal of a theory. This was a computability-theoretic method of comparing classes of structures based on the ease or difficulty of coding information into structures of the given class. In light of the results in Chapter 3, it is natural to ask what the relationship is between the Turing ordinal and the back-and-forth ordinal of a theory. In this Chapter, we will see a result of Montalbán showing that the Turing ordinal provides an upper bound for the back-and-forth ordinal, assuming the ordinals exist and are finite. In addition, we will prove a result stated by Knight that can be used to prove a similar upper bound result in the case where the ordinals are infinite.

4.1 Turing ordinal

In the previous chapter, we saw that the back-and-forth ordinal compares classes of structures by examining how difficult it is to distinguish tuples from structures in the given class. Another way to compare classes of structures is to study the collection of degrees that can be realized by a given class of structures. Recall the following definition from the introduction:

Definition 4.1.1 (Jockusch)**.** For any computable ordinal α, we say that $\mathcal{A}$ has α^{th} *jump degree* $\mathbf{d}$ if $\mathbf{d} = \min\{deg(D(\mathcal{B}))^{(\alpha)} : \mathcal{B} \cong \mathcal{A}\}$.

Given a class of structures and any computable ordinal α, one can ask what collection of degrees can be realized as α^{th} jump degrees of structures in the class. For example, in the case of $\alpha = 0$, it is easy to see that every Turing degree can be realized as the degree of a graph, while Richter showed that the only possible degree of a linear order is $\mathbf{0}$ [33]. This suggests that it is harder to code information into linear orderings than it is to code into graphs. One might ask *how much* harder it is to code into linear orderings than graphs. It turns out that we cannot code any non-trivial information into first jumps of linear orderings either [21], but any degree $\mathbf{d} \geq \mathbf{0}^{(2)}$ can be realized as the second jump degree of a linear order [1]. This idea is the motivation for the Turing ordinal defined by Jockusch and Soare in [18]:

Definition 4.1.2 (Jockusch and Soare)**.** Let T be a first order theory which has continuum many pairwise nonisomorphic countable models. We call a computable ordinal α the *Turing ordinal* of T if

(i) every degree $\geq \mathbf{0}^{(\alpha)}$ is the α^{th} jump degree of a model of T, and

(ii) for all $\beta < \alpha$, the only possible β^{th} jump degree of a model of T is $\mathbf{0}^{(\beta)}$.

From the earlier discussion, the Turing ordinal of the theory of graphs is 0 while the Turing ordinal of the theory of linear orderings is 2.

There are many natural questions that arise from this definition. One that is of particular interest in this paper is the following: Is every computable ordinal the Turing ordinal of some class of structures? And if so, how complicated must the theory of such a class be? It has been known since 1994 that, for each ordinal α satisfying $0 \leq \alpha \leq \omega$, there is a finitely axiomatizable class having Turing ordinal α. In Chapter 6, following the work of Ash, Jockusch and Knight in [1], we will define classes of structures having Turing ordinal α for all computable ordinals α. None of these classes will be finitely axiomatizable, although many will be Borel classes. For $\alpha > \omega$, it is still unknown whether or not there is a finitely axiomatizable class with Turing ordinal α.

4.2 Relating the two ordinals

If a class has finite back-and-forth ordinal, then Montalbán's work in [28] shows that the Turing ordinal provides an upper bound on the back-and-forth ordinal.

Corollary 4.2.1 (Montalbán)**.** *Let $\mathbb{K}$ be class of countable structures with $|\boldsymbol{bf}_{n+1}(\mathbb{K})| = \aleph_0$. If $\mathbb{K}$ has Turing ordinal m then $n < m$ (and hence $n + 1 \leq m$).*

Proof. Suppose that $\mathbf{bf}_{n+1}(\mathbb{K})$ is countable. Then by Theorem 3.2.1, we can only code countably many sets into the n^{th} jumps of structures in $\mathbb{K}$. It follows that structures in $\mathbb{K}$ cannot have arbitrary n^{th} jump degree. Hence the Turing ordinal (if it exists) must be strictly bigger than n by definition. □

Corollary 4.2.2. *If $\mathbb{K}$ has back-and-forth ordinal $n+1$ and the Turing ordinal of $\mathbb{K}$ is m then $n \leq m$.*

Proof. If $\mathbb{K}$ has back-and-forth ordinal $n+1$ then in particular $\mathbf{bf}_n(\mathbb{K})$ is countable. □

This work extends easily to include the case where the back-and-forth ordinal is ω.

Theorem 4.2.3. *If the back-and-forth ordinal of $\mathbb{K}$ is infinite and the Turing ordinal exists then the Turing ordinal is also infinite.*

Proof. If the back-and-forth ordinal is infinite then, in particular, we have $|\mathbf{bf}_n(\mathbb{K})| = \aleph_0$ for all $n < \omega$. By Theorem 3.2.1, for each $n > 0$, we can only code countably many sets into the $(n-1)^{st}$ jumps of structures in $\mathbb{K}$. Therefore the Turing ordinal γ (if it exists) must satisfy $n - 1 < \gamma$ for all $n > 0$ and hence $\gamma \geq \omega$. □

Corollary 4.2.4. *If $\mathbb{K}$ has back-and-forth ordinal ω and the Turing ordinal of $\mathbb{K}$ is γ then $\omega \leq \gamma$.*

The remainder of this section will be devoted to proving the following upper bound result for all infinite computable ordinals. This theorem was stated but not proved in [27].

Theorem 4.2.5. *Let $\mathbb{K}$ be a class of countable structures. If the Turing ordinal, γ, of $\mathbb{K}$ exists and satisfies $\omega \leq \gamma < \omega_1^{CK}$, then the back-and-forth ordinal of $\mathbb{K}$ is at most γ.*

To prove this theorem, we will need to extend Theorem 3.1.11 to all computable ordinals α. Recall the theorem from Chapter 3:

Theorem 3.1.11 (Knight) *Let $\mathcal{A}$ be a structure. A set $X \subseteq \omega$ is coded by the n^{th} jump of $\mathcal{A}$ if and only if X is enumeration reducible to the Σ^c_{n+1}-type of some tuple $\vec{a} \in \mathcal{A}$.*

The set X being c.e. in the n^{th} jump of some copy $\mathcal{B}$ of $\mathcal{A}$ can be rephrased as the set X being c.e. in the canonical complete Δ^0_{n+1} set relative to $\mathcal{B}$. We can extend this statement to all computable ordinals with the following definition. Recall that we defined the sets $H(a)(X)$ for all notations a and all sets X in Definition 1.1.9.

Definition 4.2.6 (Canonical complete Δ^0_α set)**.** Let $a \in \mathcal{O}$ be a notation for an ordinal α and let $\mathcal{A}$ be a structure. Then

$$\Delta^0_a(\mathcal{A}) := \begin{cases} H(a)(D(\mathcal{A})) & \text{if } |a| \geq \omega \\ H(b)(D(\mathcal{A})) \text{ where } |b|+1 = a & \text{if } 0 < |a| < \omega \end{cases}$$

If we fix a particular path in $\mathcal{O}$, then we can identify α with its notation a along that path. For this reason, we will write $\Delta^0_\alpha(\mathcal{A})$ for the complete Δ^0_α set relative to $\mathcal{A}$, namely $\Delta^0_a(\mathcal{A})$.

When dealing with infinite ordinals, we need to consider computable infinitary types instead of finitary types.

Definition 4.2.7. Given a structure $\mathcal{A}$ and a finite tuple $\vec{a}$ from $\mathcal{A}$, the Σ^c_α*-type* of $\vec{a}$ in $\mathcal{A}$ is the set of Σ^c_α formulas that are true of $\vec{a}$ in $\mathcal{A}$.

The forward direction of Theorem 3.1.11 is rephrased, for all computable ordinals, as follows:

Theorem 4.2.8 (Knight)**.** *Let α be a computable ordinal. If S is c.e. in $\Delta^0_\alpha(\mathcal{B})$ for all $\mathcal{B} \cong \mathcal{A}$ then S is enumeration reducible to the Σ^c_α-type of some tuple $\vec{a} \in \mathcal{A}$.*

Theorem 4.2.8, along with its converse, appears without proof in [20]. For completeness, we will fill in the proof of the theorem and the desired upper bound from Theorem 4.2.5 will follow. Knight's result is proven using forcing and so, before we begin, we will adapt the forcing language from [2] for our purposes.

4.2.1 Forcing Language

The content of this section is essentially presented in Chapter 10 of [2]. We will adapt the notation and fill in some details required for our particular question.

Let $\mathcal{A}$ be a structure with domain ω in a language $\mathcal{L}$ and let $B = \{b_0, b_1, b_2, \ldots\}$ be an infinite computable list of new constants. Then any bijection from B onto ω induces an isomorphic copy $\mathcal{B}$ of $\mathcal{A}$ in the natural way. We call a particular copy $\mathcal{B}$ of $\mathcal{A}$ a generic copy if $\mathcal{B}$ arises from a bijection $p = \cup_{n\in\omega} p_n$ where the sequence $(p_n)_{n\in\omega}$ consists of finite 1-1 functions from $\mathcal{B}$ to ω that decide statements about the diagram of $\mathcal{B}$ and its jumps.

Definition 4.2.9. Let F be the set of finite 1-1 functions from B to ω, called the *forcing conditions*. Then F is partially ordered by extension.

If p is a forcing condition, then we think of the range of p, $\mathrm{ran}(p)$, as a subset of the original structure $\mathcal{A}$, and the domain of p, $\mathrm{dom}(p)$, as a subset of the copy $\mathcal{B}$. As in [2], we will take our forcing language to be propositional. Let $\mathcal{P}$ be the propositional language in which the propositional symbols are the atomic sentences of the predicate language $\mathcal{L} \cup B$, and let S be the set of computable infinitary formulas in $\mathcal{P}$. Let $\mathcal{B}$ be a copy of $\mathcal{A}$ and let $\varphi \in S$. We will abuse notation a bit here and write $\mathcal{B} \models \varphi$ even though φ is not a formula in the language of $\mathcal{B}$. The atomic sentences from the diagram of $\mathcal{B}$ form a structure in the language $\mathcal{P}$, say $\mathcal{B}^*$, and we write "$\mathcal{B} \models \varphi$" when we really mean $\mathcal{B}^* \models \varphi$.

Now we will distinguish particular formulas from S that are relevant for our result [2].

Remark 4.2.10. For each $n, e \in \omega$, there is a Σ_1^c formula, ψ, in S such that, for any copy $\mathcal{B}$ of $\mathcal{A}$, we have

$$\mathcal{B} \models \psi \Leftrightarrow n \in W_e^{D(\mathcal{B})}.$$

Remark 4.2.11. For each $n, e \in \omega$ and each computable ordinal α (identified with its notation on some path in $\mathcal{O}$) we can find a Σ_α^c formula ψ in S such that, for any copy $\mathcal{B}$ of $\mathcal{A}$,

$$\mathcal{B} \models \psi \Longleftrightarrow n \in W_e^{\Delta_\alpha^0(\mathcal{B})}.$$

We will denote this formula, ψ, by $n \in W_e^{D^{(\alpha)}}$.

Now we are ready to define forcing, following the conventions from Chapter 10 of [2].

Definition 4.2.12 (Forcing)**.** Let $\psi \in S$ and let p be a forcing condition. The *forcing relation*, denoted by $p \Vdash \psi$, is defined by induction on the formula ψ as follows:

(i) If ψ is finitary, then we say $p \Vdash \psi$ and only if every $b \in B$ that appears in ψ is in $\mathrm{dom}(p)$ and if we replace each b appearing in ψ with the corresponding $p(b) \in \omega$ to obtain ψ' then $\mathcal{A} \models \psi'$.

(ii) $p \Vdash \bigvee_i \psi_i$ if and only if $p \Vdash \psi_i$ for some i.

(iii) $p \Vdash \bigwedge_i \psi_i$ if and only if for each i and each $q \supseteq p$ there exists $r \supseteq q$ such that $r \Vdash \psi_i$.

The standard results about forcing hold in this context. A full treatment appears in [2]. We present a summary of the important lemmas.

Lemma 4.2.13 (Lemma 10.2 in [2])**.** *For any forcing condition p and any $\psi \in S$ there exists $q \supseteq p$ such that q decides ψ. (I.e. either $q \Vdash \psi$ or $q \Vdash \neg\psi$.)*

Lemma 4.2.14 (Lemma 10.3 in [2]). *For all $\psi \in S$, if $p \Vdash \psi$ and $q \supseteq p$ then $q \Vdash \psi$.*

Lemma 4.2.15 (Lemma 10.4 in [2]). *For all $\psi \in S$ and all forcing conditions p we cannot have $p \Vdash \psi$ and $p \Vdash \neg\psi$.*

Let $C_\psi := \{p \in F : p \text{ decides } \psi\}$ and let $C_a := \{p \in F : a \in ran(p)\}$. Let p be any forcing condition. It is clear that for any $a \in \mathcal{A}$ there exists $q \supseteq p$ such that $q \in C_a$. By Lemma 4.2.13, for any $\psi \in S$ there exists $q \supseteq p$ such that $q \in C_\psi$. Let

$$\mathcal{C} := \{C_\psi : \psi \in S\} \cup \{C_a : a \in \omega\}.$$

Definition 4.2.16. A sequence of forcing conditions $(p_i)_{i\in\omega}$ is called a *complete forcing sequence* if it is a chain such that, for each $C \in \mathcal{C}$, there exists i such that $p_i \in C$.

Note 4.2.17. As $\mathcal{C}$ is countable, complete forcing sequences exist, and if $p = \cup_i p_i$ where (p_i) is a complete forcing sequence then, in particular, p is a bijection between B and ω.

Let (p_i) be a complete forcing sequence and let $\mathcal{B}$ be the generic copy of $\mathcal{A}$ determined by the bijection $p = \cup_i p_i$. We have the following lemma relating truth and forcing:

Lemma 4.2.18 (Lemma 10.5 in [2]). *For all $\psi \in S$, $\mathcal{B} \vDash \psi$ iff there is some i such that $p_i \Vdash \psi$.*

The next lemma asserts that, for formulas in S, forcing is definable in $\mathcal{A}$.

Lemma 4.2.19 (Lemma 10.6 in [2]). *For each $\psi \in S$ and each $\vec{b} \in \mathcal{B}$ there is a computable infinitary (predicate) formula in the language of $\mathcal{A}$, $Force_{\vec{b},\psi}(\vec{x})$, such that, for any p in F mapping $\vec{b}$ to $\vec{a}$, we have*

$$\mathcal{A} \vDash Force_{\vec{b},\psi}(\vec{a}) \iff p \Vdash \psi.$$

Moreover, if ψ is Σ^c_α (or Π^c_α) then so is $Force_{\vec{a},\psi}(\vec{x})$.

Finally, we need to formalize the notion of a set being c.e. in $\Delta^0_\alpha(\mathcal{B})$ for some generic copy $\mathcal{B}$. In other words, we need to translate statements of the form $X = W_e^{D^{(\alpha)}}$, for an arbitrary set X, into our forcing language. We follow the idea from [21].

Lemma 4.2.20. *If X is c.e. in $\Delta^0_\alpha(\mathcal{B})$ for all $\mathcal{B} \cong A$ then, for some $p \in F$ and some $e \in \omega$, we have the following:*

1. *For all $n \in X$, there is some $q \supseteq p$ such that $q \Vdash n \in W_e^{D^{(\alpha)}}$.*

2. For all $n \notin X$, there is no $q \supseteq p$ such that $q \Vdash n \in W_e^{D^{(\alpha)}}$.

Proof. Assume for a contradiction that, for all $e \in \omega$ and all $p \in F$, there is some $m(e,p) = m$ such that either

(i) $m \in X$ and there is no $q \supseteq p$ such that $q \Vdash m \in W_e^{D^{(\alpha)}}$, or

(ii) $m \notin X$ and there is some $q(e,p) = q \supseteq p$ such that $q \Vdash m \in W_e^{D^{(\alpha)}}$.

Then we can build a complete forcing sequence $(p_i)_{i\in\omega}$ such that, for the generic $\mathcal{B}$ determined by the sequence, we have $X \neq W_e^{\Delta^0_\alpha(\mathcal{B})}$ for all e. We can build such a sequence as follows: Let $p_0 = \emptyset$. Given p_{3e}, let

$$p_{3e+1} = \begin{cases} q(e, p_{3e}) & \text{if } m(e, p_{3e}) \notin X \\ p_{3e} & \text{if } m(e, p_{3e}) \in X \end{cases}$$

Select $p_{3e+2} \supseteq p_{3e+1}$ in C_ψ where ψ is the e^{th} formula in a listing of S, and $p_{3e+3} \supset p_{3e+2}$ in C_e. Then $\{p_i\}_{i\in\omega}$ is a complete forcing sequence. Fix $e \in \omega$. If we are in case (i), then $m(e, p_{3e}) \in X$ and there is no $q \supseteq p_{3e}$ such that $q \Vdash m(e, p_{3e}) \in W_e^{D^{(\alpha)}}$. It follows from Lemma 4.2.14 that we cannot have $p_i \Vdash m(e, p_{3e}) \in W_e^{D^{(\alpha)}}$ for any i. By Lemma 4.2.18, we have $m(e, p_{3e}) \notin W_e^{\Delta^0_\alpha(\mathcal{B})}$ and hence $X \neq W_e^{\Delta^0_\alpha(\mathcal{B})}$. If we are in case (ii), then $m(e, p_{3e}) \notin X$ and $p_{3e+1} = q(e, p_{3e})$ satisfies $p_{3e+1} \Vdash m(e, p_{3e}) \in W_e^{D^{(\alpha)}}$. By Lemma 4.2.18, we have $m(e, p_{3e}) \in W_e^{\Delta^0_\alpha(\mathcal{B})}$ and hence $X \neq W_e^{\Delta^0_\alpha(\mathcal{B})}$. This is a contradiction as X must be c.e. in $\Delta^0_\alpha(\mathcal{B})$. Therefore, there must be some $e \in \omega$ and $p \in F$ satisfying properties 1 and 2. □

4.2.2 Main Result

Now we are ready to prove Theorem 4.2.8 restated here:

Theorem 4.2.21. *Let α be an (infinite) computable ordinal. If X is c.e. in $\Delta^0_\alpha(\mathcal{B})$ for all $\mathcal{B} \cong \mathcal{A}$ then X is enumeration reducible to the Σ^c_α-type of some tuple $\vec{a} \in \mathcal{A}$.*

Proof. Suppose that for all $\mathcal{B} \cong \mathcal{A}$, X is c.e. relative to $\Delta^0_\alpha(\mathcal{B})$. By Lemma 4.2.20, there must exist $p \in F$ and $e \in \omega$ such that, for all $m \in \omega$,

$$(*) \qquad m \in X \Leftrightarrow (\exists q \supseteq p)\left[q \Vdash m \in W_e^{D^{(\alpha)}}\right].$$

By Remark 4.2.11, "$m \in W_e^{D(\alpha)}$" is a Σ^c_α formula in S. Suppose that p maps $\vec{b}$ to $\vec{a}$. By Lemma 4.2.19, for all $\vec{d} \in \mathcal{B}$ we can find a Σ^c_α formula $Force_{\vec{b},\vec{d},m\in W_e^{D(\alpha)}}(\vec{a},\vec{y})$ in the language of $\mathcal{A}$ such that, for all $\vec{c} \in \mathcal{A}$,

$$\mathcal{A} \vDash Force_{\vec{b},\vec{d},m\in W_e^{D(\alpha)}}(\vec{a},\vec{c}) \iff (q : \vec{b},\vec{d} \mapsto \vec{a},\vec{c}) \Vdash m \in W_e^{D(\alpha)}.$$

For each $m \in \omega$, consider the formula

$$\psi_m(\vec{x}) := \bigvee_{\vec{d}\in\mathcal{B}} \exists \vec{y}\ Force_{\vec{b},\vec{d},m\in W_e^{D(\alpha)}}(\vec{x},\vec{y}).$$

Note that, for each m, $\psi_m(\vec{x})$ is disjunction of Σ^c_α formulas. Finally, we have

$$\begin{aligned}
\mathcal{A} \vDash \psi_m(\vec{a}) \quad &\iff \quad \mathcal{A} \models \bigvee_{\vec{d}\in\mathcal{B}} \exists \vec{y}\ Force_{\vec{b},\vec{d},m\in W_e^{D(\alpha)}}(\vec{a},\vec{y}) \\
&\iff \quad \text{For some } \vec{d} \in \mathcal{B}, \mathcal{A} \vDash \exists \vec{y}\ Force_{\vec{b},\vec{d},m\in W_e^{D(\alpha)}}(\vec{a},\vec{y}) \\
&\iff \quad \text{For some } \vec{d} \in \mathcal{B} \text{ and some } \vec{c} \in \mathcal{A}, \mathcal{A} \vDash Force_{\vec{b},\vec{d},m\in W_e^{D(\alpha)}}(\vec{a},\vec{c}) \\
&\overset{\text{Lemma 4.2.19}}{\iff} \quad \text{For some } \vec{d} \in \mathcal{B} \text{ and some } \vec{c} \in \mathcal{A}, (q : \vec{b},\vec{d} \mapsto \vec{a},\vec{c}) \Vdash m \in W_e^{D(\alpha)} \\
&\iff \quad (\exists q \supseteq p) \left[q \Vdash m \in W_e^{D(\alpha)}.\right] \\
&\overset{(*)}{\iff} \quad m \in X
\end{aligned}$$

Given an enumeration of the Σ^c_α type of $\vec{a}$, we enumerate m into X if and only if one of the disjuncts of $\psi_m(\vec{x})$ appears. It follows that X is enumeration reducible to the Σ^c_α-type of $\vec{a} = \mathrm{ran}(p)$. □

Now we can prove the main result - Theorem 4.2.5 - restated here:

Theorem 4.2.22. *Let $\mathbb{K}$ be a class of countable structures. If the Turing ordinal, γ, of $\mathbb{K}$ exists and satisfies $\omega \le \gamma < \omega_1^{CK}$, then the back-and-forth ordinal of $\mathbb{K}$ is at most γ.*

Proof. Suppose that the Turing ordinal of $\mathbb{K}$ is γ. Then we can assume $\mathbb{K}$ contains uncountably many pairwise nonisomorphic models, and so the back-and-forth ordinal $\neq \infty$. From the discussion in [27], in this case, the back-and-forth ordinal is at most ω_1. Let α be the back-and-forth ordinal of $\mathbb{K}$. The case of $\alpha = \omega$ was already done in Corollary 4.2.4, so we can assume that $\alpha > \omega$. Then, by definition, we must have $|\mathbf{bf}_\beta(\mathbb{K})| = \aleph_0$ for all $\beta < \alpha$. For each $\beta < \alpha$ let

$$C_\beta(\mathbb{K}) := \{D \subseteq \omega : D \text{ is coded by the } \beta^{th} \text{ jump of some } \mathcal{A} \in \mathbb{K}\}.$$

It is enough to show that, for all infinite ordinals $\beta < \alpha$, we have $\gamma > \beta$. Fix an infinite ordinal $\beta < \alpha$ and let

$$D^e_\beta := \{D \subseteq \omega : D \leq_e \Sigma^c_\beta\text{−}tp_{\mathcal{A}}(\vec{a}) \text{ for some } (\vec{a}, \mathcal{A}) \in \mathbb{K}\}.$$

As $\mathbf{bf}_\beta(\mathbb{K})$ is countable by assumption, there are only countably many Σ_β types realized by tuples in $\mathbb{K}$ and hence D^e_β is countable. By Theorem 4.2.21, $C_\beta(\mathbb{K}) \subseteq D^e_\beta$ and so the set $C_\beta(\mathbb{K})$ is at most countable. It follows that at most countably many degrees can be realized as β^{th} jump degrees of structures in $\mathbb{K}$ and therefore $\gamma > \beta$. It follows that $\gamma > \beta$ for all (computable) ordinals $\beta < \alpha$ and hence $\alpha \leq \gamma$ as desired. □

4.3 Lower Bound

After Montalbán's paper in 2010, we had the following concrete examples of classes where both the Turing ordinals and back-and-forth ordinals were known, or easy to calculate:

Class of structures	Turing ordinal	Back-and-forth ordinal
Abelian groups	0 [33]	1
Graphs	0 [33]	1
Algebraic fields	0 [33]	1
Partial orders	0 [33]	1
Lattices	0 [33]	1
Equivalence structures	1 [33], [28]	2 [28]
Linear orders	2 [33], [21]	3 [28]
Boolean algebras	ω [18]	ω [2]

As we can see in the table, every case where the ordinals are finite satisfies that the back-and-forth ordinal is equal to the successor of the Turing ordinal. In the only infinite case, we have equality. It is natural to ask whether there is a reason for this pattern. By Theorem 4.2.2, for every finite case, the successor of the Turing ordinal is an upper bound for the back-and-forth ordinal, and by Theorem 4.2.5, in the infinite case, the Turing ordinal is an upper bound for the back-and-forth ordinal. This leads to the following questions:

Question 4.3.1. If the back-and-forth ordinal of a Borel class of structures, $\mathbb{K}$, is $n + 1$, must $\mathbb{K}$ have Turing ordinal n?

Question 4.3.2. If the back-and-forth ordinal of a Borel class of structures, $\mathbb{K}$, is $\alpha \geq \omega$, must $\mathbb{K}$ have Turing ordinal α?

The answer to each of the above questions is no in general. In Chapter 6, we will see a Borel class having Turing ordinal $\omega + 2$ and back-and-forth ordinal $\omega + 1$ which will be a counterexample to Question 4.3.2. For Question 4.3.1, we can look at a well known class. It is known that the Turing ordinal of the class of models of Peano arithmetic (PA) is 1. (A standard model of PA has degree $\mathbf{0}$ and Proposition 3.4 from [20] asserts that any nonstandard model of PA has no degree. The fact that every jump degree is realizable is explained in the Introduction of [21]). A quick analysis of the existential types of models of PA shows that the back-and-forth ordinal of the class is also 1.

The class $\mathbb{K}_W$ from Definition 3.2.3 — introduced to exhibit a class that weakly codes every set but does not code every set — was originally thought to be another counterexample to Question 4.3.1. (Note that, in contrast, models of PA *can* code every set.) It turns out that $\mathbb{K}_W$ is actually an example of a Borel class without a Turing ordinal. It follows from earlier work that the back-and-forth ordinal of $\mathbb{K}_W$ is 1 but Theorem 4.2.2 only asserts that 0 is a lower bound for the Turing ordinal and hence gives us no new information. The remainder of this chapter is devoted to analyzing the class $\mathbb{K}_W$ with respect to the notion of Turing ordinal.

Proposition 4.3.3. *The Turing ordinal of the class $\mathbb{K}_W$ (if it exists) is at least* 1*.*

Proof. Let D be the set we constructed in Theorem 3.2.6. We claim that there is no structure $\mathcal{A} \in \mathbb{K}_W$ of degree $\mathbf{d} = \deg(D)$. Suppose that we have some $\mathcal{A} \in \mathbb{K}_W$ such that $\mathbf{d}$ is the least degree in $Spec(\mathcal{A})$. Then in particular, for any structure $\mathcal{B} \cong \mathcal{A}$, we have $D \leq_T \mathcal{B}$ and hence D c.e. in $\mathcal{B}$. Therefore D is coded in the structure $\mathcal{A}$ which is a contradiction. It follows that not every degree $\mathbf{d}$ is the degree of a structure in $\mathbb{K}_W$ and hence the Turing ordinal of $\mathbb{K}_W$ (if it exists) must be strictly greater than 0. $\square$

To complete the picture, we would like to compute the Turing ordinal of $\mathbb{K}_W$, if it exists. The following proposition amends a construction of Coles, Downey and Slaman (Main theorem from [5]), and will show that the Turing ordinal of $\mathbb{K}_W$ is at most 1, if it exists.

Proposition 4.3.4. *For each $\boldsymbol{d} \geq \boldsymbol{0}'$, there is a set $A \subseteq \omega$ such that $\boldsymbol{d}$ is least in the set $\{X' : A$ is left-c.e. in $X\}$.*

Fix $D \in \mathbf{d}$. We will build two sets $A, G \subseteq \omega$ that satisfy the following:

(1) A is left-c.e. in G,

(2) If A is left-c.e. in X then $G' \leq_T X'$, and

(3) $G' \equiv_T D$

Properties (1) and (2) will ensure that the set $\{X' : A \text{ is left-c.e. in } X\}$ has a least degree, property (3) will ensure that this least degree is $\mathbf{d}$. To meet (1) we will actually build A and G such that A is c.e. in G. More precisely, we will have, for all $i \in \omega$, $i \in A$ if and only if there exists $j \in \omega$ such that $\langle i, j\rangle \in G$. To ensure that $G' \leq_T D$ our construction will be D-computable and force the jump of G. To meet (2) and ensure that $D \leq_T G'$, we will code D into both G and A.

G will be built by finite extensions $\{g_s\}_{s\in\omega}$ such that $G = \cup_s g_s$. A will be built in stages satisfying the following properties:

- $A = \lim_s A_s$,
- At each stage s, there are at most finitely many x such that $A_s(x)\downarrow$, and
- If $A_s(x)\downarrow$, then $A_t(x)\downarrow = A_s(x)$ for all $t \geq s$.

Note that we write $A_s(x)\downarrow$ to mean that the membership of x in A has been decided by stage s, else we write $A_s(x)\uparrow$.

Construction

Stage 0: Let $g_0 = \emptyset$ and let $A_0(x)\uparrow$ for all $x \in \omega$.

Stage $s+1 = e+1$: Given g_s and A_s.

$\boxed{\textit{Step 1 (forcing the jump):}}$ Determine whether the following holds:

$$(*) \qquad (\exists \sigma \supset g_s)(\exists t)\Big[\Big(\Phi^{\sigma}_{e,t}(e)\downarrow\Big) \wedge \Big(\forall \langle i, j\rangle < |\sigma|\Big)\Big(\sigma(\langle i, j\rangle) = 1 \Rightarrow A_s(i) \neq 0\Big)\Big]$$

If $(*)$ holds then let σ_{s+1} be the least such σ and define $\tilde{A}_{s+1}$ as follows:

$$\tilde{A}_{s+1}(x) = \begin{cases} A_s(x) & \text{if } A_s(x)\downarrow \\ 1 & \text{if } A_s(x)\uparrow \text{ and } (\exists j)[\sigma_{s+1}(\langle x, j\rangle) = 1] \\ \uparrow & \text{if } A_s(x)\uparrow \text{ and } (\forall j)\,[\sigma_{s+1}(\langle x, j\rangle) = 0] \end{cases}$$

If $(*)$ does not hold, then let $\sigma_{s+1} = g_s$ and $\tilde{A}_{s+1} = A_s$.

Note: At the end of this stage we have determined whether $e \in G'$ and ensured that, for all x, $(\exists j)[\sigma_{s+1}(\langle x, j\rangle) = 1] \Rightarrow \tilde{A}_{s+1}(x)\downarrow = 1$.

$\boxed{\textit{Step 2 (code D(e) into G and A):}}$ Find the least pair $\langle i, j\rangle$ such that

(i) $\sigma_{s+1}(\langle i, j\rangle)$ is undefined, and

(ii) $\tilde{A}_{s+1}(i)\uparrow$

and define f_{s+1} as follows:

$$f_{s+1}(k) = \begin{cases} \sigma_{s+1}(k) & \text{if } \sigma_{s+1}(k) \text{ is defined} \\ 0 & \text{if } \sigma_{s+1}(k) \text{ is undefined and } k < \langle i, j\rangle \\ D(e) & \text{if } k = \langle i, j\rangle \\ \uparrow & \text{Else} \end{cases}$$

Now we define A_{s+1} from $\tilde{A}_{s+1}$ to reflect the changes in G:

Let $A_{s+1}(x) = \tilde{A}_{s+1}(x)$ for all $x \neq i$. If $D(e) = 1$ then define $A_{s+1}(i){\downarrow}= 1$; if $D(e) = 0$ then define $A_{s+1}(i){\downarrow}= 0$.

This action will ensure that, for all x, $(\exists j)[f_{s+1}(\langle x, j\rangle) = 1] \Rightarrow A_{s+1}(x){\downarrow}= 1$.

Finally, if $A_{s+1}(s+1)\uparrow$ then define $A_{s+1}(s+1){\downarrow}= 0$.

Here we act to ensure that A will be total.

End Construction

Let $G = \cup_s g_s$. At the end of stage s we have defined $A_s(s) \in \{0, 1\}$ and we have $A_t(s) = A_s(s)$ for all $t \geq s$ so let $A = \lim_s A_s$.

Verification

Lemma 4.3.5. *For all $i \in \omega$, $i \in A$ if and only if there exists $j \in \omega$ such that $\langle i, j\rangle \in G$.*

Proof. This is clear from the construction. At every stage s we have

$$(\exists j)[g_s(\langle i, j\rangle) = 1] \Leftrightarrow A_s(i){\downarrow}= 1.$$

□

Lemma 4.3.6. *$G' \leq_T D$.*

Proof. The construction is $D \oplus \emptyset'$-computable and hence D-computable as $D \geq_T \emptyset'$. At stage $s+1 = e+1$, D can determine whether or not $(*)$ holds. If $(*)$ holds then, by our choice of σ_{s+1}, we will have $e \in G'$. It remains to show that if $(*)$ does not hold then we must have $e \notin G'$. If $(*)$ does not hold then we have

$$(**) \qquad (\forall \sigma \supset g_s)(\forall t)\Big[\Big(\Phi^{\sigma}_{e,t}(e)\uparrow\Big) \vee \Big(\exists \langle i,j\rangle\Big)\Big(\sigma(\langle i,j\rangle) = 1 \quad \wedge \quad A_s(i)\downarrow = 0\Big)\Big].$$

If we have $e \in G'$ then there is some string τ satisfying $g_s \subset \tau \subset G$ and a stage t such that $\Phi_{e,t}(e)^{\tau}\downarrow$. By $(**)$, we must have a pair $\langle i,j\rangle$ such that $\tau(\langle i,j\rangle) = 1$ but $A_s(i)\downarrow = 0$. This contradicts Lemma 4.3.5 as we cannot have $\langle i,j\rangle \in G$ with $i \notin A$. Therefore we have $e \notin G'$ as desired. □

Lemma 4.3.7. *$D \leq_T G'$.*

Proof. We will show that the sequence $\{f_s\}_{s\in\omega}$ is a G'-computable sequence. The result follows from this as $D(e)$ is the last bit of f_{e+1}.

As $G' \geq_T \emptyset'$, G' can determine whether or not $(*)$ holds of g_0 and find σ_1. From σ_1 we can compute $\tilde{A}_1$ and find the pair $\langle i,j\rangle$ from step 2. Using σ_1 and the appropriate bits of G we can define f_1.

Now assume that $\{f_t : t \leq s\}$ and $\{\tilde{A}_t : t \leq s\}$ are G'-computable. Given $\tilde{A}_s$ and f_s we can compute A_s and g_s. Now G' can determine whether or not $(*)$ holds of g_s and compute σ_{s+1}. From σ_{s+1} we can compute $\tilde{A}_{s+1}$ and locate the pair $\langle i,j\rangle$ from step 2. Using σ_{s+1} and G we can define f_{s+1}.

This completes the proof. □

Lemma 4.3.8. *If A is left-c.e. in X then $G' \leq_T X'$.*

Proof. If A is left-c.e. in X then $A \leq_T X'$. We will show that the construction is $A \oplus \emptyset'$-computable. This involves similar ideas as the proofs of Lemma 4.3.6 and Lemma 4.3.7. $\emptyset'$ can run most of the construction, except for defining $f_{s+1}(\langle i,j\rangle)$ in step 2. But $f_{s+1}(\langle i,j\rangle) = A(i)$ and so the construction is $A \oplus \emptyset'$-computable and hence X'-computable. □

The proof of Proposition 4.3.4 follows from Lemmas 4.3.5, 4.3.6, 4.3.7, 4.3.8.

Corollary 4.3.9. *The Turing ordinal of the class $\mathbb{K}_W$ (if it exists) is at most 1.*

Proof. This result follows directly from Proposition 4.3.3 and Proposition 4.3.4. □

It turns out that this is not another counterexample to Question 4.3.1 as $\mathbb{K}_W$ does not have a Turing ordinal. This is due to the restrictive definition of the Turing ordinal. While not all degrees can be realized as the degree of a structure in $\mathbb{K}_W$, there is (at least one) structure in $\mathbb{K}_W$ having nontrivial degree. The following example was found by Joseph Miller: Let Z be the complement of $\emptyset'$. Let $\mathcal{A}$ be the $\mathbb{K}_W$ structure with $R_{\mathcal{A}} = Z_L$. Then we have

$$\mathrm{Spec}(\mathcal{A}) = \{\deg(X) : Z_L \text{ is left-c.e. in } X\} = \{\deg(X) : X \geq_T \emptyset'\}$$

and hence $\mathcal{A}$ has degree $\mathbf{0}'$. In fact we can realize any c.e. degree in the same manner.

Recall that the example of PA provides a counterexample to Question 4.3.1, but only manages to show that we cannot replace the inequality in Theorem 4.2.2 with an equality. Perhaps, for relatively nice classes of structures, the Turing ordinal and back-and-forth ordinal still need to be close. We are left with the following question:

Question 4.3.10. Is there a class of structures $\mathbb{K}$ such that the Turing ordinal and the back-and-forth ordinal are both finite and differ by at least 2? Are arbitrarily far apart? If so, how complex is the axiomatization of such a class?

In the next two Chapters, we will explore different classes of structures, specifically classes of linear orderings, and see how they fit into the current picture.

Chapter 5

Classes of Downey and Jockusch

In this Chapter we will discuss particular classes of linear orderings, defined by Downey and Jockusch, for which the Turing ordinals are known. When introduced, these classes of orderings answered the question of whether every finite ordinal can be realized as the Turing ordinal of a finitely axiomatizable class of structures. It turns out that each of the infinitely many classes of linear orderings again satisfy that the back-and-forth ordinal of the class is equal to the successor of the Turing ordinal.

5.1 Definition and Turing ordinals

It is known that there are finitely axiomatizable theories with Turing ordinal α for $0 \leq \alpha \leq \omega$. For example, the Turing ordinal of Abelian groups is 0 [33], the Turing ordinal of equivalence structures is 1 [28], and the Turing ordinal of Boolean algebras is ω [18]. In [7], Downey and Jockusch presented a family of finitely axiomatizable theories to finish the picture.

For a linear ordering $\mathcal{L}$, let

$$\varphi(\mathcal{L}) := (\eta + 2 + \eta) \cdot \mathcal{L}$$

where η denotes the order type of the rationals. In other words, we obtain $\varphi(\mathcal{L})$ by replacing every element of $\mathcal{L}$ by a copy of $\eta + 2 + \eta$. By iterating the φ operator n times, we can define the ordering $\varphi^{(n)}(\mathcal{L})$ for any $\mathcal{L}$. The classes defined in [7] are as follows.

Definition 5.1.1. For each $n \in \omega$, consider the class of orderings defined by

$$\{\varphi^{(n)}(\mathcal{L}) : \mathcal{L} \text{ is a countable linear ordering}\}.$$

There is a theory, denoted by T_n, whose countable models are the above class. Note that T_0 is the theory of linear orderings.

In [7] the authors show that each theory T_n is finitely axiomatizable — i.e. axiomatizable via finitely many first-order formulas — and it is not hard to see that the complexity of the axiomatizations increases as a function of n. The Turing ordinals are as follows.

Theorem 5.1.2 (Downey and Jockusch)**.** *For each $n \geq 0$, the Turing ordinal of the theory T_n is $n+2$.*

In order to see how these theories fit into the picture, we need to calculate their back-and-forth ordinals. This is the topic of the next section.

Note 5.1.3. Given two orderings, $\mathcal{A}$ and $\mathcal{B}$, with tuples $\vec{a} = (a_1, \ldots, a_k) \in \mathcal{A}$ and $\vec{b} = (b_1, \ldots, b_k) \in \mathcal{B}$, we will write

$$(\mathcal{A}, \vec{a}) \hookrightarrow (\mathcal{B}, \vec{b})$$

if there is an embedding of $\mathcal{A}$ into $\mathcal{B}$ that maps a_i to b_i for all $i = 1, \ldots, k$ and we will write

$$(\mathcal{A}, \vec{a}) \cong (\mathcal{B}, \vec{b})$$

if there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ that maps a_i to b_i for all $i = 1, \ldots, k$. We will often use this notation where $\mathcal{A}$ and $\mathcal{B}$ are replaced with particular suborderings.

5.2 Back-and-forth ordinals of T_n

As the Turing ordinal of T_n is $n+2$, Corollary 4.2.2 gives us that the back-and-forth ordinal is at most $n+3$. We will show that it is exactly $n+3$. We have noted earlier that the theory of linear orderings has back-and-forth ordinal 3 and so the case of $n = 0$ is complete. To prove the general case, we need to show that, for each $n > 0$, T_n has only countably many $(n+2)$-back-and-forth types. Our first step will be to prove a main lemma, but to do so, we need to introduce some notation in order to simplify the proof. We would like a formal way of passing from tuples in $\mathcal{L}$ to tuples in $\varphi(\mathcal{L})$ and vice versa and we will do so as follows.

Let X_k be the set consisting of k-tuples of the form

$$\vec{x} = \left(\langle m_1, q_1^1, q_2^1, \ldots, q_{m_1}^1\rangle, \langle m_2, q_1^2, q_2^2, \ldots, q_{m_2}^2\rangle, \ldots, \langle m_k, q_1^k, q_2^k, \ldots, q_{m_k}^k\rangle\right)$$

where $m_1, m_2, \ldots, m_k$ are positive integers, and for each $1 \le i \le k$, we have $q_1^i < q_2^i < \ldots < q_{m_i}^i$ naming finitely many members of the ordering $(\eta + 2 + \eta)$.

Given a linear order $\mathcal{L}$, a k-tuple $\vec{a} = (a_1, \ldots, a_k)$ from $\mathcal{L}$, and a k-tuple $\vec{x}$ from X_k, we define a corresponding $(m_1 + m_2 + \ldots + m_k)$-tuple, denoted by $f_{\vec{x}}(\vec{a})$, in $\varphi(\mathcal{L})$ as follows: Let $f_{\vec{x}}(\vec{a}) := (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k)$ where the tuple $\tilde{a}_i$ is of length m_i, lies in the $(\eta + 2 + \eta)$ block corresponding to the element $a_i \in \mathcal{L}$, and there is an embedding of $\eta + 2 + \eta$ into $\varphi(\mathcal{L})$ that sends q_j^i to the j^{th} member of the tuple $\tilde{a}_i$.

For any two tuples $\vec{a} = (a_1, \ldots, a_k)$ and $\vec{b} = (b_1, \ldots, b_l)$, let $\vec{a} \cup \vec{b}$ denote the concatenation of $\vec{a}$ and $\vec{b}$, namely $\vec{a} \cup \vec{b} = (a_1, \ldots, a_k, b_1, \ldots, b_l)$. Observe that, given k-tuples $\vec{a} \in \mathcal{L}$ and $\vec{x} \in X_k$, for any decomposition $\vec{a} = \vec{c} \cup \vec{d}$, there are corresponding tuples $\vec{y} \in X_{|\vec{c}|}$ and $\vec{z} \in X_{|\vec{d}|}$ such that $f_{\vec{y}}(\vec{c}) \cup f_{\vec{z}}(\vec{d}) = f_{\vec{x}}(\vec{a})$. And, conversely, given an n-tuple $\vec{c}$ and an m-tuple $\vec{d}$ from $\mathcal{L}$ and tuples $\vec{y} \in X_n$ and $\vec{z} \in X_m$, there is a tuple $\vec{a}$ in $\mathcal{L}$ with

$$f_{\vec{y} \cup \vec{z}}(\vec{a}) = f_{\vec{y}}(\vec{c}) \cup f_{\vec{z}}(\vec{d}).$$

With this notation, we can formulate the needed lemma:

Lemma 5.2.1. *For all $n > 0$ and all (infinite) linear orderings $\mathcal{L}_1$ and $\mathcal{L}_2$, if $\vec{a} \in \mathcal{L}_1$ and $\vec{b} \in \mathcal{L}_2$ are both of length k and $\vec{x} \in X_k$ then*

$$(\mathcal{L}_1, \vec{a}) \le_{n-1} (\mathcal{L}_2, \vec{b}) \qquad \Longrightarrow \qquad (\varphi(\mathcal{L}_1), f_{\vec{x}}(\vec{a})) \le_n (\varphi(\mathcal{L}_2), f_{\vec{x}}(\vec{b})).$$

Proof. We proceed by induction on n, for all orderings and tuples of all lengths at once. For $n = 1$: Suppose that $(\mathcal{L}_1, \vec{a}) \le_0 (\mathcal{L}_2, \vec{b})$. Then $\vec{a}$ and $\vec{b}$ are ordered in the same way in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. We need to show that $(\varphi(\mathcal{L}_1), f_{\vec{x}}(\vec{a})) \le_1 (\varphi(\mathcal{L}_2), f_{\vec{x}}(\vec{b}))$. Fix $\vec{c} \in \varphi(\mathcal{L}_2)$. By how $f_{\vec{x}}(\cdot)$ was defined, we have $f_{\vec{x}}(\vec{a}) = (\tilde{a}_1, \ldots, \tilde{a}_k)$ and $f_{\vec{x}}(\vec{b}) = (\tilde{b}_1, \ldots, \tilde{b}_k)$ ordered in the same way in $\varphi(\mathcal{L}_1)$ and $\varphi(\mathcal{L}_2)$ respectively. Moreover, we have

$$(\eta + 2 + \eta, \tilde{a}_i) \cong (\eta + 2 + \eta, \vec{b}_i)$$

as suborderings. As such, if there are finitely many elements between any two consecutive members of $f_{\vec{x}}(\vec{a})$, then there is the same number of elements between the corresponding members of $f_{\vec{x}}(\vec{b})$. Else there are infinitely many elements between the two members of $f_{\vec{x}}(\vec{a})$ and corresponding members of $f_{\vec{x}}(\vec{b})$. In linear orderings, this is sufficient to show that $(\varphi(\mathcal{L}_1), f_{\vec{x}}(\vec{a})) \le_1 (\varphi(\mathcal{L}_2), f_{\vec{x}}(\vec{b}))$ as desired. This completes the proof of the base case.

Now fix $n > 1$. Let $\vec{x} \in X_k$ and suppose that $(\mathcal{L}_1, \vec{a}) \leq_{n-1} (\mathcal{L}_2, \vec{b})$. To show that $(\varphi(\mathcal{L}_1), f_{\vec{x}}(\vec{a})) \leq_n (\varphi(\mathcal{L}_2), f_{\vec{x}}(\vec{b}))$, we need to show that for any choice of $\vec{u} \in \varphi(\mathcal{L}_2)$, there is a tuple $\vec{w} \in \varphi(\mathcal{L}_1)$ such that

$$(\varphi(\mathcal{L}_2), f_{\vec{x}}(\vec{b}), \vec{u}) \leq_{n-1} (\varphi(\mathcal{L}_1), f_{\vec{x}}(\vec{a}), \vec{w}).$$

Given $\vec{u} \in \varphi(\mathcal{L}_2)$, there are tuples $\vec{c} \in \mathcal{L}_2$ and $\vec{y} \in X_{|\vec{c}|}$ such that $\vec{u} = f_{\vec{y}}(\vec{c})$. By definition of $\leq_{n-1}$, there is a tuple $\vec{d} \in \mathcal{L}_1$ such that $(\mathcal{L}_2, \vec{b}, \vec{c}) \leq_{n-2} (\mathcal{L}_1, \vec{a}, \vec{d})$. By the induction hypothesis, for any $\vec{z} \in X_{|\vec{b} \cup \vec{c}|}$, we have $(\varphi(\mathcal{L}_2), f_{\vec{z}}(\vec{b} \cup \vec{c})) \leq_{n-1} (\varphi(\mathcal{L}_1), f_{\vec{z}}(\vec{a} \cup \vec{d}))$. In particular, for $z = \vec{x} \cup \vec{y}$, we have

$$(\varphi(\mathcal{L}_2), f_{\vec{x}}(\vec{b}), f_{\vec{y}}(\vec{c})) \leq_{n-1} (\varphi(\mathcal{L}_1), f_{\vec{x}}(\vec{a}), f_{\vec{y}}(\vec{d}))$$

and hence

$$(\varphi(\mathcal{L}_2), f_{\vec{x}}(\vec{b}), \vec{u})) \leq_{n-1} (\varphi(\mathcal{L}_1), f_{\vec{x}}(\vec{a}), \vec{w})$$

for $\vec{w} := f_{\vec{y}}(\vec{d})$. This completes the proof of the Lemma. $\square$

Now we will prove the desired result:

Theorem 5.2.2. *For each $n \geq 0$, there are countably many $(n+2)$-back-and-forth types of the form $(\mathcal{A}, \vec{a})$ where $\mathcal{A} \cong \varphi^{(n)}(\mathcal{L})$ for some ordering $\mathcal{L}$, and $\vec{a}$ is a tuple from $\mathcal{A}$.*

Proof. We prove this by induction on n. For $n = 0$ we are asserting that there are countably 2-back-and-forth types of tuples from linear orderings. This was proven in [28]. Fix $n > 1$ and suppose that there are countably many pairs of the form $(\mathcal{A}, \vec{a})$ where $\mathcal{A} \cong \varphi^{(n-1)}(\mathcal{L})$ for some $\mathcal{L}$, up to $\equiv_{n+1}$-equivalence. Let's choose a representative from each of the countably many $\equiv_{n+1}$-classes and list them as follows: $\{(\mathcal{A}_i, \vec{a}_i)\}_{i \in \omega}$ where $\vec{a}_i$ is of length k_i. We claim that the list of $\equiv_{n+2}$-classes is contained in the following countable set:

$$B := \bigcup_{i \in \omega} \bigcup_{\vec{x} \in X_{k_i}} \left\{ \Big(\varphi(\mathcal{A}_i), f_{\vec{x}}(\vec{a}_i) \Big) \right\}$$

Fix an ordering $\varphi^{(n)}(\mathcal{L})$ and a tuple $\vec{b} \in \varphi^{(n)}(\mathcal{L})$. Then we have $\vec{b} = f_{\vec{x}}(\vec{a})$ for some $\vec{a} \in \varphi^{(n-1)}(\mathcal{L})$ and some $\vec{x} \in X_{|\vec{a}|}$. By assumption, there must exist some i such that $(\varphi^{(n-1)}(\mathcal{L}), \vec{a}) \equiv_{n+1} (\mathcal{A}_i, \vec{a}_i)$. By Lemma 5.2.1, for the given $\vec{x}$ we must have

$$\Big(\varphi\left(\varphi^{(n-1)}(\mathcal{L})\right), f_{\vec{x}}(\vec{a})\Big) \equiv_{n+2} \Big(\varphi(\mathcal{A}_i), f_{\vec{x}}(\vec{a}_i)\Big)$$

and hence

$$(\varphi^{(n)}(\mathcal{L}), \vec{b}) \equiv_{n+2} (\varphi(\mathcal{A}_i), f_{\vec{x}}(\vec{a}_i)).$$

This shows that our pair $(\varphi^{(n)}(\mathcal{L}), \vec{b})$ falls into one of the equivalence classes in B. Since B is countable, we conclude that there are at most countably many $\equiv_{n+2}$-classes as desired. $\square$

We now have infinitely many new theories that fit the same pattern as our previous results. Before we move on, let us have a look at the current information:

Class of structures	Turing ordinal	Back-and-forth ordinal
Abelian groups	0	1
Graphs	0	1
Algebraic fields	0	1
Partial orders	0	1
Lattices	0	1
Models of PA	1	1
$\mathbb{K}_W$	DNE	1
Equivalence structures	1	2
Linear orders	2	3
T_n	$n+2$	$n+3$
Boolean algebras	ω	ω

Chapter 6

Orderings of Ash, Jockusch and Knight

In the previous section we saw that any ordinal between 0 and ω can be realized as the Turing ordinal of a finitely axiomatizable class of structures. It is unknown whether there exists a finitely axiomatizable class with Turing ordinal strictly greater than ω. If we relax the condition of finitely axiomatizable to Borel, then we can produce a class with the desired property in some cases. In this Chapter we will introduce linear orderings defined by Ash, Jockusch and Knight in [1] and prove that, for each computable ordinal α, there is a class of linear orderings with Turing ordinal α.

6.1 Classes with Turing ordinal α

As in the previous chapter, the classes will be particular subcollections of linear orderings. First we will describe the various building blocks of the desired orderings. For any linear ordering $\mathcal{L} = (\omega, \leq_{\mathcal{L}})$, let $\mathcal{L}^* = (\omega, \leq^*)$ denote the reverse ordering of $\mathcal{L}$, i.e. $x \leq^* y$ if and only if $y \leq_{\mathcal{L}} x$.

Definition 6.1.1. The orderings $\mathbb{Z}^\alpha$ for all ordinals α are defined inductively as follows:

(i) $\mathbb{Z}^0 := \mathbf{1}$,

(ii) For successor ordinals, $\mathbb{Z}^{\beta+1} := \mathbb{Z}^\beta \cdot \omega^* + \mathbb{Z}^\beta + \mathbb{Z}^\beta \cdot \omega$,

(iii) For limit ordinals, $\mathbb{Z}^\alpha := \Big(\sum_{\gamma<\alpha} \mathbb{Z}^\gamma \cdot \omega\Big)^* + \mathbf{1} + \sum_{\gamma<\alpha} \mathbb{Z}^\gamma \cdot \omega$.

Definition 6.1.2. Given a countable family of orderings, F, the *shuffle sum* of F, denoted by $\sigma(F)$, consists of densely many copies of each ordering in F. To build a copy of $\sigma(F)$, partition the rational numbers into dense sets $Q_{\mathcal{A}}$, one for each $\mathcal{A} \in F$, and replace each rational in $Q_{\mathcal{A}}$ with a copy of $\mathcal{A}$.

Definition 6.1.3. Given a set $S \subseteq \omega$, let $\sigma^*(S) := \sigma(F)$ where F consists of ω and $n+1$ for $n \in S$.

Definition 6.1.4. Given a set $S \subseteq \omega$ and a computable limit ordinal α, let $\nu^\alpha(S)$ denote the sum of densely many copies of

$$r + 1 + \sum_{1 \leq \gamma < \beta} \mathbb{Z}^\gamma$$

for each $r < \omega$ and $1 \leq \beta < \alpha$ and densely many copies of

$$r + 1 + \sum_{1 \leq \gamma < \alpha} \mathbb{Z}^\gamma$$

for each $r \in S$.

Given the orderings from Definitions 6.1.1, 6.1.3, 6.1.4, Ash, Jockusch and Knight defined orderings $\mathcal{A}_\alpha(S)$ for each successor ordinal $\alpha \geq 2$ and each set $S \subseteq \omega$ in [1] as follows. The authors also defined orderings for each computable limit ordinal, but this construction is left until Section 6.1.3.

	Ordinal	$\mathcal{A}_\alpha(S)$
(1)	$\alpha = 2m+2,\ m \in \omega$	$\mathbb{Z}^m \cdot \sigma^*(S \oplus \overline{S})$
(2)	$\alpha = 2m+3,\ m \in \omega$	$\mathbb{Z}^m \cdot \sigma^*(S)$
(3)	$\alpha = \beta + 1$ (β limit)	$\nu^\beta(S \oplus \overline{S})$
(4)	$\alpha = \beta + 2$ (β limit)	$\nu^\beta(S)$
(5)	$\alpha = \beta + 2k + 3$ (β limit)	$\mathbb{Z}^{\beta+k} \cdot \sigma^*(S \oplus \overline{S})$
(6)	$\alpha = \beta + 2k + 4$ (β limit)	$\mathbb{Z}^{\beta+k} \cdot \sigma^*(S)$

The above orderings were introduced as structures having α^{th} jump degree sharply. (A structure having an α^{th} jump degree is said to have α^{th} jump degree *sharply* if the structure does not have a β^{th} jump degree for any $\beta < \alpha$.) We will review the work from [1] with an eye for building classes of orderings with particular Turing ordinals. For a given computable ordinal α, consider the class of structures defined by

$$\{\mathcal{A}_\alpha(S) : S \subseteq \omega\}.$$

The hope is that this class of structures — or at least some related class — has Turing ordinal α. We will include a sketch of certain proofs from [1] as these will suggest exactly what collection of orderings to consider as a candidate for having Turing ordinal α. We will work with a different definition of generic sets than that found in [1] and so we present the preliminaries here.

Definition 6.1.5. ([19]) Let α be a computable ordinal. A set $S \subseteq \omega$ is *α-generic* if for each Σ^0_α set $X \subseteq 2^{<\omega}$ there is some $\sigma \subset S$ such that either $\sigma \in X$ or else there is no $\tau \supseteq \sigma$ such that $\tau \in X$.

We will require the following facts about generic sets.

Lemma 6.1.6 (Macintyre [25])**.** *For any computable ordinal α and any set X such that $X \geq_T 0^{(\alpha)}$, there exists an α-generic set S such that $S \oplus 0^{(\alpha)} \equiv_T S^{(\alpha)} \equiv_T X$.*

Remark 6.1.7. Let $S \subseteq \omega$ and let $S_n = \{k : \langle n, k\rangle \in S\}$. If S is an $(\alpha + 1)$-generic set then S is not c.e. in $\mathbf{0}^{(\alpha)}$, and for any $k \in \omega$ and any $\beta < \alpha + 1$ we have that S_{k+1} is not computable relative to $(S_0 \oplus S_1 \oplus \cdots \oplus S_k)^{(\beta)}$.

6.1.1 Successor ordinals - Type I

For the following results, fix a computable ordinal α that is of the form in (1), (3) or (5). The authors of [1] characterize the degree spectra of the orderings $\mathcal{A}_\alpha(S)$ as follows.

Lemma 6.1.8 (Ash, Jockusch and Knight)**.** *For each $S \subseteq \omega$, we have*

$$\mathit{Spec}(\mathcal{A}_\alpha(S)) = \{\deg(D) : S \leq_T D^{(\alpha)}\}.$$

The following result from [1] shows that the class consisting of the $\mathcal{A}_\alpha(S)$ orderings has Turing ordinal at most α, if it exists.

Theorem 6.1.9 (Ash, Jockusch and Knight)**.** *Let $\alpha \geq 2$ be a computable ordinal. Then for every degree $\boldsymbol{d} \geq \boldsymbol{0}^{(\alpha)}$, there exists a set S such that $\mathcal{A}_\alpha(S)$ has α^{th} jump degree $\boldsymbol{d}$.*

Proof. Fix $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$. We claim that, for any set $S \in \mathbf{d}$, the structure $\mathcal{A}_\alpha(S)$ has α^{th} jump degree $\mathbf{d}$. By Lemma 6.1.8, we need to show that $\mathbf{d}$ is the least element in the set $\mathcal{C} := \{\deg(D)^{(\alpha)} : S \leq_T D^{(\alpha)}\}$. By Lemma 6.1.6 there is a set D_0 such that $S \equiv_T D_0^{(\alpha)}$ and so $D_0^{(\alpha)} \equiv_T S \in \mathcal{C}$. It is clear that S is a lower bound for the degrees in $\mathcal{C}$ by definition. $\square$

With the following result, we can show that the collection of structures of the form $\mathcal{A}_\alpha(S)$ forms a class with Turing ordinal α.

Theorem 6.1.10 (Ash, Jockusch and Knight)**.** *Let γ be a computable ordinal and let $S \subseteq \omega$. If $B \leq_T D^{(\gamma)}$ for all D satisfying $S \leq_T D^{(\gamma+1)}$ then $B \leq_T \mathbf{0}^{(\gamma)}$. Hence if $S \not\leq_T 0^{(\gamma+1)}$, then the set $\{D^{(\gamma)} : S \leq_T D^{(\gamma+1)}\}$ has no element of least degree.*

Corollary 6.1.11. *If α is of the form in (1), (3) or (5) then the class $\{\mathcal{A}_\alpha(S) : S \subseteq \omega\}$ has Turing ordinal α.*

Proof. As such an α is a successor ordinal, let $\alpha = \gamma + 1$. By Theorem 6.1.9, the given class satisfies part (i) of Definition 4.1.2 and so it remains to show that part (ii) is satisfied. In other words, we need to show that if $\mathcal{C} := \{D^{(\gamma)} : S \leq_T D^{(\alpha)}\}$ has an element of least degree then it is $\mathbf{0}^{(\gamma)}$. If $S \leq_T \mathbf{0}^{(\alpha)} = \mathbf{0}^{(\gamma+1)}$ then $\mathbf{0}^{(\gamma)}$ is least in $\mathcal{C}$. By Theorem 6.1.10, if $S \not\leq_T \mathbf{0}^{(\gamma+1)}$ then $\mathcal{C}$ has no element of least degree. $\square$

6.1.2 Successor ordinals - Type II

For the following results, fix a computable ordinal α that is of the form in (2), (4) or (6). Again, the authors of [1] characterize the degree spectra the orderings $\mathcal{A}_\alpha(S)$:

Lemma 6.1.12 (Ash, Jockusch and Knight)**.** *For each $S \subseteq \omega$, we have*

$$Spec(\mathcal{A}_\alpha(S)) = \{\deg(D) : S \textit{ is c.e. in } D^{(\alpha-1)}\}.$$

The following result from [1] shows that the class consisting of the $\mathcal{A}_\alpha(S)$ orderings has Turing ordinal at most α, if it exists.

Theorem 6.1.13 (Ash, Jockusch and Knight)**.** *Let $\alpha \geq 2$ be a computable ordinal. Then for every degree $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$, there exists a set S such that $\mathcal{A}_\alpha(S)$ has α^{th} jump degree $\mathbf{d}$.*

Proof. Fix $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$. By Lemma 6.1.6, we can choose a set S such that S is α-generic and $S \oplus 0^{(\alpha)} \equiv_T S^{(\alpha)} \in \mathbf{d}$. Again, we claim $\mathcal{A}_\alpha(S)$ has α^{th} jump degree $\mathbf{d}$. By Lemma 6.1.12 we need to show that $\mathbf{d}$ is the least element in the set $\mathcal{C} := \{\deg(D)^{(\alpha)} : S \text{ is c.e. in } D^{(\alpha-1)}\}$. As S is c.e. in $S^{(\alpha-1)}$ we have $\deg(S)^{(\alpha)} = \mathbf{d} \in \mathcal{C}$. Now suppose that S is c.e. in $D^{(\alpha-1)}$ for some set D. It follows that $S \leq_T D^{(\alpha)}$ and since $0^{(\alpha)} \leq_T D^{(\alpha)}$ we have $S^{(\alpha)} \equiv_T S \oplus 0^{(\alpha)} \leq_T D^{(\alpha)}$. Therefore $\mathbf{d} = \deg(S)^{(\alpha)}$ is least in $\mathcal{C}$. $\square$

In this case, the Turing ordinal of the class $\{\mathcal{A}_\alpha(S) : S \subseteq \omega\}$ is not clear. We will show that the class

$$\{\mathcal{A}_\alpha(S) : S \text{ is } \alpha\text{-generic}\}$$

has Turing ordinal α. We need the following result from [1]:

Theorem 6.1.14 (Ash, Jockusch and Knight)**.** *Let γ be a computable ordinal and let $S \subseteq \omega$ be $(\gamma+1)$-generic. If $B \leq D^{(\gamma)}$ for all D such that S is c.e. in $D^{(\gamma)}$, then $B \leq_T \boldsymbol{0}^{(\gamma)}$. Hence, since S is not c.e. in $\boldsymbol{0}^{(\gamma)}$ (by Remark 6.1.7), the set $\{D^{(\gamma)} : S \text{ is c.e. in } D^{(\gamma)}\}$ has no element of least degree.*

Corollary 6.1.15. *If α is of the form in (2), (4) or (6), then the class*

$$\{\mathcal{A}_\alpha(S) : S \textit{ is } \alpha\textit{-generic}\}$$

has Turing ordinal α.

Proof. As the proof of Theorem 6.1.13 uses orderings of the form $\mathcal{A}_\alpha(S)$ for α-generic S, the given class of orderings satisfies part (i) of Definition 4.1.2. Theorem 6.1.14 completes the proof. If S is α-generic then, by Theorem 6.1.14, the set

$$\{D^{(\alpha-1)} : S \text{ is c.e. in } D^{(\alpha-1)}\}$$

has no element of least degree. It follows that no ordering in the given class can have β^{th} jump degree for any $\beta < \alpha$. ☐

In Section 6.3, we will axiomatize the classes from Corollary 6.1.11 and Corollary 6.1.15 for all ordinals $\alpha \leq \omega + 2$ using $\mathcal{L}_{\omega_1,\omega}$ sentences, and hence show that each of these classes is Borel. The classes corresponding to $\alpha = 2m + 3$ for each $m \in \omega$ will provide extra counterexamples to Question 4.3.1 and the class for $\alpha = \omega + 2$ will provide a negative answer for Question 4.3.2.

6.1.3 Limit ordinals

Now that we have defined the orderings $\mathcal{A}_\alpha(S)$ for all successor ordinals, we are ready to define the corresponding orderings for all computable limit ordinals.

Definition 6.1.16. Let α be a countable limit ordinal. A *fundamental sequence of α* is an ω-sequence which converges to α.

For the following defintion, let α be a computable limit ordinal and let $(\alpha_n)_{n\in\omega}$ be the fundamental sequence with limit α picked out of some notation for α.

Definition 6.1.17. The *adjusted* fundamental sequence for α, denoted by $(\alpha'_n)_{n\in\omega}$ is defined as follows:

(a) If α_n is finite then $\alpha'_n := \min\{k : k \text{ even}, k \geq 4, k \geq \alpha_n, k > \alpha'_{n-1}\}$

(b) If α_n is infinite then let β be the greatest limit ordinal $\leq \alpha_n$ and we define
$\alpha'_n := \min\{\gamma : \gamma = \beta + k, k \text{ odd}, \gamma \geq \alpha_n, \gamma > \alpha'_{n-1}\}$

Given a fundamental sequence $(\alpha_n)_{n\in\omega}$ for α picked out of a notation for α, note that, from the definition above, every member of the corresponding adjusted fundamental sequence $(\alpha'_n)_{n\in\omega}$ is a successor ordinal. Hence for any limit ordinal α, the orderings $\mathcal{A}_{\alpha'_n}(S)$, for all n, were defined at the beginning of this section. To form the ordering $\mathcal{A}_\alpha(S)$, the authors of [1] combined the $\mathcal{A}_{\alpha'_n}(S)$ orderings as follows.

Definition 6.1.18 (Ash, Jockusch and Knight)**.** Let α be a computable ordinal and $(\alpha_n)_{n\in\omega}$ a fundamental sequence for α. Then, for any $S \subseteq \omega$, let

$$\mathcal{A}_\alpha(S) := \sum_{n\in\omega}\Big(1 + \eta + 1 + \mathcal{A}_{\alpha'_n}(S_n)\Big)$$

where $S_n = \{k : \langle n, k\rangle \in S\}$ and α'_n is n^{th} member of the corresponding adjusted fundamental sequence.

Using the degree spectra results from the previous sections, the authors of [1] found the following:

Lemma 6.1.19 (Lemma 4.5 (3) from [1])**.** *Let α be a computable limit ordinal with fundamental sequence $(\alpha_n)_{n\in\omega}$ picked out of a notation for α. Then for any set $S \subseteq \omega$, we have:*

$$Spec\big(\mathcal{A}_\alpha(S)\big) = \Big\{ \deg(D) : S_n \leq_T D^{(\alpha'_n)} \textit{ uniformly in } n\Big\}.$$

The following result from [1] shows that the class consisting of the $\mathcal{A}_\alpha(S)$ orderings has Turing ordinal at most α, if it exists.

Theorem 6.1.20 (Ash, Jockusch and Knight)**.** *Let $\alpha \geq 2$ be a computable limit ordinal. Then for every degree $\boldsymbol{d} \geq \mathbf{0}^{(\alpha)}$, there exists a set S such that $\mathcal{A}_\alpha(S)$ has α^{th} jump degree $\boldsymbol{d}$.*

Proof. Let $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$. By Lemma 6.1.6, we can choose an α-generic set S with $S^{(\alpha)}$ of degree $\mathbf{d}$. We claim that $\mathcal{A}_\alpha(S)$ has α^{th} jump degree $\mathbf{d}$. By Lemma 6.1.19, we need to show that $\mathbf{d}$ is least in the set $\{\deg(D)^{(\alpha)} : S_n \leq_T D^{(\alpha_n')} \text{ uniformly in } n\}$,

Clearly $S_k \leq_T S^{(\alpha_k')}$ uniformly in k, and so $S^{(\alpha)}$ is a member of the set. Suppose, for some X, that $S_k \leq_T X^{(\alpha_k')}$ uniformly in k. Then $S \leq_T X^{(\alpha)}$. Since $0^{(\alpha)} \leq_T X^{(\alpha)}$ as well, we have $S^{(\alpha)} \equiv_T 0^{(\alpha)} \oplus S \leq_T X^{(\alpha)}$. As $S^{(\alpha)} \in \mathbf{d}$, $\mathbf{d}$ is the α^{th} jump degree of $\mathcal{A}_\alpha(S)$. $\square$

Another result from [1] will show that a particular subcollection of structures of the form $\mathcal{A}_\alpha(S)$ forms a class with Turing ordinal α.

Theorem 6.1.21 (Lemma 1.4 from [1])**.** *Let α be a computable limit ordinal and let $(\alpha_n)_{n\in\omega}$ be a fundamental sequence with limit α that is picked out by a notation for α. Let $S \subseteq \omega$. Define*

$$\mathcal{C} := \{D : S_n \leq_T D^{(\alpha_n)} \text{ uniformly in } n\}$$

and suppose that, for some $\beta < \alpha$, $B \leq_T D^{(\beta)}$ for all $D \in \mathcal{C}$. Then

$$\beta < \alpha_n \implies B \leq_T (S_0 \oplus \dots S_{n-1})^{(\beta)}.$$

Hence if $\beta < \alpha_n$ and $S_n \not\leq_T (S_0 \oplus \dots S_{n-1})^{(\alpha_n)}$ then the set $\{D^{(\beta)} : D \in \mathcal{C}\}$ has no element of least degree.

Theorem 6.1.22. *Let α be a computable limit ordinal. Then for any fundamental sequence $(\alpha_n)_{n\in\omega}$ of α, the class*

$$\{\mathcal{A}_\alpha(S) : S_k \not\leq_T (S_0 \oplus S_1 \oplus \dots \oplus S_{k-1})^{(\alpha_k)} \text{ for all } k\}$$

has Turing ordinal α.

Proof. Fix $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$. By the proof of Theorem 6.1.20, there is an α-generic set S such that $\mathcal{A}_\alpha(S)$ has α^{th} jump degree $\mathbf{d}$. As S is α-generic, by Remark 6.1.7 we have $S_k \not\leq_T (S_0 \oplus S_1 \oplus \dots \oplus S_{k-1})^{(\alpha_k)}$ for all k and hence $\mathcal{A}_\alpha(S)$ is in the given class.

Now fix any $\mathcal{A}_\alpha(S)$ in the class, and fix $\beta < \alpha$. Then we must have $\beta < \alpha_n$ for some $n \in \omega$ and $S_n \not\leq_T (S_0 \oplus S_1 \oplus \dots \oplus S_{n-1})^{(\alpha_n)}$. By Theorem 6.1.21, $\mathcal{A}_\alpha(S)$ has no β^{th} jump degree. Therefore the class has Turing ordinal α as desired. $\square$

Here we were able to avoid the α-generic condition needed in Section 6.1.2.

6.2 A selection of back-and-forth ordinals

In this section we will compute the back-and-forth ordinals of the classes defined in Section 6.1 associated to all finite ordinals and the ordinals $\omega+1$ and $\omega+2$. We will deal with the limit case of $\alpha=\omega$ in Chapter 7.

6.2.1 Describing the $\mathbb{Z}$ powers

First we will define formulas that will axiomatize orderings of the form $\mathbb{Z}^k$ for all $k\in\omega$. The complexities of the formulas will be displayed immediately following each formula.

$\boxed{\text{Consider the relevant formulas for } k=0\text{:}}$

Here are the formulas needed to describe $\mathbb{Z}$-chains:

- Let $S(x,y) := (x<y)\wedge(\forall z)[z\leq x \vee z\leq y]$. $(\mathbf{\Pi_1^c})$.

  Then for any linear ordering $\mathcal{A}$ and $a,b\in\mathcal{A}$, we have $\mathcal{A}\models S(a,b)$ if and only if b is the successor of a in $\mathcal{A}$.

- For $n>1$, let $S^n(x,y){:=}(\exists x_1,\dots,x_{n-1})\,[S(x,x_1)\wedge S(x_1,x_2)\wedge\cdots\wedge S(x_{n-1},y)]$. $(\mathbf{\Sigma_2^c})$

  Then $\mathcal{A}\models S^n(a,b)$ if and only if b is the n^{th} successor of a in $\mathcal{A}$. For completeness, let $S^0(x,y):=(x=y)$.

- Let $SP(x):=(\exists y_1)(\exists y_2)[S(y_1,x)\wedge S(x,y_2)]$. $(\mathbf{\Sigma_2^c})$

  Then $\mathcal{A}\models SP(a)$ if and only if a has a predecessor and a successor in $\mathcal{A}$.

$\boxed{\text{Consider the following formulas for } k=1\text{:}}$

- Let $\varphi_{\mathbb{Z}}(x) := SP(x)\wedge \bigwedge_{n\in\omega}(\forall y)\Big[\ \big(S^n(x,y)\vee S^n(y,x)\big)\rightarrow SP(y)\ \Big]$. $(\mathbf{\Pi_3^c})$

  Then $\mathcal{A}\models\varphi_{\mathbb{Z}}(a)$ if and only if a lies in a $\mathbb{Z}$-chain in $\mathcal{A}$.

- Let $\varphi_{\mathbb{Z}}(x,y) := \varphi_{\mathbb{Z}}(x)\wedge\varphi_{\mathbb{Z}}(y)\wedge\left[\bigvee_{n\in\omega}S^n(x,y)\vee\bigvee_{n\in\omega}S^n(y,x)\right]$. $(\mathbf{\Pi_3^c})$

  Then $\mathcal{A}\models\varphi_{\mathbb{Z}}(a,b)$ if and only if a and b lie in the same $\mathbb{Z}$-chain in $\mathcal{A}$.

- Let $S_{\mathbb{Z}}(x,y) := \varphi_{\mathbb{Z}}(x) \ \wedge \ \varphi_{\mathbb{Z}}(y) \ \wedge \ (x < y) \ \wedge \ \neg\varphi_{\mathbb{Z}}(x,y)$
$$\wedge \ (\forall z)\Big[x \leq z \leq y \ \rightarrow \ \varphi_{\mathbb{Z}}(x,z) \vee \varphi_{\mathbb{Z}}(z,y)\Big]. \ \ (\mathbf{\Delta^c_4})$$
Then $\mathcal{A} \models \varphi_{\mathbb{Z}}(a,b)$ if and only if a and b are in successive $\mathbb{Z}$-chains in $\mathcal{A}$.

- Let $S^n_{\mathbb{Z}}(x,y) := (\exists x_1,\dots,x_{n-1})\,[S_{\mathbb{Z}}(x,x_1) \wedge S_{\mathbb{Z}}(x_1,x_2) \wedge \cdots \wedge S_{\mathbb{Z}}(x_{n-1},y)]\,.$ $(\mathbf{\Sigma^c_4})$
Then $\mathcal{A} \models S_{\mathbb{Z}}(a,b)$ if and only if the $\mathbb{Z}$-chain containing b in $\mathcal{A}$ is the n^{th} successor of the $\mathbb{Z}$-chain containing a. For completeness, let $S^0_{\mathbb{Z}}(x,y) := \varphi_{\mathbb{Z}}(x,y)$.

- Let $\varphi_{\mathbb{Z}} := (\forall x_1)(\forall x_2)\varphi_{\mathbb{Z}}(x_1,x_2)$. $(\mathbf{\Pi^c_3})$
Then $\mathcal{A} \models \varphi_{\mathbb{Z}} \Leftrightarrow \mathcal{A} \cong \mathbb{Z}$

- Let $SP_{\mathbb{Z}}(x) := \varphi_{\mathbb{Z}}(x) \wedge (\exists y_1)(\exists y_2)\Big[S_{\mathbb{Z}}(y_1,x) \wedge S_{\mathbb{Z}}(x,y_2)\Big].$ $(\mathbf{\Sigma^c_4})$
Then we have $\mathcal{A} \models SP_{\mathbb{Z}}(a)$ if and only if the $\mathbb{Z}$-chain containing x has immediately preceding and succeeding $\mathbb{Z}$-chains in $\mathcal{A}$.

Using this pattern, we can define the formulas for all k by induction. Suppose that we have formulas $\varphi_{\mathbb{Z}^k}$, $\varphi_{\mathbb{Z}^k}(x)$, $\varphi_{\mathbb{Z}^k}(x,y)$, $S_{\mathbb{Z}^k}(x,y)$, $S^n_{\mathbb{Z}^k}(x,y)$ and $SP_{\mathbb{Z}^k}(x)$ with complexities Π^c_{2k+1}, Π^c_{2k+1}, Δ^c_{2k+2}, Σ^c_{2k+2}, Π^c_{2k+1} and Σ^c_{2k+2} respectively. We define the formulas for $k+1$ as follows:

$\boxed{\text{Formulas for } k+1\text{:}}$

- $\varphi_{\mathbb{Z}^{k+1}}(x) := \varphi_{\mathbb{Z}^k}(x) \wedge SP_{\mathbb{Z}^k}(x) \wedge \bigwedge_{n\in\omega} (\forall y)\,[\ (S^n_{\mathbb{Z}^k}(x,y) \vee S^n_{\mathbb{Z}^k}(y,x)) \rightarrow SP_{\mathbb{Z}^k}(y)\ ].$ $(\mathbf{\Pi^c_{2k+3}})$
Then $\mathcal{A} \models \varphi_{\mathbb{Z}^{k+1}}(a)$ if and only if a lies in a $\mathbb{Z}^{k+1}$-chain in $\mathcal{A}$.

- $\varphi_{\mathbb{Z}^{k+1}}(x,y) := \varphi_{\mathbb{Z}^{k+1}}(x) \wedge \varphi_{\mathbb{Z}^{k+1}}(y) \wedge \left[\bigvee_{n\in\omega} S^n_{\mathbb{Z}^k}(x,y) \vee \bigvee_{n\in\omega} S^n_{\mathbb{Z}^k}(y,x)\right].$ $(\mathbf{\Pi^c_{2k+3}})$
Then $\mathcal{A} \models \varphi_{\mathbb{Z}^{k+1}}(a,b)$ if and only if a and b lie in the same $\mathbb{Z}^{k+1}$-chain in $\mathcal{A}$.

- $S_{\mathbb{Z}^{k+1}}(x,y) := \varphi_{\mathbb{Z}^{k+1}}(x) \wedge \varphi_{\mathbb{Z}^{k+1}}(y) \wedge (x < y) \wedge \neg\varphi_{\mathbb{Z}^{k+1}}(x,y)$
$$\wedge \ (\forall z)\Big[x \leq z \leq y \ \rightarrow \ \varphi_{\mathbb{Z}^{k+1}}(x,z) \vee \varphi_{\mathbb{Z}^{k+1}}(z,y)\Big]. \ (\mathbf{\Delta^c_{2k+4}})$$
Then $\mathcal{A} \models \varphi_{\mathbb{Z}^{k+1}}(a,b)$ if and only if a and b are in successive $\mathbb{Z}^{k+1}$-chains in $\mathcal{A}$.

- Let $S^n_{\mathbb{Z}^{k+1}}(x,y) := (\exists x_1,\ldots,x_{n-1})\left[S_{\mathbb{Z}^{k+1}}(x,x_1) \wedge S_{\mathbb{Z}^{k+1}}(x_1,x_2) \wedge \cdots \wedge S_{\mathbb{Z}^{k+1}}(x_{n-1},y)\right]$. $(\boldsymbol{\Sigma^c_{2k+4}})$

  Then $\mathcal{A} \models S^n_{\mathbb{Z}^{k+1}}(a,b)$ if and only if the $\mathbb{Z}^{k+1}$-chain containing b in $\mathcal{A}$ is the n^{th} successor of the $\mathbb{Z}^{k+1}$-chain containing a. Let $S^0_{\mathbb{Z}^{k+1}}(x,y) := \varphi_{\mathbb{Z}^{k+1}}(x,y)$.

- $\varphi_{\mathbb{Z}^{k+1}} := (\forall x)(\forall y)\varphi_{\mathbb{Z}^{k+1}}(x,y)$. $(\boldsymbol{\Pi^c_{2k+3}})$

  Then $\mathcal{A} \models \varphi_{\mathbb{Z}^{k+1}} \Leftrightarrow \mathcal{A} \cong \mathbb{Z}^{k+1}$.

- $SP_{\mathbb{Z}^{k+1}}(x) := \varphi_{\mathbb{Z}^{k+1}}(x) \wedge (\exists y_1)(\exists y_2)\Big[S_{\mathbb{Z}^{k+1}}(y_1,x) \wedge S_{\mathbb{Z}^{k+1}}(x,y_2)\Big]$. $(\boldsymbol{\Sigma^c_{2k+4}})$

  Then we have $\mathcal{A} \models SP_{\mathbb{Z}^{k+1}}(a)$ if and only if the $\mathbb{Z}^{k+1}$-chain containing a has immediately succeeding and preceding $\mathbb{Z}^{k+1}$-chains in $\mathcal{A}$.

Here is a summary of the formulas and their complexities:

Formula	Meaning	Complexity
$\varphi_{\mathbb{Z}^n}(x)$	x lies in a $\mathbb{Z}^n$ block	Π^c_{2n+1}
$\varphi_{\mathbb{Z}^n}(x,y)$	x and y lie in the same $\mathbb{Z}^n$ block	Π^c_{2n+1}
$\varphi_{\mathbb{Z}^n}$	The ordering is isomorphic to $\mathbb{Z}^n$	Π^c_{2n+1}
$S_{\mathbb{Z}^n}(x,y)$	x and y lie in successive $\mathbb{Z}^n$ blocks	Δ^c_{2n+2}
$S^k_{\mathbb{Z}^n}(x,y)$	The $\mathbb{Z}^n$ block of y is the k^{th} successor of the $\mathbb{Z}^n$ block of x	Σ^c_{2n+2}
$SP_{\mathbb{Z}^n}(x)$	The $\mathbb{Z}^n$ block of x has preceding and succeeding $\mathbb{Z}^n$ blocks	Σ^c_{2n+2}

6.2.2 Finite ordinals

In this section, we will find the back-and-forth ordinals of the following classes:

$$\mathbb{K}_{2m+2} := \{\mathcal{A}_{2m+2}(S) : S \subseteq \omega\}$$

and

$$\mathbb{K}_{2m+3} := \{\mathcal{A}_{2m+3}(S) : S \text{ is } (2m+3)\text{-generic}\}.$$

Recall from Section 6.1 that the Turing ordinals of the two classes are $2m+2$ and $2m+3$ respectively. We will show that, in both cases, the back-and-forth ordinal is $2m+3$. Recall that $\mathcal{A}_{2m+2}(S) := \mathbb{Z}^m \cdot \sigma^*(S \oplus \overline{S})$ and $\mathcal{A}_{2m+3}(S) := \mathbb{Z}^m \cdot \sigma^*(S)$.

First, consider the following formula:

$$(\exists x_0,x_1,\ldots,x_r)\left[\bigwedge_{i=0}^{r-1} S_{\mathbb{Z}^m}(x_i,x_{i+1}) \;\wedge\; (\forall y)\big(\neg S_{\mathbb{Z}^m}(y,x_0) \wedge \neg S_{\mathbb{Z}^m}(x_r,y)\big)\right]$$

asserting the existence of a $\mathbb{Z}^m \cdot (r+1)$ block in an ordering. By the work in Section 6.2.1, this formula is Σ^c_{2m+3} and, by the definition of $\mathbb{Z}^m \cdot \sigma^*(X)$, this formula will distinguish orderings of the form $\mathbb{Z}^m \cdot \sigma^*(X)$ and $\mathbb{Z}^m \cdot \sigma^*(Y)$ provided that $X \neq Y$. It follows that the back-and-forth ordinal of each of the two classes is *at most* $2m+3$.

It remains to show that there are countably many $\equiv_{2m+2}$-classes the form $(\mathcal{A}, \vec{a})$ where $\mathcal{A} \in \mathbb{K}_{2m+2}$ or $\mathcal{A} \in \mathbb{K}_{2m+3}$. Before we state our theorem, we need to recall some notation from Chapter 5. In this formulation, we will use similar notation to that appearing in Section 5.2. This time, we need a formal way of passing from tuples in $\mathcal{L}$ to tuples in $\mathbb{Z} \cdot \mathcal{L}$ and vice versa and we will do so as follows.

Let Y_k be a set consisting of k-tuples of the form

$$\vec{x} = (\langle m_1, z^1_1, z^1_2, \dots z^1_{m_1}\rangle, \langle m_2, z^2_1, z^2_2, \dots z^2_{m_2}\rangle, \dots, \langle m_k, z^k_1, z^k_2, \dots z^k_{m_k}\rangle)$$

where $m_1, m_2, \dots, m_k$ are positive integers, and for each $1 \leq i \leq k$, we have $z^i_1 < z^i_2 < \dots < z^i_{m_i}$, all picking elements of $\mathbb{Z}$.

Given a linear order $\mathcal{L}$, a k-tuple $\vec{a} = (a_1, \dots, a_k)$ from $\mathcal{L}$, and a k-tuple $\vec{x}$ from Y_k, we define a corresponding $(m_1+m_2+\dots+m_k)$-tuple, denoted by $g_{\vec{x}}(\vec{a})$, in $\varphi(\mathcal{L})$ as follows: Let $g_{\vec{x}}(\vec{a}) := (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k)$ where the tuple $\tilde{a}_i$ is of length m_i, lies in the $\mathbb{Z}$ block corresponding to the element $a_i \in \mathcal{L}$, and there is an embedding of $\mathbb{Z}$ into $\mathbb{Z} \cdot \mathcal{L}$ that sends z^i_j to the j^{th} member of the tuple $\tilde{a}_i$.

Now we are ready to characterize the pairs $(\mathbb{Z}^m \cdot \sigma^*(S), \vec{a})$ up to $\equiv_{2n+2}$-equivalence.

Theorem 6.2.1. *For all $m \geq 1$, if $\vec{a} \in \mathcal{A} \cong \mathbb{Z}^{m-1} \cdot \sigma^*(S)$ and $\vec{b} \in \mathcal{B} \cong \mathbb{Z}^{m-1} \cdot \sigma^*(R)$ are tuples of the same size such that*

(i) $\vec{a}$ lies entirely in a $\mathbb{Z}^{m-1} \cdot \alpha$ block for some $\alpha \in \{\omega\} \cup \{n+1\}_{n \in \omega}$,

(ii) $\vec{b}$ lies entirely in a $\mathbb{Z}^{m-1} \cdot \beta$ block with $\beta = \alpha$, and

(iii) as suborderings, we have $(\mathbb{Z}^{m-1} \cdot \alpha, \vec{a}) \cong (\mathbb{Z}^{m-1} \cdot \alpha, \vec{b})$,

then for any $\vec{x} \in Y_{|\vec{a}|}$ we have

$$(\mathbb{Z}^m \cdot \sigma^*(S), g_{\vec{x}}(\vec{a})) \leq_{2m+2} (\mathbb{Z}^m \cdot \sigma^*(R), g_{\vec{x}}(\vec{b})).$$

This theorem amounts to showing that

$$(\forall \vec{b}_1)(\exists \vec{a}_1)(\forall \vec{a}_2)(\exists \vec{b}_2)\cdots(\forall \vec{b}_{2m+1})(\exists \vec{a}_{2m+1})\Big[\big(\mathcal{B}, g_{\vec{x}}(\vec{b}), \vec{b}_1, \ldots, \vec{b}_{2m+1}\big) \leq_1 \big(\mathcal{A}, g_{\vec{x}}(\vec{a}), \vec{a}_1, \ldots, \vec{a}_{2m+1}\big)\Big]$$

where each $\vec{b}_i \in \mathcal{B}$ and each $\vec{a}_i \in \mathcal{A}$. To make our way through the back-and-forth argument needed for the theorem, we will use the following lemma.

Lemma 6.2.2. *Fix $m \geq 1$ and let $\mathcal{A} \cong \mathbb{Z}^m \cdot \sigma^*(S)$ and $\mathcal{B} \cong \mathbb{Z}^m \cdot \sigma^*(R)$ for $S, R \subseteq \omega$. Suppose that, for some $0 \leq k < m-1$, we have performed $2k+2$ back and forth steps resulting in the tuples*

$$\vec{b}_1, \vec{a}_1;\ \vec{a}_2, \vec{b}_2;\ \ldots\ ; \vec{b}_{2k+1}, \vec{a}_{2k+1};\ \textit{and } \vec{a}_{2k+2}, \vec{b}_{2k+2}$$

with each $\vec{a}_i \in \mathcal{A}$ and each $\vec{b}_i \in \mathcal{B}$, and suppose that the tuples chosen satisfy the following: Let $\vec{c} = \vec{a}_1 \cup \ldots \cup \vec{a}_{2k+2}$ and $\vec{d} = \vec{b}_1 \cup \ldots \cup \vec{b}_{2k+2}$ and write

$$\vec{c} = \vec{c}_1 < \ldots < \vec{c}_l$$

and

$$\vec{d} = \vec{d}_1 < \ldots < \vec{d}_r$$

where the tuples $\vec{c}_1$, $\vec{c}_2, \ldots, \vec{c}_l$ each lie in distinct $\mathbb{Z}^{m-k}$ blocks in $\mathcal{A}$ and the tuples $\vec{d}_1$, $\vec{d}_2, \ldots, \vec{d}_r$ each lie in distinct $\mathbb{Z}^{m-k}$ blocks in $\mathcal{B}$ and are such that

$$(*)_k \qquad l = r \textit{ and } (\mathbb{Z}^{m-k}, \vec{c}_i) \cong (\mathbb{Z}^{m-k}, \vec{d}_i) \textit{ for all } i = 1, \ldots, l.$$

Then $(\forall \vec{b}_{2k+3})(\exists \vec{a}_{2k+3})(\forall \vec{a}_{2k+4})(\exists \vec{b}_{2k+4})$ such that $(*)_{k+1}$ holds for the "new" tuples

$$\tilde{c} = \vec{a}_1 \cup \ldots \vec{a}_{2k+2} \cup \vec{a}_{2k+3} \cup \vec{a}_{2k+4}$$

and

$$\tilde{d} = \vec{b}_1 \cup \ldots \cup \vec{b}_{2k+2} \cup \vec{b}_{2k+3} \cup \vec{b}_{2k+4}$$

Remark 6.2.3. Let $\vec{c}_i$ and $\vec{d}_i$ be the tuples described in the above theorem. Note that, for each $i = 1, \ldots, l$, there is a subordering of the form $\mathbb{Z}^{m-k-1} \cdot \omega^*$ in between the rightmost member of $\vec{d}_{i-1}$ and the leftmost member of $\vec{d}_i$, and similarly for $\vec{c}_{i-1}$ and $\vec{c}_i$. (We will use the case of $i = 1$ to mean "to the left of the tuple $\vec{d}_1$ and $\vec{c}_1$".)

Before we prove the above lemma, we will see how this lemma will give us the desired result.

Proof. (of Theorem 6.2.1 using Lemma 6.2.2) Fix $m \geq 1$ and let $\vec{a} \in \mathcal{A} \cong \mathbb{Z}^{m-1} \cdot \sigma^*(S)$ and $\vec{b} \in \mathcal{B} \cong \mathbb{Z}^{m-1} \cdot \sigma^*(R)$ be tuples of the same size such that

(i) $\vec{a}$ lies entirely in a $\mathbb{Z}^{m-1} \cdot \alpha$ block for some $\alpha \in \{\omega\} \cup \{n+1\}_{n \in \omega}$,

(ii) $\vec{b}$ lies entirely in a $\mathbb{Z}^{m-1} \cdot \beta$ block with $\alpha = \beta$, and

(iii) as suborderings, we have $(\mathbb{Z}^{m-1} \cdot \alpha, \vec{a}) \cong (\mathbb{Z}^{m-1} \cdot \alpha, \vec{b})$.

Fix $\vec{x} \in Y_{|\vec{a}|}$. We need to show that $\big(\mathbb{Z}^m \cdot \sigma^*(S), g_{\vec{x}}(\vec{a})\big) \leq_{2m+2} \big(\mathbb{Z}^m \cdot \sigma^*(R), g_{\vec{x}}(\vec{b})\big)$. By properties (i)-(iii), and by the definition of $g_{\vec{x}}(\cdot)$, we have that

$$\big(\mathbb{Z}^m \cdot \alpha, g_{\vec{x}}(\vec{a})\big) \cong \big(\mathbb{Z}^m \cdot \alpha, g_{\vec{x}}(\vec{b})\big)$$

when we restrict the orderings $\mathbb{Z}^m \cdot \sigma^*(S)$ and $\mathbb{Z}^m \cdot \sigma^*(R)$ to only the blocks containing the tuples $g_{\vec{x}}(\vec{a})$ and $g_{\vec{x}}(\vec{b})$ respectively.

Fix $\vec{b}_1 \in \mathbb{Z}^m \cdot \sigma^*(R)$ and decompose $g_{\vec{x}}(\vec{b}) \cup \vec{b}_1$ as $g_{\vec{x}}(\vec{b}) \cup \vec{b}_1 = g_{\vec{x}}(\vec{b}) \cup \vec{b}_1(1) \cup \vec{b}_1(2) \cup \ldots \cup \vec{b}_1(k)$ where the tuples $\vec{b}_1(1) < \vec{b}_1(2) < \ldots < \vec{b}_1(k)$ lie in distinct $\mathbb{Z}^m \cdot \alpha_i$ blocks where

$$\alpha_i \in \{\omega\} \cup \{r+1\}_{r \in R}.$$

If any of $\vec{b}_1$ lies in the same $\mathbb{Z}^m \cdot \alpha$ block as $g_{\vec{x}}(\vec{b})$ then, without loss, we will include this part of $\vec{b}_1$ in the tuple $g_{\vec{x}}(\vec{b})$. We will let $g_{\vec{x}}(\vec{a}) \cup \vec{a}_1 = g_{\vec{x}}(\vec{a}) \cup \vec{a}_1(1) \cup \vec{a}_1(2) \cup \ldots \cup \vec{a}_1(k)$ where each tuple $\vec{a}_1(i)$ is chosen in a distinct $\mathbb{Z}^m \cdot \omega$ block in $\mathbb{Z}^m \cdot \sigma^*(S)$ so that the resulting tuples, $g_{\vec{x}}(\vec{b}) \cup \vec{b}_1$ and $g_{\vec{x}}(\vec{a}) \cup \vec{a}_1$, have the same atomic type, and so that, as suborderings, we have

$$\big(\mathbb{Z}^m \cdot \alpha_i, \vec{b}_1(i)\big) \hookrightarrow \big(\mathbb{Z}^m \cdot \omega, \vec{a}_1(i)\big).$$

This can be done as there are densely many $\mathbb{Z}^m \cdot \omega$ blocks.

Note that if $\alpha_i = \omega$ then the embedding is really an isomorphism.

Now fix $\vec{a}_2$ in $\mathbb{Z}^m \cdot \sigma^*(S)$. Again, decompose $\vec{a}_2$ as $\vec{a}_2 = \vec{a}_2(1) < \vec{a}_2(2) < \ldots < \vec{a}_2(l)$ where each $\vec{a}_2(j)$ lies in a distinct $\mathbb{Z}^m \cdot \beta_j$ block where $\beta_j \in \{\omega\} \cup \{s+1\}_{s \in S}$. Fix some j with $1 \leq j \leq l$. If $\vec{a}_2(j)$ lies in the same block as $g_{\vec{x}}(\vec{a})$ then the choice of corresponding tuple $\vec{b}_2(j)$ is obvious, so we consider the two remaining cases:

Case (1): $\vec{a}_2(j)$ lies in the same $\mathbb{Z}^m \cdot \omega$ block as $\vec{a}_1(i)$ for some $1 \leq i \leq k$.

If $\alpha_i = \omega$ (i.e. the corresponding tuple $\vec{b}_1(i)$ lies in a $\mathbb{Z}^m \cdot \omega$ block), then we can choose $\vec{b}_2(j)$ in the $\mathbb{Z}^m \cdot \omega$ block containing $\vec{b}_1(i)$ so that, as suborderings, we have

$$(\mathbb{Z}^m \cdot \omega, \vec{b}_1(i), \vec{b}_2(j)) \cong (\mathbb{Z}^m \cdot \omega, \vec{a}_1(i), \vec{a}_2(j)).$$

If $\alpha_i = r+1$ (i.e. the corresponding tuple $\vec{b}_1(i)$ lies in a $\mathbb{Z}^m \cdot (r+1)$ block), then we decompose the $\mathbb{Z}^m \cdot \omega$ block containing $\vec{a}_1(i)$ and $\vec{a}_2(j)$ as follows:

$$\mathbb{Z}^m \cdot \omega = \Big[\mathbb{Z}^m \cdot (r+1)\Big] + \Big[\mathbb{Z}^m \cdot \omega\Big] = \mathcal{L}_1 + \mathcal{L}_2.$$

We know $\vec{a}_1(i)$ lies entirely in the first summand $\mathcal{L}_1$. Write $\vec{a}_2(j)$ as $\vec{e}_1 \cup \vec{e}_2$ where $\vec{e}_i \in \mathcal{L}_i$ (one may be the empty tuple). Here is the configuration in $\mathbb{Z}^m \cdot \sigma^*(S)$:

$$\begin{array}{ccccccccc} \mathbb{Z}^m \cdot \omega & = & \Big[\mathbb{Z}^m \cdot (r+1)\Big] & + & \Big[\mathbb{Z}^m \cdot \omega\Big] & = & \mathcal{L}_1 & + & \mathcal{L}_2 \\ & & \uparrow \qquad \uparrow & & \uparrow & & \uparrow \quad \uparrow & & \uparrow \\ & & \vec{a}_1(i) \quad \vec{e}_1 & & \vec{e}_2 & & \vec{a}_1(i) \;\; \vec{e}_1 & & \vec{e}_2 \end{array}$$

If $\vec{e}_1 \neq \emptyset$ then we choose a corresponding $\vec{f}_1$ in the $\mathbb{Z}^m \cdot (r+1)$ block containing $\vec{b}_1(i)$ so that, as suborderings, we have

$$\big(\mathbb{Z}^m \cdot (r+1), \vec{b}_1(i), \vec{f}_1\big) \cong \big(\mathcal{L}_1, \vec{a}_1(i), \vec{e}_1\big).$$

If $\vec{e}_2 \neq \emptyset$, then fix a "new" $\mathbb{Z}^m \cdot \omega$ block in $\mathbb{Z}^m \cdot \sigma^*(R)$ that lies between $\vec{b}_1(i)$ and $\vec{b}_1(i+1)$. (*If $i = k$ then this condition reduces to being to the right of $\vec{b}_1(k)$.*) This block must exist as $\vec{b}_1(i)$ and $\vec{b}_1(i+1)$ lie in distinct $\mathbb{Z}^m \cdot \alpha_i$ blocks by assumption and hence there are densely many $\mathbb{Z}^m \cdot \omega$ blocks in between the two tuples.

Choose $\vec{f}_2$ in this $\mathbb{Z}^m \cdot \omega$ block so that, as suborderings, we have $\big(\mathbb{Z}^m \cdot \omega, \vec{f}_2\big) \cong \big(\mathcal{L}_2, \vec{e}_2\big)$. Let $\vec{b}_2(j) = \vec{f}_1 \cup \vec{f}_2$.

Case (2): $\vec{a}_2(j)$ does not lie in the same $\mathbb{Z}^m \cdot \omega$ block as any $\vec{a}_1(i)$.

Let's assume $\vec{a}_2(j)$ lies strictly in between the $\mathbb{Z}^m \cdot \omega$ blocks of $\vec{a}_1(i)$ and $\vec{a}_1(i+1)$ for some $0 \leq i \leq k$ (where $i = 0$ is interpreted as "to the left of $\vec{a}_1(1)$" and $i = k$ is interpreted as "to the right of $\vec{a}_1(k)$"). Fix a "new" $\mathbb{Z}^m \cdot \omega$ block in $\mathbb{Z}^m \cdot \sigma^*(R)$ that lies between $\vec{b}_1(i)$ and $\vec{b}_1(i+1)$. Choose $\vec{b}_2(j)$ in this block so that, as suborderings, we have

$$(\mathbb{Z}^m \cdot \beta_j, \vec{a}_2(j)) \hookrightarrow (\mathbb{Z}^m \cdot \omega, \vec{b}_2(j)).$$

Having chosen $\vec{b}_2(j)$ for $1 \leq j \leq k$, we let $\vec{b}_2 = \vec{b}_2(1) < \vec{b}_2(2) < \ldots < \vec{b}_2(k)$.

Now, let's stop to examine the current situation. Let $\vec{d} := g_{\vec{x}}(b) \cup \vec{b}_1 \cup \vec{b}_2$, the current tuple in $\mathbb{Z} \cdot \sigma^*(R)$, and $\vec{c} := g_{\vec{x}}(a) \cup \vec{a}_1 \cup \vec{a}_2$, the current tuple in $\mathbb{Z} \cdot \sigma^*(S)$. Decompose the two current tuples as

$$\vec{c} = \vec{c}_1 < \ldots < \vec{c}_l$$

and

$$\vec{d} = \vec{d}_1 < \ldots < \vec{d}_r$$

where the tuples $\vec{c}_1, \ldots, \vec{c}_l$ lie in distinct $\mathbb{Z}^m$ blocks in their respective orderings. Then we have $l = r$ and $(\mathbb{Z}^m, \vec{d}_i) \cong (\mathbb{Z}^m, \vec{c}_i)$ for all $i = 1, \ldots, l$. Note also that, for each $i = 1, \ldots, l$, there is a subordering of the form $\mathbb{Z}^{m-1} \cdot \omega^*$ in between the rightmost member of $\vec{d}_{i-1}$ and the leftmost member of $\vec{d}_i$ and similarly for $\vec{c}_{i-1}$ and $\vec{c}_i$. (*Again, $i = 1$ is interpreted as "to the left of $\vec{d}_1$ and $\vec{c}_1$."*) We have shown that

$$(\forall \vec{b}_1)(\exists \vec{a}_1)(\forall \vec{a}_2)(\exists \vec{b}_2)\Big[g_{\vec{x}}(a) \cup \vec{a}_1 \cup \vec{a}_2 \text{ and } g_{\vec{x}}(b) \cup \vec{b}_1 \cup \vec{b}_2 \text{ satisfy } (*)_0 \text{ of Lemma 6.2.2}\Big].$$

By iterating Lemma 6.2.2 $m-1$ times, resulting in $2(m-1)$ additional back-and-forth steps, we arrive at the following: For tuples

$$\vec{u} = g_{\vec{x}}(\vec{a}) \cup \vec{a}_1 \cup \vec{a}_2 \cup \ldots \cup \vec{a}_{2m-1} \cup \vec{a}_{2m} \in \mathbb{Z}^m \cdot \sigma^*(S)$$

and

$$\vec{w} = g_{\vec{x}}(\vec{b}) \cup \vec{b}_1 \cup \vec{b}_2 \cup \ldots \cup \vec{b}_{2m-1} \cup \vec{b}_{2m} \in \mathbb{Z}^m \cdot \sigma^*(R)$$

such that, when each tuple is decomposed as

$$\vec{u} = \vec{u}_1 < \vec{u}_2 < \ldots < \vec{u}_l,$$

and

$$\vec{w} = \vec{w}_1 < \vec{w}_2 < \ldots < \vec{w}_r$$

where each tuple $\vec{u}_i$ and $\vec{w}_i$ lies in a distinct $\mathbb{Z}$ block in its respective ordering, we have $l = r$ and $(\mathbb{Z}, \vec{u}_i) \cong (\mathbb{Z}, \vec{w}_i)$ for all $i = 1, \ldots, l$. Observe that, for $i = 1, \ldots, l$, there is a subordering of the form ω^* between the rightmost member of $\vec{w}_{i-1}$ and the leftmost member of $\vec{w}_i$ and similarly for $\vec{u}_{i-1}$ and $\vec{u}_i$. So we have shown that

$$(\forall \vec{b}_1)(\exists \vec{a}_1)(\forall \vec{a}_2)(\exists \vec{b}_2) \cdots (\forall \vec{b}_{2m-1})(\exists \vec{a}_{2m-1})(\forall \vec{a}_{2m})(\exists \vec{b}_{2m})$$

$$\Big[g_{\vec{x}}(a) \cup \vec{a}_1 \cup \ldots \cup \vec{a}_{2m} \text{ and } g_{\vec{x}}(b) \cup \vec{b}_1 \cup \ldots \cup \vec{b}_{2m} \text{ satisfy } (*)_{m-1} \text{ of Lemma 6.2.2}\Big].$$

Now fix $\vec{b}_{2m+1} \in \mathbb{Z}^m \cdot \sigma^*(R)$ and let $\vec{b}_{2m+1} = \vec{b}_{2m+1}(1) < \vec{b}_{2m+1}(2) < \ldots < \vec{b}_{2m+1}(r)$ where each $\vec{b}_{2m+1}(i)$ lies in a distinct $\mathbb{Z}$ block. If $\vec{b}_{2m+1}(i)$ lies in the same $\mathbb{Z}$ block as some existing

$\vec{w}_j$, then the choice for corresponding $\vec{a}_{2m+1}(i) \in \mathbb{Z}^m \cdot \sigma^*(S)$ is natural. Let j the the least index such that $\vec{b}_{2m+1} \leq \vec{w}_j$. (If no such j exists then we will follow the same procedure as follows, but to the right of all the $\vec{w}_j$'s.) Let the corresponding $\vec{a}_{2m+1}(i) \in \mathbb{Z}^m \cdot \sigma^*(S)$ be $|\vec{b}_{2m+1}(i)|$-many consecutive elements in the ω^* block that lies between the tuples $\vec{u}_{j-1}$ and $\vec{u}_j$. Finally let $\vec{a}_{2m+1} = \vec{a}_{2m+1}(1) < \vec{a}_{2m+1}(2) < \ldots < \vec{a}_{2m+1}(r)$.

We now claim that $\big(\mathbb{Z}^m \cdot \sigma^*(R), \vec{w}, \vec{b}_{2m+1}\big) \leq_1 \big(\mathbb{Z}^m \cdot \sigma^*(S), \vec{u}, \vec{a}_{2m+1}\big)$. We can justify this last claim as follows. Let

$$\vec{c} = \vec{u} \cup \vec{a}_{2m+1} = g_{\vec{x}}(\vec{a}) \cup \bigcup_{i=1}^{2m+1} \vec{a}_i = c_1 < c_2 < \ldots < c_l$$

and

$$\vec{d} = \vec{w} \cup \vec{b}_{2m+1} = g_{\vec{x}}(\vec{b}) \cup \bigcup_{i=1}^{2m+1} \vec{b}_i = d_1 < d_2 < \ldots < d_l$$

Then for any pair (c_i, c_{i+1}) and corresponding (d_i, d_{i+1}), either the number of elements between d_i and d_{i+1} in $\mathcal{B}$ equals the number of elements between c_i and c_{i+1} in $\mathcal{A}$, or there are infinitely many elements between d_i and d_{i+1}. In linear orderings, this is sufficient to show that $(\mathcal{B}, \vec{d}) \leq_1 (\mathcal{A}, \vec{c})$ as desired.

Therefore we have shown that

$$(\forall \vec{b}_1)(\exists \vec{a}_1)(\forall \vec{a}_2)(\exists \vec{b}_2) \cdots (\forall \vec{a}_{2m})(\exists \vec{b}_{2m})(\forall \vec{b}_{2m+1})(\exists \vec{a}_{2m+1})$$
$$\Big[\big(\mathcal{B}, g_{\vec{x}}(\vec{b}), \vec{b}_1, \vec{b}_2 \ldots, \vec{b}_{2m}, \vec{b}_{2m+1}\big) \leq_1 \big(\mathcal{A}, g_{\vec{x}}(\vec{a}), \vec{a}_1, \vec{a}_2 \ldots, \vec{a}_{2m}, \vec{a}_{2m+1}\big)\Big]$$

and hence $\big(\mathbb{Z}^m \cdot \sigma^*(S), g_{\vec{x}}(\vec{a})\big) \leq_{2m+2} \big(\mathbb{Z}^m \cdot \sigma^*(R), g_{\vec{x}}(\vec{b})\big)$. □

Observe that this Theorem is still true if the original tuple $\vec{a}$ decomposes into finitely many pieces all lying in distinct $\mathbb{Z}^{m-1} \cdot \alpha_i$ blocks (for various α_i's) in $\mathbb{Z}^{m-1} \cdot \sigma^*(S)$ and similarly for $\vec{b}$. Keeping track of the notation is a bit harder, but the proof is identical. Now we will finish the result by proving Lemma 6.2.2.

Proof. (of Lemma 6.2.2) Fix $m \geq 1$ and assume that we have $\vec{c} \in \mathcal{A} \cong \mathbb{Z}^m \cdot \sigma^*(S)$ and $\vec{d} \in \mathcal{B} \cong \mathbb{Z}^m \cdot \sigma^*(R)$ satisfying property $(*)_k$ for some $0 \leq k < m-1$. Fix $\vec{b}_{2k+3} \in \mathbb{Z}^m \cdot \sigma^*(R)$. If some part of $\vec{b}_{2k+3}$ appears in the same $\mathbb{Z}^{m-k}$ blocks as current $\vec{d}_i$'s then the choice of the corresponding part of $\vec{a}_{2k+3}$ is clear based on property $(*)_k$. So let's assume that $\vec{b}_{2k+3}$ appears in entirely "new" $\mathbb{Z}^{m-k}$ blocks.

Let $\vec{b}_{2k+3}(i)$ be the portion of $\vec{b}_{2k+3}$ that lies in to the right of $\vec{d}_{i-1}$ and to the left of $\vec{d}_i$, if such a tuple exists.

Remark 6.2.4. *As $\vec{b}_{2k+3}$ has no members in the $\mathbb{Z}^{m-k}$ block containing $\vec{d}_{i-1}$ by assumption, there are infinitely many (entire) copies of $\mathbb{Z}^{m-k-1}$ strictly between the $\mathbb{Z}^{m-k}$ block of $\vec{d}_{i-1}$ and the leftmost member of the tuple $\vec{b}_{2k+3}(i)$. In the case of $i = 1$, this result is obvious as every copy of $\mathbb{Z}^{m-k-1}$ to the left of the tuple $\vec{b}_{2k+3}(1)$ is "free".*

Decompose $\vec{b}_{2k+3}(i)$ as $\vec{b}_{2k+3}(i) := \vec{r}_1 < \vec{r}_2 < \ldots < \vec{r}_s$ where the tuples $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_s$ lie in distinct $\mathbb{Z}^{m-k-1}$ blocks. Then we have the following configuration in $\mathbb{Z}^m \cdot \sigma^*(R)$:

$$\begin{array}{ccccccc} \overbrace{[\cdots+\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots]}^{\mathbb{Z}^{m-k}} & < \mathbb{Z}^{m-k-1} & <\cdots< & \mathbb{Z}^{m-k-1} & < & \overbrace{[\cdots+\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots]}^{\mathbb{Z}^{m-k}} \\ \uparrow & \uparrow & & \uparrow & & \uparrow \\ \vec{d}_{i-1} & \vec{r}_1 & \cdots & \vec{r}_s & & \vec{d}_i \end{array}$$

We will choose the corresponding tuple $\vec{a}_{2k+3}(i)$ in the $\mathbb{Z}^{m-k-1} \cdot \omega^*$ block that appears between the tuples $\vec{c}_{i-1}$ and $\vec{c}_i$. Let $\vec{a}_{2k+3}(i) = \vec{q}_1 < \ldots < \vec{q}_s$ where the tuples $\vec{q}_1, \ldots, \vec{q}_s$ are chosen in distinct, consecutive $\mathbb{Z}^{m-k-1}$ blocks so that, as suborderings, we have

$$(*) \quad (\mathbb{Z}^{m-k-1}, \vec{r}_j) \cong (\mathbb{Z}^{m-k-1}, \vec{q}_j)$$

for $j = 1, \ldots, s$. The configuration in $\mathbb{Z}^m \cdot \sigma^*(S)$ is as follows:

$$\begin{array}{ccccc} \overbrace{[\cdots+\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots]}^{\mathbb{Z}^{m-k}} & < & \overbrace{[\cdots+\underset{\uparrow\atop \vec{q}_1}{\mathbb{Z}^{m-k-1}}+\cdots+\underset{\uparrow\atop \vec{q}_s}{\mathbb{Z}^{m-k-1}}+\underset{\uparrow\atop \vec{c}_i}{\underbrace{\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\ldots}}]}^{\mathbb{Z}^{m-k}} \\ \uparrow & & \\ \vec{c}_{i-1} & & \end{array}$$

Putting all the parts together we get $\vec{a}_{2k+3} = \vec{a}_{2k+3}(1) < \vec{a}_{2k+3}(2) < \ldots < \vec{a}_{2k+3}(l)$.

Let $\vec{a}$ be the entire tuple from $\mathcal{A}$ that has been chosen so far and similarly for $\vec{b}$ in $\mathcal{B}$. Observe that, based on our actions in the previous step, we may no longer have a nice correspondence between tuples in distinct $\mathbb{Z}^{m-k}$ blocks in $\mathcal{A}$ and $\mathcal{B}$; however, we do have an adequate matching if we break our tuples into distinct $\mathbb{Z}^{m-k-1}$ blocks. More precisely, if a portion, say $\vec{a}(0)$, of $\vec{a}$ lies in a distinct $\mathbb{Z}^{m-k-1}$ block in $\mathcal{A}$ then there is a corresponding portion, say $\vec{b}(0)$, of $\vec{b}$ such that $(\mathbb{Z}^{m-k-1}, \vec{a}(0)) \cong (\mathbb{Z}^{m-k-1}, \vec{b}(0))$. This makes the second back-and-forth step straightforward.

Fix $\vec{a}_{2k+4} \in \mathbb{Z}^m \cdot \sigma^*(S)$. Let $\vec{a}_{2k+4}(i)$ be the portion of $\vec{a}_{2k+4}$ that lies in between the tuples $\vec{c}_{i-1}$ and and $\vec{c}_i$. Our choice of corresponding $\vec{b}_{2k+4}(i)$ will depend on the positioning of $\vec{a}_{2k+4}(i)$ relative to the tuples $\vec{c}_i$ and $\vec{q}_1, \ldots, \vec{q}_s$. For the time being we will just refer to $\vec{a}_{2k+4}(i)$ as $\vec{a}$. There are three cases to consider:

(a) $\vec{a}$ lies to the right of the $\mathbb{Z}^{m-k-1}$-copy containing $\vec{q}_s$. In our picture, we have:

$$\begin{array}{ccccccc}
\overbrace{[\cdots+\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots]}^{\mathbb{Z}^{m-k}} & < & [\cdots+\mathbb{Z}^{m-k-1} & +\cdots+ & \mathbb{Z}^{m-k-1} & + & \underbrace{\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots}]\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
\vec{c}_{i-1} & & \vec{q}_1 & \cdots & \vec{q}_s & & \boldsymbol{\vec{a}}\ \ \vec{c}_i
\end{array}$$

Then we can choose $\vec{b}$ in the obvious way in $\mathbb{Z}^m \cdot \sigma^*(R)$:

$$\begin{array}{ccccccc}
\overbrace{[\cdots+\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots]}^{\mathbb{Z}^{m-k}} & < & \mathbb{Z}^{m-k-1} & <\cdots< & \mathbb{Z}^{m-k-1} & < & \overbrace{[\cdots+\underbrace{\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots}]}^{\mathbb{Z}^{m-k}}\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
\vec{d}_{i-1} & & \vec{r}_1 & \cdots & \vec{r}_s & & \boldsymbol{\vec{b}}\ \ \vec{d}_i
\end{array}$$

(b) $\vec{a}$ lies in the same $\mathbb{Z}^{m-k-1}$ copy as some $\vec{q}_j$. Here is the picture for $j = 1$, the others are similar:

$$\begin{array}{ccccccc}
\overbrace{[\cdots+\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots]}^{\mathbb{Z}^{m-k}} & < & [\cdots+\mathbb{Z}^{m-k-1} & +\ldots+ & \mathbb{Z}^{m-k-1} & + & \underbrace{\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\ldots}]\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
\vec{c}_{i-1} & & \vec{q}_1\ \ \boldsymbol{\vec{a}} & \cdots & \vec{q}_s & & \vec{c}_i
\end{array}$$

Then we can choose $\vec{b}$ in the obvious way in the $\mathbb{Z}^{m-k-1}$ block of $\vec{r}_j$, in this example, $\vec{r}_1$:

$$\begin{array}{ccccccc}
\overbrace{[\cdots+\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots]}^{\mathbb{Z}^{m-k}} & < & \mathbb{Z}^{m-k-1} & <\cdots< & \mathbb{Z}^{m-k-1} & < & \overbrace{[\cdots+\underbrace{\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots}]}^{\mathbb{Z}^{m-k}}\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
\vec{d}_{i-1} & & \vec{r}_1\ \ \boldsymbol{\vec{b}} & \cdots & \vec{r}_s & & \vec{d}_i
\end{array}$$

(c) $\vec{a}$ lies to the left of the copy of $\mathbb{Z}^{m-k-1}$ block containing $\vec{q}_1$. Here is the corresponding picture (note that $\vec{a}$ could still lie in the same $\mathbb{Z}^{m-k}$ block as $\vec{c}_i$):

$$\overbrace{[\cdots+\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\cdots]}^{\mathbb{Z}^{m-k}} \quad < \quad \overbrace{[\cdots+\mathbb{Z}^{m-k-1}+\ldots+\mathbb{Z}^{m-k-1}+\underbrace{\mathbb{Z}^{m-k-1}+\mathbb{Z}^{m-k-1}+\ldots}]}^{\mathbb{Z}^{m-k}}$$

$$\begin{array}{cccccc} \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow \\ \vec{c}_{i-1} & \boldsymbol{\vec{a}} & \vec{q}_1 & \cdots & \vec{q}_s & \vec{c}_i \end{array}$$

Then we can choose $\vec{b}$ in the infinitely many (entire) copies of $\mathbb{Z}^{m-k-1}$ strictly between the $\mathbb{Z}^{m-k}$ block of $\vec{d}_{i-1}$ and the tuple $\vec{r}_1$. These copies exist by Remark 6.2.4. Let

$$\vec{a} = \vec{a}(1) < \vec{a}(2) < \ldots < \vec{a}(p)$$

where each $\vec{a}(j)$ lies together in a $\mathbb{Z}^{m-k-1}$ block. We pick $\vec{b} = \vec{b}(1) < \vec{b}(2) < \ldots < \vec{b}(p)$ where each $\vec{b}(j)$ is in a distinct free copy of $\mathbb{Z}^{m-k-1}$ and is such that

$$\big(\mathbb{Z}^{m-k-1}, \vec{a}(j)\big) \cong \big(\mathbb{Z}^{m-k-1}, \vec{b}(j)\big).$$

In any of these three cases, we chose an appropriate tuple $\vec{b} = \vec{b}_{2k+4}(i)$. Putting all the parts of $\vec{b}_{2k+4}$ together we get $\vec{b}_{2k+4} = \vec{a}_{2k+4}(1) < \ldots < \vec{b}_{2k+4}(l)$.

Now let $\tilde{a} = \vec{a}_1 \cup \vec{a}_2 \cup \ldots \cup \vec{a}_{2k+2} \cup \vec{a}_{2k+3} \cup \vec{a}_{2k+4}$ and $\tilde{b} = \vec{b}_1 \cup \vec{b}_2 \cup \ldots \cup \vec{b}_{2k+2} \cup \vec{b}_{2k+3} \cup \vec{b}_{2k+4}$, the current tuples in $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $\tilde{c} = \tilde{c}_1 < \ldots < \tilde{c}_{\tilde{l}}$ and $\tilde{d} = \tilde{d}_1 < \ldots < \tilde{d}_{\tilde{l}}$ where each $\tilde{c}_i$ and $\tilde{d}_i$ lie in a distinct $\mathbb{Z}^{m-k-1}$ block in their respective orderings. Observe that, by construction, we have $(\mathbb{Z}^{m-k-1}, \tilde{c}_i) \cong (\mathbb{Z}^{m-k-1}, \tilde{d}_i)$ for all $i = 1, \ldots \tilde{l}$. This is property $(*)_{k+1}$. Note that, as the tuples $\tilde{d}_{i-1}$ and $\tilde{d}_i$ are finite and lie in distinct $\mathbb{Z}^{m-k-1}$ blocks, we have a subordering of the form $\mathbb{Z}^{m-k-2} \cdot \omega^*$ in between the rightmost member of $\vec{d}_{i-1}$ and the leftmost member of $\vec{d}_i$. Similarly for the $\vec{c}_i$'s.

This completes the proof of the Lemma. □

Corollary 6.2.5. *For each $m \geq 0$, there are countably many $(2m+2)$-back-and-forth types of the form $(\mathcal{L}, \vec{a})$ where $\mathcal{L} \cong \mathbb{Z}^m \cdot \sigma^*(S)$ for some $S \subseteq \omega$.*

Proof. Let $A := \{\omega\} \cup \{n+1\}_{n\in\omega}$. For $m = 0$ we need to show that there are countably many pairs of the form $(\sigma^*(S), \vec{a})$ up to $\equiv_2$-equivalence. Consider the following countable set of tuples:

$$\{(k, \alpha_1, \alpha_2, \ldots, \alpha_k) : k \in \omega, \alpha_i \in A\}.$$

Given any ordering $\sigma^*(S)$ with $\vec{a} \in \sigma^*(S)$, we identify $\vec{a}$ with tuple $(k, \alpha_1, \alpha_2, \ldots, \alpha_k)$ if $\vec{a}$ decomposes into k parts, $\vec{a}(1) < \vec{a}(2) < \ldots < \vec{a}(k)$, where each tuple $\vec{a}(i)$ lies in a distinct α_i block in $\sigma^*(S)$. If $\vec{a} \in \sigma^*(S)$ and $\vec{b} \in \sigma^*(R)$ are both identified with the same tuple $(k, \alpha_1, \alpha_2, \ldots, \alpha_k)$ and, as suborderings, we have $(\alpha_i, \vec{a}(i)) \cong (\alpha_i, \vec{b}(i))$ then we have

$$(\sigma^*(S), \vec{a}) \equiv_2 (\sigma^*(R), \vec{b}).$$

This is not hard to see. As there are countably many tuples of the form $(k, \alpha_1, \alpha_2, \ldots, \alpha_k)$, and for each α_i there are countably many finite tuples $\vec{a} \in \alpha_i$, we have that there are at most countably many $\equiv_2$-classes as desired.

For $m > 0$ we can use Theorem 6.2.1. Again, for each $\vec{a}$ in some $\mathbb{Z}^{m-1} \cdot \sigma^*(S)$, we identify $\vec{a}$ with tuple $(k, \alpha_1, \alpha_2, \ldots, \alpha_k)$ if $\vec{a}$ decomposes into k parts, $\vec{a}(1) < \vec{a}(2) < \ldots \vec{a}(k)$, where each tuple $\vec{a}(i)$ lies in a distinct $\mathbb{Z}^{m-1} \cdot \alpha_i$ block in $\mathbb{Z}^{m-1} \cdot \sigma^*(S)$. If $\vec{a} \in \mathbb{Z}^{m-1}\sigma^*(S)$ and $\vec{b} \in \mathbb{Z}^{m-1}\sigma^*(R)$ are both identified the the same tuple $(k, \alpha_1, \alpha_2, \ldots, \alpha_k)$ and, as suborderings, we have $\big(\mathbb{Z}^{m-1} \cdot \alpha_i, \vec{a}(i)\big) \cong \big(\mathbb{Z}^{m-1} \cdot \alpha_i, \vec{b}(i)\big)$ for each i, then by Theorem 6.2.1, for any $\vec{x} \in Y_{|\vec{a}|}$, we have

$$\big(\mathbb{Z}^m \cdot \sigma^*(S), g_{\vec{x}}(\vec{a})\big) \leq_{2m+2} \big(\mathbb{Z}^m \cdot \sigma^*(R), g_{\vec{x}}(\vec{b})\big).$$

By symmetry, we really have $\equiv_{2m+2}$. Recall that any $\vec{c} \in \mathbb{Z}^m \cdot \sigma^*(S)$, can be written in the form $\vec{c} = g_{\vec{x}}(\vec{a})$ for an appropriate choice of $\vec{a} \in \mathbb{Z}^{m-1} \cdot \sigma^*(S)$ and $\vec{x} \in Y_{|\vec{x}|}$. As there are countably many tuples of the form $(k, \alpha_1, \alpha_2, \ldots, \alpha_k)$, each $\mathbb{Z}^m \cdot \alpha_i$ countable, and Y_k for each k is countable, we have that there are countably many pairs $(\mathbb{Z}^m \cdot \sigma^*(S), \vec{c})$ up to $\equiv_{2m+2}$ equivalence. $\square$

Finally, we have our desired result:

Corollary 6.2.6. *For each $m \geq 0$, the classes of structures*

$$\mathbb{K}_{2m+2} = \{\mathbb{Z}^m \cdot \sigma^*(X \oplus \overline{X}) : X \subseteq \omega\} \text{ and } \mathbb{K}_{2m+3} = \{\mathbb{Z}^m \cdot \sigma^*(X) : X \text{ is } (2m+3)\text{-generic}\}$$

both have back-and-forth ordinal equal to $2m + 3$.

Proof. It follows from Corollary 6.2.5 that the back-and-forth ordinal of each of the two classes is at least $2m + 3$. As there are uncountably many $(2m + 3)$-generic sets, and any two orderings $\mathbb{Z}^m \cdot \sigma^*(R)$ and $\mathbb{Z}^m \cdot \sigma^*(S)$ with $R \neq S$ can be distinguished by a Σ_{2m+3} sentence, the back-and-forth ordinal of each is exactly $2m + 3$. $\square$

6.2.3 Classes for $\omega+1$, $\omega+2$

In this section we will compute the back-and-forth ordinals of the following classes:

$$\mathbb{K}_{\omega+1} := \{A_{\omega+1}(S) : S \subseteq \omega\}$$

and

$$\mathbb{K}_{\omega+2} := \{A_{\omega+2}(S) : S \text{ is } (\omega+2)\text{-generic}\}.$$

From Section 6.1, we know that the Turing ordinals of $\mathbb{K}_{\omega+1}$ and $\mathbb{K}_{\omega+2}$ are $\omega+1$ and $\omega+2$ respectively. By Lemma 4.2.5, we know that the back-and-forth ordinal of $\mathbb{K}_{\omega+1}$ is at most $\omega+1$ and the back-and-forth ordinal of $\mathbb{K}_{\omega+2}$ is at most $\omega+2$. We will show that the back-and-forth ordinal is exactly $\omega+1$ in both cases.

Recall that $\mathcal{A}_{\omega+1}(S) = \nu^\omega(S \oplus \overline{S})$ and $\mathcal{A}_{\omega+2}(S) = \nu^\omega(S)$ where $\nu^\omega(X)$, for some set X, is a sum of densely many copies of the orderings

$$r+1+\sum_{1\leq i\leq M} \mathbb{Z}^i \qquad \text{for each } r\in\omega \text{ and } M\geq 1$$

and densely many copies of the orderings

$$r+1+\sum_{1\leq i<\omega} \mathbb{Z}^i \qquad \text{for each } r\in X.$$

To simplify the notation, we will denote the orderings

$$r+1+\sum_{1\leq i\leq M} \mathbb{Z}^i = r+1+\mathbb{Z}+\mathbb{Z}^2+\ldots+\mathbb{Z}^M, \text{ and}$$

$$r+1+\sum_{1\leq i<\omega} \mathbb{Z}^i = r+1+\mathbb{Z}+\mathbb{Z}^2+\ldots+\mathbb{Z}^i+\ldots$$

by $\sum \mathbb{Z}_r^M$ and $\sum \mathbb{Z}_r^\infty$ respectively.

First we will show, that there is a $\Sigma^c_{\omega+1}$ formula that distinguishes two orderings $\nu^\omega(R)$ and $\nu^\omega(S)$ for $R \neq S$.

Describing $\sum \mathbb{Z}_r^M$ blocks

We would like a formula $\chi^{r,M}(x)$ such that, for any linear ordering $\mathcal{A}$ and any $a\in\mathcal{A}$, $\mathcal{A}\models\chi_{r,M}(a)$ if and only if a lies in a copy of $\sum\mathbb{Z}_r^M$ in $\mathcal{A}$. We will first define preliminary formulas:

1. To define an $(r+1)$ block $\vec{x}$ in an orderings let

$$B_r(x_0, \ldots, x_r) := \bigwedge_{i=1}^{r} S(x_i, x_{i+1}) \wedge (\forall y)[\neg S(y, x_0) \wedge \neg S(x_r, y)].$$

2. To define the initial $(r+1)$-segment of a $\sum \mathbb{Z}_r^M$ block we need the following family of formulas, in addition to $B_r(\vec{x})$:

   - $\theta_{\mathbb{Z}}(y_1, \vec{x} = (x_0, \ldots, x_r)) := \varphi_{\mathbb{Z}}(y_1) \wedge (\forall z)[\ x_r < z \leq y_1 \quad \Rightarrow \quad \varphi_{\mathbb{Z}}(z, y_1)\ ]$.

   - $\theta_{\mathbb{Z}^i}(y_{i-1}, y_i) := \varphi_{\mathbb{Z}^i}(y_i) \wedge (\forall z)[\ y_{i-1} \leq z \leq y_i \quad \Rightarrow \quad \varphi_{\mathbb{Z}^{i-1}}(y_{i-1}, z) \vee \varphi_{\mathbb{Z}^i}(z, y_i)\ ]$ for $i = 2, \ldots, M$.

   Then the formula

$$\theta^{r,M}(\vec{x}, \vec{y}) := B_r(\vec{x}) \wedge \Big[(y_1 > x_r) \wedge \theta_{\mathbb{Z}}(\vec{x}, y_1)\Big] \wedge \bigwedge_{i=1}^{M-1} \Big[(y_{i+1} > y_i) \wedge \theta_{\mathbb{Z}^{i+1}}(y_i, y_{i+1})\Big]$$

   defines a $\sum \mathbb{Z}_r^M$ block with initial segment $\vec{x}$ and y_i in $\mathbb{Z}^i$ for each $1 \leq i \leq M$.

   Finally, let

$$\chi_k^{r,M}(x) := (\exists x_0, \ldots, x_k, \ldots, x_r)(\exists \vec{y})\left[x_k = x \wedge \theta^{r,M}(\vec{x}, \vec{y})\right].$$

   Then we have $\mathcal{A} \models \chi_k^{r,M}(a)$ if and only if a is the $(k+1)^{st}$ member of the $r+1$ block in $\sum \mathbb{Z}_r^M$.

3. To define membership in the $\mathbb{Z}^i$ portion of a $\sum \mathbb{Z}_r^M$ block for $1 \leq i \leq M$, we need the following formula:

$$\chi_{\mathbb{Z}^i}^{r,M}(z) := (\exists \vec{x})(\exists y_1, \ldots, y_i, \ldots, y_M)[y_i = z \wedge \theta^{r,M}(\vec{x}, \vec{y})].$$

Now we are ready to define the formula of interest. Let

$$\chi^{r,M}(x) := \bigvee_{k=0}^{r} \chi_k^{r,M}(x) \ \vee \ \bigvee_{i=1}^{M} \chi_{\mathbb{Z}^i}^{r,M}(x),$$

Then for any linear ordering $\mathcal{A}$ and any $a \in \mathcal{A}$, we have $\mathcal{A} \models \chi^{r,M}(a)$ if and only if a lies in a copy of $\sum \mathbb{Z}_r^M$.

Describing $\sum \mathbb{Z}_r^\infty$ blocks

Fix $r \in \omega$ and consider the formula:

$$\chi^r(x) := \bigvee_{M \geq 1} \left[\bigwedge_{k \geq M} \chi^{r,k}(x) \right].$$

Then $A \models \chi^r(a)$ if and only if a lies in a copy of $\sum \mathbb{Z}_r^\infty$.

Here is a summary of the formulas from this section, along with their complexities:

Formula	Meaning	Complexity
$\chi_k^{r,M}(x)$	x is the $(k+1)^{st}$ member of the $(r+1)$ block in a copy of $\sum \mathbb{Z}_r^M$	Σ_{2M+2}^c
$\chi_{\mathbb{Z}^i}^{r,M}(x)$	x is in the $\mathbb{Z}^i$ block of some copy of $\sum \mathbb{Z}_r^M$	Σ_{2M+2}^c
$\chi^{r,M}(x)$	x lies in a copy of $\sum \mathbb{Z}_r^M$	Σ_{2M+2}^c
$\chi^r(x)$	x lies in a copy of $\sum \mathbb{Z}_r^\infty$	$\Sigma_{\omega+1}^c$

Now we can define our formulas separating orderings of the form $\nu^\omega(X)$.

Theorem 6.2.7. *The back-and-forth ordinals of $\mathbb{K}_{\omega+1}$ and $\mathbb{K}_{\omega+2}$ are at most $\omega+1$.*

Proof. Fix $R, S \subseteq \omega$ such that $R \neq S$. Without loss of generality assume there is some $r_0 \in \omega$ such that $r_0 \in R$ and $r_0 \notin S$. Then, by definition, the structure $\mathcal{A}_{\omega+1}(R) = \nu^\omega(R \oplus \overline{R})$ will have densely many copies of $\sum \mathbb{Z}_{2r_0}^\infty$ while the structure $\mathcal{A}_{\omega+1}(S) = \nu^\omega(S \oplus \overline{S})$ will have none. Clearly these linear orderings are not isomorphic, but moreover, they have different $\Sigma_{\omega+1}^c$ types. This is because

$$\mathcal{A}_{\omega+1}(R) \models (\exists x)\chi^{2r_0}(x) \text{ and } \mathcal{A}_{\omega+1}(S) \not\models (\exists x)\chi^{2r_0}(x)$$

where $(\exists x)\chi^{2r_0}(x)$ is a $\Sigma_{\omega+1}^c$ sentence. It follows that

$$|\mathbf{bf}_{\omega+1}(\mathbb{K}_{\omega+1})| \geq |\mathcal{P}(\omega)| = 2^{\aleph_0}$$

Similarly, for any two $(\omega+2)$-generic sets R and S such that $R \neq S$, the corresponding structures $A_{\omega+2}(R) = \nu^\omega(R)$ and $A_{\omega+2}(S) = \nu^\omega(S)$ have different $\Sigma_{\omega+1}^c$ types. As there are uncountably many distinct generic sets, we have that

$$|\mathbf{bf}_{\omega+1}(\mathbb{K}_{\omega+2})| \geq 2^{\aleph_0}$$

as well. Therefore the back-and-forth ordinals of $\mathbb{K}_{\omega+1}$ and $\mathbb{K}_{\omega+2}$ are at most $\omega+1$. $\square$

Now it remains to examine the Σ_ω types of these two classes. We will prove that there is exactly one model of $\mathbb{K}_{\omega+1}$ and one model of $\mathbb{K}_{\omega+2}$ up to $\equiv_\omega$-equivalence. It will follow that the back-and-forth ordinal of each class is at least $\omega+1$. Roughly speaking, the reason the back-and-forth ordinal will be so high is the difficulty one has in differentiating an ordering that has densely many copies of $\sum \mathbb{Z}_r^\infty$ from an ordering that only has densely many copies of $\sum \mathbb{Z}_r^M$ for arbitrarily large M. More precisely, fix $M \geq 1$ and $r \geq 0$ and answer the following question: For which $n \geq 0$ do we have $\sum \mathbb{Z}_r^\infty \equiv_n \sum \mathbb{Z}_r^M$?

Theorem 6.2.8. *For all $m \geq 1$, we have $\sum \mathbb{Z}_r^\infty \equiv_m \sum \mathbb{Z}_r^{m+1}$.*

To get to this result, we need a few general results about linear orderings involving the powers of $\mathbb{Z}$. Recall the notation from Section 6.2.2 that allows us to pass from tuples in $\mathcal{L}$ to tuples in $\mathbb{Z} \cdot \mathcal{L}$.

Let Y_k be a set consisting of k-tuples of the form

$$\vec{x} = (\langle m_1, z_1^1, z_2^1, \dots z_{m_1}^1\rangle, \langle m_2, z_1^2, z_2^2, \dots z_{m_2}^2\rangle, \dots, \langle m_k, z_1^k, z_2^k, \dots z_{m_k}^k\rangle)$$

where $m_1, m_2, \dots, m_k$ are positive integers, and for each $1 \leq i \leq k$, we have $z_1^i < z_2^i < \dots < z_{m_i}^i$, all picking out integers.

Given a linear order $\mathcal{L}$, a k-tuple $\vec{a} = (a_1, \dots, a_k)$ from $\mathcal{L}$, and a k-tuple $\vec{x}$ from Y_k, we define a corresponding $(m_1+m_2+\dots+m_k)$-tuple, denoted by $g_{\vec{x}}(\vec{a})$, in $\varphi(\mathcal{L})$ as follows: Let $g_{\vec{x}}(\vec{a}) := (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k)$ where the tuple $\tilde{a}_i$ is of length m_i, lies in the $\mathbb{Z}$ block corresponding to the element a_i, and there is an embedding of $\mathbb{Z}$ into $\mathbb{Z}\cdot\mathcal{L}$ that sends z_j^i to the j^{th} member of the tuple $\tilde{a}_i$.

Consider the following lemma:

Lemma 6.2.9. *For all $n > 0$ and for all (infinite) linear orderings $\mathcal{L}_1$ and $\mathcal{L}_2$, if $\vec{a} \in \mathcal{L}_1$ and $\vec{b} \in \mathcal{L}_2$ are both of length k then, for any $\vec{x} \in Y_k$, we have*

$$\Big(\mathcal{L}_1, \vec{a}\Big) \leq_{n-1} \Big(\mathcal{L}_2, \vec{b}\Big) \Longrightarrow \Big(\mathbb{Z}\cdot\mathcal{L}_1, g_{\vec{x}}(\vec{a})\Big) \leq_n \Big(\mathbb{Z}\cdot\mathcal{L}_2, g_{\vec{x}}(\vec{b})\Big).$$

Proof. This proof is very similar to the proof of Lemma 5.2.1.

Base case: $n = 1$.

Suppose that $(\mathcal{L}_1, \vec{a}) \leq_0 (\mathcal{L}_2, \vec{b})$. Then $\vec{a}$ and $\vec{b}$ must be ordered in the same way in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Fix $\vec{c} \in \mathbb{Z}\cdot\mathcal{L}_2$. By how $g_{\vec{x}}(\cdot)$ was defined, we have $g_{\vec{x}}(\vec{a}) = (\tilde{a}_1, \dots, \tilde{a}_k)$ and $g_{\vec{x}}(\vec{b}) = (\tilde{b}_1, \dots, \tilde{b}_k)$ ordered in the same way in $\mathbb{Z}\cdot\mathcal{L}_1$ and $\mathbb{Z}\cdot\mathcal{L}_2$ respectively. Moreover, we have

$$\Big(\mathbb{Z}, \tilde{a}_i\Big) \cong \Big(\mathbb{Z}, \vec{b}_i\Big)$$

when we restrict to the suborderings $\mathbb{Z}$ containing $\tilde{a}_i$ and $\tilde{b}_i$ for each i. It follows that, for any two elements $c_1, c_2 \in g_{\vec{x}}(\vec{a})$ appearing consecutively in $\mathbb{Z} \cdot \mathcal{L}_1$ and the corresponding elements $d_1, d_2 \in g_{\vec{x}}(\vec{b})$ in $\mathbb{Z} \cdot \mathcal{L}_2$, there are exactly the same number of elements between the pair c_1 and c_2 as the pair d_1 and d_2. This is sufficient to show that

$$(\mathbb{Z} \cdot \mathcal{L}_1, g_{\vec{x}}(\vec{a})) \leq_1 (\mathbb{Z} \cdot \mathcal{L}_2, g_{\vec{x}}(\vec{b}))$$

as desired.

The inductive step proceeds exactly as in the proof of Lemma 5.2.1 with the ordering $\mathbb{Z} \cdot \mathcal{L}_i$ in the place of the ordering $\varphi(\mathcal{L}_i)$, the set Y_k in the place of the set X_k, and the maps $g_{\vec{x}}(\cdot)$ in the place of the maps $f_{\vec{x}}(\cdot)$. We will omit the proof here. $\square$

Now we are ready to prove the following back-and-forth equivalence.

Proposition 6.2.10. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two countable linear orderings satisfying $\mathcal{L}_1 \equiv_n \mathcal{L}_2$, then we have $\mathbb{Z} \cdot \mathcal{L}_1 \equiv_{n+1} \mathbb{Z} \cdot \mathcal{L}_2$.*

Proof. Fix $\vec{b} \in \mathbb{Z} \cdot \mathcal{L}_2$ and $l < n+1$. Decompose $\vec{b} = \vec{b_1} \cup \ldots \cup \vec{b_k}$ into the tuples $\vec{b_i}$ lying in distinct $\mathbb{Z}$ blocks. Then for some $\vec{x} \in Y_k$ and some $\vec{d} \in \mathcal{L}_2$ we have $\vec{b} = g_{\vec{x}}(\vec{d})$. As $\vec{d} \in \mathcal{L}_2$, $l-1 < n$, and $\mathcal{L}_1 \equiv_n \mathcal{L}_2$, there must be some tuple $\vec{c} \in \mathcal{L}_1$ such that

$$(\mathcal{L}_2, \vec{d}) \leq_{l-1} (\mathcal{L}_1, \vec{c}).$$

By Lemma 6.2.9, we have $(\mathbb{Z} \cdot \mathcal{L}_2, g_{\vec{x}}(\vec{d})) \leq_l (\mathbb{Z} \cdot \mathcal{L}_1, g_{\vec{x}}(\vec{c}))$ and hence, for $\vec{a} := g_{\vec{x}}(\vec{c}) \in \mathbb{Z} \cdot \mathcal{L}_1$, we have $(\mathbb{Z} \cdot \mathcal{L}_2, \vec{b}) \leq_l (\mathbb{Z} \cdot \mathcal{L}_1, \vec{a})$. Therefore $\mathbb{Z} \cdot \mathcal{L}_1 \equiv_{n+1} \mathbb{Z} \cdot \mathcal{L}_2$ as desired. $\square$

Corollary 6.2.11. *For all $m \geq 1$ we have $\mathbb{Z}^m \equiv_m \mathbb{Z}^{m+k}$ for all $k \geq 0$.*

Proof. We prove this corollary by induction on m. For $m = 1$, we have $\mathbb{Z} \equiv_1 \mathbb{Z}^{1+k}$ for all $k \geq 0$ as all countably infinite orderings are $\equiv_1$-equivalent. Now suppose that $\mathbb{Z}^m \equiv_m \mathbb{Z}^{m+k}$ for all $k \geq 0$. By Proposition 6.2.10, we have that

$$\mathbb{Z} \cdot \mathbb{Z}^m \equiv_{m+1} \mathbb{Z} \cdot \mathbb{Z}^{m+k}$$

for all $k \geq 0$ or, equivalently, $\mathbb{Z}^{m+1} \equiv_{m+1} \mathbb{Z}^{(m+1)+k}$ for all $k \geq 0$. This completes the proof. $\square$

Now we will turn our attention to the orderings $\sum \mathbb{Z}_r^\infty$ and $\sum \mathbb{Z}_r^M$. First, we note the following result.

Lemma 6.2.12 (Lemma 3.1 from [21]). *Let* $\mathcal{A} = \sum_{i \in I} \mathcal{A}_i$ *and* $\mathcal{B} = \sum_{i \in I} \mathcal{B}_i$ *where all the* $\mathcal{A}_i$*'s and* $\mathcal{B}_i$*'s are linear orderings. Let* $\vec{a} \in \mathcal{A}$ *and* $\vec{b} \in \mathcal{B}$*. Let* $\vec{a}_i$ *be the portion of* $\vec{a}$ *that lies in* $\mathcal{A}_i$ *and let* $\vec{b}_i$ *be the portion of* $\vec{b}$ *that lies in* $\mathcal{B}_i$*. If* $(\mathcal{A}_i, \vec{a}_i) \leq_n (\mathcal{B}_i, \vec{b}_i)$ *for all* $i \in I$*, then* $(\mathcal{A}, \vec{a}) \leq_n (\mathcal{B}, \vec{b})$*. The result also holds for* ω *in place of* n*.*

This can be easily proven by induction. Lemma 6.2.12 will allow us to consider the summands of our orderings separately. We now prove Theorem 6.2.8 restated here:

Theorem 6.2.13. *For all* $m \geq 1$ *we have* $\sum \mathbb{Z}_r^{\infty} \equiv_m \sum \mathbb{Z}_r^{m+1}$.

Proof. Consider the two orderings

$$\begin{array}{lcccccccccccccccc}
\sum \mathbb{Z}_r^{\infty} & = & (r+1) & + & \mathbb{Z} & + & \mathbb{Z}^2 & + & \dots & + & \mathbb{Z}^m & + & \mathbb{Z}^{m+1} & + & \mathbb{Z}^{m+2} & + & \dots \\
\sum \mathbb{Z}_r^{m+1} & = & (r+1) & + & \mathbb{Z} & + & \mathbb{Z}^2 & + & \dots & + & \mathbb{Z}^m & + & \mathbb{Z}^{m+1} & & & &
\end{array}$$

and decompose them as follows:

$$\begin{array}{lcccccccccccc}
\sum \mathbb{Z}_r^{\infty} & = & (r+1) & + & \mathbb{Z} & + & \mathbb{Z}^2 & + & \dots & + & \mathbb{Z}^m & + & \Big(\mathbb{Z}^{m+1} + \mathbb{Z}^{m+2} + \dots\Big) \\
\sum \mathbb{Z}_r^{m+1} & = & (r+1) & + & \mathbb{Z} & + & \mathbb{Z}^2 & + & \dots & + & \mathbb{Z}^m & + & \Big(\dots + \mathbb{Z}^m + \mathbb{Z}^m + \dots\Big)
\end{array}$$

By Lemma 6.2.12, it suffices to show that

$$\Big(\mathbb{Z}^{m+1} + \mathbb{Z}^{m+2} + \dots\Big) \equiv_m \Big(\dots + \mathbb{Z}^m + \mathbb{Z}^m + \dots\Big)$$

By Corollary 6.2.11, and Lemma 6.2.12, we have that

$$(\mathbb{Z}^{m+1} + \mathbb{Z}^{m+2} + \dots\Big) \equiv_m \Big(\mathbb{Z}^m + \mathbb{Z}^m + \dots\Big)$$

so, by transitivity of $\leq_m$, it remains to show that

$$\Big(\mathbb{Z}^m + \mathbb{Z}^m + \dots\Big) = \mathbb{Z}^m \cdot \omega \equiv_m \mathbb{Z}^m \cdot \mathbb{Z} = \Big(\dots + \mathbb{Z}^m + \mathbb{Z}^m + \dots\Big).$$

We will prove this by induction on m:

It is clear that $\mathbb{Z} \cdot \omega \equiv_1 \mathbb{Z} \cdot \mathbb{Z}$ as they are both infinite orderings. Now assume that $\mathbb{Z}^m \cdot \omega \equiv_m \mathbb{Z}^m \cdot \mathbb{Z}$. Then by Proposition 6.2.10, we have that

$$\mathbb{Z} \cdot (\mathbb{Z}^m \cdot \omega) \equiv_{m+1} \mathbb{Z} \cdot (\mathbb{Z}^m \cdot \mathbb{Z})$$

and hence, as multiplication is associative,

$$\mathbb{Z}^{m+1} \cdot \omega \equiv_{m+1} \mathbb{Z}^{m+1} \cdot \mathbb{Z}$$

as desired. $\square$

Finally, we return to our discussion of the classes $\mathbb{K}_{\omega+1}$ and $\mathbb{K}_{\omega+2}$, more precisely, to the orderings $\nu^\omega(S)$ for some set $S \subseteq \omega$. Suppose that we have two orderings $\mathcal{A} \cong \nu^\omega(S)$ and $\mathcal{B} \cong \nu^\omega(R)$. We would like to show that we must have $\mathcal{A} \equiv_\omega \mathcal{B}$, independent of the choice of sets S and R. To get an idea of what we would like to prove consider the following. Fix an element $a \in \mathcal{A}$ and $n \in \omega$. We would like to find an element $b \in \mathcal{B}$ such that $(\mathcal{A}, a) \leq_n (\mathcal{B}, b)$. We know that a lies in a block of the form $\sum \mathbb{Z}_r^M$ for some $M \in \{1, 2, \ldots, \infty\}$ and $r \in \omega$. We would like to choose $b \in \mathcal{B}$ as similar as possible, so ideally, we'd like to choose b to be an exact copy of a in some $\sum \mathbb{Z}_r^M$ block on the $\mathcal{B}$ side. This strategy will fail if $M = \infty$ and r is in the set R but not the set S. We would then like to amend this strategy and choose a block of the form $\sum \mathbb{Z}_r^N$ in $\mathcal{B}$ with $N < \infty$ but such that $\sum \mathbb{Z}_r^N \equiv_{n+1} \sum \mathbb{Z}_r^\infty$. Then we know we can find b in the ordering $\sum \mathbb{Z}_r^N$ such that $\left(\sum \mathbb{Z}_r^\infty, a\right) \leq_n \left(\sum \mathbb{Z}_r^N, b\right)$. It turns out that this will be a winning strategy when we put all the pieces together.

Definition 6.2.14. Fix $a \in \nu^\omega(S)$ and suppose that a lies in a block of the form $\sum \mathbb{Z}_r^M$ for some $M \in \{1, 2, \ldots, \infty\}$ and $r \in \omega$. As the $\sum \mathbb{Z}_r^M$ block is itself a linear ordering, there is a natural embedding from $\sum \mathbb{Z}_r^M$ into $\nu^\omega(S)$ that maps $\sum \mathbb{Z}_r^M$ onto the $\sum \mathbb{Z}_r^M$ block in $\nu^\omega(S)$ containing a. We will denote the inverse image of a under this embedding by a^*. Conversely, if we have an element a^* in an ordering $\sum \mathbb{Z}_r^M$, then once we have fixed a single $\sum \mathbb{Z}_r^M$ block in $\nu^\omega(S)$, we let a denote the image of a^* under the natural embedding from $\sum \mathbb{Z}_r^M$ into $\nu^\omega(S)$. We will use the same notation, $\vec{a}^*$, for tuples $\vec{a}$ that lie together in a $\sum \mathbb{Z}_r^M$ block.

The following Lemma will relate the properties of a^* in $\sum \mathbb{Z}_r^M$ to the properties of a in $\nu^\omega(S)$.

Lemma 6.2.15. *For all $n \geq 0$ and all orderings $\mathcal{A} \cong \nu^\omega(S)$ and $\mathcal{B} \cong \nu^\omega(R)$, if $\vec{a} = \vec{a}_1 \cup \vec{a}_2 \cup \ldots \cup \vec{a}_k \in \mathcal{A}$ and $\vec{b} = \vec{b}_1 \cup \vec{b}_2 \cup \ldots \cup \vec{b}_k \in \mathcal{B}$ satisfy*

- *(i)* $\vec{a}_i < \vec{a}_{i+1}$ *and* $\vec{b}_i < \vec{b}_{i+1}$ *for* $1 \leq i \leq k-1$
- *(ii)* $|\vec{a}_i| = |\vec{b}_i|$ *for* $i = 1, \ldots, k$, *and*
- *(iii) Each of the tuples $\vec{a}_1, \ldots, \vec{a}_k$ lie in a distinct $\sum \mathbb{Z}_{r_i}^{M_i}$ block in $\mathcal{A}$ and each of the tuples $\vec{b}_1, \ldots, \vec{b}_k$ lie in a distinct $\sum \mathbb{Z}_{s_i}^{N_i}$ block in $\mathcal{B}$ with $M_i, N_i \in \{1, 2, \ldots, \infty\}$ and $s_i = r_i$, and*
- *(iv)* $\left(\sum \mathbb{Z}_{r_i}^{N_i}, \vec{b}_i^*\right) \leq_n \left(\sum \mathbb{Z}_{r_i}^{M_i}, \vec{a}_i^*\right)$ *for* $1 \leq i \leq k$

then $(\mathcal{B}, \vec{b}) \leq_n (\mathcal{A}, \vec{a})$.

Proof. We will prove the lemma by induction on n for all orderings and tuples at once. Let $\vec{a}$ and $\vec{b}$ be as above.

For $n = 0$: Since $\left(\sum \mathbb{Z}_{r_i}^{N_i}, \vec{b}_i^*\right) \leq_0 \left(\sum \mathbb{Z}_{r_i}^{M_i}, \vec{a}_i^*\right)$, the tuples $\vec{a}_i^*$ and $\vec{b}_i^*$ are ordered the same way in $\sum \mathbb{Z}_{r_i}^{M_i}$ and $\sum \mathbb{Z}_{s_i}^{N_i}$ respectively. As $\vec{a}_1 < \vec{a}_2 < \ldots < \vec{a}_k$ and $\vec{b}_1 < \vec{b}_2 < \ldots < \vec{b}_k$, we have that $\vec{a} = \vec{a}_1 \cup \vec{a}_2 \cup \ldots \cup \vec{a}_k$ and $\vec{b} = \vec{b}_1 \cup \vec{b}_2 \cup \ldots \cup \vec{b}_k$ are ordered in the same way in $\mathcal{A}$ and $\mathcal{B}$ respectively and hence $(\mathcal{B}, \vec{b}) \leq_0 (\mathcal{A}, \vec{a})$.

Now assume the result holds for some $n > 0$ and suppose that we have

$$\left(\sum \mathbb{Z}_{r_i}^{N_i}, \vec{b}_i^*\right) \leq_{n+1} \left(\sum \mathbb{Z}_{r_i}^{M_i}, \vec{a}_i^*\right)$$

for $1 \leq i \leq k$. We wish to show that $(\mathcal{B}, \vec{b}) \leq_{n+1} (\mathcal{A}, \vec{a})$. Fix $\vec{c} \in \mathcal{A}$. We need to find $\vec{d} \in \mathcal{B}$ such that $(\mathcal{A}, \vec{a}, \vec{c}) \leq_n (\mathcal{B}, \vec{b}, \vec{d})$. For each i, let $\vec{c}_i$ be the portion of $\vec{c}$ that lies in the same $\sum \mathbb{Z}_{r_i}^{M_i}$ block as $\vec{a}_i$. Note that we could have $\vec{c}_i = \emptyset$. Consider the corresponding tuple $\vec{c}_i^*$ in the ordering $\sum \mathbb{Z}_{r_i}^{M_i}$. Since

$$\left(\sum \mathbb{Z}_{r_i}^{N_i}, \vec{b}_i^*\right) \leq_{n+1} \left(\sum \mathbb{Z}_{r_i}^{M_i}, \vec{a}_i^*\right)$$

by assumption, there exists a tuple $\vec{d}_i^* \in \sum \mathbb{Z}_{r_i}^{N_i}$ such that

$$\left(\sum \mathbb{Z}_{r_i}^{M_i}, \vec{a}_i^*, \vec{c}_i^*\right) \leq_n \left(\sum \mathbb{Z}_{r_i}^{N_i}, \vec{b}_i^*, \vec{d}_i^*\right).$$

Consider the copy of $\sum \mathbb{Z}_{r_i}^{N_i}$ in $\mathcal{B}$ containing the tuple $\vec{b}_i$. Let $\vec{d}_i$ be the image of $\vec{d}_i^*$ in this copy of $\sum \mathbb{Z}_{r_i}^{N_i}$. Now we have $\vec{b}_i \cup \vec{d}_i$ in the same $\sum \mathbb{Z}_{r_i}^{N_i}$ block in $\mathcal{B}$, and $\vec{a}_i \cup \vec{c}_i$ in the same $\sum \mathbb{Z}_{r_i}^{M_i}$ block in $\mathcal{A}$ and $(\sum \mathbb{Z}_{r_i}^{M_i}, \vec{a}_i^*, \vec{c}_i^*) \leq_n (\sum \mathbb{Z}_{r_i}^{N_i}, \vec{b}_i^*, \vec{d}_i^*)$.

Any part of $\vec{c}$ that does not lie in the same $\sum \mathbb{Z}_{r_i}^{M_i}$ block as one of the $\vec{a}_i$'s we deal with separately. Let $\vec{e}_j$ be the portion of $\vec{c}$ that lies together in some "new" $\sum \mathbb{Z}_{s_j}^{P_j}$ block in $\mathcal{A}$. Define $\vec{e}_j^*$ in the ordering $\sum \mathbb{Z}_{s_j}^{P_j}$ as usual. By Theorem 6.2.13, we can pick Q_j such that $\sum \mathbb{Z}_{s_j}^{P_j} \equiv_{n+1} \sum \mathbb{Z}_{s_j}^{Q_j}$. If P_j is finite, then we can let $Q_j = P_j$; if $P_j = \infty$ then we just need to pick Q_j large enough. By the theorem, $Q_j = n+2$ will suffice. Then there is some tuple $\vec{f}_j^* \in \sum \mathbb{Z}_{s_j}^{Q_j}$ such that $(\sum \mathbb{Z}_{s_j}^{P_j}, \vec{e}_j^*) \leq_n (\sum \mathbb{Z}_{s_j}^{Q_j}, \vec{f}_j^*)$. Now we need to select an appropriate copy of $\sum \mathbb{Z}_{s_j}^{Q_j}$ in $\mathcal{B}$ in which to choose our corresponding tuple $\vec{f}_j$. If $\vec{a}_i <_{\mathcal{A}} \vec{e}_j <_{\mathcal{A}} \vec{a}_{i+1}$ then we select our copy of $\sum \mathbb{Z}_{s_j}^{Q_j}$ to lie to the right of $\sum \mathbb{Z}_{r_i}^{N_i}$ and to the left of $\sum \mathbb{Z}_{r_{i+1}}^{N_{i+1}}$ in $\mathcal{B}$. (If $\vec{e}_j$ lies to the left or to the right of all the tuples $\vec{a}_i$, then we proceed as follows but to the left of $\sum \mathbb{Z}_{r_1}^{N_1}$ or to the right of $\sum \mathbb{Z}_{r_k}^{N_k}$.) Let $\vec{f}_j$ be the tuple corresponding to $\vec{f}_j^*$ in this copy of $\sum \mathbb{Z}_{s_j}^{Q_j}$ in $\mathcal{B}$ and so we will have $\vec{b}_i <_{\mathcal{B}} \vec{f}_j <_{\mathcal{B}} \vec{b}_{i+1}$.

Now we have chosen $\vec{d}_1 < \vec{d}_2 < \ldots < \vec{d}_k$ and $\vec{f}_1 < \vec{f}_2 < \ldots < \vec{f}_l$ corresponding to $\vec{c}_1 < \vec{c}_2 < \ldots < \vec{c}_k$ and $\vec{e}_1 < \vec{e}_2 < \ldots < \vec{e}_l$ where l is some natural number and some of the

$\vec{c}_i$ (and corresponding $\vec{d}_i$) may be the empty tuple. Arrange the tuples $\{(\vec{a}_i \cup \vec{c}_i), \vec{e}_j\}_{i=1,j=1}^{k \ \ l}$ in $\mathcal{A}$ and $\{(\vec{b}_i \cup \vec{d}_i), \vec{f}_j\}_{i=1,j=1}^{k \ \ l}$ in $\mathcal{B}$ so that they satisfy property (i). (Note: We already have properties (ii) and (iii) by construction.) Recall that we have

$$\left(\sum \mathbb{Z}_{r_i}^{M_i}, \vec{a}_i^*, \vec{c}_i^*\right) \leq_n \left(\sum \mathbb{Z}_{s_i}^{N_i}, \vec{b}_i^*, \vec{d}_i^*\right) \text{ for } 1 \leq i \leq k$$

and

$$\left(\sum \mathbb{Z}_{s_j}^{P_j}, \vec{e}_j^*\right) \leq_n \left(\sum \mathbb{Z}_{s_j}^{Q_j}, \vec{f}_j^*\right) \text{ for } 1 \leq j \leq l$$

which is property (iv). Let the tuple $\vec{d}$ include all $\vec{d}_i$'s and $\vec{f}_j$'s ordered correctly relative to the corresponding $\vec{c}_i$'s and $\vec{e}_j$'s in $\mathcal{A}$. By the induction hypothesis, we have $(\mathcal{A}, \vec{a}, \vec{c}) \leq_n (\mathcal{B}, \vec{b}, \vec{d})$. This proves that $(\mathcal{B}, \vec{b},) \leq_{n+1} (\mathcal{A}, \vec{a})$ as desired. $\square$

Now for our main result:

Theorem 6.2.16. *For any two orderings $\mathcal{A} \cong \nu^\omega(S)$ and $\mathcal{B} \cong \nu^\omega(R)$ we have $\mathcal{A} \equiv_\omega \mathcal{B}$.*

Proof. We will show that $\mathcal{B} \leq_\omega \mathcal{A}$, the other inequality is symmetric. Fix $\vec{a} \in \mathcal{A}$ and $n < \omega$. Decompose $\vec{a}$ as $\vec{a}_1 \cup \vec{a}_2 \cup \ldots \cup \vec{a}_k$ where each tuple $\vec{a}_i$ lies in a distinct $\sum \mathbb{Z}_{r_i}^{M_i}$ block in $\mathcal{A}$. For $i = 1$, pick $N_1 \geq 1$ (including ∞) such that $\sum \mathbb{Z}_{r_1}^{M_1} \equiv_{n+1} \sum \mathbb{Z}_{r_1}^{N_1}$. Then there exists a tuple $\vec{b}_1^* \in \sum \mathbb{Z}_{r_1}^{N_1}$ such that $(\sum \mathbb{Z}_{r_1}^{M_1}, \vec{a}_1^*) \leq_n (\sum \mathbb{Z}_{r_1}^{N_1}, \vec{b}_1^*)$. Pick any copy of $\sum \mathbb{Z}_{r_1}^{N_1}$ in $\vec{B}$ and let $\vec{b}_1$ be tuple corresponding to $\vec{b}_1^*$ in this copy.

For $i = 2$, pick N_2 such that $\sum \mathbb{Z}_{r_2}^{M_2} \equiv_{n+1} \sum \mathbb{Z}_{r_2}^{N_2}$. Then, similarly, there is some tuple $\vec{b}_2^*$ such that $\left(\sum \mathbb{Z}_{r_2}^{M_2}, \vec{a}_2\right) \leq_n \left(\sum \mathbb{Z}_{r_2}^{N_2}, \vec{b}_2^*\right)$. Pick a copy of $\sum \mathbb{Z}_{r_2}^{N_2}$ in $\mathcal{B}$ such that $\sum \mathbb{Z}_{r_1}^{N_1} <_{\mathcal{B}} \sum \mathbb{Z}_{r_2}^{N_2}$ if and only if $\sum \mathbb{Z}_{r_1}^{M_1} <_{\mathcal{A}} \sum \mathbb{Z}_{r_2}^{M_2}$. Let $\vec{b}_2$ be the tuple corresponding to $\vec{b}_2^*$ in this copy. We continue in this way to find $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k$ such that $\vec{b} = \vec{b}_1 \cup \vec{b}_2 \cup \ldots \cup \vec{b}_k$ and $\vec{a}$ satisfy properties (i)-(iv) from Lemma 6.2.15. Thus $(\mathcal{A}, \vec{a}) \leq_n (\mathcal{B}, \vec{b})$ and hence $\mathcal{B} \leq_\omega \mathcal{A}$. $\square$

Corollary 6.2.17. *Let $\vec{a} \in \nu^\omega(R)$ and $\vec{b} \in \nu^\omega(S)$ satisfy the following:*

1. *The tuples decompose as $\vec{a} = \vec{c}_1 < \vec{c}_2 < \ldots < \vec{c}_k$ where each tuple $\vec{c}_i$ lies in a distinct $\sum \mathbb{Z}_{r_i}^{M_i}$ block, and $\vec{b} = \vec{d}_1 < \vec{d}_2 < \ldots < \vec{d}_l$ where each tuple $\vec{d}_j$ lies in a distinct $\sum \mathbb{Z}_{s_j}^{N_j}$ block, for $M_i, N_j \in \{1, 2, \ldots, \infty\}$ and $r_i, s_j \in \omega$.*

2. *$k = l$ and for each $i = 1, \ldots, k$, $M_i = N_i$ and $r_i = s_i$.*

3. *For each $i = 1, \ldots, k$, as suborderings, we have $\left(\sum \mathbb{Z}_{r_i}^{M_i}, \vec{c}_i\right) \cong \left(\sum \mathbb{Z}_{s_i}^{N_i}, \vec{d}_i\right)$.*

Then $(\nu^\omega(R), \vec{a}) \leq_\omega (\nu^\omega(S), \vec{b})$.

Proof. Let $\vec{a} \in \nu^\omega(R)$ and $\vec{b} \in \nu^\omega(S)$ satisfy the given properties. Since $M_i = N_i$ and $r_i = s_i$ we will only refer to M's and r's from now on. First note that we can decompose $\nu^\omega(R)$ as

$$\nu^\omega(R) = \nu^\omega(R) + \sum \mathbb{Z}_{r_1}^{M_1} + \nu^\omega(R) + \sum \mathbb{Z}_{r_2}^{M_2} + \ldots + \nu^\omega(R) + \sum \mathbb{Z}_{r_k}^{M_k} + \nu^\omega(R)$$

and similarly,

$$\nu^\omega(S) = \nu^\omega(S) + \sum \mathbb{Z}_{r_1}^{M_1} + \nu^\omega(S) + \sum \mathbb{Z}_{r_2}^{M_2} + \ldots + \nu^\omega(S) + \sum \mathbb{Z}_{r_k}^{M_k} + \nu^\omega(S).$$

By Theorem 6.2.16, we have $\nu^\omega(R) \leq_\omega \nu^\omega(S)$.

For each i, let $\vec{c_i} = c_{i,1} < c_{i,2} < \ldots < c_{i,n_i}$ and $\vec{d_i} = d_{i,1} < d_{i,2} < \ldots < d_{i,n_i}$. Then in each of $\nu^\omega(R)$ and $\nu^\omega(S)$, $\sum \mathbb{Z}_{r_i}^{M_i}$ decomposes as

$$\sum \mathbb{Z}_{r_i}^{M_i} = \mathcal{A}_{i,0} + \{c_{i,1}\} + \mathcal{A}_{i,1} + \{c_{i,2}\} + \ldots + \mathcal{A}_{i,n_i-1} + \{c_{i,n_i}\} + \mathcal{A}_{i,n_i}$$

for some linear orderings $\mathcal{A}_{i,j}$ and

$$\sum \mathbb{Z}_{r_i}^{M_i} = \mathcal{B}_{i,0} + \{d_{i,1}\} + \mathcal{B}_{i,1} + \{d_{i,2}\} + \ldots + \mathcal{B}_{i,n_i-1} + \{d_{i,n_i}\} + \mathcal{B}_{i,n_i}$$

for some linear orderings $\mathcal{B}_{i,j}$. By assumption, we have $\mathcal{A}_{i,j} \cong \mathcal{B}_{i,j}$ for all $i = i, \ldots, k$ and all $j = 1, \ldots n_i$ and hence $\mathcal{A}_{i,j} \leq_\omega \mathcal{B}_{i,j}$. Now, it follows from Lemma 6.2.12 that

$$(\nu^\omega(R), \vec{a}) \leq_\omega (\nu^\omega(S), \vec{b}).$$

□

Corollary 6.2.18. *The back-and-forth ordinal of $\mathbb{K}_{\omega+1}$ and $\mathbb{K}_{\omega+2}$ is $\omega + 1$.*

Proof. By Theorem 6.2.7, the back and forth ordinal of each theory is at most $\omega + 1$. By Corollary 6.2.17, there are only countably many $\equiv_\omega$-classes of pairs of the form $(\nu^\omega(S), \vec{a})$ where $S \subseteq \omega$ and $\vec{a} \in \nu^\omega(S)$, and hence the back-and-forth ordinal of both theories is exactly $\omega + 1$. □

The class $\mathbb{K}_{\omega+2}$ is our first example of a class of structures with infinite Turing ordinal and back-and-forth ordinal that are not equal. In fact, this class is Borel which will be discussed in the next section.

6.3 Axiomatizing the theories

Many of the linear orderings defined in this Chapter can be axiomatized by computable $\mathcal{L}_{\omega_1,\omega}$ formulas. In this section we will provide an axiomatization for a selection of the classes from Section 6.1.

6.3.1 Finite ordinals

In this section we will axiomatize orderings of the form $\mathbb{Z}^m \cdot \sigma^*(X)$ for all sets X. Recall that we have the following formulas from Section 6.2.1:

Formula	Meaning	Complexity
$\varphi_{\mathbb{Z}^m}(x)$	x lies in a $\mathbb{Z}^m$ block	Π^0_{2m+1}
$\varphi_{\mathbb{Z}^m}(x,y)$	x and y lie in the same $\mathbb{Z}^m$ block	Π^0_{2m+1}
$S_{\mathbb{Z}^m}(x,y)$	x and y lie in successive $\mathbb{Z}^m$ blocks	Δ^0_{2m+2}
$S^k_{\mathbb{Z}^m}(x,y)$	The $\mathbb{Z}^m$ block of y is the k^{th} successor of the $\mathbb{Z}^m$ block of x	Σ^0_{2m+2}

We will use these formulas to describe the basic properties of the given orderings.

Blocks of $\mathbb{Z}^m \cdot (r+1)$

Consider the following formula:

$$\theta_{r,\mathbb{Z}^m}(x_0, x_1, \ldots, x_r) := \bigwedge_{i=0}^{r-1} S_{\mathbb{Z}^m}(x_i, x_{i+1}) \ \wedge\ \forall y \Big(\neg S_{\mathbb{Z}^m}(y, x_0) \wedge \neg S_{\mathbb{Z}^m}(x_r, y) \Big)$$

Then $\mathcal{A} \models \theta_{r,\mathbb{Z}^m}(a_0, a_2, \ldots, a_r)$ if and only if $a_0, a_1, \ldots, a_r$ lie in $r+1$ successive $\mathbb{Z}^m$ blocks in $\mathcal{A}$ and this "discrete block" of $\mathbb{Z}^m$'s is maximal. Similarly consider the following formula:

$$\theta_{r,\mathbb{Z}^m}(y) := (\exists x_0, \ldots, x_r) \left(\theta_{r,\mathbb{Z}^m}(x_0, \ldots, x_r) \wedge \bigvee_{i=0}^{r} \varphi_{\mathbb{Z}^m}(x_i, y) \right)$$

Then $\mathcal{A} \models \theta_{r,\mathbb{Z}^m}(a)$ if and only if a lies in a $\mathbb{Z}^m \cdot (r+1)$ block in $\mathcal{A}$ and this "discrete block" of $\mathbb{Z}^m$'s is maximal.

Blocks of $\mathbb{Z}^m \cdot \omega$

Consider the following formula:

$$\theta^k_{\omega,\mathbb{Z}^m}(x) := \varphi_{\mathbb{Z}^m}(x) \wedge \exists y\ S^{k-1}_{\mathbb{Z}^m}(y,x) \wedge \forall y\ \neg S^k_{\mathbb{Z}^m}(y,x) \wedge \bigwedge_{l=1}^{\infty} \exists y\ S^l_{\mathbb{Z}^m}(x,y)$$

Then $\mathcal{A} \models \theta^k_{\omega,\mathbb{Z}^m}(a)$ if and only if a lies in the k^{th} copy of $\mathbb{Z}^m$ in a $\mathbb{Z}^m \cdot \omega$ block in $\mathcal{A}$. We can also define the formula

$$\theta_{\omega,\mathbb{Z}^m}(x) := \bigvee_{k=1}^{\infty} \theta^k_{\omega,\mathbb{Z}^m}(x)$$

with the property that $\mathcal{A} \models \theta_{\omega,\mathbb{Z}^m}(a)$ if and only if a lies in a copy of $\mathbb{Z}^m \cdot \omega$ in $\mathcal{A}$.

Describing the $\mathbb{Z}^m \cdot \sigma^*(X)$ orderings

Consider the following formulas.

- Let $B_m := (\forall z)\left[\theta_{\omega,\mathbb{Z}^m}(z) \ \vee \bigvee_{r=0}^{\infty} \theta_{r,\mathbb{Z}^m}(z)\right]$.

  Then $\mathcal{A} \models B_m$ if and only if every element of $\mathcal{A}$ either lies in a $\mathbb{Z}^m \cdot (r+1)$ block for some $r \in \omega$, or lies in a $\mathbb{Z}^m \cdot \omega$ block.

- Let $SB_m(x,y) := \bigvee_{k \in \omega} S^k_{\mathbb{Z}^m}(x,y)$.

  Then, for any $\mathcal{A}$ such that $\mathcal{A} \models B_m$, we have $\mathcal{A} \models SB_m(a,b)$ if and only if a and b lie in the same $\mathbb{Z}^m \cdot \alpha$ block for some $\alpha \in \{1,2,\ldots,\omega\}$.

Now we proceed to defining a shuffle sum of orderings of the form $\mathbb{Z}^m \cdot \alpha$ for $\alpha \in \{\omega, 0, 1, \ldots\}$. For the following, x and y will represent elements in different $\mathbb{Z}^m \cdot \alpha$ blocks in our ordering. We need to ensure that there is a copy of $\mathbb{Z}^m \cdot \omega$ between x and y and, if $r \in X$, a copy of $\mathbb{Z}^m \cdot (r+1)$ between x and y. Consider the following preliminary formulas:

$$B_{\omega,\mathbb{Z}^m}(x,u,y) := (x < u < y) \wedge \neg SB_m(x,u) \wedge \neg SB_m(u,y) \wedge \theta_{\omega,\mathbb{Z}^m}(u)$$

and

$$B_{r,\mathbb{Z}^m}(x,u,y) := (x < u < y) \wedge \neg SB_m(x,u) \wedge \neg SB_m(u,y) \wedge \theta_{r,\mathbb{Z}^m}(u).$$

Roughly speaking, these formulas assert the existence of a $\mathbb{Z}^m \cdot \omega$ block or a $\mathbb{Z}^m \cdot (r+1)$ block between x and y. Similarly, let

$$\begin{aligned}
R_{\omega,\mathbb{Z}^m}(x,u) &:= (x < u) \wedge \neg SB_m(x,u) \wedge \theta_{\omega,\mathbb{Z}^m}(u)\\
L_{\omega,\mathbb{Z}^m}(u,y) &:= (u < y) \wedge \neg SB_m(u,y) \wedge \theta_{\omega,\mathbb{Z}^m}(u)\\
R_{r,\mathbb{Z}^m}(x,u) &:= (x < u) \wedge \neg SB_m(x,u) \wedge \theta_{r,\mathbb{Z}^m}(u)\\
L_{r,\mathbb{Z}^m}(u,y) &:= (u < y) \wedge \neg SB_m(u,y) \wedge \theta_{\omega,\mathbb{Z}^m}(u)
\end{aligned}$$

to assert blocks to the right of x and to the left of y. Now let

$$D_{\omega,m}(x,y) := (\exists u) R_{\omega,\mathbb{Z}^m}(x,u) \wedge (\exists u) L_{\omega,\mathbb{Z}^m}(u,y) \wedge (\neg SB_m(x,y) \to (\exists u) B_{\omega,\mathbb{Z}^m}(x,u,y))$$

and

$$D_{r,m}(x,y) := (\exists u) R_{r,\mathbb{Z}^m}(x,u) \wedge (\exists u) L_{r,\mathbb{Z}^m}(u,y) \wedge (\neg SB_m(x,y) \to (\exists u) B_{r,\mathbb{Z}^m}(x,u,y))$$

then we can define the density property using the following formula

$$D_m := (\forall x < y) \left[D_{\omega,m}(x,y) \wedge \bigwedge_{r \in \omega} \Big((\exists v) \theta_{r,\mathbb{Z}^m}(v) \to D_{r,m}(x,y) \Big) \right].$$

The first half of the formula will ensure that there are densely many copies of $\mathbb{Z}^m \cdot \omega$ between x and y, and the second half of the formula will ensure that, for each $r \in \omega$, either there are no blocks of $\mathbb{Z}^m \cdot (r+1)$ in the ordering, or there are densely many.

For structures in $\mathbb{K}_{2m+2}$, we need the set X to be of the form $X = S \oplus \overline{S}$ and so we need the following additional formula

$$J_m := \bigwedge_{s \in \omega} \left[\Big((\exists u)\ \theta_{2s,\mathbb{Z}^m}(u)\ \wedge\ (\forall v)\ \neg\theta_{2s+1,\mathbb{Z}^m}(v) \Big)\ \vee\ \Big((\exists u)\ \theta_{2s+1,\mathbb{Z}^m}(u)\ \wedge\ (\forall v)\ \neg\theta_{2s,\mathbb{Z}^m}(v) \Big) \right]$$

to ensure that, for each $r \in \omega$, exactly one of $2r$ and $2r+1$ appears in X.

Finally, consider the Π^c_{2m+5} formula $\chi_{2m+2} := B_m \wedge D_m \wedge J_m$. Then for all linear orderings $\mathcal{A}$, we have

$$\mathcal{A} \models \chi_{2m+2} \iff \mathcal{A} \in \{\mathbb{Z}^m \cdot \sigma^*(S \oplus \overline{S}) : S \subseteq \omega\} = \mathbb{K}_{2m+2}$$

and hence the class $\mathbb{K}_{2m+2}$ is Borel.

To axiomatize the class $\mathbb{K}_{2m+3}$, we need to axiomatize the generic property of the set S. Observe that for any $\mathcal{A} \cong \mathbb{Z}^m \cdot \sigma^*(S)$ we have

$$\begin{aligned} r \in S \quad &\Leftrightarrow \quad \text{There is a "maximal" } \mathbb{Z}^m \cdot (r+1) \text{ block in } \mathcal{A} \\ &\Leftrightarrow \quad \mathcal{A} \models (\exists \vec{x})\ \theta_{r,\mathbb{Z}^m}(\vec{x}). \end{aligned}$$

Then, for any $\sigma \in 2^{<\omega}$, we have

$$\begin{aligned} \sigma \subset S \quad &\Leftrightarrow \quad \mathcal{A} \models \bigwedge_{\sigma(r)=1} r \in S \wedge \bigwedge_{\sigma(r)=0} r \notin S \\ &\Leftrightarrow \quad \mathcal{A} \models \bigwedge_{\sigma(r)=1} (\exists \vec{x})\ \theta_{r,\mathbb{Z}^m}(\vec{x}) \wedge \bigwedge_{\sigma(r)=0} \neg(\exists \vec{x})\ \theta_{r,\mathbb{Z}^m}(\vec{x}). \end{aligned}$$

By the definition of α-genericity, S is $(2m+3)$-generic if and only if

$$\mathcal{A} \models \bigwedge_{X \in \Sigma^0_{2m+3}} \left[\bigvee_{\sigma \in X} \text{“}\sigma \subset S\text{”} \quad \wedge \quad \bigvee_{\sigma \in 2^{<\omega}} \left(\text{“}\sigma \subset S\text{”} \wedge \bigvee_{\tau \supseteq \sigma, \tau \in X} (\exists x)(x < x) \right) \right].$$

Since this is an $\mathcal{L}_{\omega_1,\omega}$ sentence, the class $\mathbb{K}_{2m+3}$ is also a Borel class.

6.3.2 Classes for $\omega + 1$ and $\omega + 2$

How do we axiomatize orderings of the form $\nu^\omega(X)$ for some $X \subseteq \omega$? They must satisfy the following properties:

1. Every element lies in a copy of either $r + 1 + \mathbb{Z} + \mathbb{Z}^2 \ldots + \mathbb{Z}^M$ for some r, M, or $r + 1 + \mathbb{Z} + \mathbb{Z}^2 \ldots + \mathbb{Z}^i \ldots$ for some r.
2. There are densely many copies of $r + 1 + \mathbb{Z} + \mathbb{Z}^2 \ldots + \mathbb{Z}^M$ for each pair r, M.
3. For each r, either there is no copy of $r + 1 + \mathbb{Z} + \ldots + \mathbb{Z}^i \ldots$ or there are densely many.

Restricting to sets of the form $X = S \oplus \overline{S}$ is not difficult. We then also require that exactly one of $(2s) + 1 + \mathbb{Z}^1 + \ldots + \mathbb{Z}^i \ldots$ or $(2s+1) + 1 + \mathbb{Z}^1 + \ldots + \mathbb{Z}^i \ldots$ appears.

Recall the following formulas from Section 6.2.3.

Formula	Meaning
$\theta^{r,M}(\vec{x}, \vec{y})$	$\vec{x} \cup \vec{y}$ lies in a $\sum \mathbb{Z}^M_r$ block with initial segment $\vec{x}$ and each $y_i \in \mathbb{Z}^i$
$\chi^{r,M}(x)$	x lies in a $\sum \mathbb{Z}^M_r$ block
$\chi^r(x)$	x lies in a $\sum \mathbb{Z}^\infty_r$ block

Property 1

Using the work from Section 6.2.3, we can immediately describe Property 1 with the formula

$$(\forall x) \left[\bigvee_{r \geq 0, M \geq 1} \chi^{r,M}(x) \vee \bigvee_{r \in \omega} \chi^r(x) \right].$$

We need a bit more work for Properties 2 and 3.

When $\tilde{x}$ and $\tilde{y}$ are in the same copy of $\sum \mathbb{Z}_r^M$

To complete the description of the ordering, we need to define a formula that can check whether two elements are in the same copy of some $\sum \mathbb{Z}_r^M$. Recall that, for any linear ordering $\mathcal{A}$ and tuples $\vec{a} = (a_0, \ldots, a_r)$ and $\vec{b} = (b_1, \ldots, b_M)$ in $\mathcal{A}$, we have

$$\mathcal{A} \models \theta^{r,M}(\vec{a}, \vec{b}) \iff \vec{a} \cup \vec{b} \text{ lies in a single } \sum \mathbb{Z}_r^M \text{ block with initial segment } \vec{a} \text{ and } b_i \in \mathbb{Z}^i.$$

Consider the following formula in two free variables $\tilde{x}$ and $\tilde{y}$:

$$\begin{aligned}
\chi^{r,M}(\tilde{x}, \tilde{y}) \quad &:= \quad S(\tilde{x}, \tilde{y}) \vee S(\tilde{y}, \tilde{x}) \vee \bigvee_{i=1}^{M} \varphi_{\mathbb{Z}^i}(\tilde{x}, \tilde{y}) \\
&\vee \quad \bigvee_{k=0}^{r} \bigvee_{i=1}^{M} (\exists \vec{x})(\exists \vec{y})\big[\tilde{x} = x_k \wedge \tilde{y} = y_i \wedge \theta^{r,M}(\vec{x}, \vec{y})\big] \\
&\vee \quad \bigvee_{i,l=1}^{M} (\exists \vec{x})(\exists \vec{y})\big[\tilde{x} = y_i \wedge \tilde{y} = y_l \wedge \theta^{r,M}(\vec{x}, \vec{y})\big]
\end{aligned}$$

Then for any ordering $\mathcal{A}$ and any $a, b \in \mathcal{A}$, we have $\mathcal{A} \models \chi^{r,M}(a, b)$ if and only if a and b lie in the same block of the form $\sum \mathbb{Z}_r^M$ in $\mathcal{A}$.

When $\tilde{x}$ and $\tilde{y}$ are in the same copy of $\sum \mathbb{Z}_r^\infty$

Using the above formula, we can extend this to the case of infinitely many summands. Let

$$\chi^r(\tilde{x}, \tilde{y}) := \bigvee_{M \geq 1} \left[\bigwedge_{k \geq M} \chi^{r,k}(\tilde{x}, \tilde{y}) \right]$$

then $\mathcal{A} \models \chi^r(a, b)$ if and only if a and b lie in the same copy of $\sum \mathbb{Z}_r^\infty$.

When $\tilde{x}$ and $\tilde{y}$ are in different blocks in $\nu^\omega(S)$

Consider the following formula

$$\neg\chi(\tilde{x}, \tilde{y}) := \neg \left[\bigvee_{r \geq 0, M \geq 1} \chi^{r,M}(\tilde{x}, \tilde{y}) \vee \bigvee_{r \geq 0} \chi^r(\tilde{x}, \tilde{y}) \right].$$

Then $\mathcal{A} \models \neg\chi(a, b)$ if and only if a and b do not lie in the same copy of $\sum \mathbb{Z}_r^M$ for any r, M or $\sum \mathbb{Z}_r^\infty$ for any r.

Property 2

To describe Property 2, for each pair (r, M), define the sentence $D(r, M)$ as:

$$(\forall x, y)\Big[\neg\chi(x, y)\rightarrow(\exists z)\big(x < z < y \wedge \neg\chi(x, z) \wedge\ \neg\chi(z, y) \wedge \chi^{r,M}(z) \wedge \neg\chi^{r,M+1}(z)\big)\Big]$$

that says that blocks of the form $\sum \mathbb{Z}_r^M$ are dense in a given ordering.

Property 3

For the final property, let $\mathrm{Set}(r)$ be the following formula

$$(\exists z)\Big[\chi^r(z)\Big]\rightarrow(\forall x, y)\Big[\neg\chi(x, y)\rightarrow(\exists z)\big(x < z < y\ \wedge\ \neg\chi(x, z)\ \wedge\ \neg\chi(z, y)\ \wedge \chi^r(z)\big)\Big]$$

then $\mathcal{A} \models Set(r)$ if and only if there are either densely many copies of $\sum \mathbb{Z}_r^\infty$ or no copies.

Theory of $\mathbb{K}_{\omega+1}$

Recall the class $\mathbb{K}_{\omega+1} = \{\nu^\omega(S \oplus \overline{S}) : S \subset \omega\}$. This class can be axiomatized by the axioms of linear orderings and the following computable infinitary sentences:

$$(\forall x)\left[\bigvee_{r\geq 0, M\geq 1} \chi^{r,M}(x) \vee \bigvee_{r\in\omega} \chi^r(x)\right] \wedge \left[\bigwedge_{r\geq 0, M\geq 1} D(r, M)\right] \wedge \left[\bigwedge_{r\geq 0} Set(r)\right] \tag{1}$$

and

$$\bigvee_{r\in\omega} \big[(\exists x)\chi^{2r}(x) \Leftrightarrow \neg(\exists x)\chi^{2r+1}(x)\big] \tag{2}$$

Sentence (1) ensures that each linear ordering is of the form $\nu^\omega(X)$ for some set X and sentence (2) ensures that X is of the form $S \oplus \overline{S}$. These are computable $\mathcal{L}_{\omega_1,\omega}$ sentence and hence this class is Borel.

Theory of $\mathbb{K}_{\omega+2}$

Recall the class $\mathbb{K}_{\omega+2} = \{\nu^\omega(S) : S \text{ is } (\omega + 2)\text{-generic}\}$. We axiomatized the notion of $(2m + 3)$-genericity Section 6.3.1 and we will use a similar axiomatization here. In this case, we need to take our conjunction over all $\Sigma^0_{\omega+2}$ sets of strings. This is again an $\mathcal{L}_{\omega_1,\omega}$ sentence and therefore $\mathbb{K}_{\omega+2}$ provides our first counterexample to Question 4.3.2.

Chapter 7

Separating the ordinals

The goal of this chapter is to define Borel classes of structures for which the Turing ordinal and back-and-forth ordinal are far apart. First let us recall the current picture:

Class of structures	Turing ordinal	Back-and-forth ordinal
Abelian groups	0	1
Graphs	0	1
Algebraic fields	0	1
Partial orders	0	1
Lattices	0	1
Models of PA	1	1
$\mathbb{K}_W$	DNE	1
Equivalence structures	1	2
Linear orders	2	3
T_n	$n+2$	$n+3$
$\mathbb{K}_{2m+2}$	$2m+2$	$2m+3$
$\mathbb{K}_{2m+3}$	$2m+3$	$2m+3$
Boolean algebras	ω	ω
$\mathbb{K}_{\omega+1}$	$\omega+1$	$\omega+1$
$\mathbb{K}_{\omega+2}$	$\omega+2$	$\omega+1$

7.1 Infinite Turing ordinal, finite back-and-forth ordinal

In this section, we will describe classes of orderings $\mathbb{K}_N$, one for each positive integer N, having Turing ordinal ω and finite back-and-forth ordinal $2N+3$. These classes will provide the first examples having finite back-and-forth ordinal but infinite Turing ordinal.

Given a positive integer N, there is a notation for ω with adjusted fundamental sequence

$$\left(\alpha'_k\right)_{k\in\omega} = \left(2(N+k)+2\right)_{k\in\omega}.$$

Recall from Chapter 6, that, for all $S \subseteq \omega$,

$$\mathcal{A}_{2(N+k)+2}(S) := \mathbb{Z}^{N+k} \cdot \sigma^*(S \oplus \overline{S})$$

and so the corresponding ordering, $\mathcal{A}_\omega(S)$, for this fundamental sequence is defined as

$$\sum_{k=0}^{\infty} \left(1 + \eta + 1 + \mathbb{Z}^{N+k} \cdot \sigma^*(S_k \oplus \overline{S_k})\right)$$

Building on the work from [1], we can form a subcollection of these orderings and get an example of a Borel class with finite back-and-forth ordinal but infinite Turing ordinal.

This Section will be devoted to proving the following theorem and axiomatizing the given classes.

Theorem 7.1.1. *Fix $N \geq 1$. The class of orderings defined by*

$$\mathbb{K}_N := \left\{ \sum_{k=0}^{\infty} \left(1+\eta+1+\mathbb{Z}^{N+k}\cdot\sigma^*(S_k \oplus \overline{S_k})\right) \middle| S_k \not\leq_T (S_0 \oplus \ldots \oplus S_{k-1})^{2(N+k)+2} \text{ for all } k \in \omega \right\}$$

has Turing ordinal ω and back-and-forth ordinal $2N+3$.

For notation, let $\mathcal{L}(N,S) := \sum_{k=0}^{\infty}\left(1+\eta+1+\mathbb{Z}^{N+k}\cdot\sigma^*(S_k \oplus \overline{S_k})\right)$.

The fact that the Turing ordinal of each class is ω follows directly from Theorem 6.1.22.

In Section 7.1.1 we will discuss the back-and-forth ordinals of the classes and in Section 7.1.2, we will provide an axiomatization of each theory.

7.1.1 The back-and-forth ordinal of $\mathbb{K}_N$

As the building blocks of the $\mathcal{L}(N,S)$ orderings are orderings of the form $\mathbb{Z}^m \cdot \sigma^*(X)$ for various sets X, we will use the results from Section 6.2.2 to help us examine the back-and-forth types of the desired tuples.

Proposition 7.1.2. *For all positive integers N, there are only countably many pairs of the form $(\mathcal{L}(N,S), \vec{a})$, where $S \subseteq \omega$, up to $\equiv_{2N+2}$-equivalence.*

Proof. Fix $N \geq 1$. Recall that

$$\mathcal{L}(S,N) = \sum_{k=0}^{\infty} \left(1 + \eta + 1 + \mathbb{Z}^{N+k} \cdot \sigma^*(S_k \oplus \overline{S_k})\right).$$

It follows from Corollary 6.2.5 that, for all $k \geq 0$, there are at most countably many pairs of the form $(\mathbb{Z}^{N+k} \cdot \sigma^*(X), \vec{a})$, up to $\equiv_{2N+2}$ equivalence. This follows from the fact that $\equiv_{2(N+k)+2}$-equivalence for any $k \geq 0$ implies $\equiv_{2N+2}$-equivalence. Observe that there are also at most countably many pairs of the form $(1 + \eta + 1, \vec{c})$ up to $\equiv_{2N+2}$-equivalence. So, by Lemma 6.2.12, for each $k \geq 0$, there are countably many pairs $(\mathcal{A}, \vec{a})$ where

$$\mathcal{A} \cong 1 + \eta + 1 + \mathbb{Z}^{N+k} \cdot \sigma^*(X),$$

up to $\equiv_{2N+2}$-equivalence. For each $k \geq 0$, we will list the equivalence classes as follows:

$$(\mathcal{A}_0^k, \vec{a}_0^k), (\mathcal{A}_1^k, \vec{a}_1^k), \ldots$$

Now, given any set S and any finite tuple $\vec{b} \in \mathcal{L}(N,S)$, we can decompose $\vec{b}$ as

$$\vec{b} = \vec{b}_1 < \vec{b}_2 < \ldots < \vec{b}_p$$

where each $\vec{b}_i$ lies in a distinct $\left(1 + \eta + 1 + \mathbb{Z}^{N+k_i} \cdot \sigma^*(S_{k_i} \oplus \overline{S_{k_i}})\right)$ block for some k_i. Then each $\vec{b}_i$ is $\equiv_{2N+2}$-equivalent to some tuple $\vec{a}_{j_i}^{k_i}$ from our above list. We will identify $\vec{b}$ with the tuple

$$(\vec{a}_{j_1}^{k_1}, \vec{a}_{j_2}^{k_2}, \ldots, \vec{a}_{j_p}^{k_p}).$$

By Lemma 6.2.12, if we have $\vec{b} \in \mathcal{L}(N,S)$ and $\vec{d} \in \mathcal{L}(N,R)$ that are both identified with the same tuple $(\vec{a}_{j_1}^{k_1}, \vec{a}_{j_2}^{k_2}, \ldots, \vec{a}_{j_p}^{k_p})$, then we have

$$(\mathcal{L}(N,S), \vec{b}) \equiv_{2N+2} (\mathcal{L}(N,R), \vec{d}).$$

As there are only countably many such tuples, there are at most countably many pairs $(\mathcal{L}(N,S), \vec{b})$ up to $\equiv_{2N+2}$-equivalence. $\square$

Corollary 7.1.3. *The back-and-forth ordinal of $\mathbb{K}_N$ is at least $2N+3$.*

It remains to show that the back-and-forth ordinal is exactly $2N+3$.

Proposition 7.1.4. *If $R, S \subseteq \omega$ satisfy $R_0 \neq S_0$ then $\mathcal{L}(N,R) \not\equiv_{2N+3} \mathcal{L}(N,S)$.*

Proof. Suppose that $R_0 \neq S_0$. If there is some $a \in R_0 - S_0$ then $2a \in R_0 \oplus \overline{R_0}$ and $2a \notin S_0 \oplus \overline{S_0}$. If there is some $a \in S_0 - R_0$ then $2a+1 \in R_0 \oplus \overline{R_0}$ and $2a+1 \notin S_0 \oplus \overline{S_0}$. In either case, we have some $r \in R_0 \oplus \overline{R_0}$ that is not in $S_0 \oplus \overline{S_0}$. We will show that this is enough to distinguish $\mathcal{L}(N,R)$ and $\mathcal{L}(N,S)$ at the $2N+3$ level.

Fix $r \in \omega$ and positive integer N and consider the following sentence:

$$\varphi(r,N) := \exists x_0, x_1, \ldots, x_r \left[\bigwedge_{i=0}^{r-1} S_{\mathbb{Z}^N}(x_i, x_{i+1}) \ \wedge \ \forall y \Big(\neg S_{\mathbb{Z}^N}(y, x_0) \wedge \neg S_{\mathbb{Z}^N}(y, x_r) \Big) \right]$$

As the formula $S_{\mathbb{Z}^N}(x,y)$ is Δ^c_{2N+2}, the sentence $\varphi(r,N)$ is Σ^c_{2N+3}. Moreover, $\varphi(r,N)$ has the following property: For any linear ordering $\mathcal{L}$,

$$\mathcal{L} \models \varphi(r,N) \iff \mathcal{L} \text{ has a maximal } \mathbb{Z}^N \cdot (r+1) \text{ block.}$$

Now observe that, by definition, we have the following equivalence for all $R \subseteq \omega$ and all $r \in \omega$:

$$\mathcal{L}(N,R) \text{ has a maximal } \mathbb{Z}^N \cdot (r+1) \text{ block} \qquad \iff \qquad r \in R_0 \oplus \overline{R_0}$$

Therefore, as $r \in R_0 \oplus \overline{R_0}$, we have $\mathcal{L}(N,R) \models \varphi(r,N)$ and, since $r \notin S_0 \oplus \overline{S_0}$, we have $\mathcal{L}(N,S) \not\models \varphi(r,N)$. So $\mathcal{L}(N,R) \not\equiv_{2N+3} \mathcal{L}(N,S)$ as desired. ☐

Corollary 7.1.5. *The back-and-forth ordinal of $\mathbb{K}_N$ is at most $2N+3$.*

Proof. There are uncountably many sets $R_0 \subset \omega \times \omega$. and we can choose $R_1, R_2, \ldots$ such that, for each $k \geq 1$, $R_k \not\leq_T (R_0 \oplus \ldots R_{k-1})^{2(N+k)+2}$. It follows that there are uncountably many orderings $\mathcal{L}(N,S) \in \mathbb{K}_N$ with distinct 0-slices and hence, by Proposition 7.1.4, $\mathbb{K}_N$ has uncountably many $\equiv_{2N+3}$-equivalence classes. So the back-and-forth ordinal of $\mathbb{K}_N$ is at most $2N+3$. ☐

Now we restate the main result from the start of the section:

Theorem 7.1.6. *Fix $N \geq 1$. The class of orderings defined by*

$$\mathbb{K}_N := \left\{ \mathcal{L}(N,S) \;\middle|\; S_k \not\leq_T (S_0 \oplus \ldots \oplus S_{k-1})^{2(N+k)+2} \textit{ for all } k \in \omega \right\}$$

has Turing ordinal ω and back-and-forth ordinal $2N+3$.

Proof. By Theorem 6.1.22, Corollary 7.1.3 and Corollary 7.1.5. □

7.1.2 Axiomatizing the theories

In this section we will axiomatize the classes $\mathbb{K}_N$ for all $N \geq 1$ using an $\mathcal{L}_{\omega_1,\omega}$ sentence in the language of linear orderings and hence show that the given classes are Borel. Recall that we axiomatized the basic building blocks, $\mathbb{Z}^m \cdot \sigma^*(S \oplus \overline{S})$ in Section 6.3.1. We have formulas $\theta_{r,\mathbb{Z}^m}(x)$ such that $\mathcal{A} \models \theta_{r,\mathbb{Z}^m}(a)$ if and only if a lies in a $\mathbb{Z}^m \cdot (r+1)$ block in $\mathcal{A}$ and this "discrete block" of $\mathbb{Z}^m$'s is maximal. And formulas $\theta_{\omega,\mathbb{Z}^m}(x)$ such that $\mathcal{A} \models \theta_{\omega,\mathbb{Z}^m}(a)$ if and only if a lies in a copy of $\mathbb{Z}^m \cdot \omega$ in $\mathcal{A}$. We need one more building block before we start to define the $\mathcal{L}(N,S)$ orderings.

Blocks of $1 + \eta + 1$

We would like a formula in two variables x and y with the meaning that x and y are the end points of a block of the form $1 + \eta + 1$. Define this formula, which we will denote by $\varphi_\eta(x,y)$, as follows:

$$(x < y) \;\wedge\; (\forall z)(\forall w)\Big[x < z < w < y \;\rightarrow\; (\exists u\; z < u < w) \wedge (\exists u \exists v\; x < u < z < v < y)\Big]$$

Describing the $\mathbb{Z}^{N+k} \cdot \sigma^*(X)$ orderings

Here, we need to amend our formulas from Section 6.3.1 to describe the $\mathbb{Z}^n \cdot \sigma^*(X)$ orderings as suborderings of the larger ordering $\mathcal{L}(N,S)$. We alter the formula B_m from section 6.3.1 by adding parameters that will act as end points:

$$B_n(x,y) := (\forall z)\left[x < z < y \;\longrightarrow \left(\theta_{\omega,\mathbb{Z}^n}(z) \;\vee \bigvee_{r=0}^{\infty} \theta_{r,\mathbb{Z}^n}(z)\right)\right].$$

Then $\mathcal{A} \models B_n(a,b)$ if and only if every element of $\mathcal{A}$ between a and b lies in either a $\mathbb{Z}^n \cdot (r+1)$ block for some $r \in \omega$, or lies in a $\mathbb{Z}^n \cdot \omega$ block.

Now let's assume that a and b are parameters representing the endpoints of the $\mathbb{Z}^n \cdot \sigma^*(X)$ ordering as follows:

$$a + \mathbb{Z}^n \cdot \sigma^*(X) + b$$

The variables x and y will represent elements in different $\mathbb{Z}^n \cdot \alpha$ blocks satisfying $a < x < y < b$. We need to ensure that there is a copy of $\mathbb{Z}^n \cdot \omega$ between x and y and, if $r \in X$, a copy of $\mathbb{Z}^n \cdot (r+1)$ between x and y. Consider the following preliminary formulas:

$$D_{\omega,\mathbb{Z}^n}(x,u,y) := (x < u < y) \wedge \neg SB_n(x,u) \wedge \neg SB_n(u,y) \wedge \theta_{\omega,\mathbb{Z}^n}(u)$$

and

$$D_{r,\mathbb{Z}^n}(x,u,y) := (x < u < y) \wedge \neg SB_n(x,u) \wedge \neg SB_n(u,y) \wedge \theta_{r,\mathbb{Z}^n}(u)$$

Then we can describe density using the following formula:

$$D_n(a,x,y,b) := (\exists u)\, D_{\omega,\mathbb{Z}^n}(x,u,y) \wedge \bigwedge_{r\in\omega} \Big[(\exists v)\, D_{r,\mathbb{Z}^n}(a,v,b) \rightarrow (\exists u)\, D_{r,\mathbb{Z}^n}(x,u,y) \Big]$$

The first half of the formula will ensure that there are densely many copies of $\mathbb{Z}^n \cdot \omega$ between a and b, and the second half of the formula will ensure that, for each $r \in \omega$, either there are no blocks of $\mathbb{Z}^n \cdot (r+1)$ between a and b, or there are densely many.

To describe the $\mathbb{Z}^n \cdot \sigma^*(X)$ ordering between a and b we have the following formula:

$$\Sigma_n(a,b) := B_n(a,b) \wedge (\forall x)(\forall y)\Big[\big(a \le x < y \le b \wedge \neg SB_n(x,y)\big) \rightarrow D_n(a,x,y,b)\Big]$$

To ensure that the set X is of the form $S \oplus \overline{S}$, we need to add the formula

$$J_n := \bigwedge_{s\in\omega} \Bigg[\Big((\exists u)\ \theta_{2s,\mathbb{Z}^n}(u)\ \wedge\ (\forall v)\ \neg\theta_{2s+1,\mathbb{Z}^n}(v)\Big)\ \vee\ \Big((\exists u)\ \theta_{2s+1,\mathbb{Z}^n}(u)\ \wedge\ (\forall v)\ \neg\theta_{2s,\mathbb{Z}^n}(v)\Big)\Bigg]$$

and so our final formula is as follows:

$$\Sigma_n^{\oplus}(a,b) := \Sigma_n(a,b) \wedge J_n \wedge (\exists y)(a < y < b)$$

The last part of the formula is needed in order to ensure that the ordering between a and b is non-empty. For any ordering $\mathcal{A}$ we have $\mathcal{A} \models \Sigma_n^{\oplus}(a,b)$ if and only if between a and b lies an ordering of the form $\mathbb{Z}^n \cdot \sigma^*(S \oplus \overline{S})$ for some set S.

Axiomatizing $\mathcal{L}(N, S)$

Fix integers $N \geq 1$ and $k \geq 0$. We will first axiomatize all orderings of the form

$$1 + \eta + 1 + \mathbb{Z}^{N+k} \cdot \sigma^*(S_k \oplus \overline{S_k}) + 1 + \eta + 1 \ + \ \mathcal{B}$$

where S can be any subset of natural numbers and $\mathcal{B}$ can be any linear ordering. The free variables x_0, y_0, x_1, y_1 in our formula will denote the "1"s in the $(1+\eta+1)$ blocks, ordered from left to right. Then we can describe the ordering by the following formula:

$$(x_0 < y_0 < x_1 < y_1) \wedge (\forall y)\,(x_0 \leq y) \wedge \varphi_\eta(x_0, y_0) \wedge \Sigma^{\oplus}_{N+k}(y_0, x_1) \wedge \varphi_\eta(x_1, y_1)$$

Similarly, we can describe an ordering of the form

$$\sum_{k=0}^{K} \left(1 + \eta + 1 + \mathbb{Z}^{N+k}(S_k \oplus \overline{S_k})\right) + 1 + \eta + 1 \ + \ \mathcal{B}$$

with the following formula in free variables $x_0, y_0, \ldots, x_K, y_K, x_{K+1}, y_{K+1}$:

$$(x_0 < y_0 < \ldots < x_{K+1} < y_{K+1}) \wedge \bigwedge_{k=0}^{k=K} \Big(\varphi_\eta(x_k, y_k) \wedge \Sigma^{\oplus}_{N+k}(y_k, x_{k+1})\Big) \ \wedge \ \varphi_\eta(x_{K+1}, y_{K+1})$$

Let $\vec{x}_K := (x_0, y_0, \ldots, x_K, y_K, x_{K+1}, y_{K+1})$. Let's denote this formula by

$$\varphi^K_N(x_0, y_0, \ldots, x_K, y_K, x_{K+1}, y_{K+1}) = \varphi^K_N(\vec{x}_K).$$

Let $\mathcal{C}_K$ be the collection of orderings of the above form. More precisely, let

$$\mathcal{C}_K := \left\{ \sum_{k=0}^{K} \left(1 + \eta + 1 + \mathbb{Z}^{N+k}(S_k \oplus \overline{S_k})\right) + 1 + \eta + 1 \ + \ \mathcal{B} \ \middle|\ S \subseteq \omega, \mathcal{B} \text{ a lin. order} \right\}.$$

Then, for any linear ordering $\mathcal{A}$, we have

$$\mathcal{A} \models \exists \vec{x}_K \ \varphi^K_N(\vec{x}_K) \iff \mathcal{A} \in \mathcal{C}_K$$

By taking the infinite conjunction of φ^K_N for all K, we can axiomatize orderings with initial segments isomorphic to $\mathcal{L}(N, S)$ for some S. More precisely, for any linear ordering $\mathcal{A}$, we have

$$\mathcal{A} \models \bigwedge_{K=0}^{\infty} \exists \vec{x}_K \ \varphi^K_N(\vec{x}_K) \iff \mathcal{A} \cong \mathcal{L}(N, S) + \mathcal{B}$$

for some set S and some linear ordering $\mathcal{B}$. The infinite conjunction ensures that arbitrarily large initial segments of $\mathcal{A}$ must be isomorphic to corresponding initial segments of some $\mathcal{L}(N,S)$, but the formula does not exclude a non-empty ordering $\mathcal{B}$ appearing to the right.

Our last step is to define a formula that will guarantee $\mathcal{B}=\emptyset$. Informally, we would like a formula in one free variable z that will force z to lie in the initial segment of the form $\mathcal{L}(N,S)$. For each $K\geq 0$, define the formula $\psi_N^K(z)$, such that for any ordering $\mathcal{L}$ and any a from $\mathcal{L}$ we have

$$\mathcal{L}\models\psi_N^K(a) \iff \mathcal{L}\in\mathcal{C}_K \text{ where } a \text{ lies strictly to the left of the ordering } \mathcal{B}.$$

For each $K\geq 0$, consider the following collection of formulas in one free variable z:

$$\begin{aligned}(\exists\vec{x}_K)\big[\varphi_N^K(\vec{x}_K)\wedge(z=x_i)\big] &\quad \text{for}\quad i=0,\ldots,K+1\\ (\exists\vec{x}_K)\big[\varphi_N^K(\vec{x}_K)\wedge(z=y_i)\big] &\quad \text{for}\quad i=0,\ldots,K+1\\ (\exists\vec{x}_K)\big[\varphi_N^K(\vec{x}_K)\wedge(x_i<z<y_i)\big] &\quad \text{for}\quad i=0,\ldots,K+1\\ (\exists\vec{x}_K)\big[\varphi_N^K(\vec{x}_K)\wedge(y_i<z<x_{i+1})\big] &\quad \text{for}\quad i=0,\ldots,K\end{aligned}$$

Let $\psi_N^K(z)$ be the disjunction of all of the above formulas, and define the following formula:

$$\psi_N:=\left[\bigwedge_{K=0}^{\infty}\exists\vec{x}_K\ \varphi_N^K(\vec{x}_K)\right]\wedge\left[(\forall z)\bigvee_{K=0}^{\infty}\psi_N^K(z)\right]$$

Then, for any linear ordering $\mathcal{A}$, we have

$$\mathcal{A}\models\psi_N \iff \mathcal{A}\cong\mathcal{L}(N,S) \text{ for some } S\subseteq\omega.$$

Observe that, for each N, the formula ψ_N is a computable $\mathcal{L}_{\omega_1,\omega}$ formula. For the various classes $\mathbb{K}_N$, we will need the sentences ψ_N for all $N\geq 1$.

Axiomatizing $S_k\not\leq_T(S_0\oplus\ldots\oplus S_{k-1})^{2(N+k)+2}$

Let's suppose we have an ordering $\mathcal{A}$ such that $\mathcal{A}\cong\mathcal{L}(N,S)$ for some $N\geq 1$ and some $S\subseteq\omega$. For each $k\geq 1$ we need a formula $\varphi_{N,k}$ such that

$$\mathcal{A}\models\varphi_{N,k} \text{ if and only if } S_k\not\leq_T(S_0\oplus\ldots\oplus S_{k-1})^{2(N+k)+2}.$$

Observe that, from the definition of $\mathcal{L}(N, S)$, for any $r, m \in \omega$ and any positive integer N, we have

$$\begin{aligned} r \in S_m &\iff 2r \in S_m \oplus \overline{S_m} \\ &\iff \text{There is a maximal } \mathbb{Z}^{N+m} \cdot (2r+1) \text{ block in } \mathcal{L}(N,S). \\ &\iff \mathcal{L}(N,S) \models \exists \vec{x}\ \theta_{2r,\mathbb{Z}^{N+m}}(\vec{x}) \end{aligned}$$

and

$$\begin{aligned} r \notin S_m &\iff 2r+1 \in S_m \oplus \overline{S_m} \\ &\iff \text{There is a maximal } \mathbb{Z}^{N+m} \cdot (2r+2) \text{ block in } \mathcal{L}(N,S). \\ &\iff \mathcal{L}(N,S) \models \exists \vec{x}\ \theta_{2r+1,\mathbb{Z}^{N+m}}(\vec{x}) \end{aligned}$$

Let's look at how to axiomatize" $X \leq_T Y^{(M)}$" for sets X and Y, and some $M \geq 0$. We have "$X \leq_T Y^{(M)}$" if and only if there is some index e such that for all $r \in \omega$, $X(r) = \Phi_e^{Y^{(M)}}(r)$. So, as a formula, we have

$$\begin{aligned} \text{“}X \leq_T Y^{(M)}\text{”} &\iff \text{There is some } e \text{ such that for all } r \in \omega, \\ &\qquad \left[r \in X \Leftrightarrow \left(\exists \sigma \subset Y^{(M)}\right)(\exists s)\left(\Phi^{\sigma}_{e,s}(r)\downarrow = 1\right)\right], \text{ and} \\ &\qquad \left[r \notin X \Leftrightarrow \left(\exists \sigma \subset Y^{(M)}\right)(\exists s)\left(\Phi^{\sigma}_{e,s}(r)\downarrow = 0\right)\right] \\ &\iff \text{There is some } e \text{ such that for all } r \in \omega, \\ &\qquad \left[r \in X \Leftrightarrow \bigvee_{(\sigma,s):\Phi^{\sigma}_{e,s}(r)\downarrow=1} \sigma \subset Y^{(M)}\right], \text{ and} \\ &\qquad \left[r \notin X \Leftrightarrow \bigvee_{(\sigma,s):\Phi^{\sigma}_{e,s}(r)\downarrow=0} \sigma \subset Y^{(M)}\right] \\ &\iff \bigvee_{e\in\omega}\bigwedge_{r\in\omega}\left[\left(r \in X \Leftrightarrow \bigvee_{(\sigma,s):\Phi^{\sigma}_{e,s}(r)\downarrow=1} \sigma \subset Y^{(M)}\right) \wedge \right. \\ &\qquad\qquad \left. \left(r \notin X \Leftrightarrow \bigvee_{(\sigma,s):\Phi^{\sigma}_{e,s}(r)\downarrow=0} \sigma \subset Y^{(M)}\right)\right] \end{aligned}$$

Now modulo the statements "$r \in X$", "$r \notin X$", and "$\sigma \subset Y^{(M)}$", this is a computable $\mathcal{L}_{\omega_1,\omega}$ formula. In our particular case we want a statement of the form

$$\text{“}S_k \leq_T (S_0 \oplus \ldots \oplus S_{k-1})^{2(N+k)+2}\text{”}$$

and so we have $X = S_k$, $Y = S_0 \oplus \ldots \oplus S_{k-1}$ and $M = 2(N+k)+2$. Based on the work above, to axiomatize this statement we need only produce formulas with the meaning "$r \in S_k$", "$r \notin S_k$" for all $k \in \omega$ and "$\sigma \subset (S_0 \oplus \ldots \oplus S_{k-1})^{2(N+k)+2}$" for all $k \geq 1$ and all positive integers N. We already have formulas meaning "$r \in S_k$", "$r \notin S_k$" for all $k \in \omega$, so it remains to find a formula for "$\sigma \subset (S_0 \oplus \ldots \oplus S_{k-1})^{2(N+k)+2}$" for each $k \geq 1$ and positive integer N.

Observe that, if we have a formula for "$\sigma \subset (S_0 \oplus \ldots \oplus S_{k-1})$", then, by induction, we can build formulas for "$\tau \subset (S_0 \oplus \ldots \oplus S_{k-1})^{(M)}$" for all $M \geq 0$. This follows from the fact that, for $M \geq 1$,

$$\text{“}\tau \subset X^{(M)}\text{”} \Leftrightarrow \bigwedge_{\tau(i)=1} \left[\bigvee_{(\sigma,s):\Phi^{\sigma}_{i,s}(i)\downarrow} \text{“}\sigma \subset X^{(M-1)}\text{”} \right] \wedge \bigwedge_{\tau(i)=0} \left[\bigwedge_{(\sigma,s):\Phi^{\sigma}_{i,s}(i)\downarrow} \neg\text{“}\sigma \subset X^{(M-1)}\text{”} \right].$$

So it suffices to construct a formula with the meaning "$\sigma \subset (S_0 \oplus \ldots \oplus S_{k-1})$ for each $k \geq 1$.

Proposition 7.1.7. *For each triple (N, k, τ) where $N, k \geq 1$ and $\tau \subset \omega$, there is a computable $\mathcal{L}_{\omega_1,\omega}$ formula, $\alpha(N, k, \tau)$, in the language of linear orderings such that for any set $S \subseteq \omega$,*

$$\tau \subset S_0 \oplus S_1 \oplus \cdots \oplus S_{k-1} \Longleftrightarrow \mathcal{L}(N, S) \models \alpha(N, k, \tau).$$

Proof. We will construct the desired formulas by describing how statements of the form $r \in S_0 \oplus S_1 \oplus \cdots \oplus S_{k-1}$ manifest in the the ordering $\mathcal{L}(N, S)$. Note here that we write $S_0 \oplus S_1 \oplus \cdots \oplus S_{k-1}$ for $(\ldots((S_0 \oplus S_1) \oplus S_2) \oplus \ldots \oplus S_k)$. For $k = 1$, the statement reduces to $r \in S_0$, for which a formula has already been defined, and so we will take the base case as $k = 2$, the first time the join appears.

$\boxed{k = 2\text{: } r \in S_0 \oplus S_1}$

If r is even, then

$$\begin{aligned} r \in S_0 \oplus S_1 &\Longleftrightarrow r/2 \in S_0 \\ &\Longleftrightarrow r \in S_0 \oplus \overline{S_0} \\ &\Longleftrightarrow \mathcal{L}(N, S) \models \exists \vec{x}\; \theta_{r,\mathbb{Z}^N}(\vec{x}) \end{aligned}$$

If r is odd, then

$$\begin{aligned} r \in S_0 \oplus S_1 &\Longleftrightarrow (r-1)/2 \in S_1 \\ &\Longleftrightarrow r - 1 \in S_1 \oplus \overline{S_1} \\ &\Longleftrightarrow \mathcal{L}(N, S) \models \exists \vec{x}\; \theta_{r-1,\mathbb{Z}^{N+1}}(\vec{x}) \end{aligned}$$

Let $\chi_{2^1}(N,r,1) := \exists \vec{x}\ \theta_{r,\mathbb{Z}^N}(\vec{x})$ and $\chi_{2^0}(N,r,1) := \exists \vec{x}\ \theta_{r-1,\mathbb{Z}^{N+1}}(\vec{x})$. These formulas are computable $\mathcal{L}_{\omega_1,\omega}$ formulas in the language of linear orderings satisfying:

1. If $2^1|r$, then $r \in S_0 \oplus S_1 \iff \mathcal{L}(N,S) \models \chi_{2^1}(N,r,1)$, and
2. If $2^0|r$ but $2^1 \nmid r$, then $r \in S_0 \oplus S_1 \iff \mathcal{L}(N,S) \models \chi_{2^0}(N,r,1)$.

For the following, let $\mathrm{ord}_{2,k}(r) := \max\{i \leq k : 2^i | r\}$.

$\boxed{\text{Inductive Hypothesis}}$ Assume that for $k > 2$ we have k formulas

$$\begin{gathered} \chi_{2^{k-1}}(N,r,k-1) \\ \chi_{2^{k-2}}(N,r,k-1) \\ \vdots \\ \chi_{2^0}(N,r,k-1) \end{gathered}$$

with the property that, if $\mathrm{ord}_{2,k-1}(r) = l$, then

$$r \in S_0 \oplus S_1 \oplus \cdots \oplus S_{k-1} \Leftrightarrow \mathcal{L}(N,S) \models \chi_{2^l}(N,r,k-1).$$

We will describe how to obtain the formulas for $k+1$:

If $\mathrm{ord}_{2,k}(r) = l \geq 1$, then r is even and $\mathrm{ord}_{2,k-1}(r/2) = l-1 \geq 0$. Therefore we have

$$\begin{aligned} r \in (S_0 \oplus \cdots S_{k-1}) \oplus S_k &\iff r/2 \in S_0 \oplus \cdots \oplus S_{k-1} \\ &\iff \mathcal{L}(N,S) \models \chi_{2^{l-1}}(N,r/2,k-1) \end{aligned}$$

If $\mathrm{ord}_{2,k}(r) = 0$ then r is odd and therefore we have

$$\begin{aligned} r \in (S_0 \oplus \cdots S_{k-1}) \oplus S_k &\iff (r-1)/2 \in S_k \\ &\iff r-1 \in S_k \oplus \overline{S_k} \\ &\iff \mathcal{L}(N,S) \models \exists \vec{x}\ \theta_{r-1,\mathbb{Z}^{N+k}}(\vec{x}) \end{aligned}$$

For $1 \leq i \leq k$, we let $\chi_{2^i}(N,r,k) := \chi_{2^{i-1}}(N,r/2,k-1)$ and let $\chi_{2^0}(N,r,k) := \exists \vec{x}\, \theta_{r-1,\mathbb{Z}^{N+k}}(\vec{x})$. Then we have $k+1$ formulas

$$\begin{gathered} \chi_{2^k}(N,r,k) \\ \chi_{2^{k-1}}(N,r,k) \\ \vdots \\ \chi_{2^0}(N,r,k) \end{gathered}$$

with the property that, if $\mathrm{ord}_{2,k}(r) = l$, then

$$r \in S_0 \oplus S_1 \oplus \cdots \oplus S_k \iff \mathcal{L}(N,S) \models \chi_{2^l}(N,r,k).$$

Now we are ready to define the desired formulas $\alpha(N,k,\tau)$. For any N, τ, define

$$\begin{aligned}
\alpha(N,1,\tau) &= \left[\bigwedge_{\tau(r)=1} \text{“}r \in S_0\text{”} \wedge \bigwedge_{\tau(r)=0} \text{“}r \notin S_0\text{”}\right] \\
&= \left[\bigwedge_{\tau(r)=1} \exists\vec{x}\ \theta_{2r,\mathbb{Z}^N}(\vec{x}) \wedge \bigwedge_{\tau(r)=0} \exists\vec{x}\ \theta_{2r+1,\mathbb{Z}^N}(\vec{x})\right].
\end{aligned}$$

Given a positive integer N, $k \geq 2$ and $\tau \subset \omega$ let

$$\alpha(N,k,\tau) := \left[\bigwedge_{\tau(r)=1} \chi_{2^{\mathrm{ord}_{2,k-1}(r)}}(N,r,k-1) \wedge \bigwedge_{\tau(r)=0} \neg\chi_{2^{\mathrm{ord}_{2,k-1}(r)}}(N,r,k-1)\right].$$

Then for any $S \subseteq \omega$,

$$\tau \subset S_0 \oplus S_1 \oplus \cdots \oplus S_{k-1} \iff \mathcal{L}(N,S) \models \alpha(N,k,\tau).$$

$\square$

Corollary 7.1.8. *For each tuple (N,k,τ,M) where $N \geq 1$, $k \geq 0$, $\tau \subset \omega$ and $M \geq 0$, there is a computable $\mathcal{L}_{\omega_1,\omega}$ formula $\alpha(N,k,\tau)^{(M)}$ such that for all $S \subseteq \omega$,*

$$\tau \subset (S_0 \oplus S_1 \oplus \cdots \oplus S_{k-1})^{(M)} \iff \mathcal{L}(N,S) \models \alpha(N,k,\tau)^{(M)}.$$

Corollary 7.1.9. *Given a set $S \subseteq \omega$ a positive integer N and $k \geq 1$, there is a computable $\mathcal{L}_{\omega_1,\omega}$ formula $\varphi_{N,k}$ such that*

$$\mathcal{L}(N,S) \models \varphi_{N,k} \iff S_k \not\leq_T (S_0 \oplus \ldots \oplus S_{k-1})^{2(N+k)+2}$$

Proof. Let $M = 2(N+k)+2$. Recall that, speaking informally, the statement

$$\text{“}S_k \leq_T (S_0 \oplus \cdots \oplus S_{k-1})^{(M)}\text{”}$$

can be formulated by

$$\bigvee_{e\in\omega}\bigwedge_{r\in\omega}\left[\left(\text{“}r \in S_k\text{”} \Leftrightarrow \bigvee_{\substack{(\sigma,s):\\ \Phi^{\sigma}_{e,s}(r)\downarrow=1}} \text{“}\sigma \subset Y^{(M)}\text{”}\right) \wedge \left(\text{“}r \notin S_k\text{”} \Leftrightarrow \bigvee_{\substack{(\sigma,s):\\ \Phi^{\sigma}_{e,s}(r)\downarrow=0}} \text{“}\sigma \subset Y^{(M)}\text{”}\right)\right]$$

where $Y = S_0 \oplus \cdots \oplus S_{k-1}$. Now let $\varphi_{N,k}$ be the negation of the following formula:

$$\bigvee_{e\in\omega}\bigwedge_{r\in\omega}\left[\left(\exists\vec{x}\,\theta_{2r,\mathbb{Z}^{N+k}}(\vec{x}) \longleftrightarrow \bigvee_{\Phi^{\sigma}_{e,s}(r)\downarrow=1}^{(\sigma,s):} \alpha(N,k,\sigma)^{(M)}\right)\wedge\left(\exists\vec{x}\,\theta_{2r+1,\mathbb{Z}^{N+k}}(\vec{x}) \longleftrightarrow \bigvee_{\Phi^{\sigma}_{e,s}(r)\downarrow=0}^{(\sigma,s):} \alpha(N,k,\sigma)^{(M)}\right)\right]$$

Then $\varphi_{N,k}$ is a computable $\mathcal{L}_{\omega_1,\omega}$ formula in the language of linear orderings with the desired property. $\square$

Main Result

Fix any positive integer N. We would like to axiomatize the following class of orderings:

$$\mathbb{K}_N := \left\{\mathcal{L}(N,S) :\ S_k \not\leq_T (S_0 \oplus \ldots \oplus S_{k-1})^{2(N+k)+2} \text{ for all } k \geq 1\right\}$$

Let

$$\Psi_N := \psi_N \wedge \bigwedge_{k\geq 1} \varphi_{N,k}$$

Then for any linear ordering $\mathcal{A}$, we have $\mathcal{A} \in \mathbb{K}_N \iff \mathcal{A} \models \Psi_N$.

Corollary 7.1.10. *For all positive integers N, $\mathbb{K}_N$ is a Borel class.*

Corollary 7.1.11. *For every odd integer $m \geq 5$, there is a Borel class of countable linear orderings having Turing ordinal ω and back-and-forth ordinal m.*

Proof. Given m odd, $m \geq 5$, we can write $m = 2N+3$ for some positive integer N. Take the class to be $\mathbb{K}_N$. $\square$

7.2 Arbitrary finite difference

In this section we will prove that there exist Borel classes of structures where the back-and-forth and Turing ordinals are an arbitrarily large finite distance apart. More precisely, for each $0 \leq N < M$, we will define a $\Pi^c_{2(N+M)+7}$-axiomatizable class with back-and-forth ordinal $2N+3$ and Turing ordinal $2M+2$.

7.2.1 Classes of orderings

For the following, let $\sigma^*(X)$ for some set X denote the shuffle sum of the orderings ω and $r+2$ for each $r \in X$. This differs slightly from how the σ^* operator was defined in the previous chapter.

Definition 7.2.1. For any two sets $X, Y \subseteq \omega$ and $0 \leq N < M$ define the following linear ordering:

$$\mathcal{A}_{N,M}(X,Y) := 1 + \mathbb{Z}^N \cdot \sigma^*(X \oplus \overline{X}) + 1 + \mathbb{Z}^M \cdot \sigma^*(Y \oplus \overline{Y}) + 1.$$

Recall from Theorem 6.1.8 that for each $N \geq 0$,

$$\operatorname{Spec}\left(\mathbb{Z}^N \cdot \sigma^*(X \oplus \overline{X})\right) = \left\{\deg(D) : X \leq_T D^{(2N+2)}\right\}.$$

Corollary 7.2.2. *Spec*$(\mathcal{A}_{N,M}(X,Y)) = \left\{\deg(D) : X \leq_T D^{(2N+2)} \textit{ and } Y \leq_T D^{(2M+2)}\right\}$

Proof. Let $\mathcal{A} = \mathcal{A}_{N,M}(X,Y)$ and suppose we have $\mathcal{B} \cong \mathcal{A}$ with $\mathcal{B} \leq_T D$. Given the element separating the two orderings, by the degree spectrum result above, we have that $X \leq_T \mathcal{B}^{(2N+2)}$ and $Y \leq_T \mathcal{B}^{(2M+2)}$. As $\mathcal{B} \leq_T D$, we have $X \leq_T D^{(2N+2)}$ and $Y \leq_T D^{(2M+2)}$.

Now suppose that $X \leq_T D^{(2N+2)}$ and $Y \leq_T D^{(2M+2)}$. By the degree spectrum result, there are copies of $\mathbb{Z}^N \cdot \sigma^*(X \oplus \overline{X})$ and $\mathbb{Z}^M \cdot \sigma^*(Y \oplus \overline{Y})$ computable in D. So we can build a D-computable copy of $\mathcal{A}_{N,M}(X,Y)$. □

Given the above degree spectrum, we can amend the work from Chapter 6 and prove the following:

Theorem 7.2.3. *Fix $X, Y, B \subseteq \omega$ and $0 \leq N < M$ and let*

$$\mathcal{C} := \{D : X \leq_T D^{(2N+2)} \textit{ and } Y \leq_T D^{(2M+2)}\}.$$

If $B \leq_T D^{(2M+1)}$ for all $D \in \mathcal{C}$ then $B \leq_T X^{(2M+1)}$. Hence if $Y \not\leq_T X^{(2M+2)}$, then $\{D^{(2M+1)} : D \in \mathcal{C}\}$ has no element of least degree.

We will prove Theorem 7.2.3 using a generalization of the following claim of Ash, Jockusch and Knight from [1].

Proposition 7.2.4. *Given $Y \subseteq \omega$ and a computable ordinal α, if $B \not\leq_T \emptyset^{(\alpha)}$ then there is a set A such that*

(i) $Y \leq_T A \oplus \emptyset^{(\alpha+1)}$, *and*

(ii) $B \not\leq_T A \oplus \emptyset^{(\alpha)}$.

By relativizing this result (easily) we get the following:

Corollary 7.2.5. *Given any sets $X, Y \subseteq \omega$ and any computable ordinal α, if $B \not\leq_T X^{(\alpha)}$ then there is a set A such that*

(i) $Y \leq_T A \oplus X^{(\alpha+1)}$, *and*

(ii) $B \not\leq_T A \oplus X^{(\alpha)}$.

By relativizing Theorem 6.1.6 we get the following.

Corollary 7.2.6. *For any computable ordinal α, and any sets A, W such that $A \geq_T W^{(\alpha)}$. there exists a set $S \geq_T W$ such that $S \oplus W^{(\alpha)} \equiv_T S^{(\alpha)} \equiv_T A$.*

For our purposes, we need the following consequence of the previous corollary.

Corollary 7.2.7. *For any sets $A, X \subseteq \omega$ and any computable ordinal α, there is a set D such that $(D \oplus X)^{(\alpha)} \equiv_T A \oplus X^{(\alpha)}$.*

Proof. As $A \oplus X^{(\alpha)} \geq_T X^{(\alpha)}$, there is a set $D \geq_T X$ such that $D^{(\alpha)} \equiv_T A \oplus X^{(\alpha)}$, by relativized jump inversion. As $D \geq_T X$, we have $D^{(\alpha)} \equiv_T (D \oplus X)^{(\alpha)} \equiv_T A \oplus X^{(\alpha)}$. □

With these results in hand, we can prove the main lemma needed for Theorem 7.2.3.

Lemma 7.2.8. *Given $X, Y \subseteq \omega$ and any computable ordinal α, if $B \not\leq_T X^{(\alpha)}$ then there is a set D such that*

(i) $Y \leq_T (D \oplus X)^{(\alpha+1)}$, *and*

(ii) $B \not\leq_T (D \oplus X)^{(\alpha)}$.

Proof. Given X, Y and α, let A be as in Corollary 7.2.5. Given A, X and α, let D be a set such that $(D \oplus X)^{(\alpha)} \equiv_T A \oplus X^{(\alpha)}$, guaranteed by Corollary 7.2.7. Then we have

$$\begin{aligned} Y &\leq_T A \oplus X^{(\alpha+1)} \\ &\leq_T (A \oplus X^{(\alpha)})' \\ &\equiv_T \left((D \oplus X)^{(\alpha)}\right)' \\ &\equiv_T (D \oplus X)^{(\alpha+1)} \end{aligned}$$

and so (i) is satisfied. As $B \not\leq_T A \oplus X^{(\alpha)} \equiv_T (D \oplus X)^{(\alpha)}$ we also have (ii). $\square$

Finally we can prove Theorem 7.2.3:

Proof. (of Theorem 7.2.3) Consider the following two sets:

$$\mathcal{C} := \{D : X \leq_T D^{(2N+2)} \text{ and } Y \leq_T D^{(2M+2)}\}$$

and

$$\mathcal{C}^* := \{D : Y \leq_T (D \oplus X)^{(2M+2)}\}.$$

Suppose that $B \leq_T D^{(2M+1)}$ for all $D \in \mathcal{C}$. We claim that $B \leq_T (D \oplus X)^{(2M+1)}$ for all $D \in \mathcal{C}^*$. For any $D \in \mathcal{C}^*$ we have $Y \leq_T (D \oplus X)^{(2M+2)}$ by definition. Clearly, $X \leq_T (D \oplus X)^{(2N+2)}$ and hence $D \oplus X \in \mathcal{C}$. So, by assumption, we have $B \leq_T (D \oplus X)^{(2M+1)}$.

Now we wish to prove that $B \leq_T X^{(2M+1)}$. Assume for a contradiction that $B \not\leq_T X^{(2M+1)}$. Then by Lemma 7.2.8, there is a set D satisfying $Y \leq_T (D \oplus X)^{(2M+1+1)} = (D \oplus X)^{(2M+2)}$ and $B \not\leq_T (D \oplus X)^{(2M+1)}$. In other words, we have $D \in \mathcal{C}^*$ with $B \not\leq_T (D \oplus X)^{(2M+1)}$ which is a contradiction. Therefore we must have $B \leq_T X^{(2M+1)}$ as desired.

To prove the *hence* statement: We will prove the contrapositive. Suppose that the set $\{D^{(2M+1)} : D \in \mathcal{C}\}$ has an element of least degree, say $D_0^{(2M+1)}$. Then we have $X \leq_T D_0^{(2N+2)}$ and $Y \leq_T D_0^{(2M+2)}$ and, for all $D \in \mathcal{C}$ we have $D_0^{(2M+1)} \leq_T D^{(2M+1)}$. It follows from the statement of the theorem that $D_0^{(2M+1)} \leq_T X^{(2M+1)}$ and hence $D_0^{(2M+2)} \leq_T X^{(2M+2)}$. Then, by the former statement, $Y \leq_T D_0^{(2M+2)} \leq_T X^{(2M+2)}$. $\square$

With this result in hand we are ready to prove the main result:

Theorem 7.2.9. *For each $0 \leq N < M$, consider the following class of structures:*

$$\mathbb{K}_{N,M} := \{\mathcal{A}_{N,M}(X, Y) : Y \not\leq_T X^{(2M+2)}\}.$$

Then the Turing ordinal of $\mathbb{K}_{N,M}$ is $2M + 2$ and the back-and-forth ordinal of $\mathbb{K}_{N,M}$ is $2N + 3$. In particular, by choosing M and N appropriately, we can produce a class of structures such that the Turing ordinal and back-and-forth ordinal of the class differ by any odd number $d \geq 1$.

Proof. This result really has three parts so we will present each separately.

1. For any $\boldsymbol{d} \geq \mathbf{0}^{(2M+2)}$, there are sets $X, Y \subseteq$ such that $\mathcal{A}_{N,M}(X,Y) \in \mathbb{K}_M$ and $\mathcal{A}_{N,M}(X,Y)$ has $(2M+2)^{th}$ jump degree $\boldsymbol{d}$.

   Fix $\boldsymbol{d} \geq \mathbf{0}^{(2M+2)}$. We will choose our sets X and Y as follows: Let $X = \emptyset$ and by Theorem 6.1.6, choose Y to be any set such that $Y \oplus \emptyset^{(2M+2)} \equiv_T Y^{(2M+2)} \in \boldsymbol{d}$. Then clearly we have $X \leq_T Y^{(2N+2)}$ and $Y \leq_T Y^{(2M+2)}$ uniformly and hence $\deg(Y) \in \mathrm{Spec}(\mathcal{A}_{N,M}(X,Y))$. So we have $\boldsymbol{d} = \deg(Y)^{(2M+2)}$ in the $(2M+2)^{nd}$ jump spectrum of $\mathcal{A}_{N,M}(X,Y)$. Now suppose that $\deg(D) \in \mathrm{Spec}(\mathcal{A}_{N,M}(X,Y))$. By the spectrum result, we must have $Y \leq_T D^{(2M+2)}$, and since $\emptyset^{(2M+2)} \leq_T D^{(2M+2)}$ as well, we have $Y^{(2M+2)} \equiv_T Y \oplus \emptyset^{(2M+2)} \leq_T D^{(2M+2)}$. So $\boldsymbol{d} = \deg(Y)^{(2M+2)}$ is a lower bound for the $(2M+2)^{nd}$ jump spectrum of $\mathcal{A}_{N,M}(X,Y)$.

2. No $\mathcal{A} \in \mathbb{K}_{N,M}$ can have a $(2M+1)^{st}$ jump degree (and hence a k^{th} jump degree for any $k < 2M+2$).

   Fix $\mathcal{A} \in \mathbb{K}_{N,M}$. Then $\mathcal{A} \cong \mathcal{A}_{N,M}$ for some X, Y satisfying $Y \not\leq_T X^{(2M+2)}$. It follows from Theorem 7.2.3 that the set $\{D^{(2M+1)} : D \in \mathrm{Spec}(\mathcal{A})\}$ cannot have a least degree and hence the structure $\mathcal{A}$ cannot have a $(2M+1)^{st}$ jump degree.

3. The back-and-forth ordinal of $\mathbb{K}_{N,M}$ is $2N+3$.

   It follows directly from Corollary 6.2.5 that there are only countably many pairs of the form $(\mathbb{Z}^K \cdot \sigma^*(X \oplus X), \vec{a})$ up to $\equiv_k$ equivalence for any $k \leq 2K+3$. It follows that there are only countably many pairs of the form $(\mathcal{A}_{N,M}(X,Y), \vec{a})$ up to $\equiv_{2N+2}$ equivalence. So the back-and-forth ordinal of the class is at least $2N+3$.

   If $X_1 \neq X_2$ then there is a Σ^c_{2N+2} formula that separates $\mathcal{A}_{N,M}(X_1,Y)$ and $\mathcal{A}_{N,M}(X_2,Y)$ for any choice of $Y \subseteq \omega$. As there are uncountably many orderings in $\mathbb{K}_{N,M}$ having different "first sets X", it follows that the back-and-forth ordinal of the class is exactly $2N+3$.

This completes the proof of the main statement of the Theorem. If we fix any odd integer $d \geq 1$ then there are infinitely many choices for a class where the two ordinals differ by d. We will choose the least complicated one: Let $N = 0$ and let $M = (d+1)/2$. Then the back-and-forth ordinal of $\mathbb{K}_{0,\frac{d+1}{2}}$ is 3 and the Turing ordinal is $d+3$.

□

7.2.2 Axiomatizations

Each of the classes defined in the previous section are Borel. Here we will provide an axiomatization of each class.

Recall that for each $n \in \omega$ we have a formula, denoted by $\Sigma_n^{\oplus}(x, y)$, such that, for any linear ordering $\mathcal{A}$ and any $a, b \in \mathcal{A}$ we have $\mathcal{A} \models \Sigma_n^{\oplus}(a, b)$ if and only if between a and b there lies an ordering of the form $\mathbb{Z}^n \cdot \sigma^*(X \oplus \overline{X})$ for some $X \subseteq \omega$. We can axiomatize an ordering of the form $\mathcal{A}_{N,M}(X, Y)$ with the Σ_{2M+5}^c formula $\big(\exists x_0 < x_1 < x_2\big)\Theta_{N,M}(x_0, x_1, x_2)$ where

$$\Theta_{N.M} := (\forall y)(x_0 \leq y \leq x_2) \wedge \bigwedge_{i=0,1,2} (\forall y)(\neg S(y, x_i) \wedge \neg S(x_i, y)) \wedge \Sigma_N^{\oplus}(x_0, x_1) \wedge \Sigma_M^{\oplus}(x_1, x_2).$$

To axiomatize the property of $Y \not\leq_T X^{(2M+2)}$ we need the following:

Observe that for any $\mathcal{A}_{N,M}(X, Y)$, we have

$$r \in X \Leftrightarrow 2r \in X \oplus \overline{X} \Leftrightarrow \text{There is a maximal } \mathbb{Z}^N \cdot (2r + 1) \text{ block in } \mathcal{A}_{N,M}(X, Y)$$

and recall that the last statement can be described by a Σ_{2N+3}^c formula. Similarly for $r \notin X$. Similarly, there are Σ_{2M+3}^c formulas defining $r \in Y$ and $r \notin Y$ in an ordering of the form $\mathcal{A}_{N,M}(X, Y)$. Using the same techniques from section 7.1.2, we can define a $\Pi_{2(N+M)+7}^c$ formula with the meaning "$Y \not\leq_T X^{(2M+2)}$". This is limiting formula for the axiomatization in terms of complexity.

Remark 7.2.10. The class with the simplest axiomatization corresponds to $N = 0$ and $M = 1$. This produces an example of a Π_9^c-axiomatizable class with Turing ordinal 4 and back-and-forth ordinal 3. More generally, if we fix $N = 0$ and let M vary, we have a Π_{2M+7}-axiomatizable class, $\mathbb{K}_{0,M}$, having back-and-forth ordinal 3 and Turing ordinal $2M + 2$.

Chapter 8

Summary and Open Questions

For a summary of the classes of structures discussed in this thesis see the table below.

Class of structures	Turing ordinal	Back-and-forth ordinal
Models of PA	1	1
$\mathbb{K}_W$	DNE	1
Equivalence structures	1	2
Linear orderings	2	3
T_n	$n+2$	$n+3$
$\mathbb{K}_{2m+2}$	$2m+2$	$2m+3$
$\mathbb{K}_{2m+3}$	$2m+3$	$2m+3$
$\mathbb{K}_{N,M}$	$2M+2$	$2N+3$
$\mathbb{K}_N$	ω	$2N+3$
Boolean algebras	ω	ω
$\mathbb{K}_{\omega+1}$	$\omega+1$	$\omega+1$
$\mathbb{K}_{\omega+2}$	$\omega+2$	$\omega+1$

In every class of orderings presented in Chapters 5 and 6, whether finitely axiomatizable or not, a higher Turing ordinal corresponded to a more complex axiomatization of the theory. If $m < n$ then the natural axiomatization of T_n uses formulas with more quantifiers than that of T_m. For each $m \geq 0$, the orderings $\mathbb{Z}^m \cdot \sigma^*(X)$ from Section 6.3.1 are Π^c_{2m+5}-axiomatizable. The corresponding classes of orderings, $\mathbb{K}_{2m+2}$ and $\mathbb{K}_{2m+3}$, defined in Section 6.1.1 and 6.1.2 have Turing ordinals $2m+2$ and $2m+3$ respectively. Axiomatizing the orderings $\nu^\omega(X)$ from Section 6.3.2 requires Π^c_α formulas for infinite α and the Turing ordinals of the associated classes are $\omega + 1$ and $\omega + 2$. In every case, a more complex axiomatization corresponded to a higher Turing ordinal. This is not true in general as the theory of Boolean algebras is Π^c_1-axiomatizable while the Turing ordinal is infinite. In every case mentioned in Chapters 5 and 6, the rise in the Turing ordinal was coupled with a similar rise in the back-and-forth ordinal.

In Chapter 6 we proved that for each computable α there is a class of structures having Turing ordinal α. It would be of interest to know how complicated an axiomatization of a theory must be in order to realize a Turing ordinal of a particular $\alpha \geq \omega$. It is likely that all of the classes in Chapter 6 are Borel, but this needs to be checked. As with the previous examples, the classes with larger Turing ordinal will also be accompanied by more complex axiomatizations.

In Chapter 7 we set out to show that, for Borel classes of structures, the Turing ordinal and back-and-forth ordinal can be far apart. In Section 7.1 we saw the first example of a class with finite back-and-forth ordinal but infinite Turing ordinal. In Section 7.2 we proved that the Turing ordinal and the back-and-forth ordinal can be arbitrarily far apart, even for Borel classes. Given an odd integer $d \geq 1$, Theorem 7.2.9 provides a Π^c_{d+8}-axiomatizable class of linear orderings where the Turing ordinal and the back-and-forth ordinal differ by d. The success of the results in Chapter 7 relied on the fact that the structures in the classes $\mathbb{K}_{N,M}$ essentially behave as two disjoint structures: one part ensuring that we realize enough types to keep the back-and-forth ordinal low, and the other part forcing a high Turing ordinal. This method should work in more generality.

We are left with the following open questions:

Question 1: Is there a finitely axiomatizable — i.e. axiomatizable via finitely many first-order formulas — class of structures with Turing ordinal equal to some $\alpha > \omega$?

Question 2: Is there a finitely axiomatizable class of structures with the Turing ordinal strictly larger than the back-and-forth ordinal?

Question 3: What is the least $n \in \omega$ such that there is a Π^c_n-axiomatizable class of structures with the Turing ordinal strictly larger than the back-and-forth ordinal? We currently have a Π^c_9-axiomatizable class with this property.

Question 4: What conditions (if any) can one put on the complexity of the axiomatization of a class of structures in order to ensure that the Turing ordinal and the back-and-forth ordinal of the class are close?

References

[1] C. J. Ash, C. G. Jockusch, Jr., and J. F. Knight. Jumps of orderings. *Trans. Amer. Math. Soc.*, 319(2):573–599, 1990. 30, 48, 49, 50, 51, 52, 53, 54, 85, 97

[2] C. J. Ash and J. Knight. *Computable structures and the hyperarithmetical hierarchy*, volume 144 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000. 2, 4, 5, 19, 20, 22, 32, 33, 34, 37

[3] Wesley Calvert, Douglas Cenzer, Valentina Harizanov, and Andrei Morozov. Effective categoricity of equivalence structures. *Ann. Pure Appl. Logic*, 141(1-2):61–78, 2006. 8, 9

[4] Wesley Calvert, Valentina Harizanov, and Alexandra Shlapentokh. Turing degrees of isomorphism types of algebraic objects. *J. Lond. Math. Soc. (2)*, 75(2):273–286, 2007.

[5] Richard J. Coles, Rod G. Downey, and Theodore A. Slaman. Every set has a least jump enumeration. *J. London Math. Soc. (2)*, 62(3):641–649, 2000. 38

[6] S. Barry Cooper. *Computability theory*. Chapman & Hall/CRC, Boca Raton, FL, 2004. 22

[7] Rod Downey and Carl G. Jockusch. Every low Boolean algebra is isomorphic to a recursive one. *Proc. Amer. Math. Soc.*, 122(3):871–880, 1994. 43, 44

[8] Rodney Downey and Julia F. Knight. Orderings with αth jump degree $\mathbf{0}^{(\alpha)}$. *Proc. Amer. Math. Soc.*, 114(2):545–552, 1992.

[9] Andrey Frolov, Iskander Kalimullin, and Russell Miller. Spectra of algebraic fields and subfields. In *Mathematical theory and computational practice*, volume 5635 of *Lecture Notes in Comput. Sci.*, pages 232–241. Springer, Berlin, 2009.

[10] Su Gao. *Invariant descriptive set theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.

[11] S. S. Gončarov. The problem of the number of non-self-equivalent constructivizations. *Algebra and Logic*, 19:401–414, 1980. (English translation). 8

[12] S. S. Gončarov. Groups with a finite number of constructivizations. *Dokl. Akad. Nauk SSSR*, 256(2):269–272, 1981. 7

[13] S. S. Gončarov and V. D. Dzgoev. Autostability of models. *Algebra i Logika*, 19(1):45–58, 132, 1980. 7

[14] S. S. Goncharov, A. V. Molokov, and N. S. Romanovskiĭ. Nilpotent groups of finite algorithmic dimension. *Sibirsk. Mat. Zh.*, 30(1):82–88, 1989. 8

[15] Sergey S. Goncharov, Steffen Lempp, and Reed Solomon. The computable dimension of ordered abelian groups. *Adv. Math.*, 175(1):102–143, 2003. 7

[16] Kenneth Harris and Antonio Montalbán. On the n-back-and-forth types of Boolean algebras. *Trans. Amer. Math. Soc.*, 364(2):827–866, 2012.

[17] Denis R. Hirschfeldt, Bakhadyr Khoussainov, Richard A. Shore, and Arkadii M. Slinko. Degree spectra and computable dimensions in algebraic structures. *Ann. Pure Appl. Logic*, 115(1-3):71–113, 2002. 8

[18] Carl G. Jockusch, Jr. and Robert I. Soare. Boolean algebras, Stone spaces, and the iterated Turing jump. *J. Symbolic Logic*, 59(4):1121–1138, 1994. 30, 37, 43

[19] J. Johnson, J. F. Knight, V. Ocasio, and S. VanDenDriessche. An example related to Gregory's theorem. *Arch. Math. Logic*, 52(3-4):419–434, 2013. 50

[20] J. F. Knight. Degrees of models. In *Handbook of recursive mathematics, Vol. 1*, volume 138 of *Stud. Logic Found. Math.*, pages 289–309. North-Holland, Amsterdam, 1998. 32, 38

[21] Julia F. Knight. Degrees coded in jumps of orderings. *J. Symbolic Logic*, 51(4):1034–1042, 1986. 3, 30, 34, 37, 38, 73

[22] O. V. Kudinov. An integral domain with finite algorithmic dimension. Unpublished manuscript. 8

[23] P. LaRoche. Recursively presented boolean algebras. *Notices Amer. Math. Soc.*, 24:A–552, research announcement, 1977. 7

[24] Steffen Lempp, Charles McCoy, Russell Miller, and Reed Solomon. Computable categoricity of trees of finite height. *J. Symbolic Logic*, 70(1):151–215, 2005.

[25] John M. Macintyre. Transfinite extensions of Friedberg's completeness criterion. *J. Symbolic Logic*, 42(1):1–10, 1977. 50

[26] G. Metakides and A. Nerode. Effective content of field theory. *Ann. Math. Logic*, 17(3):289–320, 1979. 7

[27] Antonio Montalbán. Coding and definability in computable structures. 2010. Lecture notes, submitted for publication. 23, 31, 36

[28] Antonio Montalbán. Counting the back-and-forth types. *J. Logic Comput.*, 22(4):857–876, 2012. 19, 20, 21, 23, 30, 37, 43, 46

[29] André Nies. *Computability and randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009. 2

[30] A. T. Nurtazin. Strong and weak constructivizations, and enumerable families. *Algebra i Logika*, 13:311–323, 364, 1974. 7

[31] J. B. Remmel. Recursive isomorphism types of recursive Boolean algebras. *J. Symbolic Logic*, 46(3):572–594, 1981. 7

[32] J. B. Remmel. Recursively categorical linear orderings. *Proc. Amer. Math. Soc.*, 83(2):387–391, 1981. 7

[33] Linda Jean Richter. Degrees of structures. *J. Symbolic Logic*, 46(4):723–731, 1981. 30, 37, 43

[34] Linda Jean Chiarulli Richter. *DEGREES OF UNSOLVABILITY OF MODELS*. ProQuest LLC, Ann Arbor, MI, 1977. Thesis (Ph.D.)–University of Illinois at Urbana-Champaign.

[35] Alan L. Selman. Arithmetical reducibilities. I. *Z. Math. Logik Grundlagen Math.*, 17:335–350, 1971. 22

[36] Jack H. Silver. Counting the number of equivalence classes of Borel and coanalytic equivalence relations. *Ann. Math. Logic*, 18(1):1–28, 1980. 23

[37] Robert I. Soare. *Recursively enumerable sets and degrees*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987. A study of computable functions and computably generated sets.

[38] S. Tennenbaum. Non-archimedean models for arithmetic. *Notices Amer. Math. Soc.*, 6:270, 1959.