Control of Hysteresis in the Landau–Lifshitz Equation

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Applied Mathematics

Waterloo, Ontario, Canada, 2013

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

There are two main tools for determining the stability of nonlinear partial differential equations (PDEs): Lyapunov Theory and linearization. The former has the advantage of providing stability results for nonlinear equations directly, while the latter considers the stability of linear equations and then further justification is needed to show the linear stability implies local stability of the nonlinear equation. Linearization has the advantage of investigating stability on a simpler equation; however, the justification can be difficult to prove.

Both Lyapunov Theory and linearization are applied to the Landau–Lifshitz equation, a nonlinear PDE that describes the behaviour of magnetization inside a magnetic object. It is known that the Landau-Lifshitz equation has an infinite number of stable equilibrium points. We present a control that forces the system from one equilibrium to another. This is proved using Lyapunov Theory. The linear Landau–Lifshitz equation is also investigated because it provides insight to the nonlinear equation. The linear model is shown to be well–posed and its eigenvalue problem is solved. The resulting eigenvalues suggest an appropriate control for the nonlinear Landau–Lifshitz equation. Mathematically, the control causes the initial equilibrium to no longer be an equilibrium and the second point to be an asymptotically stable equilibrium point. This implies the magnetization has moved to the second equilibrium and hence the control objective is successfully achieved.

The existence of multiple stable equilibria is closely related to hysteresis. This is a phenomenon that is often characterized by a looping behaviour; however, the existence of a loop is not sufficient to identify hysteretic systems. A more precise definition is required, which is presented, and applied to the Landau–Lifshitz equation (both linear and nonlinear) to establish the presence of hysteresis.

Acknowledgements

First and foremost, I thank my supervisor, Professor Kirsten Morris, for providing me with the guidance, support and resources necessary throughout my degree. I would also like to thank the members of my committee, Professors Stephen Campbell, Brian Ingalls, Daniel Miller and David Siegel, for their helpful insights, support and time.

Thank you to my family, friends and colleagues for their encouragement, stimulating conversations and fun adventures. I especially want to thank the members of my research group (both past and present) for their time and many clever and fruitful ideas.

And a very special thank you to my best friend and partner in life, Andrew Beltaos, whose kindness, wisdom and humour make every day a better one.

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Notation

$\mathbb{C}$	Complex numbers.
$\mathbb{R}$	Real numbers.
$\mathbf{e}_1$	Standard basis vector, $\mathbf{e}_1 = (1, 0, 0)$.
$\mathbf{e}_2$	Standard basis vector, $\mathbf{e}_2 = (0, 1, 0)$.
$\mathbf{e}_3$	Standard basis vector, $\mathbf{e}_3 = (0, 0, 1)$.
$\mathbf{m}$	Magnetization, $\mathbf{m}(x, t) = (m_1(x, t), m_2(x, t), m_3(x, t))$.
$\mathbf{m}_x$	Partial derivative with respect to x; that is, $\mathbf{m}_x(x, t) = (m_1'(x, t), m_2'(x, t), m_3'(x, t))$.
$\dot{\mathbf{m}}$	Partial derivative with respect to t; that is, $\dot{\mathbf{m}}(x, t) = (\dot{m}_1(x, t), \dot{m}_2(x, t), \dot{m}_3(x, t))$.
$\mathcal{L}_2^3$	$\mathcal{L}_2^3 = \mathcal{L}_2[0, L] \times \mathcal{L}_2[0, L] \times \mathcal{L}_2[0, L]$ with norm $\|\| \cdot \|\|_{\mathcal{L}_2^3}$ and inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}_2^3}$.
$\mathcal{L}_2[0, L]$	Space of square integral functions with norm $\|\| \cdot \|\|_{\mathcal{L}_2}$ and inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}_2}$.
ν	Damping parameter in the Landau–Lifshitz equation.
$\times$	Cross product.
$\|\| \cdot \|\|_{op}$	Operator norm.
$\| \cdot \|$	Absolute values.
$\|\| \cdot \|\|_2$	Euclidean norm.
Y	Hilbert space.
Z	Hilbert space.

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Chapter 1

Introduction

The Landau–Lifshitz equation describes the energy interactions between a magnetic material and the effect an applied external magnetic field has on the magnetization. As a physical example, an external magnetic field is generated by wrapping some coil around a nanowire (Carbou *et al.* [26]). Nanostructures are found inside memory storage devices such as hard disks. The emergence of magnetic materials in nanostructures is of growing interest in the nanotechnology sector. Recent advances allow for more accurate experimental research with nanostructures (Cowburn *et al.* [29], Noh *et al.* [69]). Because of this, theoretical results on the control of stability is necessary, yet control results are not well developed.

The Landau–Lifshiftz equation is a nonlinear partial differential equation (PDE). It is known that the Landau–Lifshitz equation has multiple stable equilibria (Guo and Ding [40, Section 6.1.1]). We present a control which forces the dynamics in the Landau–Lifshitz equation to move from one equilibrium to another. In particular, the control causes the initial equilibrium to no longer be an equilibrium of the controlled system and the second point to be an asymptotically stable equilibrium point of the controlled system. This provides a framework for controlling mathematical models that exhibit multiple equilibria.

Investigating the linear Landau–Lifshitz equation provides insight in designing the appropriate control for the original nonlinear equation. The linear model is shown to be well–posed and its eigenvalue problem is solved. In particular, the operator associated to the linear Landau–Lifshitz equation generates an analytic semigroup and hence the eigenvalues of the linear operator determines the stability of the linear system. The eigenvalues also suggest that a constant control is appropriate for the nonlinear Landau–Lifshitz equation. This control is shown to be successful using Lyapunov theory.

Lyapunov theory applies stability analysis directly to a nonlinear partial differential equation (PDE). Lyapunov theory is commonly applied to nonlinear ordinary differential equations (ODEs) (Khalil [50, Chapter 4], Vidyasagar [89, Chapter 5]) but it can be similarly applied to PDEs, whether linear or nonlinear. We present Lyapunov theory for invariant sets, which is nearly identical to the well–known version for equilibrium points, but not as commonly applied. The one–dimensional heat equation, which we discuss in detail, is an ideal example that illustrates Lyapunov theory for PDEs and invariant sets. It also demonstrates how boundary conditions significantly affect stability. The main disadvantage of Lyapunov theory is that the result requires finding an appropriate Lyapunov function, which often requires guesswork.

An alternative approach to Lyapunov theory is linearization. It has the advantage of working with a simpler equation; namely, a linear equation. However, there needs to be a justification that the behaviour of the linearized equation implies the same behaviour as the original nonlinear equation. For ODEs or more generally, finite–dimension, the justification is well–established in the form of the Lyapunov's Indirect Method (Khalil [50, Theorem 4.7]). For PDEs or more generally, infinite–dimension, an equivalent result is not available. This justification is sometimes overlooked, dismissed or even trivialized. We describe a framework for infinite–dimensional systems that allows the stability of the linearized equation to imply the same type of stability for the original nonlinear equation. Moreover, we present a nonlinear system which is not asymptotically stable while its linearization is.

The stability for infinite–dimensional systems is far more complex than for finite–dimensional systems. For instance, the stability of infinite–dimensional systems depends heavily on the choice of the norm, while for finite–dimensional systems the choice of the norm is insignificant since all norms are equivalent in finite–dimensions. Because of these complexities, determining the stability of a PDE is challenging, especially if it is nonlinear. Furthermore, because of the nonlinearity, control results are also hard to achieve. An example which faces this challenge is the nonlinear Saint–Venant equations (Bastin *et al.* [12]). The Saint–Venant equations model the flow of water along a channel. Gates are placed along the channel to control the flow of the water as shown in Figure 1.1. The Saint–Venant equations are a system of coupled nonlinear equations where each equation represents one gated region along the channel. The gates are the physical controls and hence the controls are part of the boundary conditions of the Saint–Venant equations. Using Lyapunov theory, conditions on the control parameters are determined which provide exponential stability of the equilibrium solutions but only for the linear Saint–Venant equations. This result cannot necessarily be applied to the nonlinear Saint–Venant equations. The stability of nonlinear reaction diffusion equations are also hard to determine (Dramé *et al.* [34, 55]). In the work by Dramé *et al.*, the nonlinear equations are linearized and their eigenvalues are calculated as a way of determining the stability of the equilibrium solutions, but this does not necessarily imply the same stability of the original nonlinear equations. Our stability and control results pertain to the original nonlinear Landau–Lifshitz equation.

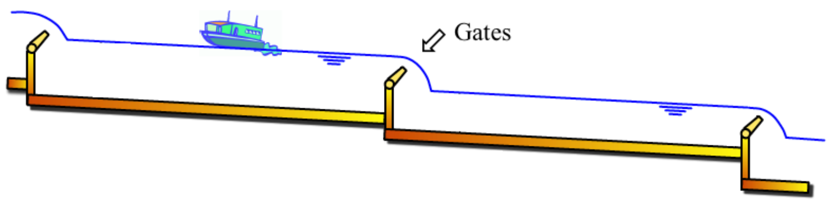


Figure 1.1: Flow of water along a channel. Gates placed along the channel control the flow of water. These dynamics are governed by the Saint–Venant equations. This illustration is taken from Bastin *et al.* [12] and has been modified.

Nonlinear PDEs often have multiple equilibria. The existence of multiple equilibria is closely related to the elusive but often occurring phenomenon known as hysteresis, which the Landau–Lifshitz equation exhibits. By elusive, we mean there is no rigorous definition of hysteresis. Loosely, hysteresis is associated with a looping behaviour in the input–output map; however, merely a looping behaviour is not sufficient to define hysteresis. The shape of the loop varies depending on the example or can even differ within the same example. This is one of the reasons why hysteresis is so difficult to define. Mathematically, hysteresis is often defined using operators (Brokate and Sprekels [20, Chapter 2], Valadkhan *et al.* [86]). The hysteresis operator is included as an additional term in the mathematical model, which can complicate the equation. As well, the mathematical definition of a hysteresis operator can include operators which are not hysteretic (Morris [67]).

It follows that a more fundamental approach in defining hysteresis is needed. We consider two such definitions. The first definition states that a system is hysteretic if it exhibits persistent looping behaviour as the frequency of the input signal approaches zero (Bernstein and Oh [70]). We show that the Landau–Lifshitz equation (both linear and nonlinear) satisfies this definition. The presence of hysteresis in the Landau–Lifshitz equation is usually demonstrated by a looping behaviour either experimentally (Noh *et al.* [69], Suess *et al.* [82]) or numerically (Wiele [87], Yang and Zhao [100]). Looping at a fixed frequency is actually not enough to define hysteresis. The second definition states a system exhibits hysteresis if it has multiple stable equilibria and fast dynamics compared to the rate at which inputs are varied (Morris [67]). This definition is closely related to the looping behaviour associated with hysteresis. Both the linear and nonlinear Landau–Lifshitz equation have multiple stable equilibria. It is widely regarded that nonlinearity of the dynamical system is essential for a system to be hysteretic (see for example Bernstein and Oh [70]). The fact that the linear Landau–Lifshitz equation exhibits hysteresis highlights that the existence of multiple equilibria equilibrium, and not nonlinearity, is crucial for hysteretic behaviour. A classic second order ODE example is also presented to illustrate this point.

We present control that forces the dynamics of the Landau–Lifshitz equation to move from one stable equilibrium to another. Essentially this means controlling the hysteresis arising in the Landau–Lifshitz equation, which is useful since the looping behaviour of hysteresis means one input can lead to more than one output. Moreover, the control can force any arbitrary magnetization to any arbitrary stable equilibrium point. This means the control results are global control results.

Our controller design can be applied to other systems that exhibit hysteresis. An example is a chemical reactor model governed by nonlinear diffusion reaction equations, which has been shown to have multiple stable equilibria and exhibit hysteresis (Jensen and Ray [45], Mancusi *et al.* [63]). Hysteresis also arises in many biological models because they often have multiple stable equilibria (Murray [68, Section 1.2]). Freezing and thawing processes are also hysteretic since freezing is not the opposite of thawing and hence follows a different path back than thawing, which leads to the formation of a loop (Alimov *et al.* [5]).

The structure of this thesis is as follows. Chapter 2 discusses the stability of PDEs. In particular, linearization and Lyapunov theory are presented. Chapter 3 defines hysteresis and how to identify it. An illustrative example of a second order ODE is considered. Chapters 4 and 5 focus on the Landau–Lifshitz equation. To begin, a literature review is presented. The thesis then focuses on the stability, hysteresis and control of the Landau–Lifshitz equation and the linear Landau–Lifshitz equation. The control chosen is shown to be successful in Chapter 5. In the final chapter, a brief summary and some extensions are discussed.

Chapter 2

Stability

Stability analysis of infinite–dimensional systems is far more complex than finite–dimensional systems. Well–known results for finite–dimensions do not always generalize to infinite–dimensions.

In finite–dimensions the local stability of a nonlinear system can often be determined from its corresponding linear system. The result which justifies this is Lyapunov's indirect method (Khalil [50, Theorem 4.7]). The theorem states that near an equilibrium point, the nonlinear system has the same stability as the linear system. This applies to all linear systems except those with eigenvalues that have zero real part. Unfortunately, Lyapunov's indirect method does not extend to infinite–dimensions. We will discuss justification of linearization for infinite–dimensional systems in Section 2.1.

In finite–dimensions, since all norms are equivalent, the choice of norm does not play a significant role in stability results; however, for infinite–dimensions, stability properties depend heavily on the choice of the norm which depends on the infinite–dimensional space. Let Z be a Hilbert space with norm $||\cdot||_Z$ and inner product $\langle \cdot, \cdot \rangle_Z$. We use this notation throughout the thesis. Consider the linear abstract Cauchy problem

$$\dot{z}(t) = Az(t), \qquad z(0) = z_0 \in D(A) \tag{2.1}$$

where A is a linear operator and $D(A)$ is the domain of A. Suppose A generates the semigroup, $T(t)$, for all $t \geq 0$. The dot notation denotes differentiation with respect to time. Define $\mathbb{R}_+ = [0, \infty)$.

Definition 2.1. (Luo *et al.* [62, Definition 2.1])
Let $T(t) : Z \to Z$ be a family of bounded linear operators for $t \in \mathbb{R}_+$. Then $T(t)$ is a

strongly continuous semigroup (C_0*-semigroup*) if

(i) $T(0) = I$

(ii) $T(t+s) = T(t)T(s)$ for $s, t \in \mathbb{R}_+$

(iii) $\lim_{t\to 0^+} ||T(t)z_0 - z_0||_Z = 0$ for all $z_0 \in Z$.

An operator $T(t)$ that satisfies (iii) is said to be *strongly continuous* at 0. In fact, $T(t)$ is strongly continuous on all $t \geq 0$ since $T(t)$ is a linear and bounded operator. A detailed proof can be found in Curtain and Zwart [31, Theorem 2.1.6b].

Semigroups are closely related to solutions of abstract Cauchy problems. If A generates a semigroup, $T(t)$, then the solution to (2.1) is

$$z(t) = T(t)z_0.$$

There are different types of semigroups. Denote the operator norm by $||\cdot||_{op}$.

Definition 2.2. (Banks [10, Definition 6.1])
A semigroup, $T(t)$, on Z is called an *analytic semigroup* if $t \to T(t)z$ is analytic for each z in Z.

Definition 2.3. (Luo *et al.* [62, Definition 2.18])
A C_0-semigroup, $T(t)$, on Z is *uniformly bounded* if $||T(t)||_{op} \leq M$ where $M \geq 1$. If $M = 1$, then $T(t)$ is a *contraction semigroup*.

Definition 2.4. (Luo *et al.* [62, Definition 3.1])
A C_0-semigroup, $T(t)$, on Z is *asymptotically stable* (or sometimes called *strongly stable*) if for every z_0, $||T(t)z_0||_Z \to 0$ as $t \to \infty$.

Definition 2.5. (Luo *et al.* [62, Definition 3.1])
A C_0-semigroup, $T(t)$, on Z is *exponentially stable* if there exists positive constants M and σ such that $||T(t)||_{op} \leq Me^{-\sigma t}$ for $t \geq 0$.

In finite–dimensions, asymptotic stability and exponential stability are equivalent. This is not necessarily true in infinite–dimensions. A semigroup that is asymptotically stable but not exponentially stable is illustrated in example 3.2 in Luo *et al.* [62].

In finite–dimensions, if all the eigenvalues of A are in the left-half plane, then A generates an exponentially stable semigroup; however, this is not always true for infinite–dimensions. Luo *et al.* [62, Example 3.3] and Pazy [71, Example 4.4.2] present an operator whose spectrum is contained in the left-half plane but the semigroup is not exponentially stable. This leads to another concept unique to infinite–dimensional space known as the *spectrum–determined growth assumption* (Curtain and Zwart [31, Section 5.1], Luo *et al.* [62, Section 3.2]). The spectrum–determined growth assumption holds if

$$\omega_0 = \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\} \tag{2.2}$$

where $\sigma(A)$ is the spectrum of A and

$$\omega_0 = \inf_{t>0} \frac{\log ||T(t)||_{op}}{t}. \tag{2.3}$$

Equation (2.2) helps to identify exponential stability via the spectrum of A for infinite–dimensional systems that satisfy the spectrum–determined growth assumption. Analytic semigroups and delay equations satisfy the spectrum–determined growth assumption (see Luo *et al* [62, Corollary 3.1.4], Curtain and Zwart [31, Theorem 5.1.7], respectively).

There are a number of theorems that show A generates a semigroup. We present two which have relatively tractable conditions. Both theorems are applied to the linear Landau–Lifshitz equation in Section 4.3 to show the problem is well–posed.

Theorem 2.6. [Lumer–Phillips Theorem] (Pazy [71, Corollary 1.4.4].)
If A is a densely defined closed linear operator satisfying

$$\text{Re}\langle Az, z\rangle_z \leq 0 \qquad \text{for all } z \in D(A) \tag{2.4}$$

$$\text{Re}\langle A^*z, z\rangle_z \leq 0 \qquad \text{for all } z \in D(A^*), \tag{2.5}$$

then it generates a contraction semigroup.

The requirement on A and its adjoint, A^*, described in (2.4),(2.5) means that A and A^* are *dissipative* (Pazy [71, Definition 1.4.1]).

Theorem 2.7. (Banks [10, Theorem 6.1], Showalter [78, Theorem 6.1])
Let Y be another Hilbert space with $Y \subset Z$ and, norm and inner product, $|| \cdot ||_Y$ and $\langle \cdot, \cdot \rangle_Y$, respectively. Suppose Y is a dense subset of Z and

$$||y||_Z \leq k||y||_Y \qquad \text{for all } y \in Y$$

for some positive constant k. Suppose $\sigma : Y \times Y \to \mathbb{C}$ satisfies

$$|\sigma(\phi, \psi)| \leq \gamma||\phi||_Y||\psi||_Y \qquad \text{for all } \phi, \psi \in Y \tag{2.6}$$

and

$$\text{Re}\,\sigma(\phi, \phi) \geq \delta||\phi||_Y^2 \qquad \text{for all } \phi \in Y \tag{2.7}$$

for some constants $\gamma, \delta > 0$. Define $A : D(A) \subset Y \to Z$ by

$$D(A) = \{\phi \in Y : \text{there exists } K_\phi > 0 \text{ depending on } \phi \\ \text{such that } |\sigma(\phi, \psi)| \leq K_\phi ||\psi||_Z \text{ for all } \psi \in Y\}$$

and

$$\sigma(\phi, \psi) = \langle -A\phi, \psi \rangle_Z \qquad \text{for all } \phi \in D(A), \psi \in Y.$$

Then $D(A)$ is dense in Z and A generates an analytic semigroup on Z.

The conditions in Theorem 2.7 are also sufficient to show A generates a contraction semigroup. The proof of the theorem is found in Showalter [78, Theorem 6.1]). An example of the heat equation with boundary conditions $w(0,t) = 0, w'(L,t) = 0$ is illustrated in Banks [10, example 6.1]. An immediate consequence of Theorem 2.7 is presented in the following corollary.

Corollary 2.8. (Banks [10, Theorem 6.2])
Suppose all the assumptions of Theorem 2.7 hold except that (2.7) for σ is replaced by

$$\operatorname{Re} \sigma(\phi, \phi) + \lambda_0 ||\phi||_Z^2 \geq \delta ||\phi||_Y^2 \qquad \text{for all } \phi \in Y \tag{2.8}$$

for some $\lambda_0 > 0$, $\delta > 0$. Then defining A as in Theorem 2.7, we have that A is densely defined and is the infinitesimal generator of an analytic semigroup in Z.

Semigroups associated to nonlinear Cauchy problems are defined similarly to their linear counterparts. Consider the nonlinear abstract Cauchy problem

$$\dot{z}(t) = f(z(t)), \qquad z(0) = z_0 \tag{2.9}$$

where $f : Z \to Z$ is a nonlinear operator. Assume (2.9) is well–posed; that is, f generates a semigroup. We also consider semilinear problems

$$\dot{z}(t) = Az + f(z), \qquad z(0) = z_0 \in D(A). \tag{2.10}$$

The domain of A and f are denoted, $D(f)$ and $D(A)$, respectively. Suppose $A + f(\cdot)$ generates the nonlinear semigroup, $F(t)$, then

$$z(t) = F(t)z_0.$$

Definition 2.9. (Michel and Wang [65, Definition 2.9.7])
Let D be a subset of Z. A family of one-parameter nonlinear operators $F(t) : D \to D, t \in$

$\mathbb{R}_+$ is a *nonlinear semigroup* defined on D if

(i) $F(0)z_0 = z_0$ for $z_0 \in D$

(ii) $F(t+s)z_0 = F(t)F(s)z_0$ for $t, s \in \mathbb{R}_+$, $z_0 \in D$

(iii) $F(t)z_0$ is continuous in t on $\mathbb{R}_+$

(iv) $F(t)z_0$ is continuous in z_0 on D.

There are generation theorems available for the nonlinear problem described in (2.9) such as the Crandall–Liggett Theorem (Luo *et al.* [62, Theorem 2.115]). We will not go into further details but for more on nonlinear semigroups, see Luo *et al.* [62, Section 2.9] and Barbu [11].

The stability of semigroups is related to the stability of equilibrium points. Suppose $z(t) \in D(f)$ is a solution of (2.9) and $\tilde{z} \in D(f)$ is an equilibrium point of (2.9); that is, $\tilde{z}$ such that $f(\tilde{z}) = 0$. The equilibrium point can also be defined using the semigroup, $F(t)$, as $\tilde{z} = F(t)\tilde{z}$ for all t.

Definition 2.10. (Walker [93, Definition 3.2])
The equilibrium point $\tilde{z}$ is *(locally) stable* if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $||z(0) - \tilde{z}||_Z < \delta$ implies $||z(t) - \tilde{z}||_Z < \epsilon$ for all $t \geq 0$.

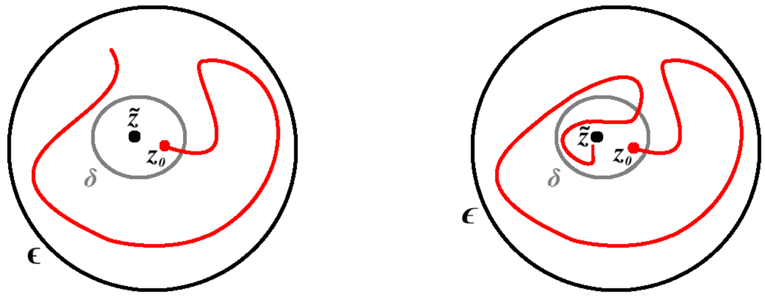
Definition 2.11. (Walker [93, Definition 3.2])
The equilibrium point $\tilde{z}$ is *(locally) asymptotically stable* if $\tilde{z}$ is stable and there exists a $\delta > 0$ such that $||z(0) - \tilde{z}||_Z < \delta$ implies

$$\lim_{t\to\infty} ||z(t) - \tilde{z}||_Z = 0. \tag{2.11}$$

The finite–dimensional versions of Definition 2.10 and 2.11 can be found in Khalil [50, Definition 4.1] and are identical. Clearly, asymptotically stable implies stable but the converse is not true. A pictorial description highlighting the difference between stable and asymptotically stable is displayed in Figure 2.1. Another common type of stability is exponential stability.

Definition 2.12. (Smoller [81, Definition 11.21])
The equilibrium point $\tilde{z}$ is *(locally) exponentially stable* if there exists a $\delta > 0$ and positive numbers k and α such that $||z(0) - \tilde{z}||_Z < \delta$ implies $||z(t) - \tilde{z}||_Z < ke^{-\alpha t}||z(0) - \tilde{z}||_Z$ for all $t \geq 0$.

The stability of invariant sets can also be considered.



(a) Stable (b) Asymptotically Stable

Figure 2.1: Pictorial descriptions of Definition 2.10 and 2.11. Let $\tilde{z}$ and z_0 be an equilibrium and initial condition of (2.9), respectively. (a) If $\tilde{z}$ is stable, then solutions of (2.9) stay within an ϵ–ball for all time. (b) If $\tilde{z}$ is stable and satisfies (2.11), then $\tilde{z}$ is asymptotically stable.

Definition 2.13. (Xu and Yung [98, Definition 2.4])
A set $M \subset Z$ is called an *invariant set* with respect to (2.9) if for each $z(0) \in M$, it follows that $z(t) \in M$ for all $t \geq 0$.

Define $\operatorname{dist}(z, M)$ as the minimum distance from $z \in Z$ to a point in M; that is, for any $z \in Z$,

$$\operatorname{dist}(z, M) = \inf\{||z - y||_Z : y \in M\}.$$

Definition 2.14. (Xu and Yung [98, Definition 2.6])
The invariant set M is *(locally) stable* if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\operatorname{dist}(z(0), M) < \delta$ implies $\operatorname{dist}(z(t), M) < \epsilon$ for all $t \geq 0$.

Definition 2.15. (Xu and Yung [98, Definition 2.6])
The invariant set M is *(locally) asymptotically stable* if M is stable and there exists a $\delta > 0$ such that $\operatorname{dist}(z(0), M) < \delta$ implies

$$\lim_{t \to \infty} \operatorname{dist}(z(t), M) = 0. \tag{2.12}$$

In the above stability definitions, the initial condition depends on δ and hence these notions of stability are local. If stability occurs for any initial state, then we say it is *globally stable* or *globally asymptotically stable.*

Define a set of equilibrium points as an *equilibrium set*, E. By definition, an equilibrium point does not change with respect to time and hence remains in E. It follows that E is an invariant set; that is, every equilibrium set is an invariant set.

There are two main methods for examining the stability of equilibrium points and equilibrium sets: linearization and Lyapunov Theory. The discussion on the linearization of nonlinear infinite–dimensional systems is based on the semilinear form described in (2.10). The justification is not trivial and many examples do not satisfy the necessary conditions. The Landau–Lifshitz equation is one such example. Fortunately, Lyapunov Theory provides a successful alternative.

2.1 Linearization

Requirements on how the stability of a nonlinear infinite–dimensional system can be determined by its corresponding linear system are presented. Three main issues need to be addressed. First, how to differentiate the nonlinear operator. We discuss Fréchet and Gateaux differentiability. Second, a relationship between the semigroup of the linear system and the semigroup of the nonlinear system must be made. This allows for a connection between the stability of a nonlinear equation and its corresponding linearized equation.

We begin by defining Fréchet and Gateaux differentiability.

Definition 2.16. (Hutson *et al.* [43, Definition 4.4.1])
Let $D(f)$ be an open subset of Z and $f : D(f) \to Z$ be an operator. For $h, a \in D(f)$, an operator f is *Fréchet differentiable* at a if there exists a bounded linear operator $df_a : Z \to Z$ such that

$$\lim_{||h||_Z \to 0} \frac{||f(a+h) - f(a) - df_a h||_Z}{||h||_Z} = 0. \tag{2.13}$$

The operator, df_a, is the *Fréchet derivative* of f at a.

Definition 2.17. (Lebedev and Vorovich [60, Definition 3.1.2])
The operator $G : D(G) \subset Z \to Z$ is *Gateaux differentiable at* $a \in D(G)$ if there exists a linear operator $\partial G_a : Z \to Z$ such that

$$\lim_{\epsilon \to 0} \frac{||G(a+\epsilon v) - G(a) - \partial G_a v||_Z}{\epsilon} = 0 \tag{2.14}$$

for every $a + \epsilon v, v \in D(G)$. The operator ∂G_a is called the *Gateaux derivative of* G *at* a.

The precise definition of Gateaux differentiability varies depending on the reference. For example, the Gateaux derivative may be defined as bounded and linear (Joshi and Bose [47, Definition 2.1.1], Kato [48, Section 1]). In Banks [10, Definition 1.1], the Gateaux derivative need not be linear. In comparison, the definition of Fréchet differentiable is consistent. Fréchet differentiability requires uniform convergence while convergence for Gateaux differentiability depends on the direction v. This suggests Fréchet differentiable implies Gateaux differentiable and this is easily proven by setting $f = G$ and $h = \epsilon v$ in equation (2.13) which leads to

$$\lim_{||\epsilon v||_Z \to 0} \frac{||G(a+\epsilon v) - G(a) - dG_a \epsilon v||_Z}{||\epsilon v||_Z} = 0.$$

Since the limit is independent of v,

$$\frac{1}{||v||_Z} \lim_{\epsilon \to 0} \frac{||G(a+\epsilon v) - G(a) - dG_a \epsilon v||_Z}{\epsilon} = 0.$$

Multiplying by $||v||_Z$ and with $\partial G_a v = dG_a v$, we recover (2.14). Therefore, Fréchet differentiable implies Gateaux differentiable and their derivatives are equal (Lebedev and Vorovich [60, Theorem 3.1.1]); however, the converse is not true. For example, the derivative of a linear differentiable operator is just itself; however on the space of square integrable functions, the derivative operator is unbounded and hence is a Gateaux derivative but not a Fréchet derivative.

Consider the nonlinear problem

$$\dot{z}(t) = G(z(t)) \tag{2.15}$$

where G is Gateaux differentiable with Gateaux derivative, ∂G_z, for some $z \in D(G)$. Let $\tilde{z}$ be an equilibrium solution of (2.15) and

$$B_r(\tilde{z}) = \{z \in Z : ||z - \tilde{z}||_Z < r\}$$

for some constant $r > 0$. Define the corresponding linear problem to be

$$\dot{z}(t) = \partial G_{\tilde{z}} z(t)$$

and suppose it generates an exponentially stable semigroup. If there exists a $\lambda_{\tilde{z}} > 0$ and nondecreasing function $L_{\tilde{z}} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$||(I + \lambda \partial G_z)^{-1} v - (I + \lambda \partial G_y)^{-1} v||_Z \leq \lambda ||z_0 - y_0||_Z L_{\tilde{z}}(||v||_Z) \tag{2.16}$$

for $0 < \lambda < \lambda_{\tilde{z}}$ with $y, z \in B_r(\tilde{z}) \cap D(G)$ and $v \in Z$, then $\tilde{z}$ is an exponentially stable equilibrium of (2.15). This result and its proof are found in Kato [48, Theorem 2.1]. In general, the condition described in (2.16) is not easy to check.

We now consider the semilinear equation (2.10), which generates the semigroup, $F(t)$. Temam [84, Section VI.8] investigates when $F(t)$ is Fréchet differentiable, which is useful for showing the stability of the linearized equation of (2.10) has similar stability to (2.10) (see Theorem 2.24 below and al Jamal [3]). The linear operator, $A : D(A) \to Z$ of (2.10) is assumed to be self-adjoint and negative. The nonlinear operator is defined as f mapping from Y to Y' where $Y = D((-A)^{1/2})$ and Y' is its corresponding dual space. For $y \in Y$, define the norms $||y||_Y = ||(-A)^{1/2}y||_Z$ and $||y||_{Y'} = ||(-A)^{-1/2}y||_Z$, write

$$f(z(t)) - f(w(t)) = L(z(t) - w(t)) + Q(z(t) - w(t)) \tag{2.17}$$

where L is a linear bounded operator on Y to Y'. For some $0 < \epsilon \leq 1$ and positive constant c_ϵ that depends on ϵ, assume L satisfies

$$|\langle Lv, v\rangle_Z| \leq (1-\epsilon)||y||_{\mathrm{Y}}^2 + c_\epsilon ||y||_Z^2 \tag{2.18}$$

for all $y \in Y$ and assume Q satisfies

$$||Q(z(t) - w(t))||_{\mathrm{Y'}} \leq k_1 ||z - w||_{\mathrm{Y}}^{1+\sigma_1}$$

for some $k_1 > 0$ and $\sigma_1 > 0$. For every $R > 0$, suppose there exists $0 < \sigma_2 \leq 1$ and constant k_R depending on R such that

$$|\langle f(z) - f(w), z - w\rangle_Z| \leq k_R ||z - w||_Z^{\sigma_2} ||z - w||_V^{2-\sigma_2}$$

for all $z, w \in D(A^{1/2})$ with $||z||_Z \leq R$ and $||w||_Z \leq R$. Given these conditions, $F(t)$ is Fréchet differentiable and its derivative is equal to the semigroup of the linearized equation of (2.10).

In what is to follow, we consider Fréchet differentiability of the nonlinear operator. Consider the semilinear equation in (2.10), which generates the nonlinear semigroup $F(t)$. Suppose the nonlinear operator, f, in (2.10) is Fréchet differentiable. Define the corresponding linear problem to (2.10) as

$$\frac{d\psi}{dt} = A\psi + df_z\psi \tag{2.19}$$

where df_z is the Fréchet derivative of f at z. We are interested in knowing the conditions required on A and f such that the stability of (2.19) implies similar stability of (2.10). We begin by presenting the Mean Value Theorem.

Theorem 2.18. ([Mean Value Theorem] Hutson *et al.* [43, Lemma 4.4.7])
Let $f : Z \to Z$ be an operator. Define for any $z \in Z$ and some positive constant r,

$$N_{z,r} = \{p \in Z : ||p - z||_Z \leq r\}. \tag{2.20}$$

If f is Fréchet differentiable on $N_{z,r}$, then

$$||f(y) - f(z)||_Z \leq \sup_{\eta \in N_{z,r}} ||df_\eta||_{op}||y - z||_Z \qquad \text{for all } y, z \in N_{z,r}$$

where df_η is the Fréchet derivative of f at η.

We will be referring to the neighbourhood, $N_{z,r}$, of z throughout this section.

There are a number of results in the literature that show if the linear system in (2.19) generates an exponentially stable semigroup, then the original semilinear system (2.10) is locally exponentially stable (Kato [48, Corollary 2.2], Henry [41, Section 5.1] and Smoller [81, Section 11.B]). These results require the nonlinear operator, f, in (2.10) to be Fréchet differentiable. Furthermore, f or its derivative is often required to be locally Lipschitz continuous, which is defined as follows.

Definition 2.19. (Belleni-Morante and McBride [13, Definition 3.6])
The operator $f : Z \to Z$ is *locally Lipschitz continuous* on $N_{z,r}$, defined in (2.20), if

$$||f(p) - f(q)||_Z \leq K_{z,r}||p - q||_Z, \qquad \text{for all } p, q \in N_{z,r}.$$

The bound $K_{z,r}$ is called the *Lipschitz constant* and depends on r and z.

Let $\tilde{z}$ be an equilibrium point of (2.10). In Kato [48, Corollary 2.2], A in (2.10) is assumed to generate an exponentially stable semigroup and the nonlinear operator, f, is assumed to be Fréchet differentiable in $D(f)$. For all $z, y \in D(f)$ with $||z||_Z \leq r, ||y||_Z \leq r$ for some $r > 0$, the Fréchet derivative, df, is assumed to satisfy

$$||df_y - df_z||_{op} \leq d(r)||y - z||_Z$$

where $d : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous increasing function. Given these assumptions, Corollary 2.2 in Kato [48] establishes that $\tilde{z}$ is a locally exponentially stable equilibrium of (2.10).

Smoller [81, Section 11.B] shows a similar result to Corollary 2.2 in Kato but with different assumptions. He assumes A in (2.10) generates a semigroup on Z and f is locally

Lipschitz continuous; that is, for all $y, z \in Z$

$$||f(y) - f(z)||_Z \leq k(||y||_Z, ||z||_Z)||y - z||_Z$$

where $k(s_1, s_2)$ is a continuous nonnegative real–valued function that is increasing in s_1 and s_2. Define df as the Fréchet derivative of f. Smoller also requires the mapping $z \to df_z$ to be continuous on Z and the following inequality to hold,

$$||f(z) - f(w) - df_w(z - w)||_Z \leq c||z - w||_Z^2 \qquad \text{for all } z, w \in N_{z,r} \tag{2.21}$$

for some positive constant, c. Given these assumptions, Smoller [81, Theorem 11.17] proves the semigroup, $F(t)$, of (2.10) is Fréchet differentiable with its derivative equal to the semigroup of (2.19). This result is used to show $\tilde{z}$ is a locally exponentially stable equilibrium point of (2.10) if equation (2.19) generates an exponentially stable semigroup (Smoller [81, Theorem 11.22]).

In Theorem 2.23 below, we also prove the semigroup of $F(t)$ of (2.10) is Fréchet differentiable and its Fréchet derivative is equal to the semigroup of (2.19). We require A in (2.10) to be nonpositive and the derivative of f, denoted df, to be locally Lipschitz continuous on $N_{z,r}$ and that df satisfies

$$\sup_{\eta \in N_{z,r}} ||df_\eta||_{op} = K_{z,r} < \infty \tag{2.22}$$

for some constant $K_{z,r} > 0$ that depends on z and r. Given these assumptions, f is locally Lipschitz continuous on $N_{z,r}$ by the Mean Value Theorem, which is a required condition of f in Smoller [81, Section 11.B] and our assumptions on f imply Smoller's condition in (2.21) as illustrated in Lemma 2.21. Furthermore, if df is continuous on Z, which is an assumption in Smoller, then (2.22) is satisfied. The result in Theorem 2.23 and the result in Smoller [81, Section 11.B] require similar assumptions. Depending on the equation, it may be easier to satisfy the conditions in Smoller or those in Theorem 2.23.

A pictorial representation of Theorem 2.23 is depicted in Figure 2.2. Before proving Theorem 2.23, we prove several lemmas needed in the proof. Recall the neighbourhood N of z defined in (2.20) is $N_{z,r}$, which is used throughout the proofs.

Lemma 2.20. (Lebedev and Vorowich [60, Lemma 3.1.2])
Suppose f is Fréchet differentiable on $N_{z,r}$, then

$$f(y) - f(z) = \int_0^1 df_{z+s(y-z)}(y - z)ds \qquad \text{for all } y \in N_{z,r}.$$

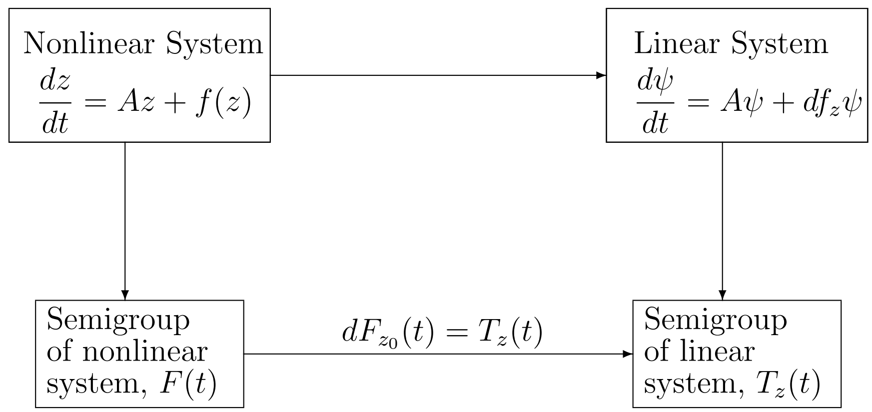


Figure 2.2: Consider the semilinear equation (2.10) with its nonlinear operator, f, Fréchet differentiable, and which generates the semigroup, $F(t)$. The corresponding linearized problem is (2.19). Then $F(t)$ is Fréchet differentiable and its derivative is equal to the semigroup generated by (2.19). See Theorem 2.23.

Proof of Lemma 2.20: Let $q(s) = z + s(y - z)$. Taking the derivative of $f(q(s))$ with respect to s leads to

$$\frac{d}{ds} f(q(s)) = df_{q(s)}(y - z).$$

This is a consequence of the chain rule for Fréchet differentiability (Lebedev and Vorowich [60, Lemma 3.1.1]). Integrating from 0 to 1, we obtain

$$\int_0^1 \frac{df(q(s))}{ds} ds = \int_0^1 df_{q(s)}(y - z) ds.$$

It follows that

$$f(q(1)) - f(q(0)) = \int_0^1 df_{q(s)}(y - z) ds$$

and substituting in q leads to the desired result. □

A similar result and proof of the following lemma can be found in Siegel [79, Section 3.12].

Lemma 2.21. Suppose f is Fréchet differentiable on $N_{z,r}$ and its derivative is locally

Lipschitz continuous on $N_{z,r}$; that is,

$$||df_y - df_z||_Z \leq L_{z,r}||y - z||_Z \text{ for all } y \in N_{z,r}$$

where $L_{z,r}$ is the Lipschitz constant that depends on z and some $r > 0$. Then

$$||f(y) - f(z) - df_z(y - z)||_Z \leq \frac{L_{z,r}}{2}||y - z||_Z^2 \text{ for all } y \in N_{z,r}.$$

Proof of Lemma 2.21: Consider the function $g : [0, 1] \to Z$,

$$g(s) = f(z + s(y - z)) - sdf_z(y - z).$$

By the chain rule,

$$g'(s) = df_{z+s(y-z)}(y - z) - df_z(y - z). \tag{2.23}$$

Consider

$$\int_0^1 g'(s)ds = \int_0^1 df_{z+s(y-z)}(y - z)ds - \int_0^1 df_z(y - z)ds.$$

The second integral evaluates to $f(y) - f(z)$ from Lemma 2.20. The integrand in the third integral is constant with respect to s and hence the integral evaluates to $df_z(y - z)$. It follows that

$$\int_0^1 g'(s)ds = f(y) - f(z) - df_z(y - z)$$

and hence

$$\begin{aligned} ||f(y) - f(z) - df_z(y - z)||_Z &\leq \int_0^1 ||g'(s)||_Z dt \\ &\leq \int_0^1 ||df_{z+s(y-z)} - df_z||_{op} \, ||y - z||_Z ds \qquad \text{by (2.23)}. \end{aligned}$$

Since df_z is locally Lipschitz continuous on $N_{z,r}$, then

$$||f(y) - f(z) - df_z(y - z)||_Z \leq L_{z,r}||y - z||_Z^2 \int_0^1 sds. \quad \square$$

Lemma 2.22. Consider the semilinear equation in (2.10). Suppose the nonlinear operator, f, of (2.10) satisfies the assumptions in Lemma 2.21. For some positive constant $K_{z,r}$ that depends on z and r, assume

$$\sup_{\eta \in N_{z,r}} ||df_\eta||_{op} = K_{z,r} < \infty$$

where df is the Fréchet derivative of f. Assume $Re\langle Az, z\rangle_Z \leq 0$ for all $z \in N_{z,r}$. Let $y, z \in N_{z,r}$ be solutions to (2.10) with corresponding y_0, z_0 initial conditions. Then

$$||y(t) - z(t)||_Z^2 \leq ||y_0 - z_0||_Z^2 e^{2K_{z,r}t}.$$

Proof of Lemma 2.22: Let $y, z \in N_{z,r}$ be solutions to (2.10) and define $h(t) := y(t) - z(t)$. It follows that

$$\frac{dh}{dt} = Ah + f(y) - f(z), \quad h(0) = y_0 - z_0. \tag{2.24}$$

Taking the inner product of (2.24) with h yields

$$\langle \frac{dh}{dt}, h\rangle_Z = \langle Ah, h\rangle_Z + \langle f(y) - f(z), h\rangle_Z.$$

For any $z \in \mathbb{C}$ and its conjugate, $\bar{z}$, we have $z + \bar{z} = 2\text{Re}(z)$ and hence

$$\begin{aligned} 2\text{Re}\langle \frac{dh}{dt}, h\rangle_Z &= \langle \frac{dh}{dt}, h\rangle_Z + \overline{\langle h, \frac{dh}{dt}\rangle}_Z \\ &= \frac{d}{dt}\langle h, h\rangle_Z \\ &= \frac{d}{dt}||h||_Z^2. \end{aligned}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}||h||_Z^2 = \text{Re}\langle Ah, h\rangle_Z + \text{Re}\langle f(y) - f(z), h\rangle_Z.$$

Since $\text{Re}\langle Ah, h\rangle_Z \leq 0$, then

$$\frac{1}{2}\frac{d}{dt}||h||_Z^2 \leq \text{Re}\langle f(y) - f(z), h\rangle_Z$$

and by the Cauchy-Schwarz inequality

$$\frac{1}{2}\frac{d}{dt}||h||_Z^2 \leq ||f(y) - f(z)||_Z ||h||_Z. \tag{2.25}$$

From the Mean Value Theorem (Theorem 2.18), we have

$$||f(y) - f(z)||_Z \leq \sup_{z \in N_{z,r}} ||df_z||_{op} ||y - z||_Z.$$

It follows that

$$||f(y) - f(z)||_Z \leq K_{z,r} ||y - z||_Z \text{ for all } y, z \in N_{z,r}$$

since $K_{z,r} = \sup_{\eta \in N_{z,r}} ||df_\eta||_{op} < \infty$. Equation (2.25) then becomes

$$\frac{d}{dt} ||h||_Z^2 \leq 2K_{z,r} ||h||_Z^2.$$

Integrating with respect to t yields

$$||h(t)||_Z^2 \leq ||h(0)||_Z^2 e^{2K_{z,r} t}.$$

Substituting in $h(t) = y(t) - z(t)$ leads to the desired result. □

Theorem 2.23. Consider the semilinear equation in (2.10), which generates the semigroup $F(t)$, and its corresponding linearized problem in (2.19). Suppose the same assumptions in Lemma 2.22 hold and also that A generates a semigroup. Then (2.19) generates the semigroup, $T_z(t)$, and furthermore,

$$T_z(t) = dF_{z_0}(t)$$

where $dF_{z_0}(t)$ is the Fréchet derivative of $F(t)$ at $z(0) = z_0$ for all $0 \leq t \leq t_f$ for some positive t_f.

Proof of Theorem 2.23: Recall the linearized equation of (2.10) is

$$\frac{d\psi}{dt} = A\psi + df_z \psi, \qquad \psi(0) = y_0 - z_0 =: \psi_0 \tag{2.26}$$

where df_z is the Fréchet derivative of f at z. Since A generates a semigroup and df_z is bounded, then $A + df_z$ generates a C_0-semigroup, $T_z(t)$, and

$$\psi(t) = T_z(t)\psi_0 \tag{2.27}$$

is the unique solution to (2.26) (see Curtain and Zwart [31, Theorem 3.2.1]).

Define $\phi(t) := y(t) - z(t) - \psi(t)$. Taking the time derivative of ϕ leads to

$$\frac{d\phi}{dt} = \frac{dy}{dt} - \frac{dz}{dt} - \frac{d\psi}{dt}.$$

Substituting in (2.10) and (2.26) yields

$$\frac{d\phi}{dt} = Ay + f(y) - Az - f(z) - A\psi - df_z\psi.$$

Replacing with $\phi = y - z - \psi$, it follows that

$$\frac{d\phi}{dt} = A\phi + df_z\phi + f(y) - f(z) - df_z(y - z), \qquad \phi(0) = 0. \tag{2.28}$$

Taking the inner product of (2.28) with ϕ yields

$$\frac{1}{2}\frac{d}{dt}||\phi||_Z^2 = \mathrm{Re}\langle A\phi, \phi\rangle_Z + \mathrm{Re}\langle df_z\phi, \phi\rangle_Z + \mathrm{Re}\langle f(y) - f(z) - df_z(y - z), \phi\rangle_Z$$

and applying Cauchy-Schwarz leads to

$$\frac{1}{2}\frac{d}{dt}||\phi||_Z^2 \leq \mathrm{Re}\langle A\phi, \phi\rangle_Z + ||df_z\phi||\,||\phi||_Z + ||f(y) - f(z) - df_z(y - z)||_Z||\phi||_Z.$$

Applying $\mathrm{Re}\langle A\phi, \phi\rangle_Z \leq 0$ and Lemma 2.21 implies,

$$\frac{1}{2}\frac{d}{dt}||\phi||_Z^2 \leq ||df_z\phi||\,||\phi||_Z + \frac{L_{z,r}}{2}||y - z||_Z^2||\phi||_Z$$

where $L_{z,r}$ is the Lipschitz constant. Since df_z is a linear bounded operator on Z, then

$$\frac{1}{2}\frac{d}{dt}||\phi||_Z^2 \leq M_z||\phi||_Z^2 + \frac{L_{z,r}}{2}||y - z||_Z^2||\phi||_Z$$

where M_z is a positive constant that depends on z and $M_z \neq K_{z,r}$. From this, we have

$$\frac{d}{dt}||\phi||_Z \leq 2M_z||\phi||_Z + L_{z,r}||y - z||_Z^2.$$

Applying Gronwall's inequality (Robinson [73, Lemma 2.8]) with $\phi(0) = 0$ implies

$$||\phi||_Z \leq L_{z,r} e^{2M_z t} \int_0^t e^{-2M_z s} ||y(s) - z(s)||_Z^2 ds.$$

Applying Lemma 2.22 yields

$$||\phi||_Z \leq L_{z,r} e^{2M_z t} \int_0^t e^{2(K_{z,r} - M_z)s} ||y_0 - z_0||_Z^2 ds.$$

For $t \in [0, t_f]$ with t_f any positive constant, it follows that

$$||\phi||_Z \leq L_{z,r} e^{2M_z t_f} ||y_0 - z_0||_Z^2 \int_0^{t_f} e^{2(K_{z,r} - M_z)s} ds.$$

and solving the integrals leads to

$$||\phi(t)||_Z \leq k(t_f) ||y_0 - z_0||_Z^2, \qquad \text{for } t \in [0, t_f]$$

where

$$k(t_f) = \frac{L_{z,r}}{2\,(K_{z,r} - M_z)} (e^{2K_{z,r} t_f} - e^{2M t_f}).$$

Since $\phi = y - z - \psi$, we have

$$||y - z - \psi||_Z \leq k(t_f) ||y_0 - z_0||_Z^2.$$

From (2.27) and recalling that $y(t) = F(t) y_0$ and $z(t) = F(t) z_0$, we obtain

$$||F(t)y_0 - F(t)z_0 - T_z(t)\psi_0||_Z \leq k(t_f) ||y_0 - z_0||_Z^2.$$

Defining $h_0 = y_0 - z_0$ leads to

$$||F(t)(h_0 + z_0) - F(t)z_0 - T_z(t)h_0||_Z \leq k(t_f) ||h_0||_Z^2$$

and hence

$$\lim_{||h_0||_Z \to 0} \frac{||F(t)(h_0 + z_0) - F(t)z_0 - T_z(t)h_0||_Z}{||h_0||_Z} = 0$$

for all $t \in [0, t_f]$. Therefore, (2.13) is satisfied which means $F(t)$ is Fréchet differentiable at z_0 and its derivative is $dF_{z_0}(t) = T_z(t)$ for $t \in [0, t_f]$. $\square$

Recall the usefulness of Theorem 2.23. It can allow the stability of a nonlinear infinite–

dimensional system to be determined by its corresponding linear system. Such a result is found in Desch and Schappacher [33, Theorem 2.1], which we state for completeness.

Theorem 2.24. (Desch and Schappacher [33, Theorem 2.1])
Let $F(t)$ be a nonlinear semigroup and let $\tilde{z}$ be an equilibrium point. Suppose $F(t)$ is Fréchet differentiable at $\tilde{z}$ with Fréchet derivative $dF_{\tilde{z}}(t)$ and $dF_{\tilde{z}}(t)$ is an exponentially stable semigroup, then $\tilde{z}$ is an exponentially stable equilibrium with respect to $F(t)$.

The assumptions needed to satisfy Theorem 2.24 are quite strong. Furthermore, it is not always true that the stability of a linearized equation implies the same stability of the original nonlinear equation. In the following example, we consider a nonlinear system which does not have an asymptotically stable solution while its linearization does. The example is due to Hans Zwart.

Example 2.25. Let ℓ_2 be the space of square summable sequences and $\mathbb{N}$ the set of natural numbers. For any $z(t) = (z_1(t), z_2(t), \dots, z_n(t), \dots) \in \ell_2$ with $n \in \mathbb{N}$, consider

$$\dot{z}_n = -\frac{1}{n} z_n + z_n^2. \tag{2.29}$$

Denote $\tilde{z}_n$ as the equilibrium solutions of (2.29). Then solving

$$0 = -\frac{1}{n}\tilde{z}_n + \tilde{z}_n^2,$$

we find that $\tilde{z}_n = 0$ and $\frac{1}{n}$. This means the full system has an infinite number of equilibria of the form $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \dots) \in \ell_2$ where $\tilde{z}_n = 0$ or $\frac{1}{n}$. That is, the equilibrium set is

$$\left\{ \tilde{z} = (\tilde{z}_1, \tilde{z}_2, \dots) \in \ell_2 \,\middle|\, \tilde{z}_n \in \{0, \frac{1}{n}\} \text{ for } n \in \mathbb{N} \right\}.$$

We show that the linearization of (2.29) at $\tilde{z} = (0, \dots, 0, \dots)$ is asymptotically stable. Denote the nonlinear term in (2.29) as

$$f_n(z_n) = z_n^2,$$

then $f_n'(0) = 0$ and hence the linearization is

$$\dot{w}_n = -\frac{1}{n} w_n. \tag{2.30}$$

Solving, the solution is $w_n(t) = w_n(0)e^{-\frac{1}{n}t}$. It follows that

$$w(t) = (w_1(0)e^{-t}, w_2(0)e^{-\frac{1}{2}t}, \dots) \tag{2.31}$$

and hence for any $w_0 = (w_1(0), w_2(0), \dots) \in \ell_2$,

$$\lim_{t\to\infty} ||w(t)||_{\ell_2}^2 = \lim_{t\to\infty} \sum_{n=1}^{\infty} w_n^2(0)e^{-\frac{2}{n}t} = 0.$$

By Definition 2.4, this implies $w(t)$ is an asymptotically stable solution.

Observe that $w(t)$ in (2.31) is not exponentially stable. Let $T(t)$ be the semigroup of (2.30); that is, $w(t) = T(t)w_0$. For any $t \geq 0$,

$$\begin{aligned} ||T(t)||_{op} &= \sup_{||w_0||_{\ell_2}=1} ||w(t)||_{\ell_2} \\ &= \sup_{||w_0||_{\ell_2}=1} \left(\sum_{n=1}^{\infty} w_n^2(0)e^{-\frac{2}{n}t} \right)^{1/2} \qquad \text{(by (2.31))} \\ &= \sup_{||w_0||_{\ell_2}=1} \left(\lim_{k\to\infty} \sum_{n=1}^{k} w_n^2(0)e^{-\frac{2}{n}t} \right)^{1/2}. \end{aligned}$$

Since $n \leq k$, then $e^{-\frac{2}{n}t} \leq e^{-\frac{2}{k}t}$ for all $t \geq 0$ and hence the exponential is independent of the sum. It follows that

$$\begin{aligned} ||T(t)||_{op} &= \sup_{||w_0||_{\ell_2}=1} \left(\sum_{n=1}^{\infty} w_n^2(0) \lim_{k\to\infty} e^{-\frac{2}{k}t} \right)^{1/2} \\ &= \lim_{k\to\infty} e^{-\frac{1}{k}t} \sup_{||w_0||_{\ell_2}=1} \left(\sum_{n=1}^{\infty} w_n^2(0) \right)^{1/2} \\ &= \lim_{k\to\infty} e^{-\frac{1}{k}t} \\ &= 1 \end{aligned}$$

and hence $T(t)$ is not an exponentially stable semigroup.

We now show the solution to the original nonlinear problem (2.29) is not asymptotically stable. For $z_n \neq 0$, define $y_n = z_n^{-1}$, then $\dot{y}_n = -z_n^{-2}\dot{z}_n$ and equation (2.29) can be rewritten as

$$\dot{y}_n - \frac{1}{n} y_n = -1.$$

Multiplying this equation by $e^{-\frac{1}{n}t}$ leads to

$$\frac{d}{dt}\left(e^{-\frac{1}{n}t} y_n\right) = -e^{-\frac{1}{n}t}.$$

Solving,

$$y_n = n + c_n e^{\frac{1}{n}t}$$

where c_n is a constant of integration. Replacing y_n with z_n^{-1} yields

$$z_n(t) = \frac{1}{n + c_n e^{\frac{1}{n}t}}.$$

This is the nontrivial solution to (2.29).

For the initial condition $z_0 = (0, \ldots, 0, \frac{1}{n}, 0, \ldots) \in \ell_2$ where the nonzero entry is at the n-th position, we have $c_n = 0$ and hence $z_n = 1/n$, while $z_i(t) = 0$ for $i \neq n$. The solution to (2.29) is

$$z(t) = (0, \ldots, 0, \frac{1}{n}, 0, \ldots).$$

For any $\delta > 0$, pick n such that $\frac{1}{n} < \delta$. For $\tilde{z} = (0, \ldots, 0, \ldots)$, $||z_0 - \tilde{z}||_{\ell_2} = \frac{1}{n} < \delta$ but

$$\lim_{t\to\infty} ||z(t) - \tilde{z}||_{\ell_2} = \lim_{t\to\infty} ||z(t)||_{\ell_2} = \frac{1}{n}.$$

That is, there exists an initial condition, z_0, such that $\lim_{t\to\infty} ||z(t)||_{\ell_2} \neq 0$. This implies $z(t)$ is not an asymptotically stable solution. □

Example 2.25 illustrates the limitation of linearization for infinite–dimensions and suggests that exponential stability is a necessary requirement for linear stability to imply the same stability of the corresponding nonlinear equation and that asymptotic stability is not sufficient. For a large class of systems, Theorem 2.24 demonstrates that exponential stability is a sufficient condition.

Example 2.25 also illustrates the significant difference between stability for finite–dimensions compared to infinite–dimensions. Consider the system described by (2.29) for $n \leq N$; that is, finite–dimensions. The solution to (2.29) for any initial condition,

$z_0 = (z_{10}, z_{20}, \ldots, z_{N0})$, is

$$z(t) = \left(\frac{z_{10}}{z_{10} + (1 - z_{10})e^t}, \frac{z_{20}}{2z_{20} + (1 - 2z_{20})e^{\frac{1}{2}t}}, \ldots, \frac{z_{N0}}{Nz_{N0} + (1 - Nz_{N0})e^{\frac{1}{N}t}} \right).$$

This implies

$$\lim_{t\to\infty} ||z(t)||_{\ell_2}^2 = \lim_{t\to\infty} \sum_{n=1}^{N} \left| \frac{z_{n0}}{nz_{n0} + (1 - nz_{n0})e^{\frac{1}{n}t}} \right|^2 = \sum_{n=1}^{N} \lim_{t\to\infty} \left| \frac{z_{n0}}{nz_{n0} + (1 - nz_{n0})e^{\frac{1}{n}t}} \right|^2 = 0$$

and hence the solution is asymptotically stable. Recall from example 2.25 that (2.29) is not asymptotically stable in infinite–dimensions.

2.2 Lyapunov Theory

Lyapunov Theory is useful for determining the stability of equilibrium points or invariant sets. The latter is not as well–known but is nearly identical to the version for points. Lyapunov Theory can be applied to finite–dimensions or infinite–dimensions and to linear or nonlinear systems. It also has tractable conditions; however, finding an appropriate Lyapunov function, which is vital in the application of Lyapunov Theory, can be a challenge. Lyapunov Theory also provides global stability, which is proved by showing the Lyapunov function is radially unbounded.

Definition 2.26. (Michel and Wang [65, Definition 5.1.11])
Let $V : Z \to \mathbb{R}$ be a continuous function and suppose there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that ψ is strictly increasing on $\mathbb{R}_+$, $\psi(0) = 0$ and $\lim_{z\to\infty} \psi(z) = \infty$. Then V is *radially unbounded* if $V(0) = 0$ and $V(z) \geq \psi(||z||_Z)$ for all $z \in Z$.

This definition of radially unboundness in Michel and Wang [65, Definition 5.1.11] is in finite–dimensions and then remarked on page 320 to be identical for infinite–dimensions after replacing the spaces appropriately. From the definition, it is clear that $V(z)$ is radially unbounded if $V(||z||_Z) \to \infty$ as $||z||_Z \to \infty$.

Lyapunov stability theory for (2.9) is discussed in Michel and Wang [65, Chapter 6] and Xu and Yung [98]. To be precise, it is a time–varying system,

$$\dot{z}(t) = f(t, z(t)), \tag{2.32}$$

that is investigated in these references but no information is lost as (2.9) is clearly a subclass of (2.32). Walker [93] investigates the Lyapunov stability of (2.9) for f linear.

Theorem 2.27. [Lyapunov Theorem for Equilibrium Points] (Michel and Wang [65, Theorem 6.2.13])
Suppose $z(t)$ is a (strong) solution to (2.9). Let $\tilde{z}$ be an equilibrium point of (2.9) and $D \subset Z$ containing $\tilde{z}$. Let $V : D \to \mathbb{R}$ be a continuously differentiable functional such that $V(\tilde{z}) = 0$, $V(z(t)) > 0$ for all $z \in D \backslash \{\tilde{z}\}$, and

$$\frac{dV(z(t))}{dt} \leq 0 \quad \text{for all } z(t) \in D,$$

then $\tilde{z}$ is stable. Moreover, if

$$\frac{dV(z(t))}{dt} < 0 \quad \text{for all } z(t) \in D \backslash \{\tilde{z}\},$$

then $\tilde{z}$ is asymptotically stable. In addition, if V is radially unbounded, then $\tilde{z}$ is globally asymptotically stable.

Consider the heat equation with Dirichlet boundary conditions. Define $\mathcal{L}_2[0, L]$ to be the space of real square–integrable functions with norm $||w(t)||^2_{\mathcal{L}_2} = \int_0^L w^2(x,t)dx$. The one dimensional heat equation is

$$\dot{w}(x,t) = w''(x,t) \tag{2.33a}$$

$$w(0,t) = w(L,t) = 0 \tag{2.33b}$$

$$w(x,0) = w_0(x) \tag{2.33c}$$

The solution to (2.33a) describes the distribution of temperature of an object, say a rod of length L, at position $x \in [0, L]$ and time $t \geq 0$. Equation (2.33) has an unique equilibrium at $\tilde{w} = 0$.

Define $A : D_1 \subset \mathcal{L}_2[0, L] \to \mathcal{L}_2[0, L]$,

$$Aw = w''$$

with

$$D_1 = \{w : w \in \mathcal{L}_2[0, L],\ w' \in \mathcal{L}_2[0, L],\ w'' \in \mathcal{L}_2[0, L],\ w(0,t) = 0 = w(L,t)\}.$$

It is known that A with domain D_1 generates a semigroup and is proved using the

Lumer–Phillips Theorem (see Theorem 2.6). We apply Theorem 2.27 to determine the stability of $\tilde{w} = 0$. Poincare's inequality is needed in establishing the stability of $\tilde{w} = 0$. It is stated and proved for completeness.

Lemma 2.28. [Poincare's Inequality](Krstic and Smyshlyaev [54, Lemma 2.1])
For any w continuously differentiable on $[0, L]$,

$$\int_0^L w^2 dx \leq 2w^2(L,t)L + 4L^2 \int_0^L w'^2 dx. \tag{2.34}$$

Proof of Lemma 2.28: For any continuously differentiable w,

$$\begin{aligned}
\int_0^L w^2 dx &= w^2(L,t)L - \int_0^L 2ww'x dx && \text{(integration by parts)} \\
&\leq w^2(L,t)L + \frac{1}{2}\int_0^L w^2 dx + \int_0^L 2w'^2 x^2 dx && \text{(Young's Inequality)} \\
&\leq w^2(L,t)L + \frac{1}{2}\int_0^L w^2 dx + L^2 \int_0^L 2w'^2 dx && (x \in [0, L]).
\end{aligned}$$

Rearranging leads to

$$\frac{1}{2}\int_0^L w^2 dx \leq w^2(L,t)L + 2L^2 \int_0^L w'^2 dx. \quad \square$$

Theorem 2.29. The equilibrium point, $\tilde{w} = 0$, of (2.33) is globally asymptotically stable in the $\mathcal{L}_2$–norm.

Proof of Theorem 2.29: Consider the Lyapunov candidate, $V_1(w) = \frac{1}{2}||w||^2_{\mathcal{L}_2}$. It is clear that $V_1(0) = 0$ and when $V_1(w) = 0$ then $w = 0$, which is the equilibrium point. Therefore, $V_1(w) > 0$ for all w except $w = 0$.

The derivative of V_1 is

$$\begin{aligned}
\frac{dV_1}{dt} &= \frac{1}{2}\frac{d}{dt}\int_0^L w^2 dx \\
&= \int_0^L w\dot{w} dx && \text{(chain rule)} \\
&= \int_0^L w w'' dx && \text{(substituting in (2.33a))} \\
&= -\int_0^L {w'}^2 dx && \text{(integration by parts with (2.33b))} \\
&= -||w'||^2_{\mathcal{L}_2}.
\end{aligned}$$

Applying the boundary conditions in (2.33b) to Poincare's inequality in (2.34) leads to

$$||w||^2_{\mathcal{L}_2} \leq 4L^2 ||w'||^2_{\mathcal{L}_2}$$

and hence

$$\frac{dV_1}{dt} \leq -\frac{1}{4L^2}||w||^2_{\mathcal{L}_2}. \tag{2.35}$$

If $\frac{dV_1}{dt} = 0$, then $||w||_{\mathcal{L}_2} = 0$ and hence $w = 0$. Therefore, $\frac{dV_1}{dt} < 0$ for all $w \neq 0$.

Since V_1 is radially unbounded, $V_1(0) = 0$ and for all $w \neq 0$, we have $V_1 > 0$ and $\frac{dV_1}{dt} < 0$, Theorem 2.27 implies $\tilde{w} = 0$ is a globally asymptotically stable equilibrium of (2.33). □

It turns out that $\tilde{w} = 0$ in problem (2.33) is exponentially stable.

Theorem 2.30. (Krstic and Smyshlyaev [54, Section 2.2])
The equilibrium point $\tilde{w} = 0$ for (2.33) is globally exponentially stable in the $\mathcal{L}_2$–norm.

Proof of Theorem 2.30: It follows immediately from (2.35) that

$$\frac{d||w||^2_{\mathcal{L}_2}}{dt} \leq -\frac{1}{2L^2}||w||^2_{\mathcal{L}_2}.$$

Solving yields $||w(x,t)||_{\mathcal{L}_2} \leq ||w(x,0)||_{\mathcal{L}_2} e^{-\frac{1}{2L^2}t}$ and hence by Definition 2.12, $\tilde{w} = 0$ is an exponentially stable equilibrium. This is true for any initial condition $w(x,0)$ and hence global stability is obtained. □

Due to Theorem 2.30, the result in Theorem 2.29 is redundant since exponential stability implies asymptotic stability. As well, it is shown in Michel and Wang [65, Example 6.2.14] that $\tilde{w} = 0$ is a globally exponentially stable equilibrium point in the norm, $||w||^2_{\mathcal{L}_2} + ||w'||^2_{\mathcal{L}_2}$.

Now consider Neumann boundary conditions rather than the Dirichlet boundary conditions in (2.33b). This leads to

$$\dot{w}(x,t) = w''(x,t) \tag{2.36a}$$
$$w'(0,t) = w'(L,t) = 0 \tag{2.36b}$$
$$w(x,0) = w_0(x). \tag{2.36c}$$

Define $A : D_2 \subset \mathcal{L}_2[0,L] \to \mathcal{L}_2[0,L]$,

$$Aw = w''$$

with

$$D_2 = \{w : w \in \mathcal{L}_2[0,L],\ w' \in \mathcal{L}_2[0,L],\ w'' \in \mathcal{L}_2[0,L],\ w'(0,t) = 0 = w'(L,t)\}.$$

It is known that A with domain D_2 generates a semigroup. This is proved using the Lumer–Phillips Theorem (see Theorem 2.6).

Equation (2.36) has equilibrium $w = c$ where c is a constant function. That is, the equilibrium is not unique; in fact, there are an infinite number of equilibria. Denote the set of equilibrium points of (2.36) by

$$E = \{w \in D_2 : w = c \in \mathbb{R}\}. \tag{2.37}$$

Our objective is to establish asymptotic stability of the set E (see Definition 2.15). Lyapunov stability theory can easily be applied to equilibrium sets. This is not surprising since we can define an equilibrium point, $\tilde{z}$, as an invariant set $\{\tilde{z}\}$. This is precisely how Michel and Wang [65, Definition 3.1.8] define equilibrium points.

Theorem 2.31. [Lyapunov Theorem for Invariant Sets] (Xu and Yang [98, Theorem 4.3]) Suppose $z(t)$ is a (strong) solution of (2.9). Let M be a invariant set of (2.9) and $D \subset Z$ be a neighbourhood of M. Let $V : D \to \mathbb{R}$ be a continuously differentiable functional such that $V(z(t)) = 0$ for all $z(t) \in M$ and $V(z(t)) > 0$ for all $z(t) \in D \backslash M$. If

$$\frac{dV(z(t))}{dt} \leq 0 \quad \text{for all } z(t) \in D,$$

then M is a stable invariant set. Moreover, if

$$\frac{dV(z(t))}{dt} < 0 \quad \text{for all } z(t) \in D \backslash M,$$

then M is an asymptotically stable invariant set. In addition, if V is radially unbounded, then M is a globally asymptotically stable invariant set.

We can apply Theorem 2.31 to the heat equation described in (2.36) since the problem is well-posed and the required Lyapunov function exists. Set $Z = \mathcal{L}_2[0, L]$ and $D = D_2$ in Theorem 2.31. Since we are interested in the stability of the equilibrium set, then $M = E$ defined in (2.37).

Theorem 2.32. The equilibrium set E defined in (2.37) is locally asymptotically stable in the $\mathcal{L}_2$–norm.

Proof of Theorem 2.32: Consider the Lyapunov candidate

$$V_2(w) = \frac{1}{2}||w'||^2_{\mathcal{L}_2} = \frac{1}{2}\int_0^L {w'}^2 dx.$$

It is clear that $V_2 \geq 0$ for all $w \in D_2$, and $V_2(c) = 0$ whenever $c \in E$. If $V_2(w) = 0$, then $w' = 0$ and hence w must be a constant; that is, $w \in E$. Therefore, $V_2(w) > 0$ for all $w \in D_2 \backslash E$. (Notice if the Lyapunov candidate had been $V_1(w) = \frac{1}{2}||w - c||^2_{\mathcal{L}_2}$, then $V_1(w)$ would be zero only for a particular constant and not for the entire set E.)

The derivative of V_2 is

$$\begin{aligned}
\frac{dV_2}{dt} &= \frac{1}{2}\frac{d}{dt}\int_0^L {w'}^2 dx \\
&= \int_0^L w'\dot{w}' dx && \text{(chain rule)} \\
&= -\int_0^L w''\dot{w} dx && \text{(integration by parts with (2.36b))} \\
&= -\int_0^L {w''}^2 dx && \text{(substituting in (2.36a))} \\
&= -||w''||^2_{\mathcal{L}_2}.
\end{aligned}$$

It follows from Poincare's Inequality (Lemma 2.28) that

$$||w'||^2_{\mathcal{L}_2} \leq 2w'^2(L,t)L + 4L^2||w''||^2_{\mathcal{L}_2}$$

and applying the boundary conditions in (2.36b) gives $||w'||^2_{\mathcal{L}_2} \leq 4L^2||w''||^2_{\mathcal{L}_2}$. Substituting into the derivative of V_2 yields

$$\frac{dV_2}{dt} \leq -\frac{1}{4L^2}||w'||^2_{\mathcal{L}_2}.$$

It is clear that $\frac{dV_2}{dt} \leq 0$ for all $w \in D_2$. Furthermore, if $\frac{dV_2}{dt} = 0$, we obtain $w' = 0$ and hence w is any constant. This implies $\frac{dV_2}{dt} < 0$ for all $w \in D_2 \backslash E$. By Theorem 2.31, E is an asymptotically stable equilibrium set of (2.36) in the $\mathcal{L}_2$–norm. □

Recall that any constant, c, is an equilibrium of (2.36). The equilibrium is uniquely determined by the initial condition, $w_0(x)$ and in particular

$$c = \frac{1}{L}\int_0^L w_0(x)dx; \tag{2.38}$$

that is, the equilibrium is the average of the initial distribution (Boyce and DiPrima [19, Section 10.6, Equation (38)]). Based on the boundary conditions in (2.36b), the rate of change at the endpoints is zero and hence there is no heat flowing in or out of the rod. Therefore, the heat distribution of the object will settle to the average of the initial temperature.

We saw in Theorem 2.32 that E is an asymptotically stable equilibrium set. Is a particular $c \in E$ an asymptotically stable equilibrium point? The answer is that c is stable but not asymptotically stable.

Theorem 2.33. Any equilibrium point of (2.36) is locally stable in the $\mathcal{L}_2$–norm.

Proof of Theorem 2.33: Let c be an equilibrium point of (2.36). Consider the Lyapunov candidate

$$V_1(w) = \frac{1}{2}||w-c||^2_{\mathcal{L}_2} = \frac{1}{2}\int_0^L (w-c)^2 dx.$$

It is clear that $V_1 \geq 0$ for all w and $V_1(c) = 0$. If $V_1(w) = 0$, then $w = c$ and therefore, $V_1(w) > 0$ for all w except when $w = c$. (If the Lyapunov candidate had been $V_2(w) = \frac{1}{2}||w'||^2_{\mathcal{L}_2}$ instead, then for $V_2(w) = 0$, it follows that $w' = 0$ and hence $w = d$, where d is any constant. Since d is not necessarily equal to c, then it is not true that $V_2(w) > 0$ for all w not equal to c and hence V_2 is not a Lyapunov function.)

The derivative of V_1 is

$$
\begin{aligned}
\frac{dV_1}{dt} &= \frac{1}{2}\frac{d}{dt}\int_0^L (w-c)^2 dx \\
&= \int_0^L (w-c)\dot{w}dx && \text{(chain rule)} \\
&= \int_0^L (w-c)w'' dx && \text{(substituting in (2.36a))} \\
&= -\int_0^L {w'}^2 dx && \text{(integration by parts with (2.36b))} \\
&= -||w'||^2_{\mathcal{L}_2}.
\end{aligned}
$$

Clearly, $\frac{dV_1}{dt} \leq 0$. Therefore, from Theorem 2.27, c is a stable equilibrium of (2.36).

If $\frac{dV_1}{dt} = 0$, then $w' = 0$ and hence $w = d$ where d is a constant. Since d is not necessarily equal to c, we cannot conclude that $\frac{dV_1}{dt} < 0$ for all w except when $w = c$. We cannot make any conclusions from V_1 about the asymptotic stability of c.

It turns out that c is not asymptotically stable. In order for c to be asymptotically stable requires that (2.11) be satisfied; however, the following shows (2.11) is not satisfied. Let $L = 1$ and $w_0(x)$ be a constant, say $\hat{w}_0$. From (2.38), it is clear that $c = \hat{w}_0$. For any $\delta > 0$, if the system begins at $\hat{w}_0 + \delta/2$, which is within a δ–ball of the initial condition $\hat{w}_0$, then since any constant is an equilibrium and $\hat{w}_0 + \delta/2$ is a constant, solutions will go to $\hat{w}_0 + \delta/2$ and not c. This means (2.11) is not satisfied and hence c is not asymptotically stable. □

The result in Theorem 2.33 that c is not asymptotically stable, does not pertain specifically to the heat equation. That is, no matter which equilibrium is chosen and no matter how small $\delta > 0$ is chosen, we can *always* find another equilibrium within the δ–ball, which implies that none of the equilibria can be asymptotically stable. If the set of all equilibria is disconnected, then asymptotic stability may be possible because there can exist a δ–ball around c such that no other equilibrium points are in the δ–ball.

The solution to the heat equation with Neumann boundary in (2.36) is

$$
w(x,t) = c + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L}t} \tag{2.39}
$$

where c is defined in (2.38) and

$$c_n = \frac{2}{L}\int_0^L w_0(x)\cos\left(\frac{\pi x}{L}\right)dx$$

(see Boyce and DiPrima [19, Section 10.6, Equation (35)]). It is clear from (2.39) that $w(x,t) \to c$ exponentially fast as $t \to \infty$. This does not contradict the result in Theorem 2.33 which shows c is stable but not asymptotically stable because the solution in (2.39) depends on a particular initial condition and hence Definitions 2.10 and 2.11 are no longer applicable.

We chose the heat equation as our illustrative example because the Landau–Lifshitz equation behaves similar to the heat equation. This similarity is useful when analyzing the stability and control of the Landau–Lifshitz equation.

Chapter 3

Hysteresis

Hysteresis is a phenomenon that occurs in nature and man–made processes. It often appears in engineering applications such as magnetization (Bertotti [15], Morris [67], Schneider and Winchell [76]) and smart materials (Valadkhan *et al.* [85], Smith [80, Section 2.4]). As well, many diffusion problems, such as freezing and thawing processes (Alimov *et al.* [5, 4], Christenson [28], Petrov and Furo [72]), chemical reactors (Jensen and Ray [45], Mancusi *et al.* [63]) and saturation of porous media (Bagagiolo and Visintin [8, 9], Kordulova [52, 53], Visintin [92, Section 1.11], Wu *et al.* [97]), exhibit hysteresis. Biological examples, such as predator and prey relationships (Aiki and Minchev [1], Murray [68, Section 1.3]), also display hysteretic behaviour. A man–made process that exhibits hysteresis is the dynamics of a thermostat. From this small collection of examples, it should be clear that hysteresis occurs in a number of different applications.

Hysteresis is difficult to define precisely; however, a common theme is the notion of a looping behaviour. One reason it is challenging to rigorously define hysteresis is that the shape of the loop can appear quite different even within the same problem (see Figure 3.5 for an example). There is still no consensus on how best to describe hysteresis from a mathematical viewpoint and there are several definitions of hysteresis available.

In much of the literature, hysteresis is represented by an operator (Aiki and Minchev [1], Brokate and Sprekels [20], Carbou *et al.* [21], Eleuteri and Krejci [35], Jayawardhana *et al.* [44], Valadkhan *et al.* [86], Visintin [90], [92, Section 1.1]). A hysteresis operator relates an input to the output of a system. One input can lead to more than one possible output. Therefore, past output is required to determine present output. For this reason, hysteresis is often said to require a memory. Another feature of hysteresis is that the output is independent of the speed of the input. This is known as *rate independence*. For a precise definition of hysteresis operators, see Brokate and Sprekels [20, Definition 2.2.8].

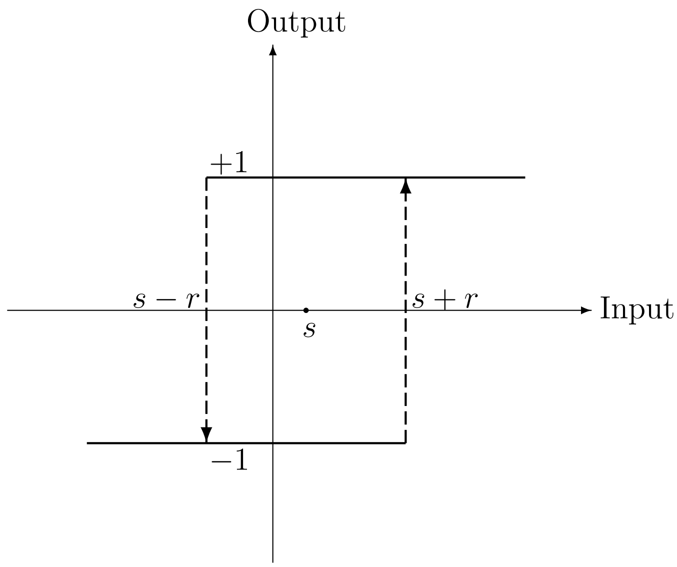


Figure 3.1: The relay operator has exactly two outputs, $\{-1, +1\}$, and two threshold input values, $s + r$ and $s - r$.

There are different types of hysteresis operators. They differ in the shape of the loop they govern. For example, Figure 3.1 shows a relay operator which is distinguished as having two outputs and two "threshold" inputs (Brokate and Sprekels [20, Example 2.1.1]). The dynamics of a thermostat exhibit hysteresis and the resulting shape of the looping behaviour is governed by a relay operator. In particular, the thermostat has on and off as its two outputs and the range of the desired room temperature provides a minimum and maximum value for the two threshold inputs. Notice from Figure 3.1, if the desired temperature is between $s - r$ and $s + r$, then it is unknown whether the thermostat is off or on, unless the previous output is given. This is the memory behaviour of hysteresis previously stated. More hysteresis operators are discussed in Brokate and Sprekels [20, Section 2.1].

Hysteresis operators are sometimes included as an additional term in a mathematical equation and hence increase the complexity of the equation. As well, not all hysteretic systems are rate independent and the definition of hysteresis operator can include operators that are not hysteretic (Morris [67]). These shortcomings are avoided if hysteresis is considered from a more fundamental perspective as is the approach in the following definition. This definition of hysteresis considers the properties of the dynamical system to establish hysteretic behaviour, whereas, considering a hysteresis operator assumes the model already exhibits hysteresis.

Definition 3.1. (Morris [67, Definition 3])
A system exhibits *hysteresis* if it has
(a) multiple stable equilibrium points and
(b) dynamics that are considerably faster than the time scale at which inputs are varied.

Definition 3.1 is related to the looping behaviour often associated with hysteresis. Consider a system with two stable equilibria (Figure 3.2) and suppose the system is initially at the left equilibrium (Figure 3.2a). If the input is increased, the system will tend to stay in equilibrium (Figure 3.2b) with only a small move upward along the hysteresis curve. When the input is increased enough such that the equilibrium disappears, the system moves to the right equilibrium (Figure 3.2c). This corresponds to moving along the steepest portion of the hysteresis loop. The system stays at the right equilibrium (Figure 3.2d). For systems which move to equilibrium faster than changes in the input, the transition from one equilibrium to the other is nearly instantaneous (Morris [67]). Physically, this means the system is observed only in equilibrium. If the input decreases enough so that the right equilibrium disappears (Figure 3.2e), the system moves back to the left equilibrium (Figures 3.2f, 3.2g). For a specific example, see the discussion on the Schmitt trigger in Morris [67], which explains how Definition 3.1 applies to the relay operator depicted in Figure 3.1.

However, the existence of a loop is not enough to identify hysteretic systems. Consider the following second order ODE

$$\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t) \tag{3.1}$$

where c and k are constants. This ODE often describes oscillatory motion such as the motion of a cart attached to a spring (see Figure 3.3). The displacement of the cart is denoted by $y(t)$, the force exerted by the spring is governed by $ky(t)$, and $c\dot{y}(t)$ describes damping which is proportional to velocity. In this case, k is the spring constant and c is the damping constant. The forcing term, $u(t)$, is usually a sinusoid. Writing (3.1) in first-order form with $\mathbf{x} = [y, \dot{y}]$ leads to

$$\dot{x}_1(t) = x_2(t) \tag{3.2a}$$

$$\dot{x}_2(t) = -cx_2(t) - kx_1(t) + u(t). \tag{3.2b}$$

A short MATLAB code (Appendix A) provides the appropriate input–output curves. The input is chosen to be $u(t) = \sin(\omega t)$ where ω is the frequency and the constants are set

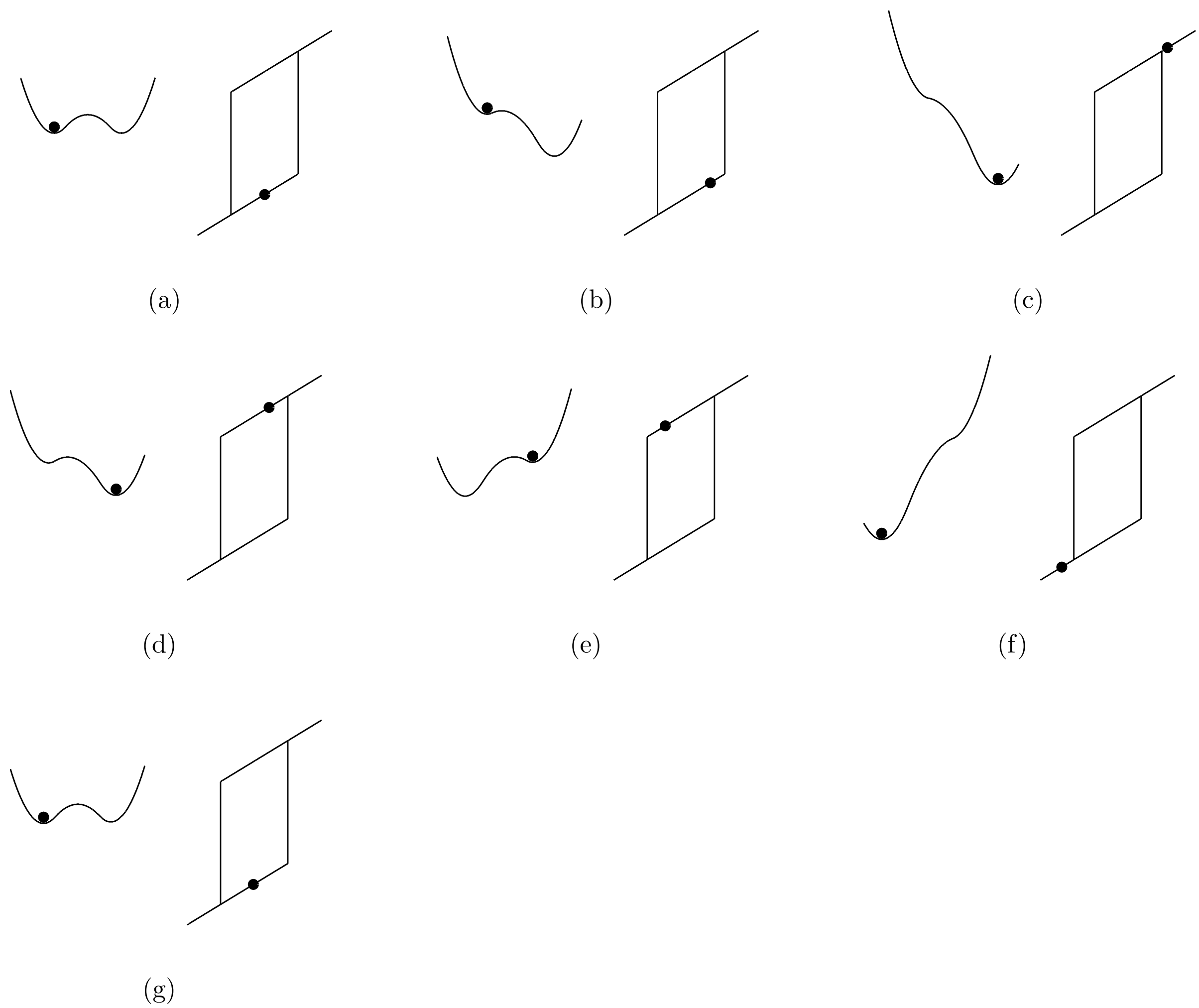


Figure 3.2: Dynamics of a system with two stable equilibria and its corresponding hysteresis loop. For the left curves, the horizontal axis is the output of the system and the vertical axis is the state of the system. For the right curves, the horizontal axis is the input while the vertical axis represents the output. As input varies, the dynamics of the system move from (a) to (g). (a) The system is initially at equilibrium. (b) As input increases, the system will tend to stay in equilibrium with only a small move upward along the hysteresis curve. (c) When input increases enough, the equilibrium disappears and the system moves to the right equilibrium and (d) stays there. This corresponds to moving along the steepest portion of the hysteresis loop. (e)–(g) Moving back to the left equilibrium is simply the reverse procedure.

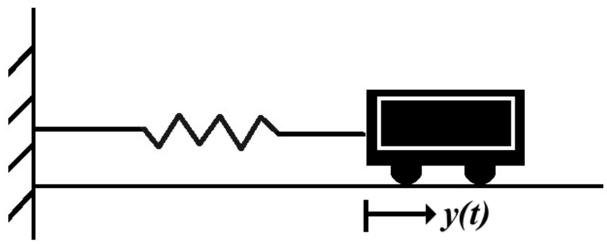


Figure 3.3: A cart attached to a spring.

to $c = 15$ and $k = 1$. When the forcing term is zero, the equilibrium is $(0,0)$ and the eigenvalues are

$$\lambda_1 = \frac{-c + \sqrt{c^2 - 4k}}{2} = \frac{-\sqrt{225} + \sqrt{221}}{2} \tag{3.3a}$$

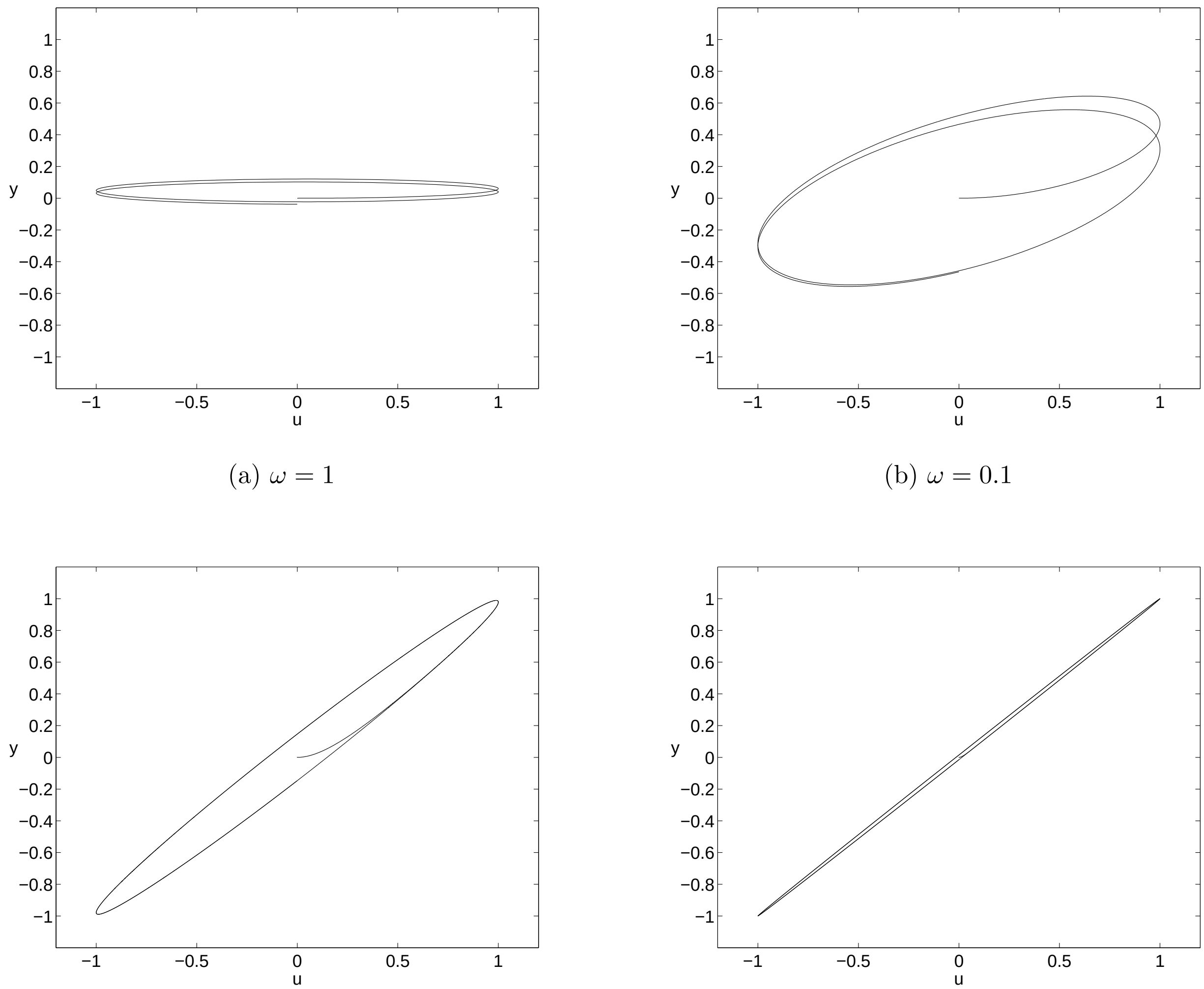
$$\lambda_2 = \frac{-c - \sqrt{c^2 - 4k}}{2} = \frac{-\sqrt{225} - \sqrt{221}}{2}. \tag{3.3b}$$

Both eigenvalues are negative and hence the equilibrium is stable. Suppose the system begins at equilibrium. It is clear from Figure 3.4 that for large ω a loop appears. However, since (3.2) has exactly one equilibrium, then from Definition 3.1 the system does not exhibit hysteresis. Therefore, the existence of an input-output loop is not sufficient to indicate the existence of hysteresis. Upon further examination of Figure 3.4, as ω approaches zero, the loop begins to vanish. Therefore, the notion of a persistent loop provides another interpretation of hysteresis.

Definition 3.2. (Oh and Bernstein [70])
A system exhibits *hysteresis* if a nontrivial closed curve in the input–output map persists for a periodic input as the frequency component of the input signal approaches zero.

For arbitrary initial conditions, $y(0) = y_0$ and $\dot{y}(0) = y_1$, the solution to (3.1) with $u(t) = \sin(\omega t)$ is

$$y(t) = \frac{1}{\sqrt{221}} \left(\frac{(-\lambda_1 - \lambda_2\omega^2)\, y_0 + \omega}{\omega^2 - 1 - 15\lambda_1} + y_1 \right) e^{\lambda_1 t} - \frac{1}{\sqrt{221}} \left(\frac{(-\lambda_2 - \lambda_1\omega^2)\, y_0 + \omega}{\omega^2 - 1 - 15\lambda_2} + y_1 \right) e^{\lambda_2 t} + \frac{15\omega\cos(\omega t) + (\omega^2 - 1)\sin(\omega t)}{2\omega^2 - 1 - \omega^2(15^2 + \omega^2)}.$$



(a) $\omega = 1$ (b) $\omega = 0.1$

(c) $\omega = 0.01$ (d) $\omega = 0.001$

Figure 3.4: Input–output curves for equation (3.1) with $c = 15$ and $k = 1$. The initial position is $y(0) = 0, \dot{y}(0) = 0$ and the input is $u(t) = \sin(\omega t)$ for various ω. Looping behaviour is observed for large ω but the loops do not persist as ω approaches 0. By Definition 3.2, equation (3.1) does not exhibit hysteresis.

It follows that

$$\lim_{\omega\to 0} y(t) = \frac{1}{\sqrt{221}}\left(\frac{\lambda_1 y_0}{1+15\lambda_1}+y_1\right)e^{\lambda_1 t} - \frac{1}{\sqrt{221}}\left(\frac{\lambda_2 y_0}{1+15\lambda_2}+y_1\right)e^{\lambda_2 t}$$

and since both eigenvalues are negative (see equation (3.3)), then $y(t)\to 0$ as $t\to\infty$, which means regardless of the initial condition, the steady state is zero as $\omega\to 0$ and $t\to\infty$. This supports the fact that as $\omega\to 0$, the input–output dynamics of (3.1) degenerate to a function as illustrated in Figure 3.4.

Consider now a nonlinear equation of motion,

$$\ddot{y}(t) + c\dot{y}(t) + k\left(y(t) - y^3(t)\right) = u(t). \tag{3.4}$$

This is equation (3.1) with an additional nonlinear term. The parameter $c = 15$ as before but k is set to -1. The initial condition is $y(0) = 0$ and $\dot{y}(0) = 0$. These curves are depicted in Figure 3.5. It is clear from the figure that even for low ω a loop persists, which by Definition 3.2 indicates the presence of hysteresis.

The dynamics governed by (3.4) also satisfy the requirements for hysteresis described in Definition 3.1. Rewrite (3.4) as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -cx_2(t) - kx_1(t) + kx_1^3(t) + u(t).\end{aligned}$$

For the unforced system, the equilibrium points are $(x_{eq}, 0)$ where x_{eq} satisfies

$$x_1 - x_1^3 = 0.$$

Therefore, equation (3.4) has equilibrium points, $(0,0)$ and $(\pm 1, 0)$. The corresponding Jacobians are

$$\begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 2k & -c \end{bmatrix}$$

for the $(0,0)$ and $(\pm 1, 0)$ equilibriums, respectively. The eigenvalues of $(0,0)$ are

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4k}}{2}$$

and the eigenvalues of $(\pm 1, 0)$ are

$$\lambda_{3,4} = \frac{-c \pm \sqrt{c^2 + 8k}}{2}.$$

For $c = 15$ and $k = 1$, we have $\lambda_{3,4} < 0$ and hence $(\pm 1, 0)$ are stable equilibrium points. This satisfies the first condition in Definition 3.1. (The equilibrium $(0, 0)$ is unstable since one of $\lambda_{1,2}$ is positive.) Figure 3.6 demonstrates that the dynamics in equation (3.4) are independent of the rate at which inputs are varied given the range of frequencies. This satisfies the second condition in Definition 3.1. For larger ω, the distinctly different shaped hysteresis loops in Figure 3.5 show that dynamics are rate dependent. A similar example illustrating the displacement of a magnetic beam is presented in Morris [67].

The dynamics in (3.4) illustrate it is crucial to have multiple stable equilibria and fast dynamics in comparison to the rate of change of inputs when defining hysteresis. Although hysteresis is synonymous with nonlinear systems (Bernstein and Oh [70]), linear systems can exhibit hysteresis. This is because linear systems can have multiple stable equilibria. Consider again equation (3.1) but with $k = 0$; that is,

$$\ddot{y}(t) + c\dot{y}(t) = u(t). \tag{3.5}$$

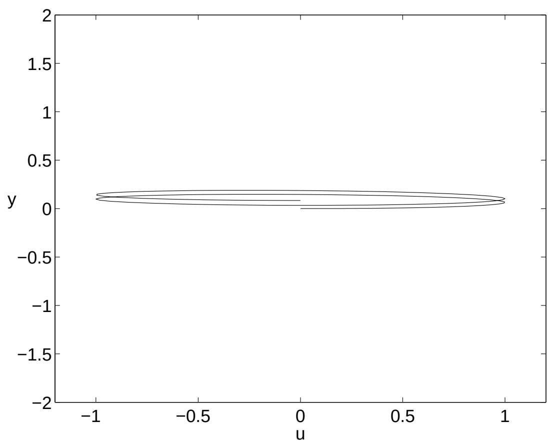
From (3.2), this leads to

$$\dot{x}_1(t) = x_2(t) \tag{3.6a}$$
$$\dot{x}_2(t) = -cx_2(t) + u(t). \tag{3.6b}$$

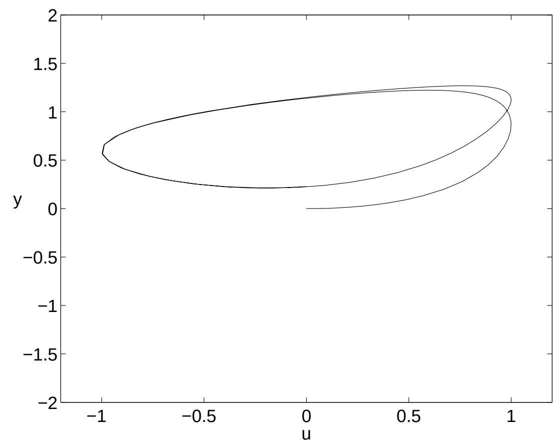
Suppose the forcing term is zero, then (3.6) has an infinite number of equilibria given by $(x_{eq}, 0)$ where x_{eq} is any constant function. Moreover, the eigenvalues of (3.6) are 0 and $-c$, which implies the equilibrium points are stable for $c > 0$. The input-output curves are displayed in Figure 3.7 with $c = 15$, $u(t) = \sin(\omega t)$ and initial condition $(x_1, x_2) = (0, 0)$. It is clear from the figure that a loop persists as ω approaches 0. Furthermore, since the solution to (3.5) is

$$\begin{aligned} y(t) &= y_0 + \frac{y_1}{c} + \frac{1}{\omega c} - \frac{1}{c}\left(y_1 + \frac{\omega}{\omega^2 + c^2}\right)e^{-ct} - \frac{\omega \sin(\omega t) + c\cos(\omega t)}{\omega\left(\omega^2 + c^2\right)} \\ &= y_0 + \frac{y_1}{c}\left(1 - e^{-ct}\right) - \frac{\sin(\omega t)}{\omega^2 + c^2} + \frac{\omega^2 + c^2 - c^2\cos(\omega t) - \omega^2 e^{-ct}}{\omega c\left(\omega^2 + c^2\right)}, \end{aligned}$$

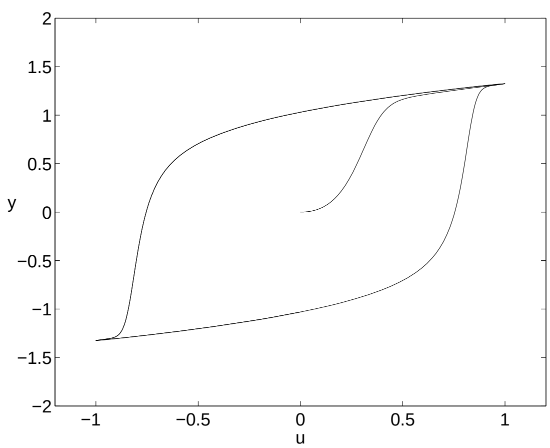
then we have



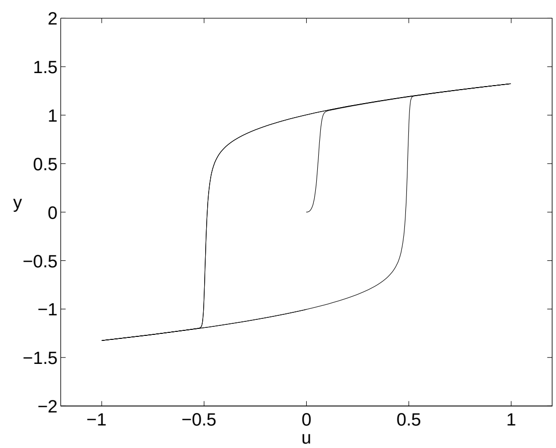
(a) $\omega = 1$



(b) $\omega = 0.1$



(c) $\omega = 0.01$



(d) $\omega = 0.001$

Figure 3.5: Input–output curves for equation (3.4) with $c = 15$ and $k = -1$. The initial position is $y(0) = 0, \dot{y}(0) = 0$ and the input is $u(t) = \sin(\omega t)$ for various ω. As ω approaches 0, looping behaviour persists. By Definition 3.2, equation (3.4) exhibits hysteresis. Notice that the loops are differently shaped, which implies rate dependence of the system.

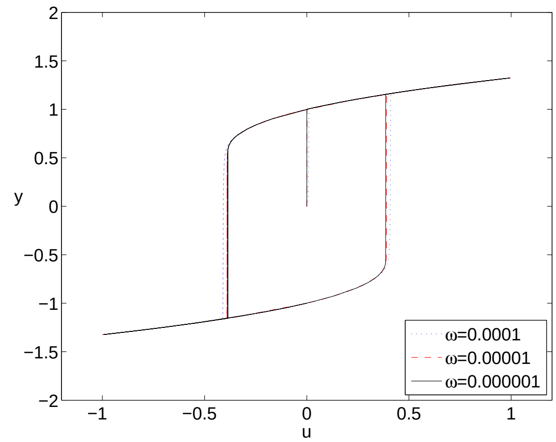


Figure 3.6: Input–output diagrams for the nonlinear equation (3.4) with $c = 15$, $k = -1$. The initial position is $y(0) = 0, \dot{y}(0) = 0$ and the input is $u(t) = \sin(\omega t)$ for $\omega \leq 0.0001$. At these low frequencies, the system exhibits rate independence.

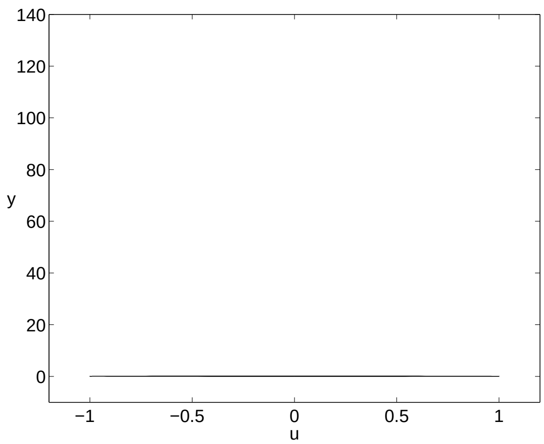
$$\lim_{\omega \to 0} y(t) = y_0 + \frac{y_1}{c}(1 - e^{-ct}).$$

It follows that

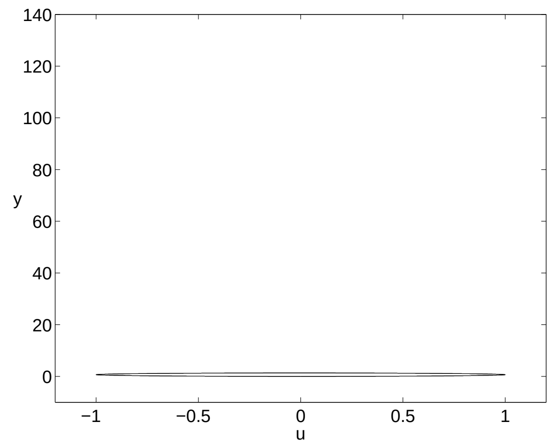
$$\lim_{t \to \infty} y(t) = y_0 + \frac{y_1}{c}$$

which means the equilibria of (3.5) depends on the initial conditions as $\omega \to 0$. This leads to a looping behaviour in the input–output map as $\omega \to 0$, which from Definition 3.2 illustrates the presence of hysteresis in the dynamics of (3.5).

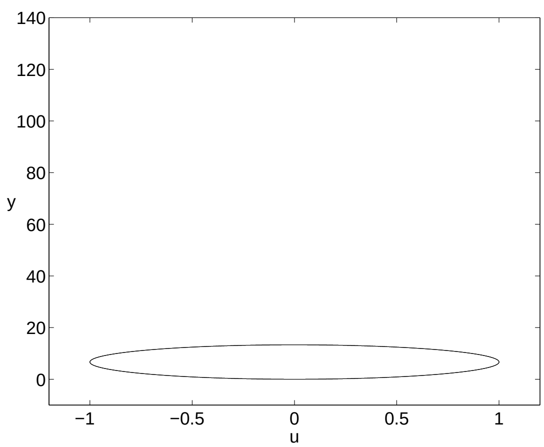
The arbitrary closeness of the equilibrium points of (3.5) explains the absence of sharp jumps that usually appear in hysteresis loops (see Figure 3.7). For the nonlinear example in (3.4), there are three distinct equilibrium points, $(-1, 0)$, $(0, 0)$ and $(1, 0)$. Due to hysteresis the system moves virtually instantaneously from one equilibrium to another which leads to the sharp jumps (see Figure 3.5). In the case of arbitrarily close equilibrium points, the jumps do not occur. We will see in Section 4.4 that hysteresis in the Landau–Lifshitz equation behaves similarly.



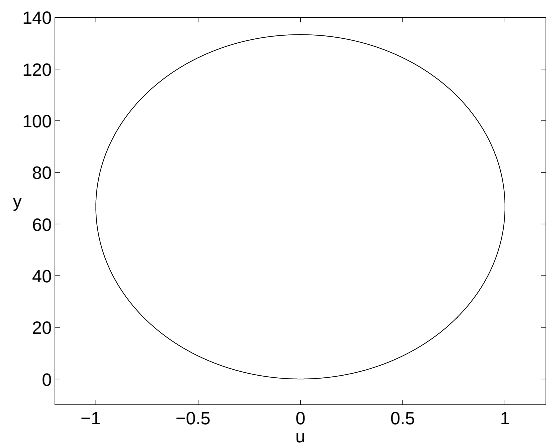
(a) $\omega = 1$



(b) $\omega = 0.1$



(c) $\omega = 0.01$



(d) $\omega = 0.001$

Figure 3.7: Input–output curves for (3.5) with $c = 15$. The initial condition is $y(0) = 0, \dot{y}(0) = 0$ and the input is $u(t) = \sin(\omega t)$ for various ω. It is clear the loops persist as ω approaches 0 and hence, by Definition 3.2 the system is hysteretic. The lack of jumps in the loop is also clear and is due to the arbitrary closeness of the equilibria of (3.5).

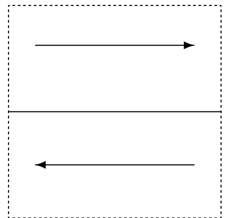
Chapter 4

Landau–Lifshitz Equation

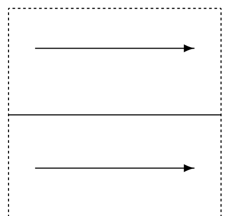
The most common type of magnetism is *ferromagnetism*. It is responsible for the magnetism observed in everyday objects such as refrigerator magnets. Examples of pure ferromagnets include iron, cobalt and nickel. Ferromagnets created from a combination of iron, cobalt and nickel are known as ferromagnetic alloys (Cullity and Graham [30, Section 4.5]). The Landau–Lifshitz equation describes the magnetization dynamics within a ferromagnetic object. On a molecular level, ferromagnets are divided into regions called *magnetic domains* or simply *domains*. Domains are separated by *domain walls*. Landau and Lifshitz derived their equation in 1935 as a way to model the motion of domain walls [58]. They were particularly interested in the velocity of the domain walls. In recent years, with the advancement of nanotechnology, the Landau–Lifshitz equation has been used to describe the dynamics of magnetization inside nanostructures such as nanowires (Carbou *et al.* [23, 26, 24], Noh *et al* [69], Gou *et al.* [39]), nanomagnets (Cowburn *et al.* [29]) and nanoparticles (Mayergoyz *et al.* [64]).

Each domain in a ferromagnet is magnetized to the same saturation, M_s. That is, every domain has the same magnetization magnitude but their directions may differ. Furthermore, the direction of the magnetization of each domain is such that the sum of the magnetizations is zero. When a magnetic field is applied to a ferromagnetic object, the domains align in the direction of the applied magnetic field and hence its net magnetization becomes nonzero. This is illustrated in Figure 4.1. For more details, see Cullity and Graham [30, Section 4.1].

Mathematically, the characteristic that each domain is magnetized to saturation throughout the ferromagnet is governed by $||\mathbf{m}(x,t)||_2 = M_s$ where $\mathbf{m}(x,t)$ is the magnetization in the ferromagnet and $||\cdot||_2$ is the Euclidean norm. In much of the literature, M_s is set



(a) external magnetic field absent, net magnetization is zero



(b) external magnetic field present, net magnetization is nonzero

Figure 4.1: Two magnetic domains separated by a domain wall (solid line). The vectors represent the direction and magnitude of each domain. (a) In the absence of an applied magnetic field, the magnitude of both domains is the same while the directions are opposite. This implies a net magnetization of zero. (b) When a magnetic field is applied, the magnetic domains orient themselves parallel to the magnetic field and hence the net magnetization is no longer zero.

to 1 and we follow the same convention; that is,

$$||\mathbf{m}(x,t)||_2 = 1. \tag{4.1}$$

For $x \in [0, L]$ where L is a positive length and time $t \in \mathbb{R}_+$, the magnetization,

$$\mathbf{m}(x,t) = (m_1(x,t), m_2(x,t), m_3(x,t)),$$

in a ferromagnetic object is described by the one–dimensional Landau–Lifshitz equation,

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \mathbf{H}_{eff} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{eff}) \tag{4.2}$$

where $\times$ denotes cross product and $\nu \geq 0$ is the damping parameter. The effective field,

$\mathbf{H}_{eff}$, governs the various energy interactions within a ferromagnet. It is the sum of the external applied magnetic field, $\mathbf{H}_a$, the demagnetization field, $\mathbf{H}_d$, exchange energy $\mathbf{H}_{ex}$, anisotropy energy $\mathbf{H}_{an}$, and magnetoelastic energy $\mathbf{H}_{me}$ (Gilbert [37]); that is,

$$\mathbf{H}_{eff} = \mathbf{H}_{ex} + \mathbf{H}_{an} + \mathbf{H}_{me} + \mathbf{H}_d + \mathbf{H}_a.$$

We now define these fields and describe how they interact with each other. Suppose a ferromagnetic object is magnetized by $\mathbf{H}_a$. An external field surrounds the magnet, but there is also a field within the magnet which tends to demagnetize the magnet. This is the demagnetization field (Cullity and Thomas [30, Section 2.7]). The exchange energy, $\mathbf{H}_{ex}$, is due to the forces between magnetic molecules and can be thought of as the potential energy of the ferromagnet (Cullity and Graham [30, Section 4.3]). The anisotropy energy is the tendency of a magnetic molecule to stay in its current position (Cullity and Graham [30, Section 7.2]). It is measured by the amount of energy, usually from an external magnetic field, needed to change the molecule's position. The magnetoelastic energy originates from the stress and strains caused by $\mathbf{H}_a$ (Cullity and Graham [30, Section 8.5]). Observe that $\mathbf{H}_a$ influences the behaviour of $\mathbf{H}_d$ and $\mathbf{H}_{an}$ and $\mathbf{H}_{me}$ but not $\mathbf{H}_{ex}$.

When analyzing (4.2), often only some of these energies are included in the effective field. For example, $\mathbf{H}_d$ is sometimes omitted because its influence on the magnetization dynamics is very small, such is the case in Bertotti *et al.* [17, Section 2.1] when he derives (4.2) and hence his $\mathbf{H}_{eff}$ is

$$\mathbf{H}_{eff} = \mathbf{H}_{ex} + \mathbf{H}_{an} + \mathbf{H}_{me} + \mathbf{H}_a.$$

In another instance, Sanchez [75] considers only the exchange energy and applied external magnetic field; that is,

$$\mathbf{H}_{eff} = \mathbf{H}_{ex} + \mathbf{H}_a.$$

We consider $\mathbf{H}_{eff} = \mathbf{H}_{ex}$ because given this effective field, analytical stability results for (4.2) are available (see Section 4.2). The exchange energy in one–dimension is $\mathbf{H}_{ex} = \mathbf{m}_{xx}$ where the subscript partial derivative notation means

$$\mathbf{m}_x(x,t) = (m_1'(x,t), m_2'(x,t), m_3'(x,t))$$
$$\mathbf{m}_{xx}(x,t) = (m_1''(x,t), m_2''(x,t), m_3''(x,t)).$$

The prime represents differentiation with respect to x. The Landau–Lifshitz equation with effective field, $\mathbf{H}_{eff} = \mathbf{H}_{ex}$, is investigated in Alouges and Soyeur [7], Carbou *et al.* [22], Guo and Ding [40, Chapter 6], Fuwa *et al.* [36]. Therefore, the form of the (uncontrolled)

Landau–Lifshitz equation we will be investigating in this chapter is

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}), \qquad \mathbf{m}(x,0) = \mathbf{m}_0(x). \tag{4.3}$$

Note that

$$\mathbf{m} \times \mathbf{m}_{xx} = (m_2 m_3'' - m_3 m_2'', -m_1 m_3'' + m_3 m_1'', m_1 m_2'' - m_2 m_1''). \tag{4.4}$$

The initial condition $\mathbf{m}_0(x)$ is chosen such that $||\mathbf{m}_0(x)||_2 = 1$. This condition on $\mathbf{m}_0(x)$ is standard in the literature (Alouges and Soyeur [7], Carbou and Fabrie [22], Guo and Ding [40, Section 6.3.1]). Physically, $||\mathbf{m}_0(x)||_2 = 1$ means initially the magnetization throughout the ferromagnet is the same, which represents the property that each domain in a ferromagnet is magnetized to the same saturation. We also assume $\mathbf{m}_0(x)$ is real–valued.

We consider Neumann boundary conditions,

$$\mathbf{m}_x(0,t) = \mathbf{m}_x(L,t) = \mathbf{0}, \tag{4.5}$$

which means there is no magnetic flux at the boundaries.

The dynamics governed by (4.3) are such that (4.1) is preserved. This is proved in Lemma 4.2. Lemma 4.1 is needed in the proof of Lemma 4.2, which requires properties of cross products. Furthermore, much of our analysis of the Landau–Lifshitz equation requires properties of cross products. For convenience, Table 4.1 contains a list of cross product properties (Bernstein [14, Fact 3.5.25]) that will be useful in the analysis of the Landau–Lifshitz equation.

Lemma 4.1. For $\mathbf{m}$ a solution of (4.3),

$$\mathbf{m}^{\mathrm{T}} \frac{\partial \mathbf{m}}{\partial t} = 0$$

where T means transpose.

Proof of Lemma 4.1: Substituting (4.3) into $\mathbf{m}^{\mathrm{T}} \frac{\partial \mathbf{m}}{\partial t}$ yields

$$\begin{aligned} \mathbf{m}^{\mathrm{T}} \frac{\partial \mathbf{m}}{\partial t} &= \mathbf{m}^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx}) - \nu \mathbf{m}^{\mathrm{T}} [\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx})] \\ &= \mathbf{m}_{xx}^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}) - \nu (\mathbf{m} \times \mathbf{m}_{xx})^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}) && \text{(Table 4.1)} \\ &= 0. && \text{(Table 4.1)} \quad \square \end{aligned}$$

1.	$\mathbf{p} \times \mathbf{p} = \mathbf{0}$
2.	$\mathbf{q} \times \mathbf{p} = -\mathbf{p} \times \mathbf{q}$
3.	$\mathbf{p} \times (\mathbf{q} + \mathbf{y}) = (\mathbf{p} \times \mathbf{q}) + (\mathbf{p} \times \mathbf{y})$
4.	$(\mathbf{p} + \mathbf{q}) \times \mathbf{y} = (\mathbf{p} \times \mathbf{y}) + (\mathbf{q} \times \mathbf{y})$
5.	$\mathbf{p}^{\mathrm{T}} (\mathbf{q} \times \mathbf{y}) = \mathbf{q}^{\mathrm{T}} (\mathbf{y} \times \mathbf{p}) = \mathbf{y}^{\mathrm{T}} (\mathbf{p} \times \mathbf{q})$
6.	$(\mathbf{p} \times \mathbf{q}) \times \mathbf{y} = \left(\mathbf{p}^{\mathrm{T}}\mathbf{y}\right) \mathbf{q} - \left(\mathbf{q}^{\mathrm{T}}\mathbf{y}\right) \mathbf{p}$
7.	$\mathbf{p} \times (\mathbf{q} \times \mathbf{y}) = \left(\mathbf{p}^{\mathrm{T}}\mathbf{y}\right) \mathbf{q} - \left(\mathbf{p}^{\mathrm{T}}\mathbf{q}\right) \mathbf{y}$
8.	$\mathbf{z} \times [\mathbf{p} \times (\mathbf{q} \times \mathbf{y})] = \mathbf{p}^{\mathrm{T}}\mathbf{y}\, (\mathbf{z} \times \mathbf{q}) - \mathbf{p}^{\mathrm{T}}\mathbf{q}\, (\mathbf{z} \times \mathbf{y})$

Table 4.1: Suppose $\mathbf{p}, \mathbf{q}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$ and $\times$ denotes cross product. The table forms a list of common cross product properties. A more detailed list is found in Bernstein [14, Fact 3.5.25].

Lemma 4.2. (Guo and Ding [40, Lemma 6.3.1])
Suppose $||\mathbf{m}(x, 0)||_2 = 1$, then the solution, $\mathbf{m}$, to (4.3) satisfies $||\mathbf{m}(x, t)||_2 = 1$.

Proof of Lemma 4.2: For $\mathbf{m}$ a solution to (4.3), it follows that

$$\frac{\partial ||\mathbf{m}||_2^2}{\partial t} = \frac{\partial (\mathbf{m}^{\mathrm{T}}\mathbf{m})}{\partial t} = \mathbf{m}^{\mathrm{T}} \frac{\partial \mathbf{m}}{\partial t} + \frac{\partial \mathbf{m}^{\mathrm{T}}}{\partial t}\mathbf{m} = 2\mathbf{m}^{\mathrm{T}} \frac{\partial \mathbf{m}}{\partial t}$$

where T means transpose. Applying Lemma 4.1 leads to

$$\frac{\partial ||\mathbf{m}||_2^2}{\partial t} = 0$$

and hence $||\mathbf{m}(x, t)||_2^2 = c(x)$. Since $||\mathbf{m}(x, 0)||_2 = 1$, then $c(x) = 1$. □

Equation (4.1) is often written as part of the Landau–Lifshitz model and plays an important role in the analysis of the Landau–Lifshitz equation. Further details are found in Bertotti *et al.* [17, Section 2.1] and Guo and Ding [40, Section 1.1].

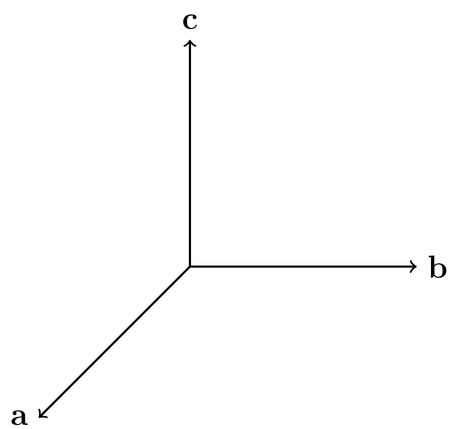


Figure 4.2: The 3-dimensional coordinate system, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, form a right handed set of orthogonal unit vectors.

The nonnegative parameter ν in (4.3) is the dimensionless *Gilbert damping constant*. It depends on the magnetization saturation and hence ν varies depending on the type of ferromagnet. Experimental values of ν are between 0.0001 and 0.03 (Widom *et al.* [95, Table 1]). When $\nu = 0$, the Landau–Lifshitz equation in (4.3) becomes

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \mathbf{m}_{xx}. \tag{4.6}$$

Equation (4.6) has been studied extensively with analytic solutions known (Bertotti *et al.* [16], Gilbert [37], Lakshmanan *et al.* [57], Guo and Ding [40, Section 2.1, Section 3.3] and Wang *et al.* [94]). One such solution is stated in Theorem 4.3.

Theorem 4.3. (Guo and Ding [40, Section 2.1])
Solutions to (4.6) are given by

$$\mathbf{m}(x,t) = \mathbf{a}\cos(\alpha) + [\mathbf{b}\cos(kx - \omega t) + \mathbf{c}\sin(kx - \omega t)]\sin(\alpha) \tag{4.7}$$

where $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ form a right–handed coordinate system of orthogonal unit vectors as illustrated in Figure 4.2, α and k are arbitrary with $\omega = k^2\cos(\alpha)$.

The solutions in (4.7) are clearly oscillatory and describe the wave motion in a ferromagnet due to the change in direction of a magnetic atom. These wave motions are called *spin waves*. The oscillations eventually die down due to internal friction (or damping) within the ferromagnet. A source of damping is the interactions between spin waves from different magnetic atoms. The damping behaviour in a ferromagnet is governed by the second term in (4.3). For more details on spin waves and magnetic damping, see Akhiezer *et al.* [2, Chapter 2] and Cullity [30, Section 12.6].

We end this section by briefly mentioning that there is another well–known equation which describes the dynamics of magnetization in a ferromagnetic object. It is called the *Landau–Lifshitz–Gilbert equation*,

$$\frac{\partial \mathbf{m}}{\partial t} = \left(1+\nu^2\right)\mathbf{m}\times\mathbf{m}_{xx} - \nu(\mathbf{m}\times\frac{\partial \mathbf{m}}{\partial t}). \tag{4.8}$$

Both equations are derived from physics but from different arguments which lead to the Landau–Lifshitz–Gilbert equation having a time dependent damping term (Gilbert [37]). A literature search of each equation will reveal that they are equally regarded by researchers. Aside from the discussion here, we consider only the Landau–Lifshitz equation.

From a mathematical viewpoint, the Landau–Lifshitz and the Landau–Lifshitz–Gilbert equations are equivalent; that is, equation (4.3) can be recovered from (4.8) and vice versa. This is illustrated in the next proposition.

Proposition 4.4. The Landau–Lifshitz equation in (4.3) is mathematically equivalent to (4.8); that is, equation (4.3) can be recovered from (4.8) and vice versa.

Proof of Proposition 4.4: Taking the cross product of $\nu\mathbf{m}$ with (4.3) leads to

$$\nu\mathbf{m}\times\frac{\partial \mathbf{m}}{\partial t} = \nu\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx}) - \nu^2\mathbf{m}\times\left[\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx})\right]. \tag{4.9}$$

From Table 4.1, we see that the second term can be rewritten as

$$\mathbf{m}\times\left[\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx})\right] = \mathbf{m}^{\mathrm{T}}\mathbf{m}_{xx}\left(\mathbf{m}\times\mathbf{m}\right) - \mathbf{m}^{\mathrm{T}}\mathbf{m}\left(\mathbf{m}\times\mathbf{m}_{xx}\right).$$

Since $\mathbf{m}^{\mathrm{T}}\mathbf{m} = 1$ from Lemma 4.2 and $\mathbf{m}\times\mathbf{m} = \mathbf{0}$, then

$$\mathbf{m}\times\left[\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx})\right] = -\mathbf{m}\times\mathbf{m}_{xx}$$

and substituting into (4.9), we obtain

$$\nu\mathbf{m}\times\frac{\partial \mathbf{m}}{\partial t} = \nu\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx}) + \nu^2\mathbf{m}\times\mathbf{m}_{xx}. \tag{4.10}$$

Adding (4.10) to (4.3) yields (4.8).

On the other hand, we can recover (4.3) from (4.8) as follows. Taking the cross product of $\mathbf{m}$ with (4.8) produces

$$\mathbf{m}\times\frac{\partial \mathbf{m}}{\partial t}=\left(1+\nu^2\right)\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx})-\nu\mathbf{m}\times(\mathbf{m}\times\frac{\partial \mathbf{m}}{\partial t}).$$

It follows that

$$\begin{aligned}\mathbf{m}\times(\mathbf{m}\times\frac{\partial \mathbf{m}}{\partial t}) &= \left(\mathbf{m}^{\mathrm{T}}\frac{\partial \mathbf{m}}{\partial t}\right)\mathbf{m}-\mathbf{m}^{\mathrm{T}}\mathbf{m}\frac{\partial \mathbf{m}}{\partial t} && \text{(Table 4.1)}\\ &= \left(\mathbf{m}^{\mathrm{T}}\frac{\partial \mathbf{m}}{\partial t}\right)\mathbf{m}-\frac{\partial \mathbf{m}}{\partial t} && \text{(Lemma 4.2)}\\ &= -\frac{\partial \mathbf{m}}{\partial t} && \text{(Lemma 4.1)}\end{aligned}$$

and hence

$$\mathbf{m}\times\frac{\partial \mathbf{m}}{\partial t}=\left(1+\nu^2\right)\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx})+\nu\frac{\partial \mathbf{m}}{\partial t}.$$

Substituting this result into (4.8) leads to

$$\frac{\partial \mathbf{m}}{\partial t}=\left(1+\nu^2\right)\mathbf{m}\times\mathbf{m}_{xx}-\nu\left(1+\nu^2\right)\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx})-\nu^2\frac{\partial \mathbf{m}}{\partial t}$$

and after rearranging, (4.3) is recovered. □

4.1 Well–Posedness

Existence and uniqueness results for the Landau–Lifshitz equation are shown in Gill and Zachary [38, Theorem 1], Li [61, Theorem 2.7, Theorem 3.1] and Carbou and Fabrie [22, Theorem 1.1, 1.2]. See also Chapter 3 in Guo and Ding [40] which includes existence and uniqueness results for nonlinear boundary conditions [40, Section 3.2] and when $\nu = 0$ [40, Section 3.3].

The uniqueness and existence results in Carbou and Fabrie [22, Theorem 1.1, 1.2] are stated below but first we establish a few notational definitions. Let $\mathcal{L}_2^3$ be the Hilbert space

$$\mathcal{L}_2^3=\mathcal{L}_2[0,L]\times\mathcal{L}_2[0,L]\times\mathcal{L}_2[0,L] \tag{4.11}$$

and

$$H_1 = \{\mathbf{m} \in \mathcal{L}_2^3,\ \mathbf{m}_x \in \mathcal{L}_2^3\}$$
$$H_2 = \{\mathbf{m} \in \mathcal{L}_2^3,\ \mathbf{m}_x \in \mathcal{L}_2^3,\ \mathbf{m}_{xx} \in \mathcal{L}_2^3\}$$
$$H_3 = \{\mathbf{m} \in \mathcal{L}_2^3,\ \mathbf{m}_x \in \mathcal{L}_2^3,\ \mathbf{m}_{xx} \in \mathcal{L}_2^3,\ \mathbf{m}_{xxx} \in \mathcal{L}_2^3\}.$$

The corresponding norms are

$$||\mathbf{m}||^2_{\mathcal{L}_2^3} = ||m_1||^2_{\mathcal{L}_2} + ||m_2||^2_{\mathcal{L}_2} + ||m_3||^2_{\mathcal{L}_2}$$
$$||\mathbf{m}||^2_{H_1} = ||\mathbf{m}||^2_{\mathcal{L}_2^3} + ||\mathbf{m}_x||^2_{\mathcal{L}_2^3}$$
$$||\mathbf{m}||^2_{H_2} = ||\mathbf{m}||^2_{\mathcal{L}_2^3} + ||\mathbf{m}_x||^2_{\mathcal{L}_2^3} + ||\mathbf{m}_{xx}||^2_{\mathcal{L}_2^3}$$
$$||\mathbf{m}||^2_{H_3} = ||\mathbf{m}||^2_{\mathcal{L}_2^3} + ||\mathbf{m}_x||^2_{\mathcal{L}_2^3} + ||\mathbf{m}_{xx}||^2_{\mathcal{L}_2^3} + ||\mathbf{m}_{xxx}||^2_{\mathcal{L}_2^3}.$$

Theorem 4.5. (Carbou and Fabrie [22, Theorem 1.1], Sanchez [74, Theorem 1.1])
If $\mathbf{m}_0(x) \in H_2, \mathbf{m}_{0x}(0) = 0, \mathbf{m}_{0x}(L) = 0$ and $||\mathbf{m}_0(x)||_2 = 1$, then there exists a time $t_f^* > 0$ depending on $\mathbf{m}_0(x)$ and there exists an unique $\mathbf{m}(x,t) \in \mathcal{C}([0,t_f]; H_2) \cap \mathcal{L}_2([0,t_f]; H_3)$ for all $t_f < t_f^*$ such that $\mathbf{m}(x,t)$ satisfies (4.3), (4.5) and $||\mathbf{m}(x,t)||_2 = 1$.

Theorem 4.6. (Carbou and Fabrie [22, Theorem 1.2])
If $\mathbf{m}_0(x) \in H_2, \mathbf{m}_{0x}(0) = 0, \mathbf{m}_{0x}(L) = 0$ and $||\mathbf{m}_0(x)||_2 = 1$, the solution given by Theorem 4.5 depends continuously on $\mathbf{m}_0(x)$ on the topology of $\mathcal{C}([0,t_f]; H_2)$.

Define $D \subset \mathcal{L}_2^3$ as

$$D = \{\mathbf{m} \in \mathcal{L}_2^3 : \mathbf{m}_x \in \mathcal{L}_2^3,\ \mathbf{m}_{xx} \in \mathcal{L}_2^3,\ \mathbf{m}_x(0,t) = \mathbf{0} = \mathbf{m}_x(L,t)\}. \tag{4.12}$$

From the existence and uniqueness of (4.3), (4.5) established in Theorem 4.5, we can define for any $\mathbf{m}_0 \in D$ an operator $F(t)$ on D by

$$F(t) : \mathbf{m}_0(x) \to \mathbf{m}(x,t). \tag{4.13}$$

Theorem 4.7. The operator, $F(t)$, defined in (4.13) is a nonlinear contraction semigroup.

Proof of Theorem 4.7: We show (4.13) satisfies condition (i) to (iv) in Definition 2.9.

Setting $t = 0$, equation (4.13) becomes $F(0)\mathbf{m}_0(x) = \mathbf{m}_0(x)$. This is true for any $\mathbf{m}_0 \in D$ and hence (i) is proved.

For any $\mathbf{m}_0 \in D$ and $t, s \in \mathbb{R}_+$, it follows from (4.13) that $\mathbf{m}(x, t+s) = F(t+s)\mathbf{m}_0(x)$ and $\mathbf{m}(x,s) = F(s)\mathbf{m}_0(x)$. Since solutions to (4.3) are unique (see Theorem 4.5), then

$\mathbf{m}(x, t+s)$ is the unique solution to (4.3) with initial condition $\mathbf{m}(x, s)$; that is, $\mathbf{m}(x, t+s) = F(t)\mathbf{m}(x, s)$. It follows that $F(t+s)\mathbf{m}_0(x) = F(t)\mathbf{m}(x, s) = F(t)F(s)\mathbf{m}_0(x)$ and hence (ii) is proved.

Since $\mathbf{m}(x, t) = F(t)\mathbf{m}_0(x)$, conditions (iii) and (iv) of Definition 2.9 follow immediately from Theorems 4.5 and 4.6, respectively. It follows that $F(t)$ is a nonlinear semigroup and from Lemma 4.2, $F(t)$ is a contraction semigroup. □

4.2 Stability

Stability behaviour of the Landau–Lifshitz equation is discussed in Carbou and Labbé *et al.* [23, 24, 56], Jizzini [46] and Gou *et al.* [39]. In Gou *et al.* [39], analytic stability results are for the associated linear equation, while stability for the full (nonlinear) equation is verified numerically. In Labbé *et al.* [56], equilibrium solutions of (4.2) with

$$\mathbf{H}_{eff} = \mathbf{m}_{xx} - m_2\mathbf{e}_2 - m_3\mathbf{e}_3,$$

where $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$, are determined by writing

$$\begin{aligned} m_1(x) &= \cos(\theta(x)) \\ m_2(x) &= \cos(\psi(x))\sin(\theta(x)) \\ m_3(x) &= \sin(\psi(x))\sin(\theta(x)). \end{aligned}$$

This defines the unit sphere on $\mathbb{R}^3$ and automatically satisfies equation (4.1). Furthermore, periodic boundary conditions are considered. It is shown that all equilibria, θ_{eq}, are solutions to

$$\theta''_{eq} - \frac{1}{2}\sin(2\theta_{eq}) = 0$$

with $\theta_{eq}(0) = \theta_{eq}(L)$ and $\theta'_{eq}(0) = \theta'_{eq}(L)$. Equilibrium solutions are in terms of elliptic functions. For more on elliptic functions, see Lawden [59]. After this, equation (4.2) with $\mathbf{H}_{eff} = \mathbf{m}_{xx} - m_2\mathbf{e}_2 - m_3\mathbf{e}_3$ is linearized around a particular equilibrium (see equation (30) in Labbé *et al.* [56]). The resulting equation is of semilinear form. The spectral properties of the linear operator are determined. The papers by Carbou and Labbé [23, 24] and Jizzini [46] are similar to Labbé *et al.* [56] in their approaches for determining stability.

In Mayergoyz *et al.* [64], the stability of magnetization in nanoparticles is investigated. The magnetization dynamics, $\mathbf{m}_{su}$, are assumed to be spatially uniform and are governed by

$$\frac{d\mathbf{m}_{su}}{dt} = -\mathbf{m}_{su} \times \mathbf{H}_{eff} - \nu \mathbf{m}_{su} \times (\mathbf{m}_{su} \times \mathbf{H}_{eff}) + \mathbf{F}(\mathbf{m}_{su}). \tag{4.14}$$

Equation (4.14) is an ODE and not a PDE. The $\mathbf{F}(\mathbf{m}_{su})$ term describes a torque–like force acting on the nanoparticles. Standard expressions for $\mathbf{F}(\mathbf{m}_{su})$ are found in Mayergoyz *et al.* [64]. The effective field, $\mathbf{H}_{eff}$, is the sum of the exchange energy, anisotropy energy and demagnetization field; that is,

$$\mathbf{H}_{eff} = \mathbf{H}_{ex} + \mathbf{H}_{an} + \mathbf{H}_{d}.$$

Neumann boundary conditions are applied. A perturbed magnetization,

$$\mathbf{m} = \mathbf{m}_{su}(t) + \mathbf{v}_{sn}(t) + \mathbf{v}_{su}(t),$$

is considered where $\mathbf{v}_{sn}(t)$ and $\mathbf{v}_{su}(t)$ are spatially nonuniform and spatially uniform perturbations, respectively. The perturbed dynamics are governed by

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{H}_{eff} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{eff}) + \mathbf{F}(\mathbf{m})$$

and from this, equations for $\partial \mathbf{v}_{sn}/\partial t$ and $d\mathbf{v}_{su}/dt$ are derived with both equations independent from one another. It is shown that

$$\frac{d||\mathbf{v}_{sn}||_{\mathcal{L}_2^3}}{dt} < 0,$$

which implies that spatially nonuniform magnetization dynamics is stable in the $\mathcal{L}_2^3$–norm with respect to any spatially uniform perturbation.

In what is to follow, we determine in detail the equilibrium solutions of (4.3), (4.5) and their stability. All of the results are analytical and pertain to the full equation in (4.3); that is, there is no linearization. These results are found in Guo and Ding [40, Section 6.1.1] with a slightly different proof than given here. Specifically, their results are for an equilibrium that is not associated to a boundary condition. The main result is in Theorem 4.10. We begin by establishing some lemmas needed in the proof of Theorem 4.10.

Lemma 4.8. For $\mathbf{m} \in \mathcal{L}_2^3$, $\mathbf{m}$ is an equilibrium solution of (4.3) if and only if $\mathbf{m} \times \mathbf{m}_{xx} = \mathbf{0}$.

Proof of Lemma 4.8: Suppose $\mathbf{m} \times \mathbf{m}_{xx} \neq \mathbf{0}$. Taking the product of (4.3) with $\mathbf{m} \times \mathbf{m}_{xx}$

leads to

$$
\begin{aligned}
(\mathbf{m}\times\mathbf{m}_{xx})^{\mathrm{T}}\frac{\partial\mathbf{m}}{\partial t} &= (\mathbf{m}\times\mathbf{m}_{xx})^{\mathrm{T}}(\mathbf{m}\times\mathbf{m}_{xx}) + \nu\,(\mathbf{m}\times\mathbf{m}_{xx})^{\mathrm{T}}(\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx}))\\
&= ||\mathbf{m}\times\mathbf{m}_{xx}||_2^2 + \nu\,(\mathbf{m}\times\mathbf{m}_{xx})^{\mathrm{T}}(\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx}))\\
&= ||\mathbf{m}\times\mathbf{m}_{xx}||_2^2 + \mathbf{m}^{\mathrm{T}}((\mathbf{m}\times\mathbf{m}_{xx})\times(\mathbf{m}\times\mathbf{m}_{xx})) && \text{(Table 4.1)}\\
&= ||\mathbf{m}\times\mathbf{m}_{xx}||_2^2 && \text{(Table 4.1)}.
\end{aligned}
$$

Since $\mathbf{m}\times\mathbf{m}_{xx}\neq\mathbf{0}$, then

$$\frac{\partial\mathbf{m}}{\partial t}\neq\mathbf{0}$$

and hence $\mathbf{m}$ is not an equilibrium. Considering the contrapositive, it follows that $\mathbf{m}$ is an equilibrium if $\mathbf{m}\times\mathbf{m}_{xx}=\mathbf{0}$. On the other hand, substituting $\mathbf{m}\times\mathbf{m}_{xx}=\mathbf{0}$ into (4.3) gives $\partial\mathbf{m}/\partial t=\mathbf{0}$ and hence is an equilibrium solution to (4.3). Therefore, $\mathbf{m}$ is an equilibrium solution of (4.3) if and only if $\mathbf{m}\times\mathbf{m}_{xx}=\mathbf{0}$. □

Lemma 4.9. If $\mathbf{m}\in\mathcal{L}_2^3$ satisfies (4.1), then $\mathbf{m}^{\mathrm{T}}\mathbf{m}_{xx}=-\,||\mathbf{m}_x||_2^2$.

Proof of Lemma 4.9: Differentiating (4.1) with respect to x produces

$$0=\frac{\partial||\mathbf{m}||_2^2}{\partial x}=\frac{\partial(\mathbf{m}^{\mathrm{T}}\mathbf{m})}{\partial x}=2\mathbf{m}^{\mathrm{T}}\mathbf{m}_x$$

and differentiating again yields

$$0=\frac{\partial}{\partial x}\left(\mathbf{m}^{\mathrm{T}}\mathbf{m}_x\right)=\mathbf{m}_x^{\mathrm{T}}\mathbf{m}_x+\mathbf{m}^{\mathrm{T}}\mathbf{m}_{xx}=||\mathbf{m}_x||_2^2+\mathbf{m}^{\mathrm{T}}\mathbf{m}_{xx}.$$

This implies $\mathbf{m}^{\mathrm{T}}\mathbf{m}_{xx}=-\,||\mathbf{m}_x||_2^2$ as desired. □

Theorem 4.10. The set of equilibrium points of (4.3), (4.5) is

$$E=\{\mathbf{a}=(a_1,a_2,a_3)\in\mathbb{R}^3 : a_1,a_2,a_3 \text{ constants and } \mathbf{a}^{\mathrm{T}}\mathbf{a}=1\}. \tag{4.15}$$

Proof of Theorem 4.10: If $\mathbf{m}$ is an equilibrium solution of (4.3), Lemma 4.8 implies $\mathbf{m}$ is parallel to $\mathbf{m}_{xx}$. That is, there exists a real function, $l(x)$, such that $\mathbf{m}_{xx}=l(x)\mathbf{m}$. Taking the scalar product with $\mathbf{m}$ yields

$$
\begin{aligned}
\mathbf{m}^{\mathrm{T}}\mathbf{m}_{xx} &= \mathbf{m}^{\mathrm{T}}(l(x)\mathbf{m})\\
&= l(x)(\mathbf{m}^{\mathrm{T}}\mathbf{m}).
\end{aligned}
$$

From Lemma 4.2, $\mathbf{m}^{\mathrm{T}}\mathbf{m} = 1$ and hence,

$$\mathbf{m}^{\mathrm{T}}\mathbf{m}_{xx} = l(x).$$

It follows from Lemma 4.9 that

$$||\mathbf{m}_x||_2^2 = -l(x). \tag{4.16}$$

Substituting (4.16) into $\mathbf{m}_{xx} = l(x)\mathbf{m}$ yields

$$\mathbf{m}_{xx} = -\,||\mathbf{m}_x||_2^2\,\mathbf{m}.$$

Taking the scalar product with $\mathbf{m}_x$, we obtain

$$\mathbf{m}_{xx}^{\mathrm{T}}\mathbf{m}_x = -\,||\mathbf{m}_x||_2^2\,\mathbf{m}^{\mathrm{T}}\mathbf{m}_x. \tag{4.17}$$

Since

$$\begin{aligned}\frac{\partial \mathbf{m}_x^{\mathrm{T}}\mathbf{m}_x}{\partial x} &= \mathbf{m}_{xx}^{\mathrm{T}}\mathbf{m}_x + \mathbf{m}_x^{\mathrm{T}}\mathbf{m}_{xx} = 2\mathbf{m}_{xx}^{\mathrm{T}}\mathbf{m}_x \\ \frac{\partial \mathbf{m}^{\mathrm{T}}\mathbf{m}}{\partial x} &= \mathbf{m}_x^{\mathrm{T}}\mathbf{m} + \mathbf{m}^{\mathrm{T}}\mathbf{m}_x = 2\mathbf{m}^{\mathrm{T}}\mathbf{m}_x,\end{aligned}$$

equation (4.17) becomes

$$\frac{\partial \mathbf{m}_x^{\mathrm{T}}\mathbf{m}_x}{\partial x} = -\,||\mathbf{m}_x||_2^2\,\frac{\partial \mathbf{m}^{\mathrm{T}}\mathbf{m}}{\partial x}.$$

Applying equation (4.1) leads to

$$\frac{\partial}{\partial x}\,||\mathbf{m}_x||_2^2 = 0.$$

Equation (4.16) implies

$$\frac{\partial\, l(x)}{\partial x} = 0.$$

This means that $l(x)$ is a constant; that is, independent of x. Define this constant to be $-k^2$ which we know is negative or zero from (4.16). Substituting $l(x) = -k^2$ into $\mathbf{m}_{xx} = l(x)\mathbf{m}$ leads to $\mathbf{m}_{xx} = -k^2\mathbf{m}$. This is a system of linear second order ODEs,

$$\begin{aligned}m_1''(x) &= -k^2 m_1(x) \\ m_2''(x) &= -k^2 m_2(x) \\ m_3''(x) &= -k^2 m_3(x).\end{aligned}$$

These ODES are easily solved and have solutions

$$\begin{aligned} m_1(x) &= a_1 \cos(kx) + b_1 \sin(kx) \\ m_2(x) &= a_2 \cos(kx) + b_2 \sin(kx) \\ m_3(x) &= a_3 \cos(kx) + b_3 \sin(kx) \end{aligned}$$

where $a_i, b_i \in \mathbb{R}$ for $i = 1, 2, 3$ are constants.

Equation (4.1) requires that $||\mathbf{m}||_2^2 = m_1^2 + m_2^2 + m_3^2 = 1$ which forces

$$\begin{aligned} &a_1^2 + a_2^2 + a_3^2 = 1 \\ &b_1^2 + b_2^2 + b_3^2 = 1 \\ &a_1 b_1 + a_2 b_2 + a_3 b_3 = 0. \end{aligned}$$

Therefore, the equilibrium solutions to (4.3) are of the form

$$\mathbf{m}(x) = \mathbf{a} \cos(kx) + \mathbf{b} \sin(kx)$$

for any $k \in \mathbb{R}$ with $||\mathbf{a}||_2 = 1$, $||\mathbf{b}||_2 = 1$ and $\mathbf{a}^{\mathrm{T}}\mathbf{b} = 0$ where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. Given Neumann boundary conditions, we have $\mathbf{m}_x(0) = k\mathbf{b} = 0$ which implies either $k = 0$ or $\mathbf{b} = \mathbf{0}$; however, since $||\mathbf{b}||_2 = 1$, then $k = 0$. Therefore, the equilibrium solution to (4.3), (4.5) is $\mathbf{m}(x) = \mathbf{a}$ which also satisfies the second boundary condition $\mathbf{m}_x(L) = 0$. That is, the equilibrium is not unique; in fact, there are an infinite number of equilibria. We denote the set of equilibrium solutions associated to (4.3) as

$$E = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1, a_2, a_3 \text{ constants}\}$$

with $||\mathbf{a}||_2 = 1$. □

In the following theorem, E is shown to be an asymptotically stable equilibrium set. A similar result to is shown in Guo and Ding [40, Proposition 6.2.1]; however, the authors only establish the stability of the equilibrium points, and do not prove the equilibrium set is asymptotically stable. We apply Lyapunov's Theorem for sets (see Theorem 2.31) to show E is asymptotically stable.

Theorem 4.11. The equilibrium set in (4.15) is asymptotically stable in the $\mathcal{L}_2^3$–norm.

Proof of Theorem 4.11: The Lyapunov candidate is

$$V(\mathbf{m}) = \frac{1}{2} ||\mathbf{m}_x||_{\mathcal{L}_2^3}^2 \tag{4.18}$$

where $\mathbf{m}(x,t)$ is the solution to (4.3). It is clear that $V \geq 0$ for all $\mathbf{m} \in D$ in (4.12) and $V(\mathbf{a}) = 0$ for any $\mathbf{a} \in E$. If $V(\mathbf{m}) = \mathbf{0}$, we have $\mathbf{m}_x = 0$ which implies $\mathbf{m} = \mathbf{a}$ for some $\mathbf{a} \in E$. Therefore, $V(\mathbf{m}) > 0$ for all $\mathbf{m} \in D \backslash E$ and $V(\mathbf{a}) = 0$ for any $\mathbf{a} \in E$.

The derivative of V is

$$\frac{dV(\mathbf{m})}{dt} = \frac{1}{2}\frac{d}{dt}||\mathbf{m}_x||^2_{\mathcal{L}_2^3} = \int_0^L \mathbf{m}_x{}^{\mathrm{T}}\dot{\mathbf{m}}_x dx.$$

Applying integration by parts with the Neumann boundary conditions in (4.5) produces

$$\frac{dV(\mathbf{m})}{dt} = -\int_0^L \mathbf{m}_{xx}^{\mathrm{T}}\dot{\mathbf{m}} dx. \tag{4.19}$$

Substituting in (4.3) yields

$$\frac{dV(\mathbf{m})}{dt} = -\int_0^L \mathbf{m}_{xx}^{\mathrm{T}}\left(\mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx})\right) dx$$

and rearranging leads to

$$\frac{dV(\mathbf{m})}{dt} = \int_0^L -\mathbf{m}^{\mathrm{T}}\left(\mathbf{m}_{xx} \times \mathbf{m}_{xx}\right) + \nu \left(\mathbf{m} \times \mathbf{m}_{xx}\right)^{\mathrm{T}}\left(\mathbf{m}_{xx} \times \mathbf{m}\right) dx$$

The first term in the integral is zero since any element cross product with itself is zero (Table 4.1). The second term can be written as

$$\frac{dV(\mathbf{m})}{dt} = -\nu \int_0^L ||\mathbf{m} \times \mathbf{m}_{xx}||_2^2 \, dx = -\nu \, ||\mathbf{m} \times \mathbf{m}_{xx}||^2_{\mathcal{L}_2^3} .$$

Since $\nu \geq 0$, it is clear that $\frac{dV}{dt} \leq 0$ for all $\mathbf{m} \in D$. Furthermore, if $\frac{dV}{dt} = 0$, then $\mathbf{m} \times \mathbf{m}_{xx} = 0$. From Lemma 4.8, $\mathbf{m} \times \mathbf{m}_{xx} = 0$ if and only if $\mathbf{m}$ is an equilibrium solution and since E is the set of all equilibria (see Theorem 4.10), then $\frac{dV}{dt} < 0$ for all $\mathbf{m} \in D \backslash E$. It follows from Theorem 2.31 that E is an asymptotically stable equilibrium set of (4.3), (4.5). □

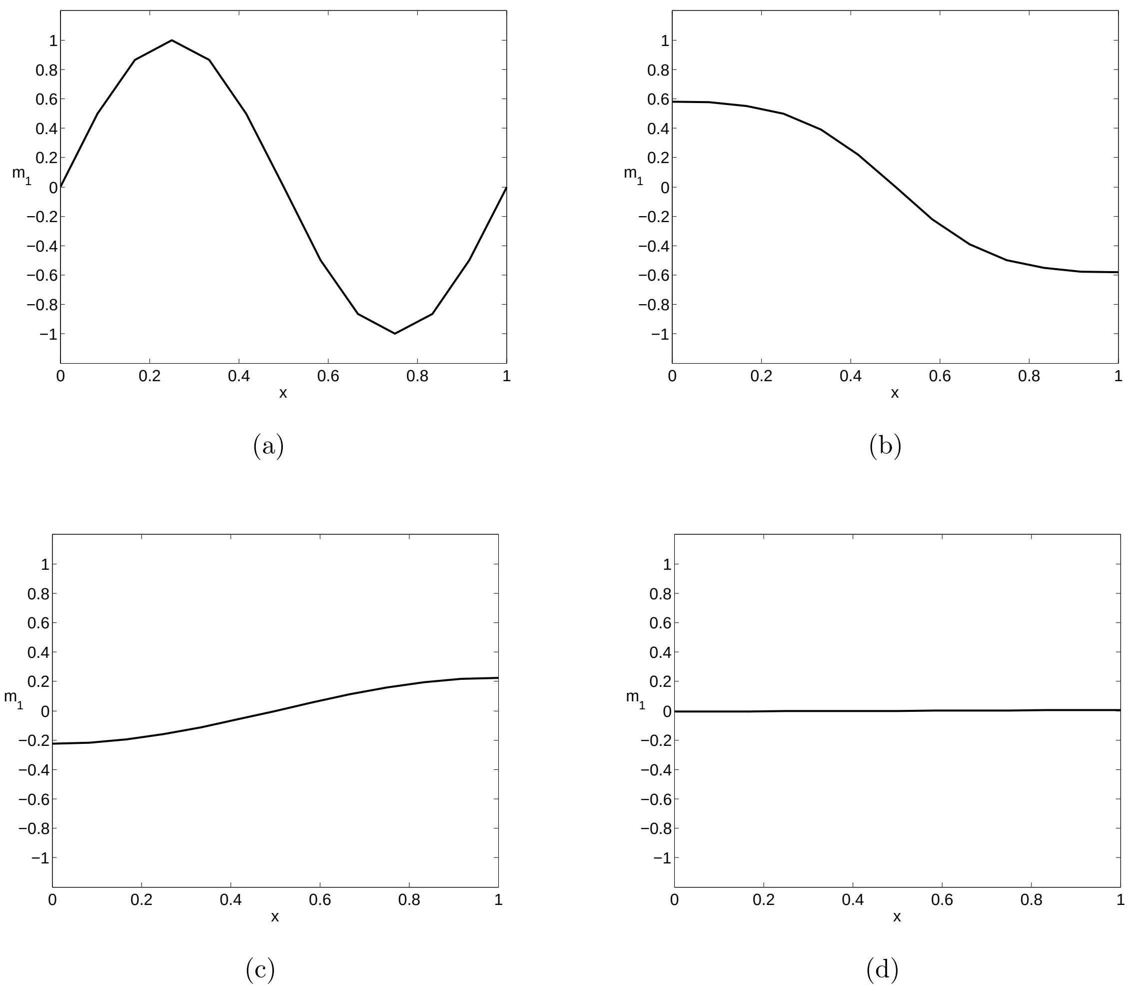
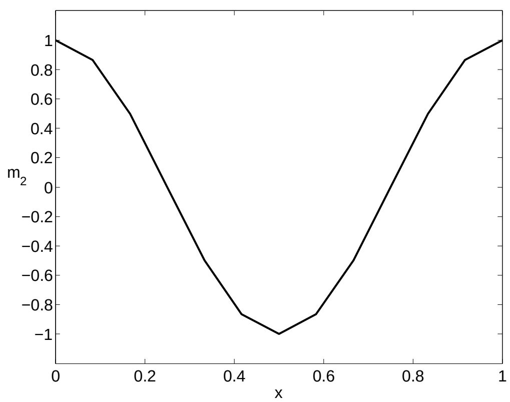
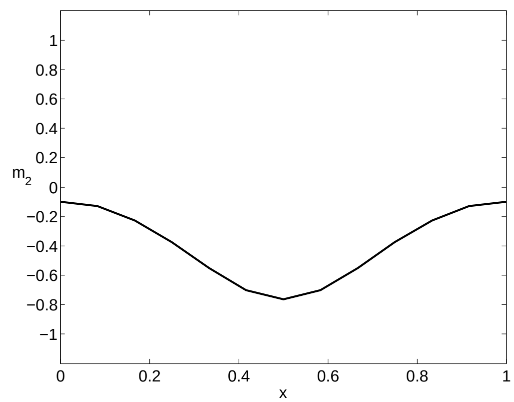


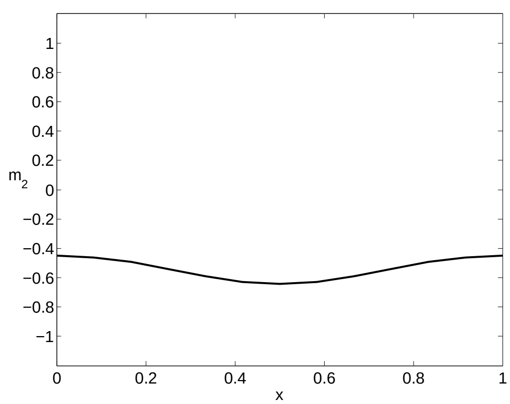
Figure 4.3: Numerical solution of the Landau–Lifshitz equation on $[0, 1]$ in the m_1 direction. The initial condition is $\mathbf{m}_0(x) = (\sin(2\pi x), \cos(2\pi x), 0)$ and ν is 0.02. On the horizontal axis is the spatial variable, $x \in [0, 1]$, and on the vertical axis is the magnetization. The initial magnetization is depicted in (a) and as time progresses, the magnetization evolves from (a) to (d). It is clear that the dynamics eventually settle to 0. For a three–dimensional depiction, see Figure 4.6a



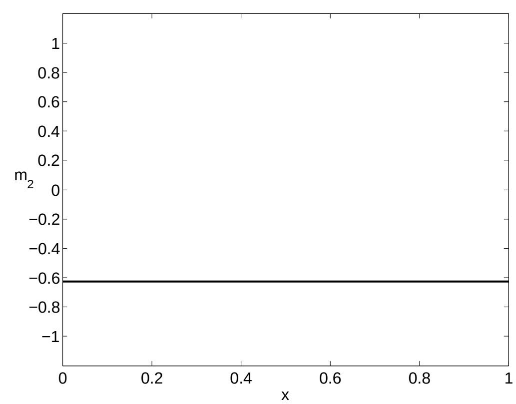
(a)



(b)



(c)



(d)

Figure 4.4: Numerical solution of the Landau–Lifshitz equation on $[0, 1]$ in the m_2 direction. The initial condition is $\mathbf{m}_0(x) = (\sin(2\pi x), \cos(2\pi x), 0)$ and ν is 0.02. On the horizontal axis is the spatial variable, $x \in [0, 1]$, and on the vertical axis is the magnetization. The initial magnetization is depicted in (a) and as time progresses, the magnetization evolves from (a) to (d). It follows that the dynamics eventually settle to -0.6. For a three–dimensional depiction, see Figure 4.6b

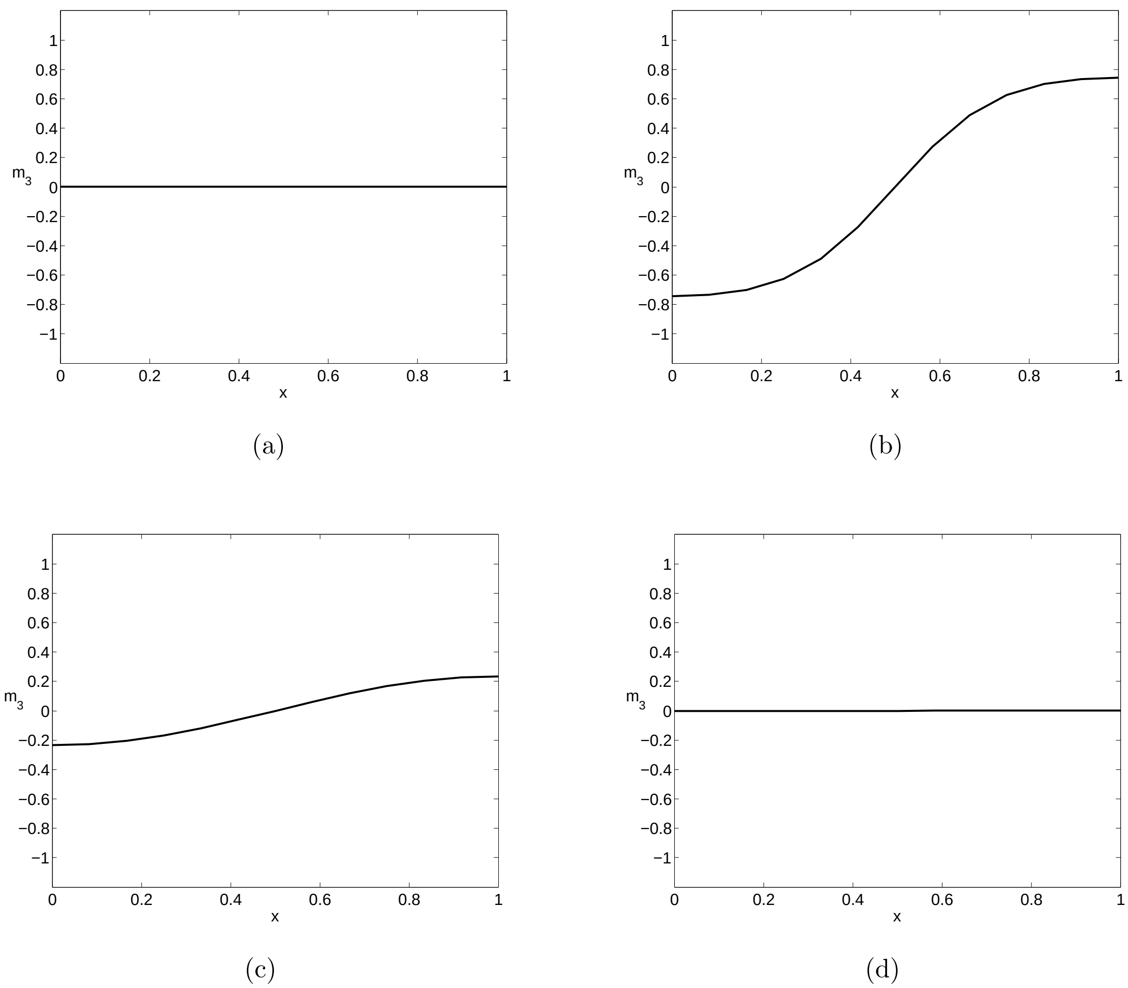
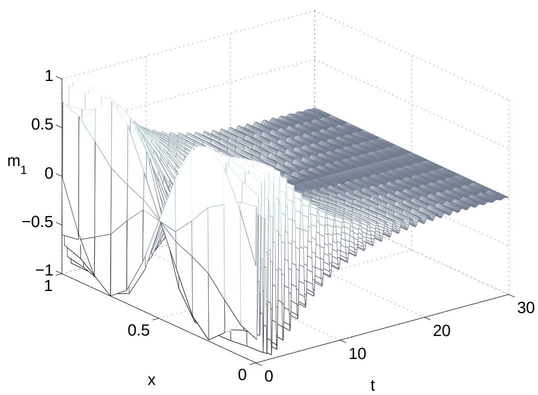
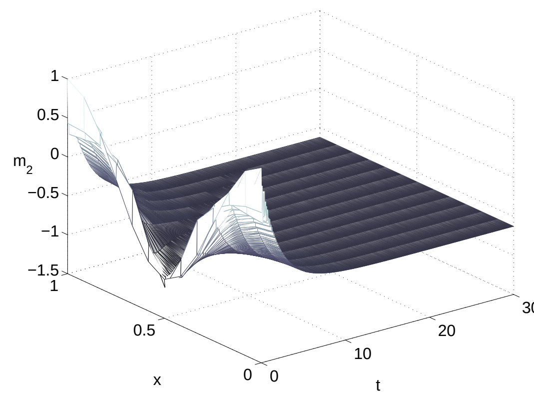


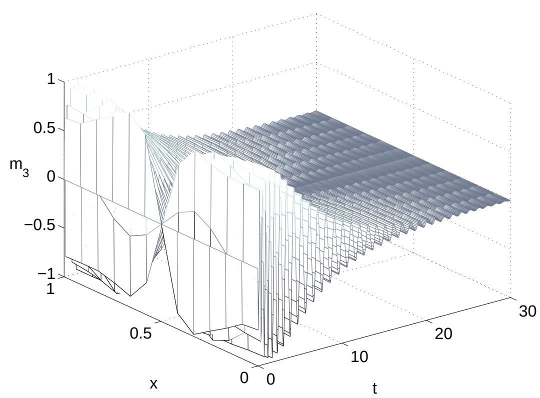
Figure 4.5: Numerical solution of the Landau–Lifshitz equation on $[0, 1]$ in the m_3 direction. The initial condition is $\mathbf{m}_0(x) = (\sin(2\pi x), \cos(2\pi x), 0)$ and ν is 0.02. On the horizontal axis is the spatial variable, $x \in [0, 1]$, and on the vertical axis is the magnetization. The initial magnetization is depicted in (a) and as time progresses, the magnetization evolves from (a) to (d). It follows that the dynamics eventually settle back to 0. For a three–dimensional depiction, see Figure 4.6c



(a) $m_1(x,t)$



(b) $m_2(x,t)$



(c) $m_3(x,t)$

Figure 4.6: Magnetization dynamics to the Landau–Lifshitz equation as x, t varies. The initial condition is $\mathbf{m}_0(x) = (\sin(2\pi x), \cos(2\pi x), 0)$ with $\nu = 0.02$ and $L = 1$. The magnetizations settle to $(0, -0.6, 0)$.

Figures 4.3 to 4.6 demonstrate that solutions to the Landau–Lifshitz equation naturally settle to a constant. This supports the analytical results in Theorem 4.10 and Theorem 4.11. The initial condition is chosen to be $\mathbf{m}_0(x) = (\sin(2\pi x), \cos(2\pi x), 0)$, which satisfies $||\mathbf{m}_0(x)||_2 = 1$. The parameter, ν, is chosen to be 0.02 and L is set to 1. It is clear from the figures that the magnetization settles to $(0, -0.6, 0)$. Comparing all three figures, the influence of m_1 and m_2 on m_3 is apparent. That is, even though m_3 begins at a constant equilibrium of 0, which is already stable by Theorem 4.11, the dynamics require some time to settle back to 0. This is due to the influence of m_1 and m_2. See Appendix B for the numerical approximation.

4.3 Linear Landau–Lifshitz Equation

To obtain the linear Landau–Lifshitz equation, we perturb (4.3) with

$$\mathbf{m}(x,t) = \tilde{\mathbf{m}}(x) + \mathbf{v}(x,t)$$

where $\tilde{\mathbf{m}}$ is an equilibrium of (4.3) and $\mathbf{v} \in \mathcal{L}_2^3$ is a small perturbation.

To begin, rewrite the last term in (4.3) as follows

$$\begin{aligned} -\nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) &= -\nu \left(\mathbf{m}^{\mathrm{T}} \mathbf{m}_{xx}\right) \mathbf{m} + \nu \left(\mathbf{m}^{\mathrm{T}} \mathbf{m}\right) \mathbf{m}_{xx} && \text{(Table 4.1)} \\ &= -\nu \left(\mathbf{m}^{\mathrm{T}} \mathbf{m}_{xx}\right) \mathbf{m} + \nu \mathbf{m}_{xx} && \text{(applying (4.1))} \\ &= \nu \left|\left|\mathbf{m}_x\right|\right|_2^2 \mathbf{m} + \nu \mathbf{m}_{xx} && \text{(Lemma 4.9)} \end{aligned}$$

and substituting into (4.3) implies

$$\frac{\partial \mathbf{m}}{\partial t} = \nu \mathbf{m}_{xx} + \mathbf{m} \times \mathbf{m}_{xx} + \nu \left|\left|\mathbf{m}_x\right|\right|_2^2 \mathbf{m}. \tag{4.20}$$

Equation (4.20) is the semilinear form of the Landau–Lifshitz equation. Substituting $\mathbf{m}(x,t) = \tilde{\mathbf{m}}(x) + \mathbf{v}(x,t)$ into (4.20) yields

$$\frac{\partial \mathbf{v}}{\partial t} = \nu \tilde{\mathbf{m}}_{xx} + \nu \mathbf{v}_{xx} + (\tilde{\mathbf{m}} + \mathbf{v}) \times (\tilde{\mathbf{m}}_{xx} + \mathbf{v}_{xx}) + \nu \left|\left|\tilde{\mathbf{m}}_x + \mathbf{v}_x\right|\right|_2^2 (\tilde{\mathbf{m}} + \mathbf{v}).$$

Expanding leads to

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} = &\ \nu \tilde{\mathbf{m}}_{xx} + \nu \mathbf{v}_{xx} + \tilde{\mathbf{m}} \times \tilde{\mathbf{m}}_{xx} + \tilde{\mathbf{m}} \times \mathbf{v}_{xx} + \mathbf{v} \times \tilde{\mathbf{m}}_{xx} + \mathbf{v} \times \mathbf{v}_{xx} \\ &+ \nu \left(\left|\left|\tilde{\mathbf{m}}_x\right|\right|_2^2 + 2\tilde{\mathbf{m}}_x^{\mathrm{T}} \mathbf{v}_x + \left|\left|\mathbf{v}_x\right|\right|_2^2\right) (\tilde{\mathbf{m}} + \mathbf{v}) \end{aligned}$$

and considering only the terms that are linear in $\mathbf{v}$, we have

$$\frac{\partial \mathbf{v}}{\partial t} = \nu \mathbf{v}_{xx} + \tilde{\mathbf{m}} \times \mathbf{v}_{xx} + \mathbf{v} \times \tilde{\mathbf{m}}_{xx} + 2\nu \left(\tilde{\mathbf{m}}_x^{\mathrm{T}} \mathbf{v}_x\right) \tilde{\mathbf{m}} + \nu \left|\left|\tilde{\mathbf{m}}_x\right|\right|_2^2 \mathbf{v}. \tag{4.21}$$

From (4.15), $\tilde{\mathbf{m}} = \mathbf{a}$, which is a constant vector and hence (4.21) becomes

$$\frac{\partial \mathbf{v}}{\partial t} = \nu \mathbf{v}_{xx} + \mathbf{a} \times \mathbf{v}_{xx}. \tag{4.22}$$

Equation (4.22) is the Landau-Lifshtiz equation in (4.3),(4.5) linearized at $\mathbf{a} = (a_1, a_2, a_3)$ where $a_2, a_2, a_3 \in \mathbb{R}$ with $||\mathbf{a}||_2 = 1$.

Let $\mathbf{z} = (z_1, z_2, z_3) \in \mathcal{L}_2^3$. Consider the linear Landau–Lifshitz equation in state–space form

$$\dot{\mathbf{z}} = A\mathbf{z}, \qquad \mathbf{z}(0) = \mathbf{z}_0 \tag{4.23}$$

for all $t \geq 0$ where $A : D(A) \to \mathcal{L}_2^3$ is defined as

$$A\mathbf{z} = \nu \mathbf{z}_{xx} + \mathbf{a} \times \mathbf{z}_{xx} = \begin{bmatrix} \nu z_1'' - a_3 z_2'' + a_2 z_3'' \\ a_3 z_1'' + \nu z_2'' - a_1 z_3'' \\ -a_2 z_1'' + a_1 z_2'' + \nu z_3'' \end{bmatrix} \tag{4.24}$$

and

$$D(A) = \{\mathbf{z} : \mathbf{z} \in \mathcal{L}_2^3,\ \mathbf{z}_x \in \mathcal{L}_2^3,\ \mathbf{z}_{xx} \in \mathcal{L}_2^3,\ \mathbf{z}_x(0,t) = \mathbf{0} = \mathbf{z}_x(L,t)\}. \tag{4.25}$$

4.3.1 Well–Posedness

We apply the Lumer–Phillips Theorem (see Theorem 2.6) to show the linear operator A defined in (4.24) generates a contraction semigroup. The main result is in Theorem 4.14; however, we first show A is closed in Lemma 4.12 and dissipativity in Lemma 4.13. These lemmas are required in the proof of Theorem 4.14.

Lemma 4.12. The operator, A, defined in (4.24) is closed.

Proof of Lemma 4.12: Let A_0 be the second derivative operator on $D(A_0) = D(A)$. It is shown in Curtain and Zwart [31, Example A.3.48] that A_0 with $D(A_0)$ is closed. Define $A_1 : \mathcal{L}_2^3 \to \mathcal{L}_2^3$ as

$$A_1 = \begin{bmatrix} \nu & -a_3 & a_2 \\ a_3 & \nu & -a_1 \\ -a_2 & a_1 & \nu \end{bmatrix}.$$

It is clear that A_1 is a linear bounded operator; that is, $||A_1\mathbf{z}||_{\mathcal{L}_2^3} \leq M_{\mathbf{a},\nu}||\mathbf{z}||_{\mathcal{L}_2^3}$ for all $\mathbf{z} \in \mathcal{L}_2^3$ where $M_{\mathbf{a},\nu}$ is a positive constant that depends on $\mathbf{a}$ and ν. Furthermore, the determinant of A_1 is $\det(A_1) = \nu^3 + \nu\,(a_1^2 + a_2^2 + a_3^2) = \nu^3 + \nu > 0$ since $a_i \in \mathbb{R}$ with $||\mathbf{a}||_2 = 1$ and $\nu > 0$. Therefore, A_1 is invertible. For all $\mathbf{z} \in D(A)$, it follows that $A\mathbf{z} = A_0 A_1 \mathbf{z}$.

For $n \in \mathbb{N}$, let $\mathbf{z}_n \in D(A)$ with $\lim_{n\to\infty} \mathbf{z}_n = \mathbf{z}$ and

$$\lim_{n\to\infty} A\mathbf{z}_n = \mathbf{y}. \tag{4.26}$$

Since A_1 is a linear bounded operator, then A_1 is continuous and hence, $\lim_{n\to\infty} A_1\mathbf{z}_n = A_1\mathbf{z}$. Define $\mathbf{x}_n = A_1\mathbf{z}_n$ and $\mathbf{x} = A_1\mathbf{z}$. It follows from (4.26) with $A = A_0A_1$ that $\lim_{n\to\infty} A_0\mathbf{x}_n = \mathbf{y}$. Since A_0 is a closed operator, then $\mathbf{x} \in D(A_0)$ and $A_0\mathbf{x} = \mathbf{y}$. This implies $A_1\mathbf{z} \in D(A_0) = D(A)$ and $A_0A_1\mathbf{z} = \mathbf{y}$; and hence $\mathbf{z} \in D(A)$ since A_1 is invertible and $A\mathbf{z} = \mathbf{y}$, which means A is a closed operator. □

A similar result to Lemma 4.12 can be found in Hundertmark [42, Proposition A.9]. It is written for a general closed linear operator and a general linear bounded operator.

Lemma 4.13. The operator, A, defined in (4.24) and its adjoint are dissipative.

Proof of Lemma 4.13: For all $\mathbf{z} \in D(A)$,

$$\langle A\mathbf{z}, \mathbf{z}\rangle_{\mathcal{L}_2^3} = \langle \nu z_1'' - a_3 z_2'' + a_2 z_3'', z_1\rangle_{\mathcal{L}_2} + \langle a_3 z_1'' + \nu z_2'' - a_1 z_3'', z_2\rangle_{\mathcal{L}_2} + \langle -a_2 z_1'' + a_1 z_2'' + \nu z_3'', z_3\rangle_{\mathcal{L}_2}$$

and applying integration by parts yields

$$\begin{aligned}\langle A\mathbf{z}, \mathbf{z}\rangle_{\mathcal{L}_2^3} = &- \nu\langle z_1', z_1'\rangle_{\mathcal{L}_2} + a_3\langle z_2', z_1'\rangle_{\mathcal{L}_2} - a_2\langle z_3', z_1'\rangle_{\mathcal{L}_2}\\ &- a_3\langle z_1', z_2'\rangle_{\mathcal{L}_2} - \nu\langle z_2', z_2'\rangle_{\mathcal{L}_2} + a_1\langle z_3', z_2'\rangle_{\mathcal{L}_2}\\ &+ a_2\langle z_1', z_3'\rangle_{\mathcal{L}_2} - a_1\langle z_2', z_3'\rangle_{\mathcal{L}_2} - \nu\langle z_3', z_3'\rangle_{\mathcal{L}_2}.\end{aligned}$$

Upon rearranging, it follows that

$$\begin{aligned}\langle A\mathbf{z}, \mathbf{z}\rangle_{\mathcal{L}_2^3} = &- \nu||z_1'||^2_{\mathcal{L}_2} + a_3\overline{\langle z_1', z_2'\rangle}_{\mathcal{L}_2} - a_2\overline{\langle z_1', z_3'\rangle}_{\mathcal{L}_2}\\ &- a_3\langle z_1', z_2'\rangle_{\mathcal{L}_2} - \nu||z_2'||^2_{\mathcal{L}_2} + a_1\overline{\langle z_2', z_3'\rangle}_{\mathcal{L}_2}\\ &+ a_2\langle z_1', z_3'\rangle_{\mathcal{L}_2} - a_1\langle z_2', z_3'\rangle_{\mathcal{L}_2} - \nu||z_3'||^2_{\mathcal{L}_2}.\end{aligned}$$

Since $\nu > 0$, then

$$\mathrm{Re}\langle A\mathbf{z}, \mathbf{z}\rangle_{\mathcal{L}_2^3} = -\nu||z_1'||^2_{\mathcal{L}_2} - \nu||z_2'||^2_{\mathcal{L}_2} - \nu||z_3'||^2_{\mathcal{L}_2} \le 0.$$

Before establishing the adjoint of A is dissipative, we first calculate A^*. For $\mathbf{z}, \mathbf{w} \in D(A)$

$$\begin{aligned}\langle A\mathbf{z}, \mathbf{w}\rangle_{\mathcal{L}_2^3} &= \langle \nu z_1'' - a_3 z_2'' + a_2 z_3'', w_1\rangle_{\mathcal{L}_2} \\ &+ \langle a_3 z_1'' + \nu z_2'' - a_1 z_3'', w_2\rangle_{\mathcal{L}_2} \\ &+ \langle -a_2 z_1'' + a_1 z_2'' + \nu z_3'', w_3\rangle_{\mathcal{L}_2}\end{aligned}$$

and applying integration by parts twice,

$$\begin{aligned}\langle A\mathbf{z}, \mathbf{w}\rangle_{\mathcal{L}_2^3} &= \nu\langle z_1, w_1''\rangle_{\mathcal{L}_2} - a_3\langle z_2, w_1''\rangle_{\mathcal{L}_2} + a_2\langle z_3, w_1''\rangle_{\mathcal{L}_2} \\ &+ a_3\langle z_1, w_2''\rangle_{\mathcal{L}_2} + \nu\langle z_2, w_2''\rangle_{\mathcal{L}_2} - a_1\langle z_3, w_2''\rangle_{\mathcal{L}_2} \\ &- a_2\langle z_1, w_3''\rangle_{\mathcal{L}_2} + a_1\langle z_2, w_3''\rangle_{\mathcal{L}_2} + \nu\langle z_3, w_3''\rangle_{\mathcal{L}_2}.\end{aligned}$$

Rearranging leads to

$$\begin{aligned}\langle A\mathbf{z}, \mathbf{w}\rangle_{\mathcal{L}_2^3} &= \langle z_1, \nu w_1''\rangle_{\mathcal{L}_2} + \langle z_1, a_3 w_2''\rangle_{\mathcal{L}_2} + \langle z_1, -a_2 w_3''\rangle_{\mathcal{L}_2} \\ &+ \langle z_2, -a_3 w_1''\rangle_{\mathcal{L}_2} + \langle z_2, \nu w_2''\rangle_{\mathcal{L}_2} + \langle z_2, a_1 w_3''\rangle_{\mathcal{L}_2} \\ &+ \langle z_3, a_2 w_1''\rangle_{\mathcal{L}_2} + \langle z_3, -a_1 w_2''\rangle_{\mathcal{L}_2} + \langle z_3, \nu w_3''\rangle_{\mathcal{L}_2}\end{aligned}$$

and hence the adjoint of A is

$$A^*\mathbf{w} = \begin{bmatrix} \nu w_1'' & a_3 w_2'' & -a_2 w_3'' \\ -a_3 w_1'' & \nu w_2'' & a_1 w_3'' \\ a_2 w_1'' & -a_1 w_2'' & \nu w_3'' \end{bmatrix}$$

with $D(A) = D(A^*)$. For all $\mathbf{z} \in D(A^*)$,

$$\langle A^*\mathbf{z}, \mathbf{z}\rangle_{\mathcal{L}_2^3} = \langle \nu z_1'' + a_3 z_2'' - a_2 z_3'', z_1\rangle_{\mathcal{L}_2} + \langle -a_3 z_1'' + \nu z_2'' + a_1 z_3'', z_2\rangle_{\mathcal{L}_2} + \langle a_2 z_1'' - a_1 z_2'' + \nu z_3'', z_3\rangle_{\mathcal{L}_2}$$

and applying integration by parts, we obtain

$$\begin{aligned}\langle A^*\mathbf{z}, \mathbf{z}\rangle_{\mathcal{L}_2^3} &= -\nu\langle z_1', z_1'\rangle_{\mathcal{L}_2} - a_3\langle z_2', z_1'\rangle_{\mathcal{L}_2} + a_2\langle z_3', z_1'\rangle_{\mathcal{L}_2} \\ &+ a_3\langle z_1', z_2'\rangle_{\mathcal{L}_2} - \nu\langle z_2', z_2'\rangle_{\mathcal{L}_2} - a_1\langle z_3', z_2'\rangle_{\mathcal{L}_2} \\ &- a_2\langle z_1', z_3'\rangle_{\mathcal{L}_2} + a_1\langle z_2', z_3'\rangle_{\mathcal{L}_2} - \nu\langle z_3', z_3'\rangle_{\mathcal{L}_2}.\end{aligned}$$

It follows that

$$\begin{aligned}\langle A^*\mathbf{z},\mathbf{z}\rangle_{\mathcal{L}_2^3} = &-\nu||z_1'||^2_{\mathcal{L}_2} - a_3\langle\overline{z_1', z_2'}\rangle_{\mathcal{L}_2} + a_2\langle\overline{z_1', z_3'}\rangle_{\mathcal{L}_2}\\ &+ a_3\langle z_1', z_2'\rangle_{\mathcal{L}_2} - \nu||z_2'||^2_{\mathcal{L}_2} - a_1\langle\overline{z_2', z_3'}\rangle_{\mathcal{L}_2}\\ &- a_2\langle z_1', z_3'\rangle_{\mathcal{L}_2} + a_1\langle z_2', z_3'\rangle_{\mathcal{L}_2} - \nu||z_3'||^2_{\mathcal{L}_2}\end{aligned}$$

and since $\nu > 0$, then

$$\mathrm{Re}\langle A^*\mathbf{z},\mathbf{z}\rangle_{\mathcal{L}_2^3} = -\nu||z_1'||^2_{\mathcal{L}_2} - \nu||z_2'||^2_{\mathcal{L}_2} - \nu||z_3'||^2_{\mathcal{L}_2} \leq 0. \quad \square$$

Theorem 4.14. The operator, A, defined in (4.24) generates a linear contraction semigroup.

Proof of Theorem 4.14: Since $\mathcal{L}_2^3$ is dense and boundary conditions do not affect denseness, then $D(A)$ is dense in $\mathcal{L}_2^3$. It is easy to see that A is a linear operator and from Lemma 4.12, A is closed. We also have from Lemma 4.13 that A and its adjoint are dissipative operators. By the Lumer–Phillips Theorem, A generates a contraction semigroup. □

The next main result is in Theorem 4.16, which shows A defined in (4.24) generates a linear analytic semigroup. The proof of Theorem 4.16 relies on the following lemma and Corollary 2.8.

Lemma 4.15. If $\mathbf{a} \in E$ where E is defined in (4.15), then $||\mathbf{a}\times\mathbf{z}||_{\mathcal{L}_2^3} \leq 2||\mathbf{z}||_{\mathcal{L}_2^3}$ for all $\mathbf{z} \in \mathcal{L}_2^3$.

Proof of Lemma 4.15: Consider

$$||\mathbf{a}\times\mathbf{z}||^2_{\mathcal{L}_2^3} = ||a_2z_3 - a_3z_2||^2_{\mathcal{L}_2} + ||-a_1z_3 + a_3z_1||^2_{\mathcal{L}_2} + ||a_1z_2 - a_2z_1||^2_{\mathcal{L}_2}.$$

It follows that

$$||\mathbf{a}\times\mathbf{z}||^2_{\mathcal{L}_2^3} \leq \left(||a_2z_3||_{\mathcal{L}_2} + ||a_3z_2||_{\mathcal{L}_2}\right)^2 + \left(||a_1z_3||_{\mathcal{L}_2} + ||a_3z_1||_{\mathcal{L}_2}\right)^2 + \left(||a_1z_2||_{\mathcal{L}_2} + ||a_2z_1||_{\mathcal{L}_2}\right)^2.$$

Expanding the square and then applying Young's inequality implies

$$||\mathbf{a}\times\mathbf{z}||^2_{\mathcal{L}_2^3} \leq 2||a_2z_3||^2_{\mathcal{L}_2} + 2||a_3z_2||^2_{\mathcal{L}_2} + 2||a_1z_3||^2_{\mathcal{L}_2} + 2||a_3z_1||^2_{\mathcal{L}_2} + 2||a_1z_2||^2_{\mathcal{L}_2} + 2||a_2z_1||^2_{\mathcal{L}_2}.$$

Since a_1, a_2, a_3 are constants,

$$||\mathbf{a}\times\mathbf{z}||^2_{\mathcal{L}_2^3} \leq 2a_2^2||z_3||^2_{\mathcal{L}_2} + 2a_3^2||z_2||^2_{\mathcal{L}_2} + 2a_1^2||z_3||^2_{\mathcal{L}_2} + 2a_3^2||z_1||^2_{\mathcal{L}_2} + 2a_1^2||z_2||^2_{\mathcal{L}_2} + 2a_2^2||z_1||^2_{\mathcal{L}_2}$$

and rewriting under the $\mathcal{L}_2^3$–norm leads to

$$||\mathbf{a} \times \mathbf{z}||_{\mathcal{L}_2^3}^2 \leq 4\left(a_1^2 + a_2^2 + a_3^2\right)||\mathbf{z}||_{\mathcal{L}_2^3}^2.$$

We have $a_1^2 + a_2^2 + a_3^2 = 1$ since $\mathbf{a} \in E$ and hence $||\mathbf{a} \times \mathbf{z}||_{\mathcal{L}_2^3} \leq 2||\mathbf{z}||_{\mathcal{L}_2^3}$ as desired. □

Theorem 4.16. The operator, A, defined in (4.24) generates a linear analytic semigroup.

Proof of Theorem 4.16: This proof relies on Corollary 2.8. The state space for the linear Landau–Lifshitz equation is $\mathcal{L}_2^3$. To determine a dense space $Y \subset \mathcal{L}_2^3$, multiply (4.24) by a test function, $\boldsymbol{\phi}$, and then integrate,

$$\begin{aligned}
\int_0^L (A\mathbf{z})^{\mathrm{T}} \boldsymbol{\phi} dx &= \int_0^L \nu \mathbf{z}_{xx}^{\mathrm{T}} \boldsymbol{\phi} dx + \int_0^L (\mathbf{a} \times \mathbf{z}_{xx})^{\mathrm{T}} \boldsymbol{\phi} dx \\
&= -\int_0^L \nu \mathbf{z}_x^{\mathrm{T}} \boldsymbol{\phi}_x dx + \left[\mathbf{z}_x^{\mathrm{T}} \boldsymbol{\phi}\right]_0^L - \int_0^L (\mathbf{a} \times \mathbf{z}_x)^{\mathrm{T}} \boldsymbol{\phi}_x dx + \left[(\mathbf{a} \times \mathbf{z}_x)^{\mathrm{T}} \boldsymbol{\phi}\right]_0^L \\
&= -\int_0^L \nu \mathbf{z}_x^{\mathrm{T}} \boldsymbol{\phi}_x dx - \int_0^L (\mathbf{a} \times \mathbf{z}_x)^{\mathrm{T}} \boldsymbol{\phi}_x dx.
\end{aligned}$$

Set $Y = H_1 = \{\boldsymbol{\phi} \in \mathcal{L}_2^3, \boldsymbol{\phi}_x \in \mathcal{L}_2^3\}$ and define $\sigma : H_1 \times H_1 \to \mathbb{C}$ as

$$\sigma(\boldsymbol{\phi}, \boldsymbol{\psi}) = \langle \nu \boldsymbol{\phi}_x, \boldsymbol{\psi}_x \rangle_{\mathcal{L}_2^3} + \langle \mathbf{a} \times \boldsymbol{\phi}_x, \boldsymbol{\psi}_x \rangle_{\mathcal{L}_2^3}$$

for all $\boldsymbol{\phi}, \boldsymbol{\psi} \in H_1$. The H_1 inner product and norm is as usual; that is,

$$\begin{aligned}
\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{H_1} &= \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{\mathcal{L}_2^3} + \langle \boldsymbol{\phi}_x, \boldsymbol{\psi}_x \rangle_{\mathcal{L}_2^3} \\
||\boldsymbol{\phi}||_{H_1}^2 &= ||\boldsymbol{\phi}||_{\mathcal{L}_2^3}^2 + ||\boldsymbol{\phi}_x||_{\mathcal{L}_2^3}^2.
\end{aligned}$$

To show σ satisfies (2.6), consider for all $\boldsymbol{\phi}, \boldsymbol{\psi} \in H_1$,

$$\begin{aligned}
|\sigma(\boldsymbol{\phi}, \boldsymbol{\psi})| &= |\langle \nu \boldsymbol{\phi}_x, \boldsymbol{\psi}_x \rangle_{\mathcal{L}_2^3} + \langle \mathbf{a} \times \boldsymbol{\phi}_x, \boldsymbol{\psi}_x \rangle_{\mathcal{L}_2^3}| \\
&\leq |\langle \nu \boldsymbol{\phi}_x, \boldsymbol{\psi}_x \rangle_{\mathcal{L}_2^3}| + |\langle \mathbf{a} \times \boldsymbol{\phi}_x, \boldsymbol{\psi}_x \rangle_{\mathcal{L}_2^3}| \\
&\leq \nu ||\boldsymbol{\phi}_x||_{\mathcal{L}_2^3} ||\boldsymbol{\psi}_x||_{\mathcal{L}_2^3} + ||\mathbf{a} \times \boldsymbol{\phi}_x||_{\mathcal{L}_2^3} ||\boldsymbol{\psi}_x||_{\mathcal{L}_2^3} && \text{(Cauchy-Schwarz Inequality)} \\
&\leq (\nu + 2)\, ||\boldsymbol{\phi}_x||_{\mathcal{L}_2^3} ||\boldsymbol{\psi}_x||_{\mathcal{L}_2^3} && \text{(Lemma 4.15)} \\
&\leq (\nu + 2) \left(||\boldsymbol{\phi}||_{\mathcal{L}_2^3} + ||\boldsymbol{\phi}_x||_{\mathcal{L}_2^3}\right) \left(||\boldsymbol{\psi}||_{\mathcal{L}_2^3} + ||\boldsymbol{\psi}_x||_{\mathcal{L}_2^3}\right) \\
&= (\nu + 2)\, ||\boldsymbol{\phi}||_{H_1} ||\boldsymbol{\psi}||_{H_1},
\end{aligned}$$

which satisfes (2.6) with $\gamma = \nu + 2$.

To show σ satisfies (2.8), consider for all $\boldsymbol{\phi} \in H_1$,

$$\sigma(\boldsymbol{\phi}, \boldsymbol{\psi}) + \nu||\boldsymbol{\phi}||^2_{\mathcal{L}^2_3} = \langle \nu\boldsymbol{\phi}_x, \boldsymbol{\phi}_x\rangle_{\mathcal{L}^3_2} + \langle \mathbf{a} \times \boldsymbol{\phi}_x, \boldsymbol{\phi}_x\rangle_{\mathcal{L}^3_2} + \nu||\boldsymbol{\phi}||^2_{\mathcal{L}^2_3}.$$

The term $\langle \mathbf{a} \times \boldsymbol{\phi}_x, \boldsymbol{\phi}_x\rangle_{\mathcal{L}^3_2}$ is zero since

$$\langle \mathbf{a} \times \boldsymbol{\phi}_x, \boldsymbol{\phi}_x\rangle_{\mathcal{L}^3_2} = \int_0^L (\mathbf{a} \times \boldsymbol{\phi}_x)^{\mathrm{T}} \boldsymbol{\phi}_x dx = \int_0^L (\boldsymbol{\phi}_x \times \boldsymbol{\phi}_x)^{\mathrm{T}} \mathbf{a}\, dx = 0$$

by Table 4.1. It follows that

$$\sigma(\boldsymbol{\phi}, \boldsymbol{\phi}) + \nu||\boldsymbol{\phi}||_{\mathcal{L}^2_3} = \nu||\boldsymbol{\phi}_x||^2_{\mathcal{L}^2_3} + \nu||\boldsymbol{\phi}||^2_{\mathcal{L}^2_3} = \nu||\boldsymbol{\phi}||^2_{H_1}.$$

and hence (2.8) is satisfied with $\lambda_0 = \delta = \nu$.

If $\boldsymbol{\phi}_{xx} \in \mathcal{L}^3_2$, we obtain

$$\begin{aligned}
|\sigma(\boldsymbol{\phi}, \boldsymbol{\psi})| &= |\langle -(\nu\boldsymbol{\phi}_{xx} + \mathbf{a} \times \boldsymbol{\phi}_{xx}), \boldsymbol{\psi}\rangle_{\mathcal{L}^3_2}| \\
&\leq ||\nu\boldsymbol{\phi}_{xx} + \mathbf{a} \times \boldsymbol{\phi}_{xx}||_{\mathcal{L}^3_2}||\boldsymbol{\psi}||_{\mathcal{L}^3_2} && \text{(Cauchy Schwarz inequality)} \\
&\leq \left(||\nu\boldsymbol{\phi}_{xx}||_{\mathcal{L}^3_2} + ||\mathbf{a} \times \boldsymbol{\phi}_{xx}||_{\mathcal{L}^3_2}\right) ||\boldsymbol{\psi}||_{\mathcal{L}^3_2} \\
&\leq \left(\nu||\boldsymbol{\phi}_{xx}||_{\mathcal{L}^3_2} + 2||\boldsymbol{\phi}_{xx}||_{\mathcal{L}^3_2}\right) ||\boldsymbol{\psi}||_{\mathcal{L}^3_2} && \text{(Lemma 4.15)} \\
&\leq (\nu + 2)\, ||\boldsymbol{\phi}_{xx}||_{\mathcal{L}^3_2}||\boldsymbol{\psi}||_{\mathcal{L}^3_2}.
\end{aligned}$$

Let $K_{\boldsymbol{\phi}} = (\nu + 2)\, ||\boldsymbol{\phi}_{xx}||_{\mathcal{L}^3_2}$, then $0 \leq K_{\boldsymbol{\phi}} < \infty$ since $\boldsymbol{\phi}_{xx} \in \mathcal{L}^3_2$. Define

$$A\boldsymbol{\phi} = \nu\boldsymbol{\phi}_{xx} + \mathbf{a} \times \boldsymbol{\phi}_{xx}$$

on $D(A)$ where

$$D(A) = \{\boldsymbol{\phi} \in H_1 : \boldsymbol{\phi}_{xx} \in \mathcal{L}^3_2, \boldsymbol{\phi}_x(0) = \boldsymbol{\phi}_x(L) = 0\}$$

and

$$\sigma(\boldsymbol{\phi}, \boldsymbol{\psi}) = \langle -A\boldsymbol{\phi}, \boldsymbol{\psi}\rangle_{\mathcal{L}^3_2} \qquad \text{for all } \phi \in D(A), \psi \in H_1.$$

This is the operator, A, and domain, $D(A)$, given in (4.24) and (4.25), respectively. By Corollary 2.8, A generates an analytic semigroup. □

Let the semigroup generated by A defined in (4.24) be denoted $T(t)$. Since $T(t)$ is a linear analytic semigroup, then the spectrum determined growth assumption is satisfied (see equation (2.2)). This implies $||T(t)||_{OP} \leq e^{\omega_0 t}$ for all t where ω_0 is defined in (2.3). We will see in the next section that the eigenvalues of (4.23) are either zero or have negative real part. Therefore, $\omega_0 = 0$ and hence $||T(t)||_{OP} \leq 1$. This means $T(t)$ is a contraction semigroup which was already established in Theorem 4.14 using the Lumer–Phillips Theorem.

4.3.2 Eigenvalues

Eigenvalues play an important role in establishing the stability of semigroups and equilibrium points and helping to determine the appropriate controller. This is made even more significant since the linear Landau–Lifshitz equation satisfies the spectrum determined growth assumption (Section 4.3.1). Therefore, in this section we evaluate the eigenvalues of the linear Landau–Lifshitz equation.

Let $\lambda \in \mathbb{C}$. The eigenvalue problem of (4.23) is

$$\lambda v_1 = \nu v_1'' - a_3 v_2'' + a_2 v_3'' \tag{4.27a}$$
$$\lambda v_2 = a_3 v_1'' + \nu v_2'' - a_1 v_3'' \tag{4.27b}$$
$$\lambda v_3 = -a_2 v_1'' + a_1 v_2'' + \nu v_3'' \tag{4.27c}$$

with boundary conditions

$$v_1'(0) = v_1'(L) = 0 \tag{4.27d}$$
$$v_2'(0) = v_2'(L) = 0 \tag{4.27e}$$
$$v_3'(0) = v_3'(L) = 0 \tag{4.27f}$$

where $\mathbf{v} \in \mathcal{L}_2^3$. The eigenvalue problem is solved in Maple. See Appendix C for the code.

For the zero eigenvalue $\lambda_1 = 0$, the corresponding eigenvector is $\mathbf{v}_1 = (c_1, c_2, c_3)$ for any constants $c_1, c_2, c_3 \in \mathbb{C}$. The choice $c_1 = c_2 = c_3 = 0$ is excluded since this leads to the zero vector. The remaining eigenvalues and eigenvectors are

$$\lambda_2^+ = \frac{-(1+2n)^2\pi^2\nu}{L^2} + i\frac{(1+2n)^2\pi^2}{L^2}, \quad \mathbf{v}_2^+ = 2c_4 \begin{bmatrix} i\frac{-a_3 - ia_1a_2}{a_2^2+a_3^2} \\ \frac{a_2 - ia_1a_3}{a_2^2+a_3^2} \end{bmatrix} \cos\left(\frac{(1+2n)\pi}{L}x\right)$$

$$\lambda_2^- = \frac{-(1+2n)^2\pi^2\nu}{L^2} - i\frac{(1+2n)^2\pi^2}{L^2}, \quad \mathbf{v}_2^- = 2c_5 \begin{bmatrix} i \\ \frac{a_3 - ia_1a_2}{a_2^2+a_3^2} \\ \frac{-a_2 - ia_1a_3}{a_2^2+a_3^2} \end{bmatrix} \cos\left(\frac{(1+2n)\pi}{L}x\right)$$

$$\lambda_3 = \frac{-(1+2n)^2\pi^2\nu}{L^2}, \quad \mathbf{v}_3 = 2ic_6 \begin{bmatrix} 1 \\ \frac{a_2}{a_1} \\ \frac{a_3}{a_1} \end{bmatrix} \cos\left(\frac{(1+2n)\pi}{L}x\right)$$

$$\lambda_4^+ = \frac{-(2n)^2\pi^2\nu}{L^2} + i\frac{(2n)^2\pi^2}{L^2}, \quad \mathbf{v}_4^+ = 2c_7 \begin{bmatrix} i \\ \frac{-a_3-ia_1a_2}{a_2^2+a_3^2} \\ \frac{a_2-ia_1a_3}{a_2^2+a_3^2} \end{bmatrix} \cos\left(\frac{2n\pi}{L}x\right)$$

$$\lambda_4^- = \frac{-(2n)^2\pi^2\nu}{L^2} - i\frac{(2n)^2\pi^2}{L^2}, \quad \mathbf{v}_4^- = 2c_8 \begin{bmatrix} i \\ \frac{a_3-ia_1a_2}{a_2^2+a_3^2} \\ \frac{-a_2-ia_1a_3}{a_2^2+a_3^2} \end{bmatrix} \cos\left(\frac{2n\pi}{L}x\right)$$

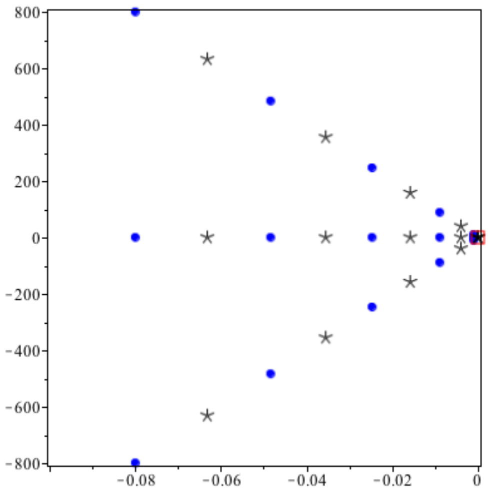
$$\lambda_5 = \frac{-(2n)^2\pi^2\nu}{L^2}, \quad \mathbf{v}_5 = 2ic_9 \begin{bmatrix} 1 \\ \frac{a_2}{a_1} \\ \frac{a_3}{a_1} \end{bmatrix} \cos\left(\frac{2n\pi}{L}x\right)$$

where $c_i \in \mathbb{C}$ for $i = 4, 5, \ldots, 9$ are nonzero constants and $n \in \mathbb{Z}$. Figure 4.7 depicts plots of the eigenvalues for various values of ν with $L = 1$ and $n = 0, \pm1, \pm2, \pm3, \pm4$.

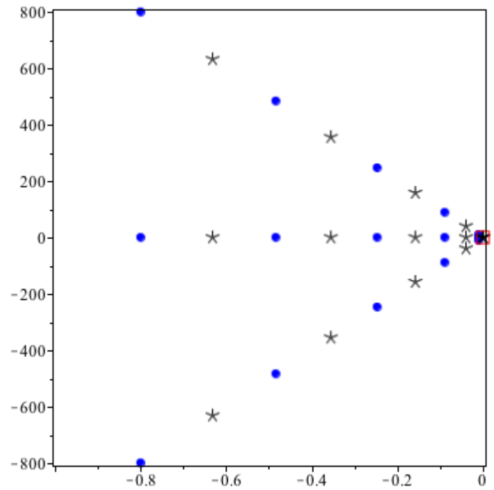
4.4 Hysteresis

It is well known that the Landau–Lifshitz equation exhibits hysteretic behaviour. The shape of the hysteresis loop is governed by the composition and structure of the magnet and the input. Physically, the input is the applied external magnetic field acting on the ferromagnet. Cowburn *et al.* [29] investigated via experiments the shape change of the hysteresis loop as the thickness and diameter of a circular–shaped nanomagnet varies. Experiments conducted on nanowires also demonstrate hysteresis loops (Noh *et al.* [69]). Suess *et al.* [83] investigates how the surface structure of a nanomagnet affects the behaviour of hysteresis. They present numerical simulations illustrating the magnetization "jumping" from one equilibrium to another. Numerical simulations illustrating hysteresis loops is also found in Wiele *et al.* [87], and Yang and Zhao [100]. Carbou *et al.* [21] model the dynamics of hysteresis by adding a hysteresis operator term to the effective field in (4.2). Visintin [91] also considers a hysteresis operator.

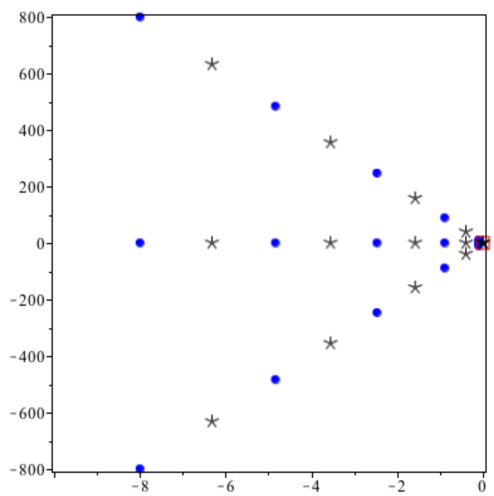
In much of the aforementioned literature, the presence of hysteresis in the Landau–Lifshitz equation is identified by the fact that input–output curves exhibit a looping behaviour. We demonstrated in Chapter 3 that this alone is not enough to characterize hysteresis. In the following, we establish that the input–output curves of the Landau–



(a) $\nu = 0.0001$, $L = 1$, $n = 0, \pm1, \pm2, \pm3, \pm4$



(b) $\nu = 0.001$, $L = 1$, $n = 0, \pm1, \pm2, \pm3, \pm4$

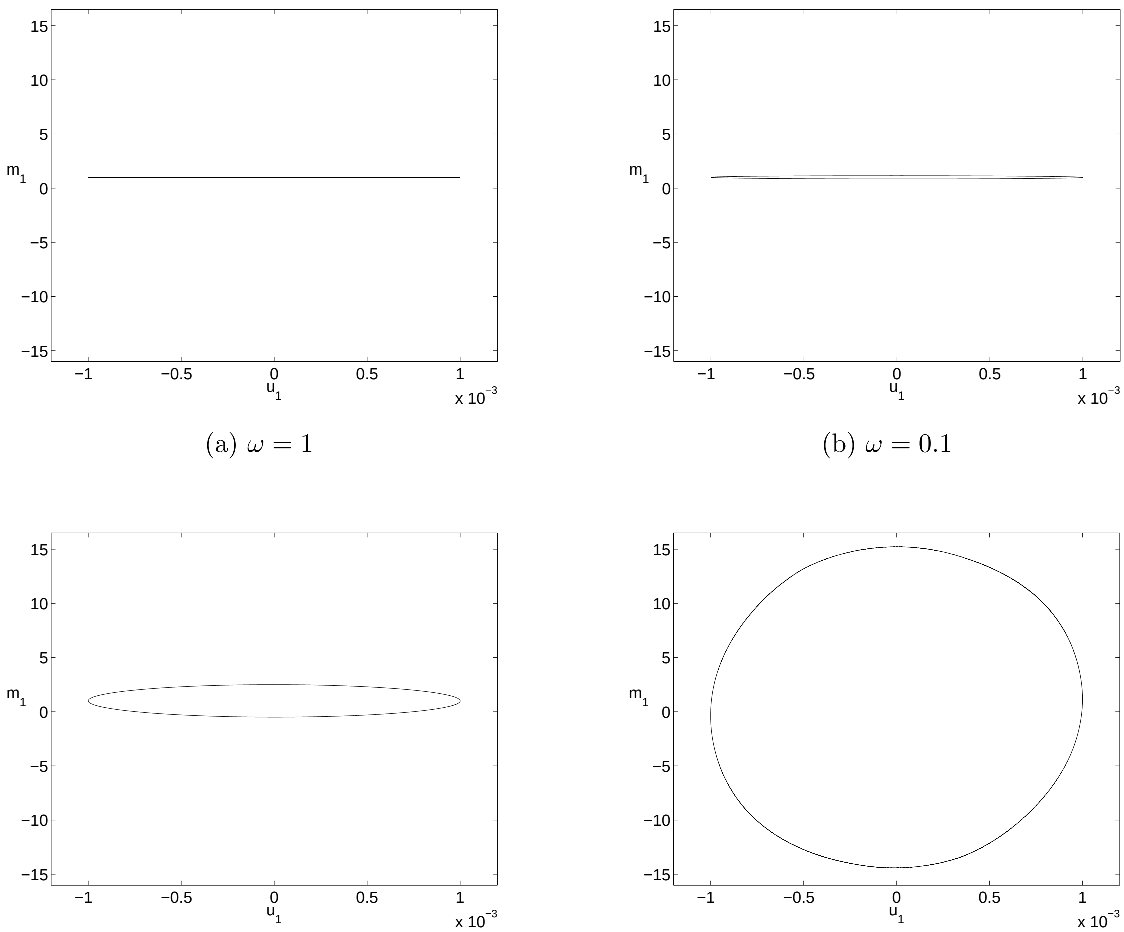


(c) $\nu = 0.01$, $L = 1$, $n = 0, \pm1, \pm2, \pm3, \pm4$

Figure 4.7: Plots of the eigenvalues $(\lambda_2, \lambda_3^+, \lambda_3^-, \lambda_4, \lambda_5^+, \lambda_5^-, \lambda_6)$ for various values of ν with $L = 1$ and $n = -4, \dots, 4$. The real and imaginary axis are on the horizontal and vertical axis, respectively. The red square represents $\lambda_2 = 0$, the blue dots are the eigenvalues with real part $-(1 + 2n)^2\pi^2\nu/L^2$ (namely, $\lambda_3^+, \lambda_3^-, \lambda_4$) and the black asterisks are the eigenvalues with real part $-(2n)^2\pi^2\nu/L^2$ (namely, $\lambda_5^+, \lambda_5^-, \lambda_6$).

Lifshitz equation has persistent loops as the frequency of the input approaches 0. This satisfies Definition 3.2 and hence the dynamics governed by the Landau–Lifshitz equation exhibits hysteresis. This is not surprising as the equation has multiple stable equilibria.

The input–output curves for the Landau–Lifshitz equation is governed by



(a) $\omega = 1$ (b) $\omega = 0.1$

(c) $\omega = 0.01$ (d) $\omega = 0.001$

Figure 4.8: Hysteresis loops for $m_1(x,t)$ of the Landau–Lifshitz equation with x fixed and $\nu = 0.02$, $L = 1$. The input is $\mathbf{u}(t) = (0.001\cos(\omega t), 0, 0)$. The initial condition is $\mathbf{m}_0(x) = (1, 0, 0)$. It is clear loops persist as ω approaches 0.

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) + \mathbf{u}(t)$$

where $\mathbf{u}(t)$ is the input. We have selected the input to be $\mathbf{u}(t) = (0.001\cos(\omega t), 0, 0)$; that is, there is a time–varying magnetic field in the m_1 direction and no magnetic field in the m_2 and m_3. The associated hysteresis loops for $m_1(x, t)$ with x fixed is illustrated in Figure 4.8. A constant initial condition, $\mathbf{m}_0(x) = (1, 0, 0)$, is chosen since for the Landau–Lifshitz equation, any constant is an equilibrium (Theorem 4.11) and hence the system begins at an equilibrium. The parameters ν and L are chosen to be 0.02 and 1, respectively. It is clear from Figure 4.8 that the input–output curves of the Landau–Lifshitz equation exhibit persistent loops. Furthermore, the hysteresis loops have the same appearance as those governed by (3.6). This is because in both examples the equilibrium points consist of the entire real line.

Hysteresis loops for $m_2(x, t)$ and $m_3(x, t)$ with initial conditions, $\mathbf{m}_0(x) = (0, 1, 0)$ and $\mathbf{m}_0(x) = (0, 0, 1)$, and input $\mathbf{u}(t) = (0, 0.001\cos(\omega t), 0)$ and $\mathbf{u}(t) = (0, 0, 0.001\cos(\omega t))$ are illustrated in Figure 4.9 and 4.10, respectively. They are nearly identical to the hysteresis loops for $m_1(x, t)$. This is due to the symmetric structure of the three differential equations in the Landau–Lifshitz model.

Recall that in Section 4.3, we investigated the Landau–Lifshitz equation linearized at a constant equilibrium, $\mathbf{a}$. In particular, the equation is described by (4.23). The equilibrium solutions of the linear Landau–Lifshitz equation is determined by

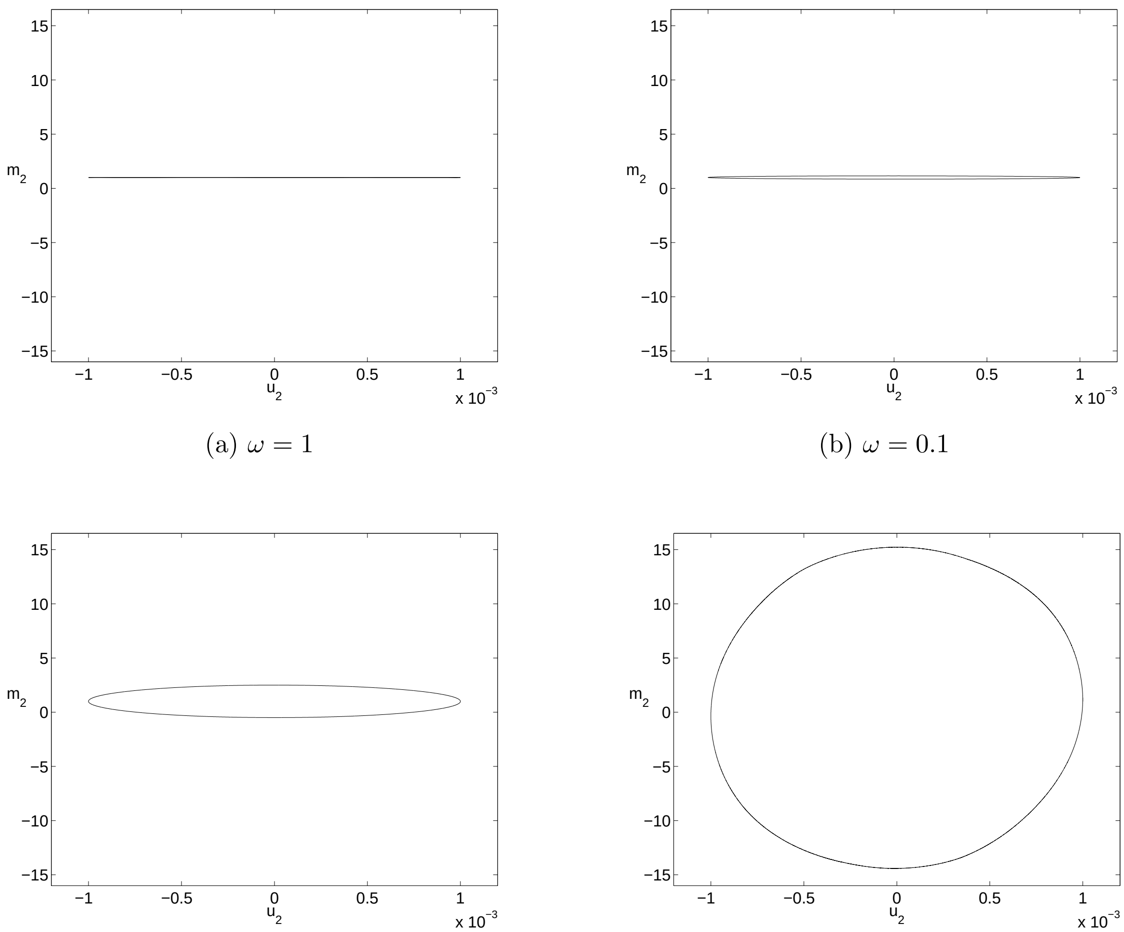
$$\mathbf{0} = \nu \mathbf{z}_{xx} + \mathbf{a} \times \mathbf{z}_{xx}.$$

It is clear that any constant is an equilibrium solution of (4.23) and also satisfy the boundary conditions, $\mathbf{z}_x(0) = \mathbf{z}_x(L) = \mathbf{0}$. Furthermore, we established in Section 4.3.2 that the linear Landau–Lifshitz equation satisfies the spectrum determined growth assumption. This means its eigenvalues can determine the stability of the equilibria. Since the eigenvalues of the linear Landau–Lifshitz equation have nonpositive real part (see Section 4.3.2), the equilibrium solutions are stable. From Definition 3.1, this suggests the linear Landau–Lifshitz equation exhibits hysteresis. The input–output curves are governed by

$$\dot{\mathbf{z}} = \nu \mathbf{z}_{xx} + \mathbf{a} \times \mathbf{z}_{xx} + \mathbf{u}(t)$$

where $\mathbf{u}(t)$ is the input.

Figures 4.11 to 4.13 depict the input–output curves for the linear Landau–Lifshitz equation. As in the nonlinear case, the parameters ν and L are 0.02 and 1, respectively, and the same cosine input is applied. It follows from the figures that a loop persists as the frequency of the input approaches zero. From Definition 3.2, the system is hysteretic. Notice again that the hysteresis loop is smooth; that is, it does not have jumps. This is reminiscent of the loops for the full (nonlinear) Landau–Lifshitz equation and the dynamics described by (3.6), both of which have arbitrary constants as its equilibrium.



(a) $\omega = 1$

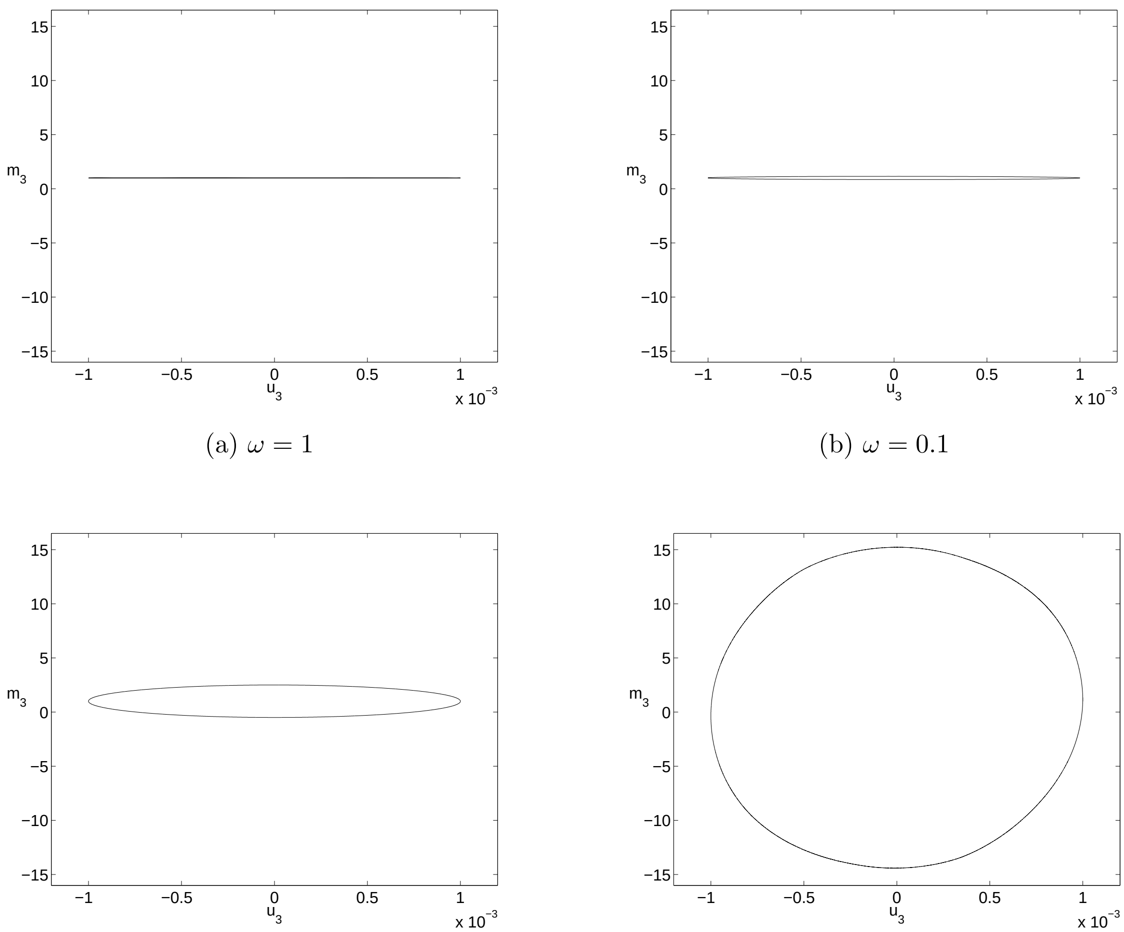
(b) $\omega = 0.1$

(c) $\omega = 0.01$

(d) $\omega = 0.001$

Figure 4.9: Hysteresis loops for $m_2(x,t)$ of the Landau–Lifshitz equation with x fixed and $\nu = 0.02$, $L = 1$. The input is $\mathbf{u}(t) = (0, 0.001\cos(\omega t), 0)$ and the initial condition is $\mathbf{m}_0(x) = (0, 1, 0)$.

In general linear systems are not believed to exhibit hysteresis because they often have only one equilibrium solution. However, we have demonstrated that the linear Landau–Lifshitz equation and equation (3.6) have persistent loops in the input–output maps as the frequency of the input approaches zero. By Definition 3.2, the two systems exhibit hysteresis. This is because both linear examples have multiple stable equilibrium, which is crucial for systems to display hysteresis (see Definition 3.1).



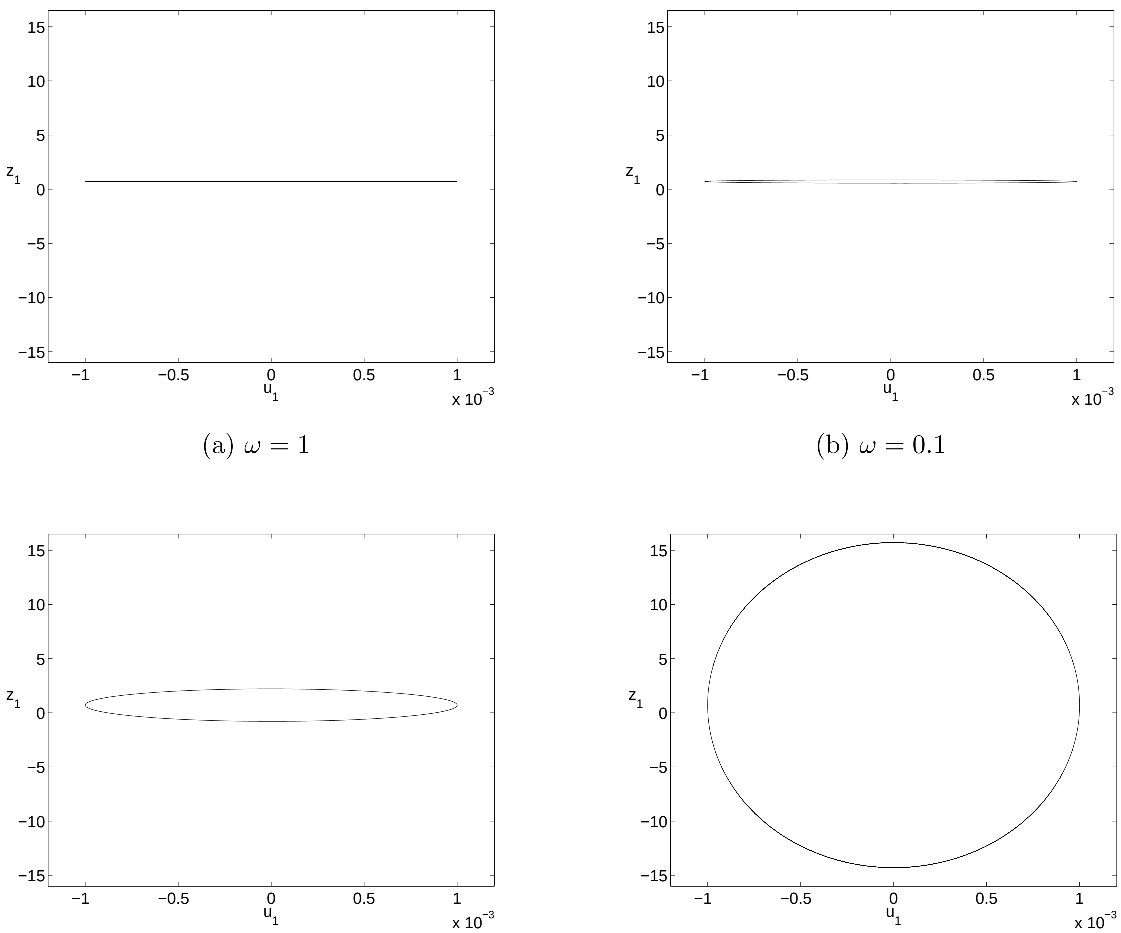
(a) $\omega = 1$

(b) $\omega = 0.1$

(c) $\omega = 0.01$

(d) $\omega = 0.001$

Figure 4.10: Hysteresis loops for $m_3(x,t)$ of the Landau–Lifshitz equation with x fixed and $\nu = 0.02$, $L = 1$. The input is $\mathbf{u}(t) = (0, 0, 0.001\cos(\omega t))$ and the initial condition is $\mathbf{m}_0(x) = (0, 0, 1)$.



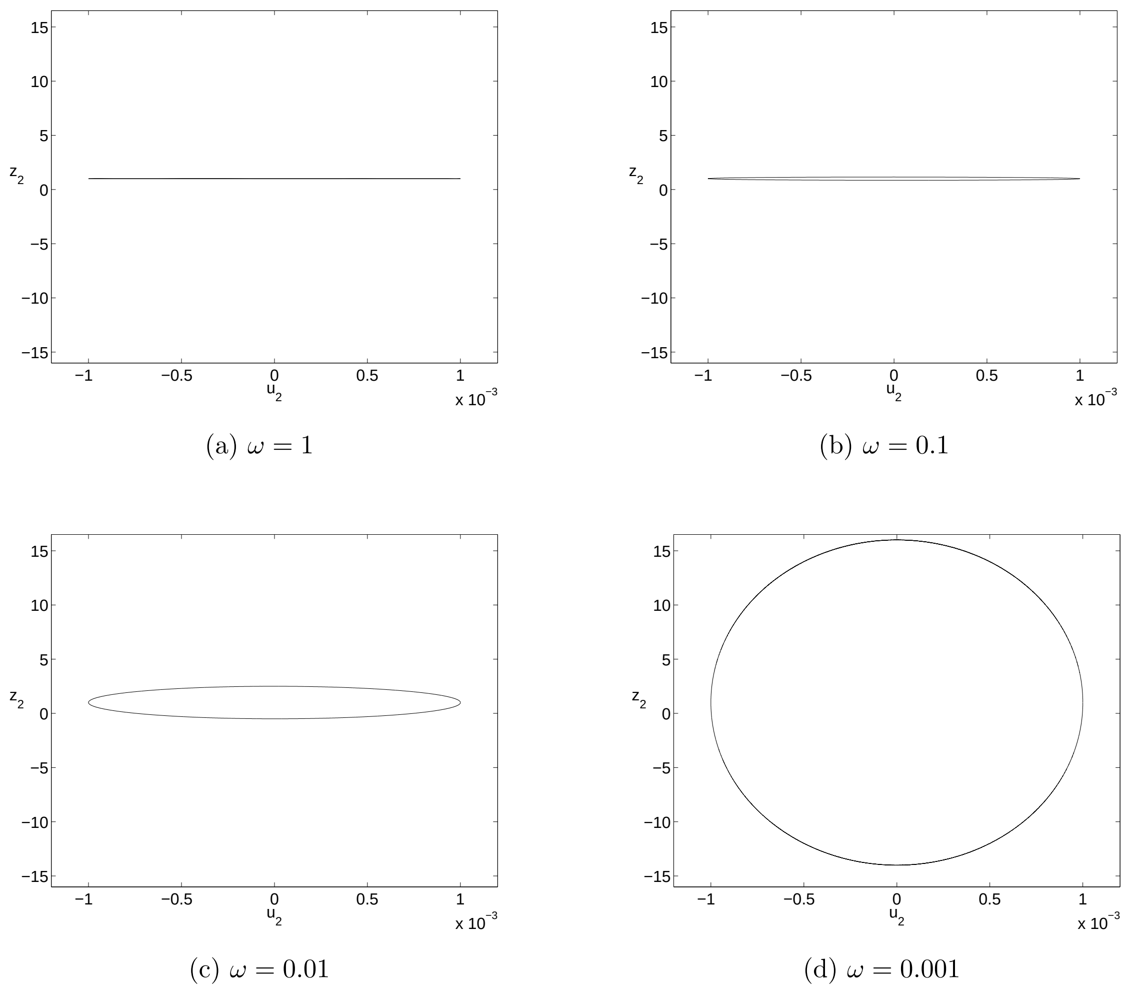
(a) $\omega = 1$

(b) $\omega = 0.1$

(c) $\omega = 0.01$

(d) $\omega = 0.001$

Figure 4.11: Hysteresis loops for $z_1(x,t)$ of the linear Landau–Lifshitz equation with x fixed and $\nu = 0.02$. The linearization is at $\mathbf{a} = (1,0,0)$. The input is $\mathbf{u}(t) = (0.001\cos(\omega t), 0, 0)$ and the initial condition is $\mathbf{z}_0(x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.



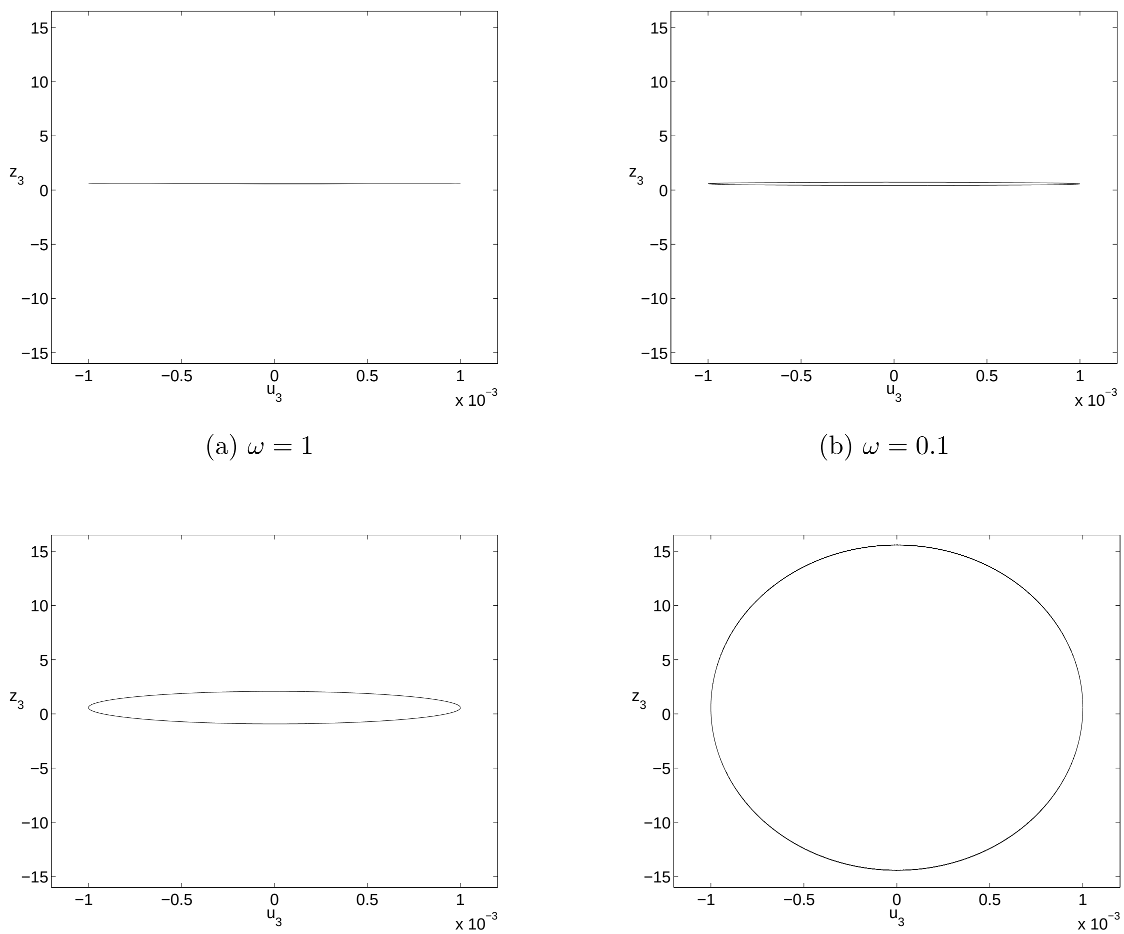
(a) $\omega = 1$

(b) $\omega = 0.1$

(c) $\omega = 0.01$

(d) $\omega = 0.001$

Figure 4.12: Hysteresis loops for $z_2(x,t)$ of the linear Landau–Lifshitz equation with x fixed and $\nu = 0.02$. The linearization is at $\mathbf{a} = (0,1,0)$. The input is $\mathbf{u}(t) = (0, 0.001\cos(\omega t), 0)$ and the initial condition is $\mathbf{z}_0(x) = (0,1,0)$.



(a) $\omega = 1$

(b) $\omega = 0.1$

(c) $\omega = 0.01$

(d) $\omega = 0.001$

Figure 4.13: Hysteresis loops for $z_3(x, t)$ of the linear Landau–Lifshitz equation with x fixed and $\nu = 0.02$. The linearization is at $\mathbf{a} = (0, 0, 1)$. The input is $\mathbf{u}(t) = (0, 0, 0.001 \cos(\omega t))$ and the initial condition is $\mathbf{z}_0(x) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Chapter 5

Control of the Landau–Lifshitz Equation

Ferromagnets are often found in memory storage devices such as hard disks, credit cards or tape recordings. Each set of data stored in a memory device is uniquely assigned to a specific stable magnetic state of the ferromagnet. As research and technology advance, the amount of data and the speed at which they can be stored increases while their physical size decreases. Today, their size is on the magnitude of nanoscales, which is why there is a renewed interest in studying the Landau–Lifshitz equation.

The magnetic state of a ferromagnet can be changed by an applied magnetic field, $\mathbf{H}_a$, which is viewed as the control (Carbou *et al.* [25, 26, 27], Alouges and Beauchard [6], Noh *et al.* [69]). Being able to control the stable states allows for exact storage of data. This is difficult due to the presence of hysteresis because more than one stable equilibrium is possible for a particular input. We are able to present a controller design that forces the system to move from one arbitrary stable equilibrium to another. Essentially, this controls the effect of hysteresis that arises in the Landau–Lifshitz equation and hence allows for a more precise retrieval of the stored data.

Results on the control of magnetization described by the Landau–Lifshitz equation are not well–developed. Most articles have been published in the last five years and many are either experimental, numerical or simplify the Landau–Lifshitz equation. For example, in Alouges and Beauchard [6], the controlled Landau–Lifshitz equation is considered on a special domain which allows the spatial variable to be fixed and hence the control problem simplifies to an ODE model. Experiments demonstrating the control of domain walls in a nanowire is presented in Noh *et al.* [69]. Numerical simulations have also been conducted on the control of domain walls in nanowires (Wieser *et al.* [96]).

Much of the theoretical results on the control of the Landau–Lifshitz have been by Carbou *et al.* [25, 26, 27]. In particular, they investigate the control of domain walls in nanowires using the external applied magnetic, $\mathbf{H}_a$, as the control mechanism. The effective field of the Landau–Lifshitz equation (4.2) is defined as the sum of the exchange field, demagnetization field and applied field; that is,

$$\mathbf{H}_{eff} = \mathbf{m}_{xx} + \mathbf{H}_d + \mathbf{H}_a.$$

It follows that the form of the Landau–Lifshitz equation considered by Carbou *et al.* [25, 26, 27] is

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{H}_d + \mathbf{H}_a) - \nu \mathbf{m} \times (\mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{H}_d + \mathbf{H}_a)). \tag{5.1}$$

In Carbou *et al.* [25], equation (5.1) is linearized with $\mathbf{H}_a = \mathbf{0}$ and shown to have an unstable equilibrium, $\mathbf{m}_*$. Setting $\mathbf{H}_a$ to be the average of the m_1–magnetization; that is,

$$\mathbf{H}_a = \frac{1}{L} \int_0^L m_1(r,t) dr \mathbf{e}_1,$$

$\mathbf{m}_*$ is shown to be stable for (5.1) in Carbou *et al.* [25, Theorem 1.4].

In Carbou *et al.* [26, 27], $\mathbf{H}_a$ is chosen to be a constant and is only applied in the m_1-direction; that is, $\mathbf{H}_a = (d, 0, 0) = d\mathbf{e}_1$ where d is a constant. The authors are interested in controlling solutions of (5.1) to special domain walls, $\mathbf{m}_{dw}$, called *Bloch walls*. They show solutions of (5.1) with $\mathbf{H}_a = (d, 0, 0) = d\mathbf{e}_1$ can be controlled to these walls $\mathbf{m}_{dw}$.

From a physical perspective, the external applied field, $\mathbf{H}_a$, can be viewed as the control because it can change the magnetization within a ferromagnet. Rearranging (5.1) leads to

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{H}_d) - \nu \mathbf{m} \times (\mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{H}_d)) + \mathbf{m} \times \mathbf{H}_a - \nu (\mathbf{m} \times \mathbf{H}_a).$$

The terms depending on $\mathbf{H}_a$ are the control terms, which are nonlinear with respect to $\mathbf{H}_a$. If these terms are linearized, then the control is linear. This is the approach we consider and hence we include a control, $\mathbf{u}(t)$, as follows

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) + b\mathbf{u}(t).$$

Let $\mathbf{a}, \mathbf{r} \in E$ be stable equilibria of (4.3), (4.5) with $\mathbf{r} \neq \mathbf{a}$. The goal is to choose $\mathbf{u}(t)$ such that the system moves from $\mathbf{a}$ to $\mathbf{r}$. From a mathematical viewpoint, the control causes $\mathbf{a}$ to no longer be an equilibrium and $\mathbf{r}$ to be an equilibrium that is asymptotically stable. This indicates the system will reach the second equilibrium.

The controller design is a closed–loop with a proportional control (Figure 5.1). The existence of a zero eigenvalue for the linear Landau–Lifshitz equation (see Section 4.3.2) suggests that a proportional control is sufficient for the output, $\mathbf{y}$, to track to $\mathbf{r}$. We will show this is in fact true. It follows that the controller design for the Landau–Lifshitz equation is governed by

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) + b\mathbf{u}(t) \tag{5.2a}$$

$$\mathbf{y} = \mathbf{m} \tag{5.2b}$$

$$\mathbf{e} = \mathbf{r} - \mathbf{m} \tag{5.2c}$$

$$\mathbf{u}(t) = k_p \mathbf{e}(t) \tag{5.2d}$$

where k_p and b are nonzero real constants. Therefore, the controlled Landau–Lifshitz equation is

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) + bk_p (\mathbf{r} - \mathbf{m}), \qquad \mathbf{m}(x, 0) = \mathbf{m}_0(x) \tag{5.3}$$

with Neumann boundary conditions $\mathbf{m}_x(0, t) = \mathbf{m}_x(L, t) = \mathbf{0}$.

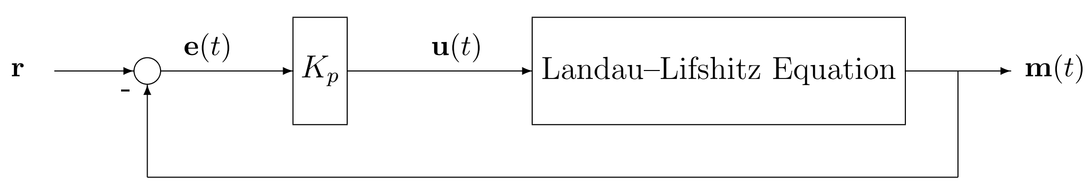


Figure 5.1: Closed-loop system for the controlled Landau–Lifshitz equation. The dynamics are described in equation (5.2).

In the following theorem, we establish that the controlled Landau–Lifshitz equation described in (5.3) is well–posed; that is, (5.3) has a strong solution.

Theorem 5.1. For $\mathbf{m} \in D = D(B + f)$, where D is defined in (4.12), let

$$f(\mathbf{m}) = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) \tag{5.4a}$$

$$B\mathbf{m} = bk_p(\mathbf{r} - \mathbf{m}). \tag{5.4b}$$

For $bk_p \geq 0$, the nonlinear operator $B + f$ generates a nonlinear contraction semigroup.

Proof of Theorem 5.1: We show that $B+f$ is dissipative and the range of $\mathrm{I} - \alpha(B+f)$ for all $\alpha > 0$ is the entire space $\mathcal{L}_2^3$.

For any $\mathbf{m}, \mathbf{y} \in D(f + B) = D(f)$,

$$\begin{aligned}\langle f(\mathbf{m}) + B\mathbf{m} - (f(\mathbf{y}) + B\mathbf{y}), \mathbf{m} - \mathbf{y}\rangle_{\mathcal{L}_2^3} =& \langle f(\mathbf{m}) - f(\mathbf{y}) + B\mathbf{m} - B\mathbf{y}, \mathbf{m} - \mathbf{y}\rangle_{\mathcal{L}_2^3} \\ =& \langle f(\mathbf{m}) - f(\mathbf{y}), \mathbf{m} - \mathbf{y}\rangle_{\mathcal{L}_2^3} + \langle B\mathbf{m} - B\mathbf{y}, \mathbf{m} - \mathbf{y}\rangle_{\mathcal{L}_2^3}.\end{aligned}$$

Since f generates a nonlinear contraction semigroup (see Theorem 4.7), it is dissipative (Luo *et al.* [62, Proposition 2.98]) and hence

$$\langle f(\mathbf{m}) - f(\mathbf{y}), \mathbf{m} - \mathbf{y}\rangle_{\mathcal{L}_2^3} \leq 0.$$

It follows that

$$\begin{aligned}\langle f(\mathbf{m}) + B\mathbf{m} - (f(\mathbf{y}) + B\mathbf{y}), \mathbf{m} - \mathbf{y}\rangle_{\mathcal{L}_2^3} &\leq \langle B\mathbf{m} - B\mathbf{y}, \mathbf{m} - \mathbf{y}\rangle_{\mathcal{L}_2^3} \\ &= \langle bk_p(\mathbf{r} - \mathbf{m}) - bk_p(\mathbf{r} - \mathbf{y}), \mathbf{m} - \mathbf{y}\rangle_{\mathcal{L}_2^3} \\ &= \langle -bk_p\mathbf{m} + bk_p\mathbf{y}, \mathbf{m} - \mathbf{y}\rangle_{\mathcal{L}_2^3} \\ &= -bk_p\langle \mathbf{m} - \mathbf{y}, \mathbf{m} - \mathbf{y}\rangle_{\mathcal{L}_2^3} \\ &= -bk_p||\mathbf{m} - \mathbf{y}||_{\mathcal{L}_2^3} \\ &< 0\end{aligned}$$

and hence $f + B$ is dissipative.

To show the range of $\mathrm{I} - \alpha(B + f)$ is the entire space $\mathcal{L}_2^3$, consider the following. Let $\mathbf{y}_1 \in \mathcal{L}_2^3$ and define

$$\mathbf{y}_2 = \frac{\mathbf{y}_1}{1 + \alpha bk_p} + \frac{\alpha bk_p\mathbf{r}}{1 + \alpha bk_p} \in \mathcal{L}_2^3$$

for some $\alpha > 0$. Since f generates a nonlinear contraction semigroup (see Theorem 4.7), then it is m-dissipative and hence, $\mathrm{ran}(\mathrm{I} - \hat{\alpha} f) = \mathcal{L}_2^3$ for any $\hat{\alpha} > 0$ (see Kato [49, Lemma 2.2]). This means there exists $\mathbf{m} \in D(f)$ such that $\mathbf{m} - \hat{\alpha} f(\mathbf{m}) = \mathbf{y}_2$. Let

$$\hat{\alpha} = \frac{\alpha}{1 + \alpha b k_p}.$$

It follows that

$$\mathbf{m} - \frac{\alpha}{1 + \alpha b k_p} f(\mathbf{m}) = \frac{\mathbf{y}_1}{1 + \alpha b k_p} + \frac{\alpha b k_p \mathbf{r}}{1 + \alpha b k_p}$$

and solving for $\mathbf{y}_1$ leads to

$$\mathbf{y}_1 = (1 + \alpha b k_p)\mathbf{m} - \alpha f(\mathbf{m}) - \alpha b k_p \mathbf{r}$$

and then rearranging, we have

$$\mathbf{y}_1 = \mathbf{m} - \alpha(b k_p(\mathbf{r} - \mathbf{m}) + f(\mathbf{m}))$$

This means for any $\mathbf{y}_1 \in \mathcal{L}_2^3$, there exists $\mathbf{m} \in D(f)$ such that $\mathbf{y}_1 = (\mathrm{I} - \alpha(B + f))\mathbf{m}$ and hence $\mathrm{ran}\,(\mathrm{I} - \alpha(B + f)) = \mathcal{L}_2^3$ for some $\alpha > 0$. By Kato [49, Lemma 2.2], this is true for all $\alpha > 0$.

Since $B + f$ is dissipative and $\mathrm{ran}(\mathrm{I} - \alpha(B + f)) = \mathcal{L}_2^3$, then $B + f$ generates a nonlinear contraction semigroup (Luo *et al.* [62, Proposition 2.114]). □

Substituting $\mathbf{m} = \mathbf{r}$ into (5.3) gives $\partial \mathbf{m}/\partial t = \mathbf{0}$ and hence $\mathbf{r}$ is an equilibrium point of (5.3). Moreover, it is clear that $\mathbf{a}$ is no longer an equilibrium of the controlled Landau–Lifshitz equation. This is due to the addition of the control term. Lyapunov's theorem is used to establish asymptotic stability of $\mathbf{r}$. Exponential stability is also shown in a similar manner.

The main result is in Theorem 5.6, which shows that any initial magnetization can be controlled to any arbitrary stable equilibrium point. The following lemmas are needed in the proof of Theorem 5.6.

Lemma 5.2. For $\mathbf{m} \in \mathcal{L}_2^3$, the derivative of $\mathbf{g} = \mathbf{m} \times \mathbf{m}_x$ is $\mathbf{g}_x = \mathbf{m} \times \mathbf{m}_{xx}$.

Proof of Lemma 5.2: Recall from (4.4) that

$$\mathbf{m} \times \mathbf{m}_x = (m_2 m_3' - m_3 m_2', -m_1 m_3' + m_3 m_1', m_1 m_2' - m_2 m_1').$$

Taking the derivative of each component yields

$$\begin{aligned}\mathbf{g}_x = (\,& m_2'm_3' + m_2m_3'' - m_3'm_2' - m_3m_2'',\\ & -m_1'm_3' - m_1m_3'' + m_3'm_1' + m_3m_1'',\\ & m_1'm_2' + m_1m_2'' - m_2'm_1' - m_2m_1''\,).\end{aligned}$$

Simplifying leads to

$$\mathbf{g}_x = (m_2m_3'' - m_3m_2'', -m_1m_3'' + m_3m_1'', m_1m_2'' - m_2m_1'') = \mathbf{m} \times \mathbf{m}_{xx}. \quad \square$$

Lemma 5.3. For $\mathbf{m} \in \mathcal{L}_2^3$, the derivative of $f = (\mathbf{m} \times \mathbf{m}_x)^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_x)$ is

$$f' = 2\,(\mathbf{m} \times \mathbf{m}_x)^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx})\,.$$

Lemma 5.3 is a simple consequence of the product rule and Lemma 5.2.

Lemma 5.4. For $\mathbf{m} \in \mathcal{L}_2^3$ satisfying $\mathbf{m}_x(0) = \mathbf{m}_x(L) = 0$, then

$$\int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx}) dx = 0.$$

Proof of Lemma 5.4: From Lemma 5.2, applying integration by parts to the integral yields

$$\int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx}) dx = \left[(\mathbf{m} - \mathbf{r})^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_x)\right]_0^L - \int_0^L \mathbf{m}_x^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_x) dx.$$

Because of the boundary conditions, the first term is zero and hence

$$\int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx}) dx = -\int_0^L \mathbf{m}_x^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_x) dx.$$

From Table 4.1,

$$\mathbf{m}_x^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_x) = \mathbf{m}^{\mathrm{T}} (\mathbf{m}_x \times \mathbf{m}_x) = 0,$$

and hence the integral is zero. $\square$

Lemma 5.5. For $\mathbf{m} \in \mathcal{L}_2^3$ satisfying $\mathbf{m}_x(0) = \mathbf{m}_x(L) = 0$, then

$$||\mathbf{m} \times \mathbf{m}_x||_{\mathcal{L}_2^3} \leq 4L^2 ||\mathbf{m} \times \mathbf{m}_{xx}||_{\mathcal{L}_2^3}$$

Proof of Lemma 5.5: For $\mathbf{m} \in \mathcal{L}_2^3$,

$$||\mathbf{m} \times \mathbf{m}_x||_{\mathcal{L}_2^3}^2 = \int_0^L (\mathbf{m} \times \mathbf{m}_x)^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_x)\, dx,$$

then applying integration by parts and using Lemma 5.3 yields

$$||\mathbf{m} \times \mathbf{m}_x||_{\mathcal{L}_2^3}^2 = \left[(\mathbf{m} \times \mathbf{m}_x)^2 x\right]_0^L - \int_0^L 2\,(\mathbf{m} \times \mathbf{m}_x)^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx})\, x dx.$$

The first term is zero from the boundary conditions. It follows from Young's inequality that

$$\begin{aligned}
||\mathbf{m} \times \mathbf{m}_x||_{\mathcal{L}_2^3}^2 &= -\int_0^L 2\,(\mathbf{m} \times \mathbf{m}_x)^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx})\, x dx \\
&\leq \frac{1}{2} \int_0^L (\mathbf{m} \times \mathbf{m}_x)^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_x)\, dx + \int_0^L 2\,(\mathbf{m} \times \mathbf{m}_{xx})^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx})\, x^2 dx
\end{aligned}$$

and since $x \in [0, L]$, then

$$\begin{aligned}
||\mathbf{m} \times \mathbf{m}_x||_{\mathcal{L}_2^3}^2 &\leq \frac{1}{2}||\mathbf{m} \times \mathbf{m}_x||_{\mathcal{L}_2^3}^2 + \int_0^L 2\,(\mathbf{m} \times \mathbf{m}_{xx})^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx})\, L^2 dx, \qquad (x \in [0, L]) \\
&= \frac{1}{2}||\mathbf{m} \times \mathbf{m}_x||_{\mathcal{L}_2^3}^2 + 2L^2||\mathbf{m} \times \mathbf{m}_{xx}||_{\mathcal{L}_2^3}^2.
\end{aligned}$$

Rearranging gives the desired inequality. □

Lemma 5.5 can be thought of as a Poincare's inequality for cross products.

Theorem 5.6. Let $\mathbf{r}$ be any equilibrium point of (5.3). For any nonzero constants, b and k_p, such that $k_p \geq 16\nu L^4/b$, $\mathbf{r}$ is globally asymptotically stable in the $\mathcal{L}_2^3$–norm.

Proof of Theorem 5.6: The Lyapunov candidate is

$$V(\mathbf{m}) = \frac{1}{2}\,||\mathbf{m} - \mathbf{r}||_{\mathcal{L}_2^3}^2 + \frac{1}{2}\,||\mathbf{m}_x||_{\mathcal{L}_2^3}^2$$

which is clearly nonegative. Furthermore, since $V = 0$ if and only if $\mathbf{m} = \mathbf{r}$, then V is positive for all $D\backslash\{\mathbf{r}\}$. Taking the derivative of V leads to

$$\frac{dV}{dt} = \int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}} \dot{\mathbf{m}} dx + \int_0^L \mathbf{m}_x^{\mathrm{T}} \dot{\mathbf{m}}_x dx.$$

Applying integration by parts on the second integral we obtain

$$\frac{dV}{dt} = \int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}} \dot{\mathbf{m}} dx - \int_0^L \mathbf{m}_{xx}^{\mathrm{T}} \dot{\mathbf{m}} dx.$$

Substituting in (5.3) leads to

$$\begin{aligned}\frac{dV}{dt} &= \int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx})\, dx - \nu \int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}} (\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}))\, dx \\ &+ bk_p \int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}} (\mathbf{r} - \mathbf{m}) dx - \int_0^L \mathbf{m}_{xx}^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx})\, dx \\ &+ \nu \int_0^L \mathbf{m}_{xx}^{\mathrm{T}} (\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}))\, dx - bk_p \int_0^L \mathbf{m}_{xx}^{\mathrm{T}} (\mathbf{r} - \mathbf{m}) dx.\end{aligned}$$

From Lemma 5.4, we have that the first integral is zero. Furthermore, since

$$\mathbf{m}_{xx}^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx}) = \mathbf{m}^{\mathrm{T}} (\mathbf{m}_{xx} \times \mathbf{m}_{xx}) = 0,$$

then

$$\int_0^L \mathbf{m}_{xx}^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx})\, dx = 0.$$

It follows that

$$\begin{aligned}\frac{dV}{dt} = &- \nu \int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}} (\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}))\, dx - bk_p ||\mathbf{m} - \mathbf{r}||_{\mathcal{L}_2^3}^2 \\ &- \nu ||\mathbf{m} \times \mathbf{m}_{xx}||_{\mathcal{L}_2^3}^2 - bk_p \int_0^L \mathbf{m}_{xx}^{\mathrm{T}} (\mathbf{r} - \mathbf{m}) dx.\end{aligned}$$

Applying integration by parts to the last integral leads to

$$\begin{aligned}\frac{dV}{dt} &= -\nu \int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}} (\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}))\, dx - bk_p ||\mathbf{m} - \mathbf{r}||_{\mathcal{L}_2^3}^2 - \nu ||\mathbf{m} \times \mathbf{m}_{xx}||_{\mathcal{L}_2^3}^2 - bk_p ||\mathbf{m}_x||_{\mathcal{L}_2^3}^2 \\ &= -\nu \int_0^L ((\mathbf{m} - \mathbf{r}) \times \mathbf{m})^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx})\, dx - bk_p ||\mathbf{m} - \mathbf{r}||_{\mathcal{L}_2^3}^2 - \nu ||\mathbf{m} \times \mathbf{m}_{xx}||_{\mathcal{L}_2^3}^2 - bk_p ||\mathbf{m}_x||_{\mathcal{L}_2^3}^2 \\ &= \nu \int_0^L (\mathbf{r} \times \mathbf{m})^{\mathrm{T}} (\mathbf{m} \times \mathbf{m}_{xx})\, dx - bk_p ||\mathbf{m} - \mathbf{r}||_{\mathcal{L}_2^3}^2 - \nu ||\mathbf{m} \times \mathbf{m}_{xx}||_{\mathcal{L}_2^3}^2 - bk_p ||\mathbf{m}_x||_{\mathcal{L}_2^3}^2.\end{aligned} \tag{5.5}$$

Applying integration by parts with Lemma 5.2 to the integral implies

$$\int_0^L (\mathbf{r}\times\mathbf{m})^{\mathrm{T}} (\mathbf{m}\times\mathbf{m}_{xx})\, dx = \Big[(\mathbf{r}\times\mathbf{m})^{\mathrm{T}} (\mathbf{m}\times\mathbf{m}_{x})\Big]_0^L - \int_0^L (\mathbf{r}\times\mathbf{m}_x)^{\mathrm{T}} (\mathbf{m}\times\mathbf{m}_{x})\, dx$$

and substituting in the boundary conditions leads to

$$\int_0^L (\mathbf{r}\times\mathbf{m})^{\mathrm{T}} (\mathbf{m}\times\mathbf{m}_{xx})\, dx = -\langle \mathbf{r}\times\mathbf{m}_x, \mathbf{m}\times\mathbf{m}_x\rangle_{\mathcal{L}_2^3}.$$

Then from Cauchy-Schwarz and Lemma 5.5 we have

$$\begin{aligned}\int_0^L (\mathbf{r}\times\mathbf{m})^{\mathrm{T}} (\mathbf{m}\times\mathbf{m}_{xx})\, dx &\le ||\mathbf{r}\times\mathbf{m}_x||_{\mathcal{L}_2^3} ||\mathbf{m}\times\mathbf{m}_x||_{\mathcal{L}_2^3} \\ &\le 4L^2 ||\mathbf{r}\times\mathbf{m}_x||_{\mathcal{L}_2^3} ||\mathbf{m}\times\mathbf{m}_{xx}||_{\mathcal{L}_2^3}.\end{aligned}$$

It follows from Young's Inequality that

$$\int_0^L (\mathbf{r}\times\mathbf{m})^{\mathrm{T}} (\mathbf{m}\times\mathbf{m}_{xx})\, dx \le 8L^4 ||\mathbf{r}\times\mathbf{m}_x||^2_{\mathcal{L}_2^3} + \frac{1}{2} ||\mathbf{m}\times\mathbf{m}_{xx}||^2_{\mathcal{L}_2^3}$$

and from Lemma 4.15 we obtain

$$\int_0^L (\mathbf{r}\times\mathbf{m})^{\mathrm{T}} (\mathbf{m}\times\mathbf{m}_{xx})\, dx \le 16L^4 ||\mathbf{m}_x||^2_{\mathcal{L}_2^3} + \frac{1}{2} ||\mathbf{m}\times\mathbf{m}_{xx}||^2_{\mathcal{L}_2^3}$$

Substituting this result into (5.5) leads to

$$\frac{dV}{dt} \le -\left(bk_p - 16\nu L^4\right) ||\mathbf{m}_x||^2_{\mathcal{L}_2^3} - \frac{\nu}{2} ||\mathbf{m}\times\mathbf{m}_{xx}||^2_{\mathcal{L}_2^3} - bk_p ||\mathbf{m}-\mathbf{r}||^2_{\mathcal{L}_2^3}. \tag{5.6}$$

The derivative is nonpositive since $bk_p \ge 16\nu L^4$. It follows that

$$\frac{dV}{dt} \le -bk_p ||\mathbf{m}-\mathbf{r}||^2_{\mathcal{L}_2^3}.$$

It is clear that

$$\frac{dV}{dt} = 0 \text{ if and only if } \mathbf{m} = \mathbf{r}.$$

Therefore, $dV/dt < 0$ for all $\mathbf{m} \neq \mathbf{r}$ and from Theorem 2.27, $\mathbf{r}$ is an asymptotically stable equilibrium of (5.3). Since $V(\mathbf{m}) \ge \frac{1}{2}||\mathbf{m}-\mathbf{r}||^2_{\mathcal{L}_2^3}$, then $V \to \infty$ as $||\mathbf{m}-\mathbf{r}|| \to \infty$ and hence global stability is obtained. $\square$

Numerical simulations for (5.3) are considered. The numerical code (see Appendix B) for the uncontrolled Landau–Lifshitz equation is used; some small adjustments to the code are required. The initial condition is chosen to be $\mathbf{m}(0) = (\sin(2\pi x), \cos(2\pi x), 0)$ for $x \in [0, L]$ and $\nu = 0.02$, $L = 1$, which is the same as the uncontrolled case and recall that the uncontrolled Landau–Lifshitz equation naturally settles to the stable equilibrium, $(0, -0.6, 0)$. We choose $\mathbf{r}$ to be $(1, 0, 0)$; that is, the control forces the magnetization from the stable equilibrium $(0, -0.6, 0)$ to another stable equilibrium, $(1, 0, 0)$. These control dynamics are illustrated in Figures 5.2 to 5.5. The control parameters, b and k_p, are chosen to be 1 and 0.5, respectively, which satisfies the constraint $k_p \geq 16\nu L^4/b$.

Figures 5.2 to 5.5 illustrate how the control can force the magnetization from one stable equilibrium to another. However, since the result in Theorem 5.6 does not depend on a specific initial condition, the control can actually force any initial magnetization to an arbitrary stable equilibrium point. We illustrate this in Figures 5.6 to 5.9 with $\mathbf{m}(0) = (\sin(2\pi x), \cos(2\pi x), 0)$ for $x \in [0, L]$, $\nu = 0.02$, $L = 1$ and $\mathbf{r} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. The control parameters are again chosen to be $b = 1$ and $k_p = 0.5$.

We now show that $\mathbf{r}$ is exponentially stable for the H_1–norm. Recall the H_1–norm is $||\mathbf{m}||^2_{H_1} = ||\mathbf{m}||^2_{\mathcal{L}_2^3} + ||\mathbf{m}_x||^2_{\mathcal{L}_2^3}$.

Theorem 5.7. Let $\mathbf{r}$ be an equilibrium point of (5.3). For any nonzero constants, b and k_p, such that $k_p \geq 16\nu L^4/b$, $\mathbf{r}$ is globally exponentially stable in the H_1–norm.

Proof of Theorem 5.7: In the proof of Theorem 5.6, we have from (5.6) that the derivative of V satisfies

$$\frac{dV}{dt} \leq -\left(bk_p - 16\nu L^4\right) ||\mathbf{m}_x||^2_{\mathcal{L}_2^3} - \frac{\nu}{2}||\mathbf{m} \times \mathbf{m}_{xx}||^2_{\mathcal{L}_2^3} - bk_p||\mathbf{m} - \mathbf{r}||^2_{\mathcal{L}_2^3}$$

and hence

$$\begin{aligned}\frac{dV}{dt} &\leq -\left(bk_p - 16\nu L^4\right) ||\mathbf{m}_x||^2_{\mathcal{L}_2^3} - bk_p||\mathbf{m} - \mathbf{r}||^2_{\mathcal{L}_2^3} \\ &\leq -\left(bk_p - 16\nu L^4\right) \left(||\mathbf{m}_x||^2_{\mathcal{L}_2^3} + ||\mathbf{m} - \mathbf{r}||^2_{\mathcal{L}_2^3}\right) \\ &= -2\left(bk_p - 16\nu L^4\right) V.\end{aligned}$$

Integrating with respect to time

$$||\mathbf{m}_x||^2_{\mathcal{L}_2^3} + ||\mathbf{m} - \mathbf{r}||^2_{\mathcal{L}_2^3} \leq e^{-2\left(bk_p - 16\nu L^4\right)t} \left(||\mathbf{m}_x(0, t)||^2_{\mathcal{L}_2^3} + ||\mathbf{m}(0, t) - \mathbf{r}||^2_{\mathcal{L}_2^3}\right).$$

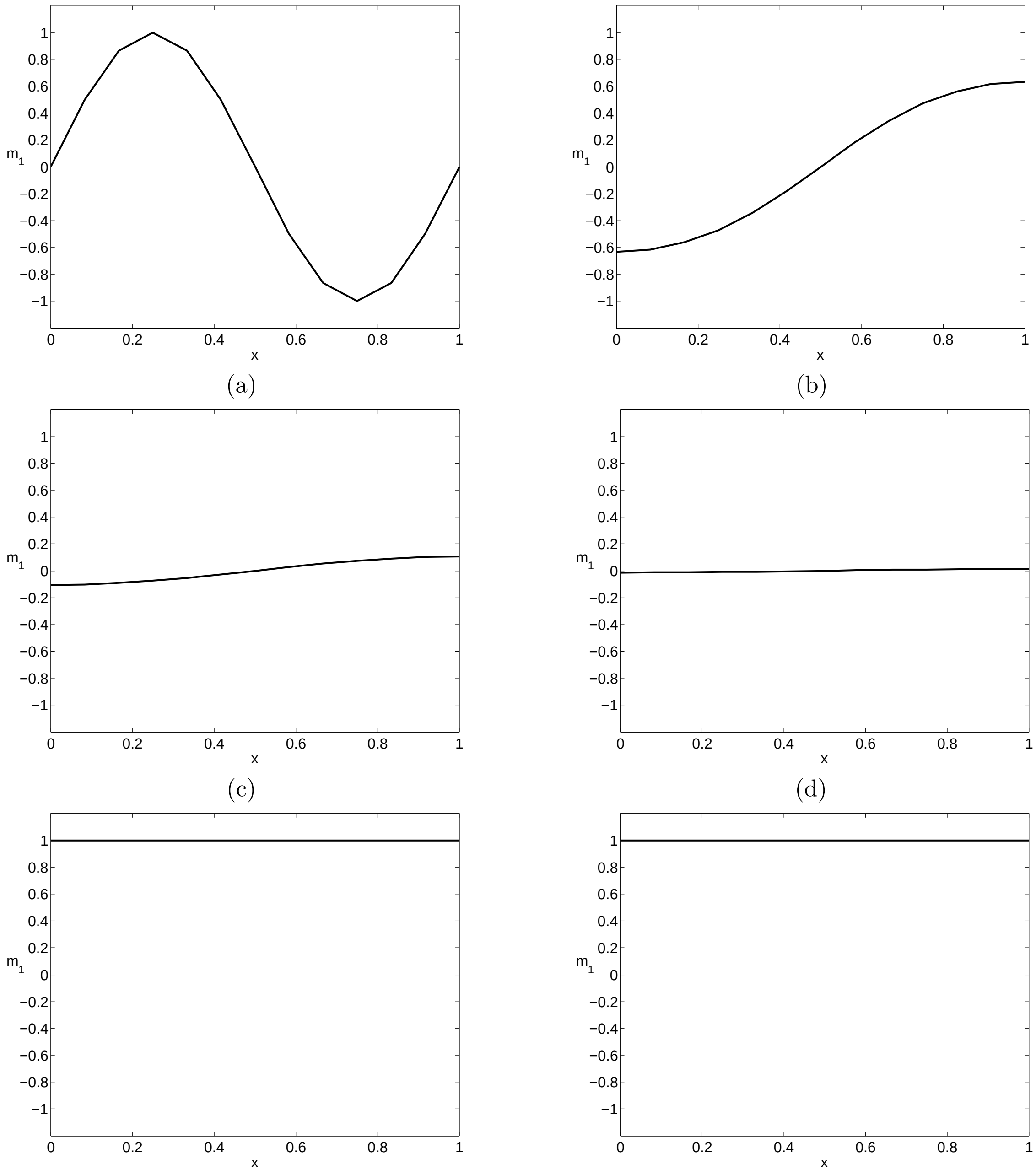


Figure 5.2: Dynamics of (5.3) for $m_1(x,t)$ where $b = 1$, $k_p = 0.5$, $\nu = 0.02$ and $L = 1$ with $\mathbf{m}(0) = (\sin(2\pi x), \cos(2\pi x), 0)$ and $\mathbf{r} = (1, 0, 0)$. On the horizontal axis is the spatial variable, $x \in [0, 1]$, and on the vertical axis is the magnetization. The magnetization evolves from (a) to (f). Initially the magnetization begins at $m_1(x, 0) = \sin(2\pi x)$ as shown in (a), which naturally settles to 0 (d). The control forces m_1 from 0 to 1 (f). A three–dimensional depiction of these dynamics is illustrated in Figure 5.5a.

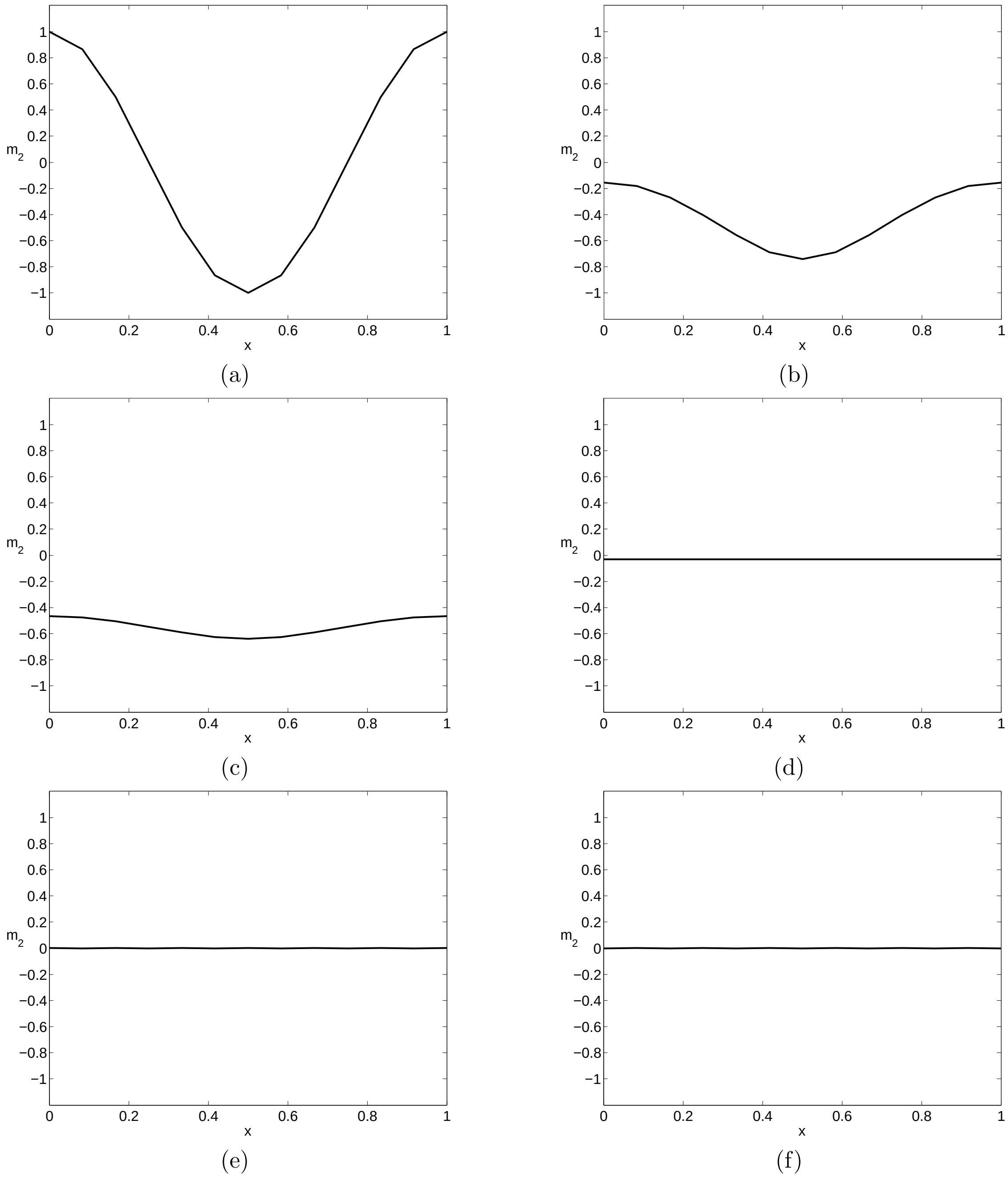


Figure 5.3: Dynamics of (5.3) for $m_2(x,t)$ where $b = 1$, $k_p = 0.5$, $\nu = 0.02$ and $L = 1$ with initial condition $\mathbf{m}(0) = (\sin(2\pi x), \cos(2\pi x), 0)$ and $\mathbf{r} = (1, 0, 0)$. On the horizontal axis is the spatial variable, $x \in [0, 1]$, and on the vertical axis is the magnetization. The magnetization evolves from (a) to (f). The system begins at $m_2(x, 0) = \cos(2\pi x)$ as depicted in (a) and eventually settles back to -0.6 (d). The control forces the magnetization from -0.6 to 0 (f). A three–dimensional depiction of these dynamics is illustrated in Figure 5.5b.

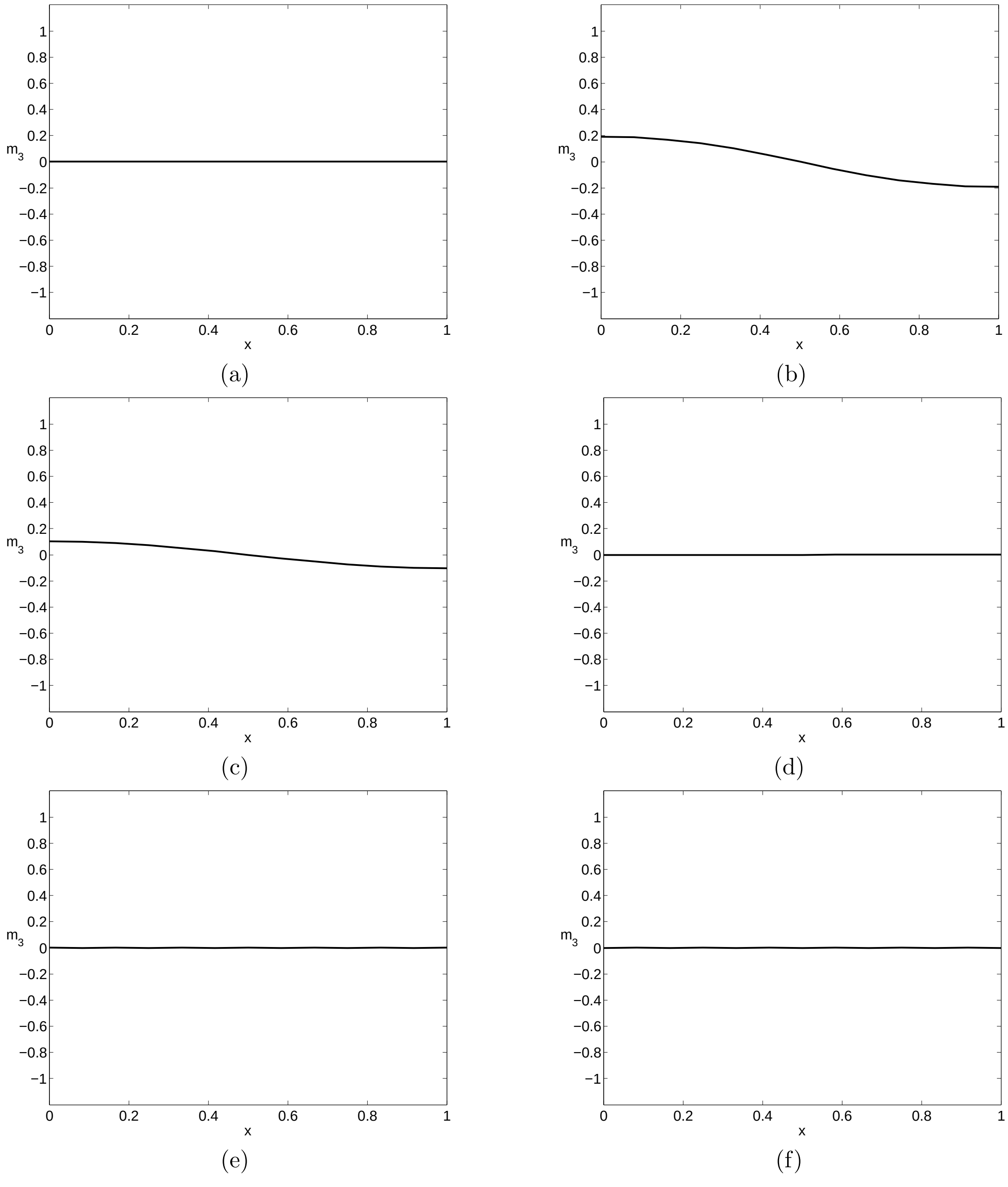
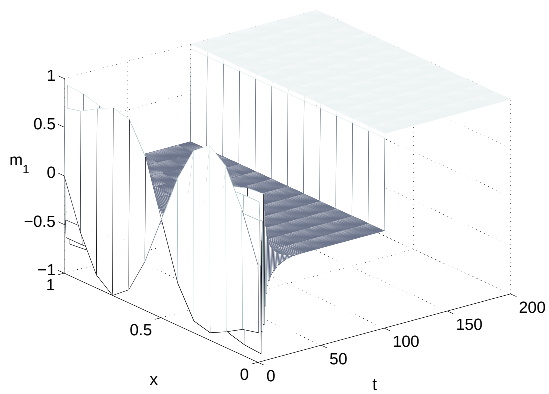
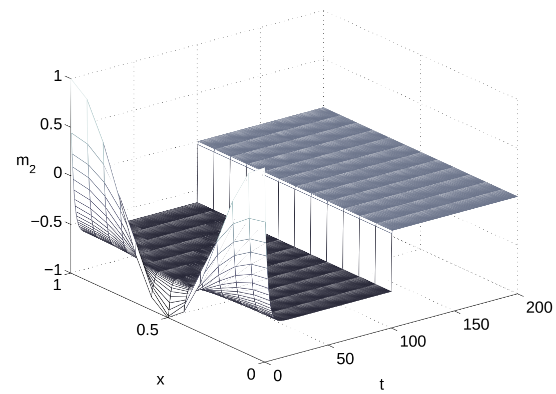


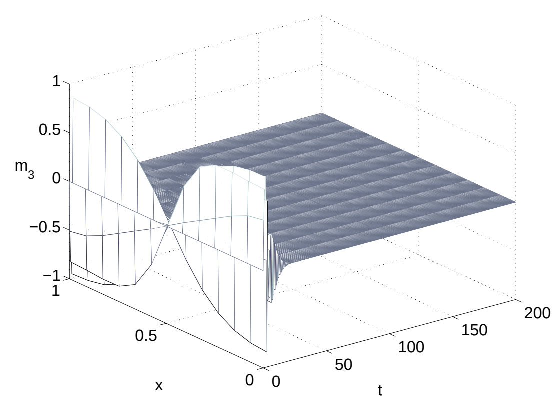
Figure 5.4: Dynamics of (5.3) for $m_3(x,t)$ where $b = 1$, $k_p = 0.5$, $\nu = 0.02$ and $L = 1$ with initial condition $\mathbf{m}(0) = (\sin(2\pi x), \cos 2(\pi x), 0)$ and $\mathbf{r} = (1, 0, 0)$. On the horizontal axis is the spatial variable, $x \in [0, 1]$, and on the vertical axis is the magnetization. The magnetization evolves from (a) to (f). The system begins at $m_3(x, 0) = 0$ as shown in (a), which naturally settles to 0 (d). The control forces the magnetization from 0 to 0 (f) and hence the control is not needed in this case. A three–dimensional depiction is illustrated in Figure 5.5c.



(a) $m_1(x,t)$



(b) $m_2(x,t)$



(c) $m_3(x,t)$

Figure 5.5: The magnetization dynamics of the controlled Landau–Lifshitz equation as x, t varies. The initial condition is $\mathbf{m}_0(x) = (\sin(2\pi x), \cos(2\pi x), 0)$ with $\nu = 0.02$ and $L = 1$. The magnetization is allowed to naturally settle to $(0, -0.6, 0)$ (see Figure 4.6), after which the control forces the magnetization to settle to $\mathbf{r} = (1, 0, 0)$.

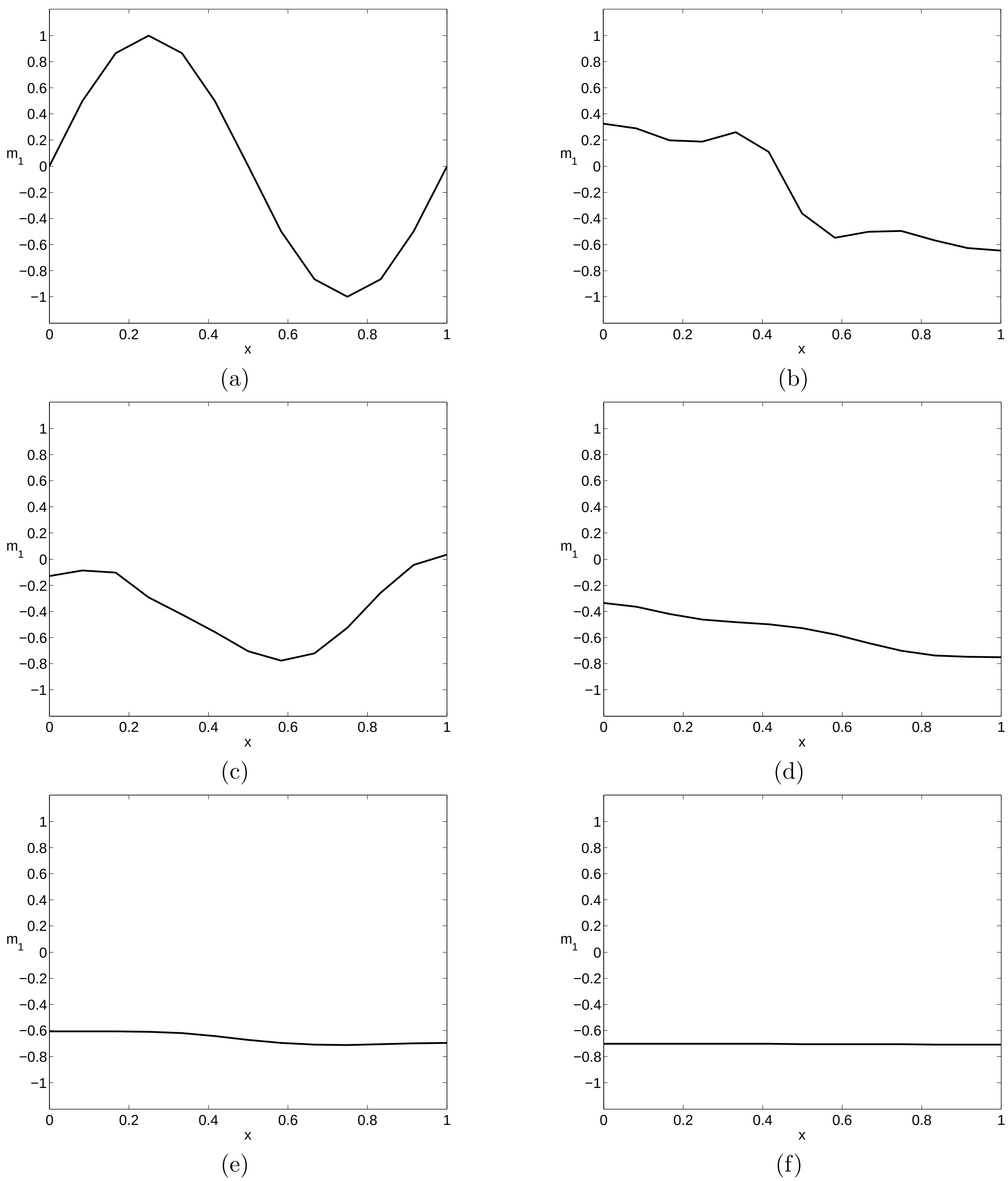


Figure 5.6: Dynamics of (5.3) for $m_1(x,t)$ where $b = 1$, $k_p = 0.5$, $\nu = 0.02$ and $L = 1$ with initial condition $\mathbf{m}(0) = (\sin(2\pi x), \cos 2(\pi x), 0)$ and $\mathbf{r} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. On the horizontal axis is the spatial variable, $x \in [0,1]$, and on the vertical axis is the magnetization. The magnetization evolves from (a) to (f). Initially the magnetization begins at $m_1(x,0) = \sin(2\pi x)$ and then the control forces the magnetization to $-\frac{1}{\sqrt{2}}$. Without the control, the magnetization naturally settles to 0 (see Figure 4.3 or 4.6a). A three–dimensional depiction is illustrated in Figure 5.9a.

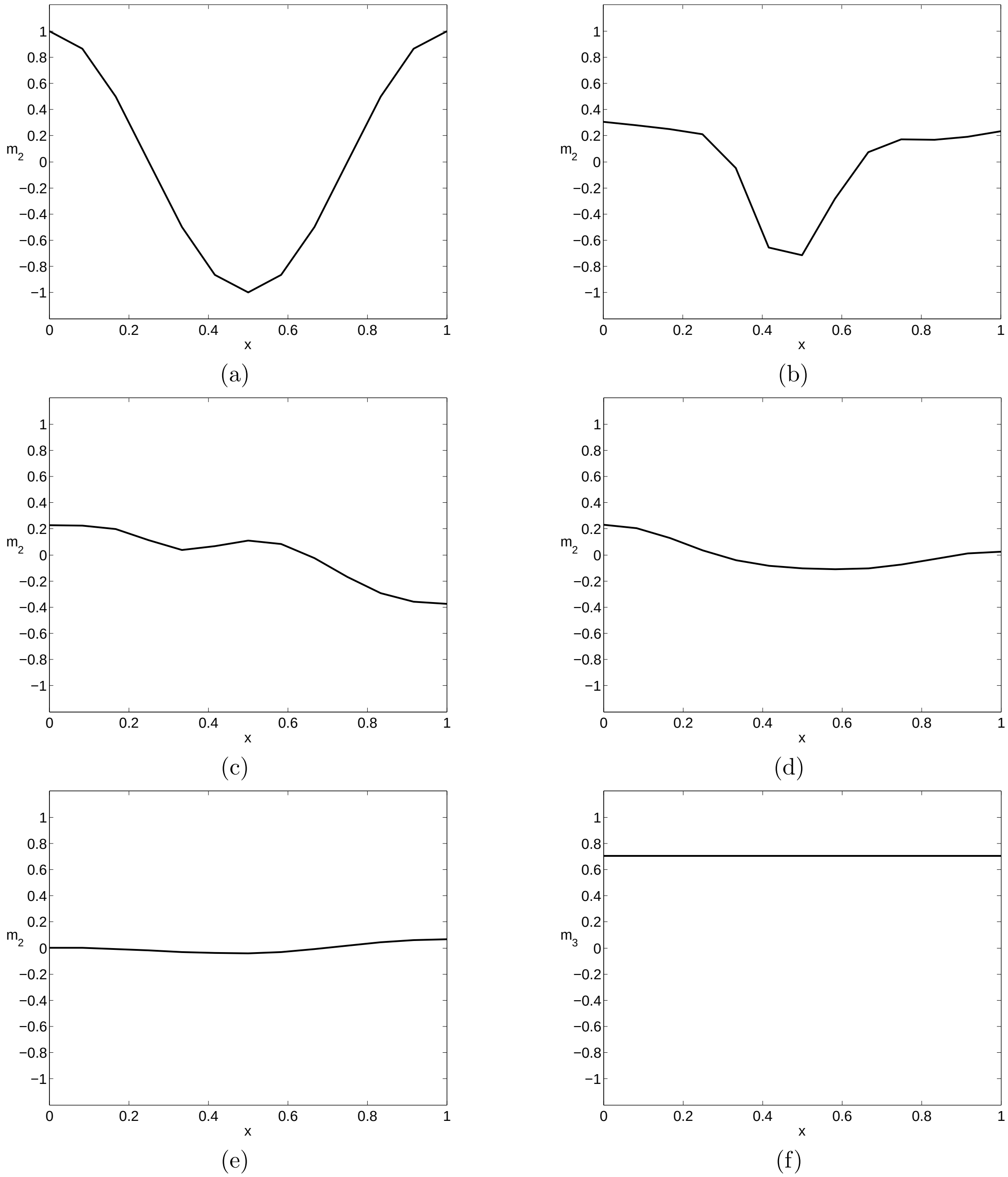


Figure 5.7: Dynamics of (5.3) for $m_2(x,t)$ where $b = 1$, $k_p = 0.5$, $\nu = 0.02$ and $L = 1$ with initial condition $\mathbf{m}(0) = (\sin(2\pi x), \cos 2(\pi x), 0)$ and $\mathbf{r} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. On the horizontal axis is the spatial variable, $x \in [0,1]$, and on the vertical axis is the magnetization. The magnetization evolves from (a) to (f). Initially the magnetization begins at $m_2(x,0) = \cos(2\pi x)$ and then the control forces the magnetization to 0. Without the control, the magnetization naturally settles to -0.6 (see Figure 4.4 or 4.6b). A three–dimensional depiction is illustrated in Figure 5.9b.

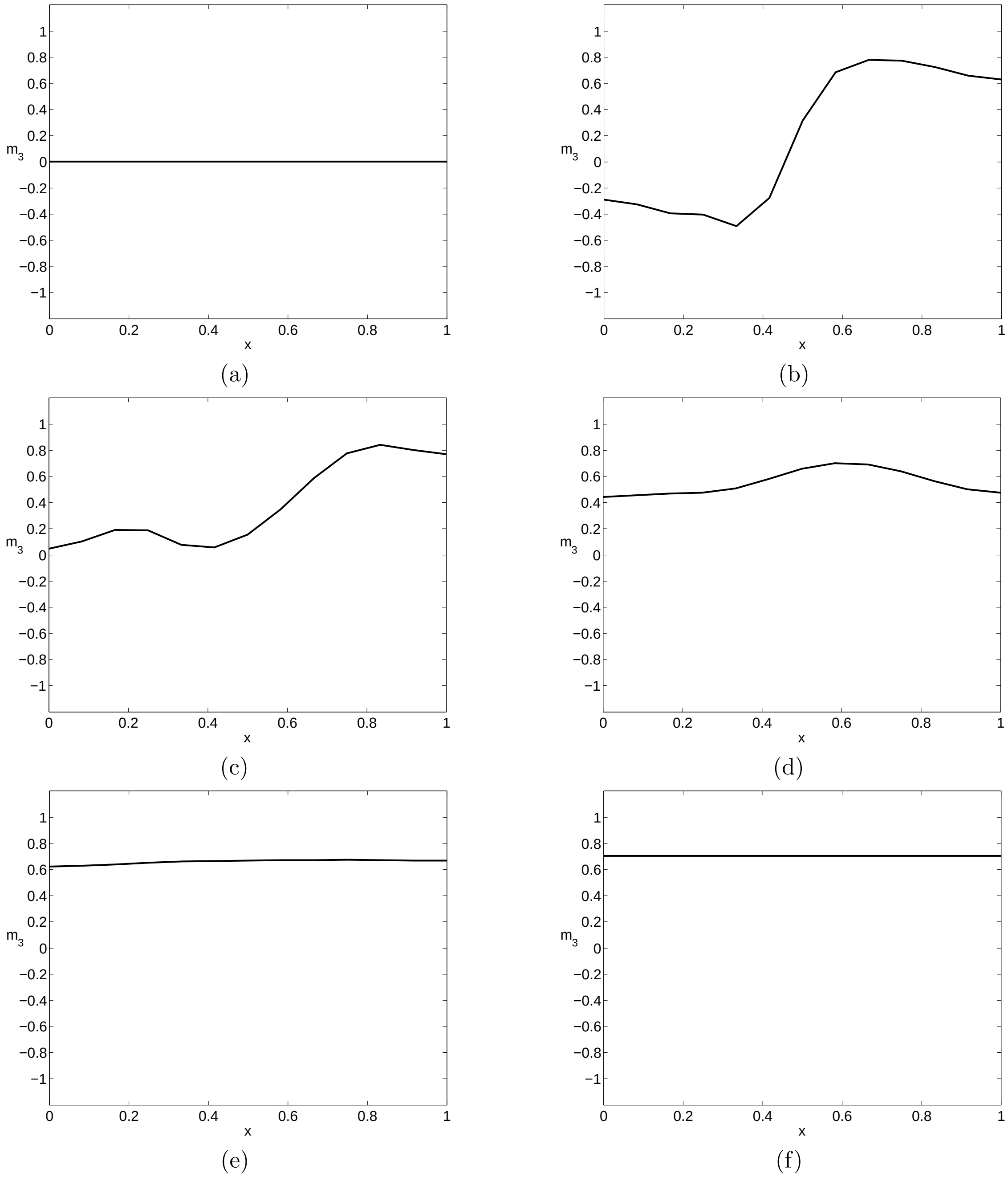
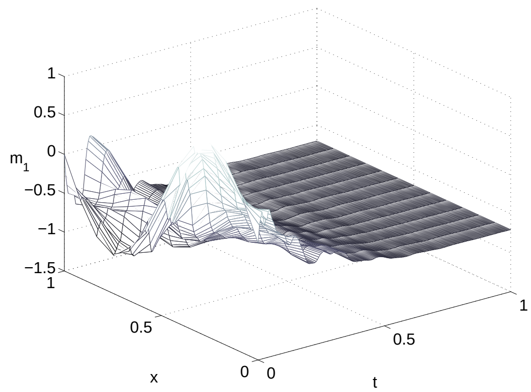
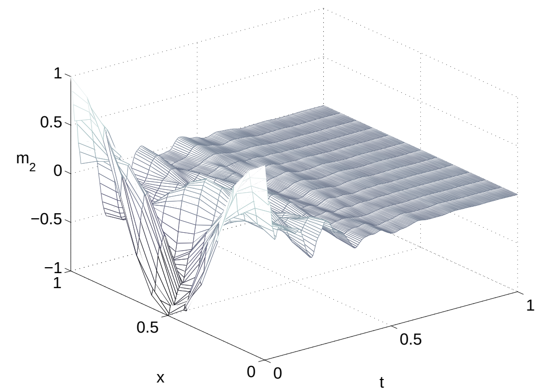


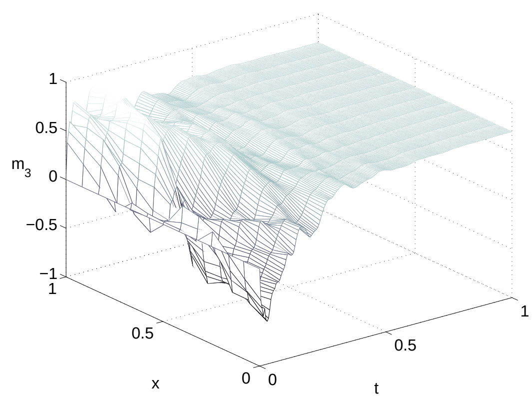
Figure 5.8: Dynamics of (5.3) for $m_3(x,t)$ where $b = 1$, $k_p = 0.5$, $\nu = 0.02$ and $L = 1$ with initial condition $\mathbf{m}(0) = (\sin(2\pi x), \cos 2(\pi x), 0)$ and $\mathbf{r} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. On the horizontal axis is the spatial variable, $x \in [0,1]$, and on the vertical axis is the magnetization. The magnetization evolves from (a) to (f). Initially the magnetization begins at $m_3(x,0) = 0$ and then the control forces the magnetization to $\frac{1}{\sqrt{2}}$. Without the control, the magnetization naturally settles to 0 (see Figure 4.5 or 4.6c). A three–dimensional depiction is illustrated in Figure 5.9c.



(a) $m_1(x,t)$



(b) $m_2(x,t)$



(c) $m_3(x,t)$

Figure 5.9: Magnetization dynamics of the controlled Landau–Lifshitz equation as x, t varies. The initial condition is $\mathbf{m}_0(x) = (\sin(2\pi x), \cos(2\pi x), 0)$ with $\nu = 0.02$ and $L = 1$. The control forces the magnetization to settle to $\mathbf{r} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. Without the control, the magnetization naturally settles to $(0, -0.6, 0)$ (see Figure 4.6).

Since $\mathbf{r}$ does not depend on x,

$$||\left(\mathbf{m}-\mathbf{r}\right)_x||^2_{\mathcal{L}_2^3} + ||\mathbf{m}-\mathbf{r}||^2_{\mathcal{L}_2^3} \le e^{-2\left(bk_p - 16\nu L^4\right)t}\left(||\left(\mathbf{m}(0,t)-\mathbf{r}\right)_x||^2_{\mathcal{L}_2^3} + ||\mathbf{m}(0,t)-\mathbf{r}||^2_{\mathcal{L}_2^3}\right).$$

Therefore,

$$||\mathbf{m}-\mathbf{r}||^2_{H_1} \le e^{-2\left(bk_p - 16\nu L^4\right)t}||\mathbf{m}(0,t)-\mathbf{r}||^2_{H_1}$$

and hence by Definition 2.12, since $bk_p - 16\nu L^4 > 0$, $\mathbf{r}$ is an exponentially stable equilibrium point of (5.3). This is true for any initial condition and hence global stability is obtained. □

Recall the linear Landau–Lifshitz equation in (4.23). We show that $\mathbf{r}$ is an exponentially stable equilibrium of the controlled linear Landau–Lifshitz equation using the same controller design as for the nonlinear case; that is,

$$\begin{aligned}\frac{\partial \mathbf{z}}{\partial t} &= \nu\mathbf{z}_{xx} + \mathbf{a}\times\mathbf{z}_{xx} + b\mathbf{u}(t)\\ \mathbf{y} &= \mathbf{z}\\ \mathbf{e} &= \mathbf{r}-\mathbf{y}\\ \mathbf{u}(t) &= k_p\mathbf{e}\end{aligned}$$

It follows that the controlled linear Landau–Lifshitz equation is

$$\frac{\partial \mathbf{z}}{\partial t} = \nu\mathbf{z}_{xx} + \mathbf{a}\times\mathbf{z}_{xx} + bk_p\left(\mathbf{r}-\mathbf{z}\right), \qquad \mathbf{z}(0) = \mathbf{z}_0 \tag{5.7}$$

with Neumann boundary conditions $\mathbf{z}_x(0) = \mathbf{z}_x(L) = \mathbf{0}$. Since the uncontrolled linear Landau–Lifshitz equation generates a linear semigroup (see Theorems 4.14 and 4.16) and $bk_p\left(\mathbf{r}-\mathbf{z}\right)$ is a bounded linear (affine) operator, then the operator in (5.7) generates a semigroup (Curtain and Zwart [31, Theorem 3.2.1]). Substituting $\mathbf{z}=\mathbf{r}$ into (5.7) leads to $\partial\mathbf{z}/\partial t = \mathbf{0}$ and hence $\mathbf{r}$ is an equilibrium point of (5.7).

Theorem 5.8. Let $\mathbf{r}$ be an equilibrium point of (5.7). For any nonzero constants b and k_p such that $bk_p > 0$, $\mathbf{r}$ is globally exponentially stable in the $\mathcal{L}_2^3$–norm.

Proof of Theorem 5.8 For $\mathbf{z}\in D(A)$, where $D(A)$ is defined in (4.25), consider the Lyapunov candidate

$$V(\mathbf{z}) = \frac{1}{2}||\mathbf{z}-\mathbf{r}||^2_{\mathcal{L}_2^3}.$$

It is clear that $V \ge 0$ for all $\mathbf{z}\in D(A)$ and furthermore, $V(\mathbf{z}) = 0$ only when $\mathbf{z}=\mathbf{r}$. Therefore, $V(\mathbf{z}) > 0$ for all $\mathbf{z}\in D(A)\backslash\{\mathbf{r}\}$.

Taking the derivative of $V(\mathbf{z})$ implies

$$\frac{dV}{dt} = \int_0^L (\mathbf{z}-\mathbf{r})^{\mathrm{T}}\dot{\mathbf{z}}dx.$$

Substituting in (5.7) yields

$$\frac{dV}{dt} = \nu\int_0^L (\mathbf{z}-\mathbf{r})^{\mathrm{T}}\mathbf{z}_{xx}dx + \int_0^L (\mathbf{r}-\mathbf{z})^{\mathrm{T}}\left(\mathbf{a}\times\mathbf{z}_{xx}\right)dx + bk_p\int_0^L (\mathbf{z}-\mathbf{r})^{\mathrm{T}}(\mathbf{r}-\mathbf{z})dx.$$

By Lemma 5.4, the middle term is zero and applying integration by parts the first term becomes

$$-\nu\int_0^L \mathbf{z}_x^{\mathrm{T}}\mathbf{z}_x dx.$$

It follow that

$$\frac{dV}{dt} = -\nu||\mathbf{z}_x||^2_{\mathcal{L}_2^3} - bk_p||\mathbf{z}-\mathbf{r}||^2_{\mathcal{L}_2^3}$$

and since $\nu \geq 0$,

$$\frac{dV}{dt} \leq -bk_p||\mathbf{z}-\mathbf{r}||^2_{\mathcal{L}_2^3} = -2bk_pV.$$

Solving yields

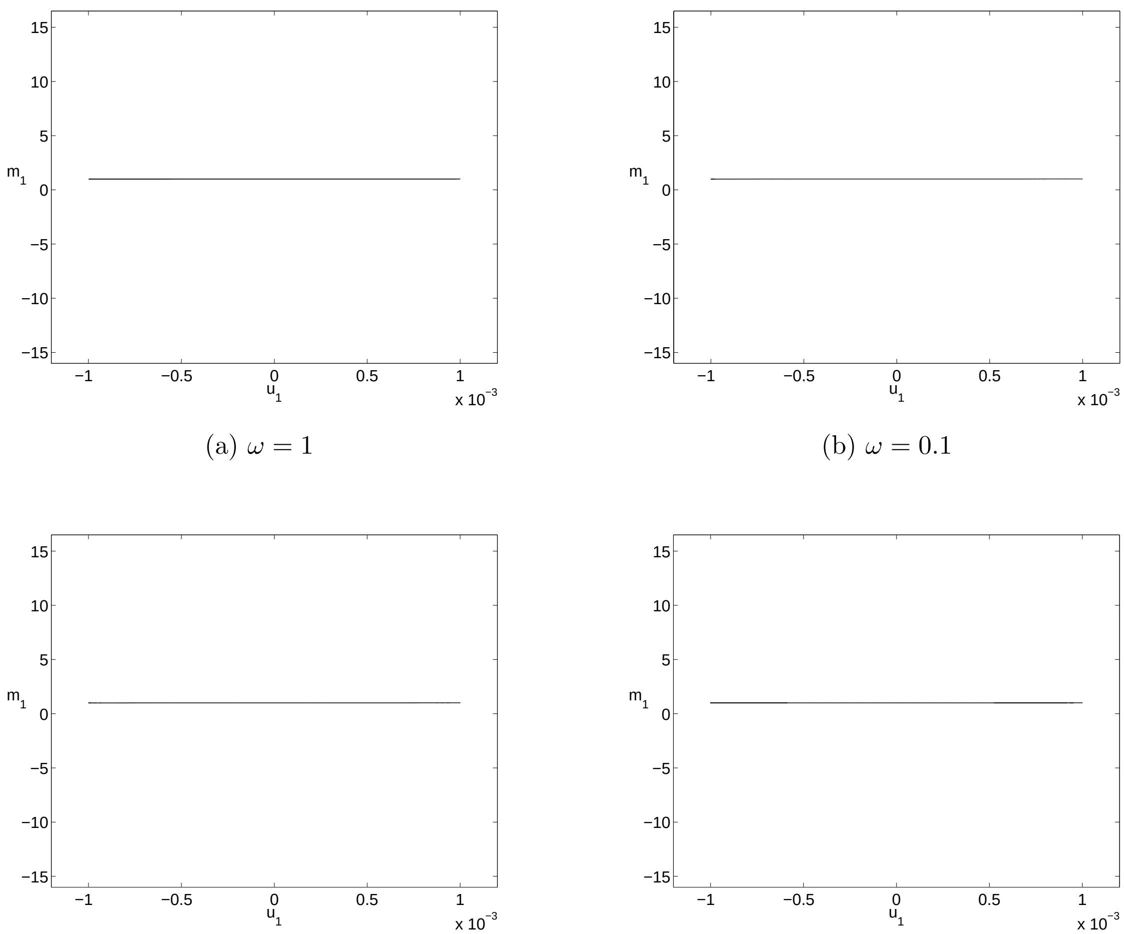
$$||\mathbf{z}-\mathbf{r}||^2_{\mathcal{L}_2^3} \leq e^{-2bk_pt}||\mathbf{z}_0-\mathbf{r}||^2_{\mathcal{L}_2^3}.$$

By Definition 2.12, if $bk_p > 0$ then $\mathbf{r}$ is a locally exponentially stable equilibrium point of (5.7). This is true for any initial condition and hence global stability is obtained. □

Theorem 5.8 suggests that the equilibrium point in Theorem 5.6 for the controlled nonlinear Landau–Lifshitz equation (5.3) is exponentially stable in the $\mathcal{L}_2^3$–norm.

We now consider the input–output dynamics for the controlled Landau–Lifshitz equation in (5.3). Figure 5.10 illustrates the input–output dynamics for $m_1(x,t)$ with x fixed, $L = 1$ and $\nu = 0.02$. The input is $\mathbf{u}(t) = (0.001\cos(\omega t), 0, 0)$ and the initial condition is $\mathbf{m}_0(x) = (1, 0, 0)$. The control parameters are chosen to be $b = 1$, $k_p = 0.5$ and $\mathbf{r} = (1, 0, 0)$. It is clear from Figure 5.10 that persistent looping behaviour does not occur. Similar behaviour is observed for $m_2(x,t)$ and $m_3(x,t)$ (see Figures 5.11 and 5.12, respectively). This suggests from Defintion 3.2 that the control term in (5.3) removes the presence of hysteresis in the Landau–Lifshitz equation. Furthermore, the result in Theorem 5.6 shows that $\mathbf{r}$ is globally stable, which means $\mathbf{r}$ is the only equilibrium point of the controlled Landau–

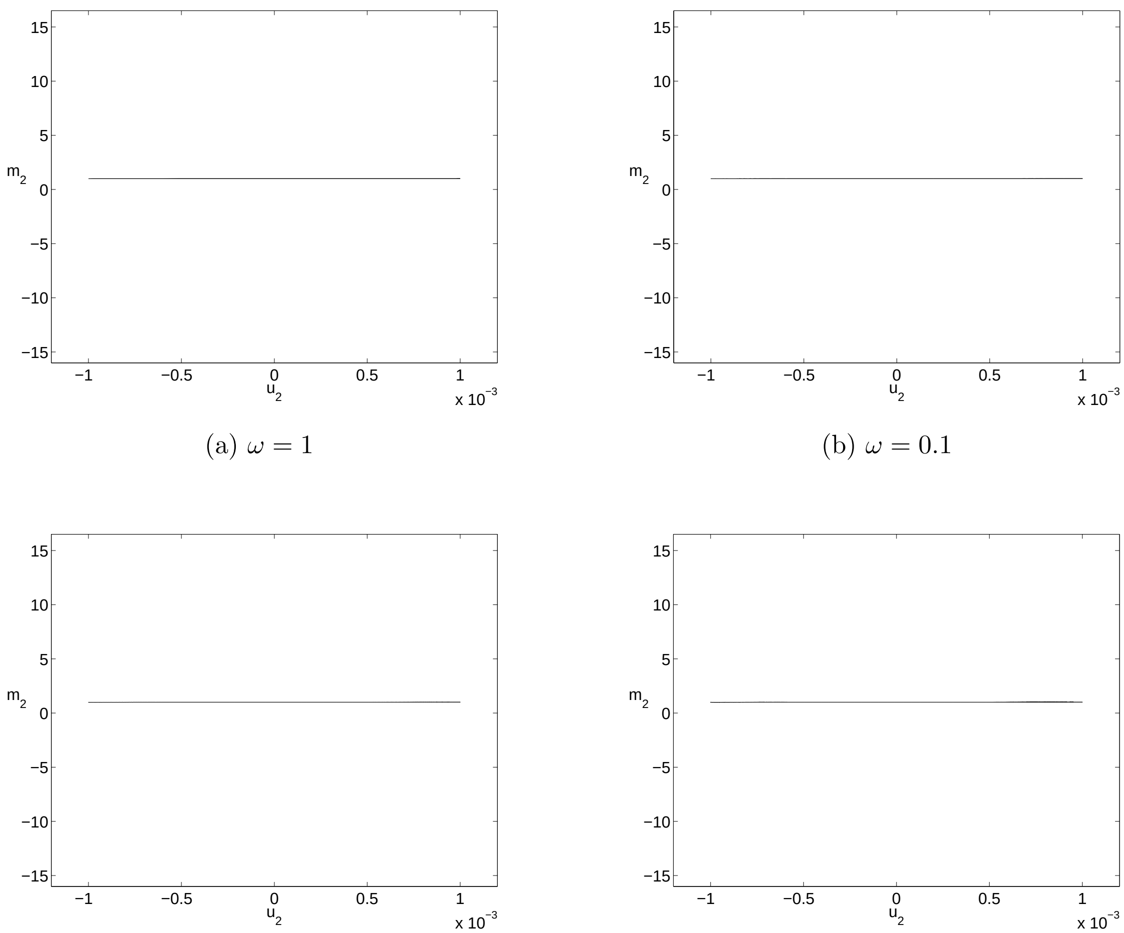
Lifshitz equation (5.3). It follows from Definition 3.1 that the dynamics described by the controlled Landau–Lifshitz equation does not exhibit hysteresis. The lack of hysteresis implies controlling the magnetization from **a** to **r** is the same dynamics as controlling from **r** to **a** but in reverse.



(a) $\omega = 1$ (b) $\omega = 0.1$

(c) $\omega = 0.01$ (d) $\omega = 0.001$

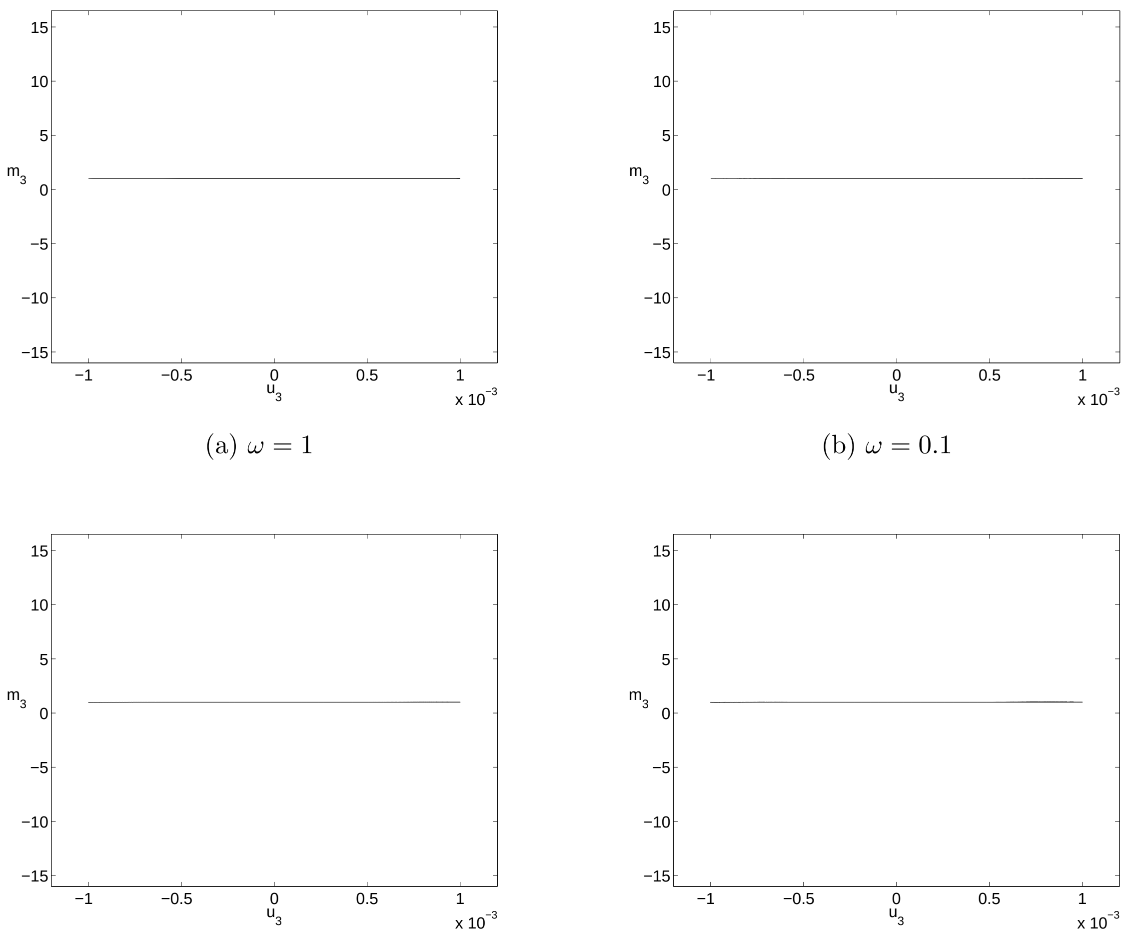
Figure 5.10: Input–output dynamics for $m_1(x,t)$ of the controlled Landau–Lifshitz equation in (5.3) with x fixed and $\nu = 0.02$, $L = 1$. The input is $\mathbf{u}(t) = (0.001\cos(\omega t), 0, 0)$ and the initial condition is $\mathbf{m}_0(x) = (1, 0, 0)$. The control parameters are chosen to be $b = 1$, $k_p = 0.5$, $\mathbf{r} = (1, 0, 0)$. It is clear loops do not persist as ω approaches 0, which suggests the controlled Landau–Lifshitz equation does not exhibit hysteresis.



(a) $\omega = 1$ (b) $\omega = 0.1$

(c) $\omega = 0.01$ (d) $\omega = 0.001$

Figure 5.11: Input–output dynamics for $m_2(x,t)$ of the controlled Landau–Lifshitz equation described in (5.3) with x fixed and $\nu = 0.02$, $L = 1$. The input is $\mathbf{u}(t) = (0, 0.001\cos(\omega t), 0)$. The initial condition is $\mathbf{m}_0(x) = (0,1,0)$ and the control parameters are $b = 1$, $k_p = 0.5$, $\mathbf{r} = (0,1,0)$. It is clear loops do not persist as ω approaches 0, which suggests the control in (5.3) removes hysteresis in the Landau–Lifshitz equation.



(a) $\omega = 1$

(b) $\omega = 0.1$

(c) $\omega = 0.01$

(d) $\omega = 0.001$

Figure 5.12: Input–output dynamics for $m_3(x,t)$ of the controlled Landau–Lifshitz equation described in (5.3) with x fixed and $\nu = 0.02$, $L = 1$. The input is $\mathbf{u}(t) = (0, 0, 0.001 \cos(\omega t))$. The initial condition is $\mathbf{m}_0(x) = (0, 0, 1)$ and the control parameters are $b = 1$, $k_p = 0.5$, $\mathbf{r} = (0, 0, 1)$. It is clear loops do not persist as ω approaches 0, which suggests the controlled Landau–Lifshitz equation does not exhibit hysteresis.

Chapter 6

Conclusion and Future Research

The research in this thesis was initially motivated by the notion of controlling hysteresis arising in nonlinear PDEs. This is of interest because of the presence of hysteresis in many natural processes and the lack of research regarding hysteresis in PDEs and its control. Eventually this led to exploring methods for determining the stability of equilibrium sets and points in PDEs, and the implementation of a controller design to force systems to move from one stable equilibrium to another. The Landau–Lifshitz equation is of interest because it is a nonlinear PDE for which few control results are known. It models magnetization in nanostructures which is of continued interest in technological sectors. A discussion of the Landau–Lifshitz equation was presented with an emphasis on its stability, control of its equilibrium points and the presence of hysteresis.

The Landau–Lifshitz equation has an asymptotically stable equilibrium set (Theorem 4.11). An open question is whether global stability holds for this equilibrium set and also if the equilibrium set is exponentially stable. Furthermore, investigating the stability of nonlinear PDEs raises the question: for a given nonlinear infinite–dimensional system, can the stability of the linearized system be applied to the original nonlinear system? The answer is only partially known and so far all the results require the linear system to exhibit exponential stability. Furthermore, based on example 2.25, it appears exponential stability is a necessary requirement. Therefore, an open question remains: is the exponential stability of a linearized infinite–dimensional system a necessary condition to imply the same stability of the original nonlinear system? Moreover, what conditions, in addition to exponential stability, are needed?

The limitations of linearization means that another stability technique is needed. This led to Lyapunov theory, which was relied on significantly to provide analytical control and stability results to the Landau–Lifshitz equation. We also emphasized that Lyapunov Theory can be applied to invariant sets, which is not very common but virtually identical to the well–known version for equilibrium points.

We discussed hysteresis from a more fundamental and precise approach and demonstrated that the existence of multiple equilibria; and not nonlinearity, is crucial for systems to exhibit hysteresis. Based on this, both the linear and original (nonlinear) Landau–Lifshitz equations exhibit hysteretic behaviour. Determining a rigorous definition of hysteresis is an issue because of the complex nature of hysteresis. There is also virtually no literature for defining hysteretic systems that have arbitrarily close equilibrium points, which is the case for the Landau–Lifshitz equation.

Our main result is the control from one stable equilibrium to another in the Landau–Lifshitz equation. A feedback controller design with a proportional control was shown to successfully achieve this. The control causes the initial equilibrium to no longer be an equilibrium of the controlled system, and ensures the second equilibrium is an asymptotically stable equilibrium point of the controlled system (Theorem 5.6). It is still unknown whether the second equilibrium is exponentially stable, but analysis of the corresponding linear Landau–Lifshitz equation, which shows the second equilibrium is exponentially stable (Theorem 5.8), suggests this may be true for the original nonlinear Landau–Lifshitz equation. Moreover, we showed the linear Landau–Lifshitz equation has an analytic semigroup (Theorem 4.16) and hence satisfies the spectrum determined growth assumption.

Our control can also force any magnetization to any arbitrary stable equilibrium, $\mathbf{r}$. This is because the stability of $\mathbf{r}$ in Theorem 5.6 is a global result and the proof of the theorem does not rely on a specific initial magnetization. For the damping constant ν and length L, the control parameters, b and k_p, in Theorem 5.6 must be chosen such that $bk_p \geq 16\nu L^4$; however, numerical simulations suggest that $bk_p > 0$ is a sufficient condition. It follows that future research could explore weakening the requirement that $bk_p \geq 16\nu L^4$.

Currently, the control result in Theorem 5.6 applies to any equilibrium magnetization only, and not an arbitrary magnetization. Controlling to any point would require significant modification of the work presented in Chapter 5, as the results rely on the equilibria being constant. Other future work on the control includes reducing the necessity of full state feedback and implementing our control and stability framework to related Landau–Lifshitz equations.

APPENDICES

Appendix A

Matlab Code for Equation (3.1)

Equation (3.1) is solved in Matlab using the following function file:

```
function [t,x,input]=Dynamics(tspan,x0,w)
%tspan is the interval of time
%x0 is the initial condition
%w is the frequency of the input

c=15; k=1;

[t,x]=ode45(@linearequations,tspan,x0);
input=u(t);

function dx=linearequations(t,x)
dx1=x(2);
dx2=-c*x(2)-k*x(1)+u(t);
%dx2=-c*x(2)-k*x(1)+k*x(1)^3+u(t); %Nonlinear Equation
dx=[dx1;dx2];
end;

function f=u(t)
f=sin(w*t);
end;

end
```

Appendix B

Numerical Approximations for the Landau–Lifshitz Equation

Two common numerical approaches for the Landau–Lifshitz equation are a finite–difference computation (Fuwa *et al.* [36], Serpico *et al.* [77] and Wiele *et al.* [88]) or a finite–element method (Bottauscio *et al.* [18], and Wiele *et al.* [88]). We apply a Galerkin finite–element scheme with linear spline elements to approximate the weak form of the Landau–Lifshitz equation. Galerkin approximations are demonstrated to be successful for the controlled heat equation where approximations are shown to converge asymptotically (Morris [66, example 4.12]); that is,

$$\lim_{N\to\infty} ||w_N - w||_H = 0$$

where w_N and w is the approximate and exact solution to the heat equation, respectively.

To begin, recall the semilinear form of the Landau–Lifshitz equation from (4.20):

$$\frac{\partial \mathbf{m}}{\partial t} = \nu \mathbf{m}_{xx} + \mathbf{m} \times \mathbf{m}_{xx} + \nu \, ||\mathbf{m}_x||_2^2 \, \mathbf{m}. \tag{B.1}$$

This form of the Landau–Lifshitz equation allows for a weak formulation that consists of only first order derivatives.

The magnetization, $\mathbf{m}(x,t)$, is approximated as

$$\mathbf{M}(x,t) = \sum_{i=0}^{N} \mathbf{c}_i(t)\phi_i(x) \tag{B.2}$$

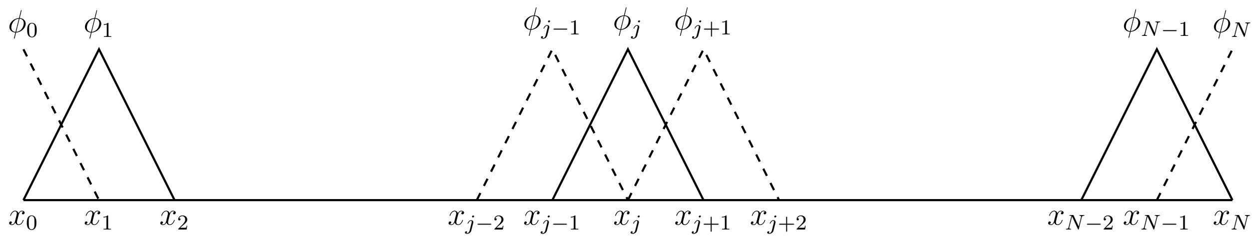


Figure B.1: Linear spline functions.

where $\mathbf{c}_i(t) = (c_i^1(t), c_i^2(t), c_i^3(t))$; that is,

$$\mathbf{M}(x,t) = \left(\sum_{i=0}^{N} c_i^1(t)\phi_i(x), \sum_{i=0}^{N} c_i^2(t)\phi_i(x), \sum_{i=0}^{N} c_i^3(t)\phi_i(x) \right).$$

The approximation in (B.2) is the same one chosen by Yang and Fredkin [99]. The functions, $\phi_i(x)$, are chosen to be simple linear spline functions (Figure B.1),

$$\phi_0(x) = \begin{cases} \frac{1}{h}(x_1 - x), & x_0 \le x \le x_1 \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_i(x) = \begin{cases} \frac{1}{h}(x - x_{i-1}), & x_{i-1} \le x \le x_i \\ \frac{1}{h}(x_{i+1} - x), & x_i \le x \le x_{i+1} \\ 0, & \text{otherwise}, \end{cases} \qquad \text{for } i = 1, 2, \ldots, N-2, N-1$$

$$\phi_N(x) = \begin{cases} \frac{1}{h}(x - x_{N-1}), & x_{N-1} \le x \le x_N \\ 0, & \text{otherwise} \end{cases}$$

where $h = L/(N+1)$. The one–dimensional spatial domain is uniformly discretized with $x_i = ih$.

Taking the scalar product of (B.1) with a test function, $\mathbf{v} = (v_1, v_2, v_3)$, then applying integration by parts and Lemma 5.2, we obtain the weak form of the Landau–Lifshitz equation,

$$\int_0^L \frac{\partial \mathbf{m}}{\partial t}^{\mathrm{T}} \mathbf{v} dx = -\nu \int_0^L \mathbf{m}_x^{\mathrm{T}} \mathbf{v}_x dx - \int_0^L (\mathbf{m} \times \mathbf{m}_x)^{\mathrm{T}} \mathbf{v}_x dx + \nu \int_0^L ||\mathbf{m}_x||_2^2 \, \mathbf{m}^{\mathrm{T}} \mathbf{v} dx.$$

The components of the test function are made to be equivalent; that is, $v_1 = v_2 = v_3$. As well, the test function is chosen to be the same as ϕ_i, which is a standard approach. Therefore, substituting in (B.2) into the weak form of the Landau–Lifshitz equation leads to

$$\sum_{i=0}^{N} K_{li} \dot{\mathbf{c}}_i = -\nu \sum_{i=0}^{N} P_{li} \mathbf{c}_i - \sum_{i=0}^{N} \sum_{j=0}^{N} Q_{lij} (\mathbf{c}_i \times \mathbf{c}_j) + \nu \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{n=0}^{N} S_{lijn} \mathbf{c}_i^{\mathrm{T}} \mathbf{c}_j \mathbf{c}_n \tag{B.3}$$

for $l = 0, 1, \ldots, N$, where

$$\begin{aligned} K_{li} &= \int_0^L \phi_l \phi_i dx \\ P_{li} &= \int_0^L \phi_l' \phi_i' dx \\ Q_{lij} &= \int_0^L \phi_l' \phi_i \phi_j' dx \\ S_{lijn} &= \int_0^L \phi_l \phi_i' \phi_j' \phi_n dx. \end{aligned}$$

Notice from the choice of ϕ_i that many of the integrals are zero. Equation (B.3) is a system of $3(N+1)$ coupled nonlinear ODEs with unknowns

$$\begin{aligned} \mathbf{c} &= \{\mathbf{c}_0, \mathbf{c}_1, \cdots, \mathbf{c}_{N-1}, \mathbf{c}_N\} \\ &= \{c_0^1, c_0^2, c_0^3, c_1^1, c_1^2, c_1^3, \cdots, c_{N-1}^1, c_{N-1}^2, c_{N-1}^3, c_N^1, c_N^2, c_N^3\}. \end{aligned}$$

These ODEs are solved using an explicit fourth and fifth order Runge–Kutta method (Kharab and Guenther [51, Chapter 12.4]). The corresponding MATLAB solver is ODE45.

The numerical approximations described in (B.3) is coded in Matlab using the following function files. The main function file is FEMLL and it calls the functions LoadVectorEquation and MassMatrix, which create the system of ODEs in (B.3).

```
function [t,m,N,L]=FEMLL
clc

N=10; %number of elements
L=1; %length of interval
v=0.02; %damping parameter
h=L/N;
endtime=20;%end time for ODE solver

%initial condition: m0=(sin(pi*x),cos(pi*x),0)
m0=zeros(3*(N+1),1);
%for m1 component
for i=1:(N+1)
    x=(i-1)*h;
    y(i)=sin(pi*x);
end
j=1;
for i=1:(N+1)
   m0(j)=y(i);
   j=j+3;
end
%for m2 component
for i=1:(N+1)
    x=(i-1)*h;
    y(i)=cos(pi*x);
end
j=2;
for i=1:(N+1)
   m0(j)=y(i);
   j=j+3;
end

%solve ODE using built in Matlab ODE solver
%solves ODE of the form m=K\dg
%where K is the mass matrix and dg is the load vector
options = odeset('Mass',@mass);
[t,m]=ode45(@equationsMassForm,[0:0.1:endtime],m0,options);
```

```
%Calling Load vector function
function dg=equationsMassForm(t,m)
    dg=LoadVectorEquation(N,h,v,m);
end

%Calling Mass matrix function
function K = mass(t,m)
   K=MassMatrix(N,h,v);
end

end

%Load vector function
function dg=LoadVectorEquation(N,h,v,m)
%N is the number of elements
%L is the length of space interval
%h=L/N
%v is the damping parameter
%m is the solution vector

%at the x=0 boundary
dg(1)=-v*(m(1)-m(4))+(m(2)*m(6)-m(3)*m(5))+v*((m(1)^2+m(2)^2+m(3)^2)*(1/3*m(1)...
     +1/6*m(4))-(m(1)*m(4)+m(2)*m(5)+m(3)*m(6))*(2/3*m(1)+1/3*m(4))...
     +(m(4)^2+m(5)^2+m(6)^2)*(1/3*m(1)+1/6*m(4)));

dg(2)=-v*(m(2)-m(5))+(-m(1)*m(6)+m(3)*m(4))+v*((m(1)^2+m(2)^2+m(3)^2)*(1/3*m(2)...
    +1/6*m(5))-(m(1)*m(4)+m(2)*m(5)+m(3)*m(6))*(2/3*m(2)+1/3*m(5))...
    +(m(4)^2+m(5)^2+m(6)^2)*(1/3*m(2)+1/6*m(5)));

dg(3)=-v*(m(3)-m(6))+(m(1)*m(5)-m(2)*m(4))+v*((m(1)^2+m(2)^2+m(3)^2)*(1/3*m(3)...
    +1/6*m(6))-(m(1)*m(4)+m(2)*m(5)+m(3)*m(6))*(2/3*m(3)+1/3*m(6))...
    +(m(4)^2+m(5)^2+m(6)^2)*(1/3*m(3)+1/6*m(6)));

%dg(4),dg(5),dg(6),...,dg(3*(N+1)-4),dg(3*(N+1)-3)
j=4;
for i=1:(N-1)
```

```
dg(j)=-v*(-m(j-3)+2*m(j)-m(j+3))-(m(j-2)*m(j+2)-m(j-1)*m(j+1))+(m(j+1)*m(j+5)...
    -m(j+2)*m(j+4))+v*((m(j-3)^2+m(j-2)^2+m(j-1)^2)*(1/6*m(j-3)+1/3*m(j))...
    -(m(j-3)*m(j)+m(j-2)*m(j+1)+m(j-1)*m(j+2))*(1/3*m(j-3)+2/3*m(j))...
    +(m(j)^2+m(j+1)^2+m(j+2)^2)*(1/6*m(j-3)+2/3*m(j)+1/6*m(j+3))...
    -(m(j)*m(j+3)+m(j+1)*m(j+4)+m(j+2)*m(j+5))*(2/3*m(j)+1/3*m(j+3)) ...
    +(m(j+3)^2+m(j+4)^2+m(j+5)^2)*(1/3*m(j)+1/6*m(j+3)));

dg(j+1)=-v*(-m(j-2)+2*m(j+1)-m(j+4))-(-m(j-3)*m(j+2)+m(j-1)*m(j))+(-m(j)*m(j+5)...
    +m(j+2)*m(j+3))+v*((m(j-3)^2+m(j-2)^2+m(j-1)^2)*(1/6*m(j-2)+1/3*m(j+1))...
    -(m(j-3)*m(j)+m(j-2)*m(j+1)+m(j-1)*m(j+2))*(1/3*m(j-2)+2/3*m(j+1))...
    +(m(j)^2+m(j+1)^2+m(j+2)^2)*(1/6*m(j-2)+2/3*m(j+1)+1/6*m(j+4))...
    -(m(j)*m(j+3)+m(j+1)*m(j+4)+m(j+2)*m(j+5))*(2/3*m(j+1)+1/3*m(j+4)) ...
    +(m(j+3)^2+m(j+4)^2+m(j+5)^2)*(1/3*m(j+1)+1/6*m(j+4)));

dg(j+2)=-v*(-m(j-1)+2*m(j+2)-m(j+5))-(m(j-3)*m(j+1)-m(j-2)*m(j))+(m(j)*m(j+4)...
    -m(j+1)*m(j+3))+v*((m(j-3)^2+m(j-2)^2+m(j-1)^2)*(1/6*m(j-1)+1/3*m(j+2))...
    -(m(j-3)*m(j)+m(j-2)*m(j+1)+m(j-1)*m(j+2))*(1/3*m(j-1)+2/3*m(j+2))...
    +(m(j)^2+m(j+1)^2+m(j+2)^2)*(1/6*m(j-1)+2/3*m(j+2)+1/6*m(j+5))...
    -(m(j)*m(j+3)+m(j+1)*m(j+4)+m(j+2)*m(j+5))*(2/3*m(j+2)+1/3*m(j+5)) ...
    +(m(j+3)^2+m(j+4)^2+m(j+5)^2)*(1/3*m(j+2)+1/6*m(j+5)));

    j=j+3;
end

%at the x=L boundary
dg(3*(N+1)-2)=-v*(-m(3*(N+1)-5)+m(3*(N+1)-2))-(m(3*(N+1)-4)*m(3*(N+1))...
    -m(3*(N+1)-3)*m(3*(N+1)-1))+v*((m(3*(N+1)-5)^2+m(3*(N+1)-4)^2...
    +m(3*(N+1)-3)^2)*(1/6*m(3*(N+1)-5)+1/3*m(3*(N+1)-2))...
    -(m(3*(N+1)-5)*m(3*(N+1)-2)+m(3*(N+1)-4)*m(3*(N+1)-1)....
    +m(3*(N+1)-3)*m(3*(N+1)))*(1/3*m(3*(N+1)-5)+2/3*m(3*(N+1)-2))...
    +(m(3*(N+1)-2)^2+m(3*(N+1)-1)^2+m(3*(N+1))^2)*(1/6*m(3*(N+1)-5)...
    +1/3*m(3*(N+1)-2)));

dg(3*(N+1)-1)=-v*(-m(3*(N+1)-4)+m(3*(N+1)-1))-(-m(3*(N+1)-5)*m(3*(N+1))...
    +m(3*(N+1)-3)*m(3*(N+1)-2))+v*((m(3*(N+1)-5)^2+m(3*(N+1)-4)^2...
    +m(3*(N+1)-3)^2)*(1/6*m(3*(N+1)-4)+1/3*m(3*(N+1)-1))...
    -(m(3*(N+1)-5)*m(3*(N+1)-2)+m(3*(N+1)-4)*m(3*(N+1)-1)...
    +m(3*(N+1)-3)*m(3*(N+1)))*(1/3*m(3*(N+1)-4)+2/3*m(3*(N+1)-1))...
```

```
    +(m(3*(N+1)-2)^2+m(3*(N+1)-1)^2+m(3*(N+1))^2)*(1/6*m(3*(N+1)-4)...
    +1/3*m(3*(N+1)-1)));

dg(3*(N+1))=-v*(-m(3*(N+1)-3)+m(3*(N+1)))-(m(3*(N+1)-5)*m(3*(N+1)-1)...
    -m(3*(N+1)-4)*m(3*(N+1)-2))+v*((m(3*(N+1)-5)^2+m(3*(N+1)-4)^2...
    +m(3*(N+1)-3)^2)*(1/6*m(3*(N+1)-3)+1/3*m(3*(N+1)))...
    -(m(3*(N+1)-5)*m(3*(N+1)-2)+m(3*(N+1)-4)*m(3*(N+1)-1)...
    +m(3*(N+1)-3)*m(3*(N+1)))*(1/3*m(3*(N+1)-3)+2/3*m(3*(N+1)))...
    +(m(3*(N+1)-2)^2+m(3*(N+1)-1)^2+m(3*(N+1))^2)*(1/6*m(3*(N+1)-3)...
    +1/3*m(3*(N+1))));

dg=1/h*dg';

end

%Mass Matrix function
function K=MassMatrix(N,h,v)
%N is the number of elements
%L is the length of space interval
%h=L/N
%v is the damping parameter
%m is the solution vector

%K at the x=0 boundary
%1st row
K(1,1)=1/3;
K(1,4)=1/6;
%2nd row
K(2,2)=1/3;
K(2,5)=1/6;
%3rd row
K(3,3)=1/3;
K(3,6)=1/6;

  j=4;
    for i=1:N-1
        %jth row
        K(j,j-3)=1/6;
```

```
        K(j,j)=2/3;
        K(j,j+3)=1/6;
        %j+1th row
        K(j+1,j-2)=1/6;
        K(j+1,j+1)=2/3;
        K(j+1,j+4)=1/6;
        %j+2th row
        K(j+2,j-1)=1/6;
        K(j+2,j+2)=2/3;
        K(j+2,j+5)=1/6;
        j=j+3;
    end


%K at the x=L boundary
%3rd last row
K(3*(N+1)-2,3*(N+1)-5)=1/6;
K(3*(N+1)-2,3*(N+1)-2)=1/3;
%2nd last row
K(3*(N+1)-1,3*(N+1)-4)=1/6;
K(3*(N+1)-1,3*(N+1)-1)=1/3;
%3rd last row
K(3*(N+1),3*(N+1)-3)=1/6;
K(3*(N+1),3*(N+1))=1/3;
 K=h*K;
end
```

Appendix C

Maple Code to Solve the Eigenvalue Problem of the Linear Landau–Lifshitz equation

The eigenvalue problem for the linear Landau–Lifshitz equation described in (4.27) is solved in Maple. To make the computations a little easier, we rescale the problem with $y = x/L$. Then (4.27) becomes

$$\lambda v_1 = \rho v_1'' - b_3 v_2'' + b_2 v_3'' \tag{C.1a}$$

$$\lambda v_2 = b_3 v_1'' + \rho v_2'' - b_2 v_3'' \tag{C.1b}$$

$$\lambda v_3 = -b_2 v_1'' + b_1 v_2'' + \rho v_3'' \tag{C.1c}$$

with boundary conditions

$$v_1'(0) = v_1'(1) = 0 \tag{C.1d}$$

$$v_2'(0) = v_2'(1) = 0 \tag{C.1e}$$

$$v_3'(0) = v_3'(1) = 0 \tag{C.1f}$$

where the prime notation is with respect to y and

$$\rho = \frac{\mu}{L^2}, \qquad b_1 = \frac{a_1}{L^2}, \qquad b_2 = \frac{a_2}{L^2}, \qquad b_3 = \frac{a_3}{L^2}.$$

The following commands in Maple are used:

```
> #eigenvalue problem
> eq1:= lambda*v1(x)=rho*diff(v1(x),x,x)-b3*diff(v2(x),x,x)+b2*diff(v3(x),x,x):
> eq2:= lambda*v2(x)=b3*diff(v1(x),x,x)+rho*diff(v2(x),x,x)-a1*diff(v3(x),x,x):
> eq3:= lambda*v3(x)=-b2*diff(v1(x),x,x)+b1*diff(v2(x),x,x)+rho*diff(v3(x),x,x):
>
> #boundary conditions
> ics1 := D(v1)(0)=0, D(v1)(1)=0:
> ics2 := D(v2)(0)=0, D(v2)(1)=0:
> ics3 := D(v3)(0)=0, D(v3)(1)=0:
>
> #solving in terms of eigenvectors and eigenvalues
> dsolve([eq1,eq2,eq3, ics1,ics2,ics3], [v1(x),v2(x),v3(x), lambda]);
```

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