Orthogonal Separation of The Hamilton-Jacobi Equation on Spaces of Constant Curvature

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

What is in common between the Kepler problem, a Hydrogen atom and a rotating black-hole? These systems are described by different physical theories, but much information about them can be obtained by separating an appropriate Hamilton-Jacobi equation. The separation of variables of the Hamilton-Jacobi equation is an old but still powerful tool for obtaining exact solutions.

The goal of this thesis is to present the theory and application of a certain type of conformal Killing tensor (hereafter called concircular tensor) to the separation of variables problem. The application is to spaces of constant curvature, with special attention to spaces with Euclidean and Lorentzian signatures. The theory includes the general applicability of concircular tensors to the separation of variables problem and the application of warped products to studying Killing tensors in general and separable coordinates in particular. Our first main result shows how to use these tensors to construct a special class of separable coordinates (hereafter called Kalnins-Eisenhart-Miller (KEM) coordinates) on a given space. Conversely, the second result generalizes the Kalnins-Miller classification to show that any orthogonal separable coordinates in a space of constant curvature are KEM coordinates. A closely related recursive algorithm is defined which allows one to intrinsically (coordinate independently) search for KEM coordinates which separate a given (natural) Hamilton-Jacobi equation. This algorithm is exhaustive in spaces of constant curvature. Finally, sufficient details are worked out, so that one can apply these procedures in spaces of constant curvature using only (linear) algebraic operations. As an example, we apply the theory to study the separability of the Calogero-Moser system.

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Acronyms

KV Killing vector 46
KT Killing tensor 46, 57, 80
ChKT characteristic Killing tensor 70, 87
CKT conformal Killing tensor 48, 85
CV concircular vector 98, 225
CT concircular tensor also called a C-tensor 83
OCT orthogonal concircular tensor 90, 149
ICT irreducible concircular tensor 91, 151
KS-space Killing-Stackel space 10, 72, 85
KEM Kalnins-Eisenhart-Miller 101
KBD Killing Bertrand-Darboux 112, 213
KBDT Killing Bertrand-Darboux tensor 92
WP-net warped product net 40
TP-net twisted product net 40
HJ Hamilton-Jacobi 61

List of Notations

$\mathbb{E}^n_\nu$ pseudo-Euclidean space, an n-dimensional vector space equipped with a metric with signature ν xii, 136, 260

$\mathbb{E}^n_\nu(\kappa)$ A hyperquadric of pseudo-Euclidean space. More precisely the central hyperquadric of $\mathbb{E}^n_\nu$ with curvature κ. 144

$\ominus$ The orthogonal direct sum. 4, 245

sgn Given a real number a, $\operatorname{sgn} a$ is the sign of a if $a \neq 0$ and 0 if $a = 0$. 144

$\mathcal{F}(M)$ The set of functions defined on the manifold M. 6

$\mathfrak{X}(M)$ The set of vector fields defined on the manifold M. 6

$\Gamma(E)$ The set of vector fields tangent to the distribution E. 6, 25

$S^p(M)$ The set of symmetric contravariant tensors of valence p defined on the manifold M. 6

$\mathrm{C}^p(M)$ The vector space of concircular contravariant tensors of valence p defined on the manifold M. 83

$\mathrm{C}^p_0(M)$ The vector space of covariantly constant contravariant tensors of valence p defined on the manifold M. 83

$\odot$ The symmetric product of two tensors, i.e. if u, v are tensors then $u \odot v$ is the symmetrization of $u \otimes v$. 5, 150

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Chapter 1

Introduction

The method of separation of variables for the Hamilton-Jacobi equation is considered to be a powerful tool for obtaining exact solutions. Classically, this method was one of the only known methods for obtaining exact solutions. Until recently (last few decades), it was not known how to fully exploit this method to its maximum potential.

We say coordinates are *separable* if they are orthogonal and they separate the geodesic Hamilton-Jacobi equation. Separable coordinates can be used to integrate the geodesic equations by quadratures. Additionally, these coordinates are important to mathematical physics for other reasons. Assuming the Ricci tensor is diagonalized in them, one can show that they separate the Laplace equation and the Klein-Gordon equation from relativity.

The *fundamental problems* concerning the separation of the Hamilton-Jacobi equation are the following:

1. Give a (pseudo-)Riemannian manifold, what are the "inequivalent" coordinate systems that separate the geodesic Hamiltonian?

2. How do we determine, intrinsically (coordinate-independently), the "inequivalent" coordinate systems in which a given natural Hamiltonian separates?

3. If we have determined that the natural Hamiltonian is separable in coordinates $(u^1, \ldots, u^n)$, what is the transformation to these coordinates from the original coordinates $(q^1, \ldots, q^n)$ in which the natural Hamiltonian is defined?

Concircular tensors can be used to obtain an elegant solution to these problems in spaces of constant curvature, as we shall show throughout this thesis. In Chapter 2 we will give an overview of this solution.

1.1 Preface

We give a textbook like account of the modern geometric theory of orthogonal separation of variables for the Hamilton-Jacobi equation, with special attention to spaces of constant curvature. We first present the general theory in the first part and then specialize it to spaces of constant curvature in the second. The second part, for the most part, can be read separately from the first. The chapters tend to provide greater details and generalizations on certain aspects of the theory. Hence it is necessary to tie them together, this is done in Chapter 2. In this chapter, we also give an overview of the main results.

The thesis is organized so that it can be a useful reference on several subtopics and related topics. Although the thesis is not completely self-contained, most proofs are given, especially when they are difficult to find elsewhere, or are important for understanding the theory. Sections marked with an asterisk (*) are mostly optional. They can be skipped with little to no loss of continuity until their results are referenced elsewhere, which is usually a rare occurrence.

We assume the reader has some familiarity with differential geometry, including the notion of distributions (plane fields) and the related Frobenius theorem (e.g. see [Lee12]). It is assumed the reader has a sufficient knowledge of (pseudo-)Riemannian geometry to have a basic understanding of general relativity (e.g. see [O'N83]).

By using the notions and tools of differential geometry to solve this problem, we provide a fairly general setting in which our results are applicable. We are able to generalize previous results given for Riemannian spaces of constant curvature in [Kal86] to arbitrary signature. Furthermore we are able to present solutions to problems, in a single framework, which are usually solved separately.

Much of the content of this thesis comes from other articles. The content from [RM14b] is split up into Sections 4.4, 4.5, 5.3.1, 5.4.1, 6.5 and 6.7. The content from [RM14c] makes up Chapter 9 and Sections 10.2 and 10.3. The content from [RM14a] makes up Chapter 7 and also appears in Section 6.5. The content from [Raj14a] is summarized in Section 8.2, and it makes up Appendix C. The content from [Raj14b] is summarized in Section 8.4, and it makes up Appendix D and Sections 3.1 and 3.2.

1.2 Historical Outline

The theory presented in this thesis is a synthesis of decades of research in this area. The seminal result from which we start is the intrinsic characterization of separation for geodesic Hamiltonians given by Eisenhart in [Eis34]. This result can be deduced from the Levi-Civita equations originally given in [LC04].

This problem has a long history, for the work preceding Eisenhart see [Kal86] and references therein. Recent interest in the subject was due to the discovery of the separability of the Hamilton-Jacobi equation for the geodesics in the Kerr solution from general relativity [Car68]. Some of this research culminated in the Kalnins-Miller classification of separable coordinates in Riemannian spaces of constant curvature. This classification, which was based on Eisenhart's results, was originally presented in the articles [KM86; KM82], and then combined in the book [Kal86].

The next result was due to Benenti in [Ben92a], where he obtained an intrinsic method to calculate Killing tensors associated with certain separable coordinates in Euclidean space such as elliptic and parabolic coordinates. This was done with the help of a certain torsionless conformal Killing tensor.

Around the same time, Benenti had come up with the intrinsic characterization of separation for natural Hamiltonians presented in [Ben93]. These results were eventually generalized to general (possibly non-orthogonal) separation in [Ben97].

The results given by Benenti in [Ben92a] were further refined by Crampin in [Cra03], and concircular tensors were first formally introduced into separation of variables. By then it was known that certain separable coordinates in Euclidean space, such as elliptic coordinates, could be intrinsically characterized using concircular tensors.

A recursive algorithm for separating natural Hamiltonians in Euclidean and spherical space was given by Waksjo and Wojciechowski in [WW03]. It turns out that this algorithm, which is based on the Kalnins-Miller classification, can be intrinsically defined using concircular tensors.

Further research on concircular tensors was done by Benenti in [Ben05] and Crampin et al. in [TCS05; Cra07].

Motivated by the desire to obtain an intrinsic algorithm to separate natural Hamiltonians in spaces of constant curvature, the author developed a more general theory on the application of concircular tensors to the separation of variables problem in [RM14b]. This theory was then applied to spaces of constant curvature in [RM14c].

1.3 Summary of Main Results

The main purpose of this thesis is to present the theory of concircular tensors (CTs) and their application to the separation of variables problem in general, and in spaces of constant curvature in particular.

The first result given in Section 6.5 shows how to use CTs to construct a special class of separable coordinates, hereafter called Kalnins-Eisenhart-Miller (KEM) coordinates, in a given pseudo-Riemannian manifold. The second result is a converse to the first result, which shows that all orthogonal separable coordinates in spaces of constant curvature are KEM coordinates (see Chapter 7). This result generalizes the corresponding result by

Kalnins and Miller in [Kal86] and serves as an independent verification of it.

A recursive algorithm is given (see Section 6.7) which allows one to intrinsically (coordinate independently) search for KEM coordinates which separate a given (natural) Hamilton-Jacobi equation. This algorithm is exhaustive in spaces of constant curvature.

Results given originally by Benenti in [Ben92a] are generalized to give a recursive procedure to construct the KS-space associated with KEM coordinates in Section 6.6.

Finally, sufficient details are worked out in Chapter 9, so that one can apply these procedures in spaces of constant curvature using only (linear) algebraic operations. Further details are worked out in Chapter 10 and then applied to study the separability properties of the Calogero-Moser system.

Additionally, various generalizations are given. The most significant is the study of Killing tensors and separation in warped products. See Sections 4.5 and 5.3.1 for the former and Section 5.4.1 for the latter.

1.4 Notations and Conventions

Our notations and conventions build on those in [O'N83] and [Lee12].

1.4.1 pseudo-Euclidean spaces

Suppose V is a *vector space* over a field $\mathbb{F}$ (which for us is $\mathbb{R}$ or $\mathbb{C}$). A *symmetric bilinear form* on V is bilinear function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$. A symmetric bilinear form $\langle \cdot, \cdot \rangle$ is called *non-degenerate* if for a fixed $x \in V$, $\langle x, y \rangle = 0$ for all $y \in V$ implies $x = 0$. If $x \in V$ then we denote $x^2 := \langle x, x \rangle$ and $\|x\| := \sqrt{|x^2|}$. A vector $x \in V$ is called a *unit vector* if $\|x\| = 1$.

Given a non-zero vector $x \in V$, it is classified as follows:

timelike If $\langle x, x \rangle < 0$

lightlike (null) If $\langle x, x \rangle = 0$

spacelike If $\langle x, x \rangle > 0$

We define a *scalar product (metric)* on a vector space V as a non-degenerate symmetric bilinear form on V. A real vector space V equipped with a scalar product is called a *scalar product space (pseudo-Euclidean space)*. The index of a real scalar product space V, denoted $\operatorname{ind} V$, is defined as the number of timelike basis vectors in an orthogonal basis for the scalar product, which is an invariant of the scalar product by Sylvester's law of inertia. For all notions related to the index, we will assume the scalar product space is real. The invariant $\operatorname{ind} V$ is also called the signature of the metric $\langle \cdot, \cdot \rangle$. A *pseudo-Euclidean space* of dimension n and signature ν is often denoted $\mathbb{E}^n_\nu$.

The *Euclidean metric* given as follows is an example of a non-degenerate scalar product:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

An n-dimensional real vector space equipped with the Euclidean metric is called *Euclidean space* and denoted $\mathbb{E}^n$. The standard example of a non-degenerate scalar product with non-zero signature is the *Minkowski metric* given as follows:

$$\langle x, y \rangle = \sum_{i=2}^{n} x_i y_i - x_1 y_1$$

A n-dimensional real vector space equipped with the Minkowski metric is called *Minkowski space* and sometimes denoted M^n.

Given a subspace $H \subseteq V$, we denote the *orthogonal subspace* of H as $H^\perp$ which is defined as follows:

$$H^\perp = \{x \in V : \langle x, y \rangle = 0 \quad \text{for all } y \in H\}$$

$H^\perp$ is complementary to H (i.e. $V = H \oplus H^\perp$) iff the restriction of the scalar product to H is non-degenerate [O'N83, P. 49]. One can also show that for a non-degenerate subspace H, $\operatorname{ind} V = \operatorname{ind} H + \operatorname{ind} H^\perp$. If U, W are subspaces of V, then $V = U \ominus W$ means that $V = U \oplus W$ and $U \perp W$.

Tensors We now discuss notation related to tensors on a vector space V. Let T be a type $\binom{a}{b}$ tensor on V. If $b = 0$ (resp. $a = 0$) we say T is a *contravariant* (resp. *covariant*) tensor of valence a (resp. b). Now suppose V is a scalar product space. Without further specification, *tensor* is short for valence 2-tensor and the type depends on the context.

Let T be an endomorphism (i.e. $\binom{1}{1}$ tensor) of V. A subspace D is called T-*invariant* if $TD \subseteq D$. T is said to have a *simple eigenvalue* λ, if λ is real and has algebraic multiplicity equal to 1. T is said to have *simple eigenvalues* if all its eigenvalues are simple. T is called *self-adjoint* if

$$\langle Tx, y \rangle = \langle x, Ty \rangle \qquad \text{for all } x, y \in V$$

The above condition is equivalent to requiring T to be metrically equivalent to a symmetric contravariant (covariant) tensor. When V is non-Euclidean a self-adjoint endomorphism is not necessarily diagonalizable. Hence, by an *orthogonal tensor*, we mean a symmetric contravariant tensor whose associated endomorphism is diagonalizable with real eigenvalues. One can check that the eigenspaces of such an endomorphism are necessarily pair-wise orthogonal non-degenerate subspaces. Finally given a subspace $W \leq V$, the restriction of T to W is denoted $T|_W$.

Index Notation We will occasionally use index notation for calculations. Suppose V is a vector space and let T be a type $\binom{a}{b}$ tensor on V. Let $v_1, \ldots, v_n$ be a basis for V with dual basis $f^1, \ldots, f^n$ for the dual space V^*, which satisfy:

$$f^i(v_j) = \delta^i{}_j$$

where $\delta^i{}_j$ is the Kronecker delta. Index notation is defined by:

$$T^{i_1 \cdots i_a}{}_{j_1 \cdots j_b} = T(f^{i_1}, \cdots, f^{i_a}, v_{j_1}, \cdots, v_{j_b})$$

We will make use of the Einstein summation convention, which is illustrated with the following example:

$$S^i T_i = \sum_{j=1}^{n} S^j T_j$$

Furthermore, if the abbreviation (n.s.) appears beside an equation, it means "no sum". The *symmetrization* of a type $\binom{a}{0}$ tensor T is defined as follows:

$$T^{(i_1,\ldots,i_a)} = \frac{1}{a!} \sum_{\sigma \in S_a} T^{i_{\sigma(1)},\ldots,i_{\sigma(a)}}$$

where the sum is over all elements of the symmetric group S_a on a elements. In this notation, the *symmetric product* takes the form:

$$(T \odot S)^{i_1,\ldots,i_a,j_1,\ldots,j_b} = T^{(i_1,\ldots,i_a} S^{j_1,\ldots,j_b)}$$

Similarly, the *anti-symmetrization* of a type $\binom{0}{a}$ tensor T is defined as follows:

$$T_{[i_1,\ldots,i_a]} = \frac{1}{a!} \sum_{\sigma \in S_a} (\operatorname{sgn} \sigma) T_{i_{\sigma(1)},\ldots,i_{\sigma(a)}}$$

where $\operatorname{sgn} \sigma = \pm 1$ denotes the sign of the permutation σ. In this notation, the *wedge product* takes the form:

$$(T \wedge S)_{i_1,\ldots,i_a,j_1,\ldots,j_b} = \frac{(a+b)!}{a!b!} T_{[i_1,\ldots,i_a} S_{j_1,\ldots,j_b]}$$

1.4.2 pseudo-Riemannian manifolds

All differentiable structures are assumed to be smooth (class C^∞). Let M be a *pseudo-Riemannian manifold* of dimension n equipped with covariant metric g. By a *Riemannian manifold*, we mean a pseudo-Riemannian manifold with a positive-definite metric. Unless specified otherwise, it is assumed that M is connected and $n \geq 2$. The contravariant metric is usually denoted by G and $\langle \cdot, \cdot \rangle$ plays the role of the covariant and contravariant metric depending on the arguments. We denote $S^p(M)$ (resp. $A^p(M)$) as the set of symmetric (resp. anti-symmetric) contravariant tensor fields of valence p on M and $S(M) = \bigcup_{p \geq 0} S^p(M)$. Furthermore $\mathcal{F}(M) = S^0(M)$ is the set of functions from M to $\mathbb{R}$ and $\mathfrak{X}(M) = S^1(M)$ denotes the set of vector fields over M. If $f \in \mathcal{F}(M)$ then $\nabla f \in \mathfrak{X}(M)$ denotes the gradient of f, i.e. the vector field metrically equivalent to the exterior derivative $\mathrm{d}f$.

We assume the reader is familiar with the concept of a distribution, foliation, and the (local) Frobenius theorem (see, for example, [Lee12]). A distribution E naturally induces a subspace of $\mathfrak{X}(M)$, denoted $\Gamma(E)$. More precisely, $v \in \Gamma(E)$ if $v \in \mathfrak{X}(M)$ and for every $p \in M$ we have $v|_p \in E_p$.

We will also use the existence and uniqueness theorem for the following class of PDEs:

$$\frac{\partial y_i}{\partial x^j} = A^i_j(x^1, \ldots, x^p, y^1, \ldots, y^q) \quad i = 1, \ldots, q,\ j = 1, \ldots, p$$

It can be deduced from the Frobenius theorem (see [AMR01, Section 7.4B] or [Lee12, Proposition 19.29]) that the above system of PDEs has a complete solution $y = f(x, c)$ where $c = f(0, c)$ are the initial conditions iff the mixed partials commute, i.e. $\dfrac{\partial^2 y_i}{\partial x^k \partial x^j} = \dfrac{\partial^2 y_i}{\partial x^j \partial x^k}$.

All notions from pseudo-Euclidean space generalize point-wise to pseudo-Riemannian manifolds. Definitions with subspaces in pseudo-Euclidean space naturally generalize to distributions in pseudo-Riemannian manifolds. For example, given a distribution E, the orthogonal distribution $E^\perp$ is defined at each point $p \in M$ by $(E^\perp)_p = (E_p)^\perp$. All definitions are only required to hold locally. For example, given a self-adjoint $\binom{1}{1}$-tensor T on M, we say it is an *orthogonal tensor* if it is point-wise diagonalizable on some (non-empty) open subset of M and we tacitly work on this subset. Similarly we say T is not an orthogonal tensor on M if T is not point-wise diagonalizable on a open dense subset of M. Similar definitions apply to other notions such as constancy of functions on M.

Smoothness of Eigenvalues and Eigenvectors Suppose T is a $\binom{1}{1}$-tensor on M. The question arises weather the eigenfunctions of T and the eigenvector (fields) are smooth.

First of all, we should make clear that we always work in an open subset of M where the Jordan form is of a fixed type. For example, if T is point-wise diagonalizable, we assume the multiplicities of the eigenvalues are constants and the eigenvectors corresponding to a given eigenfunction form a smooth distribution. Now, suppose T is a $\binom{1}{1}$-tensor of class C^p such that for some $q \in M$, T_q has simple eigenvalues. Then one can show that there exists a neighborhood of q in which T has simple eigenfunctions of class C^p, and T admits a basis of eigenvector fields of class C^p. The proof is an application of the implicit function theorem (see, for example [Die08, Theorems 10.2.1-10.2.4]). Details can be found in [Kaz98], see also [Lax07].

If we relax the condition that the eigenfunctions of T are simple, then the problem inevitably gets more complex (see [Kaz98] for some examples). For our applications though, we will not need such general results. When stating general results involving the eigenfunctions/eigenvector fields of a $\binom{1}{1}$-tensor, we will always assume the eigenfunctions/eigenvector fields are smooth.

Riemann curvature tensor Our sign convention for the Riemann curvature tensor, R, is the opposite of that in [O'N83]. Hence it is defined by:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad X, Y, Z \in \mathfrak{X}(M)$$

A *space of constant curvature* κ is intrinsically defined by the following condition on the Riemann curvature tensor R [O'N83, Corollary 3.43]:

$$R(X, Y, V, W) = \kappa(\langle V, X\rangle \langle Y, W\rangle - \langle V, Y\rangle \langle X, W\rangle) \quad X, Y, V, W \in \mathfrak{X}(M) \tag{1.4.1}$$

or in index notation $R_{ijkl} = \kappa(g_{ik}g_{jl} - g_{jk}g_{il})$. We will be working with specific models of these spaces which will be introduced when needed.

Chapter 2

Overview of Chapters and Theory

In this chapter we will connect the following chapters by giving an overview of the theory. References are given to the appropriate chapters for more details, proofs and generalizations. As much of the chapters in this thesis attempt to be somewhat comprehensive and contain generalizations, this chapter is crucial for understanding the theory. This chapter motivates many of the generalizations presented in later chapters and some key notions such as concircular tensors and warped products. In this chapter we basically show, fairly abstractly, and with some examples, how one can use concircular tensors to solve the fundamental problems (given in the introduction) for spaces of constant curvature. For illustrative purposes, we will sketch how one obtains the separable coordinate systems for the Calogero-Moser system.

Some proofs are included when they are simple and allow the reader to understand the theory. However, the chapter is recommended to be read lightly at first, and in more detail after reading the subsequent chapters in the first part of the thesis. We emphasize this point for Sections 2.3.2 and 2.4 which are more technical, and so the proofs can be skipped on the first reading. We also note that some earlier review articles written by Benenti may complement our exposition given here, see [Ben04; Ben93].

In this chapter (M, g) is a pseudo-Riemannian manifold and T^*M denotes the cotangent bundle of M. If (q, p) denote the canonical (position-momenta) coordinates on T^*M, then the (natural) *Hamiltonian* H with potential $V \in \mathcal{F}(M)$ is defined by:

$$H(q, p) := \frac{1}{2} \langle p, p \rangle + V(q)$$

The *geodesic Hamiltonian* is obtained by setting $V \equiv 0$ in the above equation. The Hamilton-Jacobi equation is a PDE defined on M in terms of the Hamiltonian. Coordinates (q^i) (for M) are called *separable* if they are orthogonal and the Hamilton-Jacobi equation separates in them. This is all one needs to know about the Hamilton-Jacobi equation to understand the theory which we are about to present.

2.1 The Intrinsic Characterization of Separation

The first cruical result is due to Stäckel in 1893 [Sta93]. Stäckel showed that if the Hamilton-Jacobi equation of a natural Hamiltonian is orthogonally separable then it admits n quadratic first integrals $F_1, \ldots, F_n$ (where $F_1 := H$) each having the following form in canonical coordinates (q^i, p_j) [Sta93] (see [Kal86, P. 9] for English readers):

$$F = \frac{1}{2} K^{ij} p_i p_j + U(q^i) \tag{2.1.1}$$

with:

$$\{F_i, F_j\} = 0, \quad \mathrm{d}F_1 \wedge \ldots \wedge \mathrm{d}F_n \neq 0$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. One can show that the condition $\{F, H\} = 0$ is equivalent to the following equations on M (see Theorem 4.2.2 and Eq. (4.2.7)):

$$\nabla_{(i} K_{jk)} = 0 \tag{2.1.2a}$$

$$\mathrm{d}U = K \mathrm{d}V \tag{2.1.2b}$$

where ∇ is the Levi-Civita connection induced by g. The first of the above equations, Eq. (2.1.2a), shows the important fact that K is a Killing tensor (KT) on (M, g). Using this fact Eisenhart was the first to obtain an intrinsic characterization of separation for geodesic Hamiltonians [Eis34]. In order to present this theorem, we first need a definition. A *characteristic Killing tensor (ChKT)* is a Killing tensor which has simple eigenfunctions and admits coordinates in which it is diagonalized.

Theorem 2.1.1 (Orthogonal Separation of Geodesic Hamiltonians [Eis34]) *The geodesic Hamiltonian is separable in orthogonal coordinates (q^i) iff there exists a ChKT which is diagonalized in these coordinates.* □

Given a ChKT, K, let $\mathcal{E} = (E_1, \ldots, E_n)$ denote the collection of eigenspaces of K. The above theorem shows that any coordinates (q^i) with the property that $\text{span}\{\partial_i\} = E_i$ are separable. Hence we call the collection $\mathcal{E}$ a *separable web*. More generally, any collection $\mathcal{E} = (E_1, \ldots, E_n)$ of pair-wise orthogonal non-degenerate 1-distributions which admit local coordinates (q^i) satisfying $\text{span}\{\partial_i\} = E_i$ is called an *(orthogonal) web*. Since separable webs are uniquely determined by ChKTs, we will often work with them instead of coordinates.

The second equation, Eq. (2.1.2b), is a compatibility condition between the KT K and potential V. The structure of Eq. (2.1.2) hints at the fact that the separation of the geodesic Hamiltonian is a necessary condition for the separation of a natural Hamiltonian. Benenti was the first to obtain the intrinsic characterization of separation for natural Hamiltonians [Ben97].

Theorem 2.1.2 (Orthogonal Separation of Natural Hamiltonians [Ben97]) *A natural Hamiltonian with potential V is separable in orthogonal coordinates (q^i) iff there exists a ChKT K diagonalized in these coordinates which satisfies:*

$$d(K\,dV) = 0 \qquad \square$$

The above equation is called the *dKdV equation* associated with the KT K and potential V.

For geodesic separation, each first integral given by Eq. (2.1.1) has a corresponding KT K. It can be deduced from Stäckel's theorem that the n KTs are point-wise independent on M and span an n-dimensional vector space of KTs which are simultaneously diagonalized in the separable coordinates. This vector space of KTs is called the *Killing-Stackel space (KS-space)* associated with a separable web.

We conclude with the following observations. Firstly, Theorem 2.1.1 implies that the problem of classifying separable coordinates for a geodesic Hamiltonian is equivalent to the problem of classifying ChKTs. Secondly, Theorem 2.1.2 shows that the problem of classifying ChKTs is important for separating natural Hamiltonians as well. Killing tensors are studied in greater detail in Chapter 4. The Hamilton-Jacobi equation, its separation, and the intrinsic characterization of separation is studied in greater detail in Chapter 5.

2.2 Concircular tensors

In the previous section we have given an intrinsic characterization of separation, which allows one to, in principle, obtain all separable coordinates systems defined on a given pseudo-Riemannian manifold. There are several problems one confronts when trying to apply the theory, particularly to spaces of constant curvature. We list some of these problems, assuming $n > 2$.

- It is difficult to obtain an algebraic expression for the general ChKT in a space of constant curvature.
- Given a ChKT, it's hard to obtain the transformation from separable coordinates $(u^i) \to (q^i)$ to the standard coordinates.
- It's also hard to find canonical forms for ChKTs modulo the action of the isometry group.

When $n = 3$, one can manage with these difficulties. Indeed, in [HMS05] (resp. [HM08; HMS09]), building on results from [Eis34], canonical forms for the isometrically inequivalent ChKTs in $\mathbb{E}^3$ (resp. M^3) were given together with the corresponding transformations from separable coordinates. Furthermore, the authors were able to solve all the fundamental problems given in the introduction. However, generalizing their solution to higher dimensions seems intractable.

Kalnins and Miller in [Kal86] were able to devise a procedure to construct the transformation from separable coordinates for all isometrically inequivalent ChKTs on Riemannian spaces of constant curvature, thereby solving the first fundamental problem (1). Waksjo and Wojciechowski in [WW03] used this procedure to solve the last two fundamental problems (2 and 3) for Euclidean and spherical spaces. A careful study of these solutions and works by others (e.g. [Ben05; Cra03]) show that concircular tensors have a fundamental role to play in these solutions.

A *concircular tensor* (CT), $L \in S^2(M)$, is defined by the following equation:

$$\nabla_k L_{ij} = \alpha_{(i} g_{j)k}$$

for some covector α. One can obtain a general solution to the above equation in $\mathbb{E}^n$. First, define the *dilatational vector field* in $\mathbb{E}^n$ in Cartesian coordinates (x^i) by $r := x^i \partial_i$. Then the general solution is given as follows (see Proposition 6.4.4):

$$L = A + 2w \odot r + mr \odot r$$

where A is a symmetric and constant matrix, w is a constant vector and m is a constant scalar. We denote the unit sphere in $\mathbb{E}^n$ by $\mathbb{S}^n$. Then the restriction of the above tensor to $\mathbb{S}^n$ gives the general CT (see Proposition 9.3.2). CTs solve the problems confronted with ChKTs listed above. Indeed, in this thesis, we will show that CTs can be used to solve the fundamental problems in spaces of constant curvature.

We say a CT is an *orthogonal concircular tensor (OCT)* if it is point-wise diagonalizable. An important property of OCTs is that they always admit local coordinates which diagonalize them. More precisely, suppose L is an OCT, then there exist local coordinates (x^i) such that L has the following form (see Propositions 6.3.1 and 6.3.6):

$$L = \sum_{a \in M} \sigma_a \partial_a \otimes \mathrm{d}x^a + \sum_{I \in P} e_I \sum_{i \in I} \partial_i \otimes \mathrm{d}x^i \tag{2.2.1}$$

where $\{1, \ldots, n\} = M \cup (\cup_{I \in P} I)$ is a partition (here P is an index set and each $I \in P$ is a subset of $\{1, \ldots, n\}$), the $\sigma_a(x^a)$ are non-constant and the e_I are constants.

Another important property of CTs is that some special KTs can be constructed using them. Indeed, if L is a CT, it can be shown that the following sequence of tensors are KTs (see Section 6.6):

$$K_0 = G, \qquad K_a = \frac{1}{a}\operatorname{tr}(K_{a-1}L)G - K_{a-1}L \quad 1 < a < n \tag{2.2.2}$$

The KT K_1 is so special, it deserves a name. If L is a CT, then the tensor

$$K = \operatorname{tr}(L)G - L$$

is a KT, called the *Killing Bertrand-Darboux tensor (KBDT)* associated with L. This KT will be useful for connecting CTs with the general theory of separation given in the previous section. An important observation is that it has the same eigenspaces as L.

We conclude by noting that concircular tensors in general are studied in Chapter 6 while those in spaces of constant curvature are studied in Chapter 9.

2.3 Separation of Geodesic Hamiltonians

2.3.1 Benenti tensors

We say a CT L is a *Benenti tensor* if it has simple eigenfunctions. A key observation made by Benenti is that any Benenti tensor induces a separable web [Ben92a]. Indeed, since the KBDT is a KT with simple eigenfunctions and can be diagonalized in a coordinate system (see Eq. (2.2.1)), it's a ChKT, hence the result follows by Theorem 2.1.1. Furthermore it can be shown that the KTs given by Eq. (2.2.2) form a basis for the KS-space associated with this separable web.

An important class of Benenti tensors are the *irreducible concircular tensors (ICTs)*. A CT L is called irreducible if it's a Benenti tensor and its eigenfunctions are functionally independent. By Eq. (2.2.1) any Benenti tensor with non-constant eigenfunctions is irreducible. This class of CTs are of interest, because in this case, by Eq. (2.2.1) the eigenfunctions can be used as separable coordinates! We will see shortly that ICTs can be used as building blocks to construct more general classes of separable coordinates. The following is the prototypical example of an ICT:

Example 2.3.1 (Elliptic coordinates in $\mathbb{E}^2$)
Let $M = \mathbb{E}^2$ and fix an orthonormal basis $\{d, e\}$ for this Euclidean space. Let (x, y) be Cartesian coordinates for $\mathbb{E}^2$ so that $d = \partial_x$ and $e = \partial_y$. Then consider the following CT:

$$L = \lambda_1 d \odot d + \lambda_2 e \odot e + r \odot r$$

WLOG we can assume $\lambda_1 < \lambda_2$. We will show how to obtain the transformation from separable to Cartesian coordinates after showing that L is a Benenti tensor. The characteristic polynomial of L is given as follows:

$$p(z) = \det(zI - L) = (z - \lambda_1)(z - \lambda_2) - x^2(z - \lambda_2) - y^2(z - \lambda_1)$$

From the above equation, we note the following:

$$p(\lambda_1) = x^2(\lambda_2 - \lambda_1) \qquad\qquad p(\lambda_2) = y^2(\lambda_1 - \lambda_2) \tag{2.3.1}$$

Now, assume that $x, y \neq 0$. Then we observe that $p(\lambda_1) > 0$, $p(\lambda_2) < 0$ and $\lim_{z\to\infty} p(z) = \infty$. Hence by the intermediate value theorem, at each point, $p(z)$ has two distinct roots $u^1 < u^2$ satisfying:

$$\lambda_1 < u^1 < \lambda_2 < u^2$$

Thus L is a Benenti tensor. Since $\mathrm{d}p \neq 0$, it follows that L cannot have constant eigenfunctions, thus from the preceding discussion we see that L is an ICT. Now observe that we can write $p(z) = (z - u^1)(z - u^2)$. Then Eq. (2.3.1) can be used to obtain the transformation from the separable coordinates (u^1, u^2) to Cartesian coordinates (x, y):

$$x^2 = \frac{(\lambda_1 - u^1)(\lambda_1 - u^2)}{(\lambda_2 - \lambda_1)} \qquad\qquad y^2 = \frac{(\lambda_2 - u^1)(\lambda_2 - u^2)}{(\lambda_1 - \lambda_2)} \qquad \square$$

The above example will be generalized to higher dimensions and signatures later on, see Example 9.4.11. Proceeding as in the above example and using additional results from Chapter 9, one can classify all (isometrically inequivalent) separable coordinates associated with Benenti tensors in $\mathbb{E}^2$, including polar and Cartesian coordinates. The results of this classification are summarized in Table 9.1.

We conclude by introducing a diagram for Benenti tensors (see Fig. 2.1) which will be used later on. It represents the structure of the separable web associated with the Benenti tensor, which is the simplest possible.

Figure 2.1: Concircular tensor with simple eigenspaces $E_1, \ldots, E_n$

$$\boxed{E_1} \quad \cdots \quad \boxed{E_n}$$

2.3.2 Concircular tensors with Multidimensional Eigenspaces and KEM webs

More generally, any orthogonal concircular tensor can (possibly) be used to construct separable webs, as we will see in this section.

Suppose L is a non-trivial[1] concircular tensor with a single multidimensional eigenspace D and denote by $D^\perp$ the distribution orthogonal to D. Then one can show that (see Proposition 6.3.6):

- There is a local product manifold $B \times F$ of (pseudo-)Riemannian manifolds (B, g_B) and (F, g_F) such that:
  $\{p\} \times F$ is an integral manifold of D for any $p \in B$ and
  $B \times \{q\}$ is an integral manifold of $D^\perp$ for any $q \in F$.
- $B \times F$ equipped with the metric $\pi_B^* g_B + \rho^2 \pi_F^* g_F$ for a specific function $\rho : B \to \mathbb{R}^+$ is locally isometric to (M, g).

Such a product manifold is called a *warped product* and is denoted $B \times_\rho F$. The manifold B is called the *geodesic factor* and F is called the *spherical factor* of the warped product. We also say that the warped product $B \times_\rho F$ is *adapted* to the splitting $(D^\perp, D)$, which is often called a *warped product net (WP-net)*. When a distribution D admits an adapted warped product as above, it is called a *Killing distribution*. See Chapter 3 for more details on these matters.

We note here that warped products are rigid. For example, in Euclidean space, it can be shown that if an open connected subset U is isometric to a warped product with a single spherical factor, then the warped product must have one of the following forms:

1. $\mathbb{E}^m \times_\rho \mathbb{S}^r$

2. $\mathbb{E}^m \times_1 \mathbb{E}^r$

The warped products in more general spaces of constant curvature are systematically obtained in Appendix D, see Section 8.4 for a summary sufficient for the purposes of this thesis.

Now, if we enumerate the one dimensional eigenspaces of L by $E_1, \ldots, E_m$ and denote the multidimensional eigenspace of L by D as above, then Fig. 2.2 gives a diagram for L. In this figure, the block containing the eigenspace D represents a "degeneracy" which needs to be removed to uniquely specify a separable web. We now describe how to do this.

Figure 2.2: Concircular tensor with eigenspaces $E_1, \ldots, E_m, D$

$$\boxed{E_1} \quad \cdots \quad \boxed{E_m} \quad \boxed{D}$$

A remarkable property of the warped product decomposition is the following. Let $\tilde{K}$ be a ChKT on F, this can be canonically lifted to a tensor, $\tilde{K} \in S^2(B \times_\rho F)$, which is

[1] A CT is called non-trivial if its not a multiple of the metric.

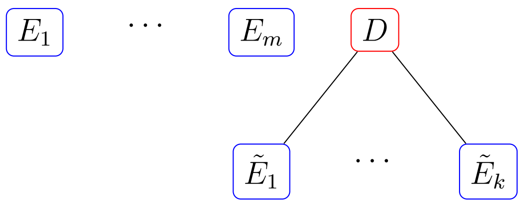
in fact a KT on $B \times_\rho F$! Hence if K' is the KBDT associated with L, then locally we can assume that $K' + \tilde{K}$ is a ChKT on $B \times_\rho F$. Indeed, one can show that L induces a Benenti tensor, $\tilde{L}$, on B by restriction. Let (x^i) be any coordinates on B which diagonalize $\tilde{L}$. Note that we observed in the previous section that these coordinates are separable on B. Suppose (y^j) are coordinates on F which diagonalize $\tilde{K}$, hence are separable (see Theorem 2.1.1). Then since the product coordinates (x^i, y^j) diagonalize $K' + \tilde{K}$ (see Eq. (2.3.2)), Theorem 2.1.1 implies that $K' + \tilde{K}$ is a ChKT[2] and that these coordinates are separable. Note that in these coordinates $K' + \tilde{K}$ have the following form:

$$K' + \tilde{K} = \sum_i (\mathrm{tr}(L) - \lambda_i)\partial_i \otimes \mathrm{d}x^i + \sum_j (\mathrm{tr}(L) - c + \tilde{\lambda}_j)\partial_j \otimes \mathrm{d}y^j \tag{2.3.2}$$

where λ_i are the eigenfunctions of $\tilde{L}$, c is the constant eigenfunction of L associated with D and $\tilde{\lambda}_j$ are the eigenfunctions of $\tilde{K}$. In conclusion, we have shown how to construct separable coordinates (x^i, y^j) using the CT L and ChKT $\tilde{K}$.

Now take $\tilde{K}$ to be the KBDT associated with a Benenti tensor on F which has eigenspaces $\tilde{E}_1, \ldots, \tilde{E}_k$. Then Fig. 2.3 is a diagram for the above construction applied to $\tilde{K}$, which represents the tree-like structure of the constructed separable web. It should be interpreted as a tree diagram, where the one dimensional eigenspaces are the leaves. We illustrate this construction with two simple examples in $\mathbb{E}^3$, both of which are depicted by Fig. 2.3 with $m = 1$ and $k = 2$.

Figure 2.3: KEM web I



Example 2.3.2 (Cylindrical coordinates in $\mathbb{E}^3$)
Fix a unit vector $d \in \mathbb{E}^3$ and consider the following CT:

$$L = d \odot d$$

The eigenspaces of L are $\mathrm{span}\{d\}$ and $d^\perp$. Identify $\mathbb{E} = \mathrm{span}\{d\}$ and $\mathbb{E}^2 = d^\perp$, then the warped product $\psi : \mathbb{E} \times_1 \mathbb{E}^2 \to \mathbb{E}^3$ given by $(q, p) \to q + p$ is adapted to the eigenspaces of L. We can construct separable coordinates in $\mathbb{E}^3$ by parameterizing $\mathbb{E}^2$ with any of the separable coordinates from Table 9.1. For example, let e, f be an orthonormal basis for $d^\perp$, let $q = x\, d$ and $p = \rho \cos\theta\, e + \rho \sin\theta\, f$, then we obtain cylindrical coordinates:

[2] The eigenfunctions may not exactly be simple, but one can modify $\tilde{K}$ so that they are locally simple.

$$\psi(q,p) = x\,d + \rho\cos\theta\,e + \rho\sin\theta\,f \qquad \square$$

The following is a more interesting example of this construction.

Example 2.3.3 (Spherical coordinates in $\mathbb{E}^3$)
Consider the following CT in $\mathbb{E}^3$:

$$L = r \odot r$$

The eigenspaces of L are span$\{r\}$ and $r^\perp$. Fix a unit vector $a \in \mathbb{E}^3$, identify $\mathbb{E} = \mathbb{R}^+ a$, let $\mathbb{S}^2$ be the unit sphere in $\mathbb{E}^3$ and $\rho_1 := \langle q, a\rangle$ for $q \in \mathbb{E}$. Then the warped product $\psi : \mathbb{E} \times_{\rho_1} \mathbb{S}^2 \to \mathbb{E}^3$ given by $(q,p) \to \rho_1 p$ is adapted to the eigenspaces of L. We can construct separable coordinates in $\mathbb{E}^3$ by parameterizing $\mathbb{S}^2$ with any of the separable coordinates defined in it.

For example, one can take spherical coordinates on $\mathbb{S}^2$. Indeed, fix a unit vector $d \in \mathbb{E}^3$. Then one can show that the restriction of $d \odot d$ to $\mathbb{S}^2$ is a Benenti tensor diagonalized in spherical coordinates (see Example 9.6.3), which are given as follows:

$$p = \cos(\phi)d + \sin(\phi)(\cos(\theta)e + \sin(\theta)f)$$

where e, f is any orthonormal basis for $d^\perp$. Hence the above coordinates are separable in $\mathbb{S}^2$. If we let $q = \rho a$ where $\rho > 0$ and take p as above, then we obtain spherical coordinates in $\mathbb{E}^3$:

$$\psi(q,p) = \rho(\cos(\phi)d + \sin(\phi)(\cos(\theta)e + \sin(\theta)f)) \qquad \square$$

More examples can be found in Section 9.6.2.

This construction procedure can be generalized in two ways. Firstly, we can recursively apply this procedure, by treating $B \times_\rho F$ as the spherical factor of a larger warped product and use $K + \tilde{K}$ in place of $\tilde{K}$. Figure 2.4 depicts such a construction where the CT L has eigenspaces E_1' and D'. Again, this figure depicts the tree-like structure of the KEM web where the leaves are the one dimensional eigenspaces of the CTs that make it up.

Secondly, we can allow L to have multiple distinct multidimensional eigenspaces. These procedures can also be combined to create even more complex webs, as the following example will show. Figure 2.5 depicts the natural generalization of the above construction procedure to CTs with multiple multidimensional eigenspaces. In this case, the CT L has only multidimensional eigenspaces $D_1, \ldots, D_r$.

We emphasize here that in each case, the constructed web is separable. Any coordinates constructed using this procedure are called *Kalnins-Eisenhart-Miller (KEM) coordinates*

Figure 2.4: KEM web II

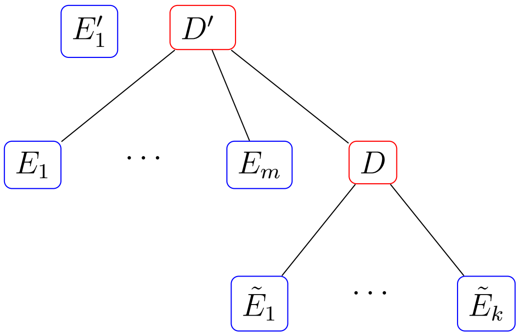
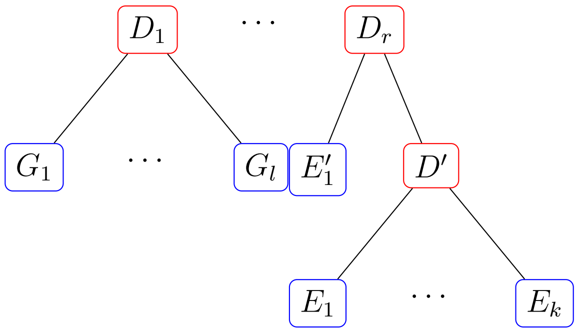


Figure 2.5: KEM Web III



and the associated webs are called KEM webs. We will show that KEM webs are always separable.

We've shown how CTs can be used to construct a special class of separable webs called KEM webs. A significant advantage of KEM webs is that we can reduce the problem of classifying isometrically inequivalent KEM webs to the similar problem for CTs. We will see that the problem of classifying isometrically inequivalent CTs in spaces of constant curvature can be reduced to problems in linear algebra (see Chapter 9).

In conclusion, we mention how some of the ideas presented here are generalized. The observation that CTs (which are in fact CKTs) induce a warped product decomposition of the (pseudo-)Riemannian manifold, motivates the more systematic study of CKTs in Section 4.4. This culminates in Corollary 4.4.8 and Corollary 4.4.11. In Section 4.5 we will be able to prove that the KT $\tilde{K}$ in the above construction is a KT on the warped product. In fact, this observation will be generalized to describe KTs which are "decomposable" in a warped product. In Section 6.5 we will prove the facts on KEM webs we presented here more rigorously. In Section 6.6 we will apply the theory developed in Section 5.3.1 on KS-spaces in warped products to show how one can obtain the KS-space associated with a KEM web.

2.3.3 Necessity of KEM webs in spaces of constant curvature

In the previous section we have shown how to construct a class of separable webs called KEM webs. These webs were originally discovered by Kalnins and Miller when classifying the separable webs in Riemannian spaces of constant curvature [Kal86]. Generalizing their results, one can prove the following.

Theorem 2.3.4 (Separable Webs in Spaces of Constant Curvature)
In a space of constant curvature, every separable web is a KEM web. □

This theorem is proven in Chapter 7. It involves a long calculation in which we solve the Levi-Civita equations together with the equations satisfied by the Riemann curvature tensor in a space of constant curvature (see Eq. (1.4.1)).

The above theorem allows us to tractably solve problem (1) in spaces of constant curvature. Motivated by the above theorem, in Chapter 9 we study (orthogonal) concircular tensors in spaces of constant curvature. In that chapter, we obtain the required information to reduce problem (1) to simple problems in linear algebra.

2.4 Separation of Natural Hamiltonians

In this section we will sketch how concircular tensors can be used to separate natural Hamiltonians. We will use Theorem 2.1.2 and our knowledge of the structure of KEM webs to develop a recursive algorithm to separate natural Hamiltonians in KEM webs.

Fix some $V \in \mathcal{F}(M)$. Let L be the general concircular tensor on M and let $K := \operatorname{tr}(L)G - L$ be the KBDT associated with L. The Killing-Bertrand-Darboux (KBD) equation on M is defined as follows:

$$\mathrm{d}(K\mathrm{d}V) = 0$$

It can be shown that this equation defines a linear system of equations with at most $\frac{1}{2}(n+1)(n+2)$ unknowns, where the maximum is achieved iff the space has constant curvature.

Let L be a particular solution of the KBD equation which is point-wise diagonalizable with k distinct eigenfunctions. We analyze the following cases.

Case 1 (k = 1, i.e. all the eigenfunctions coincide)
$L = cG$ for some $c \in \mathbb{R}$. This is the trivial solution which gives no information.

Case 2 (the eigenfunctions are simple)
L has simple eigenfunctions, hence it's a Benenti tensor. Then V separates in any coordinates which diagonalize L by Theorem 2.1.2.

Case 3 (at least one eigenfunction is not simple)

Assume for convenience, that L has a single multidimensional eigenspace D. If $E_1, \dots, E_m$ denote the one dimensional eigenspaces of L, then so far we know that V is "compatible" with the partial separable web in Fig. 2.6.

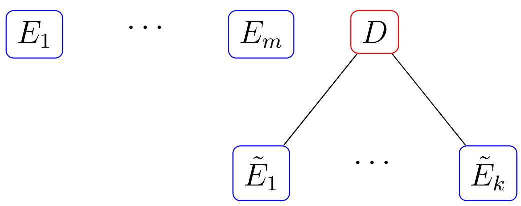
Figure 2.6: Concircular tensor with eigenspaces $E_1, \dots, E_m, D$

Now the goal is to fill in the degeneracy coming from D. This is done as follows. Let $B \times_\rho F$ be a local warped product adapted to $(D^\perp, D)$. Let $\tau : F \to B \times F$ be an embedding. Assume the natural Hamiltonian on F associated with potential $V \circ \tau$ is separable in some coordinates (y^j). Let (x^i) be separable coordinates associated with the induced Benenti tensor on B. Then one can show that the natural Hamiltonian associated with V (on $B \times_\rho F$) is separable in the product coordinates (x^i, y^j).

Indeed, this can be seen as follows. Let $\tilde{K}$ be a ChKT on F diagonalized in (y^j), and K' be the KBDT associated with L. In the discussion preceding Eq. (2.3.2), it was shown that we can assume $K := K' + \tilde{K}$ is locally a ChKT on $B \times_\rho F$ diagonalized in (x^i, y^j). Given the assumptions, one can show that V satisfies the dKdV equation with K on $B \times_\rho F$, hence by Theorem 2.1.2 it's separable in the coordinates (x^i, y^j).

In the third case, in order to obtain the separable coordinates (y^j), the idea is to apply the same procedure again on F with the potential $V \circ \tau \in \mathcal{F}(F)$. So one has to solve the KBD equation on F with the potential $V \circ \tau$ and then go through each case. This gives us a recursive algorithm for separating natural Hamiltonians, which is called the Benenti-Eisenhart-Kalnins-Miller (BEKM) separation algorithm. Figure 2.7 gives a possible KEM web that can be constructed, assuming the solution of the KBD equation on F is a Benenti tensor with eigenspaces $\tilde{E}_1, \dots, \tilde{E}_k$.

Figure 2.7: Possible KEM web that can be constructed



In principle, one can construct any KEM web using the BEKM separation algorithm. For example, just take $V = 0$. We now briefly illustrate the execution of this algorithm with the following example.

Example 2.4.1 (Calogero-Moser system)
The Calogero-Moser system is a natural Hamiltonian system with configuration manifold $\mathbb{E}^3$ given by the following potential in Cartesian coordinates (q_1, q_2, q_3):

$$V = (q_1 - q_2)^{-2} + (q_2 - q_3)^{-2} + (q_1 - q_3)^{-2}$$

First note that the constant vector $d = \frac{1}{\sqrt{3}}(\partial_1 + \partial_2 + \partial_3)$ is a symmetry of V, i.e. $\mathcal{L}_d V = 0$. One can prove that the general solution of the KBD equation associated with V is[3]:

$$L = c\, d \odot d + 2w\, d \odot r + m\, r \odot r$$

where $c, w, m \in \mathbb{R}$. We note that given a CT L, then for any $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$, the CT $aL + bG$ is a CT which is equivalent to L. After classifying the above CTs modulo this equivalence and isometric equivalence, we can obtain canonical forms. Before we present these, fix an orthonormal basis e, f for $d^\perp$. We have the following canonical forms.

Cartesian: $L = d \odot d$

From Example 2.3.2 we know that a warped product manifold adapted to L has the form $\mathbb{E} \times \mathbb{E}^2$. Let (q_1', q_2', q_3') be Cartesian coordinates adapted to this product manifold, then one can show that V takes the form:

$$V = \frac{9(q_3'^2 + q_2'^2)^2}{2q_2'^2(3q_3'^2 - q_2'^2)^2}$$

In this case V naturally restricts to a potential on $\mathbb{E}^2$ with coordinates (q_2', q_3'). In $\mathbb{E}^2$ one can apply the BEKM separation algorithm to find that the only solution of the KBD equation (up to constant multiplies) is $L = r \odot r$. One can show that polar coordinates diagonalize this CT. Hence V is separable in cylindrical coordinates:

$$x\, d + \rho \cos\theta\, e + \rho \sin\theta\, f$$

Spherical: $L = r \odot r$

From Example 2.3.3 we know that a warped product manifold adapted to L has the form $\mathbb{E} \times_\rho \mathbb{S}^2$. One can show that the restriction of V to $\mathbb{S}^2$ satisfies the KBD equation associated with the CT obtained by restricting $d \odot d$ to $\mathbb{S}^2$. Hence from Example 2.3.3, V is separable in spherical coordinates:

$$\rho(\cos(\phi)d + \sin(\phi)(\cos(\theta)e + \sin(\theta)f))$$

[3]We ignore constant multiples of the metric.

Elliptic: $L = cd \odot d + r \odot r$, $c \neq 0$

In this case L is a Benenti tensor. If we let $a := \sqrt{|c|}$, then if $c > 0$, V is separable in prolate spheroidal coordinates:

$$a \cos\phi \cosh\eta \, d + a \sin\phi \sinh\eta \, (\cos\theta \, e + \sin\theta \, f)$$

If $c < 0$, V is separable in oblate spheroidal coordinates:

$$a \sin\phi \sinh\eta \, d + a \cos\phi \cosh\eta \, (\cos\theta \, e + \sin\theta \, f)$$

Parabolic: $L = 2d \odot r$

In this case L is a Benenti tensor, and so V is separable in rotationally symmetric parabolic coordinates:

$$\frac{1}{2}(\mu^2 - \nu^2) \, d + \mu\nu(\cos\theta \, e + \sin\theta \, f) \qquad \square$$

The above example will be done in much greater detail and for a more general potential in Section 10.2. The BEKM separation algorithm is presented in more detail and with proofs in Section 6.7. It motivates the study of separation of natural Hamiltonians in warped products in Section 5.4.1.

Completeness of the BEKM separation algorithm In spaces of constant curvature, the BEKM separation algorithm gives a complete test for orthogonal separation. This is a consequence of Theorem 2.3.4. We also note that the separable coordinates can be explicitly constructed by following through the algorithm, this is shown by way of example in Section 10.2. Hence the BEKM separation algorithm solves problems (2) and (3) in spaces of constant curvature.

Spaces of constant curvature In order to apply the BEKM separation algorithm (i.e. reduce it to problems in linear algebra) in spaces of constant curvature, CTs in these spaces are studied throughly in Chapter 9. In order to do this in spaces with arbitrary signature, one needs to solve some non-standard problems in linear algebra. The prerequisite theory is covered in Appendix C, and summarized in Section 8.2. The results obtained in Chapter 9 are used in Section 10.3 to concretely carry out the BEKM separation algorithm, and in Section 10.2 to study the separability properties of a well known example, the Calogero-Moser system.

Separable potentials We also mention here that in Section 10.1, we give special potentials that can proven to be separable in KEM webs.

2.5 Conclusion

We have given an overview of how the fundamental problems are solved in this thesis, and how this solution is broken down in the various different chapters. The first part of this thesis will present the theory more rigorously, with greater detail, and present some generalizations of parts of it. The second part of this thesis will (mainly) apply the theory to spaces of constant curvature.

Part I

General Theory

Chapter 3

Warped Products

In this chapter we will present the theory of warped products, most of which will be used in later chapters. Warped products will not be defined until Section 3.5. We will first develop the necessary theory to characterize these products in terms of the distributions they induce. This characterization is better suited for applications later on. In fact, motivated by applications, we will study a more general product structure known as a twisted product. In this chapter we provide a fairly comprehensive introduction to the warped product and so it is recommended that it be read lightly at first and in more detail when the need arises.

3.1 pseudo-Riemannian Submanifolds and Foliations

In this section we will summarize the theory of pseudo-Riemannian submanifolds and foliations that will be useful to us. We can conveniently treat this as a special case of the theory of pseudo-Riemannian distributions, so we will present this first. For more details on pseudo-Riemannian submanifolds see (for example) [O'N83; Lee97]. Similarly for pseudo-Riemannian foliations see [Rov98; Ton88].

3.1.1 Brief outline of The Theory of Pseudo-Riemannian Distributions

The following brief exposition of the theory of pseudo-Riemannian distributions is a combination of that given in [MRS99] and [CFS06]. Suppose E is an m-dimensional non-degenerate distribution defined on a pseudo-Riemannian manifold $\bar{M}$. Then we use the orthogonal splitting $T\bar{M} = E \oplus E^{\perp}$, $V = V^E + V^{E^{\perp}}$, to define a tensor $s^E : T\bar{M} \times E \to E^{\perp}$ and a linear connection ∇^E for E by:

$$\nabla_X Y = \nabla^E_X Y + s^E(X, Y)$$

for all $X \in \mathfrak{X}(\bar{M})$ and $Y \in \Gamma(E)$. s^E is called the *generalized second fundamental form* of E and the above equation is referred to as the *Gauss equation*. One can also check that ∇^E is metric compatible, i.e. $X\langle Y, Z\rangle = \left\langle \nabla^E_X Y, Z\right\rangle + \left\langle Y, \nabla^E_X Z\right\rangle$ for all $X \in \mathfrak{X}(\bar{M})$ and $Y, Z \in \Gamma(E)$.

For the remainder of the discussion we set $s^E := s^E|_{(E\times E)}$. For $X, Y \in \Gamma(E)$, we can further decompose $s^E(X,Y)$ into its anti-symmetric and symmetric parts

$$\begin{aligned} s^E(X,Y) &= (\nabla_X Y)^{E^\perp} = \frac{1}{2}(\nabla_X Y + \nabla_Y X)^{E^\perp} + \frac{1}{2}(\nabla_X Y - \nabla_Y X)^{E^\perp} \\ &= h^E(X,Y) + A^E(X,Y) \\ A^E(X,Y) &:= \frac{1}{2}(\nabla_X Y - \nabla_Y X)^{E^\perp} \\ h^E(X,Y) &:= \frac{1}{2}(\nabla_X Y + \nabla_Y X)^{E^\perp} \end{aligned}$$

Since ∇ is torsion-free, $A^E(X,Y) = \frac{1}{2}([X,Y])^{E^\perp}$, hence E is integrable iff $A^E \equiv 0$. h^E is called the *second fundamental form* of E. The second fundamental form can be decomposed in terms of its trace to get a further classification of E as follows:

$$\begin{aligned} h^E(X,Y) &= \langle X, Y\rangle H_E + h^E_T(X,Y) \\ H_E &= \frac{1}{m}\operatorname{tr}(h^E) \end{aligned}$$

where h^E_T is trace-less. H_E is called the *mean curvature normal* of E. E is called *minimal*, *umbilical* or *geodesic*[1] if $s^E(X,Y) = h^E_T(X,Y)$, $s^E(X,Y) = \langle X, Y\rangle H_E$ or $s^E(X,Y) = 0$ respectively for all $X, Y \in \Gamma(E)$. We add the qualification "almost" to the three definitions above by replacing s^E with h^E; this just drops the requirement that $A^E \equiv 0$. For example E is *almost umbilical* iff $h^E_T = 0$. We remark that when E is one dimensional $h^E_T = 0$ trivially, hence all one dimensional non-degenerate foliations and all one dimensional pseudo-Riemannian submanifolds are trivially umbilical. If E is umbilical and $\nabla^{E^\perp}_X H_E = 0$ for all $X \in \Gamma(E)$ then E is called *spherical*. Finally if E is spherical and $E^\perp$ is geodesic then E is called *Killing*.

We also note here that s^E and $s^{E^\perp}$ are not independent of each other:

Proposition 3.1.1
For $X, Y \in \Gamma(E)$ and $Z \in \Gamma(E^\perp)$, the following holds:

$$\left\langle s^E(X,Y), Z\right\rangle = -\left\langle Y, s^{E^\perp}(X,Z)\right\rangle \qquad \square$$

[1]Note that some authors use the name auto-parallel instead [MRS99].

PROOF

$$\begin{aligned} 0 &= \nabla_X \langle Y, Z\rangle \\ &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ &= \left\langle s^E(X,Y), Z\right\rangle + \left\langle Y, s^{E^\perp}(X,Z)\right\rangle \end{aligned}$$

■

The following proposition gives an important geometric characterization of the second fundamental form. It is the key lemma used to connect twisted products with their differential counterparts as we will see in Section 3.5. It is taken from [Rov98, Proposition 2.7] (cf. [Ton88, Theorem 5.23], [Zeg11, Lemma 2.3]).

Proposition 3.1.2 (Geometric Characterization of The Second Fundamental Form) *Let E be a non-degenerate distribution. Denote the covariant metric by g and by g_E the restriction of g to E. Suppose $U, V \in \Gamma(E)$ and $Z \in \Gamma(E^\perp)$, then the second fundamental h of E is characterized by the following equation:*

$$(\mathcal{L}_Z g_E)(U,V) = -2g(Z, h(U,V))$$

E is almost geodesic (almost umbilical) iff g_E is invariant (resp. conformal invariant) under flows of vector fields orthogonal to E, i.e. for $V \in \Gamma(E^\perp)$, $\mathcal{L}_V g_E = 0$ (resp, $\mathcal{L}_V g_E = -2g(H,V)g_E$ where H is the mean curvature normal of E). □

PROOF By definition, we have the following:

$$\begin{aligned} (\mathcal{L}_Z g_E)(U,V) &= Zg_E(U,V) - g_E([Z,U],V) - g_E(U,[Z,V]) \\ &= Zg(U,V) - g([Z,U],V) - g(U,[Z,V]) \end{aligned}$$

Now since $[Z,U] = \nabla_Z U - \nabla_U Z$ for a torsion-free connection and with a similar equation holding for $[Z,V]$, the above equation becomes

$$\begin{aligned} (\mathcal{L}_Z g_E)(U,V) &= Zg(U,V) - g([Z,U],V) - g(U,[Z,V]) \\ &= g(\nabla_U Z, V) + g(U, \nabla_V Z) \end{aligned}$$

Since $0 = Ug(Z,V) = g(\nabla_U Z, V) + g(Z, \nabla_U V)$, with a similar equation holding for $Vg(U,Z)$, the above equation becomes:

$$\begin{aligned}(\mathcal{L}_Z g_E)(U,V) &= g(\nabla_U Z, V) + g(U, \nabla_V Z)\\ &= -g(Z, \nabla_U V) - g(Z, \nabla_V U)\\ &= -2g(Z, h(U,V))\end{aligned}$$

The remaining assertions follow from the definitions of almost geodesic and almost umbilical distributions. ∎

3.1.2 Specialization to pseudo-Riemannian submanifolds

Suppose $\phi : M \to \bar{M}$ is a local embedding of (a pseudo-Riemannian submanifold) M^m inside $\bar{M}^n$. Then for any point $p \in M$, it is known that there exist local coordinates (x^i) on $\bar{M}$, such that the subset

$$\{(x^1, \dots, x^m, x^{m+1}, \dots, x^n) : x^{m+1} = c_{m+1}, \dots, x^n = c_n\}$$

for some $c_{m+1}, \dots, c_n \in \mathbb{R}$ can be identified with $\phi(U)$ where U is an open subset with $p \in U \subseteq M$. These coordinates induce a local foliation L in a neighborhood of p, with M being a leaf given by the above equation. We will refer to such a foliation as a (local) foliation of $\bar{M}$ associated with M. Now suppose L is an arbitrary foliation of $\bar{M}$ associated with M, and let E be the induced distribution. Locally we can assume L is a foliation by pseudo-Riemannian submanifolds of $\bar{M}$, hence E is non-degenerate and the discussion in the previous section applies to it. Since E is integrable, it follows that for any $X, Y \in \Gamma(E)$, that $[X, Y] \in \Gamma(E)$. Throughout this discussion, for any $X \in \Gamma(E)$, we let $\tilde{X} \in \mathfrak{X}(M)$ denote the unique vector field such that for any $p \in M$, we have $X_{\phi(p)} = \phi_* \tilde{X}_p$. Then for any $X, Y \in \Gamma(E)$ we see that

$$[X, Y]|_{\phi(p)} = \phi_*[\tilde{X}, \tilde{Y}]|_p$$

Thus $[X, Y]|_{\phi(p)}$ depends only on $[\tilde{X}, \tilde{Y}]|_p$ in M.

Now denote by ∇ (resp. $\bar{\nabla}$) the Levi-Civita connection on M (resp. $\bar{M}$). By the uniqueness properties of the Levi-Civita connection on M, it follows that for any $X, Y \in \Gamma(E)$ we have for any $p \in M$ that

$$(\bar{\nabla}^E_X Y)|_{\phi(p)} = \phi_*(\nabla_{\tilde{X}} \tilde{Y})|_p$$

Thus $(\bar{\nabla}^E_X Y)|_{\phi(p)}$ depends only on $(\nabla_{\tilde{X}} \tilde{Y})|_p$ in M. By also using the Gauss equation, we observe that for any $p \in M$, that $(\bar{\nabla}_X Y)|_{\phi(p)}$ depends only on $\tilde{X}$ and $\tilde{Y}$.

In consequence of these observations, it follows that the theory presented for pseudo-

Riemannian distributions induces a similar one for pseudo-Riemannian manifolds. We now connect this with the standard notations [Che11]; in effect this removes the appearance of the extraneous distribution, E.

In this case $s^E \equiv h^E$ and $h := (h^E)|_M$, then the Gauss equation becomes:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$. We denote the set of normal vector fields over M, i.e. the restriction of $\Gamma(E^\perp)$ to M by $\mathfrak{X}(M)^\perp$. The Gauss equation for $E^\perp$ is usually called the *Weingarten equation* and is only defined for $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(M)^\perp$. This is because in this case, $\bar{\nabla}_X Y$ depends only on the values that X and Y take on M[2]. Thus we can let $A_Y(X) := -s^{E^\perp}(X, Y)$, $\nabla^\perp_X Y := \bar{\nabla}^{E^\perp}_X Y$ and the Gauss equation (for $E^\perp$) becomes:

$$\bar{\nabla}_X Y = \nabla^\perp_X Y - A_Y(X)$$

for all $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(M)^\perp$. Note that the properties of $\bar{\nabla}^{E^\perp}$ imply that $\nabla^\perp$ is a connection[3] on $\mathfrak{X}(M)^\perp$. In this notation, the relationship between s^E and $s^{E^\perp}$ given in Proposition 3.1.1 becomes:

$$\langle h(X, Y\rangle, Z) = \langle A_Z(X), Y\rangle \tag{3.1.1}$$

for all $X, Y \in \mathfrak{X}(M)$ and $Z \in \mathfrak{X}(M)^\perp$.

Finally, we note that the definitions of minimal, umbilical, or geodesic foliations induces corresponding definitions for submanifolds. For example, a submanifold is geodesic if its second fundamental form vanishes identically.

In conclusion, we should mention that even though we have given a concise presentation of the theory, it's not useful for practical calculations. For these, one will have to evaluate these quantities in terms of curves on M. See for example, Proposition 4.8 in [O'N83].

3.2 Circles and Spheres*

In this section we will briefly overview the theory of circles and spherical submanifolds (spheres) of pseudo-Riemannian manifolds. Circles are covariantly defined using the Frenet formula, but the definition of a sphere requires more work [Nom73]. This material is optional, although it gives an application of the general theory presented in the previous section, a geometric interpretation of spherical submanifolds, and gives some background for the results on the intrinsic properties of warped products to come. We also present

[2]This is because for any $p \in \bar{M}$, $(\bar{\nabla}_X Y)|_p$ depends only on the values of Y along any curve tangent to X_p. See Lemma 4.8 in [Lee97] and the following exercise, or Proposition 3.18 (3) in [O'N83].

[3]More precisely it satisfies the properties in definition 3.9 in [O'N83] and is metric compatible.

this theory here because it's not covered in standard references, in contrast with the corresponding theory for geodesic submanifolds (see [O'N83]).

A proper circle[4] in a pseudo-Riemannian manifold is defined using the Frenet formula as a unit speed curve whose first curvature is constant and non-zero and remaining curvatures vanish. To be precise, let $\gamma(t)$ be a unit speed curve in M, i.e. $\dot{\gamma}^2 = \pm 1$. Let $X := \dot{\gamma}$. Let $\kappa(t) := \|\nabla_X X\|$ be the (first) curvature of γ. Assuming $\kappa \neq 0$, we define Y to be the unit vector field over γ derived from $\nabla_X X$, that is Y satisfies the following equation

$$\nabla_X X = \kappa Y$$

A proper circle is defined to be a curve which satisfies $\nabla_X Y = cX$ for some $c \in \mathbb{R} \setminus \{0\}$. We observe that

$$\begin{aligned} \langle \nabla_X Y, X \rangle &= - \langle Y, \nabla_X X \rangle \\ &= -\kappa \langle Y, Y \rangle \end{aligned}$$

The above equation implies that $c = -\varepsilon_0 \varepsilon_1 \kappa$ where $\varepsilon_0 := \operatorname{sgn} \langle X, X \rangle$ and $\varepsilon_1 := \operatorname{sgn} \langle Y, Y \rangle$. Thus a proper circle is defined by the equations

$$\begin{aligned} \nabla_X X &= \kappa Y \\ \nabla_X Y &= -\varepsilon_0 \varepsilon_1 \kappa X \end{aligned}$$

where $\kappa \neq 0$ is a constant. A proper circle satisfies the following third order ODE [ANY90]:

$$\nabla_X \nabla_X X = - \langle \nabla_X X, \nabla_X X \rangle \langle X, X \rangle X \tag{3.2.1}$$

Conversely we will see shortly that any unit speed curve satisfying the above equation with $\langle \nabla_X X, \nabla_X X \rangle \neq 0$ is a proper circle. We define a *circle* in a pseudo-Riemannian manifold to be a unit speed curve satisfying the above equation, hereafter called the *circle equation*. The following lemma shows that any pseudo-Riemannian manifold admits circles:

Lemma 3.2.1 (Existence and Uniqueness of Circles [NY74])
Consider the following initial conditions: $p \in M$, a unit vector $X_p \in T_p M$ and $Y_p \in X_p^{\perp}$. There exists a unique locally defined unit speed curve $\gamma(t)$ in M satisfying Eq. (3.2.1) and

[4] Sometimes these are called geodesic circles [Ami03]. This name emphasizes the fact that we due not require the image of these curves to be a compact set, i.e. homeomorphic to $\mathbb{S}^1$.

the initial conditions:

$$\begin{aligned} \gamma(0) &= p \\ \dot{\gamma}(0) &= X_p \\ (\nabla_X X)|_p &= Y_p \end{aligned}$$

where $X := \dot{\gamma}$ and $Y := \nabla_X X$. Furthermore, $\langle Y, Y \rangle$ is constant along any circle. □

Proof It follows by the existence and uniqueness theorem for ODEs that there exists a unique locally defined curve $\gamma(t)$ satisfying Eq. (3.2.1) with the above initial conditions. Then observe the following:

$$\begin{aligned} \nabla_X \langle X, X \rangle &= 2 \langle X, \nabla_X X \rangle = \langle X, Y \rangle \\ \nabla_X \langle X, Y \rangle &= \langle Y, Y \rangle + \langle X, \nabla_X Y \rangle \\ &\overset{(3.2.1)}{=} \langle Y, Y \rangle - \langle X, X \rangle^2 \langle Y, Y \rangle \\ &= \langle Y, Y \rangle (\langle X, X \rangle^2 - 1) \end{aligned}$$

The above two equations define a system of ODEs for $\langle X, X \rangle$ and $\langle X, Y \rangle$, with initial values $\langle X, X \rangle |_p = \varepsilon = \pm 1$ and $\langle X, Y \rangle |_p = 0$. Thus by the uniqueness of the solutions, it follows that $\langle X, X \rangle = \varepsilon$ and $\langle X, Y \rangle = 0$ wherever γ is defined. Hence γ is a unit speed curve.

Finally observe that

$$\begin{aligned} \nabla_X \langle Y, Y \rangle &= 2 \langle \nabla_X Y, Y \rangle \\ &\overset{(3.2.1)}{=} -2 \langle \nabla_X X, \nabla_X X \rangle \langle X, X \rangle \langle X, Y \rangle \\ &= 0 \end{aligned}$$

Hence $\langle Y, Y \rangle$ is constant. ■

Note that $k := \|Y\|$ in the above lemma is usually called the *curvature* of the circle. In Riemannian manifolds, circles are completely classified by their curvature, although this is not true for pseudo-Riemannian manifolds. Using the above lemma we can classify circles in a pseudo-Riemannian manifold as follows. Let $\gamma(t)$ be a circle in M and suppose γ satisfies the initial conditions of the above lemma. Then γ can be classified as follows depending on Y_p:

Geodesic: If $Y_p = 0$.

Proper Circle: If $\langle Y, Y\rangle |_p \neq 0$.

Null Circle: If $\langle Y, Y\rangle |_p = 0$ but $Y_p \neq 0$, i.e. Y_p is lightlike, hence Eq. (3.2.1) reduces to $\nabla_X \nabla_X X = 0$.

Note that this classification is well defined globally since $\langle Y, Y\rangle$ is a constant of a circle and the uniqueness theorem for ODEs forces any circle with $Y_p = 0$ to be a geodesic.

Example 3.2.2 (Geodesics in Spherical Submanifolds [Kas10])
Let M be a spherical submanifold of $\bar{M}$. Suppose $\gamma(t)$ is a unit speed geodesic on M. We will show that γ is a circle in $\bar{M}$. By the Gauss equation, we have the following:

$$\bar{\nabla}_X X = \langle X, X\rangle H$$

Then by the Weingarten equation and using the fact that $\bar{\nabla}^{\perp} H = 0$ where $\bar{\nabla}^{\perp}$ is the induced normal connection over M, we have the following:

$$\begin{aligned}
\bar{\nabla}_X \bar{\nabla}_X X &= \langle X, X\rangle \bar{\nabla}_X H \\
&= - \langle X, X\rangle A_H(X) \\
&= - \langle X, X\rangle \langle H, H\rangle X \\
&= - \langle X, X\rangle \left\langle \bar{\nabla}_X X, \bar{\nabla}_X X\right\rangle X
\end{aligned}$$

since for any $Z \in \mathfrak{X}(M)^{\perp}$, $\langle A_H(X), Z\rangle \overset{(3.1.1)}{=} \langle h(X, Z), H\rangle = \langle X, Z\rangle \langle H, H\rangle$. □

We note here that the above example in combination with Lemma 3.2.1 shows that the mean curvature vector field of a spherical submanifold is locally determined by its value at a single point. Also note that the proper circles in pseudo-Euclidean space are given in Example D.4.4.

We will now present some additional results that show how circles can be used to characterize spherical submanifolds. These results were first obtained for the Riemannian case by Nomizu and Yano in [NY74]. They were generalized to the Lorentzian case by Ikawa in [Ika85] and to the pseudo-Riemannian case by Abe, Nakanishi, and Yamaguchi in [ANY90].

For the following theorems we denote a pseudo-Riemannian manifold M with signature α by M_α. The following theorem characterizes spherical submanifolds in terms of circles, it is analogous to the corresponding theorem for geodesics and geodesic submanifolds (see [O'N83, section 4.4]).

Theorem 3.2.3 (Circles and Spheres [ANY90])
Let M_α be an n dimensional pseudo-Riemannian submanifold of $\bar{M}_\beta$. For any $\varepsilon_0 \in \{-1, 1\}$ and $\varepsilon_1 \in \{-1, 0, 1\}$ satisfying $2 - 2\alpha \leq \varepsilon_0 + \varepsilon_1 \leq 2n - 2\alpha - 2$ and $k \in \mathbb{R}^+$, the following are equivalent:

(a) Every circle in M_α with $\langle X, X\rangle = \varepsilon_0$ and $\langle \nabla_X X, \nabla_X X\rangle = \varepsilon_1 k^2$ is a circle in $\bar{M}_\beta$.

(b) M_α is a spherical submanifold of $\bar{M}_\beta$. □

Proof See [ANY90]. ■

More intuitively, the above theorem states that a spherical submanifold M is precisely a submanifold in which all circles in M are circles in the ambient space. Also note that the above theorem shows that a circle is precisely a spherical submanifold of dimension one. The following theorem is a variant of the above theorem which is known to hold (in full generality) only in the strictly pseudo-Riemannian case.

Theorem 3.2.4 (Circles and Spheres II [ANY90])
Let M_α be an n dimensional ($1 \leq \alpha \leq n-1$) pseudo-Riemannian submanifold of $\bar{M}_\alpha$ having the same signature α. For any $\varepsilon_0 \in \{-1, 1\}$, the following are equivalent:

(a) Every geodesic in M_α with $\langle X, X\rangle = \varepsilon_0$ is a circle in $\bar{M}_\alpha$.

(b) M_α is a spherical submanifold of $\bar{M}_\alpha$. □

These results can be further generalized by considering more general types of curves such as helices (which we will not define here). See [Nak88] where a theorem analogous to Theorem 3.2.3 is proven characterizing helices in terms of geodesic submanifolds. Also in [JF94] results relating conformal circles to umbilical submanifolds are presented.

The following lemma describes how much information is required to specify a sphere. It is a partial generalization of the corresponding lemma for the Riemannian case proven in [Kas10].

Lemma 3.2.5 (Uniqueness of Spheres)
Suppose that M and N are connected and geodesically complete spherical submanifolds of $\bar{M}$ both satisfying the following condition: For some $p \in M \cap N$, M and N are tangent and have the same mean curvature vectors. Then $M \equiv N$. □

Proof Our proof is a generalization of the proof of lemma 4.14 in [O'N83, P. 105].

Let $q \in M$ be arbitrary and suppose that $\gamma(t)$ is a geodesic segment in M running from p to q. Then observe that γ is a geodesic circle in $\bar{M}$ with velocity X_p and acceleration $\langle X, X\rangle\,|_p H^M_p$ at p where H^M is the mean curvature vector field of M. By the uniqueness of circles (see Lemma 3.2.1) and the hypothesis it follows that γ is also geodesic in N which is defined everywhere since N is geodesically complete. Note that this implies that mean curvature vector fields of M and N coincide over γ, so we denote this vector field by H.

Now suppose $Z_p \in T_pM \cap X_p^{\perp}$ and let Z be the parallel transport of Z_p over γ with respect to M. Since parallel transport is an isometry, $\langle Z, X\rangle = 0$. Thus by the Gauss equation,

$$\begin{aligned}\bar{\nabla}_X Z &= \nabla_X^M Z + \langle Z, X\rangle H \\ &= 0\end{aligned}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ and ∇^M is the induced Levi-Civita connection on M. Thus Z is also the parallel transport of Z_p over γ with respect to $\bar{M}$.

Thus the parallel transport of $T_pM \cap X_p^{\perp}$ to q on $\bar{M}$ is equal to $T_qM \cap X_q^{\perp}$. Similarly the parallel transport of $T_pN \cap X_p^{\perp}$ to q on $\bar{M}$ is equal to $T_qN \cap X_q$. Since the parallel transport on $\bar{M}$ is uniquely determined, we deduce that $T_qM \cap X_q^{\perp} = T_qN \cap X_q^{\perp}$. Since $X_q \in T_qM, T_qN$, we conclude that $T_qM = T_qN$. Thus since M is connected, one can apply this argument to an arbitrary broken geodesic (see [O'N83]) to conclude that $M \subseteq N$.

Finally by applying the argument for M interchanged with N, we see that $M \equiv N$. ∎

Let M be a space of constant curvature. We will show in this thesis that for every $p \in M$, non-degenerate subspace $V \subset T_pM$, and normal vector $H \in (T_pM)^{\perp}$ there exists a connected and geodesically complete spherical submanifold passing through p with tangent space V and mean curvature vector H at p. In the following theorem, we will show that this property characterizes Riemannian spaces of constant curvature. For the following theorem, we say a Riemannian manifold M satisfies the *axiom of r-spheres* if: for every $p \in M$ and any r dimensional subspace $V \subset T_pM$ there exists a spherical submanifold passing through p and tangent to V.

Theorem 3.2.6 (Spheres in spaces of constant curvature [LN71])
Let M be a Riemannian manifold with dimension $n \geq 3$ and fix $2 \leq r < n$. Then M is a space of constant curvature iff it satisfies the axiom of r-spheres (see above). □

Proof See [LN71]. ∎

3.3 Product Manifolds

In this section we will briefly introduce some notations used on product manifolds. Suppose $M = \prod_{i=1}^{k} M_i$ is a product of pseudo-Riemannian manifolds (M_i, g_i). We denote $M_{i\perp} := M_1 \times \cdots \times M_{i-1} \times M_{i+1} \times \cdots \times M_k$ and the canonical projections $\pi_i : M \to M_i$

by $p \to p_i$ for each i. We also let $\pi_{i\perp} : M \to M_{i\perp}$ be the canonical projection associated with the decomposition $M = M_i \times M_{i\perp}$. We denote by L_i the canonical foliation of M induced by M_i. For $\bar{p} \in M$, the leaf of $L_i(\bar{p})$ through $\bar{p}$ and the canonical embedding of M_i in M denoted τ_i are given by

$$\tau_i(p) := (\bar{p}_1, \ldots, \bar{p}_{i-1}, p, \bar{p}_{i+1}, \ldots, \bar{p}_k), \quad p \in M_i$$
$$L_i(\bar{p}) := \tau_i(M_i) = \{p \in M : p = \tau_i(p_i),\ p_i \in M_i\}$$

We let E_i denote the integrable distribution induced by L_i.

We can naturally "lift" any tensor defined on the manifolds M_i to M. For example if $\tilde{\varphi} \in \mathcal{F}(M_i)$ then the lift is $\varphi := \tilde{\varphi} \circ \pi_i \in \mathcal{F}(M)$, we denote the set of all such functions on M of this form by $\hat{\mathcal{F}}(M_i)$. For $\tilde{v} \in \mathfrak{X}(M_i)$, the lift is the unique vector field $v \in \mathfrak{X}(M)$ such that $(\pi_i)_* v = \tilde{v}$ and $(\pi_{i\perp})_* v = 0$. Analogously we denote the set of all such vector fields on M of this form by $\hat{\mathfrak{X}}(M_i)$; note that $\hat{\mathfrak{X}}(M_i)$ is in general a proper subspace of $\Gamma(E_i)$. $\hat{S}^p(M_i)$ is defined similarly.

Example 3.3.1
Suppose $M = \prod_{i=1}^k M_i$ is a product manifold. In adapted coordinates this lifting operation is very simple. Indeed, let (y_j^i) be coordinates for M_j and consider the product coordinates $(x) = (y_1, \ldots, y_k)$ for M. If $T \in S^2(M_j)$, then the lift, $\tilde{T}$, satisfies the following equation:

$$\tilde{T}(\mathrm{d}y_i^k, \mathrm{d}y_i^l) = \begin{cases} T(\mathrm{d}y_i^k, \mathrm{d}y_i^l) & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

Hence note that if E_j denotes the distribution induced by M_j, then $\tilde{T}$ is tangent to E_j, i.e. $\tilde{T}$ can locally be written as a sum of 2-fold symmetrized products of elements in $\Gamma(E_j)$. Furthermore, the non-zero components in product coordinates are functions on M_j. □

If $v \in \hat{\mathfrak{X}}(M_i)$ and $u \in \hat{\mathfrak{X}}(M_j)$, then $(\pi_i)_*[v, u] = [\tilde{v}, \tilde{u}]$ if $i = j$ and $[v, u] = 0$ if $i \neq j$. Also note that usually we will use the same symbol for a tensor and its lift. For $\varphi \in \mathcal{F}(M)$, we say that φ is independent of M_i (or E_i) if $\varphi \in \hat{\mathcal{F}}(M_{i\perp})$; if M is connected this is equivalent to $\varphi_* E_i = 0$. We say that φ depends only on M_i (or E_i) if $\varphi \in \hat{\mathcal{F}}(M_i)$.

3.4 Nets and their Integrability

The following notion of (orthogonal) nets will be useful:

Definition 3.4.1 (Nets [MRS99])
A family $\mathcal{E} = (E_i)_{i=1}^k$ of integrable distributions E_i on a manifold M is called a *net* on M if the tangent bundle TM can be decomposed as:

$$TM = \bigoplus_{i=1}^{k} E_i$$

If M is a pseudo-Riemannian manifold, and the direct sum in the above equation is replaced with the orthogonal direct sum, then $\mathcal{E}$ is called an *orthogonal net*. □

Remark 3.4.2

If M is a pseudo-Riemannian manifold, then unless specified otherwise, all nets are assumed to be orthogonal. □

A net $\mathcal{E}$ is said to be *(locally) integrable* (or *locally decomposable* in [MRS99]) if for every $p \in M$ there exists a neighborhood $U \subseteq M$ of p and a C^∞-diffeomorphism f from a product manifold $\prod_{i=1}^{k} M_i$ onto U such that for every $q \in \prod_{i=1}^{k} M_i$ and every $i = 1, ..., k$ the slice $(q_1, ..., q_{i-1}) \times M_i \times (q_{i+1}, ..., q_k)$ gets mapped into an integral manifold of E_i. In this case, the product manifold $\prod_{i=1}^{k} M_i$ is said to be (locally) *adapted* to $\mathcal{E}$. An (orthogonal) net $\mathcal{E}$ is called an *(orthogonal) web* if it is integrable and $\dim E_i = 1$ for each i. Given a collection of distributions $\mathcal{E} = (E_i)_{i=1}^{k}$ on a pseudo-Riemannian manifold, we say the collection is *orthogonally integrable* if $\mathcal{E}$ forms an integrable orthogonal net. For the following theorem, if $\mathcal{E}$ is not assumed to be orthogonal, then $E_i^\perp := \bigoplus_{j \neq i} E_j$. In [RS99, Theorem 1] the following has been shown, which justifies the term "orthogonally integrable"

Theorem 3.4.3 (Characterizations of integrable nets [RS99])

For the decomposition $TM = \bigoplus_{i=1}^{k} E_i$ by the family of distributions $\mathcal{E} = (E_i)_{i=1}^{k}$, the following are equivalent

1. *$\mathcal{E}$ is an integrable net.*
2. *The orthogonal distributions $E_i^\perp$ are integrable for $i = 1, ..., k$.*
3. *The distributions E_i and their direct sums $E_i \oplus E_j$ are integrable for $i, j = 1, ..., k$.* □

The above theorem also proves the following well known fact:

Corollary 3.4.4

Any net $\mathcal{E}$ with two factors, i.e. $\mathcal{E} = (E_1, E_2)$, is integrable. □

When the net has more than two factors, it's easy to find non-integrable cases:

Example 3.4.5 (Non-integrable nets)

Suppose M^n is a pseudo-Riemannian manifold with $n = 3$. Let u be any non-null vector field such that $u^\perp$ is not an integrable distribution. Extend this to a local orthonormal basis, $\{u, v, w\}$ for TM, then clearly these vector fields form a net which is not integrable by the above theorem.

As a concrete example, one can take $M = \mathbb{E}^3$ and u to be the Killing vector field whose integral curves are helices. □

The following example gives the simplest way to obtain integrable nets.

Example 3.4.6 (Product Nets)
Suppose $M = \prod_{i=1}^{k} M_i$ is a product of manifolds M_i. If E_i denotes the canonical foliation induced by M_i then $\mathcal{E} = (E_i)_{i=1}^{k}$ is called the *product net* of $\prod_{i=1}^{k} M_i$. Note that by definition, $\mathcal{E}$ is an integrable net. If each M_i is a pseudo-Riemannian manifold equipped with covariant metric g_i, then equipping M with the metric $g = \sum_{i=0}^{k} \pi_i^* g_i$ makes $\mathcal{E}$ into an orthogonal net. □

3.5 Warped and Twisted Products

Warped products are ubiquitous in applications of pseudo-Riemannian geometry. Most of the separable coordinate systems in spaces of constant curvature are built up using them [Kal86], and some exact solutions in general relativity are composed of them [DU05; Zeg11]. They can intuitively be thought of as a partial generalization of the spherical coordinate system to arbitrary pseudo-Riemannian manifolds. Indeed, it will eventually become clear that all the spherical coordinate systems (on any space of constant curvature) can be constructed iteratively using warped products, and that they share several properties with these coordinate systems. Similarly it will be clear that the well known Schwarzschild metric in relativity can be constructed by using warped products. In this section we will give a brief introduction to these products by studying them as special cases of twisted products. The content of this section is primarily from [MRS99] where the notion of a twisted product is studied. For more on warped products and applications see [O'N83; MRS99; Zeg11].

The following general definition of a twisted product is useful in the study of conformal Killing tensors.

Definition 3.5.1 (Warped and Twisted Products)
Let $M = \prod_{i=0}^{k} M_i$ be a product of pseudo-Riemannian manifolds (M_i, g_i) where $\dim M_i > 0$ for $i > 0$. Suppose for $i = 0, ..., k$, $\pi_i : M \to M_i$ is the projection map and $\rho_i : M \to \mathbb{R}^+$ is a function. The following metric g on M is called a *twisted product metric*

$$g(X, Y) = \sum_{i=0}^{k} \rho_i^2 g_i(\pi_{i*} X, \pi_{i*} Y) \quad \text{for } X, Y \in \mathfrak{X}(M)$$

In this case (M, g) is called a *twisted product* and is denoted by ${}^{\rho}\prod_{i=0}^{k} M_i$ where $\rho = (\rho_0, ..., \rho_k)$. Furthermore the ρ_i are called twist functions of the twisted product. If each ρ_i depends only on M_0 and $\rho_0 \equiv 1$ then g is called a *warped product metric* and (M, g) is called a *warped product*. The warped product is denoted by $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$.

M_0 is called the geodesic factor of the warped product and the M_i for $i > 0$ are called spherical factors. $\square$

Example 3.5.2
By taking M_0 to be a point and $k = 1$ in the definition of a twisted product, we get a *conformal product.* $\square$

Example 3.5.3
By taking M_0 to be a point and $k > 1$ in the definition of a warped product, we get a *pseudo-Riemannian product.* Throughout this thesis we will treat pseudo-Riemannian products as special cases of warped products this way. $\square$

Example 3.5.4
If $\dim M_i = 1$ for each i, then the twisted product metric is locally the metric of an *orthogonal coordinate system.* $\square$

Example 3.5.5 (Prototypical warped product)
The prototypical example of a warped product is the following warped product defined in (an open subset of) $\mathbb{E}^n$, which is the product manifold $\mathbb{R}^+ \times S^{n-1}$ equipped with the metric $g = \mathrm{d}\rho^2 + \rho^2 \tilde{g}$ where $\tilde{g}$ is the metric of the $(n-1)$-sphere S^{n-1}. $\square$

Note that a twist function ρ_i of a twisted product is only uniquely defined modulo products of functions $f \in \hat{\mathcal{F}}(M_i)$. To elaborate, from the above definition one sees that we can multiply ρ_i^2 by $f \in \hat{\mathcal{F}}(M_i)$ if we divide g_i by f. The geometry of the twisted product is not altered by such transformations as we will see. We say that the twist functions are *normalized* (with respect to a point $\bar{p} \in M$), if for each i, $\rho_i(p) = 1$ for all $p \in L_i(\bar{p})$.

First we give the formulas for the Levi-Civita connection and Riemann tensor of a twisted product; it is from Proposition 1 in [MRS99]. We will make use of the following notation: Given a collection of distributions $(E_i)_{i=1}^k$ satisfying $TM = \bigoplus_{i=1}^k E_i$, then for any vector $X \in \mathfrak{X}(M)$, we have the orthogonal splitting $X = \sum_i X^i$ where each $X^i \in \Gamma(E_i)$.

Proposition 3.5.6
Let ${}^{\rho}\prod_{i=0}^k M_i$ *be a twisted product with product net* $\mathcal{E} = (E_i)_{i=0}^k$*. Let* $\tilde{\nabla}$ *be the Levi-Civita connection associated with the ordinary pseudo-Riemannian product metric of* $\prod_{i=0}^k M_i$ *with Riemann tensor* $\tilde{R}$ *and* ∇ *be the Levi-Civita connection of the twisted product metric. Let* $U_i := -\nabla \log \rho_i$ *and* $X, Y \in \mathfrak{X}(M)$*, then* ∇ *is given as follows*

$$\nabla_X Y = \tilde{\nabla}_X Y + \sum_{i=0}^k (\langle X^i, Y^i \rangle U_i - \langle X, U_i \rangle Y^i - \langle Y, U_i \rangle X^i) \tag{3.5.1}$$

Note that $\tilde{\nabla}$ *satisfies* $(\tilde{\nabla}_X Y)^i = \tilde{\nabla}_X Y^i$*. The Riemann tensor* R *of the twisted product is given by:*

$$R(X,Y) = \tilde{R}(X,Y) + \sum_{i=0}^{k}((\nabla_X U_i - \langle X, U_i\rangle U_i) \wedge Y^i + X^i \wedge (\nabla_Y U_i - \langle Y, U_i\rangle U_i))$$
$$+ \sum_{i,j=0}^{k} \langle U_i, U_j\rangle X^i \wedge Y^j \tag{3.5.2}$$

for any $X, Y \in \mathfrak{X}(M)$. For $X, Y \in \mathfrak{X}(M)$, the linear operator $X \wedge Y$ is the one metrically equivalent to the bivector. □

The following corollary gives the corresponding formulas for a warped product.

Corollary 3.5.7

Let $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ be a warped product. Let $\tilde{\nabla}$ be the Levi-Civita connection associated with the ordinary pseudo-Riemannian product metric of $\prod_{i=0}^{k} M_i$ with Riemann tensor $\tilde{R}$ and ∇ be the Levi-Civita connection of the warped product metric. Let $H_i := -\nabla \log \rho_i$ and $X, Y \in \mathfrak{X}(M)$, then ∇ is given as follows

$$\nabla_X Y = \tilde{\nabla}_X Y + \sum_{i=1}^{k}(\langle X^i, Y^i\rangle H_i - \langle X, H_i\rangle Y^i - \langle Y, H_i\rangle X^i)$$

Note that $\tilde{\nabla}$ satisfies $(\tilde{\nabla}_X Y)^i = \tilde{\nabla}_X Y^i$. The Riemann tensor R of the warped product is given by:

$$R(X,Y) = \tilde{R}(X,Y) + \sum_{i=1}^{k}((\nabla_{X^0} H_i - \langle X, H_i\rangle H_i) \wedge Y^i + X^i \wedge (\nabla_{Y^0} H_i - \langle Y, H_i\rangle H_i))$$
$$- \sum_{i,j=1}^{k} \langle H_i, H_j\rangle X^i \wedge Y^j$$

for any $X, Y \in \mathfrak{X}(M)$. For $X, Y \in \mathfrak{X}(M)$, the linear operator $X \wedge Y$ is the one metrically equivalent to the bivector. Furthermore, the Riemann tensor $\tilde{R}$ satisfies $(\tilde{R}(X,Y)Z)^i = \tilde{R}(X^i, Y^i)Z^i$. □

Proof The formula for Riemann tensor follows from Eq. (3.5.2) by expanding $\nabla_X H_j$ as follows:

$$\nabla_X H_j = \nabla_{X^0} H_j - \sum_{i=1}^{k} \langle H_i, H_j\rangle X^i$$

The remaining facts follow from Proposition 3.5.6 and Corollary 2 in [MRS99]. ■

The above formula for the curvature tensor can be used to obtain general formulas for the sectional curvature of warped products. First we need some definitions. If $f \in \mathcal{F}(M)$, we denote the Hessian of f [O'N83, P. 86], by $S^f_{ij} = \nabla_i \nabla_j f$. If $X, Y \in T_pM$ span a non-degenerate 2-plane then the sectional curvature of the 2-plane, $K(X,Y)$, is given in terms of the curvature tensor R as [O'N83, lemma 3.39]:

$$K(X,Y) = \frac{\langle R(X,Y)Y, X\rangle}{\|X \wedge Y\|^2}, \qquad \|X \wedge Y\|^2 = \langle X, X\rangle \langle Y, Y\rangle - \langle X, Y\rangle^2$$

Now we have the following:

Corollary 3.5.8 (Sectional curvature of warped products)
Suppose $M = M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ is a warped product with product net $\mathcal{E} = (E_i)^k_{i=0}$. Let $X, Y \in \Gamma(E_0)$, $V \in \Gamma(E_i)$ and $U \in \Gamma(E_k)$ for $i, k > 0$. If $H_i = -\nabla \log \rho_i$ denotes the mean curvature normal of E_i, then we have the following:

$$K_{XY} = K^{M_0}_{XY}$$

$$K_{XV} = -\frac{S^{\rho_i}(X,X)}{\rho_i X^2} \tag{3.5.3}$$

$$K_{UV} = -\langle H_i, H_k\rangle \quad (i \neq k) \tag{3.5.4}$$

$$K_{UV} = \frac{K^{M_i}_{UV} - (\nabla \rho_i)^2}{\rho_i^2} \quad (i = k) \tag{3.5.5}$$

□

Proof This follows from the formula for the curvature tensor in Corollary 3.5.7. ■

The following properties of the twisted product can be found in Proposition 2 in [MRS99].

Proposition 3.5.9 (Properties of the Twisted Product [MRS99])
Let ${}^{\rho}\prod_{i=0}^{k} M_i$ be a twisted product with product net $\mathcal{E} = (E_i)^k_{i=0}$ and $U_i := -\nabla \log \rho_i$.

1. *$\mathcal{E}$ is an orthogonally integrable net.*

2. *For each i the distribution E_i is umbilical with mean curvature normal $H_i = U_i^{\perp i}$.*

3. *E_i is geodesic iff ρ_i is independent of M_j for $j \neq i$. $E_i^{\perp}$ is geodesic iff ρ_j is independent of M_i for $j \neq i$.*

4. *If ρ is independent of M_i then E_i is Killing. The converse is also true if the twisted product is normalized.*

□

The following theorem characterizes twisted and warped products in terms of the geometry of their canonical foliations.

Theorem 3.5.10 (Geometric Characterization of Twisted and Warped Products [MRS99])
Let $M = \prod_{i=0}^{k} M_i$ be a connected product manifold equipped with metric g and orthogonal product net $\mathcal{E} = (E_i)_{i=0}^{k}$. Then g is the metric of

1. *a twisted product ${}^{\rho}\prod_{i=0}^{k} M_i$ iff E_i are umbilical distributions*
2. *a warped product $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ iff E_i are Killing distributions for $i = 1, ..., k$*
3. *a pseudo-Riemannian product iff E_i are geodesic distributions* □

Proof The characterization of the twisted product follows from Proposition 4 in [MRS99]. We note here that the relationship between the second fundamental form and Lie derivatives of the metric given in Proposition 3.1.2 is crucial to the proof of this fact. The other characterizations follow from the first and Proposition 3.5.9 above. ■

Remark 3.5.11
It follows by definition of the twisted product, that they are invariant under conformal transformations. The conformal generalizations of warped and pseudo-Riemannian products and their characterizations are given in [Toj04]. □

The following notions of twisted and warped product nets will be especially useful for studying conformal Killing tensors. It was originally Definition 3 in [MRS99].

Definition 3.5.12 (Twisted and warped product nets)
Let M be a pseudo-Riemannian manifold and suppose $\mathcal{E} = (E_i)_{i=0}^{k}$ is an orthogonal net.

1. $\mathcal{E}$ is called a *twisted product net (TP-net)* if it is integrable and each distribution E_i is umbilical.
2. $\mathcal{E}$ is called a *warped product net (WP-net)* if E_i is Killing for $i = 1, ..., k$. □

Remark 3.5.13
In all applications, $\dim E_i > 0$ for $i > 0$. Although we will allow $\dim E_0 = 0$ for a WP-net since this gives us a *pseudo-Riemannian product net (RP-net)*. □

It can be shown that if $\mathcal{E}$ is a WP-net, then it is a TP-net with $E_0 = \bigcap_{i=1}^{k} E_i^{\perp}$ a geodesic distribution [MRS99, Proposition 3]. Also in the case $\mathcal{E}$ is a WP-net we refer to E_0 as the geodesic distribution of the WP-net and the E_i for $i > 0$ as the Killing distributions of the WP-net. The following theorem, which is Corollary 1 in [MRS99], gives the motivation for the above definition. It shows that every TP-net (resp. WP-net) admits a locally adapted twisted product (resp. warped product).

Theorem 3.5.14 (Twisted and warped product nets [MRS99])
Let (M, g) be a pseudo-Riemannian manifold and suppose $\mathcal{E} = (E_i)_{i=0}^k$ is a TP-net (resp. WP-net). Then for every $p \in M$ there exists an open set $U \subseteq M$ containing p and a map $f : \prod_{i=0}^k M_i \to U$ which is an isometry with respect to a twisted (resp. warped) product metric on $\prod_{i=0}^k M_i$. □

Proof The existence of the map $f : \prod_{i=0}^k M_i \to U$ which is a diffeomorphism is guaranteed by the integrability of the net $\mathcal{E}$, see Theorem 3.4.3. Once $\prod_{i=0}^k M_i$ is equipped with f^*g, the result then follows by Theorem 3.5.10. ■

Remark 3.5.15
One can also check that a similar theorem holds for a pseudo-Riemannian product net and metric. □

Now we can give some justification to the name "Killing" for a non-degenerate distribution which is spherical and has a geodesic orthogonal complement. By the above corollary, we see that a one dimensional Killing distribution is always spanned by a Killing vector field. Conversely any normal[5] non-null Killing vector field spans a Killing distribution. The following can be said about multidimensional Killing distributions via the warped products they induce [Zeg11]:

Proposition 3.5.16 (Lifting isometries from Killing distributions)
Let $M = B \times_\rho F$ be a warped product and suppose $\tilde{f} : F \to F$ is an isometry of F. Then the lift f defined by

$$f(x, y) := (x, \tilde{f}(y)), \quad (x, y) \in B \times F$$

is an isometry of M. □

[5] We say a non-null vector field is normal if the orthogonal distribution is Frobenius integrable.

Chapter 4

Killing tensors

In this chapter we thoroughly study the geometric objects encoding separation, (conformal) Killing tensors. In the first section we will first review a formalism we will use in this chapter and the next. In the following section, we introduce Killing tensors and give some equivalent definitions of them. In Section 4.3 we consider a conformal generalization of Killing tensors which will also be of use. In Section 4.4 we study orthogonal conformal Killing tensors systematically. We give a characterization of them based on the geometry of their eigenspaces and present some consequences of it. We then present other miscellaneous results which will be used later. Finally, in Section 4.5 we describe the Killing tensors which have a canonical algebraic decomposition in warped products.

4.1 Hamiltonian mechanics on the Cotangent bundle

We will be working on the cotangent bundle T^*M, which is the natural geometric setting for Hamiltonian mechanics, and Hamilton-Jacobi theory [Arn89]. We assume the reader is familiar with the basic notions of Hamiltonian mechanics on the cotangent bundle T^*M, see [Lee12] for the basics and [Arn89] for more details. We review the basics to fix our notations, following [Woo75] and [Ben89]. We denote the natural projection map by $\pi : T^*M \to M$ which acts on a point $(q, p) \in T^*M$ as $\pi(q, p) = q$. Any local coordinate system (q^i) on M induces coordinates (q^i, p_j) on T^*M, hereafter called *canonical coordinates*. The coordinates (p_j) are called *momenta*.

A *Hamiltonian* is simply a function $H \in \mathcal{F}(T^*M)$. If M is a pseudo-Riemannian manifold with metric $\langle \cdot, \cdot \rangle$, the *natural Hamiltonian* H with potential $V \in \mathcal{F}(M)$ is defined by:

$$H(q, p) := \frac{1}{2} \langle p, p \rangle + V(q) \quad (q, p) \in T^*M$$

The *geodesic Hamiltonian* is obtained by setting $V \equiv 0$ in the above equation.

The Liouville 1-form θ is defined by:

$$\theta_{(q,p)}(X) = p(\pi_* X) \quad (q,p) \in T^*M,\ X \in T_{(q,p)}T^*M$$

In canonical coordinates, $\theta = p_i \mathrm{d}q^i$. The canonical *symplectic form* on T^*M is then $\omega := \mathrm{d}\theta$. A crucial property of ω is that it's *non-degenerate*, i.e. at each point (q,p) the quadratic form ω on $T_{(q,p)}T^*M$ is non-degenerate.

This means to each function $F \in \mathcal{F}(T^*M)$, ω induces a vector field X_F, called the *Hamiltonian vector field* of F defined by the following equation:

$$\omega(X_F, Y) = \mathrm{d}F(Y) = YF, \quad Y \in \mathfrak{X}(T^*M) \tag{4.1.1}$$

If we take $F = H$ where H is the Hamiltonian, then an integral curve of X_H satisfies the classical *Hamilton's equations* [LL76] in canonical coordinates (q,p):

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad\qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

For any $F, G \in \mathcal{F}(T^*M)$, the *Poisson bracket* is defined by:

$$\{F, G\} := \omega(X_F, X_G)$$

In canonical coordinates

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p^i} \right) \tag{4.1.2}$$

If the Poisson bracket vanishes identically, then we say the functions F and G *Poisson commute*. A set of functions which Poisson commute are said to be in *involution*. We say a function $F \in \mathcal{F}(T^*M)$ is a *first integral* if it satisfies

$$\{F, H\} = 0$$

where H is the Hamiltonian. Note that it follows from Eq. (4.1.1), that first integrals are constant along the integral curves of X_H, i.e. a first integral F satisfies $X_H F = 0$. In particular we note that the Hamiltonian is a first integral, known as the *energy* for natural Hamiltonian systems.

4.2 Definition in terms of Poisson and Schouten brackets and Covariant Derivative

In this section we give three equivalent definitions of a Killing tensor on a pseudo-Riemannian manifold. We first start with the definition on the cotangent bundle.

Poisson bracket Fix canonical coordinates (q, p) for T^*M. Each $K \in S^r(M)$ induces a homogeneous polynomial of the momenta defined by

$$E_K := K^{j_1 .. j_r} p_{j_1} \dots p_{j_r} \in \mathcal{F}(T^*M) \tag{4.2.1}$$

This process can be inverted to obtain K from E_K. Indeed, if we denote the dependence of $E_K(p)$ on p explicitly, then observe that for any covectors $p_1, \dots, p_r \in T_q^*M$, we recover K using the generalized polarization identity:

$$K_{i_1 \dots i_r} p_1^{i_1} \dots p_r^{i_r} = \frac{1}{r!} \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_r} E_K(t_1 p_1 + \dots + t_r p_r) \tag{4.2.2}$$

We also note that if $K \in S^r(M)$ and $L \in S^t(M)$, then

$$E_{K \odot L} = E_K E_L \tag{4.2.3}$$

A tensor $K \in S^r(M)$ is called a *Killing tensor* on M if E_K is a first integral for the geodesic Hamiltonian on T^*M. Hence from the previous section we see that the function E_K is constant on the trajectories of the geodesic flow.

Schouten bracket We now introduce the Schouten bracket and use it to obtain a condition on M which characterizes when $K \in S^r(M)$ is a Killing tensor. In analogy with Proposition 3.1 in [Mar97], the Schouten bracket (for symmetric contravariant tensors) is defined as follows:

Theorem 4.2.1 (Schouten bracket [Sch53])
There exists a unique $\mathbb{R}$-bilinear operator, mapping $S(M) \times S(M) \to S(M)$, called the Schouten bracket, *denoted by $(P, Q) \mapsto [P, Q]$, and determined by the following properties:*

(a) For $f, g \in S^0(M)$, $[f, g] = 0$.

(b) For a vector $X \in S^1(M)$, and $Q \in S(M)$ we have $[X, Q] = \mathcal{L}_X Q$.

(c) For $P, Q \in S(M)$

$$[P, Q] = -[Q, P]$$

(d) *For $P, Q, R \in S(M)$*

$$[P, Q \odot R] = [P, Q] \odot R + Q \odot [P, R]$$ □

The proof of the above fact follows by expanding the tensors in a local basis (see [Mar97] for the details of a rigorous proof). Indeed, in local coordinates (x^i), for $P \in S^p(M)$ and $Q \in S^q(M)$ one can derive the following:

$$[P, Q]^{i_1 \dots i_p j_1 \dots j_{q-1}} = pP^{k(i_1 \dots i_{p-1}}\partial_k Q^{i_p j_1 \dots j_{q-1})} - qQ^{k(j_1 \dots j_{q-1}}\partial_k P^{i_1 \dots i_p)} \tag{4.2.4}$$

The Schouten bracket has the following fundamental property.

Theorem 4.2.2 (Schouten and Poisson brackets)
If $K \in S^p(M)$ and $G \in S^q(M)$, the Schouten bracket satisfies the following identity:

$$E_{[K,G]} = -\{E_K, E_G\} \tag{4.2.5}$$

□

Proof This result follows by a straightforward calculation using Eq. (4.1.2) and Eq. (4.2.4). ■

Due to the connection above, the Schouten bracket is often defined in terms of the Poisson bracket (for example, in [Ben89]). With this connection, the properties of the Schouten bracket can be derived from similar properties of the Poisson bracket. We will give an example of this in the proof of the following result.

Proposition 4.2.3 (Properties of the Schouten bracket)
For $P, Q, R \in S(M)$, the Schouten bracket satisfies the following:

(a) *If $P \in S^p(M)$ and $Q \in S^q(M)$ then $[P, Q] \in S^{p+q-1}(M)$*

(b) *The Jacobi identity is satisfied:*

$$[P, [Q, R]] + [R, [P, Q]] + [Q, [R, P]] = 0$$

(c) *If $P \in S^p(M)$ and $f \in \mathcal{F}(M)$ then*

$$[P, f]^{i_1 \dots i_{p-1}} = pP^{i_1 \dots i_{p-1} j}\partial_j f \tag{4.2.6}$$

□

Proof The first property follows from the coordinate formula. The second follows by a direct calculation (see [Nij55] for more details), or using the Jacobi identity for the Poisson bracket (see [Lee12]) and Theorem 4.2.2. The third property, which is a useful fact, follows immediately from the coordinate formula. ■

In analogy with the Poisson bracket, we say two tensors $K \in S^p(M)$ and $G \in S^q(M)$ *Schouten commute* if they satisfy $[K, G] = 0$. By Theorem 4.2.2 and the generalized polarization identity (Eq. (4.2.2)), we see that a tensor $K \in S^p(M)$ is a *Killing tensor (KT)* iff it satisfies:

$$[K, G] = 0$$

where G is the (inverse) contravariant metric. If $p = 1$, then K is called a *Killing vector (KV)* and the above equation reduces to $\mathcal{L}_K G = 0$ (see Theorem 4.2.1). The properties of the Schouten bracket imply that Killing tensors form a Lie algebra (with respect to the Schouten bracket) which is closed under the symmetric product.

We conclude with some historical remarks on the Schouten bracket. The Schouten bracket was discovered originally by Schouten in [Sch53]. Its properties have been studied by his student Nijenhuis in [Nij55]. In Schouten's original work he introduced a more general bracket defined on the space of contravariant tensors, also called the *Schouten bracket*. This bracket naturally breaks down into two brackets, one for symmetric tensors discussed above, and another for anti-symmetric tensors. Indeed, if P, Q are contravariant tensors with symmetric and anti-symmetric parts P_s, Q_s and P_a, Q_a respectively, then the Schouten bracket can be written [Nij55]:

$$[P, Q] = [P_s, Q_s] + [P_a, Q_a]$$

where $[P_s, Q_s]$ is the Schouten bracket for symmetric tensors and $[P_a, Q_a]$ is the one for anti-symmetric tensors. The one for anti-symmetric tensors satisfies a theorem similar to Theorem 4.2.1 (see Proposition 3.1 in [Mar97]), the main difference being that the symmetric product is replaced with the wedge product. The Schouten bracket for anti-symmetric tensors appears more often in the literature because of its role in the coordinate-independent definition of a Poisson manifold.

Levi-Civita Connection We will give our last definition of a Killing tensor in terms of the Levi-Civita connection, ∇, of the metric g. First we need the following fact [Woo75].

Proposition 4.2.4 (Schouten bracket and Connections)
If $P \in S^p(M)$ and $Q \in S^q(M)$, and ∇ is a torsion-free connection, then the Schouten bracket has the following form:

$$[P, Q]^{i_1 \ldots i_p j_1 \ldots j_{q-1}} = pP^{k(i_1 \ldots i_{p-1}} \nabla_k Q^{i_p j_1 \ldots j_{q-1})} - qQ^{k(j_1 \ldots j_{q-1}} \nabla_k P^{i_1 \ldots i_p)} \qquad \square$$

Proof The proof follows by a straightforward calculation. Fix a local coordinate system (x^i) and let ${\Gamma^i}_{kl}$ be the Christoffel symbols of the connection ∇. Then observe that for $K \in S^q(M)$ we can write:

$$\begin{aligned}\nabla_i K^{j_1\dots j_q} &= \partial_i K^{j_1\dots j_q} + \sum_r \Gamma^{j_r}{}_{il} K^{j_1\dots l\dots j_q}\\ &= \partial_i K^{j_1\dots j_q} + q\Gamma^{j_q}{}_{il} K^{j_1\dots j_{q-1}l}\end{aligned}$$

Thus

$$\begin{aligned}[P,Q]^{i_1\dots i_p j_1\dots j_{q-1}} &= pP^{k(i_1\dots i_{p-1}}\partial_k Q^{i_p j_1\dots j_{q-1})} - qQ^{k(j_1\dots j_{q-1}}\partial_k P^{i_1\dots i_p)}\\ &= pP^{k(i_1\dots i_{p-1}}\nabla_k Q^{i_p j_1\dots j_{q-1})} - qQ^{k(j_1\dots j_{q-1}}\nabla_k P^{i_1\dots i_p)}\\ &- qpP^{k(i_1\dots i_{p-1}}\Gamma^{i_p}{}_{kl} Q^{j_1\dots j_{q-1})l} + qpQ^{k(j_1\dots j_{q-1}}\Gamma^{i_p}{}_{kl} P^{i_1\dots i_{p-1})l}\\ &= pP^{k(i_1\dots i_{p-1}}\nabla_k Q^{i_p j_1\dots j_{q-1})} - qQ^{k(j_1\dots j_{q-1}}\nabla_k P^{i_1\dots i_p)}\end{aligned}$$

■

We have the following corollary.

Corollary 4.2.5
If $K \in S^q(M)$ and G is the (inverse) contravariant metric, then:

$$[K, G]^{i_1\dots i_{q-1}jk} = -2\nabla^{(j} K^{ki_1\dots i_{q-1})} \tag{4.2.7}$$

□

Equation (4.2.7) implies that $K \in S^p(M)$ is a Killing tensor iff the following is satisfied:

$$\nabla_{(i} K_{i_1\dots i_p)} = 0$$

This is the standard definition of a Killing tensor. Using the above equation, we can give another characterization of Killing tensors as follows.

Theorem 4.2.6 (Killing tensors and constants of motion)
A symmetric tensor $K \in S^p(M)$ is a Killing tensor iff for any unit speed geodesic $\gamma(t)$, the quantity

$$K_{i_1\dots i_p}\dot\gamma^{i_1}\dots\dot\gamma^{i_p}$$

is a constant along the geodesic. □

Proof This follows from a straightforward calculation. The converse follows with the help of the generalized polarization identity (Eq. (4.2.2)). ■

4.3 Conformal Killing tensors and special classes

The conformally covariant generalization of a Killing tensor is known as *conformal Killing tensor (CKT)* . A tensor $K \in S^p(M)$ is said to be a conformal Killing tensor of valence p if there exists $C \in S^{p-1}(M)$ (called the *conformal factor*) such that

$$[K, G] = -2C \odot G$$

As an immediate consequence of the definition, one can check that a Killing tensor is a conformal Killing tensor on any conformally related manifold. In analogy with Killing tensors, we have the following equivalent characterizations of a conformal Killing tensor:

Theorem 4.3.1 (Characterizations of conformal Killing tensors)
Suppose $K \in S^p(M)$, and let $E_K \in \mathcal{F}(T^*M)$ denote the corresponding homogeneous polynomial on T^*M and H be the geodesic Hamiltonian. The following are equivalent:

(a) K is a conformal Killing tensor with conformal factor $C \in S^{p-1}(M)$.

(b) On T^*M we have $\{E_K, H\} = 2E_C H$.

(c) With respect to the Levi-Civita connection, ∇, we have:

$$\nabla_{(i}K_{ji_1\ldots i_{p-1})} = C_{(i_1\ldots i_{p-1}}g_{ij)} \tag{4.3.1}$$

□

Proof The equivalence of the first and second characterizations follows from Eqs. (4.2.3) and (4.2.5). The equivalence of the first and third characterizations follows from Eq. (4.2.7).■

In the literature, CKTs are often assumed to be traceless. This is due to the fact that for any $f \in \mathcal{F}(M)$, fG is a CKT. Though we do not make this assumption.

An important class of CKTs are those of valence two for which the conformal factor $C = \nabla f$ for some $f \in \mathcal{F}(M)$. If L is such a CKT, then it is said to be of *gradient-type*, and one can show that the following tensor is a KT:

$$K = fG - L$$

If in addition, $f = \mathrm{tr}(L)$, then it said to be of *trace-type*.

4.4 Orthogonal conformal Killing tensors

In this section we study the most important class of conformal Killing tensors for our purposes, the *orthogonal conformal Killing tensors*. An orthogonal conformal Killing tensor, is a conformal Killing tensor which is also an orthogonal tensor. In the remainder of this chapter, all tensors are assumed to be of valence two.

For now, by a conformal Killing tensor we will mean an orthogonal conformal Killing tensor. In this section we first present a formulation of the conformal Killing equation in terms of the eigenspaces of a conformal Killing tensor given in [CFS06]. This formulation will be the most useful in our study. We then use the theory of twisted and warped products given in [MRS99] (which is reviewed in Chapter 3) to show that an orthogonal conformal Killing tensor naturally induces a twisted product, and then derive the well known conformal Killing equation in the eigenframe. We then give necessary and sufficient conditions on an eigenfunction of the tensor for the associated eigenspace to be geodesic or Killing. We then end the section with some miscellaneous results on CKTs which will be applied in the next chapter.

We denote $\{x, y\} := \frac{1}{2}(\nabla_x y + \nabla_y x)$ for $x, y \in \mathfrak{X}(M)$ which is sometimes called the Jordan bracket [Rov98]. Note that the following statements are made for a CKT with conformal factor t, hence the corresponding statements for KTs can be obtained by setting $t = 0$.

Lemma 4.4.1 ([CFS06])
Let E_λ be a non-degenerate eigenspace of a CKT, T, associated with eigenfunction λ. Then the following equation holds for all $x, y \in \Gamma(E_\lambda)$

$$(T - \lambda I)\{x, y\} = \frac{1}{2} \langle x, y \rangle (\nabla\lambda - t)$$

where t is the conformal factor of T. Moreover $\nabla\lambda - t \in \Gamma(E_\lambda^\perp)$. The following equation holds for eigenvectors x, y, z with different eigenfunctions

$$T(\{x, y\}, z) + T(\{z, x\}, y) + T(\{y, z\}, x) = 0 \qquad \square$$

Proof We give some details of the proof, following [CFS06], by using Eq. (4.3.1) as the defining equation of a CKT.

Suppose $x, y \in \Gamma(E_\lambda)$ and $z \in \mathfrak{X}(M)$. A direct calculation shows the following:

$$\begin{aligned} 3!\nabla_{(i}T_{jk)}x^i y^j z^k &= 2(x(\lambda)g(y, z) + y(\lambda)g(x, z) + z(\lambda)g(x, y) - 2(T_{jk} - \lambda g_{jk})\{x, y\}^j z^k) \\ 3!g \odot t(x, y, z) &= 2(g(x, y)t(z) + g(z, x)t(y) + g(y, z)t(x)) \end{aligned}$$

If we take $x = y = z$, then equating the above equations implies that:

$$x^2 g(\nabla\lambda - t, x) = 0$$

By non-degeneracy of E_λ, we see that $\nabla\lambda - t \in \Gamma(E_\lambda^\perp)$. The first equation in the lemma then follows by equating the first set of equations. The second equation in the lemma holds since if x, y, z have different eigenfunctions, then

$$\begin{aligned} 3!\nabla_{(i}T_{jk)}x^i y^j z^k &= -2(T(\{x,y\},z) + T(\{z,x\},y) + T(\{y,z\},x)) \\ 3!g \odot t(x,y,z) &= 0 \end{aligned}$$

■

The following proposition is an immediate consequence of the above lemma.

Proposition 4.4.2 ([CFS06])
Let T be an orthogonal tensor and let E_i be the eigenspaces corresponding to the eigenfunctions λ_i. Then T is a conformal Killing tensor with conformal factor t iff

1. $(T - \lambda_i I)\{x,y\} = \frac{1}{2}\langle x,y\rangle(\nabla\lambda_i - t)$ *for all* $x, y \in \Gamma(E_i)$
2. $T(\{x,y\},z) + T(\{z,x\},y) + T(\{y,z\},x) = 0$ *for eigenvectors* x, y, z *with different eigenfunctions* □

The following theorem gives an equivalent characterization of condition 1 of the above proposition which will allow us to more directly study the geometrical properties of CKTs. It was originally Theorem 2 in [CFS06]. Before we state it, we remind the reader of some notation. Given a collection of distributions $(E_i)_{i=1}^k$ satisfying $TM = \bigoplus_{i=1}^k E_i$, then for any vector $x \in \mathfrak{X}(M)$, we have the orthogonal splitting $x = \sum_i x^i$ where each $x^i \in \Gamma(E_i)$.

Theorem 4.4.3 (Geometric Characterization of Orthogonal CKTs [CFS06])
Let T be an orthogonal tensor and let E_i be the eigenspaces corresponding to the eigenfunctions λ_i. Then T is a conformal Killing tensor with conformal factor t iff

1. *The eigenspaces E_i are almost umbilical.*
2. *The mean curvature normals of the eigenspaces satisfy the following equation:*

$$H_i = -\frac{1}{2}\sum_{j\neq i}(\nabla \log|\lambda_i - \lambda_j|)^j \tag{4.4.1}$$

3. *The conformal factor satisfies the following equation:*

$$t = \sum(\nabla\lambda_i)^i$$

4. $T(\{x,y\},z) + T(\{z,x\},y) + T(\{y,z\},x) = 0$ *for eigenvectors* x, y, z *with different eigenfunctions* □

Remark 4.4.4
For a Killing tensor the second condition can be simplified to:

$$H_i = -\frac{1}{2}\sum_{j\neq i}\frac{1}{(\lambda_i-\lambda_j)}(\nabla\lambda_i)^j \qquad \square$$

PROOF By projecting condition 1 in Proposition 4.4.2 onto E_i, we see that $(\nabla\lambda_i - t)^i = 0$, thus

$$t = \sum(\nabla\lambda_i)^i \tag{4.4.2}$$

By projecting condition 1 in Proposition 4.4.2 onto every eigenspace E_j with $j \neq i$, one obtains for $x, y \in \Gamma(E_i)$

$$(\lambda_j - \lambda_i)\{x,y\}^j = \frac{1}{2}g(x,y)(\nabla\lambda_i - t)^j$$

Then summing over $j \neq i$, one obtains

$$\{x,y\}^{\perp i} = g(x,y)\sum_{j\neq i}\frac{1}{2(\lambda_j-\lambda_i)}(\nabla\lambda_i - t)^j$$

Since $t = \sum(\nabla\lambda_i)^i$ the above equation can be written

$$\begin{aligned}\{x,y\}^{\perp i} &= g(x,y)\sum_{j\neq i}\frac{1}{2(\lambda_j-\lambda_i)}(\nabla\lambda_i - \nabla\lambda_j)^j \\ &= \frac{-g(x,y)}{2}\sum_{j\neq i}(\nabla\log|\lambda_i-\lambda_j|)^j\end{aligned}$$

Thus if h_i denotes the second fundamental form of E_i, the above equation is equivalent to the following:

$$h_i(x,y) = \{x,y\}^{\perp i} = g(x,y)H_i \tag{4.4.3}$$

where H_i is given by Eq. (4.4.1). This last equation is equivalent to saying that E_i is almost umbilical with mean curvature normal H_i. Equations (4.4.2) and (4.4.3) are equivalent to condition 1 in Proposition 4.4.2 as we just projected that condition onto all the eigenspaces to derive the above equations. Hence by Proposition 4.4.2, the theorem is proven. ∎

We will now proceed to show that when the eigenspaces are orthogonally integrable, Condition 4 of the above theorem is automatically satisfied. The following lemma can be deduced from a knowledge of rotation coefficients, although we state it for completeness.

Lemma 4.4.5
Suppose $(E_i)_{i=0}^k$ is an integrable net. Then for $x \in \Gamma(E_i)$ and $y \in \Gamma(E_j)$ with $j \neq i$, $\nabla_x y \in \Gamma(E_i \oplus E_j)$. □

PROOF Suppose $z \in \Gamma(E_k)$ where k is different from i, j. Observe that

$$g(\nabla_x y, z) - g(\nabla_y x, z) = g([x, y], z) = 0$$

Also

$$g(\nabla_y x, z) + g(x, \nabla_y z) = \nabla_y g(x, z) = 0$$

The above two equations hold for all permutations of x, y, z. Thus

$$\begin{aligned} g(\nabla_x y, z) = g(\nabla_y x, z) = -g(x, \nabla_y z) = -g(x, \nabla_z y) \\ = g(\nabla_z x, y) = g(\nabla_x z, y) = -g(z, \nabla_x y) \end{aligned}$$

Thus $g(\nabla_x y, z) = 0$. ■

The following corollary gives a version of the above theorem for orthogonal tensors with orthogonally integrable eigenspaces.

Corollary 4.4.6
Suppose T is an orthogonal tensor with orthogonally integrable eigenspaces and let E_i be the eigenspaces corresponding to the eigenfunctions λ_i. Then T is a conformal Killing tensor with conformal factor t iff

1. *The eigenspaces E_i are umbilical.*

2. *The mean curvature normals of the eigenspaces satisfy the following equation:*

$$H_i = -\frac{1}{2} \sum_{j \neq i} (\nabla \log |\lambda_i - \lambda_j|)^j \tag{4.4.4}$$

3. *The conformal factor satisfies the following equation:*

$$t = \sum (\nabla \lambda_i)^i \tag{4.4.5}$$

◇

Remark 4.4.7
The Haantjes theorem (Theorem B.0.19) gives a simple necessary and sufficient condition to check if an orthogonal tensor has orthogonally integrable eigenspaces. □

PROOF Since $A^{E_i} = 0$ for each i, the eigenspaces are almost umbilical iff they are umbilical. Condition 4 of Theorem 4.4.3 is automatically satisfied due to Lemma 4.4.5, hence the result holds by Theorem 4.4.3. ■

Now we use a result from [MRS99] which characterizes twisted products to show that orthogonally integrable CKTs naturally give rise to a twisted product structure.

Corollary 4.4.8 (Conformal Killing tensors induce twisted product nets)
Suppose T is an orthogonal tensor with orthogonally integrable eigenspaces $(E_i)_{i=1}^k$ and associated eigenfunctions $(\lambda_i)_{i=1}^k$. Let $M = \prod_{i=1}^k M_i$ be a connected product manifold locally adapted to the eigenspaces of T. Then T is a CKT iff (M, g) is a twisted product with twist functions ρ_i satisfying the following equation:

$$(\nabla \log \rho_i)^{\perp i} = \frac{1}{2} \sum_{j \neq i} (\nabla \log |\lambda_i - \lambda_j|)^j \tag{4.4.6}$$

◇

PROOF This result follows from the above corollary together with Theorem 3.5.10 (1) and Proposition 3.5.9 (2). ■

Remark 4.4.9
As a direct consequence of the above two corollaries, we have the following. In local coordinates (x^i), a tensor T diagonalized in these coordinates with eigenfunctions (λ^i) (counted with multiplicity) is a conformal Killing tensor with conformal factor t iff the following equations are satisfied:

$$\partial_i \lambda_j = (\lambda_i - \lambda_j)\partial_i \log \left|g^{jj}\right| + t_i \qquad \partial_i \lambda_i = t_i \tag{4.4.7}$$

□

Later on, we will use the above corollary to show how to encode the orthogonal separation of the Hamilton-Jacobi equation in terms of Killing tensors.

The above corollary motivates us to define a *Killing net* (K-net) (resp. *Conformal Killing net* (CK-net)) as the TP-net formed by the eigenspaces of a Killing tensor (resp. conformal Killing tensor) when the eigenspaces are orthogonally integrable. The following lemma shows that CK-nets are a special class of TP-nets. In particular, it will give us a simple way to check when an eigenspace of a CKT is Killing.

Lemma 4.4.10

Suppose $(E_i)_{i=1}^k$ is an orthogonally integrable CK-net and let λ_i be the associated eigenfunctions. If $E_i^\perp$ is geodesic, then E_i is spherical. □

PROOF Suppose $x \in \hat{\mathfrak{X}}(M_i)$ and $y \in \hat{\mathfrak{X}}(M_j)$ where $j \neq i$. Recall that this implies $[x, y] = 0$. Then by Eq. (4.4.4)

$$\begin{aligned} x \langle H_i, y \rangle &= -\frac{1}{2} x \langle \nabla \log |\lambda_i - \lambda_j|, y \rangle \\ &= -\frac{1}{2} xy \log |\lambda_i - \lambda_j| \\ &= -\frac{1}{2} yx \log |\lambda_i - \lambda_j| \\ &= -\frac{1}{2} y \langle \nabla \log |\lambda_i - \lambda_j|, x \rangle \\ &= y \langle H_j, x \rangle \end{aligned}$$

Now, since $E_i^\perp$ is geodesic, one can show that $H_j^i = 0$ for $j \neq i$. This can be seen for example, by working in a local twisted product given by Corollary 4.4.8 and then using Proposition 3.5.9 (3). Hence by the above calculation, $x \langle H_i, y \rangle = y \langle H_j, x \rangle = y \langle H_j^i, x \rangle = 0$. Thus

$$\begin{aligned} \langle \nabla_x H_i, y \rangle &= x \langle H_i, y \rangle - \langle H_i, \nabla_x y \rangle \\ &= - \langle H_i, \nabla_y x \rangle \\ &= \langle \nabla_y H_i, x \rangle - y \langle H_i, x \rangle \\ &= \langle \nabla_y H_i, x \rangle \\ &= 0 \end{aligned}$$

where the last line follows since $E_i^\perp$ is geodesic. Hence $\langle \nabla_x H_i, y \rangle = 0$ for all $x \in \Gamma(E_i)$ and $y \in \Gamma(E_i^\perp)$, thus E_i is spherical. ■

The following corollary allows us to determine the geometry of the eigenspaces of a CKT with orthogonally integrable eigenspaces using its eigenfunctions.

Corollary 4.4.11

Suppose T is a CKT with conformal factor t and orthogonally integrable eigenspaces $(E_i)_{i=1}^k$.

1. *E_i is Killing iff*

$$(\nabla \lambda_j)^i = t^i \quad \textit{for all } j \neq i$$

2. E_i is geodesic iff

$$(\nabla\lambda_i)^j = t^j \quad \textit{for all } j \neq i$$

In particular for a KT, E_i is Killing iff all the eigenfunctions are independent of E_i and E_i is geodesic iff λ_i is a constant. ♢

Proof This follows from the above lemma together with Corollary 4.4.6 and the definitions of Killing and geodesic distributions. ■

From the above corollary, it follows immediately that if M admits a KT with orthogonally integrable eigenspaces $\mathcal{E} = (E_i)_{i=0}^k$ and respective eigenfunctions $(\lambda_i)_{i=0}^k$ such that λ_0 is constant and λ_i depends only on E_0 for each $i > 0$, then $\mathcal{E}$ is a WP-net. One can easily use Corollary 4.4.6 and Corollary 4.4.8 to show conversely that any WP-net admits a KT. Although in the next section, we will give a different proof of this fact (see Corollary 4.5.3).

We can also deduce the following important fact from Corollary 4.4.8 [Ben93].

Proposition 4.4.12
Let K, J be Killing tensors. Suppose there exists an orthogonal web $(E_i)_{i=1}^n$ such that K and J are simultaneously diagonalized in any coordinates adapted to this web. Then

$$[K, J] = 0$$ □

Proof The proof follows by a straightforward calculation. Let (x^i) be local coordinates adapted to $(E_i)_{i=1}^n$ and (λ_i) (resp. (μ_i)) be the eigenfunctions of K (resp. J) counted with multiplicity. In these coordinates the remark following Corollary 4.4.8 gives the equations satisfied by the eigenfunctions of these KTs. Using Eq. (4.2.4), we calculate the possibly non-zero terms of the Schouten bracket as follows[1]:

$$\begin{aligned}
K^{jj}\partial_j J^{kk} - J^{jj}\partial_j K^{kk} &= g^{jj}(\lambda_j\partial_j(\mu_k g^{kk}) - \mu_j\partial_j(\lambda_k g^{kk})) \\
&= g^{jj}(\lambda_j[(\partial_j\mu_k)g^{kk} + \mu_k\partial_j g^{kk}] - \mu_j[(\partial_j\lambda_k)g^{kk} + \lambda_k\partial_j g^{kk}]) \\
&= g^{jj}g^{kk}(\lambda_j[\partial_j\mu_k + \mu_k\partial_j \log\left|g^{kk}\right|] - \lambda_j\mu_j\partial_j \log\left|g^{kk}\right| \\
&+ \lambda_j\mu_j\partial_j \log\left|g^{kk}\right| - \mu_j[\partial_j\lambda_k + \lambda_k\partial_j \log\left|g^{kk}\right|]) \\
&= g^{jj}g^{kk}(\lambda_j[\partial_j\mu_k + (\mu_k - \mu_j)\partial_j \log\left|g^{kk}\right|] \\
&- \mu_j[\partial_j\lambda_k + (\lambda_k - \lambda_j)\partial_j \log\left|g^{kk}\right|]) \\
&\overset{(4.4.7)}{=} 0
\end{aligned}$$

[1]Note that there is no sum on the index j.

Thus it follows by Eq. (4.2.4) that K and J Schouten commute. ■

The above proposition has a converse, given as follows.

Proposition 4.4.13 ([KMJ80])
Suppose $K_1, \ldots, K_n$ are point-wise independent Killing tensors which pair-wise Schouten commute. Suppose $\mathcal{E} = (E_i)_{i=1}^n$ is an orthogonal net which simultaneously diagonalizes these tensors. Then $\mathcal{E}$ is integrable, i.e. it is an orthogonal web. □

Remark 4.4.14
This fact was originally discovered in [KMJ80]. A proof of this result can be found in [BCR02], where it was shown that the assumption that the tensors are Killing tensors is redundant. An obvious possible generalization is to replace n in the above proposition with some positive integer $k \leq n$. □

If D is a distribution then we denote by $S^p(D)$ the set of symmetric contravariant tensors of valence p over D, i.e. each element $T \in S^p(D)$ can locally be written as a sum of p-fold symmetrized products of elements in $\Gamma(D)$. The following proposition on restriction of CKTs to submanifolds will be of use later on.

Proposition 4.4.15 (Restriction of CKTs to Invariant Submanifolds)
Let T be a CKT with conformal factor t and suppose D is an integrable non-degenerate T-invariant distribution. If $\tilde{M}$ is an integral manifold of D regarded as a pseudo-Riemannian manifold with the induced metric, then T restricts to a CKT on $\tilde{M}$ with the induced conformal factor. □

Proof By hypothesis $TM = D \ominus D^\perp$, hence we can write

$$
\begin{aligned}
T &= T_D + T_{D^\perp} \\
t &= t_D + t_{D^\perp} \\
G &= G_D + G_{D^\perp}
\end{aligned}
$$

Let $\iota : \tilde{M} \to M$ be the inclusion map, then note that $T_D = \iota_* \tilde{T}$ for some $\tilde{T} \in S^2(\tilde{M})$. Similar equations hold for t_D and G_D. Thus we observe that the following equation holds over $\tilde{M}$, $[T_D, G_D] = [\iota_* \tilde{T}, \iota_* \tilde{G}] = \iota_* [\tilde{T}, \tilde{G}]$ by naturality of the Schouten bracket. In particular, we see that $[T_D, G_D] \in S^2(D)$. Now

$$
[T, G] = [T_D, G_D] + [T_D, G_{D^\perp}] + [T_{D^\perp}, G_D] + [T_{D^\perp}, G_{D^\perp}]
$$

also

$$t \odot G = t_D \odot G_D + t_D \odot G_{D^\perp} + t_{D^\perp} \odot G_D + t_{D^\perp} \odot G_{D^\perp}$$

By projecting onto $S^2(D)$ we find that $[T_D, G_D] = -2t_D \odot G_D$, thus $[\tilde{T}, \tilde{G}] = -2\tilde{t} \odot \tilde{G}$ by injectivity of ι_*. ∎

4.5 Killing tensors in Warped Products

In the previous section we have seen that a multidimensional eigenspace of an orthogonally integrable Killing tensor is necessarily umbilical. The ideal case where this eigenspace is Killing is amenable to analysis. We will also see later on that this case is important to the study of certain Killing tensors in spaces of constant curvature.

In this section we give conditions under which a tensor $K \in S^2(M)$ that admits a K-invariant Killing distribution is a KT. Our first application of this result is to find necessary and sufficient conditions for extending Killing tensors defined on the geodesic and spherical factors of a warped product. These results are very useful for constructing Killing tensors.

The following lemma won't be directly used but it's useful to keep it in mind for proofs to come.

Lemma 4.5.1 (Schouten bracket on Product Manifolds)
Let $M = B \times F$ be a product manifold and suppose $K \in \hat{S}^p(B)$, $G \in \hat{S}^q(F)$. Then the following holds:

$$[K, G] = 0$$ □

Proof This follows from the naturality of the Schouten bracket, i.e. the proof is similar to that when K and G are vector fields. ∎

In the follow proposition we will characterize KTs in warped products.

Proposition 4.5.2 (Killing tensors in Warped Products)
Suppose $K \in S^2(M)$ and D is a K-invariant Killing distribution. Let $B \times_\rho F$ be a local warped product adapted to the WP-net $(D^\perp, D)$ with contravariant metric $G = G_0 + \kappa G_1$ where $\kappa := \rho^{-2}$.

Then K is a KT iff there exist KTs $K' \in S^2(B)$, $\tilde{K} \in S^2(F)$ and $t \in \mathcal{F}(B)$ such that the following equations hold:

$$\begin{aligned} K &= K' + tG_1 + \tilde{K} \\ dt &= K' d\kappa \end{aligned} \tag{4.5.1}$$

Furthermore $\tilde{K}$ is also a KT on $B \times_\rho F$. ♢

PROOF By hypothesis, we can write $K = K_0 + K_1$ where $K_0 \in S^2(D^\perp)$ and $K_1 \in S^2(D)$. Thus,

$$\begin{aligned} [K, G] &= [K_0 + K_1, G_0 + \kappa G_1] \\ &= [K_0, G_0] + [K_0, \kappa G_1] + [K_1, G_0] + [K_1, \kappa G_1] \\ &\overset{(4.2.6)}{=} [K_0, G_0] + \kappa[K_0, G_1] + 2K_0(\mathrm{d}\kappa) \odot G_1 + [K_1, G_0] + \kappa[K_1, G_1] \end{aligned}$$

Note that $[K_0, G_0] \in S^3(D^\perp)$ (see Eq. (4.2.4)), then by linear independence, $[K, G] = 0$ iff

$$\begin{aligned} [K_0, G_0] &= 0 & (4.5.2) \\ [K_1, G_1] &= 0 \\ \kappa[K_0, G_1] + 2K_0(\mathrm{d}\kappa) \odot G_1 + [K_1, G_0] &= 0 & (4.5.3) \end{aligned}$$

Suppose $(x^i) = (x^a, x^\alpha)$ are local coordinates adapted to the warped product $B \times_\rho F$. We denote coordinates for B using Latin letters such as a, b, coordinates for F using Greek letters such as α, β and the letters i, j, k are reserved for generic indices. Let $X_i := \partial_i$, then by Eq. (4.2.4) we have:

$$\begin{aligned} [K_1, G_0] &= 2(K_1 \mathrm{d}G_0^{jk} - G_0 \mathrm{d}K_1^{jk}) \odot X_j \odot X_k \\ &= -2G_0 \mathrm{d}K_1^{\alpha\beta} \odot X_\alpha \odot X_\beta \end{aligned}$$

and

$$\begin{aligned} [K_0, G_1] &= 2(K_0 \mathrm{d}G_1^{jk} - G_1 \mathrm{d}K_0^{jk}) \odot X_j \odot X_k \\ &= -2G_1 \mathrm{d}K_0^{ab} \odot X_a \odot X_b \end{aligned}$$

Thus by linear independence, Eq. (4.5.3) is satisfied iff

$$\begin{aligned} [K_0, G_1] &= 0 \\ 2K_0(\mathrm{d}\kappa) \odot G_1 + [K_1, G_0] &= 0 \end{aligned}$$

The first of the above equations are satisfied iff $G_1 \mathrm{d}K_0^{ab} = 0$, i.e. $K_0 \in \hat{S}^2(B)$. The second becomes

$$2K_0(\mathrm{d}\kappa) \odot G_1 + [K_1, G_0] = 2(K_0(\mathrm{d}\kappa)G_1^{\alpha\beta} - G_0 \mathrm{d}K_1^{\alpha\beta}) \odot X_\alpha \odot X_\beta$$

which is identically zero iff

$$\begin{aligned} \mathrm{d}^0 K_1^{\alpha\beta} &= K_0(\mathrm{d}\kappa)G_1^{\alpha\beta} \\ \Rightarrow \mathrm{d}^0(K_0(\mathrm{d}\kappa)) &= 0 \quad \text{by non-degeneracy of } G_1 \end{aligned}$$

where d^0 is d followed by the (point-wise) orthogonal projection onto $(D^\perp)^*$. So, $K_0(\mathrm{d}\kappa) = \mathrm{d}^0 t$ for some $t \in \mathcal{F}(B)$, thus

$$\mathrm{d}^0 K_1^{\alpha\beta} = \mathrm{d}^0(tG_1^{\alpha\beta})$$

Hence $\tilde{K}^{\alpha\beta} := K_1^{\alpha\beta} - tG_1^{\alpha\beta} \in \mathcal{F}(F)$, i.e. $\tilde{K} \in \hat{S}^2(F)$. Equation (4.5.3) is satisfied iff $\tilde{K}$ is a KT on F and Eq. (4.5.2) is satisfied iff K_0 is a KT on B. Finally if we let $K' := K_0$, the result follows. The last statement that $\tilde{K}$ is a KT on $B \times_\rho F$ can be readily verified from the above equations. ∎

Two important special cases of the above proposition are the following:

1. By taking $K', t = 0$, we see that $\tilde{K} \in \hat{S}^2(F)$ is a KT on F iff it is a KT on $B \times_\rho F$.

2. By taking $\tilde{K} = 0$ we find that a necessary and sufficient condition for $K' \in S^2(B)$ to be lifted into a KT on $B \times_\rho F$ is that

$$\mathrm{d}(K'\mathrm{d}\kappa) = 0$$

We can also prove the following corollary cf. [Jel00], which shows that a WP-net is a K-net.

Corollary 4.5.3 (WP-nets always admit KTs)
A pseudo-Riemannian manifold M admits a WP-net $\mathcal{E} = (E_i)_{i=0}^k$ iff there exists a KT, K on M whose eigen-net is $\mathcal{E}$ and the corresponding eigenfunctions λ_i satisfy:

1. *λ_0 is a constant*

2. *λ_i depends only on E_0 for each $i > 0$*

Furthermore if such a KT exists, then the warping functions can locally be chosen to satisfy the following equation $\rho_i^2 = |\lambda_i - \lambda_0|$ for $i > 0$. □

Proof If M admits a KT with orthogonally integrable eigenspaces and eigenfunctions satisfying the above conditions, then it follows from Corollary 4.4.11 that its eigenspaces form a WP-net.

Conversely suppose $\mathcal{E}$ is a WP-net, and suppose $G = G_0 + \sum_{i=1}^k \kappa_i G_i$ is an adapted warped product metric. The above proposition shows that each G_i for $i > 0$ is a KT on M. Hence for each i if we choose $c_i \in \mathbb{R}$, then $K := c_0 G + \sum_{i=1}^k c_i G_i$ is a KT on M. Thus, locally we can always choose the c_i such that K is a KT with eigenspaces equal to $\mathcal{E}$ and clearly the eigenfunctions satisfy the above conditions.

Now if such a KT exists, by Eq. (4.4.6) in Corollary 4.4.8, we have for $i > 0$

$$\begin{aligned}(\nabla \log \rho_i^2)^{\perp i} &= \sum_{j \neq i} (\nabla \log |\lambda_i - \lambda_j|)^j \\ &= \nabla \log |\lambda_i - \lambda_0|\end{aligned}$$

Thus it follows that locally we can choose the warping functions as stated. ∎

The following corollary follows immediately by inductively applying Proposition 4.5.2.

Corollary 4.5.4
Suppose $\mathcal{E} = (D_i)_{i=0}^k$ is a WP-net and K is KT with D_i a K-invariant distribution for $i = 1, \ldots, k$. Let $M = M_0 \times_\rho \prod_{i=1}^k M_i$ be a local warped product adapted to $\mathcal{E}$. Then in contravariant form, K can be decomposed as follows:

$$K = K_0 + \sum_{i=1}^k K_i$$

where each $K_i \in \hat{S}^2(M_i)$ is a KT for $i = 1, .., k$. Furthermore K_0 is a KT and each D_i is an eigenspace of K_0 for $i = 1, .., k$ (see Corollary 4.4.11 for more on K_0). ♢

Chapter 5

Hamilton-Jacobi separation via Characteristic Killing tensors

In this chapter we present the geometric theory of the Hamilton-Jacobi equation, its separation, and the intrinsic characterization of separation. In the first section, we introduce Hamilton-Jacobi theory. This section is mainly included for completeness and not necessary to read later chapters. In Section 5.2 we introduce the separation of variables method for the Hamilton-Jacobi equation, and present the Levi-Civita equations which characterize separable coordinates. In the last two sections, we present the intrinsic characterization of separation for geodesic and natural Hamiltonians respectively. In these sections we also add more details for the case when the separable coordinates are decomposable in a warped product. In this chapter we will be using the formalism introduced in Section 4.1. In the first two sections of this chapter, we consider an arbitrary Hamiltonian H. Finally, we note that this chapter can be read fairly lightly if one is not particularly interested in the general theory.

5.1 Hamilton-Jacobi Theory

We present the fundamental results of Hamilton-Jacobi theory. We will not go into much detail, just presenting the results of interest to us. Motivation for this theory can be found in classical references such as [LL76]. The exposition here mainly follows [Woo75], with help from [Ben89] and [Arn89].

Given a 1-form $\phi \in A_1(M)$, we denote by $\Phi : M \to T^*M$, the associated cross-section of T^*M. We observe that by definition, $\pi \circ \Phi = \mathrm{Id}$. Furthermore ϕ is said to be *closed* if $\mathrm{d}\phi = 0$. A closed 1-form $\phi \in A_1(M)$ is called a *solution* of the Hamilton-Jacobi (HJ) *equation* if [Woo75]

$$H \circ \Phi = E$$

for some $E \in \mathbb{R}$ where H is the Hamiltonian. In canonical coordinates (q, p), locally we can assume that there exists $W \in \mathcal{F}(M)$ such that $\phi = \mathrm{d}W$, then the HJ equation takes its usual form:

$$H(q^i, \frac{\partial W}{\partial q^i}) = E$$

A *local complete solution* of the *Hamilton-Jacobi equation* is a diffeomorphism $\Psi : U \times V \subseteq M \times \mathbb{R}^n \to T^*M$ (onto some open subset of T^*M), such that for each $v \in V$, the restriction $\Psi_v : U \to T^*M$ defines a closed 1-form $\psi_v \in A_1(M)$ which is a solution of the Hamilton-Jacobi equation [Woo75].

In local canonical coordinates (q, p), Ψ takes the form

$$\Psi(q^i, c^j) = (q^i, \frac{\partial W}{\partial q^i})$$

The condition that Ψ is locally invertible is equivalent to the condition

$$\det(\frac{\partial^2 W}{\partial q^i \partial c_j}) \neq 0$$

which recovers the standard definition of a complete solution [Ben89]. We can now state the central theorem of Hamilton-Jacobi theory:

Theorem 5.1.1 (Jacobi Theorem)
Let Ψ be a local complete solution of the Hamilton-Jacobi equation as above. Then the Hamiltonian admits n functionally independent first integrals $F_1, \ldots, F_n$ which Poisson commute. In fact, if $\pi_2 : U \times V \to V$ is the projection and if $k_1, \ldots, k_n \in \mathcal{F}(V)$ are functionally independent, then the F_i are given explicitly by:

$$F_i = k_i \circ \pi_2 \circ \Psi^{-1} \qquad \square$$

For the proof we need the following lemmas from [Woo75].

Lemma 5.1.2
If X and Y are tangent vectors to T^*M at some point $p \in T^*M$, then the following hold:

$$\omega(\Phi_*\pi_*X, \Phi_*\pi_*Y) = d\phi(\pi_*X, \pi_*Y)$$
$$\omega(\Phi_*\pi_*X, Y) + \omega(X, \Phi_*\pi_*Y) = \omega(X, Y) + d\phi(\pi_*X, \pi_*Y) \qquad \square$$

Proof To prove the first equation, we note first note that $\Phi^*\theta = \phi$ (θ is the Liouville form). Indeed, for $p \in M$ and $X \in T_pM$ then

$$\Phi^*\theta(X) = \theta(\Phi_* X) = \phi(\pi_* \Phi_* X) = \phi(X)$$

where the last equality follows from the fact that $\pi_*\Phi_* = (\pi \circ \Phi)_* = \mathrm{Id}$. Thus

$$\begin{aligned} \omega(\Phi_*\pi_*X, \Phi_*\pi_*Y) &= (\Phi^*\omega)(\pi_*X, \pi_*Y) \\ &= \mathrm{d}(\Phi^*\theta)(\pi_*X, \pi_*Y) \\ &= \mathrm{d}\phi(\pi_*X, \pi_*Y) \end{aligned}$$

Now, for the second equation we first make the following observation, if $p \in T^*M$ then for any $X, Y \in T_pT^*M$ satisfying $\pi_*X = \pi_*Y = 0$, we have by definition of ω that:

$$\omega(X, Y) = 0$$

We also note that $X' := X - \Phi_*\pi_*X$ satisfies $\pi_*X' = 0$. Hence the above equation applied to the vectors X' and Y' implies:

$$\omega(X, Y) = \omega(\Phi_*\pi_*X, Y) + \omega(X, \Phi_*\pi_*Y) - \omega(\Phi_*\pi_*X, \Phi_*\pi_*Y)$$

The second equation then follows from the above and first equations. ■

Lemma 5.1.3

If $\phi \in A_1(M)$ is closed and if $F, G \in \mathcal{F}(T^*M)$ satisfy:

$$F \circ \Phi = \text{const}, \qquad G \circ \Phi = \text{const}$$

Then $\{F, G\}$ vanishes on $\Phi(M)$. □

Proof This is a consequence of the second equation in the above lemma. Indeed, since ϕ is closed, we have from the above lemma that

$$\{F, G\} = \omega(X_F, X_G) = \omega(\Phi_*\pi_*X_F, X_G) + \omega(X_F, \Phi_*\pi_*X_G) \tag{5.1.1}$$

Then by hypothesis, for points in $\Phi(M)$, we observe that

$$\omega(\Phi_*\pi_*X_F, X_G) = (\Phi_*\pi_*X_F)g = \pi_*X_F(G \circ \Phi) = 0$$

Similarly, $\omega(X_F, \Phi_* \pi_* X_G) = 0$, hence the result follows from Eq. (5.1.1). ■

We are now ready to prove the Jacobi theorem:

PROOF (THEOREM 5.1.1) Fix $f, g \in \mathcal{F}(V)$ then let $F := f \circ \pi_2 \circ \Psi^{-1}$ and $G := g \circ \pi_2 \circ \Psi^{-1}$. As usual let H be the Hamiltonian.

Fix $v \in V$. By construction it follows that both F and G are constant on $\Psi_v(M)$. Since Ψ_v is induced by a closed 1-form on M, the above lemma implies that $\{F, G\} = 0$ on $\Psi_v(M)$. By assumption, $H \circ \Psi_v$ is constant, hence the same argument shows that $\{F, H\} = 0$ on $\Psi_v(M)$. Since Ψ is a bijection onto its image, $\mathrm{Im}(\Psi)$, $\{F, G\} = \{F, H\} = 0$ on $\mathrm{Im}(\Psi)$.

The conclusions of the theorem immediately follow from these observations. ■

A natural question arises: Under what conditions does the Jacobi theorem have a converse? More precisely, if $F_1, \ldots, F_n \in \mathcal{F}(T^*M)$ are functionally independent commuting first integrals, then when do these integrals arise from a complete solution of the Hamilton-Jacobi equation?

It turns out that a necessary and sufficient condition is that $\{\pi_* X_{F_1}, \ldots, \pi_* X_{F_n}\}$ are point-wise independent [Woo75]. Motivated by this, we say that functions $F_1, \ldots, F_n \in \mathcal{F}(T^*M)$ are *vertically independent* if $\{\pi_* X_{F_1}, \ldots, \pi_* X_{F_n}\}$ are point-wise independent. Note that this condition implies the functions are functionally independent. Furthermore, in canonical coordinates, this condition is equivalent to:

$$\det(\frac{\partial F_i}{\partial p_j}) \neq 0 \tag{5.1.2}$$

To show that this is in fact necessary, suppose Ψ is a complete solution of the HJ equation. If $\varphi := \Psi^{-1} : T^*M \to M \times \mathbb{R}^n$, in canonical coordinates, we see that:

$$\varphi(q, p) = (q, F(q, p))$$

Then by the inverse function theorem, φ is locally invertible iff Eq. (5.1.2) holds.

Conversely, suppose $F_1, \ldots, F_n \in \mathcal{F}(T^*M)$ are functionally independent commuting first integrals. These functions define a foliation, L, whose leaves are given by:

$$L_v = \{\, p \in T^*M \ : \ F_i(p) = v_i, \ i = 1, \ldots, n\}$$

for a constant vector $v \in \mathbb{R}^n$. The following theorem shows that these integrals arise from a complete solution to the HJ equation:

Theorem 5.1.4 (Complete solutions via First integrals)
If $F_1, \ldots, F_n \in \mathcal{F}(T^*M)$ are vertically independent commuting first integrals, then these integrals arise from a local complete solution to the HJ equation. □

Proof Let L be the foliation of T^*M obtained from these functions as defined above. Then since $X_{F_i}F_j = \{F_i, F_j\} = 0$ and because of the functional independence condition, $\{X_{F_1}, \ldots, X_{F_n}\}$ form a (point-wise) basis for TL. Take local canonical coordinates (q, p) for T^*M, then consider the map $\varphi : T^*M \to M \times \mathbb{R}^n$ given by

$$\varphi(q, p) = (q, F_1(q, p), \ldots, F_n(q, p))$$

By the inverse function theorem, this map is invertible iff

$$\det(\frac{\partial F_i}{\partial p_j}) \neq 0$$

which is precisely the condition that $\{\pi_* X_{F_1}, \ldots, \pi_* X_{F_n}\}$ are point-wise independent. Let $\Psi : M \times \mathbb{R}^n \to T^*M$ be the inverse, then it must have the form

$$\Psi(q, F) = (q, w(q, F))$$

Now observe that for any $v \in \mathbb{R}^n$ that $L_v = \Psi_v(M)$. Hence the symplectic form ω vanishes on $\Psi_v(M)$ since $\omega(X_{F_i}, X_{F_j}) = \{F_i, F_j\} = 0$. It follows by the first equation in Lemma 5.1.2 that the form $w_i \mathrm{d}q^i$ is closed. We must have that $H \circ \Psi$ is a constant, since $X_{F_i}H = \{F_i, H\} = 0$. Thus Ψ is a (local) complete solution of the Hamilton-Jacobi equation which induces the integrals F_i. ∎

Liouville first showed that one can relax the vertical independence condition to functional independence in the hypothesis of the above theorem and obtained a method to integrate Hamilton's equations by quadratures [Arn89]. This method for integrating Hamilton's equations is known as *Liouville integrability*. Also, this formulation of a complete solution of the Hamilton-Jacobi equation is more intuitive and will help motivate the intrinsic characterization of separation which will be given later on.

We end this section with a remark on a characterization of a complete solution of the HJ equation in contemporary terms [Ben89]. First, a submanifold of T^*M is called *Lagrangian* if the symplectic form vanishes over it and it has maximal dimension n [Lee12]. The proof of the above theorem shows that the leaves of the foliation induced by the functions $F_1, \ldots, F_n \in \mathcal{F}(T^*M)$ arising from a complete solution to the HJ equation are Lagrangian, level sets of the Hamiltonian, and transverse to the fibers of T^*M. It's an easy exercise to show that this locally characterizes a complete solution to the HJ equation. From this characterization, we can define when two complete solutions of the Hamilton-Jacobi equation are equivalent [Ben91]:

Definition 5.1.5

Two complete solutions of the Hamilton-Jacobi are called *equivalent* if on the subset where they are both defined, their induced Lagrangian foliations coincide. □

5.2 Separation of the Hamilton-Jacobi equation

The standard method for solving the Hamilton-Jacobi equation is the method of separation of variables. In this section we will briefly describe precisely what this means and then obtain the Levi-Civita equations, which characterize separable coordinates. In this section we will work in canonical coordinates (q^i, p_j) on T^*M and all considerations are local. Furthermore, we use the following notations:

$$\partial_i = \frac{\partial}{\partial q^i} \qquad \qquad \partial^j = \frac{\partial}{\partial p_j}$$

In canonical coordinates, a local complete solution to the Hamilton-Jacobi equation is a (generating) function $W(q, c)$ where $c = (c_1, \ldots, c_n)$ are constants of integration, satisfying:

$$H(q^i, \frac{\partial W}{\partial q^i}) = E$$

for some $E \in \mathbb{R}$ and the completeness condition:

$$\det(\frac{\partial^2 W}{\partial q^i \partial c_j}) \neq 0$$

Such a solution is called a *separable solution* if it additionally has the form:

$$W = W_1(q^1, c_j) + W_2(q^2, c_j) + \cdots + W_n(q^n, c_j) \tag{5.2.1}$$

The idea behind this ansatz is that, if one can break up the Hamiltonian as follows:

$$H_1(q^1, \frac{\partial W_1}{\partial q^1}) + H_2(q^2, \frac{\partial W_2}{\partial q^2}) + \cdots + H_n(q^n, \frac{\partial W_n}{\partial q^n}) = E$$

then one obtains the following system of decoupled ODEs:

$$\begin{gathered} H_1(q^1, \frac{\partial W_1}{\partial q^1}) = E_1 \\ \vdots \\ H_n(q^n, \frac{\partial W_n}{\partial q^n}) = E_n \end{gathered}$$

which can be integrated by quadratures to obtain W, provided $\partial^i H \neq 0$. See [Arn89; LL76] for some classical examples on explicitly separating the HJ equation. This is only the very start of our work, and so examples at this stage are largely irrelevant.

Now the natural question is: when does the Hamilton-Jacobi equation admit a separable solution? We first need a definition:

Definition 5.2.1 (Separable Coordinates)
A coordinate system (q^i) for M is called *separable* if the Hamilton-Jacobi equation admits a separable solution in the induced canonical coordinates (q^i, p_j) on T^*M and $\partial^i H \neq 0$. These coordinates are called *orthogonally separable* if the metric is orthogonal, i.e. it satisfies $g_{ij} = 0$ for $i \neq j$. □

An important observation to be made is the following. If (q^i) are separable coordinates, then any coordinate system $(\bar{q}^i)$ having a transformation formula of the form $(\bar{q}^1, \ldots, \bar{q}^n) = (f_1(q^1), \ldots, f_n(q^n))$ is also separable. Hence the separable property is dependent only on the web formed by the coordinates (q^i). Motivated by this observation, we define a *separable web* to be the orthogonal web formed by orthogonally separable coordinates.

The next step is to obtain the Levi-Civita equations. These equations originally obtained by Levi-Civita in [LC04] give necessary and sufficient conditions to determine if a given coordinate system on M is separable.

Theorem 5.2.2 (Levi-Civita equations [LC04])
Suppose H is a Hamiltonian on T^*M. Let (q^i) be local coordinates for M and (q^i, p_j) be the induced canonical coordinates on T^*M. Then the coordinates (q^i) are separable iff the following equations[1] are satisfied:

$$\partial^i H \partial^j H \partial_{ij} H + \partial_i H \partial_j H \partial^{ij} H - \partial^i H \partial_j H \partial_i^j H - \partial^j H \partial_i H \partial_j^i H = 0 \quad (i \neq j)$$

which are called the Levi-Civita equations*.* □

Proof Our proof is a modification of that in [DR07], where it is used in a somewhat different context. See also [Ben91] or [Kal86]. Let $W(q, c)$ be a separable solution of the HJ equation. Then the following equations are satisfied:

$$H(q,p) = E \qquad\qquad p_i = \frac{\partial W}{\partial q^i}$$

Upon differentiating the first of these equations, one obtains:

$$\frac{\partial H}{\partial q^i} + \frac{\partial p_i}{\partial q^i}\frac{\partial H}{\partial p_i} = 0$$

Let $w_i := \dfrac{\partial W}{\partial q^i}$, then w_i satisfies the following system of PDEs

[1] There is no summation over the indices.

$$\frac{\partial w_i}{\partial q^i} = -\frac{\partial_i H}{\partial^i H} = R_i(q,w) \qquad\qquad \frac{\partial w_i}{\partial q^j} = 0 \quad (i \neq j) \tag{5.2.2}$$

This system has a complete solution $w_i(q,c)$ iff the integrability conditions

$$0 = \frac{dR_i}{dq^j} = \frac{\partial R_i}{\partial q^j} + \frac{\partial w_j}{\partial q^j}\frac{\partial R_i}{\partial p_j} = \frac{\partial R_i}{\partial q^j} + R_j\frac{\partial R_i}{\partial p_j} \quad (i \neq j)$$

are satisfied. Upon expanding the right hand side of the above equation, one obtains the Levi-Civita equations. Conversely, assume $w_i(q,c)$ is a complete solution of the system Eq. (5.2.2). Then clearly there exists a function $W(q,c)$ of the form in Eq. (5.2.1) such that $w_i = \frac{\partial W}{\partial q^i}$. The first of Eq. (5.2.2) implies that W is a solution of the HJ equation. Finally, w_i is a complete solution of Eq. (5.2.2) iff

$$\det\left(\frac{\partial^2 W}{\partial q^i \partial c_j}\right) \neq 0$$

i.e. W is a complete solution of the HJ equation. ∎

The Levi-Civita equations evaluated with a natural Hamiltonian in orthogonal coordinates (q^i) are equivalent to the following PDEs:

$$\partial_i\partial_j g^{kk} - \partial_i \log\left|g^{jj}\right| \partial_j g^{kk} - \partial_j \log\left|g^{ii}\right| \partial_i g^{kk} = 0 \quad (i \neq j) \tag{5.2.3a}$$

$$\partial_i\partial_j V - \partial_i \log\left|g^{jj}\right| \partial_j V - \partial_j \log\left|g^{ii}\right| \partial_i V = 0 \quad (i \neq j) \tag{5.2.3b}$$

An important observation to be made here is that separation of the geodesic Hamiltonian is necessary for the separation of a natural Hamiltonian. Thus our main focus will be on the separation of geodesic Hamiltonians. The theory for natural Hamiltonians will be added in afterwards. In the next section we will build on the Levi-Civita equations and obtain an intrinsic characterization of separation for geodesic Hamiltonians.

We now proceed to find an analogue of Theorem 5.1.4 for separable solutions, i.e. characterize these solutions in terms of the first integrals they induce. Benenti has shown in [Ben89, Theorem 2.1] that the correct additional condition is that the integrals be in separable involution. Two first integrals, $F, G \in \mathcal{F}(T^*M)$, are said to be in *separable involution* if there exists coordinates (q^i) on M such that

$$\{F,G\}_i := \partial^i F \partial_i G - \partial_i F \partial^i G = 0 \quad i = 1,\ldots,n$$

in the induced canonical coordinates on T^*M. We first show that the integrals generated from a separable solution are in separable involution:

Proposition 5.2.3 ([Ben89])
Suppose $F_1, \dots, F_n \in \mathcal{F}(T^*M)$ are the first integrals generated by a separable solution to the Hamilton-Jacobi equation. Then with respect to the associated separable coordinates (q^i), we have that

$$\{F_i, F_j\}_k = 0 \qquad \square$$

Proof See the proof of Theorem 2.1 in [Ben89]. ∎

The following theorem is the analogue of Theorem 5.1.4 for separable solutions.

Theorem 5.2.4 (Separable solutions via First integrals [Ben89])
If $F_1, \dots, F_n \in \mathcal{F}(T^*M)$ are vertically independent first integrals in separable involution with respect to coordinates (q^i) for M, then these coordinates are separable and they generate these integrals via a separable solution to the HJ equation. □

Proof See the proof of Theorem 2.1 in [Ben89]. ∎

5.3 Intrinsic characterization of Separation for geodesic Hamiltonians

In this section we consider a geodesic Hamiltonian H. We will present an intrinsic characterization of separation for this Hamiltonian. Eisenhart was the first to obtain this characterization in [Eis34]. Although we will follow a more recent proof of this fact by Benenti in [Ben97] which uses the Levi-Civita equations.

This characterization of separation is motivated by the characterization of a complete solution of the Hamilton-Jacobi equation in terms of commuting first integrals $F_1, \dots, F_n$, see Theorem 5.1.4. We assume that the Hamilton-Jacobi equation admits a separable solution. Stackel proved remarkably in [Sta93], that each of these integrals are necessarily quadratic in momenta. Hence, in the notation of Section 4.2, there exist Killing tensors $K_1, \dots, K_n$ on M such that each $F_i = \frac{1}{2}E_{K_i}$. It was additionally shown in [Sta93] that these KTs are simultaneously diagonalized in the separable coordinates. Note that all the properties satisfied by the integrals F_i in Theorem 5.1.4 translate to properties satisfied by the KTs on M. Indeed, the tensors $K_1, \dots, K_n \in S^2(M)$:

1. Are Killing tensors which pair-wise Schouten commute.

2. Are point-wise independent.

3. Are simultaneously diagonalized in a coordinate system (q^i).

Note that the last point is due to the separability condition, as mentioned before. It follows by Theorem 5.1.4 that such a set of tensors induce a complete solution to the Hamilton-Jacobi equation on T^*M. We will prove later in this section that the above conditions are sufficient to ensure that the solution is separable. But for now, the key observation to be made is that among the KTs in the vector space spanned by $K_1, \ldots, K_n$, at least one of them, say K, locally has simple eigenvalues. This follows immediately by using the point-wise independence of these tensors in the coordinate system (q^i) which diagonalizes them. Furthermore the eigenspaces of K form an orthogonal web which is identical to the separable web formed by the separable coordinates. This motivates the following definition:

Definition 5.3.1
A *characteristic Killing tensor (ChKT)* is Killing tensor with point-wise real simple eigenvalues and orthogonally integrable eigenspaces. □

The KT, K, is a ChKT. The following theorem shows that the existence of a ChKT is necessary and sufficient for separation and thereby gives an intrinsic characterization of separation.

Theorem 5.3.2 (Orthogonal Separation of Geodesic Hamiltonians [Eis34])
The geodesic Hamiltonian is separable in an orthogonal web $\mathcal{E}$ iff there exists a ChKT whose eigenspaces form $\mathcal{E}$. □

Proof Our proof follows that in [Ben97, proposition 3]. Suppose K is a ChKT and (q^i) are coordinates adapted to the eigenspaces of K. If $\lambda_1, \ldots, \lambda_n$ are the eigenfunctions of K, it follows by Eq. (4.4.7) that they satisfy the following equations in these coordinates:

$$\partial_i \lambda_j = (\lambda_i - \lambda_j)\partial_i \log \left|g^{jj}\right| \qquad \partial_i \lambda_i = 0 \tag{5.3.1}$$

The integrability conditions of the above system of PDEs are:

$$(\lambda_i - \lambda_j)(\partial_i \partial_j g^{kk} - \partial_i \log \left|g^{jj}\right| \partial_j g^{kk} - \partial_j \log \left|g^{ii}\right| \partial_i g^{kk}) = 0 \quad (i \neq j)$$

Since K has simple eigenfunctions, one observes that the above equations are identical to the Levi-Civita equations for a geodesic Hamiltonian Eq. (5.2.3a). Hence it follows by Theorem 5.2.2 that the coordinates (q^i) are separable iff there exists a ChKT diagonalized in the coordinates. ■

This gives our first and most fundamental characterization of separation. Although, this characterization is still computationally difficult to work with. One simplification is offered by Haantjes theorem (Theorem B.0.19), which gives a simpler necessary and sufficient condition on an orthogonal tensor to determine if it has orthogonally integrable eigenspaces. Also on a Riemannian manifold (or in dimensions less than four), one can use the discriminant to check if a linear operator has simple eigenvalues, see for example [Ben04, Theorem 3.6]. In spaces of constant curvature, we will not use this theorem directly to obtain parameterizations of the separable webs. This will depend on deeper insights which we will discuss in later chapters.

We have the following corollary of the proof:

Corollary 5.3.3
Suppose $\mathcal{E} = (E_i)_{i=1}^n$ is a separable web. Then there exists an n dimensional space, $\mathcal{K}$, of Killing tensors which pair-wise Schouten commute, are point-wise independent and simultaneously diagonalized in $\mathcal{E}$. A necessary and sufficient condition for an arbitrary KT, K, to be an element of $\mathcal{K}$ is that it is diagonalized in the separable web $\mathcal{E}$.

Furthermore, the induced quadratic first integrals on T^*M are precisely those guaranteed by the Jacobi theorem. □

Proof In adapted coordinates (q^i), because the integrability conditions of Eq. (5.3.1) are satisfied, it follows that there exist point-wise independent KTs $K_1, \ldots, K_n$ which are simultaneously diagonalized in (q^i). It follows by Proposition 4.4.12 that they pair-wise Schouten commute. Furthermore, because of the linearity of Eq. (5.3.1) it follows that the KTs $K_1, \ldots, K_n$ span an n dimensional space $\mathcal{K}$.

If a KT, K, is diagonalized in the coordinates (q^i), then by the uniqueness of the solutions to the PDE system (Eq. (5.3.1)) it follows that $K \in \mathcal{K}$.

To prove the last remark, we must show that the first integrals induced by elements of $\mathcal{K}$ arise from a separable solution to the HJ equation. By Theorem 5.2.4, we only need to show that these first integrals are in separable involution. Let $K, J \in \mathcal{K}$ with eigenfunctions $(\lambda_i)_{i=1}^n$ and $(\mu_i)_{i=1}^n$ respectively, and let $F, G \in \mathcal{F}(T^*M)$ be the induced first integrals (see Eq. (4.2.1)). In the induced canonical coordinates (q^i, p_j) on T^*M, we calculate

$$
\begin{aligned}
\{F, G\}_j &= \partial^j F \partial_j G - \partial_j F \partial^j G \\
&= \frac{1}{2}(\lambda_j g^{jj} p_j \sum_{i=1}^{n} \partial_j(\mu_i g^{ii}) p_i p_i - \mu_j g^{jj} p_j \sum_{i=1}^{n} \partial_j(\lambda_i g^{ii}) p_i p_i) \\
&= \frac{1}{2} \sum_{i=1}^{n} (\lambda_j g^{jj} \partial_j(\mu_i g^{ii}) - \mu_j g^{jj} \partial_j(\lambda_i g^{ii})) p_j p_i p_i \\
&= \frac{1}{2} \sum_{i=1}^{n} (K^{jj} \partial_j(J^{ii}) - J^{jj} \partial_j(K^{ii})) p_j p_i p_i \\
&= \frac{1}{4} \sum_{i=1}^{n} [K, J]^{jii} p_j p_i p_i \\
&= 0
\end{aligned}
$$

The last equality follows from Proposition 4.4.12. ∎

The vector space of KTs, $\mathcal{K}$, in the above corollary is called the *KS-space* associated with the separable web $\mathcal{E}$.

We now give an application of this characterization. It is particularly useful to prove separability of certain warped product metrics. We will consider a generalization of a well known metric from Relativity:

Example 5.3.4 (Separability of The Schwarzschild metric)
This example is from [Ben91, section 5], where more examples from Relativity can be found. Consider the *Reissner-Nordström metric*

$$
\mathrm{d}s^2 = \frac{r^2}{\Delta}\mathrm{d}r^2 - \frac{\Delta}{r^2}\mathrm{d}t^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)
$$

where $\Delta = r^2 + e^2 - 2mr$, which models the gravitational field outside a spherically symmetric body of charge e and mass m. The *Schwarzschild metric* is obtained by setting $e = 0$. We first note that this metric is a warped product

$$
\mathbb{E}^1 \times_{\frac{\Delta}{r^2}} \mathbb{E}^1_1 \times_{r^2} \mathbb{S}^2
$$

where the 2-sphere $\mathbb{S}^2$ is equipped with spherical coordinates (θ, ϕ) and metric $g_2 := \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$. It is well known that the spherical coordinate system is separable, and hence by Theorem 5.3.2 it admits a ChKT K_2 diagonalized in these coordinates. By Proposition 4.5.2, this ChKT can be lifted to KT on M. Similarly, the contravariant metric G_1 (resp. G_2), of $\mathbb{E}^1_1$ (resp. $\mathbb{S}^2$) can be lifted to a KT on M. Hence, locally one can obtain a ChKT (diagonalized in these coordinates) by taking an appropriate linear combination of the KTs in $\mathcal{K} = \mathrm{span}\{G, G_1, G_2, K_2\}$. Thus this metric is separable by

Theorem 5.3.2. We also note that $\mathcal{K}$ is the KS-space associated with these separable coordinates. □

Motivated by Corollary 5.3.3, using Proposition 4.4.13 we can obtain another intrinsic characterization of separation due originally to [KMJ80]:

Theorem 5.3.5 (Orthogonal Separation of Geodesic Hamiltonians II [KMJ80])
Suppose $K_1, \ldots, K_n$ are point-wise independent Killing tensors which pair-wise Schouten commute. Suppose there exists an orthogonal net $\mathcal{E} = (E_i)_{i=1}^n$ which simultaneously diagonalizes these tensors. Then $\mathcal{E}$ is a separable web. □

Proof From Proposition 4.4.13, we see that $\mathcal{E}$ is an orthogonal web. Using the point-wise independence condition one can construct a ChKT in a neighborhood of each point by taking a constant linear combination of the KTs $K_1, \ldots, K_n$. The eigenspaces of this ChKT locally form the net $\mathcal{E}$, hence it follows from Theorem 5.3.2 that $\mathcal{E}$ is separable. ■

In a Riemannian manifold, the above theorem can be strengthened [KMJ80]:

Corollary 5.3.6
Suppose M is a Riemannian manifold, and $K_1, \ldots, K_n$ are point-wise independent Killing tensors which pair-wise Schouten commute and commute as linear operators. Then locally there exists a separable web $\mathcal{E} = (E_i)_{i=1}^n$ which simultaneously diagonalizes these tensors. □

Proof Since the tensors $K_1, \ldots, K_n$ pair-wise commute as linear operators, they can be simultaneously diagonalized at each point. Using the point-wise independence condition, we can assume that there locally exists an orthogonal net $\mathcal{E} = (E_i)_{i=1}^n$ which simultaneously diagonalizes these tensors. Then the result follows from the above theorem. ■

The following example shows that the assumption that M is a Riemannian manifold in the above corollary is necessary:

Example 5.3.7 (Complex Separation [DR07])
Let $M = \mathbb{E}_1^2$ with coordinates (t, x). Consider the following contravariant Killing tensors:

$$K_1 := G = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad K_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

One can check that these KTs satisfy the hypothesis of Corollary 5.3.6, but they don't arise from a separable solution because the linear operator associated with K_2 is not diagonalizable. □

Note that the Killing tensors in the above example induce first integrals on T^*M which satisfy the hypothesis of Theorem 5.1.4. Hence they arise from a complete solution to the HJ equation. It was shown in [DR07] that one can define a notion of complex separation associated with this complete solution.

We end this section with a remark on the Stäckel form, which was introduced in [Sta93]. He gives a complete (non-intrinsic) characterization of orthogonal separation, by specifying the exact form of the KS-space in separable coordinates in terms of the Stäckel matrix. Thus, he has implicitly obtained the general solution of the system of PDEs given by Eq. (5.3.1). See for example [Ben91, Theorem 3.1] or [Kal86, Stäckel's Theorem] or [Par65] for details. Eisenhart's original solution in [Eis34] was based on Stäckel's work. Although, due to the non-intrinsic nature of Stäckel's results, they are not of much use for our purposes.

5.3.1 Killing-Stackel spaces in Warped Products

In this section we will study the KS-space of a separable web when it's decomposable in a warped product. We wish to further understand the structure of the KS-space associated to a ChKT K which admits a K-invariant Killing distribution. We first need a definition:

Definition 5.3.8
We define the *dKdV equation* with Killing tensor K and potential[2] $V \in \mathcal{F}(M)$ as:

$$\mathrm{d}(K\mathrm{d}V) = 0 \qquad \square$$

We will first do some calculations in a more general setting to study the dKdV equation. Suppose K is a KT with orthogonally integrable eigenspaces $(E_i)_{i=1}^k$ with associated eigenfunctions $\lambda_1, ..., \lambda_k$. We work in the local twisted product ${}^{\rho}\prod_{i=1}^{k} M_i$ adapted to the eigenspaces of K given by Corollary 4.4.8. Fix $x \in \hat{\mathfrak{X}}(M_i)$ and $y \in \hat{\mathfrak{X}}(M_j)$ such that $[x, y] = 0$, then letting $\sigma_i := \log \rho_i^2$, it follows from Eq. (4.4.6) that the eigenfunctions satisfy

$$x\lambda_j = (\lambda_j - \lambda_i)x\sigma_j$$

Fixing $V \in \mathcal{F}(M)$ and using the above equation we have

[2] The reason for this terminology will become apparent in the next section.

$$\begin{aligned}
\mathrm{d}(K\mathrm{d}V)(x,y) &= x(K(y,\nabla V)) - y(K(x,\nabla V)) \\
&= x(\lambda_j yV) - y(\lambda_i xV) \\
&= x\lambda_j yV - y\lambda_i xV + \lambda_j xyV - \lambda_i yxV \\
&= (\lambda_j - \lambda_i)x\sigma_j yV - (\lambda_i - \lambda_j)y\sigma_i xV + (\lambda_j - \lambda_i)xyV \\
&= (\lambda_j - \lambda_i)(xyV + x\sigma_j yV + y\sigma_i xV)
\end{aligned}$$

Hence we have proven the following:

Lemma 5.3.9 (The dKdV Equation in the eigenframe)
Given K and V as above, $d(K\,dV) \equiv 0$ iff for each $x \in \hat{\mathfrak{X}}(M_i)$ and $y \in \hat{\mathfrak{X}}(M_j)$ with $i \neq j$ the following holds:

$$xyV + x\log\rho_j^2 yV + y\log\rho_i^2 xV = 0 \tag{5.3.2}$$

From which we can deduce the following:

1. *If $E_i^\perp$ is geodesic, hence E_i is Killing (see Lemma 4.4.10), we have for all $y \in \hat{\mathfrak{X}}(M_{i\perp})$:*

$$y(\rho_i^2 xV) = 0 \tag{5.3.3}$$

2. *In particular, if E_i and E_j are Killing and $i \neq j$, we have for $x \in \hat{\mathfrak{X}}(M_i)$ and $y \in \hat{\mathfrak{X}}(M_j)$:*

$$xyV = 0$$

□

Proof The first equation immediately follows from the above calculations. Now for the consequences, if $E_i^\perp$ is geodesic, then $x(\log\rho_j^2) = 0$ for $j \neq i$ by Proposition 3.5.9 (3), hence

$$\begin{aligned}
xyV + x\log\rho_j^2 yV + y\log\rho_i^2 xV &= xyV + y\log\rho_i^2 xV \\
&= xyV + \frac{y\rho_i^2}{\rho_i^2}xV \\
&= \frac{1}{\rho_i^2}(\rho_i^2 xyV + y\rho_i^2 xV) \\
&= \frac{1}{\rho_i^2}y(\rho_i^2 xV)
\end{aligned}$$

Hence $y(\rho_i^2 xV) = 0$. The second statement also follows immediately. ■

We now obtain a necessary and sufficient condition for extending a Killing-Stäckel space from the geodesic factor of a warped product.

Proposition 5.3.10 (Extending a Killing-Stäckel space into a warped product) *Suppose $M = B \times_\rho F$ is a warped product and $\mathcal{K}$ is Killing-Stäckel space in B. If there exists a ChKT $K \in \mathcal{K}$ that can be extended into a KT on M (via the method of Proposition 4.5.2) then all KTs in $\mathcal{K}$ can be extended into KTs on M.* $\diamondsuit$

Proof Suppose $K \in \mathcal{K}$ is a ChKT that can be extended into a KT on M. Then from Proposition 4.5.2, K satisfies the dKdV equation with $\kappa = \rho^{-2}$. Then by Eq. (5.3.2) in Lemma 5.3.9 it follows that every $K \in \mathcal{K}$ satisfies the dKdV equation with ρ^{-2}. Hence by Proposition 4.5.2 every $K \in \mathcal{K}$ can be extended into a KT on M. ∎

The above proposition motivates the following notion of a reducible separable web, which is characterized intrinsically by the invariant distributions of an associated ChKT.

Definition 5.3.11 (Reducible separable web)
Suppose $\mathcal{E}$ is a separable web locally characterized by a ChKT, K. $\mathcal{E}$ is said to be *reducible* if it admits a K-invariant Killing distribution. □

First note that since all KTs in the KS-space of a separable web are simultaneously diagonalized, the above definition doesn't depend on the choice of the ChKT, hence is well-defined. The following proposition states clearly why we introduce the notion of a reducible separable web.

Proposition 5.3.12 (The Killing-Stäckel space of a reducible separable web) *Suppose K is a ChKT with associated KS-space $\mathcal{K}$ inducing a reducible separable web, i.e. there exists a K-invariant Killing distribution D. Let $M = B \times_\rho F$ be a local warped product adapted to the WP-net $(D^\perp, D)$ with adapted contravariant metric $G = G_B + \rho^{-2} G_F$. Then there are KS-spaces $\mathcal{K}_B$ and $\mathcal{K}_F$ on B and F respectively such that $L \in \mathcal{K}$ iff there exists $L_B \in \mathcal{K}_B$, $L_F \in \mathcal{K}_F$ and $l \in \hat{\mathcal{F}}(B)$ such that the following equations hold*

$$\begin{aligned} L &= L_B + lG_F + L_F \\ dl &= L_B d\rho^{-2} \end{aligned} \qquad \diamondsuit$$

Proof By Proposition 4.4.15 it follows that $\mathcal{K}$ induces a KS-space $\mathcal{K}_B$ in B and a KS-space $\mathcal{K}_F$ in F. If $L \in \mathcal{K}$, then it follows from Proposition 4.5.2 that L is determined up to constants by KTs in $\mathcal{K}_B$ and $\mathcal{K}_F$ satisfying the above equations. Conversely from Proposition 4.5.2 it follows that every KT in $\mathcal{K}_F$ can be extended to a KT in $\mathcal{K}$. Furthermore it follows from Proposition 4.5.2 that K can be decomposed into a KT on M to satisfy the hypothesis of the above corollary. Hence from the above corollary it follows that each $L_B \in \mathcal{K}_B$ can be extended into a KT in $\mathcal{K}$ given by the above equation by taking $L_F = 0$. ∎

One usually determines if an orthogonal separable web is reducible by inspecting the metric in adapted coordinates by using Proposition 3.5.9 (4) and keeping in mind that all KTs in the KS-space are diagonalized in adapted coordinates. We give some examples to illustrate this.

Example 5.3.13

The dimension of the Killing distribution is one in the above definition iff there is a Killing vector spanning one of the distributions of the web. This is sometimes called a *web symmetry* [HMS09]. □

Example 5.3.14

There is an abundant supply of reducible separable webs in spaces of constant curvature [Kal86]. These are a special case of KEM webs which will be introduced in Section 6.5. Concrete examples can be found in Section 9.6.2. □

5.4 Intrinsic characterization of Separation for natural Hamiltonians

In this section we consider a natural Hamiltonian H. We will present an intrinsic characterization of separation for this Hamiltonian. Following [Ben97], this will reduce to the intrinsic characterization of separation for geodesic Hamiltonians.

In order to reduce this to the geodesic case, consider the following construction. Let $V \in \mathcal{F}(M)$ be the potential function of the natural Hamiltonian and assume locally that $V \neq 0$. Consider the local warped product $\bar{M} := M \times_\rho \mathbb{E}^1_\nu$, with adapted contravariant metric $\bar{G} := G + \rho^{-1} G_1$ where ρ, ν are defined as follows[3]:

$$\frac{1}{\rho} := 2V \qquad\qquad \nu := \operatorname{sgn} V$$

We let $(\bar{q}^j) = (q^0, q^i)$ be product coordinates on $\bar{M}$, where (q^i) are coordinates for M. This warped product metric is called an *Eisenhart metric*, since Eisenhart showed that geodesics $\bar{q}^j(t)$ in this warped product with $\dot{q}^0 = 1$ project onto solutions of Hamilton's Equations for the natural Hamiltonian associated with V [Eis28].

It was a remarkable observation by Benenti in [Ben97], that showed that the Hamilton-Jacobi equation associated with potential V is separable in coordinates (q^i) on M iff the geodesic Hamilton-Jacobi equation is separable in the induced product coordinates (q^0, q^i) on $M \times_\rho \mathbb{E}^1_\nu$. This follows for example, by an inspection of the Levi-Civita equations (Eq. (5.2.3)) associated with the respective Hamiltonians. This observation allows us to prove the following theorem:

[3] Note the difference in the use of the warping function here, this will simply following calculations.

Theorem 5.4.1 (Benenti's Theorem [Ben97])
A natural Hamiltonian with potential V is separable in a web $\mathcal{E}$ iff there exists a ChKT K whose eigenspaces form $\mathcal{E}$ which satisfies the dKdV equation:

$$d(K\,dV) = 0$$

Furthermore if V separates in the separable web $\mathcal{E}$, then all K in the KS-space associated with $\mathcal{E}$ satisfy the dKdV equation with V. □

PROOF By the preceding observations, a necessary and sufficient condition for the separability of the potential V is that the geodesic Hamiltonian on $\bar{M} = M \times_\rho \mathbb{E}^1_\nu$ be separable in product coordinates. By Theorem 5.3.2, this is equivalent to the existence of a ChKT on $\bar{M}$ which has the Killing distribution, $T\mathbb{E}^1_\nu$, as an invariant distribution. It follows by Proposition 4.5.2 that any such ChKT, $\tilde{K}$, can be put into the form:

$$\tilde{K} = K + 2tG_1$$

where $K \in \hat{S}^2(M)$ and $t \in \mathcal{F}(M)$ satisfies:

$$\mathrm{d}t = K\mathrm{d}V$$

Thus it follows by Proposition 4.5.2, that a necessary and sufficient condition is the existence of a ChKT $K \in S^2(M)$ satisfying the dKdV equation with V.

The last remark follows by Eq. (5.3.2) in Lemma 5.3.9 as in the proof of Proposition 5.3.10. ■

We now examine the form of the first integrals guaranteed by the Jacobi theorem (Theorem 5.1.1). On the Eisenhart manifold $\bar{M}$, we are guaranteed $n+1$ commuting, point-wise independent Killing tensors by Corollary 5.3.3. Let (q^0, q^i) be the associated separable coordinates. By Proposition 4.5.2 each of these KTs can be put into the form:

$$\tilde{K} = K + 2UG_1$$

where $K \in \hat{S}^2(M)$ and $U \in \mathcal{F}(M)$. Choose a basis $\tilde{K}_0, \ldots, \tilde{K}_n$ for this KS-space such that $\tilde{K}_0 := G_1$ and $\tilde{K}_1 := \bar{G}$. On $T^*\bar{M}$, in the induced canonical coordinates (q^i, p_j), consider the following first integrals:

$$E_1 := H \qquad\qquad E_k := \frac{1}{2}K_k^{ij}p_ip_j + U_kp_0^2 \quad k = 2, \ldots, n$$

In the following corollary, we will show that the induced functions on T^*M, obtained by taking $p_0 = 1$, are commuting first integrals:

Corollary 5.4.2
Suppose V is a potential separable in the web associated with a ChKT K. Let (q^i) be the associated separable coordinates and (q^i, p_j) be the induced canonical coordinates on T^*M. Then there exist functionally independent commuting first integrals $F_1, \dots, F_n$ (where $F_1 := H$) each having the form:

$$F = \frac{1}{2} K^{ij} p_i p_j + U(q^i)$$

where each quadratic polynomial in the momenta is induced by a KT in the KS-space associated with the separable coordinates. Furthermore, these integrals are precisely those guaranteed by the Jacobi theorem. □

Proof Let $\tilde{K}_1, \dots, \tilde{K}_n$ be the KTs from the preceding discussion. Then it follows from Proposition 4.4.15 that the projected tensor, K_i, is a KT on M. Then note that the KTs $K_1, \dots, K_n$ form a basis for the KS-space associated with the ChKT K. Furthermore, since $\tilde{K}_i$ is a KT on $\bar{M}$, it follows from Proposition 4.5.2 that each K_i satisfies:

$$\mathrm{d}U_i = K_i \mathrm{d}V \tag{5.4.1}$$

Now, as mentioned earlier, we define each $F_k \in \mathcal{F}(T^*M)$ by:

$$F_k := E_{K_k} + U_k = \frac{1}{2} K_k^{ij} p_i p_j + U_k \quad k = 1, \dots, n$$

with $F_1 = H$. Then,

$$\begin{aligned} \{F_i, F_j\} &= \{E_{K_i}, E_{K_j}\} + \{E_{K_i}, U_j\} + \{U_i, E_{K_j}\} \\ &\overset{(4.2.5)}{=} -\frac{1}{2}\left(\frac{1}{2} E_{[K_i, K_j]} + E_{[K_i, U_j]} + E_{[U_i, K_j]}\right) \end{aligned}$$

Now note that $[K_i, K_j] = 0$. Also by Proposition 4.2.3,

$$[K_i, U_j] = 2K_i \mathrm{d}U_j$$

Thus one can immediately verify that $[K_i, U_j] = [K_j, U_i]$, due to Eq. (5.4.1) and because the KTs, K_i, commute as linear operators. Thus we conclude that $\{F_i, F_j\} = 0$. The functional independence of the integrals follows from that fact that the KTs $K_1, \dots, K_n$ are point-wise independent.

The proof of the last remark (showing that these integrals arise from a separable solution) is a simple generalization of that in Corollary 5.3.3. ■

For completeness sake, we also mention that given a ChKT K, the most general potential satisfying the dKdV equation with K is known in separable coordinates. Indeed, if (q^i) are coordinates which diagonalize K, then the dKdV equation in these coordinates is Eq. (5.2.3b). This follows from the proof of Theorem 5.4.1 or from Eq. (5.3.2). The general solution, V, of this PDE is easily obtained from Stäckel theory (see references at the end of Section 5.3 for a proof), and is given as follows:

$$V = V_i g^{ii} \tag{5.4.2}$$

where each V_i depends only on q^i.

5.4.1 Separation of natural Hamiltonians in Warped Products

In this section we are concerned with the separation of the Hamilton-Jacobi equation in reducible separable webs. So we fix a natural Hamiltonian H with potential V. K is assumed to be a KT with orthogonally integrable eigenspaces $(E_i)_{i=1}^k$ with associated eigenfunctions $\lambda_1, ..., \lambda_k$. We work in the local twisted product ${}^{\rho}\prod_{i=1}^{k} M_i$ adapted to the eigenspaces of K given by Corollary 4.4.8.

Now suppose E_i is Killing and that $\tilde{K}_i$ is a KT on M_i. Then by Proposition 4.5.2, the lift K_i, is a KT on M. The following proposition will allow us to reduce the calculation of the dKdV equation with K_i on M to the restriction of the equation on M_i. To make this precise, we fix $\bar{p} \in M$ and let $L_i(\bar{p})$ be the leaf of the canonical foliation of M_i through $\bar{p}$. Furthermore let $\tau_i : M_i \to L_i(\bar{p})$ be the embedding of M_i in M.

Proposition 5.4.3 (Reduction of The dKdV equation on warped products) *Suppose K and K_i are as above, E_i is Killing and additionally assume that M is connected. For a potential $V \in \mathcal{F}(M)$, let $V_i := \tau_i^* V \in \mathcal{F}(M_i)$. Suppose $d(K dV) = 0$ holds on M, then the following is true:*

$$d(K_i dV) = 0 \quad \Leftrightarrow \quad d(\tilde{K}_i dV_i) = 0 \qquad \diamondsuit$$

Proof The first implication follows trivially by naturality of the exterior derivative, so now we prove the converse. First we note that as endomorphisms of T^*M, $K_i = \rho_i^2 \tilde{K}_i$ where $\tilde{K}_i$ is the lift of an endomorphism of T^*M_i. We also note that for $y \in \hat{\mathfrak{X}}(M_{i\perp})$

$$\mathcal{L}_y(\rho_i^2 (\mathrm{d}V)_i) = 0$$

where $(\mathrm{d}V)_i$ is the orthogonal projection of $\mathrm{d}V$ onto T^*M_i. To prove this, we first note that since $\mathrm{d}(K\mathrm{d}V) = 0$, $y(\rho_i^2 xV) = 0$ for all $x \in \hat{\mathfrak{X}}(M_i)$ by Eq. (5.3.3) in Lemma 5.3.9. This implies that $\mathrm{d}(\rho_i^2 (\mathrm{d}V)_i) = 0$. Hence the above equation follows by Cartan's Formula which relates the exterior derivative of forms to their Lie derivatives.

Now by hypothesis, for $x \in \hat{\mathfrak{X}}(M_i)$ and $y \in \hat{\mathfrak{X}}(M_i)$ with $[x, y] = 0$ we have that $\tau_i^*(\mathrm{d}(K_i \mathrm{d}V)(x, y)) = 0$. Then for $z \in \hat{\mathfrak{X}}(M_{i\perp})$,

$$\begin{aligned} z\mathrm{d}(K_i \mathrm{d}V)(x, y) &= z[x(K_i(y, \mathrm{d}V)) - y(K_i(x, \mathrm{d}V))] \\ &= z[x(\tilde{K}_i(y, \rho_i^2(\mathrm{d}V)_i))) - y(\tilde{K}_i(x, \rho_i^2(\mathrm{d}V)_i)))] \\ &= x(\tilde{K}_i(y, \mathcal{L}_z(\rho_i^2(\mathrm{d}V)_i)))) - y(\tilde{K}_i(x, \mathcal{L}_z(\rho_i^2(\mathrm{d}V)_i)))) \\ &= 0 \end{aligned}$$

where the last equation follows since $\mathcal{L}_z(\rho_i^2(\mathrm{d}V)_i) = 0$. Thus since M is connected we conclude that $\mathrm{d}(K_i \mathrm{d}V)(x, y) = 0$ on M.

For $x \in \hat{\mathfrak{X}}(M_i)$ and $y \in \hat{\mathfrak{X}}(M_{i\perp})$

$$\begin{aligned} \mathrm{d}(K_i \mathrm{d}V)(x, y) &= x(\tilde{K}_i(y, \rho_i^2(\mathrm{d}V)_i)) - y(\tilde{K}_i(x, \rho_i^2(\mathrm{d}V)_i)) \\ &= -\tilde{K}_i(x, y(\rho_i^2(\mathrm{d}V)_i)) \\ &= 0 \end{aligned}$$

Also it easily follows that for $x \in \hat{\mathfrak{X}}(M_{i\perp})$ and $y \in \hat{\mathfrak{X}}(M_{i\perp})$, that $\mathrm{d}(K_i \mathrm{d}V)(x, y) = 0$. Thus the result is proven. ∎

We now consider the problem of separation in warped products. To be precise, suppose $N = N_0 \times_\rho \prod_{i=1}^{l} N_i$ is a warped product and $\mathcal{E} = (D_i)_{i=0}^{l}$ is the associated WP-net. Suppose K is a ChKT such that each Killing distribution defining $\mathcal{E}$ is K-invariant. According to Benenti's Theorem (Theorem 5.4.1), for a potential $V \in \mathcal{F}(M)$ to be separable in the web associated with K, we need to check that the dKdV equation is satisfied. Although in this case we have some more information. Due to Corollary 4.5.4, K can be decomposed as follows in contravariant form:

$$K = K_0 + \sum_{i=1}^{l} K_i$$

where each $K_i \in \hat{S}^2(N_i)$ is a KT for $i = 1, .., l$, each D_i is an eigenspace of K_0 for $i = 1, .., l$ and K_0 restricted to D_0 is characteristic. By Benenti's Theorem, if V satisfies the dKdV equation with K, then it must satisfy the dKdV equation with each K_i. In particular it must satisfy the dKdV equation with K_0. Since K_0 invariantly encodes the warped product through it's eigenspaces and a partial separable web on D_0, one could ask if the converse holds. If V satisfies the dKdV equation with a given KT K_0 with eigenspaces as just stated, is it possible to build up a separable web for V by reducing the problem to one on the spherical factors of N? The following theorem shows that we can.

Theorem 5.4.4 (Separation in Warped Products)
Suppose $(D_i)_{i=0}^{l}$ is a WP-net and K_0 is a KT with eigenspaces D_i for $i = 1, ..., l$ and characteristic on D_0. Fix $\bar{p} \in M$ and let $N = \prod_{i=0}^{l} N_i$ be a connected product manifold passing through $\bar{p}$ adapted to the WP-net $(D_i)_{i=0}^{l}$. Then the following holds:

Suppose $V \in \mathcal{F}(M)$ satisfies $d(K_0 dV) = 0$. Let $V_i := \tau_i^* V \in \mathcal{F}(N_i)$ and suppose for each $i = 1, ..., k$ there exists a ChKT $\tilde{K}_i$ on N_i such that $d(\tilde{K}_i dV_i) = 0$.

Then V is separable in the web formed by the simple eigenspaces of K_0 together with the lifts of the simple eigenspaces of $\tilde{K}_1, ..., \tilde{K}_l$. ◇

Proof For $i = 1, ..., l$, let K_i be the lift of $\tilde{K}_i$ to N. Consider the tensor

$$K := K_0 + \sum_{i=1}^{l} K_i$$

By Proposition 4.5.2, K is a Killing tensor on N. Let $\tilde{G}_i$ be the contravariant metric on N_i, then by replacing $\tilde{K}_i$ with $a_i \tilde{K}_i + b_i \tilde{G}_i$ for some $a_i \in \mathbb{R} \setminus \{0\}$ and $b_i \in \mathbb{R}$, we can assume K locally has simple eigenfunctions. Let q_0 be coordinates which diagonalize the ChKT induced by K_0 on N_0. Let q_j be coordinates which diagonalize $\tilde{K}_j$ on N_j for each $j > 0$. Then one can check that the product coordinates $(q_0, q_1, \ldots, q_l)$ are orthogonal and diagonalize K, hence K is a ChKT. By Proposition 5.4.3, $\mathrm{d}(K_i \mathrm{d}V) = 0$ on N for each $i > 0$, hence K satisfies the dKdV equation with V. Thus it follows by Theorem 5.4.1 that V separates in the product coordinates $(q_0, q_1, \ldots, q_l)$, which proves the claim. ■

The above theorem and the preceding discussion shows that reducible separable webs enable one to reduce the problem of separation to certain spherical submanifolds after one finds a KT with the same eigenspaces as K_0 in the above theorem.

The motivating application of the above theorem is to devise a recursive algorithm (The BEKM separation algorithm) to separate natural Hamiltonians defined on spaces of constant curvature. Before we can do this, we have to first introduce concircular tensors; this is done in the next chapter.

5.5 Notes

Much of the theory on the Hamilton-Jacobi equation and its separation that we have presented is based on contemporary formulations [Woo75; Ben89]. Most of the theory on the intrinsic characterization of separation is due to Benenti [Ben97], following Eisenhart's lead [Eis34]. Much of the recent interest in the separation of the Hamilton-Jacobi equation was due to the discovery of the separability of the Hamilton-Jacobi equation for the geodesics in the Kerr solution from general relativity [Car68].

Before the 1960s, the fundamental result was due to Stäckel in [Sta93]. He obtained the general form of the orthogonal separable metric in separable coordinates. Eisenhart's intrinsic characterization in [Eis34] is based on Stäckel's work and the proof is much more complicated than the one presented here.

We have omitted the theory for general (possibly non-orthogonal) separation which is covered in [Ben97]. Furthermore, as hinted at by Example 5.3.7, a notion of complex separation is possible on strictly pseudo-Riemannian manifolds. See [DR07] for details.

The material on warped products is new and is from the article [RM14b].

Chapter 6

Concircular tensors and KEM webs

In this chapter we study concircular tensors and the orthogonal (separable) webs which can be constructed using them: Kalnins-Eisenhart-Miller (KEM) webs. As stated in the introduction (see Section 2.2), we study these tensors because of their computational value in working with KEM webs.

$L \in S^p(M)$ is called a *concircular tensor also called a C-tensor (CT)* of valence p if there exists $C \in S^{p-1}(M)$ (called the *conformal factor*) such that

$$\nabla_x L = C \odot x \tag{6.0.1}$$

for all $x \in \mathfrak{X}(M)$. The reason for the name "concircular" will be given in Section 6.4. Sometimes we denote the space of concircular tensors of valence p by $\mathrm{C}^p(M)$ and the subspace of covariantly constant tensors by $\mathrm{C}^p_0(M)$. Concircular tensors of arbitrary valence were originally defined in [Cra08], where they were called special conformal Killing tensors. This is because concircular tensors are conformal Killing tensors as we shall show shortly.

In the first four sections, we study CTs in general. In the last three sections, we present the application of CTs to problem of orthogonal separation of the Hamilton-Jacobi equation.

6.1 General Valence

The theory of general valence CTs has been studied in [Cra08]. We give a brief outline here.

We first observe that the defining equation implies that CTs form a vector space which is closed under the symmetric product. Indeed, if L_1 and L_2 are CTs with conformal factors C_1 and C_2 respectively, then a short calculation shows that $L_1 \odot L_2$ is a CT with conformal factor $C_1 \odot L_2 + C_2 \odot L_1$.

Proposition 6.1.1 (Properties of Concircular tensors [Cra08])
Suppose L is a CT of arbitrary valence with conformal factor C. Then L is a CKT with conformal factor C and C is given as follows:

$$C = \frac{r}{n+r-1}\nabla \cdot L \qquad \square$$

Proof In coordinates the defining equation of L reads:

$$\nabla_j L_{i_1,\ldots,i_r} = C_{(i_1,\ldots,i_{r-1}}g_{i_r)j} \tag{6.1.1}$$

Thus

$$\nabla_{(j} L_{i_1,\ldots,i_r)} = C_{(i_1,\ldots,i_{r-1}}g_{i_r j)}$$

which proves that L is a CKT. Also to obtain the equation for C, we get from Eq. (6.1.1) that

$$\begin{aligned}\nabla \cdot L &= \nabla_j L^{i_1,\ldots,i_{r-1}j} \\ &= C^{(i_1,\ldots,i_{r-1}}\delta^{j)}{}_j \\ &= \frac{(n+r-1)}{r}C^{i_1,\ldots,i_{r-1}}\end{aligned} \qquad \blacksquare$$

In [Cra08], Crampin has derived structural equations for CTs of arbitrary valence and as a consequence, he has proven the following theorem:

Theorem 6.1.2 (The Vector Space of Concircular tensors [Cra08])
Suppose $n > 2$. Then the C-tensors of valence r form a finite dimensional real vector space with maximal dimension equal to the dimension of the space of constant symmetric r-tensors in $\mathbb{R}^{n+1}$. Furthermore the maximal dimension is achieved if and only if the space has constant curvature. $\square$

Remark 6.1.3
When $r \leq 2$ the above result holds for $n = 2$ as well [TCS05; Cra07]. In particular, if $r = 1$ (resp. $r = 2$) the maximal dimension is $n + 1$ (resp. $\frac{1}{2}(n + 1)(n + 2)$). $\square$

The above theorem implies the following:

Corollary 6.1.4 (Concircular tensors in spaces of constant curvature)
Suppose M^n is a space of constant curvature. Let $\beta = \{v_1, \ldots, v_{n+1}\}$ be a basis for the space of concircular vectors, then a given C-tensor of valence r can be written uniquely as a linear combination of r-fold symmetric products of the vectors in β. $\square$

6.2 Torsionless Conformal Killing tensors

Before moving on to study concircular 2-tensors (the main object of interest), we will first study torsionless (orthogonal) conformal Killing tensors. The main reason for studying torsionless orthogonal CKTs is because they will help us to study concircular tensors, as we will see in the next section.

Historically, Benenti originally showed that a torsionless CKT, L, with n functionally independent eigenfunctions can be used to generate a basis for a KS-space in a coordinate independent way [Ben92a, Proposition 2.1]. In [Ben05] he showed that his method worked when the eigenfunctions were just assumed to be simple. For this reason, we refer to any torsionless CKT with simple (real) eigenfunctions as a *Benenti tensor* (also called an L-tensor by Benenti). In [Ben05], Benenti extended his study to include general torsionless orthogonal CKTs. Most of the results in this section are based on Benenti's, but we arrive at them using the characterization of orthogonal CKTs in terms of their eigenspaces (given by Corollary 4.4.6).

In this section, L is assumed to be a torsionless orthogonal CKT unless otherwise stated. We now recall the implications of the torsionless property, which are studied in detail in Appendix B. Suppose $(E_i)_{i=1}^k$ are the eigenspaces of L and $(\lambda_i)_{i=1}^k$ are the associated eigenfunctions. Then by Theorem B.0.20, the eigenspaces are orthogonally integrable and each eigenfunction satisfies

$$(\nabla\lambda_i)^j = 0 \quad j \neq i \tag{6.2.1}$$

We will see that the above equations satisfied by the eigenfunctions make these CKTs highly amenable to analysis. Due to Eq. (4.4.5), we have to assume L is a CKT, not just a KT, in order to deal with non-trivial cases.

By Corollary 4.4.8 there is a twisted product ${}^{\rho}\prod_{i=1}^{k} M_i$ which is adapted to the eigenspaces of L. We can explicitly solve for the twist function ρ_i in this case. Indeed, from Eq. (4.4.4), we have

$$\begin{aligned}
H_i &= -\frac{1}{2}\sum_{j\neq i}(\nabla \log|\lambda_i - \lambda_j|)^j \\
&\overset{(6.2.1)}{=} -\frac{1}{2}\sum_{j\neq i}(\sum_{k\neq i}\nabla \log|\lambda_i - \lambda_k|)^j \\
&= -\frac{1}{2}\sum_{j\neq i}(\nabla \log\prod_{k\neq i}|\lambda_i - \lambda_k|)^j \\
&= -\frac{1}{2}(\nabla \log\prod_{k\neq i}|\lambda_i - \lambda_k|)^{\perp i}
\end{aligned}$$

Hence by Eq. (4.4.6), we have

$$(\nabla \log \rho_i^2)^{\perp i} = (\nabla \log \prod_{k \neq i} |\lambda_i - \lambda_k|)^{\perp i}$$

Thus $\log \rho_i^2 - \log \prod\limits_{k \neq i} |\lambda_i - \lambda_k| = f_i$ where f_i is a function of M_i only. Hence we have the following (cf. [Ben05, Theorem 18.1]):

Proposition 6.2.1 (Characterization of torsionless orthogonal CKTs) *Suppose L is a torsionless orthogonal tensor with eigenspaces $(E_i)_{i=1}^k$ and associated eigenfunctions $(\lambda_i)_{i=1}^k$. Then L is a CKT iff there is a twisted product adapted to its eigenspaces such that each twist function ρ_i can be chosen to be:*

$$\rho_i^2 = \prod_{k \neq i} |\lambda_i - \lambda_k| \qquad \square$$

Proof The characterization follows from the preceding calculation and Corollary 4.4.8. ■

We now deduce some important geometric properties of the eigenspaces of torsionless orthogonal CKTs.

Proposition 6.2.2 (Properties of torsionless orthogonal CKTs [Ben05]) *Suppose that L is a torsionless orthogonal CKT, then the following statements are true:*

1. *L is of gradient-type and the conformal factor α is given as follows:*

$$\alpha = \sum_i \nabla \lambda_i \tag{6.2.2}$$

2. *An eigenfunction λ_i is constant iff its associated eigenspace E_i is Killing.*

3. *L is of trace-type iff each multidimensional eigenspace is Killing.*

4. *If L has simple eigenfunctions, then it is of trace-type.*

5. *If $S := \bigoplus\limits_{\dim E_i = 1} E_i$ is the space of simple eigenspaces of L, then L restricted to any integral manifold of S is a Benenti tensor.* $\square$

Proof The first statement follows from condition 3 of Corollary 4.4.6 and Eq. (6.2.1). The second statement follows from Corollary 4.4.11 and Eq. (6.2.1) or from Proposition 3.5.9 (4) using the formula for the twist function from Proposition 6.2.1.

For the third statement assume L is trace-type. Due to the second statement, we need only show that λ_i is constant when $\dim E_i > 1$. The trace-type condition implies that $\alpha = \nabla \operatorname{tr}(L) = \sum_i m_i \nabla\lambda_i$ where $m_i = \dim E_i$. This together with condition 3 in Corollary 4.4.6 gives the following:

$$m_i \nabla\lambda_i = \alpha^i = \nabla\lambda_i$$

Hence when $\dim E_i > 1$, λ_i must be constant. The converse follows from Eq. (6.2.2).

The fourth statement is an immediate corollary of the third statement.

To prove the fifth statement note that due to Proposition 4.4.15, L restricts to a torsionless CKT with simple eigenfunctions on any integral manifold of S. Hence the fifth statement follows from the fourth statement. ■

Remark 6.2.3

It follows from statement 1 of the above proposition that if L is a torsionless orthogonal CKT, then it has a conformal factor ∇f for some $f \in \mathcal{F}(M)$. Then one can show that the following tensor is a KT:

$$K = fG - L$$

Note that such a KT shares the same eigenspaces as L and the above is a KT for all gradient-type CKTs; it will be useful to us later. □

Using the above remark, we can prove the following:

Corollary 6.2.4 (Benenti tensors induce Separable Webs)

If L is a Benenti tensor then the web formed by its eigenspaces is separable. □

Proof If ∇f is the conformal factor of L, then the above remark implies that $K = fG - L$ is a KT with the same eigenspaces as L, hence a ChKT. The result then follows by Theorem 5.3.2. ■

We call the separable web induced by a Benenti tensor L (as in the above corollary) the separable web generated by L. Similarly we call the associated KS-space the KS-space generated by L. Theorem 7.1 in [Ben05] shows that a basis for the KS-space can be generated using only L and the metric; a slight generalization of this result will be presented in Section 6.6.

One can show that given a separable web $\mathcal{E} = (E_i)_{i=1}^n$, there are many ChKTs whose eigenspaces form this web. In fact, Corollary 6.6.8 will show that the KS-space generated by an ICT contains an $(n-1)$-dimensional subspace of ChKTs. The following proposition shows that this is not the case when we restrict ourselves to torsionless orthogonal CKTs. It shows that for all non-trivial cases the eigenfunctions of a torsionless orthogonal CKT are essentially uniquely determined by its eigenspaces.

Proposition 6.2.5 (Equivalent Torsionless Orthogonal CKTs)
Suppose L and $\tilde{L}$ are torsionless orthogonal CKTs and assume M is connected. Suppose $(E_i)_{i=1}^k$ are the eigenspaces of L.

If L is not covariantly constant (on any open set), then: The eigenspaces of $\tilde{L}$ are the same as those of L iff there exists $a, b \in \mathbb{R}$ with $a \neq 0$ such that

$$L = a\tilde{L} + bG$$

If L is covariantly constant, then: The eigenspaces of $\tilde{L}$ are the same as those of L iff there exists $c_i \in \mathbb{R}$ for $i = 1, \ldots, k$ such that[1]

$$L = \tilde{L} + \sum_i c_i G_i$$

where G_i denotes the restriction of the metric G to E_i, which is a KT in this case. □

PROOF Suppose L and $\tilde{L}$ share the same eigenspaces $\mathcal{E} = (E_i)_{i=1}^k$ with respective eigenfunctions $(\lambda_i)_{i=1}^k$ and $(\tilde{\lambda}_i)_{i=1}^k$. Then since H_i is uniquely determined by E_i for each i, from Eq. (4.4.4) we have the following for $j \neq i$

$$\begin{aligned}(\nabla \log |\lambda_i - \lambda_j|)^j &= (\nabla \log \left|\tilde{\lambda}_i - \tilde{\lambda}_j\right|)^j \\ \Leftrightarrow (\nabla \log \left|\frac{\lambda_i - \lambda_j}{\tilde{\lambda}_i - \tilde{\lambda}_j}\right|)^j &= 0\end{aligned}$$

Similarly by permuting $i \leftrightarrow j$, we have

$$(\nabla \log \left|\frac{\lambda_i - \lambda_j}{\tilde{\lambda}_i - \tilde{\lambda}_j}\right|)^i = 0$$

Hence

$$\begin{aligned}\frac{\lambda_i - \lambda_j}{\tilde{\lambda}_i - \tilde{\lambda}_j} &\overset{(6.2.1)}{=} a_{ij} \in \mathbb{R} \\ \Rightarrow \lambda_i - a_{ij}\tilde{\lambda}_i = \lambda_j - a_{ij}\tilde{\lambda}_j &\overset{(6.2.1)}{=} b_{ij} \in \mathbb{R}\end{aligned} \tag{6.2.3}$$

If i, j, k are distinct, by differentiating the above equation we get:

[1]There are additional technical restrictions on the constants c_i which ensure that $\tilde{L}$ and L have the same eigenspaces.

$$
\begin{aligned}
a_{ij}\nabla\tilde{\lambda}_i &= \nabla\lambda_i = a_{ik}\nabla\tilde{\lambda}_i \\
\Rightarrow (a_{ij} - a_{ik})\nabla\tilde{\lambda}_i &= 0
\end{aligned}
$$

From Eq. (4.4.4) and Proposition 6.2.2 (1) and (2), we see that L (or $\tilde{L}$) is covariantly constant iff all its eigenfunctions are constants iff $\mathcal{E}$ is a pseudo-Riemannian product net. Hence if L is not covariantly constant, then there exists an i such that $\nabla\tilde{\lambda}_i \neq 0$. Then the above equation implies for each $j, k \neq i$ that $a_{ij} = a_{ik}$. So let $a := a_{ij}$ for some $j \neq i$. Then from Eq. (6.2.3), we see that

$$
b_{ij} = \lambda_i - a\tilde{\lambda}_i = \lambda_k - a\tilde{\lambda}_k = b_{ik}
$$

Thus we can let $b := b_{ij}$ for some $j \neq i$. Then Eq. (6.2.3) shows that for all i

$$
\lambda_i = a\tilde{\lambda}_i + b
$$

This proves the first part of the proposition. Finally if L is covariantly constant, then $\mathcal{E}$ is a pseudo-Riemannian product net, thus the eigenfunctions are forced to be constants and the second part follows. ∎

Remark 6.2.6
We should mention here that Theorem 10.1 in [Ben05] is incorrect as stated. The mistake can be seen by comparing the statement of Theorem 10.1 with that of the above theorem, while keeping in mind that L-tensors are torsionless CKTs with simple eigenfunctions. □

6.3 Concircular 2-tensors

Hereafter by concircular tensor, we mean a concircular 2-tensor. In this section we will develop the basic theory of concircular tensors. This class of concircular tensors are the most important for separating the Hamilton-Jacobi equation and so these tensors are the most studied. We will assume the reader is familiar with Appendix B.

Much of the theory is due to Benenti [Ben05] and Crampin [Cra03]. They were first formally introduced within the context of separation of variables by Crampin in [Cra03], where he referred to them as special conformal Killing tensors, cf. [TCS05; Cra07]. They have also been studied in [Ben05] where they are called J-tensors. The theory regarding the cases when these tensors have multidimensional eigenspaces is new and was originally presented in [RM14b].

First note that the defining equations for CTs can be written in index notation as follows:

$$\nabla_k L_{ij} = \alpha_{(i} g_{j)k}$$

When $n > 1$, we say a concircular tensor is *non-trivial* if it's not a multiple of the metric.

The following proposition presents two key properties of concircular tensors, from which much else can be deduced.

Proposition 6.3.1 (Properties of Concircular 2-tensors [Cra03])
Suppose L is a CT with conformal factor α. Then the following hold:

1. *L is a trace-type CKT, i.e. the conformal factor α is given as follows:*

$$\alpha = \nabla \operatorname{Tr} L$$

2. *The Nijenhuis tensor of L vanishes.* □

Proof The first property follows by taking the trace of the defining equation

$$\nabla_k L_{ij} = \frac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik})$$

over the indices i, j.

To show that L is torsionless, by Proposition B.0.14 (2) we only need to prove that $(\nabla_{Lu} L)v - L(\nabla_u L)v$ is symmetric with respect to u, v. Then

$$\begin{aligned}
(\nabla_{Lu} L)v - L(\nabla_u L)v &= \frac{1}{2}[(Lu \otimes \mathrm{d}\operatorname{Tr} L + \nabla \operatorname{Tr} L \otimes (Lu)^\flat)v - L(u \otimes \mathrm{d}\operatorname{Tr} L + \nabla \operatorname{Tr} L \otimes u^\flat)v] \\
&= \frac{1}{2}[(v \operatorname{Tr} L)Lu + \langle Lu, v\rangle \nabla \operatorname{Tr} L - (v \operatorname{Tr} L)Lu - \langle u, v\rangle L(\nabla \operatorname{Tr} L)] \\
&= \frac{1}{2}[\langle Lu, v\rangle \nabla \operatorname{Tr} L - \langle u, v\rangle L(\nabla \operatorname{Tr} L)]
\end{aligned}$$

which is symmetric with respect to u, v since L is self-adjoint, i.e. $\langle Lu, v\rangle = \langle u, Lv\rangle$. ■

An *orthogonal concircular tensor (OCT)* is a concircular tensor which is also an orthogonal tensor. The above proposition will allow us to study OCTs as special cases of orthogonal CKTs. Hence we can apply the results of the previous section. Another key observation is the following: by Proposition 6.2.2 (3) any multidimensional eigenspace of an OCT is Killing. This will allow us to develop a constructive theory (in Section 6.5) to separate the HJ equation using these tensors. We summarize here the results following from Proposition 6.2.2:

Corollary 6.3.2 (OCTs induce Warped Products)
Suppose L is an OCT. Let $S := \bigoplus_{\dim E_i = 1} E_i$ be the space of simple eigenspaces of L. Then S is the geodesic distribution of a warped product net with the (multidimensional) eigenspaces complementary to S as the Killing distributions. Furthermore L restricted to any integral manifold of S is a Benenti tensor. ♢

PROOF Proposition 6.2.2 shows that each eigenspace satisfying $\dim E_i > 1$ is Killing, also $\bigcap_{\dim E_i > 1} E_i^{\perp} = \bigoplus_{\dim E_i = 1} E_i = S$ is geodesic since each $E_i^{\perp}$ is. Thus S together with the complementary eigenspaces forms a warped product net. Then Proposition 6.2.2 (5) completes the proof. ■

Since Benenti tensors have been well studied in the literature (see for example [Ben05]), the above proposition implies that much of this theory can still be applied to OCTs (provided $S \neq 0$).

The following class of CTs are the basic building blocks of all OCTs.

Definition 6.3.3 (Irreducible concircular tensors)
An OC-tensor with functionally independent eigenfunctions is referred to as an *irreducible concircular tensor (ICT)* or more succinctly an IC-tensor. To be precise, an IC-tensor has real eigenfunctions $u^1, ..., u^k$ (counted without multiplicity) satisfying:

$$\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^k \neq 0$$

Furthermore an OC-tensor which is not irreducible is called reducible. □

Remark 6.3.4
IC-tensors were the class of C-tensors mainly studied in [Cra03]. □

Since by Proposition 6.2.2, the eigenfunction associated with a multidimensional eigenspace of an OCT is constant, it follows that an ICT must have simple eigenfunctions, hence ICTs are Benenti tensors. The special property that ICTs have is that their eigenfunctions can be used as (local) coordinates for the separable web they induce [Cra03]. We will refer to these coordinates as the *canonical coordinates* induced by these tensors. See Example 2.3.1 which shows how elliptic coordinates in $\mathbb{E}^2$ can be obtained from an ICT.

Locally, we can assume a reducible OC-tensor has eigenfunctions $u^1, ..., u^k$ which are functionally independent and the rest of which are constants. Hence as in the proof of Corollary 6.3.2, since each eigenspace corresponding to the constant eigenfunctions are Killing (see Proposition 6.2.2), there exists a warped product in which the functions $u^1, ..., u^k$ can be taken as coordinates on the geodesic factor.

We add one final remark. If L is a CT then the Killing tensor defined in Remark 6.2.3 is called the *Killing Bertrand-Darboux tensor (KBDT)* generated by L and has the following form:

$$K = \mathrm{tr}(L)G - L \tag{6.3.1}$$

This tensor will be useful in some constructions done later on.

6.3.1 Characterizations of OCTs

In this section, our goal is to give some characterizations of OCTs which will be useful in their application. The following proposition allows us to do this. It is a converse to Proposition 6.3.1.

Proposition 6.3.5 ([Ben05])
If L is an orthogonal trace-type CKT with vanishing torsion then L is a CT. □

Proof We prove this by generalizing a proof by Crampin of a special case when the eigenfunctions are simple [Cra03, Proposition 1] (see [Ben05, Theorem A.5.3] for an alternate proof). First we need to do some calculations in index notation.

By hypothesis, L satisfies $\nabla_{(k}L_{ij)} = \alpha_{(i}g_{jk)}$. Define T by

$$T_{ijk} = \nabla_k L_{ij} - \alpha_{(i}g_{j)k} = \nabla_k L_{ij} - \frac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik})$$

then observe that $T_{(ijk)} = 0$ and $T_{ijk} = T_{jik}$.

Now the Nijenhuis torsion N_L is given by (see Proposition B.0.14 and the following remark)

$$(N_L)_{kij} = L^l{}_i \nabla_l L_{kj} - L^l{}_j \nabla_l L_{ki} - L_k{}^l(\nabla_i L_{lj} - \nabla_j L_{li}) \tag{6.3.2}$$

First we express the first two terms in terms of T as follows:

$$\begin{aligned} L^l{}_i \nabla_l L_{kj} - L^l{}_j \nabla_l L_{ki} &= L^l{}_i(T_{kjl} + \frac{1}{2}(\alpha_k g_{jl} + \alpha_j g_{kl})) - L^l{}_j(T_{kil} + \frac{1}{2}(\alpha_k g_{il} + \alpha_i g_{kl})) \\ &= L^l{}_i T_{kjl} - L^l{}_j T_{kil} + \frac{1}{2}(L^l{}_i \alpha_j g_{kl} - L^l{}_j \alpha_i g_{kl}) \\ &= L^l{}_i T_{kjl} - L^l{}_j T_{kil} + \frac{1}{2}(L_{ki}\alpha_j - L_{kj}\alpha_i) \end{aligned}$$

Now we express the last two terms in terms of T as follows:

$$\begin{aligned}
\nabla_i L_{lj} - \nabla_j L_{li} &= T_{lji} + \frac{1}{2}(\alpha_l g_{ji} + \alpha_j g_{li}) - (T_{lij} + \frac{1}{2}(\alpha_l g_{ij} + \alpha_i g_{lj})) \\
&= T_{lji} - T_{lij} + \frac{1}{2}(\alpha_j g_{li} - \alpha_i g_{lj}) \\
\Rightarrow L_k{}^l(\nabla_i L_{lj} - \nabla_j L_{li}) &= L_k{}^l T_{lji} - L_k{}^l T_{lij} + L_k{}^l \frac{1}{2}(\alpha_j g_{li} - \alpha_i g_{lj}) \\
&= L_k{}^l T_{lji} - L_k{}^l T_{lij} + \frac{1}{2}(L_{ki}\alpha_j - L_{kj}\alpha_i)
\end{aligned}$$

Hence Eq. (6.3.2) becomes

$$(N_L)_{kij} = L^l{}_i T_{kjl} - L^l{}_j T_{kil} - L_k{}^l(T_{lji} - T_{lij}) = L^l{}_i T_{jkl} - L^l{}_j T_{ikl} + L_k{}^l(T_{lij} - T_{lji})$$

Now $T_{(lij)} = 0$ implies that $T_{lij} = -T_{jli} - T_{ijl}$, thus the vanishing of N_L implies:

$$2L_k{}^l T_{lji} = L^l{}_i T_{jkl} - L^l{}_j T_{ikl} - L_k{}^l T_{ijl}$$

Since the right hand side is symmetric in j, k it follows that $L_k{}^l T_{lji} = L_j{}^l T_{kli}$. Now, in invariant notation we evaluate T with different combinations of eigenvectors to show that it vanishes. First observe that for $x, y, z \in \mathfrak{X}(M)$ this equation takes the following form

$$T(Lx, y, z) = T(x, Ly, z)$$

Hence the above equation readily implies that for eigenvectors x, y with different eigenfunctions and any z, $T(x, y, z) = 0$.

Now suppose E is a multi-dimensional eigenspace with eigenfunction λ. Then λ must be a constant due to the trace-type condition by Proposition 6.2.2 (3). Let $x, y \in \Gamma(E)$ and $z \in \mathfrak{X}(M)$. First note that $L(z, y) = \lambda g(z, y)$. Then

$$\begin{aligned}
(\nabla_z L)(x, y) &= \nabla_z L(x, y) - L(\nabla_z x, y) - L(x, \nabla_z y) \\
&= \lambda(\nabla_z g(x, y) - g(\nabla_z x, y) - g(x, \nabla_z y)) \\
&= \lambda(\nabla_z g)(x, y) \\
&= 0
\end{aligned}$$

Let $m = \dim E$, then note that $\alpha(x) = m\, x\lambda = 0$ since λ is constant and because of the torsionless condition. Hence $T(x, y, z) = 0$ for all $x, y \in \Gamma(E)$ and $z \in \mathfrak{X}(M)$.

Now suppose E is a one dimensional eigenspace, $x \in \Gamma(E)$ and z is an eigenvector with a different eigenfunction than x. Then the cyclic condition implies that $T(x,x,z) = -2T(z,x,x) = 0$ and that $T(x,x,x) = 0$.

Thus since L has a basis of eigenvectors by hypothesis, it follows that $T \equiv 0$. ∎

As a consequence of the above result, we have our first characterization of OCTs.

Proposition 6.3.6 (Characterization of orthogonal CTs)
Suppose L is a torsionless orthogonal tensor with eigenspaces $(E_i)_{i=1}^k$ and associated eigenfunctions $(\lambda_i)_{i=1}^k$. Then L is an OCT iff there is a twisted product adapted to its eigenspaces such that each twist function ρ_i can be chosen to be:

$$\rho_i^2 = \prod_{k \neq i} |\lambda_i - \lambda_k|$$

and each multidimensional eigenspace E_i is a Killing distribution, or equivalently the eigenfunction corresponding to E_i is constant. □

Proof First note that Proposition 6.3.1 together with the above proposition shows that an orthogonal CT is precisely an orthogonal trace-type CKT with vanishing torsion. Since imposing the trace-type condition is equivalent to requiring the multidimensional eigenspaces to be Killing, the result then follows from Proposition 6.2.1. ∎

Also since any torsionless CKT with simple eigenfunctions is necessarily of trace-type (see Proposition 6.2.2), the above characterization implies that a Benenti tensor is precisely a CT with simple eigenfunctions.

The following proposition gives another characterization of OCTs which is designed to answer the following question: Given a warped product and an ICT $\tilde{L}$ on the geodesic factor of the warped product, can we extend $\tilde{L}$ to an OCT on the warped product and if so what is this extension?

Proposition 6.3.7 (Characterization of Reducible OCTs)
Suppose $L \in S^2(M)$ is an orthogonal tensor. Then L is a reducible OCT iff there exists a warped product decomposition $M = M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ with adapted contravariant metric $G = \sum_{i=0}^k G_i$ such that L has the following contravariant form:

$$L = \tilde{L} + \sum_{i=1}^k \lambda_i G_i$$

where each $\lambda_i \in \mathbb{R}$ and $\tilde{L} \in \hat{S}^2(M_0)$ is the canonical lift (see Section 3.3) of an ICT $\tilde{L} \in S^2(M_0)$ satisfying the following equation on M_0 for each $i > 0$

$$\tilde{L}(d \log \rho_i) = d(\lambda_i \log \rho_i + \frac{1}{2} \operatorname{tr}(\tilde{L})) \tag{6.3.3}$$

♢

Proof Suppose L is an OCT. Let $D_1, \ldots, D_l$ be the eigenspaces of L associated with constant eigenfunctions and let $M = M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ be a warped product adapted to $(\bigcap_{i=1}^{l} D_i^{\perp}, D_1, \ldots, D_l)$ which exists by Proposition 6.3.6. We define $\tilde{L}$ to be the restriction of L to M_0; it follows by Proposition 6.3.6 that $\tilde{L}$ is an ICT in M_0. It also follows by Proposition 6.3.6 that we can assume

$$\rho_i^2 = \prod_a |\lambda_i - \lambda_a| \tag{6.3.4}$$

where a ranges over all eigenfunctions of $\tilde{L}$. If $\dim M_0 = 0$, i.e. L induces a pseudo-Riemannian product, the conclusion follows. Otherwise, since λ_i is constant and because $\tilde{L}$ is torsionless, we see that on M_0

$$\begin{aligned}
\tilde{L}(\mathrm{d}\log\rho_i) &= \frac{1}{2}\sum_a \lambda_a \mathrm{d}\log|\lambda_i - \lambda_a| \\
&= \frac{1}{2}\sum_a \lambda_a \frac{\mathrm{d}\lambda_a}{\lambda_a - \lambda_i} \\
&= \frac{\lambda_i}{2}\sum_a \frac{\mathrm{d}\lambda_a}{\lambda_a - \lambda_i} + \frac{1}{2}\sum_a \mathrm{d}\lambda_a \\
&= \mathrm{d}(\lambda_i \log\rho_i + \frac{1}{2}\operatorname{tr}(\tilde{L}))
\end{aligned}$$

Conversely, it is easily checked that if $\tilde{L}$ is an ICT and ρ_i satisfies the above equation, then $c\rho_i$ must satisfy Eq. (6.3.4) for some $c \in \mathbb{R}^+$. Hence it follows that L defined in the statement is torsionless and then by Proposition 6.3.6 that L is a reducible OCT. ■

The above proposition will be applied to classify reducible OCTs in spaces of constant curvature (see Section 9.5).

6.3.2 Relation to Geodesically Equivalent Metrics

In this section we will briefly describe how concircular tensors appear in the study of geodesically equivalent metrics. This is an important connection as geometers have studied CTs with this interest in mind [Sha00]. We follow [BM03] in our exposition, see also [Ben05].

First, we have the following definition from [BM03].

Definition 6.3.8
Two metrics g and $\bar{g}$ are *geodesically equivalent* if they have the same geodesics (considered as unparameterized curves). □

Note that some authors use the term *projectively equivalent* instead. The study of such metrics dates back to the late nineteenth century, see [BM03] for historical references. The following theorem significantly simplifies the search for geodesically equivalent metrics [Ben05]:

Theorem 6.3.9
Given a concircular tensor L for a metric g, one can construct another metric, $\bar{g}$, which is geodesically equivalent to g. Conversely, given two geodesically equivalent metrics g and $\bar{g}$, a concircular tensor for g can be constructed from them. □

We will not get into the details of this construction, see [BM03; Ben05]. This connection will allow us to obtain results on CTs from the theory of geodesically equivalent metrics, as we will see in the next section.

On a related note, concircular tensors also appear in classical mechanics in the following way. Given a Riemannian manifold (M, g), and a force vector field $F \in \mathfrak{X}(M)$, consider the following dynamical equation:

$$\nabla_X X = F, \qquad X = \dot{\gamma}$$

where $\gamma(t)$ is a curve in M. A solution, γ, of the above equation is called a *geodesic* for the system (M, g, F). When $M = \mathbb{E}^3$, the above equation is simply Newton's equation with a position dependent force F.

A natural question in this context is: when are the geodesics of the above system equivalent to those of a natural Hamiltonian system on M? It turns out that a necessary condition is that g admits a non-singular concircular tensor L such that (see Theorem 13.2 in [Ben05]):

$$F = -A^{-1}\nabla V, \qquad A = \operatorname{cof}(L)$$

where $\operatorname{cof}(L)$ is the cofactor tensor of L (see Eq. (A.0.3)). Locally the above condition is equivalent to $\mathrm{d}(AF) = 0$. The geodesically equivalent natural Hamiltonian system is the one obtained from the geodesically equivalent metric $\bar{g}$ constructed from g and L (see Theorem 6.3.9) and the potential V in the above equation. Systems admitting such force vectors generalize conservative mechanical systems and are called *cofactor systems* in the literature; see [Ben05] and references therein for more on these systems.

Since concircular tensors appear in a few different areas of research, they go by several different names. They have been called special conformal Killing tensors in [Cra03], J-tensors in [Ben05], Sinyukov mappings in [Mat05], Benenti tensors in [BM03], and finally elliptic coordinates matrices in [Lun03].

6.3.3 Existence in arbitrary manifolds

In this section we will briefly review some results on the existence of concircular tensors in general Riemannian manifolds. The connection to the theory of geodesically equivalent metrics provides for some fruitful cross-pollination of results, the following being one of them [Cra07]:

Theorem 6.3.10 (Lacunae in the dimension of the space of CTs [Sha00])
Let M^n be a Riemannian manifold with $n > 2$ and set $m := \dim \mathrm{C}^1(M)$ which is the dimension of the space of concircular vectors. If $m \neq n+1$, then we have the following estimate:

$$\frac{1}{2}m(m+1) + 1 \leq \dim \mathrm{C}^2(M) \leq \frac{1}{2}m(m+1) + \mathrm{int}(\frac{1}{3}(n+1-m))$$

where $\mathrm{int}(r)$ *is the integer part of* r*.* □

In order for the above result to be of maximum value, it helps to have similar results for concircular vectors:

Theorem 6.3.11 (Lacunae in the dimension of the space of CVs [Cra07])
Let M^n be a Riemannian manifold with $n > 2$. Then the dimension of the space of concircular vectors, $\dim \mathrm{C}^1(M)$*, satisfies:* $\dim \mathrm{C}^1(M) \leq n-2$ *or* $\dim \mathrm{C}^1(M) = n+1$*.* □

Table 6.1: $\dim \mathrm{C}^2(M)$ if $n = 3$

$\dim \mathrm{C}^1(M)$	$\dim \mathrm{C}^2(M)$
0	1
1	2
4	10

Table 6.2: $\dim \mathrm{C}^2(M)$ if $n = 4$

$\dim \mathrm{C}^1(M)$	$\dim \mathrm{C}^2(M)$
0	1
1	2
2	4
5	15

Combining the above theorems, we have summarized their implications for low dimensions in Tables 6.1 and 6.2. One can deduce from the above table that any Riemannian 3-manifold admitting a Benenti tensor is a space of constant curvature [Cra07]. We also mention the following fact, which is Theorem 2 in [Cra03].

Theorem 6.3.12 ([Cra03])
Suppose M^n is a pseudo-Riemannian manifold with $n > 2$. If M admits two CTs, one of which is an ICT, and at each point they have no non-trivial common invariant subspaces, then M is a space of constant curvature. □

For applications to general relativity, see [Gro11].

6.4 Concircular vectors*

Concircular vectors aren't directly useful for separation of variables, so this section is mainly optional. Although one indirect use follows from the fact that concircular vectors can be used to obtain CTs in spaces of constant curvature (see Corollary 6.1.4). Suppose $r \in \mathfrak{X}(M)$ is a *concircular vector (CV)*, then Equation (6.0.1) must be satisfied, i.e

$$\nabla_x r = \phi x \tag{6.4.1}$$

for all $x \in \mathfrak{X}(M)$ and some fixed $\phi \in \mathcal{F}(M)$. The material we present here is mainly from [Cra07] where results on CTs were obtained by using the corresponding results on CVs as motivation. See also section 3.4 in [Ami03], which contain more references and other applications of these vectors.

We first explain where this object comes from, following Crampin in [Cra07]. A *concircular transformation* of a pseudo-Riemannian manifold (M, g) is a conformal transformation ($g \to \rho^2 g$) which maps circles into circles (see Section 3.2) [Yan40]. It was shown by Yano in [Yan40] that in the Riemannian case, a necessary and sufficient condition for this is that $\nabla\rho$ satisfy Eq. (6.4.1). Concircular tensors are generalizations of concircular vectors to higher valence.

We give some examples as follows:

Example 6.4.1
Any covariantly constant vector field gives a trivial example of a CV with $\phi \equiv 0$. □

Example 6.4.2 (The Dilatational vector field in $\mathbb{E}^n_\nu$)
A non-trivial prototypical example is given in pseudo-Euclidean space as follows. Set $M = \mathbb{E}^n_\nu$ and let (x^i) be Cartesian coordinates for M. Let $r := \sum_i x^i \partial_i$, then for $v \in \mathfrak{X}(M)$

$$(\nabla_v r)^j = v^i \partial_i x^j = v^j$$

Hence r is a concircular vector with $\phi \equiv 1$, which is known as the *dilitational vector field* (in $\mathbb{E}^n_\nu$). □

We can easily calculate the general CV in $\mathbb{E}^n_\nu$:

Proposition 6.4.3 (Concircular vectors in $\mathbb{E}^n_\nu$ [Cra07])
A vector $v \in \mathfrak{X}(\mathbb{E}^n_\nu)$ is a CV in $\mathbb{E}^n_\nu$ iff there exists $a \in C^0_0(\mathbb{E}^n_\nu)$ and $b \in C^1_0(\mathbb{E}^n_\nu)$ such that

$$v = ar + b$$

where r is the dilatational vector field. □

PROOF In $\mathbb{E}^n_\nu$ with canonical Cartesian coordinates (x^i), Eq. (6.4.1) becomes:

$$\frac{\partial v^i}{\partial x^j} = \phi \delta^i{}_j$$

This equation can be easily solved by observing the following:

$$\frac{\partial \phi}{\partial x^k}\delta^i{}_j = \frac{\partial^2 v^i}{\partial x^k \partial x^j} = \frac{\partial \phi}{\partial x^j}\delta^i{}_k$$

Thus taking $i = j \neq k$, we find that $\frac{\partial \phi}{\partial x^k} = 0$. Thus $\phi \in \mathbb{R}$ and we find that v must have the form given by $v^i = \phi x^i + b^i$ where each $b^i \in \mathbb{R}$. ■

Then using Corollary 6.1.4 we can deduce the general CT in $\mathbb{E}^n_\nu$:

Proposition 6.4.4 (Concircular 2-tensors in $\mathbb{E}^n_\nu$)
L is a concircular 2-tensor in $\mathbb{E}^n_\nu$ iff there exists $A \in C^2_0(\mathbb{E}^n_\nu)$, $w \in C^1_0(\mathbb{E}^n_\nu)$ and $m \in C^0_0(\mathbb{E}^n_\nu)$ such that:

$$L = A + 2w \odot r + mr \odot r$$

where r is the dilatational vector field. The tensors A, w and m are uniquely determined by L. □

The dilatational vector field in $\mathbb{E}^n_\nu$ gives us some intuition for CVs. The following proposition shows that many of the properties held by the dilatational vector field are shared by general CVs [Cra07].

Proposition 6.4.5 (Properties of Concircular vectors)
Suppose $r \in \mathfrak{X}(M)$ is a concircular vector. Let $E := r^\perp$ and $r^2 = \langle r, r\rangle$, then the following statements are true:

1. *r is a conformal Killing vector with conformal factor ϕ.*
2. *$\nabla_r r = \phi r$, so the integral curves of r are affinely parameterized geodesics.*
3. *$dr^\flat = 0$, so r is of gradient-type.*
4. *If r is non-null, then E is a Killing distribution, i.e. E is spherical with geodesic orthogonal complement.*
5. *$dr^2(E) = 0$* □

Proof The first property follows since we have shown that concircular tensors are conformal Killing tensors. The second property follows by definition.

To verify the third property, suppose $x, y \in \mathfrak{X}(M)$ and $[x, y] = 0$, then

$$\begin{aligned} \mathrm{d}r^\flat(x, y) &= x(\langle y, r\rangle) - y(\langle x, r\rangle) \\ &= \langle \nabla_x y, r\rangle + \langle y, \nabla_x r\rangle - \langle \nabla_y x, r\rangle - \langle x, \nabla_y r\rangle \\ &= \langle [x, y], r\rangle + \phi \langle y, x\rangle - \phi \langle x, y\rangle \\ &= 0 \end{aligned}$$

The last two properties follow from the fact that if r is non-null, then $r \odot r$ is an orthogonal concircular tensor. Although one can prove these properties directly, for example if $x \in \Gamma(E)$, then

$$\begin{aligned} x(r^2) &= 2 \langle \nabla_x r, r\rangle \\ &= 2 \langle \phi x, r\rangle \\ &= 0 \end{aligned}$$

thus property 5 holds even if r is null. ■

Note that property 3 implies that a non-null CV r naturally induces a warped product. In fact, if r is non-null, then $r \odot r$ is an OCT, hence from Proposition 6.3.6, we can choose an adapted warped product metric to be:

$$g = r^2 g' + r^2 \tilde{g}$$

6.5 KEM webs

Our main motivation for working with concircular tensors is because they are invariants of Kalnins-Eisenhart-Miller (KEM) webs. In this section we will define KEM webs and show that they are separable. We will see throughout this chapter that several questions concerning KEM webs can be answered using their defining concircular tensors.

Before we introduce the general notion of a KEM web, we first present the following simple motivating example:

Example 6.5.1 (KEM webs)
In this example we work in $\mathbb{E}^3$ with the CT $L = d \odot d$ where $d \neq 0$ is a constant vector. In this case, L has a simple eigenspace $S_1 := \operatorname{span}\{d\}$ and a multidimensional eigenspace

$D_1 := d^{\perp}$. Clearly a warped product manifold adapted to the WP-net (S_1, D_1) is $\mathbb{E}^1 \times \mathbb{E}^2$.

Now in $\mathbb{E}^2$ we can specify a Cartesian coordinate system via the CT $L = A$ where A is symmetric, constant and has simple eigenspaces. We can also specify polar coordinates via the CT $L = r \odot r$ where r is the dilatational vector field as in the previous section. In both cases it is well known that this defines a separable web $\mathcal{E}_1$ in $\mathbb{E}^2$.

Back in $\mathbb{E}^3$ we can define an orthogonal web, $\mathcal{E}$, formed by S_1 together with the lift of $\mathcal{E}_1$ (which is obtained by translating $\mathcal{E}_1$ along d). In the first case we obtain a web defining Cartesian coordinates and in the second case we obtain a web defining cylindrical coordinates, both of which are separable. □

We have shown two examples where an orthogonal (in fact separable) web was obtained recursively using concircular tensors. For low dimensions we define a KEM web as follows: When $n = 1$ the tangent bundle TM itself is trivially defined to be a KEM web. When $n = 2$ any non-trivial OCT has simple eigenfunctions, hence is a Benenti tensor and defines an orthogonal web. So when $n = 2$ we define a KEM web to be any orthogonal web associated with a Benenti tensor. In the general case we define recursively a KEM web as follows:

Definition 6.5.2 (KEM web)

Let L be a non-trivial OCT with simple eigenspaces $(S_i)_{i=1}^k$ and multidimensional eigenspaces $(D_i)_{i=1}^l$. For each $i = 1, ..., l$, let $\mathcal{E}_i$ be a KEM web on an integral manifold of D_i. Then the web formed by $(S_i)_{i=1}^k$ together with the lifts of $\mathcal{E}_i$ is called a *Kalnins-Eisenhart-Miller (KEM) web*. □

Remark 6.5.3

One can check that the above definition is well-defined since each D_i is necessarily integrable and the lift of $\mathcal{E}_i$ is necessarily an orthogonal web at least locally. □

Theorem 6.5.4 (KEM webs)

A KEM web is a separable web. □

Proof Suppose inductively that this theorem holds for all KEM webs with dimension $k < n$ and note that the statement trivially holds for $k = 1$ since the metric is always an OCT. Now we prove the proposition for KEM webs of dimension $n > k \geq 1$.

Let L be the OCT in the definition of the KEM web and let K be the KBDT associated with L (see Eq. (6.3.1)). Let $D_1, ..., D_l$ be the multidimensional eigenspaces of L. These are necessarily Killing distributions by Proposition 6.3.6. Then the net formed by $D_1, ..., D_l$ together with $D_0 := \bigcap_{i=1}^{l} D_i^{\perp}$ is a WP-net. So fix $\bar{p} \in M$ and let $N = \prod_{i=0}^{l} N_i$ be a connected product manifold adapted to this net and passing through $\bar{p}$. For each $i = 1, ..., l$, let K_i be a ChKT for $\mathcal{E}_i$ on N_i which is given by Theorem 5.3.2. It follows from Proposition 4.5.2 that K_i can be extended to a KT on M (which we call K_i). After adding a constant multiple of the induced metric on N_i to K_i if necessary, we can assume that $K + \sum_{i=1}^{l} K_i$ is a ChKT at least locally. Since $K + \sum_{i=1}^{l} K_i$ is a ChKT for this KEM web, it follows from Theorem 5.3.2 that this KEM web is a separable web.

Thus the result follows by induction. ∎

KEM webs are associated to the separable coordinate systems originally discovered by Kalnins and Miller (see [Kal86]) for spaces of constant curvature. We will show later on that in spaces of constant curvature, the converse of the above theorem is true (see Theorem 7.1.1). We also note here that it's clear from the proof above, that (in principle) OCTs can be used to construct separable webs which are not necessarily KEM webs. Although KEM webs will be the most straightforward to analyze, so we will work exclusively with them.

Any coordinates adapted to a KEM web are called *KEM coordinates*. We now show how to reduce the problem of obtaining KEM coordinates for a KEM web to the special case of webs associated with Benenti tensors. Fix a non-trivial OCT L as in the definition of a KEM web and assume it has a multidimensional eigenspace. Let $\psi : N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k \to M$ be a local warped product decomposition adapted to the WP-net induced by the multidimensional eigenspaces of L. Since $L|_{N_0}$ is a Benenti tensor, there exist local coordinates (x_0) on N_0 which diagonalize $L|_{N_0}$. Inductively assume for each $i > 0$ that $(x_i) = (x_i^1, \ldots, x_i^{n_i})$ are separable coordinates for N_i adapted to the KEM web $\mathcal{E}_i$. Then the above theorem shows that the product coordinates $\psi(x_0, x_1, \ldots, x_k)$ are separable coordinates for M. Some examples of this construction are given in Sections 2.3.2 and 9.6.2.

Hence constructing KEM coordinates reduces to constructing coordinates adapted to a Benenti tensor. For Benenti tensors which are also ICTs, canonical coordinates can be constructed for the associated webs (see the discussion following Definition 6.3.3). The case for more general Benenti tensors can be reduced to ICTs using appropriate warped products (see Proposition 6.2.2 (2)).

We now give another property of KEM coordinates which will be important for their further study. First we need a definition. An orthogonal coordinate system is said to have *diagonal curvature* if the Riemann curvature tensor satisfies $R_{ijik} = 0$ for $j \neq k$ in the coordinate induced basis. This definition is equivalent to requiring the curvature operator (which is a $\binom{2}{2}$-tensor associated with R which induces a map in $\mathrm{End}(\wedge^2(M))$ [Pet06]) to be diagonal in the coordinate induced basis. One can also check that the diagonal curvature condition implies that the Ricci tensor is diagonalized. Now, we have the following result.

Proposition 6.5.5

KEM coordinates have diagonal curvature, and hence they diagonalize the Ricci tensor. □

Proof Assume that (x^i) are KEM coordinates. First, observe that the metric necessarily has the following form:

$$g = \sum_{a\in M} e_a \rho_a^2 \mathrm{d}x_a^2 + \sum_{I\in P} \rho_I^2 g^I$$

where $\{1,\ldots,n\} = M \cup (\cup_{I\in P} I)$ is a partition (here P is an index set and each $I \in P$ is a subset of $\{1,\ldots,n\}$), each $e_a = \pm 1$ as the case may be and each $\rho_i(x_1,\ldots,x_m)$ is a positive valued function and each g^I has the following form:

$$g_{ij}^I = \begin{cases} f_{ij}^I(x^I) & i,j \in I \\ 0 & i \notin I \end{cases}$$

Define $\tilde{g}$ as follows:

$$\tilde{g} = \sum_{a\in M} e_a \mathrm{d}x_a^2 + \sum_{I\in P} g^I$$

Let R (resp. $\tilde{R}$) denote the Riemann curvature tensor of g (resp. $\tilde{g}$). Then for $i \in I, j \in J, k \in K$ with i,j,k distinct, it follows from Eq. (3.5.2) that

$$\left\langle (R(\partial_i,\partial_k) - \tilde{R}(\partial_i,\partial_k))\partial_j, \partial_i \right\rangle = g(\partial_i,\partial_i)(\langle \nabla_{\partial_k} U_I - \langle \partial_k, U_I\rangle U_I, \partial_j\rangle)$$

where $U_I = -\nabla \log \rho_I$ is the negative gradient of $\log \rho_I$. By using Eq. (3.5.1), we get the following:

$$\langle \nabla_{\partial_k} U_I - \langle \partial_k, U_I\rangle U_I, \partial_j\rangle = (\partial_k \log \rho_J \partial_j + \partial_j \log \rho_K \partial_k) \log \rho_I - (\partial_k \log \rho_I)(\partial_j \log \rho_I)$$

The above vanishes if either $j \notin M$ or $k \notin M$. So we can assume further that I, J, K are distinct. Then from a direct calculation using the specific form of the twist functions (see Proposition 6.3.6), it follows that the above is identically zero in this case. Thus we have proven that if i,j,k are distinct, then

$$\langle R(\partial_i,\partial_k)\partial_j, \partial_i\rangle = \left\langle \tilde{R}(\partial_i,\partial_k)\partial_j, \partial_i \right\rangle$$

First observe that $R_{ijik} = \langle R(\partial_i,\partial_k)\partial_j, \partial_i\rangle$ and we can assume i,j,k are distinct to check the diagonal curvature condition. Also note that $\tilde{R}(\partial_i,\partial_k)\partial_j$ is not necessarily zero only if $I = J = K$ or if $i,j,k \in M$ (see Proposition 3.5.6). In the later case, clearly $R_{ijik} = 0$. In the former case the result follows by induction from the above equation. ∎

Remark 6.5.6
When the KEM coordinates are generated by an ICT, this fact was originally shown by Crampin in [Cra03]. We were motivated to generalize this fact after observing that all separable coordinates derived by Kalnins and Miller (see [Kal86]) are KEM coordinates and they have diagonal curvature. □

Remark 6.5.7
When the coordinates are separable, the additional condition that the Ricci tensor is diagonal is known as the *Robertson condition* [Eis34]. It is known that if this condition is satisfied, then the free particle Schördinger equation (a.k.a Helmholtz equation) is separable [Ben02; Eis34]. Eisenhart first gave this characterization (in terms of the Ricci tensor) in [Eis34]. The above proposition shows that KEM coordinates satisfy the Robertson condition. □

Remark 6.5.8
In a space of constant curvature any orthogonal coordinates have diagonal curvature (see Eq. (1.4.1)). This is a crucial observation that enabled Eisenhart [Eis34] and then Kalnins and Miller [KM86; KM82] to classify orthogonal separable coordinates in these spaces. We will use this observation to give an independent classification of these coordinates in Chapter 7. □

The fact that KEM coordinates are orthogonal separable with diagonal curvature is almost sufficient to characterize them. We will observe this in Chapter 7 when we will solve for all such metrics. The solution from that chapter will show that the Schwarzschild metric is orthogonally separable and has diagonal curvature. Although one can deduce from Proposition 6.3.6 that the separable web associated with this metric is not a KEM web. Furthermore, not all orthogonal separable coordinates have diagonal curvature. A simple counter-example is given by the Liouville metric [Cra05]. The Liouville metric is conformally Euclidean with each $g_{ii} = \varphi_1(x^1) + \cdots + \varphi_n(x^n)$. This metric is a classic example of an orthogonally separable metric; one can verify this using the Levi-Civita equations (see Eq. (5.2.3a)). One can show that this metric has diagonal curvature iff $\varphi_j'\varphi_k' = 0$ for each $j \neq k$ (see Eq. (3.5.2)).

6.6 The Killing-Stackel space of KEM webs

In this section we show how one can obtain Killing tensors which pair-wise commute (algebraically as linear operators) from a concircular tensor. This was the original motivation for studying Benenti tensors [Ben92a]. In contrast with Benenti's approach in which elementary symmetric polynomials are used, we will make use of the coordinate-independent theory of the cofactor tensor summarized in Appendix A.

Theorem 6.6.1 (Killing tensors from Concircular tensors [Cra03])
If L is a non-singular concircular tensor then $K := \operatorname{cof}(L)$ is a Killing tensor satisfying the following equation:

$$\nabla_x K = (\nabla_x \mu) K - \nabla \mu \odot Kx$$

for all $x \in \mathfrak{X}(M)$ where $\mu = \log |\det L|$. □

Proof We follow the proof of Theorem 3.1 in [Ben05]. Recall (Eq. (A.0.3)) that $K = \operatorname{cof}(L)$ satisfies

$$KL = LK = (\det L) I \tag{6.6.1}$$

Since K is a polynomial in L (see Eq. (A.0.4)), it follows that K is self-adjoint. Also note that Proposition B.1.1 implies that

$$L\mathrm{d}(\det L) = (\det L)\mathrm{d}(\operatorname{Tr} L) \qquad \Rightarrow K\mathrm{d}(\operatorname{Tr} L) = \mathrm{d}(\det L)$$

Differentiating Eq. (6.6.1) with respect to $x \in \mathfrak{X}(M)$ gives:

$$\nabla_x K L + K \nabla_x L = \nabla_x (\det L) I$$

Right multiplying by K gives:

$$\begin{aligned}(\det L)\nabla_x K &= \nabla_x (\det L) K - K(\nabla_x L) K \\ &= \nabla_x (\det L) K - \frac{1}{2} K(x \otimes \mathrm{d} \operatorname{Tr} L + \nabla \operatorname{Tr} L \otimes x^\flat) K \\ &= \nabla_x (\det L) K - \frac{1}{2}(Kx \otimes \mathrm{d} \det L + \nabla \det L \otimes (Kx)^\flat)\end{aligned}$$

which gives us the equation for $\nabla_x K$. Symmetrizing ∇K proves that K is a Killing tensor. ■

In our discussion, for a CT L, we will let $K_a := (\wedge^{n-1} L^a)^{\wedge *}$ which has been defined in Appendix A. We call the tensors $K_0, \ldots, K_{n-1}$ the *L-sequence* generated by L. Note that these tensors satisfy a number of identities given in Appendix A. By working with an equivalent CT, $L + cG$, for some $c \in \mathbb{R}$, we can assume (locally) that the CT is non-singular. Then by Eq. (A.0.1) and the above proposition, we see that each tensor K_a is a Killing tensor. Furthermore we note that by Eq. (A.0.4), each of these tensors are polynomials in L. These observations allow us to obtain a generalization of Benenti's theorem [Ben04] for an arbitrary OCT:

Theorem 6.6.2 ([Ben04])
Suppose L is an OCT with k distinct eigenspaces. Then L generates a k dimensional space of orthogonal Killing tensors which are point-wise independent, which algebraically and Schouten commute. This space of Killing tensors is $\mathcal{K} = \operatorname{span}\{K_0, \dots, K_{n-1}\}$ where each K_a was defined above.

In particular, if L is a Benenti tensor then $\mathcal{K}$ is the KS-space associated with the separable web induced by L. □

Proof Since each tensor K_a is a polynomial in L, it follows that they commute algebraically and are simultaneously diagonalized in any coordinate system which diagonalizes L. Thus it follows by Proposition 4.4.12 that any $K, J \in \mathcal{K}$ Schouten commute.

We now calculate the dimension of this space. First note that $\dim \mathcal{K}$ is the (abstract vector space) dimension of $\mathcal{K}$, i.e. not the point-wise dimension. Fix a point $p \in M$. Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of L at p. Then by using Proposition A.0.12, we can obtain Killing tensors $\tilde{K}_1, \dots, \tilde{K}_k$ such that at p, $\tilde{K}_i$ has eigenspaces E_i and $E_i^\perp$. This implies that $\tilde{K}_1, \dots, \tilde{K}_k$ must be independent KTs in a neighborhood of p. The fact that they are elements of $\mathcal{K}$ follows from Lemma A.0.5. This result together with the fact that the elements of $\mathcal{K}$ can be simultaneously diagonalized with L implies that the point-wise dimension of $\mathcal{K}$ is k. It also implies that $\dim \mathcal{K} \geq k$. In particular when L is a Benenti tensor, $\dim \mathcal{K} = n$, and the tensors $K_0, \dots, K_{n-1}$ are independent.

We now consider the case L is a non-trivial OCT. For convenience, we assume that L has one multidimensional eigenspace D. Let $B \times_\rho F$ be a local connected warped product adapted to $(D^\perp, D)$ with adapted contravariant metric $G = G' + \rho^{-2}\tilde{G}$. By Proposition 6.2.2 (5) it follows that $\tilde{L} := L|_B$ is a Benenti tensor. If $\tilde{K}_a$ denote the Killing tensors in the associated L-sequence, it follows[2] by Proposition 5.3.10 that these tensors admit extensions $\tilde{K}_a'$ to KTs on M which have the form:

$$\tilde{K}_a' = \tilde{K}_a + t_a \tilde{G}$$

where $\mathrm{d}t_a = \tilde{K}_a \mathrm{d}\kappa$. Let K_α denote the Killing tensors in the L-sequence generated by L. Since the tensors $\tilde{K}_a$ form a basis for the KS-space generated by $\tilde{L}$, it follows by Proposition 4.5.2 that there exist constants ${A_\alpha}^a$ such that

$$K_\alpha = \sum_a {A_\alpha}^a (\tilde{K}_a + t_a \tilde{G}) + c_\alpha \tilde{G}$$

for some $c_\alpha \in \mathbb{R}$. This proves that $\dim \mathcal{K} \leq k$, hence $\dim \mathcal{K} = k$. The general case follows similarly.

[2]See the discussion following the proof for more details.

The last fact concerning the case when L is a Benenti tensor follows from Corollary 5.3.3. ■

As a corollary, we see that if L is a Benenti tensor, then the L-sequence forms a basis for the KS-space generated by L. The proof also shows us how to obtain a basis for $\mathcal{K}$ in the more general case. Indeed, suppose $M = B \times_\rho F$ is a warped product with adapted contravariant metric $G = G' + \tilde{G}$ and L is a Benenti tensor on B satisfying:

$$L(\mathrm{d}\log\rho) = \mathrm{d}(c\log\rho + \frac{1}{2}\operatorname{tr}(L)) \tag{6.6.2}$$

for some $c \in \mathbb{R}$. Then $\bar{L} := L + c\tilde{G}$ is a CT on M by Proposition 6.3.7. Hence by Remark 6.2.3 we see that $\bar{K} := \operatorname{tr}(\bar{L})G - \bar{L}$ is KT on M which pulls back to a ChKT on B. Thus by Proposition 5.3.10, the KS-space generated by L on B can be extended into M. The following proposition explicitly gives this extended KS-space.

Proposition 6.6.3 (Extension of the L-sequence)
Suppose $B \times_\rho F$ is a warped product and L is a Benenti tensor on B as above. Then the L-sequence can be extended into independent KTs on M given as follows in contravariant form:

$$\bar{K}_a := K_a + (\sum_{i=0}^{a}(-c)^i\sigma_{a-i})\tilde{G}$$

for each $0 \leq a \leq m-1$ where $m = \dim B$ and $\sigma_a := (\wedge^m L^a)^{\wedge *}$. In terms of $\bar{L}$, they have the following form:

$$\bar{K}_a = \sum_{i=0}^{a}(-1)^i\sigma_{a-i}\bar{L}^i \qquad \diamondsuit$$

Proof Throughout this proof we work exclusively on B. Denote by K_a the L-sequence generated by L. It follows by Eq. (A.0.2) that they satisfy the following equations:

$$K_0 = I, \qquad K_a = \sigma_a I - K_{a-1}L \quad 1 < a < m \tag{6.6.3}$$

Furthermore since L is torsionless, Eq. (B.1.1) from Proposition B.1.2 implies the following:

$$K_{a-1}\mathrm{d}\operatorname{tr}(L) = \mathrm{d}\sigma_a \tag{6.6.4}$$

We now proceed to calculate the function t in Eq. (4.5.1) for each K_a. For $a > 1$, by using Eqs. (6.6.2) and (6.6.3) we have

$$\begin{aligned} K_a \mathrm{d}\log\rho &= \sigma_a \mathrm{d}\log\rho - K_{a-1}\mathrm{d}(c\log\rho + \frac{1}{2}\operatorname{tr}(L)) \\ &\overset{(6.6.4)}{=} \sigma_a \mathrm{d}\log\rho - cK_{a-1}\mathrm{d}\log\rho - \frac{1}{2}\mathrm{d}\sigma_a \end{aligned}$$

Then if we let $\kappa := \rho^{-2}$, we have

$$\begin{aligned} K_a \mathrm{d}\kappa &= \kappa K_a \mathrm{d}\log\kappa \\ &= -2\kappa K_a \mathrm{d}\log\rho \\ &= \sigma_a \mathrm{d}\kappa - cK_{a-1}\mathrm{d}\kappa + \kappa \mathrm{d}\sigma_a \\ &= \mathrm{d}(\kappa\sigma_a) - cK_{a-1}\mathrm{d}\kappa \end{aligned} \tag{6.6.5}$$

which gives us a recursive equation for $K_a \mathrm{d}\kappa$. We solve it as follows:

$$\begin{aligned} K_1 \mathrm{d}\kappa &= \mathrm{d}(\kappa\sigma_1) - c\mathrm{d}\kappa \\ &= \mathrm{d}((\sigma_1 - c)\kappa) \\ K_2 \mathrm{d}\kappa &= \mathrm{d}(\kappa\sigma_2) - c\mathrm{d}((\sigma_1 - c)\kappa) \\ &= \mathrm{d}((\sigma_2 - c(\sigma_1 - c))\kappa) \\ \Rightarrow K_a \mathrm{d}\kappa &= \mathrm{d}((\sum_{i=0}^{a}(-c)^i \sigma_{a-i})\kappa) \end{aligned}$$

We check the above equation for $K_a \mathrm{d}\kappa$ using induction:

$$\begin{aligned} K_a \mathrm{d}\kappa &\overset{(6.6.5)}{=} \mathrm{d}(\kappa\sigma_a) - cK_{a-1}\mathrm{d}\kappa \\ &= \mathrm{d}((\sigma_a - c(\sum_{i=0}^{a-1}(-c)^i \sigma_{a-1-i}))\kappa) \\ &= \mathrm{d}((\sigma_a + (\sum_{i=0}^{a-1}(-c)^{i+1} \sigma_{a-1-i}))\kappa) \\ &= \mathrm{d}((\sum_{i=0}^{a}(-c)^i \sigma_{a-i})\kappa) \end{aligned}$$

Thus it follows by Proposition 4.5.2 that each $\bar{K}_a$ is a KT on M. The second formula for $\bar{K}_a$ follows from Eq. (A.0.4). ■

Remark 6.6.4

The conclusions follow as long as L is a CT. □

Thus the above two propositions together with Proposition 5.3.12 reduces the calculation of the KS-space for KEM webs to algebraic operations (provided the appropriate warped product decompositions are known). We also note that one can construct a ChKT for any KEM web using the KBDTs associated with the defining CTs. This construction is elaborated in the proof of Theorem 6.5.4.

We now end this section with the expected result that the KS-space generated by an ICT is not reducible. In order to do this, it is sufficient to first work with orthogonal CKTs. Suppose L is an orthogonal CKT with orthogonally integrable eigenspaces $E_1, ..., E_k$ and associated eigenfunctions $\lambda_1, ..., \lambda_k$. Let $D_c \subset \{1, \dots, k\}$ with $|D_c| \geq 2$ and define $D = \bigoplus_{i \in D_c} E_i$. We will need the following lemma which gives the mean curvature normal of D:

Lemma 6.6.5

Suppose $\mathcal{E} = (E_i)_{i=1}^k$ is a TP-net and let $D = \bigoplus_{i \in D_c} E_i$ (similar to above), then the mean curvature normal H_D of D is given as follows:

$$H_D = \sum_{i \in D_c} \frac{m_i}{d} H_i^{\perp D}$$

where H_i is the mean curvature normal of E_i, $m_i = \dim E_i$ and $d = \dim D$. In particular D is umbilical iff $H_i^{\perp D} = H_D = H_j^{\perp D}$ for each $i, j \in D_c$. □

Proof By Eq. (3.5.1), the second fundamental form of D is

$$h(x, y) = \sum_{i \in D_c} \langle x^i, y^i \rangle U_i^{\perp D}$$

Hence the formula for H_D follows since $H_i^{\perp D} = U_i^{\perp D}$ for each $i \in D_c$. Then D is umbilical iff $H_D = H_i^{\perp D}$ for each $i \in D_c$, hence the result follows. ■

We now assume that D is umbilical. From the above lemma, D is umbilical iff $H_i^{\perp D} = H_j^{\perp D}$ for each $i, j \in D_c$. Thus from Corollary 4.4.6, the following equation must be satisfied for $i, j \in D_c$ and $k \notin D_c$.

$$\begin{aligned} &(\nabla \log |\lambda_i - \lambda_k|)^k = (\nabla \log |\lambda_j - \lambda_k|)^k \\ \Leftrightarrow\ &(\lambda_j - \lambda_k)(\nabla(\lambda_i - \lambda_k))^k = (\lambda_i - \lambda_k)(\nabla(\lambda_j - \lambda_k))^k \end{aligned}$$

Thus $\log \left| \dfrac{\lambda_i - \lambda_k}{\lambda_j - \lambda_k} \right|$ is independent of E_k. Now if L is a torsionless orthogonal CKT this condition simplifies to

$$\begin{aligned}(\lambda_j - \lambda_k)\nabla\lambda_k = (\lambda_i - \lambda_k)\nabla\lambda_k \\ \Leftrightarrow (\lambda_j - \lambda_i)\nabla\lambda_k = 0 \\ \Leftrightarrow \nabla\lambda_k = 0 \quad \text{for each k} \notin D_c\end{aligned}$$

Hence we have the following:

Proposition 6.6.6 (Some Geometrical properties of torsionless CKTs)
Suppose L is a torsionless orthogonal CKT with orthogonally integrable eigenspaces. If a distribution D (constructed as above) containing at least two eigenspaces is umbilical, then λ_k is a constant for each $k \notin D_c$. Thus we can deduce the following:

1. *Any such distribution D induces a warped product with D as the geodesic distribution and the eigenspaces complementary to D as the Killing distributions.*
2. *In particular if $\dim D = n - 1$, then $D^\perp$ is tangent to a Killing vector field.*
3. *If the eigenfunctions of L are functionally independent then any L-invariant distribution with dimension greater than one is not umbilical.* ◊

Remark 6.6.7
This proposition contains a well known property of Benenti tensors stated in some articles [Ben05; CRA07]. Namely that a simple eigenspace of a torsionless orthogonal CKT cannot be tangent to proper CKV (a CKV which is not a KV) if its orthogonal complement contains more than one eigenspace. This follows directly from property 2 above and the fact that the orthogonal complement of normal non-null CKV is umbilical. □

Hence the above proposition shows that the KS-space generated by an ICT is not reducible. In fact, even more can be said:

Corollary 6.6.8 (The KS-space of an ICT)
Suppose L is an ICT. If K is in the Killing-Stäckel space generated by L, then K is either a constant multiple of the metric or characteristic. ◊

Proof By hypothesis K is diagonalized in any coordinate system adapted to the eigenspaces of L. If K has an eigenspace D which has dimension d satisfying $1 < d < n$ then D is umbilical by Corollary 4.4.6. Also D is a direct sum of at least two eigenspaces of L, hence by Proposition 6.6.6 at least one of the eigenfunctions of L must be a constant, a contradiction. Thus we conclude that either K is characteristic or has a single eigenspace in which case it must be a constant multiple of the metric. ■

Remark 6.6.9
This corollary highlights a weakness of the intrinsic characterization of separable webs using ChKTs (Theorem 5.3.2). The non-uniqueness of ChKTs makes it more difficult to classify the associated webs using them. This is in sharp contrast with CTs, see Proposition 6.2.5. Although the cost of working with CTs is that, in general, we have to work with multiple CTs. □

Proposition 6.6.6 can also be used to derive properties of the Killing-Stäckel space, $\mathcal{K}$, of a Benenti tensor. For example, suppose L is a Benenti tensor with only one constant eigenfunction and the rest of which are non-constant. In this case $\mathcal{K}$ contains a rank 1 KT, say V, which is the tensor product of a KV with itself by Proposition 6.2.2. Then one can use Proposition 6.6.6 to deduce that all KTs in $\mathcal{K}$ must be either a constant multiple of the metric, a KT sharing the same eigenspaces as V, or a ChKT.

6.7 Separation in KEM webs: The BEKM Separation Algorithm

In this section we will present the *Benenti-Eisenhart-Kalnins-Miller (BEKM) separation algorithm*, which is named after the researchers whose work anticipated this algorithm [Ben05; Eis34; KM86]. We fix a potential $V \in \mathcal{F}(M)$ and suppose $n = \dim M > 1$. We present a tractable intrinsically defined algorithm to determine separability of the natural Hamiltonian associated with V in a KEM web.

This algorithm is developed using the structure of KEM webs. In the proof of Theorem 6.5.4, we showed how to construct a ChKT for a KEM web using KBDTs associated with the defining CTs. We now observe that given a KEM web $\mathcal{E}$, the KBDT, K', associated with the first CT defining this web is in the KS-space associated with $\mathcal{E}$. Thus by Theorem 5.4.1 any potential separable in $\mathcal{E}$ must satisfy the dKdV equation with K'. We use these observations and the theory of the separation of the Hamilton-Jacobi equation in warped products (see Section 5.4.1) to obtain a recursive algorithm to find separable coordinates for V.

Remark 6.7.1
The authors originally discovered the necessity of KBDTs for $\mathbb{E}^n$ and $\mathbb{S}^n$ implicitly through Corollary 5.4 in [WW03]. Indeed, according to the remarks following Equation 4.2 in [Ben04], the Bertrand-Darboux equations in [WW03] are the dKdV equations generated by a KBDT. Hence Corollary 5.4 in [WW03] implies the necessity of KBDTs for the special case of $\mathbb{E}^n$. Corollary 5.4 in [WW03] also implies a similar statement for $\mathbb{S}^n$. This explains the origin of the name Bertrand-Darboux in Killing-Bertrand-Darboux tensor and one of our initial reasons for working with CTs. □

Now we present the BEKM separation algorithm, so assume M is an arbitrary pseudo-Riemannian manifold. Let L denote the general concircular tensor on (M, g) and $K := \mathrm{tr}(L)G - L$ be the KBDT generated by L. Now impose the condition:

$$\mathrm{d}(K\mathrm{d}V) = 0 \tag{6.7.1}$$

which is called the *Killing Bertrand-Darboux (KBD)* equation. The above equation defines a system of linear equations in the unspecified parameters of L. Indeed, by Theorem 6.1.2, the C-tensors form a finite-dimensional vector space. Since the KBDT is linearly related to L, it follows that the above equation defines a linear system. Furthermore by Theorem 6.1.2 the maximum number of unknowns in the above equation is $\frac{1}{2}(n+1)(n+2)$.

Suppose now that K is a particular solution of the KBD equation and let L be the associated C-tensor. We make the assumption that L is an orthogonal tensor (which is always satisfied on a Riemannian manifold). Let $(E_i)_{i=1}^k$ be the eigenspaces of L and $(\lambda_i)_{i=1}^k$ the corresponding eigenfunctions. We now classify such a solution:

Case 1 (k = 1, i.e. all the eigenfunctions coincide)

In this case $L = cG$ where $c := \lambda_1 \in \mathbb{R}$, thus the associated KBDT, $K = c(n-1)G$ is the trivial solution of Eq. (6.7.1) and so the algorithm yields no information.

Case 2 (the eigenfunctions are simple)

K is a characteristic Killing tensor, then by Benenti's theorem (Theorem 5.4.1), V is separable in the web of the eigenspaces of L.

Case 3 (at least one eigenfunction is not simple)

In this case, we enumerate the eigenspaces $D_1, ..., D_l$ with dimension greater than one. Since each D_i is Killing by Proposition 6.3.6, the net formed by $D_1, ..., D_l$ together with $D_0 := \bigcap_{i=1}^{l} D_i^{\perp}$ is a WP-net. So fix $\bar{p} \in M$ and let $N = \prod_{i=0}^{l} N_i$ be a connected product manifold adapted to this net and passing through $\bar{p}$.

If $D_0 \neq 0$, then K restricted to D_0 is characteristic by construction. Let $V_i := \tau_i^* V \in \mathcal{F}(N_i)$ and suppose for each $i = 1, ..., k$ there exists a ChKT $\tilde{K}_i$ on N_i such that $\mathrm{d}(\tilde{K}_i \mathrm{d}V_i) = 0$.

Then by Theorem 5.4.4, V is separable in the web formed by the simple eigenspaces of L together with the lifts of the simple eigenspaces of $\tilde{K}_1, ..., \tilde{K}_l$.

The algorithm can be applied recursively in the case L has a non-simple eigenfunction. In the notation of case 3 one would have to apply the algorithm to each N_i equipped with the induced metric for $i = 1, ..., l$.

Now, some remarks are in order:

Remark 6.7.2
In case 3 even if there are no ChKTs on the submanifolds N_i which satisfy the dKdV equation with V_i, one should be able to prove that the Hamilton-Jacobi equation is partially separable. By Theorem 6.6.2, one can at least obtain first integrals for the Hamiltonian. □

Remark 6.7.3
Since the metric is always a solution of the KBD equation and because the KBD equation is linear in K, we always consider a solution of the KBD equation modulo multiples of the metric. □

In the following example we will show how to use the theory just presented to show that the Calogero-Moser system is separable in cylindrical coordinates. It was originally shown to be separable in these coordinates by Calogero in [Cal69].

Example 6.7.4 (Calogero-Moser system)
The Calogero-Moser system is a natural Hamiltonian system with configuration manifold $\mathbb{E}^3$ given by the following potential in Cartesian coordinates (q_1, q_2, q_3):

$$V = (q_1 - q_2)^{-2} + (q_2 - q_3)^{-2} + (q_1 - q_3)^{-2}$$

First note that the constant vector $d = \frac{1}{\sqrt{3}}[1, 1, 1]$ is a symmetry of V, i.e. $\mathcal{L}_d V = 0$. Hence we observe that the CT $L = d \odot d$ is a solution of the KBD equation associated with V. From Example 6.5.1 we know that a warped product manifold adapted to L has the form $\mathbb{E}^1 \times \mathbb{E}^2$. One can choose Cartesian coordinates (q_1', q_2', q_3') adapted to this product manifold, such that V takes the form:

$$V = \frac{9(q_3'^2 + q_2'^2)^2}{2q_2'^2(3q_3'^2 - q_2'^2)^2}$$

In this case V naturally restricts to a potential on $\mathbb{E}^2$ with coordinates (q_2', q_3'). In $\mathbb{E}^2$ one can apply the BEKM separation algorithm to find that the only solution of the KBD equation (up to constant multiplies) is $L = r \odot r$ where r is the dilatational vector field. Hence we conclude that V is separable in cylindrical coordinates which are obtained by taking polar coordinates $(r\cos(\theta), r\sin(\theta))$ on $\mathbb{E}^2$. □

The above example will be worked out in greater detail in Section 10.2. When $n = 3$, we will show that the Calogero-Moser system is separable in four additional coordinate systems. The following example illustrates how one can obtain ChKTs when an ignorable coordinate is present.

Example 6.7.5 (Separation in Static space-times)
A static space time is the product manifold $M = B \times_\rho \mathbb{E}_1^1$ equipped with warped product metric $g = \tilde{g} - \rho^2 \mathrm{d}t^2$ where $\tilde{g}$ is a Riemannian metric. By Proposition 5.3.12, M admits a ChKT K with timelike eigenvector field $\frac{\partial}{\partial t}$ iff there exists a ChKT $\tilde{K} \in S^2(B)$ satisfying:

$$\mathrm{d}(\tilde{K}\mathrm{d}\rho^{-2}) = 0$$

This observation is a special case of the connection between separation of potentials and extensions of KTs. We note here that in order to find $\tilde{K}$, the BEKM separation algorithm can be applied on B with $V := \rho^{-2}$. In particular if B is a space of constant curvature, we will observe immediately after this example that the BEKM separation algorithm gives a complete method for determining $\tilde{K}$ satisfying the above equation if it exists. $\square$

Completeness of the algorithm It follows from the definition of the KEM web that if this algorithm is applied recursively then it will always test if the potential is separable in a KEM web. Since it will be proven in Chapter 7 that every separable web in a space of constant curvature is a KEM web, it follows that this algorithm gives a complete test for separability in spaces of constant curvature. Although if one uses a ChKT not associated with a KEM web in case 3 of the algorithm, then one can test for separability against more general separable webs.

Practical Implementation For spaces of constant curvature, we will work out sufficient details in Chapter 9 to concretely implement this algorithm in Section 10.3. To do this, the only problems that remain are the classification of OCTs modulo the action of the isometry group, then obtaining the transformation to Cartesian coordinates for their associated webs and classifying warped product decompositions on these spaces. These problems are solved in Chapter 9.

This algorithm has been implemented concretely in Euclidean and spherical space by Waksjo and Wojciechowski in their solution [WW03]. Their solution which was more classical, involved Stäckel theory and was based on the work of Kalnins-Miller [Kal86]. They made no use of Benenti's modern formulation of the separation of the Hamilton-Jacobi equation [Ben97] in terms of Killing tensors which is independent of Stäckel theory.

Like the algorithm in [WW03], in spaces of constant curvature the BEKM separation algorithm reduces to a series of problems in linear algebra. Although for hyperbolic space and Minkowski space-time, one will have to deal with finding the Jordan canonical form of non-diagonalizable (constant) matrices.

6.8 Notes

Reviews of the theory presented in this chapter are presented in Sections 2.2 to 2.4 and 9.1.1.

In this chapter we have not studied non-orthogonal CTs. Non-orthogonal CTs may be applicable to complex and non-orthogonal separation. In [BM13], a procedure is given to obtain the local canonical (normal) forms for CTs in pseudo-Riemannian manifolds. Hence this procedure may be of interest for those who wish to study non-orthogonal CTs.

Part II

Specialization to Spaces of Constant Curvature

Chapter 7

In Spaces of Constant Curvature: Separable webs are KEM webs

In this chapter we will prove the fundamental result: in spaces of constant curvature, separable webs are KEM webs. This fact can be deduced from the Kalnins-Miller classification [Kal86], but we will give an independent proof which somewhat generalizes their results. This chapter builds on Section 6.5 which we assume the reader is familiar with. The contents of this chapter are from the article [RM14a].

In order to prove this result, we will solve for all orthogonally separable metrics with diagonal curvature. This generalizes results by Eisenhart in [Eis34] and Kalnins and Miller in [Kal86]. Note that this is sufficient to solve for all orthogonal separable coordinates in spaces of constant curvature, since by Eq. (1.4.1), all orthogonal coordinates in these spaces have diagonal curvature.

We now provide an outline of this chapter. In the first section we will summarize the results. In Section 7.2 we present the first steps of the derivation of all orthogonal separable coordinates with diagonal curvature. In Section 7.3 we will finish off this derivation. Finally in Section 7.4, we will do additional calculations in order to prove that all separable coordinates in spaces of constant curvature are KEM coordinates.

7.1 Summary of Results

In this section we will present the results of this chapter in detail and combine them to prove the following fundamental result:

Theorem 7.1.1 (Separable Webs in Spaces of Constant Curvature)
In a space of constant curvature, every separable web is a KEM web. □

First we need a preliminary characterization of orthogonal concircular tensors, which is the coordinate form of Proposition 6.3.6. Suppose (x_i) are local coordinates and L is a tensor defined as follows:

$$L = \sum_{a \in M} \sigma_a \partial_a \otimes \mathrm{d}x_a + \sum_{I \in P} e_I \sum_{i \in I} \partial_i \otimes \mathrm{d}x_i \tag{7.1.1}$$

where $\{1, \dots, n\} = M \cup (\cup_{I \in P} I)$ is a partition (here P is an index set and each $I \in P$ is a subset of $\{1, \dots, n\}$), the $\sigma_a(x_a)$ are non-constant and the e_I are constants. Proposition 6.3.6 states that if L is a concircular tensor, then the metric has the following form:

$$g = \sum_{a \in M} \Phi_a \prod_{\substack{b \in M \\ b \neq a}} (\sigma_a - \sigma_b) \mathrm{d}x_a^2 + \sum_{I \in P} \left(\prod_{a \in M} (e_I - \sigma_a) \right) g^I$$

$$g^I_{ij} = \begin{cases} f^I_{ij}(x^I) & i, j \in I \\ 0 & i \notin I \end{cases} \tag{7.1.2a}$$

where Φ_a is a function of x^a only. Conversely, it follows by Proposition 6.3.6, that if the metric has the above form, then L is a CT. It will follow by the proof of our main result (see Section 7.3), that given a metric with the above form, one can construct L such that its eigenspaces are uniquely determined from the metric.

We will see that most orthogonally separable metrics with diagonal curvature have a form given by the above equation, i.e. they admit a concircular tensor diagonalized in the coordinates. We now list the general form of orthogonally separable metrics with diagonal curvature.

The ones having a form given by Eq. (7.1.2) can be divided into the following three classes. The first class are the **irreducible metrics**

$$g = \sum_{a=1}^{n} \Phi_a \prod_{b \neq a} (\sigma_a - \sigma_b) \mathrm{d}x_a^2 \tag{7.1.3}$$

which occur when the eigenfunctions of any associated concircular tensor are functionally independent. These metrics were first found by Eisenhart in his article [Eis34]. The remaining two classes of metrics are referred to as reducible metrics. The following are **product metrics**

$$g = \sum_{I=1}^{p} g^I \tag{7.1.4}$$

where each g^I is given in Eq. (7.1.2a). The final class are the **warped product metrics**

$$g = \sum_{a=1}^{m} \Phi_a \prod_{\substack{b \leq m \\ b \neq a}} (\sigma_a - \sigma_b) \mathrm{d}x_a^2 + \sum_{I=1}^{p} \left(\prod_{a \leq m} (e_I - \sigma_a) \right) g^I \tag{7.1.5}$$

where each g^I is given in Eq. (7.1.2a).

There is one class of orthogonally separable metric with diagonal curvature which is not in general associated with a concircular tensor, it is given as follows:

$$g = \Phi_1 \mathrm{d}x_1^2 + \sum_{I=1}^{p} \sigma_1^I g^I \tag{7.1.6}$$

where Φ_1, σ_1^I are functions of x_1 at most with each σ_1^I non-constant. In conclusion, every orthogonally separable metric with diagonal curvature has a form given by Eq. (7.1.2) or Eq. (7.1.6). We will show later that if g is an orthogonally separable metric with diagonal curvature, then each of the metrics g^I must also be an orthogonally separable metric with diagonal curvature. This shows why the classification is recursive: if $|I| > 1$ then our classification will tell us that each g^I must be of the form given by Eq. (7.1.2) or Eq. (7.1.6). Thus one must recursively apply this classification to obtain all orthogonally separable metrics with diagonal curvature for a given dimension.

Using the above classification, we will prove the following theorem concerning orthogonal separation in spaces of constant curvature:

Theorem 7.1.2 (KEM Separation Theorem)
Suppose (M, g) is a space of constant curvature. In orthogonal separable coordinates, g necessarily has the form given by Eq. (7.1.2).

In terms of tensors, suppose K is a characteristic Killing tensor defined on M. Then there is a non-trivial concircular tensor L defined on M such that each eigenspace of K is L-invariant, i.e. L is diagonalized in coordinates adapted to the eigenspaces of K. Furthermore, the eigenspaces of L are uniquely determined by the separable web defined by K.

□

The above theorem is a generalization of the results due to Kalnins and Miller from [Kal86]; it holds in Lorentzian spaces as well. For Riemannian spaces of constant curvature, this theorem can be proven by connecting the classification of separable metrics given by Kalnins and Miller in [Kal86] with Proposition 6.3.6. Indeed, by examining the separable metrics given in [Kal86], it can be shown that all separable metrics derived in [Kal86] have the form given by Eq. (7.1.2). Then the desired concircular tensor, L, is given by Eq. (7.1.1). For a space of constant curvature with arbitrary signature, we will generalize the classification given by Kalnins and Miller and show that all separable metrics still have the form given by Eq. (7.1.2).

We will apply the above theorem, shortly, to prove Theorem 7.1.1. But first, we note here that Theorem 7.1.1 together with the results presented in [Ben92b] (cf. [Kal86]) allow us to conclude the following:

Theorem 7.1.3 (KEM Separation Theorem II)

Suppose (M, g) is a space of constant curvature with Euclidean signature or Lorentzian[1] *signature with positive curvature. Then every separable (not necessarily orthogonal) coordinate system has an orthogonal equivalent which is a KEM coordinate system.* □

In the above theorem, the term "equivalent" is in the sense of Definition 5.1.5. Precisely, it means that in the aforementioned spaces, every separable solution to the geodesic Hamilton-Jacobi equation induces the same Lagrangian foliation as a separable solution associated with some KEM coordinate system. Now, we will need the following lemma, which will be proven in Section 7.4.

Lemma 7.1.4

In a space of constant curvature, a Killing foliation is a foliation of homothetic[2] *spaces of constant curvature.* □

In particular for $\mathbb{E}^n$ one can show that a Killing foliation is foliation by subsets of (affine) spheres or planes of lesser dimension. We are now ready to prove Theorem 7.1.1.

Proof (Theorem 7.1.1) Suppose inductively that this theorem holds for all separable webs in spaces of constant curvature of dimension $k < n$. The statement trivially holds when $k = 1$. We now show that the theorem holds when $\dim M = n$.

Suppose K is a ChKT defined on a space of constant curvature M defining a separable web. Then let L be a concircular tensor guaranteed by the KEM separation theorem.

Case 1 If L has simple eigenfunctions (i.e. is a Benenti tensor), then it follows that the separable web determined by K is a KEM web.

Case 2 Suppose L has multidimensional eigenspaces $D_1, \dots, D_l$; these must be Killing by Proposition 6.3.6. Thus each D_i induces a foliation of spherical submanifolds of M. Then it follows by Lemma 7.1.4 that this is a foliation of spaces of constant curvature of lesser dimension. Suppose N_i is an integral manifold of D_i. Then it follows from Proposition 4.4.15 that K restricts to a ChKT $\tilde{K}_i$ on N_i. Thus $\tilde{K}_i$ is a ChKT on a space of constant curvature N_i which has dimension less than n. Hence by induction hypothesis, it follows that the separable web $\mathcal{E}_i$ associated with $\tilde{K}_i$ is a KEM web. Thus by definition it follows that the separable web associated with K is a KEM web.

The result then follows by induction on n. ■

[1] We take Lorentzian signature to be $(- + \cdots +)$

[2] By homothetic pseudo-Riemannian manifolds, we mean a pair of pseudo-Riemannian manifolds whose metrics are related by the equation $h = \lambda^2 g$ where $\lambda \in \mathbb{R}^+$.

7.2 Preliminary results

In this section we will present some relevant calculations from the literature for completeness. In particular, we will partially solve for all orthogonally separable metrics with diagonal curvature. This calculation will be finished in the next section.

We now assume (x^i) are orthogonal separable coordinates with diagonal curvature. Assume the covariant metric $g = \operatorname{diag}(e_1 H_1^2, ..., e_n H_n^2)$ where each $e_i = \pm 1$ as the case may be. The assumption of orthogonal separability implies the metric satisfies the Levi-Civita equations (Eq. (5.2.3a)). They take the following form:

$$\frac{\partial^2 \log H_i^2}{\partial x_i \partial x_j} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_i} = 0 \tag{7.2.1a}$$

$$\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} - \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j} = 0 \tag{7.2.1b}$$

where i,j, and k are all distinct. We will now proceed to solve the above equations in combination with the diagonal curvature condition.

The following calculation is from [Cra03, proposition 6] which is adapted from Kalnins' book [Kal86] which is from [Eis34]. First note that in orthogonal coordinates the Riemann curvature component R_{jiik} for i,j,k distinct has the following form [Eis34]:

$$\begin{aligned} R_{jiik} = \frac{e_i H_i^2}{4}\Bigg[2\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k} \\ - \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j}\Bigg] \end{aligned}$$

In consequence of the second integrability condition, Eq. (7.2.1b), we find that:

$$R_{jiik} = \frac{3}{4} e_i H_i^2 \frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k}$$

Thus the diagonal curvature assumption implies that for i,j,k distinct:

$$\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} = 0 \tag{7.2.2}$$

Solving the above equation we find that:

$$H_i^2 = \prod_{j \neq i} \Psi_{ij}(x_i, x_j) \tag{7.2.3}$$

Now the first integrability condition, Eq. (7.2.1a), applied twice to $i \neq j$ implies that:

$$\frac{\partial^2 \log H_i^2}{\partial x_i \partial x_j} = -\frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_i} \tag{7.2.4a}$$

$$\frac{\partial^2 \log H_i^2}{\partial x_i \partial x_j} = \frac{\partial^2 \log H_j^2}{\partial x_i \partial x_j} \tag{7.2.4b}$$

If we substitute the form of H from Eq. (7.2.3) into Eq. (7.2.4b) we have that:

$$\frac{\partial^2 \log \dfrac{\Psi_{ij}}{\Psi_{ji}}}{\partial x_i \partial x_j} = 0$$

Thus

$$\frac{\Psi_{ij}}{\Psi_{ji}} = \frac{\chi_{ij}(x_i)}{\chi_{ji}(x_j)}$$

If we let $\Phi_{ij} = \Phi_{ji} = \dfrac{\Psi_{ij}}{\chi_{ij}}$ and $\Phi_i = \prod\limits_{j \neq i} \chi_{ij}$, Eq. (7.2.3) becomes:

$$H_i^2 = \Phi_i(x_i) \prod_{j \neq i} \Phi_{ij}(x_i, x_j)$$

Now if we substitute the form of H above into Eq. (7.2.4a) we have that:

$$\frac{\partial^2 \Phi_{ij}}{\partial x_i \partial x_j} = 0$$

Thus

$$\Phi_{ij}(x_i, x_j) = \sigma_{ij}(x_i) + \sigma_{ij}(x_j)$$

This gives us the following general form of H satisfying Eq. (7.2.1a) and Eq. (7.2.2):

$$H_i^2 = \Phi_i(x_i) \prod_{j \neq i} (\sigma_{ij}(x_i) + \sigma_{ji}(x_j)) \quad (i = 1, .., n) \tag{7.2.5}$$

The above equation was first derived by Eisenhart in his seminal paper [Eis34] and it was used resourcefully by Kalnins and Miller in their classification of separable coordinates systems in S^n, E^n and H^n [Kal86]. When $n = 2$, the above equation gives the general solution and it follows that the metric has the form given by Eq. (7.1.2). Thus for the remainder of this solution we assume $n > 2$. Now for i,j,k distinct we evaluate Eq. (7.2.1b) with all cyclic permutations of i,j,k using the form of H given above to get the following system of equations:

$$\sigma'_{ji}\sigma'_{ki}(\sigma_{jk}+\sigma_{kj})-\sigma'_{ji}\sigma'_{kj}(\sigma_{ki}+\sigma_{ik})-\sigma'_{ki}\sigma'_{jk}(\sigma_{ij}+\sigma_{ji})=0 \tag{7.2.6a}$$

$$\sigma'_{kj}\sigma'_{ij}(\sigma_{ki}+\sigma_{ik})-\sigma'_{kj}\sigma'_{ik}(\sigma_{ij}+\sigma_{ji})-\sigma'_{ij}\sigma'_{ki}(\sigma_{jk}+\sigma_{kj})=0 \tag{7.2.6b}$$

$$\sigma'_{ik}\sigma'_{jk}(\sigma_{ij}+\sigma_{ji})-\sigma'_{ik}\sigma'_{ji}(\sigma_{jk}+\sigma_{kj})-\sigma'_{jk}\sigma'_{ij}(\sigma_{ki}+\sigma_{ik})=0 \tag{7.2.6c}$$

where the primes indicates differentiation. Now since each $\Phi_{ij}=\sigma_{ij}+\sigma_{ji}$ is non-zero, the determinant of the above equations must vanish, this gives us the following equation:

$$\sigma'_{ij}\sigma'_{jk}\sigma'_{ki}+\sigma'_{ji}\sigma'_{kj}\sigma'_{ik}=0 \tag{7.2.7}$$

We will solve the remaining equations in the next section.

7.3 Classification of orthogonal separable coordinates with diagonal curvature

We continue the derivation started in the previous section. An important subset of coordinates are the coordinates i which satisfy $\sigma'_{ij}\neq 0 \quad \forall j\neq i$. These coordinates will be called **connecting coordinates** for reasons that will become apparent later on. The set of all connecting coordinates for a given separable metric will be denoted by M and we will assume the coordinates are chosen such that $M=\{1,...,m\}$.

First we give a rough idea of how we will do this classification. When there are no connecting coordinates, we show that metric is necessarily a product metric. When there is at least one connecting coordinate we show that the metric is any one of the other metrics listed in the introduction. In order to prove that the metric is a product metric when it has no connecting coordinates we define a relation among the coordinates. We then prove that this relation is an equivalence relation. Then we use this equivalence relation to prove that the metric has at least one connecting coordinate or is a product metric.

We now define a relation among the coordinates to distinguish between the different possible metrics that can occur. The relation is designed so that if it gives multiple partitions then these partitions are associated with a product metric. Furthermore, we should be able to conclude that the metric is connected if there is only one partition. It's easiest to first define when two coordinates i and j are inequivalent. If I and J are distinct partitions from the product metric in Eq. (7.1.4) and $i\in I$ and $j\in J$, then the first thing to notice is that $\sigma'_{ij}=\sigma'_{ji}=0$. But with this definition of in-equivalence, if there are multiple partitions, it's still possible that we're dealing with a warped product metric given by Eq. (7.1.5); we need to make sure that there is no third coordinate k such that $\sigma'_{ki}, \sigma'_{kj} \neq 0$. This gives us a definition of equivalence:

Definition 7.3.1
Two distinct variables i and j are said to be connected and denoted $i \sim j$ if one of the following conditions hold:

$$\sigma'_{ij} \neq 0$$
$$\sigma'_{ji} \neq 0$$
$$\exists\, k \neq i, j : \sigma'_{ki}, \sigma'_{kj} \neq 0$$

Also we define $\sim$ such that $i \sim i$. □

There are two special types of connectedness that arise, the first is when i,j satisfy $\sigma'_{ij} \neq 0$ or $\sigma'_{ji} \neq 0$, in this case we say that i and j are **strongly connected**. If i,j are connected but not strongly connected, we say that i and j are **weakly connected** by k or just that i and j are **weakly connected**.

Proposition 7.3.2
The relation $\sim$ defined in Definition 7.3.1 is an equivalence relation. □

Proof We check that this relation is transitive, as reflexivity and symmetry are immediately verified. So suppose that $i \sim j$ and $j \sim k$ where i,j,k are mutually distinct.

Case 1 ($\sigma'_{ji} \neq 0$ and $\sigma'_{jk} \neq 0$)
In this case i and k are weakly connected by j.

Case 2 ($\sigma'_{ij} \neq 0$ and $\sigma'_{kj} \neq 0$)
Assume to the contrary that $\sigma'_{ik} = \sigma'_{ki} = 0$, then Eq. (7.2.6b) can't be satisfied. Thus i must be strongly connected to k.

Case 3 ($\sigma'_{ij} \neq 0$ and $\sigma'_{jk} \neq 0$ or $\sigma'_{ji} \neq 0$ and $\sigma'_{kj} \neq 0$)
Assume first that $\sigma'_{ij} \neq 0$ and $\sigma'_{jk} \neq 0$ and to the contrary that $\sigma'_{ik} = 0$, then Eq. (7.2.6c) can't be satisfied. Also the case where $\sigma'_{ji} \neq 0$ and $\sigma'_{kj} \neq 0$ is just a permutation of the first, so the same argument applies. Thus in either case i must be strongly connected to k.

Case 4 ($\sigma'_{ij} \neq 0$ and j and k are weakly connected or $\sigma'_{kj} \neq 0$ and i and j are weakly connected)
Suppose first that $\sigma'_{ij} \neq 0$ and j and k are weakly connected by h. So we have that $\sigma'_{hj}, \sigma'_{hk} \neq 0$.

If $h = i$ then $\sigma'_{ik} \neq 0$, so assume that $h \neq i$. If $\sigma'_{hi} \neq 0$ then i and k are weakly connected by h, so assume that $\sigma'_{hi} = 0$. If $\sigma'_{ih} \neq 0$ then by Case 3 we get that

$\sigma'_{ik} \neq 0$, so assume further that $\sigma'_{ih} = 0$. Then after checking Eq. (7.2.6c) with the following coordinates, we get a contradiction.

$$h \to i$$
$$i \to j$$
$$j \to k$$

The case where $\sigma'_{kj} \neq 0$ and i and j are weakly connected is just a permutation of the first case. Thus we conclude that i is connected to k.

Case 5 ($\sigma'_{ji} \neq 0$ and j and k are weakly connected or $\sigma'_{jk} \neq 0$ and i and j are weakly connected)
Suppose first that $\sigma'_{ji} \neq 0$ and j and k are weakly connected by h. So we have that $\sigma'_{hj}, \sigma'_{hk} \neq 0$.

If $h = i$ then $\sigma'_{ik} \neq 0$, so assume that $h \neq i$. Since $\sigma'_{hj} \neq 0$ and $\sigma'_{ji} \neq 0$, by Case 3 we get that $\sigma'_{hi} \neq 0$. Thus i and k are weakly connected by h.

The case where $\sigma'_{jk} \neq 0$ and i and j are weakly connected is just a permutation of the first case. Thus we conclude that i is connected to k.

Case 6 (i and j are weakly connected and j and k are weakly connected)
Suppose l satisfies $\sigma'_{li}, \sigma'_{lj} \neq 0$ and h satisfies $\sigma'_{hj}, \sigma'_{hk} \neq 0$.

If $h = l$ then i and k are clearly weakly connected, so assume that $h \neq l$.

Note that l is strongly connected to j and $j \sim k$ then we can use one of the previous cases considered to find that $l \sim k$. Similarly because i is strongly connected to l and $l \sim k$ we find that $i \sim k$.

Thus we conclude that $\sim$ is transitive and thus defines an equivalence relation. ∎

Now suppose that $\sim$ gives a single partition of the coordinates, i.e. the coordinates are connected. Our goal is to show that there must be at least one connecting coordinate. First we need a definition. We define S, called the set of strongly connected coordinates as follows:

$$S \equiv \{ \; i : i \text{ is strongly connected to every } j\}$$

The reason to make this definition is because $M \subseteq S$ (this inclusion might be proper in some cases which can be observed by inspecting KEM metrics derived by Kalnins-Miller [Kal86]). So the idea is to first show that $S \neq \emptyset$ since this is easier to do using the hypothesis of connectedness. It turns out that this is possible.

Proposition 7.3.3

When the coordinates are connected, S has at least one coordinate. □

Proof Suppose to the contrary that $S = \emptyset$. Then $\forall i$, there exists j, k with i, j, k distinct such that

$$\sigma'_{ij} = \sigma'_{ji} = 0 \text{ and } \sigma'_{ki}, \sigma'_{kj} \neq 0 \tag{7.3.1}$$

So fix some i, and choose j, k satisfying Eq. (7.3.1). Then by Eq. (7.2.6a) we must have that $\sigma'_{jk} = 0$, similarly by Eq. (7.2.6b) we must have that $\sigma'_{ik} = 0$. Now let $A = \{i, j\}$ then note that $k \notin A$ and $\forall\, l \in A,\ \ \sigma'_{lk} = 0$.

Claim 7.3.3.1

Suppose we have a coordinate f and a set of coordinates $A \neq \emptyset$ such that $f \notin A$ and $\forall i \in A,\ \ \sigma'_{if} = 0$. Furthermore assume that $\{f\} \cup A \neq \{1, ..., n\}$. Then we can obtain a new set A' such that $A \cup \{f\} \subseteq A'$ and an $h \notin A'$ such that $\forall i \in A',\ \ \sigma'_{ih} = 0$. □

Proof (Proof of claim) By assumption there exists g, h satisfying Eq. (7.3.1) with f in place of i and g in place of j. Since $\sigma'_{hf} \neq 0$, $h \notin A$. If $\{f\} \cup A = \{1, ..., n\}$, then we have reached a contradiction, so assume otherwise. As we observed earlier for a similar case, we must have $\sigma'_{gh} = \sigma'_{fh} = 0$. Also $\forall i \in A$ since $\sigma'_{if} = 0$ and $\sigma'_{hf} \neq 0$ by evaluating Eq. (7.2.6b) with $i \to i, f \to j, h \to k$ we find that $\sigma'_{ih} = 0$.

Thus if we let $A' = A \cup \{g, f\}$ then $\forall i \in A', \sigma'_{ih} = 0$. Also note that $|A'| > |A|$ and $h \notin A'$. ■

Now we can inductively apply Claim 1 to get a set of coordinates $A \neq \emptyset$, an $f \notin A$ such that $\forall i \in A,\ \ \sigma'_{if} = 0$ and $\{f\} \cup A = \{1, ..., n\}$. Then by assumption there must exist a coordinate g such that f is weakly connected to g. So there is a coordinate h, with $h \neq f$, such that $\sigma'_{hf} \neq 0$. Since $\{f\} \cup A = \{1, ..., n\}$, $h \in A$, thus $\sigma'_{hf} = 0$, a contradiction.

Thus $S \neq \emptyset$. ■

Then assuming $S \neq \emptyset$ we try to prove that $M \neq \emptyset$. This is also possible.

Proposition 7.3.4

When the coordinates are connected and S has at least one coordinate then there must be at least one connecting coordinate. Thus due to the previous proposition we find that when the coordinates are connected there must be at least one connecting coordinate. □

Proof Assume to the contrary that $M = \emptyset$. Then $\forall i \in S$ there exists j such that

$$\sigma'_{ij} = 0 \text{ and } \sigma'_{ji} \neq 0 \tag{7.3.2}$$

Since $S \neq \emptyset$ by hypothesis, we can choose some $i \in S$ and some $j \neq i$ such that Eq. (7.3.2) is satisfied. Let $B = \{i\}$.

Claim 7.3.4.1

Suppose $\emptyset \neq B \subseteq S$ and there is a $j \notin B$ such that $\forall i \in B$, $\sigma'_{ij} = 0$. Furthermore assume that $\{j\} \cup B \neq \{1, ..., n\}$. Then we can obtain a new set $B' = \{j\} \cup B \subseteq S$ and a $k \notin B'$ such that $\forall i \in B'$, $\sigma'_{ik} = 0$. □

Proof (Proof of claim) Fix an $i \in B$, then $\sigma'_{ij} = 0$ and $\sigma'_{ji} \neq 0$. Now pick $k \neq i, j$, then $\sigma'_{ik} \neq 0$ or $\sigma'_{ki} \neq 0$.

If $\sigma'_{ik} \neq 0$ then by Eq. (7.2.6c) we must have that $\sigma'_{jk} \neq 0$. If $\sigma'_{ki} \neq 0$ then by Eq. (7.2.6a) either $\sigma'_{jk} \neq 0$ or $\sigma'_{kj} \neq 0$. In either case we find that j is strongly connected to k. Since k was arbitrary and because j is also strongly connected to i we find that $j \in S$. Then by Eq. (7.3.2) there exists $\exists\, k \neq i, j$ such that $\sigma'_{jk} = 0$ and $\sigma'_{kj} \neq 0$, note that $k \notin B$.

Assume $i \in B$ is arbitrary, then $\sigma'_{ij} = 0$ and $\sigma'_{kj} \neq 0$, thus by Eq. (7.2.6b) we must have that $\sigma'_{ik} = 0$.

Let $B' = B \cup \{j\} \subseteq S$ then $\forall i \in B'$ we have $\sigma'_{ik} = 0$, also note that $k \notin B'$. ■

Now we can inductively apply Claim 1 to a get set B satisfying $\emptyset \neq B \subseteq S$ and a $j \notin B$ such that $\forall i \in B$, $\sigma'_{ij} = 0$ and $\{j\} \cup B = \{1, ..., n\}$. As in the proof of the Claim 1, we find that $j \in S$. Then by Eq. (7.3.2) $\exists\, k \neq j$ such that $\sigma'_{kj} \neq 0$, but since $\{j\} \cup B = \{1, ..., n\}$ we must have that $k \in B$, then $\sigma'_{kj} = 0$, a contradiction.

Thus $M \neq \emptyset$. ■

The following proposition classifies all metrics with at least one connecting coordinate.

Proposition 7.3.5

If the metric has at least one connecting coordinate then the following statements are true. For $a \in M$:

$$H_a^2 = \Phi_a \prod_{\substack{b \in M \\ b \neq a}} (\sigma_a - \sigma_b)$$

If $m > 1$ then one can partition the coordinates in[3] *M^c such that if I is an equivalence class of this partition and $\alpha \in I$, then*

$$H_\alpha^2 = \Phi_\alpha \prod_{\substack{\beta \in I \\ \beta \neq \alpha}} (\sigma_{\alpha\beta} + \sigma_{\beta\alpha}) \prod_{a=1}^{m} (e_I - \sigma_a) \quad (m \geq 2)$$

If $m = 1$ then one can partition the coordinates in M^c such that if I is an equivalence class of this partition and $\alpha \in I$, then

[3] If $Y \subseteq X$, then we denote the complement of Y in X (elements of X not in Y) as Y^c.

$$H_\alpha^2 = \Phi_\alpha \sigma_a^I \prod_{\substack{\beta \in I \\ \beta \neq \alpha}} (\sigma_{\alpha\beta} + \sigma_{\beta\alpha}) \quad (m = 1)$$

Equations (7.2.6) are satisfied whenever at least one of i, j or k is in M if and only if the functions H_i^2 are of the form just described. Furthermore Equations (7.2.6) are satisfied whenever i, j, k are not all in the same partition. □

PROOF By hypothesis we can assume $m \geq 1$. We use Latin letters such as a to denote the connecting coordinates and the remaining coordinates are denoted with Greek letters such as α. Although i and j are reserved for arbitrary coordinates. Furthermore we denote $N = \{1, ..., n\}$. Then by definition $\forall\, a \in M, i \in N$ we have that $\sigma'_{ai} \neq 0$.

Claim 7.3.5.1

For $a \in M$ and $\alpha \in M^c$, $\sigma'_{\alpha a} = 0$. □

PROOF For any $\alpha \in M^c$, there exists $i \in N$ such that $\sigma'_{\alpha i} = 0$.

Suppose first that $i \in M$ and let $a = i$. Suppose to the contrary that there exists $b \in M \setminus \{a\}$ such that $\sigma'_{\alpha b} \neq 0$. Then Eq. (7.2.6b) can't hold with $\alpha \to i, a \to j, b \to k$. Thus the claim holds in this case.

If $i \in M^c$, let $\beta = i$. If the first case doesn't hold for α, then $\sigma'_{\alpha a} \neq 0\ \forall\, a \in M$. Fix $a \in M$, then Eq. (7.2.6b) can't hold with $\alpha \to i, \beta \to j, a \to k$. Thus the first case must hold for some $a \in M$, thus the claim must hold. ■

The proof for the following claim is mainly from [Eis34, P. 292].

Claim 7.3.5.2

For $a \in M$, the following holds

$$H_a^2 = \Phi_a \prod_{\substack{b \in M \\ b \neq a}} (\sigma_a - \sigma_b)$$

where each $\sigma_a(x_a)$. □

PROOF Suppose first that $m = 1$ and let $a \in M$. Then for $\alpha \in M^c$ by the above claim we know that $\sigma_{a\alpha} + \sigma_{\alpha a}$ only depends on the a coordinate and so these factors can be absorbed into Φ_a and Φ_α. Thus the claim holds in this case.

So for the remainder of the proof of this claim assume that $m > 1$. To prove this statement, for $a, b \in M$ our goal is to remove the b dependence from σ_{ab}. First assume $m > 2$ and let $a, b, c \in M$. From Eq. (7.2.7) evaluated with $a \to i, b \to j, c \to k$ we get:

$$\sigma'_{ab}\sigma'_{bc}\sigma'_{ca} + \sigma'_{ba}\sigma'_{cb}\sigma'_{ac} = 0$$

Since each term is non-zero, by separating variables it follows that $\dfrac{\sigma'_{ab}}{\sigma'_{ac}}$ is a constant. Thus we can set $\sigma_{ab} = a_{ab}\sigma_a$ where a_{ab} is a constant and σ_a involves x_a at most. The above equation implies that the constants must satisfy the following:

$$a_{ab}a_{bc}a_{ca} + a_{ba}a_{cb}a_{ac} = 0 \tag{7.3.3}$$

Assuming the above equation holds, it follows that all three Equations (7.2.6) are satisfied for $a \to i, b \to j, c \to k$. Now set

$$\sigma_a = a_{bc}a_{ca}\bar{\sigma}_a \qquad\qquad \sigma_b = a_{cb}a_{ac}\bar{\sigma}_b$$

Then Eq. (7.3.3) implies that $a_{ab}\sigma_a + a_{ba}\sigma_b = a_{ab}a_{bc}a_{ca}(\bar{\sigma}_a - \bar{\sigma}_b)$; in which case the constant factor may be absorbed into Φ_a and Φ_b. Thus we can assume $a_{ab} = -a_{ba} = 1$, then Eq. (7.3.3) becomes:

$$a_{bc}a_{ca} - a_{cb}a_{ac} = 0$$

Now set $a_{ca}\sigma_c = -a_{ac}\bar{\sigma}_c$, then we have:

$$a_{ca}\sigma_c + a_{ac}\bar{\sigma}_a = a_{ac}(\bar{\sigma}_a - \bar{\sigma}_c)$$

Thus we can assume $a_{ac} = -a_{ca} = 1$. Then $a_{bc}\bar{\sigma}_b + a_{cb}\bar{\sigma}_c = a_{bc}(\bar{\sigma}_b - \bar{\sigma}_c)$ and so we can assume $a_{bc} = -a_{cb} = 1$. Inductively, this process can be continued so that each $\sigma_{ab} = \pm\sigma_a$, where the sign is positive if σ_{ab} appears in H_a^2 and is negative if σ_{ab} appears in H_b^2.

If $m = 2$ then we can define $\sigma_a = \sigma_{ab}$ and $\sigma_b = -\sigma_{ba}$ without loss of consistency. For $\alpha \in M^c$, $\sigma'_{\alpha a} = 0$, so we can absorb terms of the form $\sigma_{\alpha a} + \sigma_{a\alpha}$ into Φ_a and Φ_α. Thus we have proven the following:

$$H_a^2 = \Phi_a \prod_{\substack{b \le m \\ b \neq a}} (\sigma_a - \sigma_b) \qquad \blacksquare$$

We can now assume that Equations (7.2.6) have been solved whenever $i, j, k \in M$. Thus if $m = n$ the above claim proves that the metric has the form given by Eq. (7.1.3) and so we are finished. So assume for the remainder of the proof that $m < n$.

Now fix $a, b \in M$ and $\alpha \in M^c$. Let $a_{\alpha a} = \sigma_{\alpha a} \in \mathbb{R}$ and $a_{\alpha b} = \sigma_{\alpha b} \in \mathbb{R}$. Then Eq. (7.2.6c) evaluated with $a \to i, b \to j, \alpha \to k$ gives:

$$\sigma'_{a\alpha}\sigma'_{b\alpha}(\sigma_a - \sigma_b) + \sigma'_{a\alpha}\sigma'_b(\sigma_{b\alpha} + a_{\alpha b}) - \sigma'_{b\alpha}\sigma'_a(a_{\alpha a} + \sigma_{a\alpha}) = 0$$

We now proceed to solve the above equations. First we rearrange terms to separate the variables:

$$\begin{aligned}(\sigma_a - \sigma_b) + \frac{\sigma_b'}{\sigma_{b\alpha}'}(\sigma_{b\alpha} + a_{\alpha b}) - \frac{\sigma_a'}{\sigma_{a\alpha}'}(a_{\alpha a} + \sigma_{a\alpha}) = 0 \\ \Rightarrow \sigma_a - \frac{\sigma_a'}{\sigma_{a\alpha}'}(a_{\alpha a} + \sigma_{a\alpha}) = \sigma_b - \frac{\sigma_b'}{\sigma_{b\alpha}'}(\sigma_{b\alpha} + a_{\alpha b}) = c \in \mathbb{R}\end{aligned} \tag{7.3.4}$$

Then one can show that $\frac{\sigma_a'}{\sigma_{a\alpha}'} = -d \in \mathbb{R} \setminus \{0\}$ and similarly $\frac{\sigma_b'}{\sigma_{b\alpha}'} = -f \in \mathbb{R} \setminus \{0\}$. Thus the above equation implies:

$$\begin{aligned}\sigma_a &= -d(a_{\alpha a} + \sigma_{a\alpha} - \frac{c}{d}) \\ \sigma_b &= -f(a_{\alpha b} + \sigma_{b\alpha} - \frac{c}{f})\end{aligned}$$

Now let $e_\alpha = c$ then the two equations above implies the following:

$$\begin{aligned}a_{\alpha a} + \sigma_{a\alpha} &= \frac{e_\alpha - \sigma_a}{d} \\ a_{\alpha b} + \sigma_{b\alpha} &= \frac{e_\alpha - \sigma_b}{f}\end{aligned}$$

Thus by absorbing the constants d, f into the Φ functions we can assume $a_{\alpha a} = a_{\alpha b} = e_\alpha$, $\sigma_{a\alpha} = -\sigma_a$ and $\sigma_{b\alpha} = -\sigma_b$. With these assumptions, it follows that all three Equations (7.2.6) are satisfied for $a \to i, b \to j, \alpha \to k$. Thus we conclude that Equations (7.2.6) hold whenever $i, j \in M$ and $k \in M^c$.

Suppose $m > 1$, we just observed that for $\alpha \in M^c$ and $a \in M$ that $\sigma_{\alpha a} = e_\alpha$. Thus we can partition the $\alpha \in M^c$ by the value e_α. We consider $\alpha, \beta \in M^c$ to be in the same equivalence class, say I, if $e_\alpha = e_\beta$. We define e_I such that $\alpha \in I$ implies that $e_\alpha = e_I$. We denote these equivalence classes by I and J.

Suppose $a \in M$ and $\alpha \in I, \beta \in J$. We now check Equations (7.2.6) for $a \to i, \alpha \to j, \beta \to k$. Since $\sigma_{\alpha a}' = \sigma_{\beta a}' = 0$, Eq. (7.2.6a) is satisfied. Equation (7.2.6b) and (7.2.6c) reduce to the following:

$$\sigma_{\beta\alpha}'(\sigma_{a\alpha}'(e_\beta + \sigma_{a\beta}) - \sigma_{a\beta}'(\sigma_{a\alpha} + e_\alpha)) = 0 \tag{7.3.5}$$

$$\sigma_{\alpha\beta}'(\sigma_{a\alpha}'(e_\beta + \sigma_{a\beta}) - \sigma_{a\beta}'(\sigma_{a\alpha} + e_\alpha)) = 0 \tag{7.3.6}$$

Now we can use the fact that $\sigma_{a\alpha} = \sigma_{a\beta} = -\sigma_a$ to get the following:

$$\sigma'_{\beta\alpha}(e_\alpha - e_\beta) = 0$$
$$\sigma'_{\alpha\beta}(e_\alpha - e_\beta) = 0$$

If $I = J$ then $e_\alpha = e_\beta$ and the above equations are satisfied. If $I \neq J$ then we must have that $\sigma'_{\alpha\beta} = \sigma'_{\beta\alpha} = 0$; in which case $\sigma_{\alpha\beta} + \sigma_{\beta\alpha}$ can be absorbed into the Φ functions. Thus we have proven the following: if $\alpha \in I$ then

$$H_\alpha^2 = \Phi_\alpha \prod_{\substack{\beta\in I\\ \beta\neq\alpha}} (\sigma_{\alpha\beta} + \sigma_{\beta\alpha}) \prod_{a=1}^{m} (e_I - \sigma_a) \quad (m \geq 2)$$

Now suppose $m = 1$, then we can pullback to a submanifold given by $x_1 =$ constant and then partition the coordinates in M^c into connected components. We denote these equivalence classes by I and J. Let $a \in M$ and $\alpha \in I, \beta \in J$. As for the case $m > 1$ one can see that Eq. (7.2.6a) is satisfied. Furthermore Equation (7.2.6b) and (7.2.6c) reduce to Equations (7.3.5) and (7.3.6) above. If $I \neq J$ then $\sigma'_{\beta\alpha} = \sigma'_{\alpha\beta} = 0$, thus Equations (7.3.5) and (7.3.6) are both satisfied. Otherwise assume $I = J$ and $\alpha, \beta \in I$ satisfy $\sigma'_{\alpha\beta} \neq 0$ for the moment, then Eq. (7.3.6) implies

$$\frac{\sigma'_{a\alpha}}{\sigma'_{a\beta}}(e_\beta + \sigma_{a\beta}) - \sigma_{a\alpha} + e_\alpha = 0$$

As with Eq. (7.3.4) we can deduce that $\dfrac{\sigma'_{a\alpha}}{\sigma'_{a\beta}} = d \in \mathbb{R}$. Then the above equation implies that

$$e_\beta + \sigma_{a\beta} = \frac{e_\alpha + \sigma_{a\alpha}}{d}$$

Let $\sigma_a^I = e_\alpha + \sigma_{a\alpha}$, then after absorbing d into the Φ functions and relabelling, we can assume $\sigma_{a\alpha} = \sigma_{a\beta} = \sigma_a^I$ and $a_{\alpha a} = a_{\beta b} = 0$. Since the coordinates in I are connected, we can assume that for any $\alpha, \beta \in I$ with $\alpha \neq \beta$ that $\sigma_{a\alpha} = \sigma_{a\beta} = \sigma_a^I$ and $a_{\alpha a} = a_{\beta b} = 0$ and then Equations (7.3.5) and (7.3.6) are both satisfied. Then for $\alpha \in I$ and $a \in M$, we have proven the following:

$$H_\alpha^2 = \Phi_\alpha \sigma_a^I \prod_{\substack{\beta\in I\\ \beta\neq\alpha}} (\sigma_{\alpha\beta} + \sigma_{\beta\alpha}) \quad (m = 1)$$

Also note that Equations (7.2.6) are satisfied whenever $i \in M$ and $j, k \in M^c$. Thus we can conclude that Equations (7.2.6) are satisfied whenever at least one of i, j or k is in M. Furthermore one can easily check that Equations (7.2.6) are satisfied whenever i, j, k are not all in the same partition. ∎

When the coordinates are disconnected, i.e. $\sim$ gives multiple partitions, one can easily show that the metric is a product metric.

Proposition 7.3.6

If the coordinates are disconnected, then the metric is a product metric which is given by Eq. (7.1.4). Furthermore Equations (7.2.6) are satisfied whenever i, j or k aren't in the same connected component. □

Proof Suppose $i \in I$ and $j \in J$ where I and J are a disconnected set of coordinates. Then it follows by definition that $\sigma_{ij'} = \sigma'_{ji} = 0$, thus we can absorb the factors $\sigma_{ij} + \sigma_{ji}$ into the Φ functions. Thus we have proven that for $i \in I$ the following holds:

$$H_i^2 = \Phi_i \prod_{\substack{j \in I \\ j \neq i}} (\sigma_{ij} + \sigma_{ji})$$

Thus the metric has the form given by Eq. (7.1.4). One can easily check that Equations (7.2.6) are satisfied whenever i, j or k aren't in the same connected component. ∎

Proposition 7.3.7

If g is a reducible orthogonal separable metric with diagonal curvature given by Eqs. (7.1.4) to (7.1.6) and $|I| > 1$, the metric g^I pulls back to a orthogonal separable metric with diagonal curvature on the submanifold with metric proportional to g^I. □

Proof We know that an orthogonally separable web restricted to one of the integral manifolds of its n foliations is still separable Proposition 4.4.15, and g^I still has the form given by Eq. (7.2.5) thus its Riemann curvature tensor will still satisfy $R_{ijik} = 0$ for $j \neq k$ on the integral manifolds. ∎

We can see how the classification works. If $n = 2$ then we've noted in the previous section that the general solution is given by Eq. (7.1.2). So suppose $n > 2$ and the general orthogonal separable metrics with diagonal curvature are known on manifolds with dimension $k \leq n - 1$. If the coordinates are disconnected then Proposition 7.3.6 shows us that the metric must have the form given by Eq. (7.1.4) and the only equations that haven't been solved are Equations (7.2.6) when i, j, k are inside a connected component. If the coordinates are connected then Proposition 7.3.4 in conjunction with Proposition 7.3.5 tells us that the metric must have the form given by Eq. (7.1.2) or Eq. (7.1.6). Furthermore in this case the only equations that haven't been solved are Equations (7.2.6) when $i, j, k \in I$

where $I \subseteq M^c$ is an equivalence class as given in Proposition 7.3.5. Now fix some $|I| > 1$, then by Proposition 7.3.7, g^I pulls back to an orthogonal separable metric with diagonal curvature on the submanifold with coordinates (x_I). Thus inductively we know the general form of g^I since the dimension of the submanifold is at most $n-1$. In particular if $|I| > 2$ then the components of g^I satisfy Equations (7.2.6). Thus the solution g will satisfy Equations (7.2.6) for all i, j, k distinct and so g satisfies all relevant equations. Thus we have found all the orthogonal separable metrics with diagonal curvature.

7.4 Spaces of Constant Curvature

In this section our main goal is to show that the metric given Eq. (7.1.6) can be ruled out in spaces of constant curvature. In other words, all metrics in spaces of constant curvature are given by Eq. (7.1.2) and thus are invariantly characterized by concircular tensors.

Let $M = M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ be a warped product with M a space of constant curvature κ. We now use the formulas for the sectional curvature given in Corollary 3.5.8 for the following calculations.

After applying the polarization identity to Eq. (3.5.3), we get

$$S^{\rho_i}(X, Y) + \kappa \rho_i \langle X, Y \rangle = 0 \tag{7.4.1}$$

By Eq. (3.5.5), we have:

$$K^{M_i}_{VU} = \kappa \rho_i^2 + (\nabla \rho_i)^2$$

Now,

$$\begin{aligned}
\nabla_X K^{M_i}_{VU} &= 2\kappa \rho_i \nabla_X \rho_i + 2 \langle \nabla_X \nabla \rho_i, \nabla \rho_i \rangle \\
&= 2(\kappa \rho_i \nabla_X \rho_i + S^{\rho_i}(X, \nabla \rho_i)) \\
&= 2(\kappa \rho_i \nabla_X \rho_i - \kappa \rho_i \langle X, \nabla \rho_i \rangle) \\
&= 0
\end{aligned}$$

Hence for each $i > 0$, (M_i, g_i) necessarily has constant curvature, say κ_i; this proves Lemma 7.1.4. Finally Eq. (3.5.4) gives us the following:

$$\langle \nabla \log \rho_i, \nabla \log \rho_k \rangle = -\kappa \quad (i \neq k) \tag{7.4.2}$$

Now suppose $\dim M_0 = 1$ and suppose coordinates on M_0 are chosen such that $\tilde{g} = \varepsilon dx_1^2$ where $\varepsilon = \pm 1$ as the case may be. Then Eqs. (7.4.1) and (7.4.2) imply the following:

$$\partial_1^2 \rho_i = -\kappa\varepsilon\rho_i$$
$$(\partial_1 \log \rho_i)(\partial_1 \log \rho_k) = -\varepsilon\kappa \quad (i \neq k)$$

Hereafter we denote $\omega = \kappa\varepsilon$ and let $\sigma_i = \rho_i^2$. We make exclusive use of the above two equations in the following calculations, suppose $i \neq k$, then:

$$\begin{aligned}
\frac{\sigma_i'}{\sigma_k'} &= \frac{\rho_i \rho_i'}{\rho_k \rho_k'} \\
&= \frac{\rho_i \rho_i'}{\rho_k^2}(-\omega \frac{\rho_i'}{\rho_i}) \\
&= -\omega(\frac{\rho_i'}{\rho_k})^2
\end{aligned}$$

Thus

$$\begin{aligned}
(\frac{\sigma_i'}{\sigma_k'})' &= -2\omega(\frac{\rho_i'}{\rho_k})(\frac{\rho_i'}{\rho_k})' \\
&= -2\omega(\frac{\rho_i'}{\rho_k})(\frac{\rho_i''\rho_k - \rho_i'\rho_k'}{\rho_k^2}) \\
&= -2\omega(\frac{\rho_i'}{\rho_k^2})(\rho_i'' + \omega\rho_i) \\
&= 0
\end{aligned}$$

Hence any warped product decomposition of a space of constant curvature with $\dim M_0 = 1$ has a metric given by Eq. (7.1.5). Thus we have proven Theorem 7.1.2.

7.5 Conclusion

In this chapter our main result is that in a space of constant curvature, every orthogonal separable coordinate system is a KEM coordinate system. This fact motivates the systematic study of concircular tensors in these spaces in the next section. The results in that chapter will allow us to apply the theory presented in Chapter 6.

7.6 Notes

The results of this chapter raise the following question: Are there any other spaces of interest in which one can prove that orthogonally separable coordinates have diagonal curvature, or at least admit such coordinates? This is a very nice property to have since if we can rule out the metric Eq. (7.1.6), then the separable coordinates are invariantly characterized by concircular tensors and hence highly amenable to analysis (see Chapter 6). It may also be of interest to find another condition in addition to the diagonal curvature condition in order to characterize KEM coordinates.

A related question: Given a pseudo-Riemannian manifold, what is a necessary and sufficient intrinsic condition that guarantees the existence of an orthogonal coordinate system having diagonal curvature? One can show that a necessary condition is the existence of an orthogonal coordinate system in which the Ricci tensor is diagonal.

We end with some notes on the KEM separation theorem (Theorem 7.1.2), which guarantees a non-trivial OCT associated with every ChKT in a space of constant curvature. For the Euclidean and spherical spaces, this theorem is implicitly applied in [WW03]; see Remark 6.7.1 for more details. This theorem was explicitly known for the special class of separable webs defined by Benenti tensors, see [Ben92a]. It was first stated in its present form in [RM14b], then proven in [RM14a].

Chapter 8

Preliminaries from Linear Algebra and Geometry

In this chapter we introduce the perquisite notation and results to read Chapter 9.

8.1 pseudo-Euclidean space

We will briefly review some definitions for a pseudo-Euclidean space from Section 1.4.1. First recall, that an n-dimensional vector space V equipped with metric g of signature[1] ν is denoted by $\mathbb{E}^n_\nu$ and called *pseudo-Euclidean space*. In some contexts the space is simply denoted V, and the metric $\langle\cdot,\cdot\rangle$ is called a *scalar product* (following [O'N83]). We also refer to ν as the *index* of the subspace V, denoted $\operatorname{ind} V$. We obtain Euclidean space $\mathbb{E}^n$ in the special case where $\nu = 0$. Also Minkowski space M^n is obtained by taking $\nu = 1$. Also note that since $\mathbb{E}^n_\nu$ is a vector space, for any $p \in \mathbb{E}^n_\nu$ we identify vectors in $T_p\mathbb{E}^n_\nu$ with points in $\mathbb{E}^n_\nu$. This will be done tacitly.

Recall, a set $v_1, ..., v_n$ for V is said to be *orthonormal* if $\langle v_i, v_i\rangle = \pm 1$ and $\langle v_i, v_j\rangle = 0$ for $i \neq j$. Clearly an orthonormal set forms a basis for V and the metric in this basis is $g = \operatorname{diag}(\pm 1, ..., \pm 1)$.

Now assume the scalar product $\langle\cdot,\cdot\rangle$ is (possibly) degenerate. We say a sequence of vectors $v_1, \ldots, v_p$ is a *skew-normal sequence* of (length p) and (sign $\varepsilon = \pm 1$) if $\langle v_i, v_j\rangle = \varepsilon$ when $i + j = p + 1$ and $\langle v_i, v_j\rangle = 0$ otherwise. We will show shortly that these vectors are necessarily linearly independent, so let $H = \operatorname{span}\{v_1, \ldots, v_p\}$. Then the bilinear form restricted to H is *skew-diagonal* and is given as follows:

[1]The signature is equal to the number of negative diagonal entries in a basis which diagonalizes g.

$$S_k := \begin{pmatrix} 0 & & \varepsilon \\ & \cdot^{\cdot^{\cdot}} & \\ \varepsilon & & 0 \end{pmatrix}$$

The following lemma shows that a skew-normal sequence forms a linearly independent set and it gives the index of the space spanned by such a set of vectors.

Lemma 8.1.1

Suppose V is a (possibly complex) scalar product space. Suppose $\varepsilon \in \{-1, 1\}$ and let $z_1, \ldots, z_p$ be a skew-normal sequence of sign ε.

Then $z_1, \ldots, z_p$ form a linearly independent set and the subspace H spanned by these vectors is non-degenerate and has index:

$$\operatorname{ind} H = \begin{cases} \lfloor \frac{p+1}{2} \rfloor & \text{if } \epsilon = -1 \\ p - \lfloor \frac{p+1}{2} \rfloor & \text{if } \epsilon = 1 \end{cases}$$ □

Proof Given $1 \leq i \leq p$, denote the additive conjugate of i by $i' = p + 1 - i$. Note that $1 \leq i' \leq p$ and $i + i' = p + 1$. Suppose $i < \frac{p+1}{2}$ and $j > \frac{p+1}{2}$. Define vectors v_i and v_j as follows:

$$v_i = \frac{1}{\sqrt{2}}(z_i + z_{i'})$$
$$v_j = \frac{1}{\sqrt{2}}(z_j - z_{j'})$$

If $2i = p + 1$ then let $v_i = z_i$. Now for $i < j$ suppose $i + j = p + 1$, then note that $i' = j$ and $j' = i$, observe that:

$$\begin{aligned} \langle v_i, v_j \rangle &= \frac{1}{2}(\langle z_i + z_j, z_j - z_i \rangle) \\ &= \frac{1}{2}(\langle z_j, z_j \rangle + \langle z_i, z_i \rangle) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \langle v_i, v_i \rangle &= \frac{1}{2}(\langle z_j + z_i, z_j + z_i \rangle) \\ &= \frac{1}{2}(2 \langle z_i, z_j \rangle) \\ &= \epsilon \end{aligned}$$

$$
\begin{aligned}
\langle v_j, v_j \rangle &= \frac{1}{2}(\langle z_j - z_i, z_j - z_i \rangle) \\
&= -\frac{1}{2}(2 \langle z_i, z_j \rangle) \\
&= -\epsilon
\end{aligned}
$$

Furthermore if $2i = p+1$ then $\langle v_i, v_i \rangle = \epsilon$. Now suppose $i + j \neq p+1$, then the only way in which $\langle v_i, v_j \rangle \neq 0$ if $i+j' = p+1$ or $i'+j = p+1$, but one can see immediately that $i+j' = p+1$ iff $i'+j = p+1$. So suppose $i+j' = p+1$. Then $p+1 = i+i' = i+p+1-j$, hence $i = j$, in which case $\langle v_i, v_j \rangle$ reduces to the ones examined.

Thus we conclude that $\langle v_i, v_i \rangle = \epsilon$ if $i \leq \lfloor \frac{p+1}{2} \rfloor$, $\langle v_i, v_i \rangle = -\epsilon$ if $i > \lfloor \frac{p+1}{2} \rfloor$ and $\langle v_i, v_j \rangle = 0$ if $i \neq j$. Thus the conclusions follow. ■

8.2 Self-adjoint operators in pseudo-Euclidean space

In this section we review the metric-Jordan canonical form of a self-adjoint operator on a pseudo-Euclidean space. The details of the theory behind this canonical form is given in Appendix C; these are solutions to exercises 18-19 in [O'N83, P. 260-261]. Appendix C will be useful to those who want to calculate the metric-Jordan canonical form for a given self-adjoint operator. Now recall, that a linear operator T on a scalar product space V is called self-adjoint if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in V$.

A *Jordan block* of dimension k with eigenvalue $\lambda \in \mathbb{C}$ is a $k \times k$ matrix denoted by $J_k(\lambda)$, and defined as:

$$
J_k(\lambda) := \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & \ddots & & 0 \\ & & \ddots & 1 & \\ & & & \lambda & 1 \\ & 0 & & & \lambda \end{pmatrix}
$$

Recall, the *skew-diagonal matrix* of dimension k is denoted by S_k, and defined as:

$$
S_k := \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}
$$

In order to express the metric-Jordan canonical form of a self-adjoint operator on a pseudo-Euclidean space, we use the signed integer $\varepsilon k \in \mathbb{Z}$ where $k \in \mathbb{N}$ and $\varepsilon = \pm 1$. Then the notation $J_{\varepsilon k}(\lambda)$ is short hand for the pair:

$$A = J_k(\lambda) \qquad\qquad g = \varepsilon S_k$$

Furthermore, given matrices A_1 and A_2, we denote the following block diagonal matrix by $A_1 \oplus A_2$

$$A_1 \oplus A_2 := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

A key fact used to derive the metric-Jordan canonical form is the following:

Proposition 8.2.1
Suppose V is a scalar product space and T is a self-adjoint operator on V. Suppose $H \subseteq V$ is an invariant subspace of T. Then $T(H^\perp) \subseteq H^\perp$, i.e. $H^\perp$ is an invariant subspace of T. □

First we give the complex metric-Jordan canonical form of a self-adjoint operator.

Theorem 8.2.2 (Complex metric-Jordan canonical form [O'N83])
A real operator T on a pseudo-Euclidean space $\mathbb{E}^n_\nu$ is self-adjoint iff there exists a (possibly complex) basis β such that

$$T|_\beta = J_{\varepsilon_1 k_1}(\lambda_1) \oplus \cdots \oplus J_{\varepsilon_l k_l}(\lambda_l)$$

Furthermore there exists a canonical basis such that the unordered list

$$\{J_{\varepsilon_1 k_1}(\lambda_1), \ldots, J_{\varepsilon_l k_l}(\lambda_l)\}$$

is uniquely determined by T and an invariant of T under the action of the orthogonal group $O(\mathbb{E}^n_\nu)$. This unordered list is by definition, the complex metric-Jordan canonical form *of T.* □

Remark 8.2.3
Since T is real, each Jordan block $J_{\varepsilon k}(\lambda)$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$ comes with a complex conjugate pair $J_{\varepsilon k}(\overline{\lambda})$. For complex eigenvalues, we can additionally assume that $\varepsilon = 1$. □

In order to describe the real metric-Jordan canonical form, we need some additional notation. A *real Jordan block* of dimension k with parameters $a \pm ib \in \mathbb{C}$ is a $2k \times 2k$ matrix denoted by $J_{2k}(a \pm ib)$ defined as [LR05]:

$$J_{2k}(a \pm ib) := \begin{pmatrix} a & b & 1 & 0 & & & & \\ -b & a & 0 & 1 & & & 0 & \\ & & & & & & & \\ & & & \ddots & & & & \\ & & & & a & b & 1 & 0 \\ & & & & -b & a & 0 & 1 \\ & 0 & & & & & a & b \\ & & & & & & -b & a \end{pmatrix}$$

If we denote the basis vectors by $\{u_1, v_1, ..., u_k, v_k\}$, then in addition to the above real Jordan block, we will assume the non-zero metric coefficients are given by the relations $\langle u_i, u_j \rangle = 1 = -\langle v_i, v_j \rangle$ where $i + j = k + 1$. Note that in contrast with the complex case, there is no sign associated with the real Jordan blocks. If we let $\lambda = a + ib$, then $J_{2k}(a \pm ib)$ is obtained from $J_k(\lambda) \oplus J_k(\overline{\lambda})$ by an appropriate change of basis (see the discussion preceding Lemma C.3.6).

The following crucial lemma allows one to understand and apply the (real) metric-Jordan canonical form, it will be proven in Appendix C (see Lemma C.3.6).

Lemma 8.2.4

Suppose V is a real scalar product space (where the scalar product is possibly degenerate) and T is a self-adjoint operator on V. Consider the case where $T = J_{\varepsilon k}(\lambda)$ for some $\lambda \in \mathbb{R}$, then

$$\operatorname{ind} V = \begin{cases} \lfloor \frac{k+1}{2} \rfloor & \text{if } \varepsilon = -1 \\ k - \lfloor \frac{k+1}{2} \rfloor & \text{if } \varepsilon = 1 \end{cases}$$

If $T = J_{2k}(a \pm ib)$ for $a, b \in \mathbb{R}$, then

$$\operatorname{ind} V = k$$

In particular we observe that in both cases, the scalar product is necessarily non-degenerate. □

Thus we have:

Theorem 8.2.5 (Real metric-Jordan canonical form [O'N83])

A real operator T on a pseudo-Euclidean space $\mathbb{E}^n_\nu$ is self-adjoint iff there exists a real basis β such that

$$T|_\beta = J_{\varepsilon_1 k_1}(\lambda_1) \oplus \cdots \oplus J_{\varepsilon_l k_l}(\lambda_l) \oplus J_{2\tilde{k}_1}(a_1 \pm ib_1) \oplus \cdots \oplus J_{2\tilde{k}_r}(a_r \pm ib_r)$$

where each $\lambda_i, a_j, b_j \in \mathbb{R}$. Furthermore, the unordered list

$$\{J_{\varepsilon_1 k_1}(\lambda_1), \ldots, J_{\varepsilon_l k_l}(\lambda_l), J_{2\tilde{k}_1}(a_1 \pm ib_1), \ldots, J_{2\tilde{k}_r}(a_r \pm ib_r)\}$$

is uniquely determined by T and an invariant of T under the action of the orthogonal group $O(\mathbb{E}^n_\nu)$. This unordered list is by definition, the (real) metric-Jordan canonical form *of T.* □

We will apply the above results in Section 8.2.1 to enumerate the possible metric-Jordan canonical forms in Minkowski space. For now, we give an important example which clearly distinguishes the metric-Jordan canonical form the standard Jordan canonical form.

Example 8.2.6
Suppose V is Minkowski space equipped with the standard metric

$$g = \mathrm{diag}(-1, 1, \ldots, 1)$$

For $\lambda_1 < \ldots < \lambda_n \in \mathbb{R}$ define two self-adjoint operators T_1 and T_2 as follows:

$$\begin{aligned} T_1 &= \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \\ T_2 &= \mathrm{diag}(\lambda_2, \lambda_1, \lambda_3, \ldots, \lambda_n) \end{aligned}$$

Now observe that even though T_1 and T_2 have the same eigenvalues, they have different metric-Jordan canonical forms. Hence the above theorem shows that these operators are isometrically inequivalent. □

8.2.1 Minkowski Space

Fix a self-adjoint operator T in Minkowski space. We will use Theorem 8.2.5 to enumerate the possible Jordan canonical forms of T together with the metric in an adapted basis. As a consequence of Theorem 8.2.5, we simply have to determine which combination of Jordan blocks are possible in Minkowski space by imposing the dimension and signature restrictions. This can be done with the help of Lemma 8.2.4, since it gives us the index of a given subspace associated with a Jordan block. We denote by D_k a diagonal $k \times k$ matrix and I_k the identity $k \times k$ matrix. We have the following cases.

Case 1 T is diagonalizable with real spectrum

In this case T must have a time-like eigenvector. Indeed, since each eigenspace E_λ is non-degenerate, one eigenspace, say H, must have index 1. Then by obtaining an orthonormal basis for H, we can obtain a time-like eigenvector. Thus T has the following form:

$$T = D_n \qquad\qquad g = \mathrm{diag}(-1, 1, ..., 1)$$

Case 2 T has a complex eigenvalue $\lambda = a + ib$ with $b \neq 0$

By Lemma 8.2.4 the real subspace H spanned by a complex eigenvector with eigenvalue λ and its complex conjugate must have index 1. Since this subspace is T-invariant, by Proposition 8.2.1 $H^\perp$ is a complementary invariant subspace, which must be Euclidean. Hence T must have the following form:

$$T = \begin{pmatrix} a & b & 0 \\ -b & a & \\ 0 & & D_{n-2} \end{pmatrix} \qquad\qquad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & \\ 0 & & I_{n-2} \end{pmatrix}$$

Case 3 T has real eigenvalues but is not diagonalizable

In this case we go through the possible multidimensional Jordan blocks associated to a real irreducible subspace, say H, of T. By Theorem 8.2.5, each basis for this subspace can be adapted to the scalar product, hence is non-degenerate. By Lemma 8.2.4 there are three types of Jordan blocks which have an associated subspace, H, with index one. For each of these subspaces, $H^\perp$ is a complementary T-invariant Euclidean subspace. The first two cases occur when $\dim H = 2$, and are given as follows:

$$T = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & \\ 0 & & D_{n-2} \end{pmatrix} \qquad\qquad g = \begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & \\ 0 & & I_{n-2} \end{pmatrix} \qquad \epsilon = \pm 1$$

Note that the above form contains two metric-Jordan canonical forms depending on the sign of ϵ. The third occurs when $\dim H = 3$:

$$T = \begin{pmatrix} \lambda & 1 & 0 & \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & \\ & 0 & & D_{n-3} \end{pmatrix} \qquad g = \begin{pmatrix} 0 & 0 & 1 & \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \\ & 0 & & I_{n-3} \end{pmatrix}$$

We also note that this case (T has real eigenvalues but is not diagonalizable) holds iff T has a unique lightlike eigenvector. This fact can be deduced by inspection of the above canonical forms.

Now, we collect some necessary and sufficient conditions concerning the diagonalizability of T in the following theorem. The second and third facts are from Theorem 4.1 in [Hal+96] while the last fact is from Section 9.5 in [Gre75]. All these facts can be readily deduced from the canonical forms listed above.

Theorem 8.2.7 (Properties of self-adjoint operators in Minkowski space) *Let V be a Minkowski space and T a self-adjoint operator on V. Then the following statements are true:*

1. *T is diagonalizable with a real spectrum iff T has 1 timelike eigenvector or equivalently T has $n-1$ linearly independent spacelike eigenvectors.*

2. *If T has two linearly independent null eigenvectors then T is diagonalizable with a real spectrum and T has a time-like eigenspace of dimension at least 2 containing these eigenvectors.*

3. *If T has a real spectrum, then T is diagonalizable iff it has no null eigenvectors or at least two linearly independent null eigenvectors. In other words, T is not diagonalizable iff it has a unique null eigendirection.*

4. *If $n \geq 3$ and $\langle Tx, x \rangle \neq 0$ for all null vectors x then T is diagonalizable with a real spectrum.* □

8.3 Spaces of Constant Curvature in pseudo-Euclidean space

In this section we will briefly review the models of spaces of constant curvature as subsets of pseudo-Euclidean space. It is well known that $\mathbb{E}^n_\nu$ has constant zero curvature (flat space) and signature ν. There is another useful model of flat space which we will review towards the end.

Given an open subset $U \subseteq \mathbb{E}^n_\nu$ and $\kappa \in \mathbb{R} \setminus \{0\}$, we denote by $U(\kappa)$ the *central hyperquadric* of $\mathbb{E}^n_\nu$ contained in U, which is defined by:

$$U(\kappa) = \{p \in U \mid \langle p, p\rangle = \kappa^{-1}\}$$

Usually $U = \mathbb{E}^n_\nu$ and this is denoted $\mathbb{E}^n_\nu(\kappa)$. The notation $U(\kappa)^\circ$ represents a maximal connected component of $U(\kappa)$. It is well known that $\mathbb{E}^n_\nu(\kappa)$ is a pseudo-Riemannian manifold of dimension $n-1$ with signature $\nu + \frac{(\operatorname{sgn}\kappa - 1)}{2}$ and constant curvature[2] κ [O'N83]. Since $\mathbb{E}^n_\nu(\kappa) \subset \mathbb{E}^n_\nu$, for any $p \in \mathbb{E}^n_\nu(\kappa)$ we identify vectors in $T_p\mathbb{E}^n_\nu(\kappa)$ with points in $\mathbb{E}^n_\nu$. Occasionally we use the following conventions: If $\kappa = 0$, we set $\mathbb{E}^n_\nu(0) := \mathbb{E}^n_\nu$, if $\kappa = \infty$ we set $\mathbb{E}^n_\nu(\infty)$ to be the light cone, i.e. the set of non-zero null vectors. We also use the following notations: If $\kappa > 0$ then $S^n_\nu(\kappa) := \mathbb{E}^{n+1}_\nu(\kappa)^\circ$, if $\kappa < 0$ then $H^n_\nu(\kappa) := \mathbb{E}^{n+1}_{\nu+1}(\kappa)^\circ$.

We define the parabolic embedding of $\mathbb{E}^n_\nu$ in $\mathbb{E}^{n+2}_{\nu+1}$ with mean curvature vector $-a \in \mathbb{E}^n_\nu(\infty)$ by [Toj07]

$$\mathbb{P}^n_\nu := \{p \in \mathbb{E}^{n+2}_{\nu+1}(\infty) : \langle p, a\rangle = 1\}$$

An explicit isometry with $\mathbb{E}^n_\nu$ is obtained by choosing $b \in \mathbb{P}^n_\nu$, i.e. b is lightlike and $\langle a, b\rangle = 1$. We let $V := \operatorname{span}\{a, b\}^\perp$, note that $V \cong \mathbb{E}^n_\nu$, then for $x \in V$:

$$\psi(x) = b + x - \frac{1}{2}x^2 a \in \mathbb{P}^n_\nu \tag{8.3.1}$$

For the proofs of these properties of $\mathbb{P}^n_\nu$, see Proposition D.2.2.

8.4 Warped products in Spaces of Constant Curvature

In this section we will briefly describe the warped product decompositions of spaces of constant curvature, generalizing results by Nolker in [Nol96]. This exposition will be sufficient for our applications. More information and proofs can be found in Appendix D.

We will use the notation $\mathbb{E}^n_\nu(\kappa)$ (where κ can be zero) to represent the general space of constant curvature. First we will need to know the spherical submanifolds of these spaces.

Theorem 8.4.1 (Spherical submanifolds of $\mathbb{E}^n_\nu(\kappa)$)
Let $\bar{p} \in \mathbb{E}^n_\nu(\kappa)$ be arbitrary, $V \subset T_{\bar{p}}\mathbb{E}^n_\nu(\kappa)$ a non-degenerate subspace with $m := \dim V \geq 1$, $\mu := \operatorname{ind} V$ and $z \in V^\perp \cap T_{\bar{p}}\mathbb{E}^n_\nu(\kappa)$. Let $a := \kappa\bar{p} - z$, $\tilde{\kappa} := a^2$ and $W := \mathbb{R}a \oplus V$. There is exactly one m-dimensional connected and geodesically complete spherical submanifold $\tilde{N}$ with $\bar{p} \in \tilde{N}$, $T_{\bar{p}}\tilde{N} = V$ and having mean curvature vector at $\bar{p}$, z. $\tilde{N}$ is an open submanifold

[2] See Appendix D.2.

of N; N is referred to as the spherical submanifold determined by $(\bar{p}, V, a)$*, it is geodesic iff* $z = 0$ *and is given as follows (where* $\simeq$ *means isometric to):*

(a) $a = 0$*, in this case* $N \simeq \mathbb{E}^m_\mu$

$$N = \overline{p} + V$$

(b) a *is timelike, then* $\mu \leq \nu - 1$ *and* $N \simeq H^m_\mu(\tilde{\kappa})$

(c) a *is spacelike, then* $N \simeq S^m_\mu(\tilde{\kappa})$
For cases (b) and (c), let $c = \overline{p} - \frac{a}{\tilde{\kappa}}$ *be the center of N, then N is given as follows:*

$$N = c + \{p \in W \mid p^2 = \frac{1}{\tilde{\kappa}}\}$$

(d) a *is lightlike, then* $\mu \leq \nu - 1$ *and* $N \simeq \mathbb{E}^m_\mu$

$$N = \bar{p} + \{p - \frac{1}{2}p^2 a \mid p \in V\}$$ □

Remark 8.4.2
If a is lightlike, then N is isometric to $\mathbb{P}^m_\mu$ with mean curvature vector $-a$. Furthermore, let $b \in V^\perp$ be a lightlike vector satisfying $\langle a, b\rangle = 1$. Then the orthogonal projector onto V, P, induces an isometry of $N - \bar{p} + b$ onto V. □

Remark 8.4.3
One can find more details on when N is connected in the remarks following Theorems D.4.1 and D.6.2. □

Proof See Theorems D.4.1 and D.6.2. ■

With the knowledge of these spherical submanifolds, we can now specify how to construct warped products in $\mathbb{E}^n_\nu(\kappa)$. This construction depends on the following data: A point $\bar{p} \in \mathbb{E}^n_\nu(\kappa)$, a decomposition $T_{\bar{p}}\mathbb{E}^n_\nu(\kappa) = \bigoplus_{i=0}^{k} V_i$ into non-trivial (hence non-degenerate) subspaces with $k \geq 1$, and vectors $z_1, \ldots, z_k \in V_0$ such that the vectors $a_i := \kappa\bar{p} - z_i$ are pair-wise orthogonal and independent. We call the data $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$, *initial data* for a (proper) warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$. If $\kappa = 0$, one can more generally let some of the a_i be zero, this results in Cartesian products as done in [Nol96]. Since we assume the a_i are non-zero, we sometimes use the additional qualifier "proper".

With this initial data, for $i > 0$ let N_i be the sphere in $\mathbb{E}^n_\nu(\kappa)$ determined by $(\bar{p}, V_i, a_i)$ and $\rho_i(p_0) = 1 + \langle a_i, p_0 - \overline{p}\rangle$. Let N_0 be the subset of the sphere in $\mathbb{E}^n_\nu(\kappa)$ determined by $(\bar{p}, V_0, \kappa\bar{p})$ where each $\rho_i > 0$. Then the data $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$, induces a warped product decomposition (of $\mathbb{E}^n_\nu(\kappa)$) given as follows:

$$\psi : \begin{cases} N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k & \to \mathbb{E}^n_\nu(\kappa) \\ (p_0, ..., p_k) & \mapsto p_0 + \sum_{i=1}^{k} \rho_i(p_0)(p_i - \bar{p}) \end{cases} \tag{8.4.1}$$

We note that ψ has the property that $\psi(\bar{p}, \ldots, p_i, \bar{p}, \ldots, \bar{p}) = p_i$. Often the point $\bar{p}$ doesn't enter calculations, hence we will usually omit it.

For actual calculations, it will be more convenient to work with canonical forms. The following definition will be particularly useful.

Definition 8.4.4 (Canonical form for Warped products of $\mathbb{E}^n_\nu$)
We say that a proper warped product decomposition of $\mathbb{E}^n_\nu$ determined by $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$ is in canonical form if: $\bar{p} \in V_0$ and $\langle \bar{p}, a_i \rangle = 1$. □

Any proper warped product decomposition ψ of $\mathbb{E}^n_\nu$ can be brought into canonical form, see the discussion preceding Corollary D.4.10 for details.

We will now give more information on standard warped product decompositions of $\mathbb{E}^n_\nu$ in canonical form. Suppose the initial data $(\bar{p}; V_0 \oplus V_1; a)$ is in canonical form, and let ψ be the associated warped product decomposition given by Eq. (8.4.1). Denote $\kappa := a^2$ and $\epsilon := \operatorname{sgn} \kappa$. We have two types of warped products:

non-null warped decomposition If $\kappa \neq 0$, let $W_0 := V_0 \cap a^\perp$ and $W_1 := W_0^\perp$.

null warped decomposition If $\kappa = 0$, then a is lightlike, so fix another lightlike vector $b \in V_0$ such that $\langle a, b \rangle = 1$, let $W_0 := V_0 \cap \operatorname{span}\{a, b\}^\perp$ and $W_1 := V_1$.

For $i = 0, 1$, let $P_i : \mathbb{E}^n_\nu \to W_i$ be the orthogonal projection. Then the following holds:

Theorem 8.4.5 (Standard Warped Products in $\mathbb{E}^n_\nu$ [Nol96])
Let ψ be the warped product decomposition of $\mathbb{E}^n_\nu$ determined by the initial data $(\bar{p}; V_0 \oplus V_1; a)$ given above. Then N_0 has the following form:

$$N_0 = \{p \in V_0 |\, \langle a, p \rangle > 0\}$$

and

$$\rho : \begin{cases} N_0 & \to \mathbb{R}_+ \\ p_0 & \mapsto \langle a, p_0 \rangle \end{cases}$$

The map ψ is an isometry onto the following set:

$$\mathrm{Im}(\psi) = \begin{cases} \{p \in \mathbb{E}^n_\nu \mid \operatorname{sgn}(P_1 p)^2 = \epsilon\} & \textit{non-null case} \\ \{p \in \mathbb{E}^n_\nu \mid \langle a, p\rangle > 0\} & \textit{null case} \end{cases}$$

Furthermore, the following equation holds:

$$\psi(p_0, p_1)^2 = p_0^2 \tag{8.4.2}$$

□

Proof See Corollary D.4.10. ■

In fact, for $(p_0, p_1) \in N_0 \times N_1$, ψ has one of the following forms, first if ψ is non-null:

$$\psi(p_0, p_1) = P_0 p_0 + \langle a, p_0\rangle (p_1 - c) \tag{8.4.3}$$

where $c = \bar{p} - \frac{a}{a^2}$, and if ψ is null:

$$\psi(p_0, p_1) = P_0 p_0 + (\langle b, p_0\rangle - \frac{1}{2}\langle a, p_0\rangle (P_1 p_1)^2) a + \langle a, p_0\rangle b + \langle a, p_0\rangle P_1 p_1 \tag{8.4.4}$$

The above forms are obtained from the equation for ψ from the above theorem by expanding p_0 in an appropriate basis. We note that the warped products with multiple spherical factors can be obtained using the standard ones described above. Indeed, suppose $\phi_1 : N_0' \times_{\rho_1} N_1 \to \mathbb{E}^n_\nu$ is the warped product decomposition determined by $(\bar{p}; V_0 \ominus V_1; a_1)$ as above. Since V_0 is pseudo-Euclidean, consider a warped product decomposition, $\phi_2 : \tilde{N}_0 \times_{\rho_2} N_2 \to V_0$, determined by $(\bar{p}; \tilde{V}_0 \ominus \tilde{V}_1; a_2)$ with $V_0 \cap W_0^\perp \subset \tilde{W}_0$ (hence $a_1 \in \tilde{W}_0$). Note that $\tilde{W}_0$ is the subspace W_0 from the above construction for ϕ_2. Let $N_0 := N_0' \cap \tilde{N}_0$, then one can check that the map ψ defined by:

$$\psi : \begin{cases} N_0 \times_{\rho_1} N_1 \times_{\rho_2} N_2 & \to \mathbb{E}^n_\nu \\ (p_0, p_1, p_2) & \mapsto \phi_1(\phi_2(p_0, p_2), p_1) \end{cases}$$

is a warped product decomposition of $\mathbb{E}^n_\nu$ satisfying Eq. (8.4.1). We illustrate this construction with an example.

Example 8.4.6 (Constructing multiply warped products)
Suppose ϕ_1 and ϕ_2 are given as follows:

$$\begin{aligned} \phi_1(p_0', p_1) &= P_0' p_0' + \langle a_1, p_0'\rangle (p_1 - c_1) \\ \phi_2(\tilde{p}_0, p_2) &= \tilde{P}_0 \tilde{p}_0 + \langle a_2, \tilde{p}_0\rangle (p_2 - c_2) \end{aligned}$$

Now observe that $\rho_1(\phi_2(\tilde{p}_0, p_2)) = \rho_1(\tilde{p}_0)$, which follows from the above equation for ϕ_2 and the fact that $a_1 \in \tilde{W}_0$. Then,

$$\begin{aligned}\psi(p_0, p_1, p_2) &= \phi_1(\phi_2(p_0, p_2), p_1) \\ &= P_0'\phi_2(p_0, p_2) + \langle a_1, \phi_2(p_0, p_2)\rangle(p_1 - c_1) \\ &= P_0'\tilde{P}_0 p_0 + \langle a_2, p_0\rangle\,(p_2 - c_2) + \langle a_1, p_0\rangle\,(p_1 - c_1)\end{aligned}$$

where $P_0'\tilde{P}_0$ is the orthogonal projector onto $\tilde{W}_0 \cap W_0 = \tilde{V}_0 \cap \operatorname{span}\{a_1, a_2\}^{\perp}$. A similar calculation shows that ψ satisfies Eq. (8.4.1), since ϕ_1 and ϕ_2 each satisfy it. □

This procedure can be repeated as many times as necessary to obtain the more general warped products given by Eq. (8.4.1). Hence the properties of the more general warped product decompositions of $\mathbb{E}^n_\nu$ can be deduced from Theorem 8.4.5.

The following proposition shows that any proper warped product decomposition of $\mathbb{E}^n_\nu$ in canonical form restricts to a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ where $\kappa \neq 0$. Its proof is straightforward consequence of Eq. (8.4.2).

Theorem 8.4.7 (Restricting Warped products to $\mathbb{E}^n_\nu(\kappa)$)
Let ψ be a proper warped product decomposition of $\mathbb{E}^n_\nu$ associated with $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$ in canonical form. Suppose $\kappa^{-1} := \bar{p}^2 \neq 0$ and let $N' := N_0(\kappa) \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k$. Then $\phi : N' \to \mathbb{E}^n_\nu(\kappa)$ defined by $\phi := \psi|_{N'}$ is a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ passing through $\bar{p}$. □

Proof See Theorem D.6.5. ■

Hence the details of warped product decompositions of $\mathbb{E}^n_\nu(\kappa)$ can be deduced from Theorem 8.4.5. More information on these decompositions can be found in Appendix D. The results presented here will be applied in Section 9.5, where examples can also be found.

Chapter 9

Concircular tensors in Spaces of Constant Curvature

It has been shown in Section 6.5 that any point-wise diagonalizable concircular tensor hereafter called a *OCT* can be used to recursively construct separable coordinates for the (geodesic) Hamilton-Jacobi equation. These coordinates were called Kalnins-Eisenhart-Miller (KEM) coordinates. It was shown in Chapter 7 that all orthogonal separable coordinates for the Hamilton-Jacobi equation in spaces of constant curvature occur this way. Hence the classification of OCTs in spaces of constant curvature is of fundamental importance for classifying separable coordinates in these spaces.

Specifically, OCTs have the following uses:

1. An algebraic classification of these tensors modulo the action of the isometry group can be used to obtain a notion of in-equivalence for KEM coordinate systems.

2. Crampin has shown in [Cra03] that one can obtain transformations to separable coordinates for OCTs with functionally independent eigenfunctions. It was shown in Section 6.5 that a knowledge of the warped product decompositions of the space is sufficient to obtain transformations to separable coordinates for any KEM coordinate system.

3. When concircular tensors have simple eigenfunctions, it was shown in [Ben05] (see also [Ben92a; Ben93; Ben04]) that a basis for the Killing-Stackel space can be obtained. These results have been generalized to arbitrary KEM webs in Section 6.6.

4. With a classification of concircular tensors, the BEKM separation algorithm (presented in Section 6.7), can be executed to solve the separation of variables problem for natural Hamiltonians.

In this chapter we will obtain a complete (local) classification of orthogonal concircular tensors in all spaces of constant curvature with Euclidean and Lorentzian signature[1]. This will enable one to carry out the above tasks using only (linear) algebraic operations. We note that the contents of this chapter are from the article [RM14c].

More details on our classification and the way in which it's done is given in Section 9.1.3, after we have briefly reviewed the necessary material from Chapter 6 in Sections 9.1.1 and 9.1.2. We will assume the reader is familiar with results and notations introduced in Chapter 8. Some of our results are also summarized in Section 9.1.3.

9.1 Preliminaries and Summary

9.1.1 Concircular tensors

Recall from Chapter 6 that a tensor $L \in S^p(M)$ is called a *concircular tensor (CT)* of valence p if there exists $C \in S^{p-1}(M)$ (called the *conformal factor*) such that

$$\nabla_x L = C \odot x$$

for all $x \in \mathfrak{X}(M)$. Throughout this chapter, we will simply call L a concircular tensor when $p = 2$.

We now recall some properties of OCTs from Chapter 6. First, given a $\binom{1}{1}$ tensor L, let N_L be the Nijenhuis tensor (torsion) of L (see Definition B.0.13). We say that L is *torsionless* if its Nijenhuis tensor vanishes. Then if L is a concircular tensor, the following equations hold by Proposition 6.3.1

$$\begin{aligned} [L,G] &= -2\nabla \operatorname{tr}(L) \odot G \quad ([L,G]_{abc} = -2\nabla_{(a}L_{bc)}) \\ N_L &= 0 \end{aligned}$$

Conversely, by Proposition 6.3.5, an orthogonal tensor satisfying the above equations is a C-tensor. The first of the above equations tells us that a C-tensor is a conformal Killing tensor of trace-type. The second equation can be interpreted if we assume L is an OC-tensor.

Suppose now that L is an OC-tensor with eigenspaces $(E_i)_{i=1}^k$ and corresponding eigenfunctions $\lambda^1, ..., \lambda^k$. Since an OC-tensor has Nijenhuis torsion zero, by Theorem B.0.20 the eigenspaces $(E_i)_{i=1}^k$ are orthogonally integrable and each eigenfunction λ^i depends only on E_i. Furthermore the trace-type condition implies that the eigenfunction corresponding to a multidimensional eigenspace of L is a constant (see Proposition 6.2.2 (3)).

[1]The classification for other signatures can be obtained fairly easily if one wishes.

Suppose D is a multidimensional eigenspace of a non-trivial[2] OCT L. Denote by $D^\perp$ the distribution orthogonal to D. Then one can show that (see Proposition 6.3.6):

- There is a local product manifold $B \times F$ of Riemannian manifolds (B, g_B) and (F, g_F) such that:
  $\{p\} \times F$ is an integral manifold of D for any $p \in B$ and
  $B \times \{q\}$ is an integral manifold of $D^\perp$ for any $q \in F$.

- $B \times F$ equipped with the metric $\pi_B^* g_B + \rho^2 \pi_F^* g_F$ for a specific function $\rho : B \to \mathbb{R}^+$ is locally isometric to (M, g); where π_B (resp. π_F) is the canonical projection onto B (resp. F).

Such a product manifold is called a *warped product* and is denoted $B \times_\rho F$. We also say in this case that the warped product $B \times_\rho F$ is *adapted* to the splitting $(D^\perp, D)$. The manifold F is a *spherical submanifold* and B is *geodesic submanifold* of M (see Section 3.1). An important observation is that L restricted to B is an OCT (by Proposition 4.4.15); we will use this later to construct OCTs from Benenti tensors.

In general if L has multiple multidimensional eigenspaces, we will have to consider more general warped products. So suppose $M = \prod_{i=0}^k M_i$ is a product manifold of pseudo-Riemannian manifolds (M_i, g_i) where $\dim M_i > 0$ for $i > 0$. Equip M with the metric $g = \sum_{i=0}^k \rho_i^2 \pi_i^* g_i$ where $\rho_i : M_0 \to \mathbb{R}^+$ are functions with $\rho_0 \equiv 1$ and $\pi_i : M \to M_i$ are the canonical projection maps. Additionally we assume either $\dim M_0 > 0$ or $k > 1$. Then (M, g) is called a *warped product* and the metric g is called a *warped product metric*. If $\dim M_0 = 0$ then (M, g) is called a *pseudo-Riemannian product*. The warped product is denoted by $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$. M_0 is called the geodesic factor of the warped product and the M_i for $i > 0$ are called spherical factors. See Section 3.5 and references therein for more on warped products.

The following class of OCTs are fundamental to the classification:

Definition 9.1.1 (Irreducible concircular tensors)
An OC-tensor with functionally independent eigenfunctions is referred to as an *ICT* or more succinctly an *IC-tensor*. To be precise, an IC-tensor has real eigenfunctions $u^1, \dots, u^k$ (counted without multiplicity) satisfying:

$$du^1 \wedge \cdots \wedge du^k \neq 0$$

Furthermore an OC-tensor which is not irreducible is called *reducible*. □

Since we observed earlier that the eigenfunction associated with a multidimensional eigenspace of an OCT is constant, it follows that an ICT must have simple eigenfunctions,

[2]By a non-trivial concircular tensor, we mean one which is not a multiple of the metric when $n > 1$.

hence ICTs are Benenti tensors. The special property that ICTs have is that their eigenfunctions can be used as (local) coordinates for the separable web they induce [Cra03]. We will refer to these coordinates as the *canonical coordinates* induced by these tensors.

Away from singular points, locally, we can assume a reducible OC-tensor has eigenfunctions $u^1, ..., u^k$ which are functionally independent and the rest of which are constants. Indeed, in this thesis, this is what we will mean by a reducible OC-tensor. More generally we say a CT is *reducible* if it admits a non-degenerate eigenspace with constant eigenfunction. We will outline in Section 9.1.3 how we will break down the classification in terms of irreducible and reducible OCTs.

9.1.2 Properties of OCTs

We will now list some properties of OCTs that will be used later. The following proposition gives a necessary and sufficient condition to determine when two OCTs (one of which is not covariantly constant) share the same eigenspaces.

Proposition 9.1.2
Suppose M is a connected manifold and L is an OCT on M which is not covariantly constant (around any neighborhood). Then $\tilde{L}$ is a CT sharing the same eigenspaces as L iff there exists $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that

$$\tilde{L} = aL + bG \qquad \diamondsuit$$

Proof See Proposition 6.2.5. ∎

The above proposition no longer holds if we relax the assumption that L is not covariantly constant. One can easily see why by considering any non-trivial covariantly constant symmetric tensor in Euclidean space. We now define an important notion for classifying KEM webs.

Definition 9.1.3 (Geometric Equivalence of CTs)
We say two CTs L and $\tilde{L}$ are *geometrically equivalent* if there exists $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$ and $T \in I(M)$ such that

$$\tilde{L} = aT_*L + bG \qquad \square$$

An immediate corollary of the above proposition is the following:

Corollary 9.1.4 (Geometric Equivalence of OCTs)
Suppose M is a connected manifold. Suppose L and $\tilde{L}$ are OCTs with respective eigenspaces $\mathcal{E} = (E_1, \ldots, E_k)$ and $\tilde{\mathcal{E}} = (\tilde{E}_1, \ldots, \tilde{E}_k)$. Suppose further that $\mathcal{E}$ is not a Riemannian product net, equivalently one of the CTs is not covariantly constant. Then $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are related by $T \in I(M)$, i.e. $\tilde{E}_i = T_*E_{\sigma(i)}$ for each i (where σ is a permutation of $\{1, \ldots, k\}$) iff L and $\tilde{L}$ are geometrically equivalent. $\square$

The above corollary implies that the classification of isometrically inequivalent KEM webs can be reduced to the classification of geometrically inequivalent OCTs. For the proof of the following theorem (which was already presented in Section 6.1), see [TCS05; Cra07].

Theorem 9.1.5 (The Vector Space of Concircular tensors [TCS05])
If $n > 1$, then the C-tensors of valence $r \leq 2$ form a finite dimensional real vector space with maximal dimension equal to the dimension of the space of constant symmetric r-tensors in $\mathbb{R}^{n+1}$. Furthermore the maximal dimension is achieved if and only if the space has constant curvature. □

The above theorem implies the following:

Corollary 9.1.6 (Concircular tensors in spaces of constant curvature)
Suppose M^n is a space of constant curvature with $n > 1$ and let $r \leq 2$. Let $\beta = \{v_1, \dots, v_{n+1}\}$ be a basis for the space of concircular vectors, then a given C-tensor of valence r can be written uniquely as a linear combination of r-fold symmetric products of the vectors in β. □

9.1.3 Summary of Results

We first give an overview of the classification. The classification breaks down into three parts: obtaining canonical forms for C-tensors modulo the action of the isometry group (Sections 9.2 and 9.3), classifying the webs described by IC-tensors (Section 9.4) and obtaining warped product decompositions adapted to reducible OCTs (Section 9.5).

The webs formed by IC-tensors are the basic building blocks of all separable webs. Section 9.4 is devoted to obtaining information about these webs from the corresponding IC-tensors. In that section we obtain the transformation from the canonical coordinates (u^i) induced by these tensors to Cartesian coordinates (x^i) and we obtain the metric in canonical coordinates. This is done by first calculating the characteristic polynomial of all CTs in spaces of constant curvature in a Cartesian coordinate system. In examples, we will also show how to obtain the coordinate domains for coordinate systems induced by IC-tensors.

To obtain all orthogonal separable coordinates in spaces of constant curvature, we also have to consider reducible OCTs. Let L be a non-trivial reducible OCT and suppose $\psi : N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k \to M$ is a local warped product decomposition of M adapted to the eigenspaces of L such that $L_0 := L|_{N_0}$ is an ICT[3]. Let $(x_0) = (u^1, \dots, u^{n_0})$ be the canonical coordinates induced by L_0 on some open subset of N_0. For $i > 0$ suppose $(x_i) = (x_i^1, \dots, x_i^{n_i})$ are separable coordinates for N_i, then it was shown in Section 6.5

[3]If L has only constant eigenfunctions, we can choose N_0 to be a point.

that the coordinates $\psi(x_0, x_1, \ldots, x_k)$ are separable coordinates for M. To construct the separable coordinates (x_i) on N_i where $i > 0$, one would apply this procedure again on N_i equipped with the induced metric. It was shown in Chapter 7 that all orthogonal separable coordinates for spaces of constant curvature arise this way. Hence a remaining problem is to develop a method to construct warped product decompositions which decompose a given reducible OCT as above; this is done in Section 9.5. Together with the results of Section 9.4, this gives a recursive procedure to construct the orthogonal separable coordinates of these spaces.

Finally in Section 9.6 we will show how to apply the theory developed in this chapter to solve motivating problems.

The classification generally breaks down into one for pseudo-Euclidean space $\mathbb{E}^n_\nu$ then one for its spherical submanifolds $\mathbb{E}^n_\nu(\kappa)$ (which usually reduces to a similar problem in $\mathbb{E}^n_\nu$). We give more details in the following subsections.

pseudo-Euclidean space

First we define the *dilatational vector field*, r, to be the vector field given in Cartesian coordinates (x^i) by $r = \sum_i x^i \partial_i$. The general concircular contravariant tensor in $\mathbb{E}^n_\nu$ is given as follows (see Proposition 9.2.2):

$$L = A + 2w \odot r + mr \odot r \tag{9.1.1}$$

where $A \in C^2_0(\mathbb{E}^n_\nu)$, $w \in C^1_0(\mathbb{E}^n_\nu)$ and $m \in C^0_0(\mathbb{E}^n_\nu)$. For $k \geq 0$, define constants ω_k as follows:

$$\omega_k = \begin{cases} m & \text{if } k = 0 \\ \langle w, A^{k-1} w \rangle & \text{else} \end{cases} \tag{9.1.2}$$

The above constants aren't necessarily invariant under isometries. But invariants can be defined from them.

Definition 9.1.7

Suppose L is a CT in $\mathbb{E}^n_\nu$ as defined above. Then we define the *index* of L to be the first integer $k \geq 0$ for which $\omega_k \neq 0$; L is said to be *non-degenerate* if such an integer exists. Furthermore if L is non-degenerate, it has an associated sign (characteristic):

$$\varepsilon = \begin{cases} 1 & 1 \text{ if } k \text{ is even} \\ \operatorname{sgn} \omega_k & \text{if } k \text{ is odd} \end{cases}$$

□

The following theorem which is proven in Section 9.2 summarizes our results on the canonical forms of concircular tensors; it classifies C-tensors into five disjoint classes.

Theorem 9.1.8 (Canonical forms for CTs in $\mathbb{E}^n_\nu$)
Let $\tilde{L} = \tilde{A} + mr \otimes r^\flat + w \otimes r^\flat + r \otimes w^\flat$ be a CT in $\mathbb{E}^n_\nu$. Let k be the index and ε be the sign of $\tilde{L}$ if $\tilde{L}$ is non-degenerate. These quantities are geometric invariants of $\tilde{L}$. Furthermore, after a possible change of origin and after changing to a geometrically equivalent CT, $L = a\tilde{L}$ for some $a \in \mathbb{R} \setminus \{0\}$, $\tilde{L}$ admits precisely one of the following canonical forms.

Central: *If $k = 0$*

$$L = A + r \otimes r^\flat$$

non-null Axial: *If $k = 1$, i.e. $m = 0$, and $\langle w, w\rangle \neq 0$:*

There exists a vector $e_1 \in \operatorname{span}\{w\}$ such that L has the following form:

$$L = A + e_1 \otimes r^\flat + r \otimes e_1^\flat \qquad Ae_1 = 0, \quad \langle e_1, e_1\rangle = \varepsilon$$

null Axial: *If $k \geq 2$, hence $m = 0$ and $\langle w, w\rangle = 0$:*

There exists a skew-normal sequence $\beta = \{e_1, ..., e_k\}$ with $\langle e_1, e_k\rangle = \varepsilon$ where $e_1 \in \operatorname{span}\{w\}$ which is A-invariant such that L has the following form:

$$L = A + e_1 \otimes r^\flat + r \otimes e_1^\flat$$
$$A|_\beta = J_k(0)^T = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

Cartesian: *If k doesn't exist, $m = 0$ and $w = 0$*

$$L = \tilde{A}$$

degenerate null Axial: *If k doesn't exist and $w \neq 0$* □

Remark 9.1.9
The degenerate null axial concircular tensors will be of no concern to us. In Euclidean space they don't occur and it will be proven later (see Section 9.2.3) that in Minkowski space that they are never orthogonal concircular tensors. □

Remark 9.1.10
The precise classification for Euclidean and Minkowski space can be directly inferred from the above theorem by imposing the signature of the metric. The classification for Euclidean space is clear. In Minkowski space, $k \leq 3$ and when $k = 3$ the sign of the axial CT must be positive (see Lemma 8.1.1). □

Remark 9.1.11
When $k = 0$ and 1 respectively, the translation vector v for the isometry $T : r \to r + v$ which sends $\tilde{L}$ to canonical form is given as follows:

$$v = \frac{w}{\omega_0} \qquad \text{if } k = 0 \tag{9.1.3}$$

$$v = \frac{1}{\omega_1}(Aw - \frac{1}{2}\frac{\omega_2}{\omega_1}w) \qquad \text{if } k = 1 \tag{9.1.4}$$

For the general case, see Section 9.2.3. □

One can easily deduce that in Euclidean or Minkowski space, any covariantly non-constant OCT is non-degenerate. Hence we will only be interested in non-degenerate CTs throughout this chapter.

Some notation will be useful. The matrix A will be called the *parameter matrix* and the vector w the *axial vector* of the CT. When $k \geq 1$ in the above theorem, we will refer to the CT as an *axial concircular tensor*.

Suppose L is a non-degenerate CT in the canonical form given by Theorem 9.1.8. We denote by D the A-invariant subspace spanned by $w, Aw, \ldots$. This subspace is either zero (if $w = 0$) or metrically non-degenerate. We will let $A_c := A|_{D^\perp}$, $A_d := A|_D$ and the central CT in $D^\perp$ with parameter matrix A_c by L_c. Furthermore we define the following functions:

$$p(z) := \det(zI - L)$$
$$B(z) := \det(zI - A_c)$$

where the second determinant is evaluated in $D^\perp$.

The canonical forms for non-degenerate CTs can be enumerated by choosing a non-degenerate CT from Theorem 9.1.8 then choosing a metric-Jordan canonical form for the pair $(A|_{D^\perp}, g|_{D^\perp})$. The proofs of these canonical forms, which are given in Section 9.2, can be omitted on first reading. Once these canonical forms are obtained, in Sections 9.4.1 and 9.4.2 we will calculate the characteristic polynomial for non-degenerate CTs in $\mathbb{E}^n_\nu$. Using this, for ICTs we can calculate the transformation from their canonical coordinates to Cartesian coordinates and the metric in canonical coordinates. Then in Section 9.5.1 we will show how to obtain the warped product decompositions induced by reducible OCTs.

Spherical submanifolds of pseudo-Euclidean space

First the orthogonal projection R onto the spherical distribution $r^{\perp}$ is given as follows:

$$R = I - \frac{r \otimes r^{\flat}}{r^2} \qquad\qquad R^* = I - \frac{r^{\flat} \otimes r}{r^2}$$

Then the general CT in $\mathbb{E}^n_\nu(\kappa)$ is obtained by restricting $A \in C^2_0(\mathbb{E}^n_\nu)$ to $\mathbb{E}^n_\nu(\kappa)$. It is given as follows in $\mathbb{E}^n_\nu$ in contravariant form (see Proposition 9.3.2):

$$L = RAR^* = A + \kappa^2 \langle r, Ar\rangle\, r \odot r - 2\kappa(Ar \odot r) \qquad L^{ij} = R^i{}_l A^{lk} R^j{}_k \tag{9.1.5}$$

The matrix A is called the *parameter matrix* of the CT. We denote by L_c the central CT in $\mathbb{E}^n_\nu$ with parameter matrix A. Note that $L = RL_cR^*$. We will see later that several questions concerning L can be related to similar ones concerning L_c.

The canonical forms for these CTs can be enumerated by choosing a metric-Jordan canonical form for the pair (A, g). The proofs of these canonical forms, which are given in Section 9.3, can be omitted on first reading. Once these canonical forms are obtained, in Section 9.4.3 we will calculate the characteristic polynomial for CTs in $\mathbb{E}^n_\nu(\kappa)$ by making use of the solution to the similar problem in $\mathbb{E}^n_\nu$. Using this, for ICTs we can calculate the transformation from their canonical coordinates to Cartesian coordinates and the metric in canonical coordinates. Then in Section 9.5.2 we will show how to obtain the warped product decompositions induced by reducible OCTs by making use of the solution to the similar problem in $\mathbb{E}^n_\nu$.

9.2 Canonical forms for Concircular tensors in pseudo-Euclidean space

9.2.1 Standard Model of pseudo-Euclidean space

In this section we recall the CVs and CTs in $\mathbb{E}^n_\nu$ in its standard vector space model, which were calculated in Section 6.4. These results are well known [Cra07; Ben05], but we include it here for completeness.

First we define the *dilatational vector field*, r, to be the vector field satisfying for any $p \in \mathbb{E}^n_\nu$, $r_p = p \in T_p\mathbb{E}^n_\nu$. In Cartesian coordinates (x^i), we have

$$r = \sum_i x^i \partial_i$$

The general CV in $\mathbb{E}^n_\nu$ is given by the following proposition.

Proposition 9.2.1 (Concircular vectors in $\mathbb{E}^n_\nu$ [Cra07])
A vector $v \in \mathfrak{X}(\mathbb{E}^n_\nu)$ is a CV in $\mathbb{E}^n_\nu$ iff there exists $a \in C^0_0(\mathbb{E}^n_\nu)$ and $b \in C^1_0(\mathbb{E}^n_\nu)$ such that

$$v = ar + b$$

where r is the dilatational vector field. □

PROOF See Proposition 6.4.3. ■

Then using Corollary 9.1.6 we can deduce the general CT in $\mathbb{E}^n_\nu$:

Proposition 9.2.2 (Concircular tensors in $\mathbb{E}^n_\nu$)
L is a concircular 2-tensor in $\mathbb{E}^n_\nu$ iff there exists $A \in C^2_0(\mathbb{E}^n_\nu)$, $w \in C^1_0(\mathbb{E}^n_\nu)$ and $m \in C^0_0(\mathbb{E}^n_\nu)$ such that:

$$L = A + 2w \odot r + mr \odot r$$

where r is the dilatational vector field. The tensors A, w and m are uniquely determined by L. □

9.2.2 Parabolic Model of pseudo-Euclidean space

In order to obtain canonical forms for CTs it will be useful to work with a different model of $\mathbb{E}^n_\nu$. We will refer to it as the parabolic model of $\mathbb{E}^n_\nu$, to be introduced shortly. The main reason for working with this model is because it is a spherical submanifold of the ambient space in which the isometries of $\mathbb{E}^n_\nu$ are linearized, which we will elaborate on shortly.

Recall that $\mathbb{P}^n_\nu$ was defined in Section 8.3. We stated that an explicit isometry with $\mathbb{E}^n_\nu$ can be obtained by fixing $b \in \mathbb{P}^n_\nu$, i.e. b is lightlike and $\langle a, b\rangle = 1$. If we let $V := \operatorname{span}\{a, b\}^\perp$, note that $V \cong \mathbb{E}^n_\nu$, then for $x \in V$:

$$\psi(x) = b + x - \frac{1}{2}x^2 a \in \mathbb{P}^n_\nu \tag{9.2.1}$$

gives an explicit isometry between $\mathbb{E}^n_\nu$ and $\mathbb{P}^n_\nu$. By definition of $\mathbb{P}^n_\nu$, it follows that $T_p\mathbb{P}^n_\nu = p^\perp \cap a^\perp = \operatorname{span}\{p, a\}^\perp$. Also note that for $x \in \mathbb{P}^n_\nu$

$$\psi^{-1}(x) = x - \langle x, b\rangle\, a - \langle x, a\rangle\, b$$

An important reason for working with $\mathbb{P}^n_\nu$ is the following [Nol96]:

Proposition 9.2.3 (Isometry group of $\mathbb{P}^n_\nu$)
The isometry group of $\mathbb{P}^n_\nu$ is:

$$I(\mathbb{P}^n_\nu) = \{T \in O_{\nu+1}(n+2) \mid Ta = a\}$$

Furthermore suppose we fix an isometry with $\mathbb{E}^n_\nu$ via Eq. (9.2.1) by fixing a subspace $V \subset a^\perp$ such that $V \simeq \mathbb{E}^n_\nu$, then for $p \in V$ and $\tilde{p} \in V^\perp$ we have the following Lie group isomorphism:

$$\phi : \begin{cases} O^n_\nu(V) \ltimes V & \to I(\mathbb{P}^n_\nu) \\ (B, v) & \mapsto \phi(B, v) \end{cases}$$

where

$$\phi(B, v)(p + \tilde{p}) = \tilde{p} + Bp + \langle a, \tilde{p}\rangle v - (\langle Bp, v\rangle + \frac{1}{2}\langle a, \tilde{p}\rangle v^2))a \tag{9.2.2}$$

□

Proof See Proposition D.5.2 or [Nol96, lemma 6] which covers the case when $\mathbb{E}^n_\nu$ is Euclidean. ■

Remark 9.2.4
If $\psi : \mathbb{E}^n_\nu \to \mathbb{P}^n_\nu$ is the standard embedding from Eq. (9.2.1), then ψ is equivariant. In other words, if we let $Tp := Bp + v$ for $(B, v) \in O^n_\nu(V) \ltimes V$ as above, and $\hat{T} := \phi(B, v)$ then $\psi \circ T(p) = \hat{T} \circ \psi(p)$. □

We also have the following:

Lemma 9.2.5
For $\bar{p} \in V$ and $X \in T_{\bar{p}}V$

$$\psi_* X = X - \langle X, \bar{p}\rangle a$$

For $Y \in T_{\psi(\bar{p})}\mathbb{P}^n_\nu$, the inverse of the above map is given by:

$$P_b : \begin{cases} T_{\psi(\bar{p})}\mathbb{P}^n_\nu & \to T_{\bar{p}}V \\ Y & \mapsto Y - \langle Y, b\rangle a \end{cases}$$ □

Proof The first statement is clear. First observe that $P_b\psi_* X = X$. Now,

$$\psi_* P_b Y = Y - \langle Y, b\rangle a - \langle Y, \bar{p}\rangle a$$

Now $0 = \langle Y, \psi(\bar{p})\rangle = \langle Y, b\rangle + \langle Y, \bar{p}\rangle$. Thus $\psi_* P_b Y = Y$. ■

Furthermore we denote by P_1 the orthogonal projector onto $T\mathbb{P}^n_\nu$. It is given as follows for $r \in \mathbb{E}^{n+2}_{\nu+1}$

$$P_1 : \begin{cases} T_r\mathbb{E}^{n+2}_{\nu+1} & \to T_r\mathbb{E}^{n+2}_{\nu+1} \\ V & \mapsto V - \langle V, r\rangle a - \langle V, a\rangle r \end{cases}$$

We will now calculate the CT in $\mathbb{E}^{n+2}_{\nu+1}$ which restricts to the most general CT in $\mathbb{P}^n_\nu$. Due to Corollary 9.1.6 we only need to examine how CVs restrict. By Proposition 9.2.1 and Theorem 9.1.5, the general CV in $\mathbb{E}^{n+2}_{\nu+1}$ can be written

$$v = c_0 r + \sum_{i=1}^{n} c_i a_i + c_{n+1} b + c_{n+2} a$$

where each $c_i \in \mathbb{R}$, $a_1, \dots, a_n$ is a basis for V and r is the dilatational vector field in $\mathbb{E}^{n+2}_{\nu+1}$. Then

$$\begin{aligned} P_b P_1 v &= P_b(\sum_{i=1}^{n} c_i(a_i - \langle a_i, r\rangle a) + c_{n+1}(b - \langle b, r\rangle a - r)) \\ &= \sum_{i=1}^{n} c_i a_i - c_{n+1} x \end{aligned}$$

where x is the dilatational vector field in V. Then using Corollary 9.1.6 we have proven the following:

Proposition 9.2.6

Suppose $\mathbb{P}^n_\nu$ is identified with $\mathbb{E}^n_\nu$ by the embedding in Eq. (9.2.1). Denote by $V =$ $\operatorname{span}\{a, b\}^\perp$, let $\tilde{A} \in C^2_0(V)$, $w \in C^1_0(V)$, and $m \in C^0_0(V)$. Define

$$A = \tilde{A} + mb \odot b - 2w \odot b \tag{9.2.3}$$

Then the restriction of A to V, denoted L, via the embedding in Eq. (9.2.1) is:

$$L = \tilde{A} + mr \odot r + 2w \odot r$$

□

Note that A is completely determined by the condition $Ab = 0$. Now for $A \in C^2_0(\mathbb{E}^{n+2}_{\nu+1})$, define A_b by

$$(A_b)^{ij} := (P_b)^i{}_l A^{lk} (P_b)^j{}_k$$

Note that b is an eigenvector of A_b with eigenvalue 0. Also observe that

$$P_1 P_b = P_1 - w \otimes b^\flat + w \otimes b^\flat = P_1$$

The above equation shows that A and A_b induce the same CT on $\mathbb{P}^n_\nu$. From the calculations proceeding Eq. (9.2.3) we see that

$$\{a_1 - \langle a_1, r\rangle a, \dots, a_n - \langle a_n, r\rangle a, b - \langle b, r\rangle a - r\}$$

is basis for the space of CVs on $\mathbb{P}^n_\nu$. Thus it follows from Corollary 9.1.6 and the proceeding calculations that $A, B \in C^2_0(\mathbb{E}^{n+2}_{\nu+1})$ induce the same CT on $\mathbb{P}^n_\nu$ iff for some $b \in \mathbb{P}^n_\nu$ we have

$$A_b = B_b$$

Furthermore, one should note that if $b, c \in \mathbb{P}^n_\nu$, then $(A_c)_b = A_b$. Hence it follows that if $A_b = B_b$ for some $b \in \mathbb{P}^n_\nu$ then $A_c = B_c$ for all $c \in \mathbb{P}^n_\nu$.

9.2.3 Existence of Canonical forms

In this section $A \in C^2_0(\mathbb{E}^{n+2}_{\nu+1})$. We are interested in finding canonical forms for the CT on $\mathbb{P}^n_\nu$ induced by this tensor. As it was shown in the previous section, the induced CT depends only on A_b for some $b \in \mathbb{P}^n_\nu$. Hence our goal will be to find $\tilde{b} \in \mathbb{P}^n_\nu$ such that $A_{\tilde{b}}$ is in a canonical form. Since the isometry with $\mathbb{E}^n_\nu$ (see Eq. (9.2.1)) is fixed by a vector $b \in \mathbb{P}^n_\nu$, we will then choose $T \in I(\mathbb{P}^n_\nu)$ such that $T\tilde{b} = b$. This will transform $A_{\tilde{b}}$ to $(T_* A)_b$ which can be restricted to $\mathbb{E}^n_\nu$ using Proposition 9.2.6 to obtain a canonical form for the original CT in $\mathbb{E}^n_\nu$.

To obtain the canonical choice of $b \in \mathbb{P}^n_\nu$, first note that A_b is completely determined by the fact that $A_b b = 0$. Secondly, note that since isometries of $\mathbb{P}^n_\nu$ fix a, it follows that for each $l \geq 0$, $\langle a, A^l a\rangle$ are invariants of A. Although these are in general not invariants of the CT induced by A, they will play a significant role in the classification. Thirdly, since a cannot be transformed by isometries, we will attempt to choose $b \in \mathbb{P}^n_\nu$ such that a is a basis vector in a metric-Jordan canonical basis for A_b. Since $\langle a, b\rangle = 1$, one can deduce that (using the metric-Jordan canonical form discussed in Section 8.2) in the simplest cases, a, b lie in the same eigenspace of A_b or a generates a Jordan cycle ending in a constant multiple of b. These observations motivate our search for b.

For the following calculations, $b \in \mathbb{P}^n_\nu$ is arbitrary and we let $\tilde{A} := A_b$. The following lemma will get us started:

Lemma 9.2.7
Suppose there is $k \in \mathbb{N}$ such that $\langle a, A^l a\rangle = 0$ for $0 \leq l < k$. Then for each $0 \leq l \leq k$

$$\tilde{A}^l a = A^l a - \sum_{j=0}^{l-1} \left\langle b, A^{l-j} a \right\rangle \tilde{A}^j a \tag{9.2.4}$$

Furthermore, if $0 \leq l \leq k$ *then*

$$\left\langle a, \tilde{A}^l a \right\rangle = \left\langle a, A^l a \right\rangle \tag{9.2.5}$$

So the constants $\left\langle a, A^l a \right\rangle$ *are invariants of the CT on* $\mathbb{P}^n_\nu$ *induced by* A. □

Proof We prove Eq. (9.2.4) by induction. It clearly holds for $l = 0, 1$. Now assume it holds for $l - 1$, then

$$\begin{aligned}
\tilde{A}^l a &= \tilde{A} A^{l-1} a - \sum_{j=0}^{l-2} \left\langle b, A^{l-1-j} a \right\rangle \tilde{A}^{j+1} a \\
&= A^l a - a \left\langle b, A^l a \right\rangle - \sum_{j=0}^{l-2} \left\langle b, A^{l-1-j} a \right\rangle \tilde{A}^{j+1} a \\
&= A^l a - a \left\langle b, A^l a \right\rangle - \sum_{j=1}^{l-1} \left\langle b, A^{l-j} a \right\rangle \tilde{A}^{j} a \\
&= A^l a - \sum_{j=0}^{l-1} \left\langle b, A^{l-j} a \right\rangle \tilde{A}^{j} a
\end{aligned}$$

Hence the first equation follows by induction.

Suppose $0 \leq l < k$, then

$$\left\langle a, \tilde{A}^l a \right\rangle = - \sum_{j=0}^{l-1} \left\langle b, A^{l-j} a \right\rangle \left\langle a, \tilde{A}^j a \right\rangle$$

Thus it follows by induction that $\left\langle a, \tilde{A}^l a \right\rangle = 0$. Thus $\left\langle a, \tilde{A}^k a \right\rangle = \left\langle a, A^k a \right\rangle$. ■

Now, define ω_i by

$$\omega_i := \left\langle a, A^{i+1} a \right\rangle$$

We will also need the following lemma to calculate ω_i in $\mathbb{E}^n_\nu$.

Lemma 9.2.8

Suppose A *has the form given by Eq.* (9.2.3)*, then*

$$A^l a = \begin{cases} mb - w & l = 1 \\ \left\langle w, \tilde{A}^{l-2} w \right\rangle b - \tilde{A}^{l-1} w & l > 1 \end{cases} \tag{9.2.6}$$

and ω_i is given by Eq. (9.1.2). $\square$

Using the above lemma we can also apply the definitions of index, sign and degeneracy of CTs in $\mathbb{E}^n_\nu$ from Definition 9.1.7 to CTs in $\mathbb{P}^n_\nu$.

Non-degenerate cases

Now we consider the case where there exists a least $k \in \mathbb{N}$ such that $\left\langle a, A^k a \right\rangle \neq 0$. This will be the most important case for our interests. Motivated by special cases and the metric-Jordan canonical form of $\tilde{A}$ discussed earlier, we will try to find b such that $a, \tilde{A}a, \ldots, \tilde{A}^k a$ forms a skew-normal sequence with $\left\langle a, A^k a \right\rangle b = \tilde{A}^k a$. The following lemma describes b provided it exists:

Lemma 9.2.9
Suppose there is $k \in \mathbb{N}$ such that $\left\langle a, A^l a \right\rangle = 0$ for $0 \leq l < k$ and $\left\langle a, A^k a \right\rangle \neq 0$. Assume there exists a b such that $\left\langle a, A^k a \right\rangle b = \tilde{A}^k a$ and $\left\langle \tilde{A}^j a, \tilde{A}^k a \right\rangle = 0$ for all $1 \leq j \leq k$. Then b must satisfy the following equations for each $l \in \{0, \ldots, k\}$

$$2\left\langle b, A^l a \right\rangle = \frac{\left\langle A^l a, A^k a \right\rangle}{\left\langle a, A^k a \right\rangle} - \sum_{j=1}^{l-1} \left\langle b, A^{l-j} a \right\rangle \left\langle b, A^j a \right\rangle \tag{9.2.7}$$

$\square$

Proof Suppose $0 < l \leq k$. Expanding $\tilde{A}^k a$ using Eq. (9.2.4), we have

$$\begin{aligned}\left\langle \tilde{A}^l a, \tilde{A}^k a \right\rangle &= \left\langle \tilde{A}^l a, A^k a \right\rangle - \left\langle b, A^l a \right\rangle \left\langle \tilde{A}^l a, \tilde{A}^{k-l} a \right\rangle \\ &\overset{(9.2.5)}{=} \left\langle \tilde{A}^l a, A^k a \right\rangle - \left\langle b, A^l a \right\rangle \left\langle a, A^k a \right\rangle\end{aligned}$$

By imposing the condition $\left\langle \tilde{A}^l a, \tilde{A}^k a \right\rangle = 0$, the above equation implies that:

$$\left\langle \tilde{A}^l a, A^k a \right\rangle - \left\langle b, A^l a \right\rangle \left\langle a, A^k a \right\rangle = 0 \tag{9.2.8}$$

Now expanding $\tilde{A}^l a$ using Eq. (9.2.4), the above equation becomes

$$\begin{aligned}
\left\langle \tilde{A}^l a, A^k a \right\rangle &= \left\langle A^l a - \sum_{j=0}^{l-1} \left\langle b, A^{l-j} a \right\rangle \tilde{A}^j a, A^k a \right\rangle \\
&= \left\langle A^l a, A^k a \right\rangle - \sum_{j=0}^{l-1} \left\langle b, A^{l-j} a \right\rangle \left\langle \tilde{A}^j a, A^k a \right\rangle \\
&= \left\langle A^l a, A^k a \right\rangle - \left\langle b, A^l a \right\rangle \left\langle a, A^k a \right\rangle - \sum_{j=1}^{l-1} \left\langle b, A^{l-j} a \right\rangle \left\langle \tilde{A}^j a, A^k a \right\rangle \\
&\overset{(9.2.8)}{=} \left\langle A^l a, A^k a \right\rangle - \left\langle b, A^l a \right\rangle \left\langle a, A^k a \right\rangle - \sum_{j=1}^{l-1} \left\langle b, A^{l-j} a \right\rangle \left\langle b, A^j a \right\rangle \left\langle a, A^k a \right\rangle
\end{aligned}$$

Equating the above equation with Eq. (9.2.8) and solving for $\left\langle b, A^l a \right\rangle$ proves the result. ■

Now we will use the above lemma and Eq. (9.2.4) to construct a vector b such that $\tilde{A}$ is in canonical form. First define a sequence $b_1, \ldots, b_k$ of scalars recursively as follows:

$$2b_l := \frac{\left\langle A^l a, A^k a \right\rangle}{\left\langle a, A^k a \right\rangle} - \sum_{j=1}^{l-1} b_{l-j} b_j$$

Then define vectors $s_0, s_1, \ldots, s_k$ as follows:

$$s_l := A^l a - \sum_{j=0}^{l-1} b_{l-j} s_j$$

Then define b by $b \left\langle a, A^k a \right\rangle := s_k$. The following lemma shows that this choice does work:

Proposition 9.2.10

The vectors $s_0, s_1, \ldots, s_k$ form a skew-normal sequence with $\langle s_0, s_k \rangle = \left\langle a, A^k a \right\rangle$. If $\tilde{A}^l a$ are defined as in Eq. (9.2.4) with the above vector b then $\tilde{A}^l a = s_l$. □

Proof The fact that $s_0, s_1, \ldots, s_k$ form a skew-normal sequence follows verbatim from Lemma 9.2.7 and the proceeding arguments by replacing $s_l \to \tilde{A}^l a$ and $b_l \to \left\langle b, A^l a \right\rangle$.

Suppose that $s_0, s_1, \ldots, s_k$ form a skew-normal sequence where $\langle s_0, s_k \rangle = \left\langle a, A^k a \right\rangle$. By definition of s_l, it follows that each $A^l a$ can be expanded in this basis as:

$$A^l a = s_l + \sum_{j=0}^{l-1} b_{l-j} s_j$$

Thus

$$\left\langle A^k a, a \right\rangle \left\langle b, A^l a \right\rangle = \left\langle s_k, A^l a \right\rangle = b_l \left\langle A^k a, a \right\rangle$$

Hence $b_l = \left\langle b, A^l a \right\rangle$. Then it follows by definition of s_l and $\tilde{A}^l a$ in Eq. (9.2.4) that $\tilde{A}^l a = s_l$. ∎

Now suppose A is in the canonical form stated above. Let $V = \text{span}\{a, b\}^\perp$ where b was chosen as above. Then $H = \text{span}\{a, Aa, \ldots, A^k a\}$ is a non-degenerate A-invariant subspace (see Lemma 8.2.4). Hence $H^\perp$ is a non-degenerate A-invariant subspace complementary to H. We now mention more precisely what we mean by "the" canonical form:

Definition 9.2.11
Suppose L is a CT in $\mathbb{P}^n_\nu$ with parameter matrix A as above and index $k' := k - 1 \geq 0$, i.e. L is *non-degenerate*. The *iso-canonical form* for L is the metric-Jordan canonical form for $(A|_{H^\perp}, g|_{H^\perp})$ together with the index k' and constant $\left\langle a, A^{k'+1} a \right\rangle \in \mathbb{R} \setminus \{0\}$. □

We will prove later on that this canonical form is uniquely determined by L. But for now we will examine this canonical form further. Let $\tilde{A} := A|_{H^\perp}$, then we can write:

$$A = \tilde{A} + \omega_0 b \odot b - 2w \odot b$$

where $w = \omega_0 b - Aa$.

If $\omega_0 \neq 0$ then it follows that $w = 0$ and it follows by Proposition 9.2.6 that the induced CT on V is

$$\tilde{A} + \omega_0 r \odot r$$

Thus after dividing by ω_0 we get the central CT from Theorem 9.1.8. If $\omega_0 = 0$, one can check that $w, \tilde{A}w, \ldots, \tilde{A}^{k-2} w \in V$ form a skew-normal sequence with $\left\langle w, \tilde{A}^{k-2} w \right\rangle = \omega_{k-1}$. It follows by Proposition 9.2.6 that the induced CT on V is

$$\tilde{A} + 2w \odot r$$

This CT is a constant multiple of a (null) axial CT with the same index and sign from Theorem 9.1.8 (after an appropriate choice of basis).

Transformation to Canonical form: We now denote by $\tilde{b}$ the vector b obtained above which puts A into a canonical form. The vector $b \in \mathbb{P}^n_\nu$ is fixed by an isometry with $\mathbb{E}^n_\nu$ (see Eq. (9.2.1)), furthermore we let $V = \text{span}\{a, b\}^\perp$. We can assume A has the form given by Eq. (9.2.3). The last problem is to choose $T \in I(\mathbb{P}^n_\nu)$ such that $T\tilde{b} = b$. We can obtain a unique transformation if we require T to induce a translation in V. Indeed, by Eq. (9.2.2) the most general transformation of this type is

$$T = I - a \otimes (\frac{1}{2}v^2 a^\flat + v^\flat) + v \otimes a^\flat$$

where $v \in V$ is arbitrary. The unique transformation with the above form satisfying $T\tilde{b} = b$ is obtained by taking

$$v = b - \tilde{b} + a\left\langle \tilde{b}, b\right\rangle$$

We now proceed to calculate v. First we can write

$$\tilde{b} = \frac{1}{\omega_{k-1}} \sum_{i=0}^{k} c_i A^i a$$

Since $\left\langle b, A^l a\right\rangle = 0$ for any $l > 0$, we see that

$$\tilde{b} - a\left\langle \tilde{b}, b\right\rangle = \frac{1}{\omega_{k-1}} \sum_{i=1}^{k} c_i A^i a$$

Since for $0 < l < k$, $\left\langle a, A^l a\right\rangle = 0$ it follows by Eq. (9.2.6) that $A^l a = -\tilde{A}^{l-1} w$. Thus

$$\begin{aligned} v &= -\frac{1}{\omega_{k-1}} \sum_{i=1}^{k} c_i A^i a + b \\ &= \frac{1}{\omega_{k-1}} \sum_{i=1}^{k} c_i \tilde{A}^{i-1} w \end{aligned}$$

where the last equation follows from the fact that $c_k = 1$. We have calculated the first four coefficients (which are sufficient for Euclidean and Minkowski space):

$$\begin{aligned} c_k &= 1 \\ c_{k-1} &= -\frac{1}{2}\frac{\omega_k}{\omega_{k-1}} \\ c_{k-2} &= \frac{1}{16}\frac{(-8\omega_{k-1}\omega_{k+1} + 6\omega_k^2)}{\omega_{k-1}^2} \\ c_{k-3} &= \frac{1}{16}\frac{(-8\omega_{k-1}^2\omega_{k+2} + 12\omega_{k-1}\omega_k\omega_{k+1} - 5\omega_k^3)}{\omega_{k-1}^3} \end{aligned}$$

In particular when $k = 1$ and 2 respectively we have the following:

$$
\begin{aligned}
v &= \frac{w}{\omega_0} \\
v &= \frac{1}{\omega_1}(\tilde{A}w - \frac{1}{2}\frac{\omega_2}{\omega_1}w)
\end{aligned}
$$

Finally, we note that by equivariance of the map ψ (see remark after Proposition 9.2.3), one only needs to apply the isometry $T : V \to V$ given by $r \mapsto r + v$ to send the induced CT in V into canonical form. Hence in practice one does not need to work in $\mathbb{P}^n_\nu$.

Degenerate cases

We now consider the case where $\langle a, A^l a \rangle = 0$ for every $l \in \mathbb{N}$. First note that the dimension of the subspace spanned by $a, Aa, \ldots$ must be at most $n-1$ by non-degeneracy of the scalar product. So there exists a least $l \leq n-1$ such that $\{a, Aa, \ldots, A^l a\} \subseteq a^\perp$ is a linearly independent set but $A^{l+1}a \in \operatorname{span}\{a, Aa, \ldots, A^l a\}$. Thus it follows that $A^m a \in \operatorname{span}\{a, Aa, \ldots, A^l a\}$ for all $m > l$. Also note by Lemma 9.2.7 it follows that these properties are invariant under the transformation $A \to A_b$.

Case 1 $l = 0$

In this case a is an eigenvector of A. After transforming A to A_b (if necessary), we can assume that $Aa = 0$. Also $Ab = 0$, then since $\langle a, b \rangle = 1$ it follows that $\operatorname{span}\{a, b\}$ is a non-degenerate A-invariant subspace. Hence after identifying $\mathbb{E}^n_\nu \simeq \operatorname{span}\{a, b\}^\perp$, it follows by Proposition 9.2.6 that A restricts to a Cartesian CT on $\mathbb{E}^n_\nu$.

Case 2 $l \geq 1$

Fix $b \in \mathbb{P}^n_\nu$, let $V = \operatorname{span}\{a, b\}^\perp$ and assume $Ab = 0$. Then we can write:

$$
A = \tilde{A} + 2w \odot b
$$

Now note that for any $j \in \mathbb{N}$, $\langle a, A^j a \rangle = 0$. Suppose inductively that for all $1 \leq j \leq i$ that $A^j a \in V$ then

$$
AA^i a = \tilde{A}A^i a \in V
$$

since $\langle A^i a, w \rangle = \langle A^i a, Aa \rangle = 0$ and $\langle A^i a, b \rangle = 0$. Hence by induction for any $j \in \mathbb{N}$, $\langle b, A^j a \rangle = 0$. Thus $Aa, \ldots, A^l a, A^{l+1}a \in V$.

In particular, when $l = 1$ we see that w is a lightlike eigenvector of $\tilde{A}$. Then by Proposition 9.2.6, A induces the following CT in $\mathbb{E}^n_\nu$

$$L = \tilde{A} - 2w \odot r$$

Observe that w is a lightlike eigenvector of L with non-constant eigenfunction. Thus L is never an OC-tensor because lightlike eigenvectors of OC-tensors must have constant eigenfunctions.

If $l > 1$, we see that $Aa, A^2a \in V$ are linearly independent orthogonal lightlike vectors. Thus this case can't occur in Euclidean or Minkowski case, so we ignore it.

9.2.4 Uniqueness of Canonical Forms

In this section we will show that the canonical forms obtained in the previous section are uniquely determined by a given CT in $\mathbb{P}^n_\nu$. As a consequence of this we will show that the different canonical forms divide the CTs into isometrically inequivalent classes. We will be working with the case when the CT is non-degenerate as the other cases are either straightforward or uninteresting.

Suppose L and M are CTs in $\mathbb{P}^n_\nu$ with parameter matrices A and B respectively. We observed at the end of Section 9.2.2 that $L = M$ iff for one (hence all) $b \in \mathbb{P}^n_\nu$:

$$A_b = B_b$$

Thus it follows that $L = T_* M$ for some $T \in I(\mathbb{P}^n_\nu)$ iff for one (hence all) $b \in \mathbb{P}^n_\nu$:

$$A_b = (T_* B)_b$$

Lemma 9.2.12

Suppose A_2 is a parameter matrix, and $A_1 = (A_2)_b$ for some $b \in \mathbb{P}^n_\nu$. Assume each A_i have the same index and admit a vector b_i which transforms it to canonical form according to Proposition 9.2.10. Then $b_1 = b_2$. □

PROOF Let $A_0 = (A_2)_{b_2}$, then $A_1 = (A_0)_b$. Since A_0 is in canonical form, $a, A_0 a, \cdots, A_0^k a$ forms an adapted cycle of generalized eigenvectors for A_0 with eigenvalue 0. In this case $\left\langle a, A_0^k a \right\rangle \in \mathbb{R} \setminus \{0\}$.

Let b_1 be the vector admitted by A_1 and let $A_3 := (A_1)_{b_1} = (A_0)_{b_1}$. Now by Proposition 9.2.10 and Lemma 9.2.7, b_1 satisfies:

$$\left\langle a, A_1^k a \right\rangle b_1 = A_3^k a = A_0^k a - \sum_{j=0}^{k-1} \left\langle b_1, A_0^{k-j} a \right\rangle A_3^j a \tag{9.2.9}$$

Since A_3 is in canonical form, it follows for each $l \in \{1, \cdots, k\}$, $\left\langle b_1, A_0^l a \right\rangle$ satisfies Eq. (9.2.7). Then since A_0 is in canonical form, we have $\left\langle b_1, A_0^l a \right\rangle = 0$ for $l \in \{1, \cdots, k\}$. Thus Eq. (9.2.9) shows that

$$\langle a, A_1^k a \rangle b_1 = A_0^k a = \langle a, A_1^k a \rangle b_2$$

Hence $b_1 = b_2$. ■

In the following theorem we will show that the iso-canonical form defined in Definition 9.2.11 for non-degenerate CTs is uniquely determined by the CT.

Theorem 9.2.13 (Isometric Equivalence of CTs in $\mathbb{E}^n_\nu$)
Suppose L and M are CTs in $\mathbb{P}^n_\nu$ such that M has an index $k \geq 0$. Then $L = T_* M$ for some $T \in I(\mathbb{P}^n_\nu)$ iff L and M have the same iso-canonical form. □

PROOF Assume that $L = T_* M$ for some $T \in I(\mathbb{P}^n_\nu)$. Then for some $b \in \mathbb{P}^n_\nu$:

$$A_b = (T_* B)_b$$

By the above equation and Lemma 9.2.7 it follows that the index of L is also k. Let b_2 be the vector which puts B in canonical form given by Proposition 9.2.10. Then Tb_2 sends $T_* B$ to canonical form. By Lemma 9.2.12, Tb_2 is the vector obtained from Proposition 9.2.10 which puts A in canonical form. Let $\tilde{b} := Tb_2$ then

$$A_{\tilde{b}} = (T_* B)_{\tilde{b}} = T_*(B_{b_2})$$

Hence B_{b_2} is isometric to $A_{\tilde{b}}$. Then it follows from the uniqueness of the metric-Jordan canonical form (see Theorem 8.2.5) that $A_{\tilde{b}}$ and B_{b_2} have the same iso-canonical form.

Conversely suppose L and M have the same iso-canonical form. Then A (resp. B) each admit a vector $b_1 \in \mathbb{P}^n_\nu$ (resp. $b_2 \in \mathbb{P}^n_\nu$) such that A_{b_1} and B_{b_2} have the same iso-canonical form. Then one can easily construct $T \in I(\mathbb{P}^n_\nu)$ which transforms a metric-Jordan canonical basis of B_{b_2} into A_{b_1}, so that $A_{b_1} = T_*(B_{b_2})$. Thus

$$\begin{aligned} &\Rightarrow T(B_{b_2})^k a = (A_{b_1})^k a \\ &\Rightarrow Tb_2 = b_1 \end{aligned}$$

Note that in the last equation we have used the fact that $\langle a, B^k a \rangle = \langle a, A^k a \rangle$. Then

$$A_{b_1} = T_*(B_{b_2}) = (T_* B)_{b_1}$$

Thus $L = T_* M$, which proves the converse. ■

Geo-Canonical forms We now give a geo-canonical form for non-degenerate CTs in $\mathbb{P}^n_\nu$. Suppose L is such a CT with index k and parameter matrix A in iso-canonical form. Then for $c \in \mathbb{R}$, cL has parameter matrix cA and

$$\langle a, (cA\rangle^{k+1} a) = c^{k+1} \langle a, A^{k+1} a \rangle$$

Hence after an appropriate transformation $L \to cL$, we can assume

$$\langle a, A^{k+1}a \rangle = \begin{cases} 1 & 1 \text{ if } k \text{ is even} \\ \pm 1 & \text{if } k \text{ is odd} \end{cases}$$

Note that when k is odd, c is only determined up to sign. Hence there are two possible geo-canonical forms in this case. Now, if L is an axial CT, we can fix $d \in \mathbb{R}$ by requiring that $(A + dI)^k a \in \operatorname{span}\{a, b\}$. This condition is satisfied in the iso-canonical form. If L is central, we choose d such that the real part of the smallest eigenvalue (see Definition E.0.9) of $A|_{H^\perp}$ is zero.

9.3 Canonical forms for Concircular tensors in Spherical submanifolds of pseudo-Euclidean space

9.3.1 Obtaining concircular tensors in umbilical submanifolds by restriction

Let $\tilde{M}$ be a pseudo-Riemannian submanifold of M with Levi-Civita connections $\tilde{\nabla}$ and ∇ respectively. We say $\tilde{M}$ is an umbilical submanifold (see Section 3.1 for more details) if there exists $H \in \mathfrak{X}(\tilde{M})^\perp$ (i.e. H is orthogonal to $T\tilde{M}$) called the *mean curvature normal* of $\tilde{M}$ such that

$$\nabla_x y = \tilde{\nabla}_x y + \langle x, y \rangle H$$

for all $x, y \in \mathfrak{X}(\tilde{M})$. By generalizing an observation made in [Cra03] one can deduce the following:

Proposition 9.3.1 (Restriction of CTs to umbilical submanifolds [Cra03]) *Suppose $\tilde{M}$ is an umbilical submanifold of M with mean curvature normal H and L is a concircular r-tensor on M with conformal factor C in covariant form. Then the pullback of L to $\tilde{M}$ is a concircular r-tensor with conformal factor equal to the pullback of $C + rL(H)$, where in components, $L(H)_{i_1,\dots,i_{r-1}} = L_{i_1,\dots,i_{r-1}j} H^j$.* □

Since spherical submanifolds are umbilical submanifolds and $\mathbb{E}^n_\nu(\kappa)$ is a spherical submanifold (see Section 3.1), the above proposition allows us to obtain CTs on $\mathbb{E}^n_\nu(\kappa)$. We will do this in the following section.

9.3.2 Concircular tensors in Spherical submanifolds of pseudo-Euclidean space

In this section we study the CTs in $\mathbb{E}^n_\nu(\kappa)$ via the canonical embedding in $\mathbb{E}^n_\nu$. Let r denote the dilatational vector field, we work on the subset of $\mathbb{E}^n_\nu$ for which $r^2 \neq 0$. Let $E := r^\perp$ and let L be a CT on M. To obtain the CT on $\mathbb{E}^n_\nu(\frac{1}{r^2})$ (which is an integral manifold of E), we first let $R := I - \dfrac{r^\flat \otimes r}{r^2}$ where I is the identity endomorphism then $L_E := L|_E$ is given as follows:

$$(L_E)^{ij} = R^i{}_l L^{lk} R^j{}_k$$

Now we will calculate the general CT on $\mathbb{E}^n_\nu(\kappa)$.

Proposition 9.3.2 (Concircular tensors in $\mathbb{E}^n_\nu(\kappa)$)
$\tilde{L}$ is a concircular tensor in $\mathbb{E}^n_\nu(\frac{1}{r^2})$ where $n > 2$ iff there exists $A \in C^2_0(\mathbb{E}^n_\nu)$ such that $\tilde{L}$ has the following form embedded in $\mathbb{E}^n_\nu$:

$$L = A_E = A + \frac{\langle r, Ar\rangle}{r^4} r \odot r - \frac{2}{r^2}(Ar \odot r)$$

A is uniquely determined by $\tilde{L}$. Furthermore $\tilde{L}$ is covariantly constant iff its a constant multiple of the metric on $\mathbb{E}^n_\nu(\frac{1}{r^2})$, i.e. $A = cG$ for some $c \in \mathbb{R}$ where G is the metric of $\mathbb{E}^n_\nu$. ♢

Proof Fix $\tilde{L} \in S^2(\mathbb{E}^n_\nu(\frac{1}{r^2}))$. Choose an orthonormal basis $a_1, \ldots, a_n$ for $\mathbb{E}^n_\nu$. Let $R^* = I - \dfrac{r \otimes r^\flat}{r^2}$, then it follows from Proposition 9.3.1 that the vectors

$$R^* a_i = a_i - \frac{\langle r, a_i\rangle}{r^2} r \quad i = 1, \ldots, n$$

are CVs on $\mathbb{E}^n_\nu(\frac{1}{r^2})$. Furthermore one can check that these vectors are linearly independent. Thus by Corollary 9.1.6 every CT can be written uniquely as a linear combination of symmetric products of the above CVs. Thus it follows that we can choose a unique $A \in C^2_0(\mathbb{E}^n_\nu)$ such that $\tilde{L} = A_E$ on $\mathbb{E}^n_\nu(\frac{1}{r^2})$. In $\mathbb{E}^n_\nu$, A_E is given as follows:

$$\begin{aligned} A_E &= R^* A R \\ &= A + A(r^\flat, r^\flat)\frac{r \odot r}{r^4} - \frac{2}{r^2} A(r^\flat) \odot r \\ &= A + \langle r, Ar\rangle \frac{r \odot r}{r^4} - \frac{2}{r^2} Ar \odot r \end{aligned}$$

Conversely by Corollary 9.1.6 it follows that for any $A \in C^2_0(\mathbb{E}^n_\nu)$, A_E corresponds to CT on $\mathbb{E}^n_\nu(\frac{1}{r^2})$.

The last statement follows from Proposition 9.3.1. ∎

Remark 9.3.3
The general CT in $\mathbb{E}^n_\nu(\kappa)$ has been obtained in [TCS05, Section 3] with respect to certain canonical coordinates for these spaces. They use a different method for obtaining these tensors based on the theory developed in their article. □

For the remainder of this chapter we will always work with CTs in $\mathbb{E}^n_\nu(\kappa)$ via the tensor L defined in $\mathbb{E}^n_\nu$ in the above proposition.

Definition 9.3.4
Suppose L is a CT in $\mathbb{E}^n_\nu(\kappa)$ with parameter matrix $A \in S^2(\mathbb{E}^n_\nu)$ as above. The *iso-canonical form* for L is the metric-Jordan canonical form for (A, g). □

Except for hyperbolic space H^{n-1}_0 and the space anti-isomorphic to it S^{n-1}_{n-1}, uniqueness of the iso-canonical form follows from the uniqueness of the metric-Jordan canonical form and the fact that $I(\mathbb{E}^n_\nu(\kappa)) = O(\mathbb{E}^n_\nu)$ [O'N83]. For H^{n-1}_0, $I(H^{n-1}_0)$ is the subset of $O(\mathbb{E}^n_1)$ that preserves time orientation [O'N83]. In this case, minor modifications of the proof of the uniqueness of the metric-Jordan canonical form will show that it holds true with $I(H^{n-1}_0)$ in place of of $O(\mathbb{E}^n_1)$. A similar argument goes for S^{n-1}_{n-1}. Hence we have proven the following:

Theorem 9.3.5 (Isometric Equivalence of CTs in $\mathbb{E}^n_\nu(\kappa)$)
Suppose L and M are CTs in $\mathbb{E}^n_\nu(\kappa)$. Then $L = T_* M$ for some $T \in I(\mathbb{E}^n_\nu(\kappa))$ iff L and M have the same iso-canonical form. □

Geo-Canonical forms By definition, the restriction of G to $\mathbb{E}^n_\nu(\kappa)$ is the metric on $\mathbb{E}^n_\nu(\kappa)$. Hence we see that if $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$ and $A \in C^2_0(\mathbb{E}^n_\nu)$, then A and $aA + bG$ induce geometrically equivalent CTs on $\mathbb{E}^n_\nu(\kappa)$ (see Proposition 9.1.2). We now show how to obtain the geo-canonical forms. Suppose $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ are the distinct eigenvalues of A. Let $|\cdot|$ denote the modulus of a complex number, then define:

$$|a| := \min_{i,j} |\lambda_i - \lambda_j| > 0$$

Note that this quantity is invariant under geometric equivalence. By making the transformation $\lambda_i \to \frac{\lambda_i}{|a|}$, we can assume $|a| = 1$. Furthermore we choose $b \in \mathbb{R}$ such that the real part of the smallest eigenvalue (see Definition E.0.9) of A is zero. Since its not possible to specify the sign of a, we conclude that there are (in general) two geo-canonical forms for CTs in $\mathbb{E}^n_\nu(\kappa)$. Although in practice one can often use more information from the metric-Jordan canonical form of A to obtain a single geo-canonical form, as the following example shows:

Example 9.3.6 (Separable coordinates in hyperbolic space)
Consider $H^{n-1} = \mathbb{E}^n_1(-1)$ with the standard metric:

$$g = \operatorname{diag}(-1, 1, \dots, 1)$$

For $\lambda_1 < \cdots < \lambda_n \in \mathbb{R}$ define two linear operators A_1 and A_2 as follows:

$$\begin{aligned} A_1 &= \operatorname{diag}(\lambda_1, \dots, \lambda_n) \\ A_2 &= \operatorname{diag}(-\lambda_1, \dots, -\lambda_n) \end{aligned}$$

These two operators are isometrically inequivalent since they have different metric-Jordan canonical forms. The timelike eigenvalue of the first is the smallest, while that of the second is the largest. Although $-A_2 = A_1$ and hence the CT on H^{n-1} induced by these operators are geometrically equivalent. So, in H^{n-1} we can work with inequivalent CTs (under change of sign) by working with those whose parameter matrix has a timelike eigenvalue which is less than or equal to $\lfloor \frac{n}{2} \rfloor$ spacelike eigenvalues.

Thus the set of eigenvalues $\lambda_1 < \cdots < \lambda_n \in \mathbb{R}$ induce $\lceil \frac{n}{2} \rceil$ inequivalent separable coordinates in H^{n-1}; in contrast with the n inequivalent separable coordinates in $\mathbb{E}^n_1$ induced by central CTs. □

9.4 Properties of Concircular tensors in Spaces of Constant Curvature

In this section we will assume that each CT in $\mathbb{E}^n_\nu$ or $\mathbb{E}^n_\nu(\kappa)$ is in a canonical form listed in Section 9.1.3. Furthermore we will assume that the Cartesian coordinates are chosen such that the parameter matrix A_c is in the complex metric-Jordan canonical form stated in Theorem 8.2.2 (see Appendix C for details). We now describe how to transform to real Cartesian coordinates such that A_c obtains the real metric-Jordan canonical form given by Theorem 8.2.5. Suppose $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and (A, g) is given as follows:

$$A = J_k(\lambda) \oplus J_k(\overline{\lambda}) \qquad\qquad g = S_k \oplus S_k$$

in coordinates $(x^1, \dots, x^k, \overline{x}^1, \dots, \overline{x}^k)$. Define real coordinates $(s^1, t^1, \dots, s^k, t^k)$ implicitly as follows:

$$x^j = \frac{1}{\sqrt{2}}(s^j - it^j)$$

$$\overline{x}^j = \frac{1}{\sqrt{2}}(s^j + it^j)$$

These coordinates were chosen so that the pair (A, g) are in the real metric-Jordan canonical form in the real coordinates $(s^1, t^1, \ldots, s^k, t^k)$ after applying the appropriate tensor transformation law.

In Cartesian coordinates (x^i), we will use the convention that $x_i := g_{ij}x^j$; this is the only case where the Einstein summation convention is used in this section.

We now list some generic facts about tensors and C-tensors that will be used. We first recall some facts about $\binom{1}{1}$-tensors which were first stated in Section 1.4.2. In the following proposition, we use the notation C^p to denote the differentiability class of a geometric object, where $p \in \mathbb{N} \cup \{\infty, \omega\}$, and C^ω denotes the analytic class.

Proposition 9.4.1

Suppose T is a $\binom{1}{1}$-tensor of class C^p and fix $q \in M$.

Let λ_0 be a simple eigenvalue of T_q. Then there exists a neighborhood of q in which T has a simple eigenfunction λ with a corresponding eigenvector field which are both of class C^p, and $\lambda(q) = \lambda_0$.

If T_q has simple eigenvalues, then there exists a neighborhood of q in which T has simple eigenfunctions of class C^p, and T admits a basis of eigenvector fields of class C^p. □

The above proposition shows that Benenti tensors necessarily locally admit a smooth basis of eigenvectors with corresponding smooth eigenfunctions. The following proposition gives necessary and sufficient conditions to determine when a given Benenti tensor is an IC-tensor.

Proposition 9.4.2

Suppose L is a Benenti tensor in a neighbourhood U of a point p. If the eigenfunctions of L are not constant in U, then the eigenfunctions are functionally independent, i.e. L is an IC-tensor in a dense open subset of U. □

PROOF This is a direct consequence of the torsionless property of these tensors. Since in this case there are coordinates (q^i) such that L is diagonal and each eigenfunction $u^i(q^i)$. Then

$$\begin{aligned}
\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^n &= \frac{du^1}{dq^1}\mathrm{d}q^1 \wedge \cdots \wedge \frac{du^n}{dq^n}\mathrm{d}q^n \\
&= (\prod_{i=1}^{n} \frac{du^i}{dq^i})\mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n
\end{aligned}$$

Hence if $\mathrm{d}u^i \neq 0$ for each i, the eigenfunctions are functionally independent. If the u^i are analytic functions of q^i, then by assumption it follows that L is an IC-tensor in a dense open subset of U. ■

Proposition 9.4.3
Suppose L is an OCT and $p(z) = \det(zI - L)$ is its characteristic polynomial. Suppose u^i is a simple eigenfunction of L and $du^i \neq 0$, then the corresponding eigenform is given by:

$$du^i = -\frac{(dp)|_{z=u^i}}{p'(u^i)}$$

where dp is the exterior derivative of p with respect to the ambient coordinates and p' is the partial derivative of p with respect to z. Furthermore if L is an IC-tensor, then the metric in the coordinates induced by the eigenfunctions of L is:

$$g^{ij} = \begin{cases} (p'(u^i))^{-2} \left\langle (dp)|_{z=u^i}, (dp)|_{z=u^i} \right\rangle & \text{if } i = j \\ 0 & \text{else} \end{cases} \qquad \diamondsuit$$

Proof Since $p(z) = (z - u^i)f(z)$ for a smooth function $f(z)$. By taking the exterior derivative, we get:

$$\mathrm{d}p = -f\mathrm{d}u^i + (z - u^i)\mathrm{d}f$$

Then by L'Hopital's rule, we find that:

$$(\mathrm{d}p)|_{z=u^i} = -p'(u^i)\mathrm{d}u^i$$

which can be solved for $\mathrm{d}u^i$ since u^i is a simple eigenfunction. The fact that $L\mathrm{d}u^i = u^i\mathrm{d}u^i$ follows from the fact that L is torsionless.

To calculate the metric, first it follows that $g^{ij} = 0$ when $i \neq j$ since L is self-adjoint and has simple eigenfunctions. For the remaining component:

$$\begin{aligned} g^{ii} &= \left\langle \mathrm{d}u^i, \mathrm{d}u^i \right\rangle \\ &= (p'(u^i))^{-2} \left\langle (\mathrm{d}p)|_{z=u^i}, (\mathrm{d}p)|_{z=u^i} \right\rangle \end{aligned} \qquad \blacksquare$$

Remark 9.4.4
The assumption that L is a concircular tensor can be replaced with any symmetric contravariant tensor whose associated endomorphism is torsionless. □

The following lemma on determinants will be used several times.

Lemma 9.4.5
Suppose $T = A + v \otimes x$ where $A = [a_1, ..., a_n]$ is an $n \times n$ matrix, $v \in \mathbb{F}^n$ and $x \in \mathbb{F}^n$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). Then $\det T$ is given as follows:

$$\det T = \bigwedge_{i=1}^{n}(a_i + x_i v) = \bigwedge_{i=1}^{n} a_i + \sum_{i=1}^{n} a_1 \wedge \cdots \wedge x_i v \wedge \cdots \wedge a_n \qquad \square$$

PROOF The formula clearly holds for $n = 1$, so inductively suppose the formula holds for $k = n - 1$, then:

$$\begin{aligned}
\bigwedge_{i=1}^{n}(a_i + x_i v) &= \bigwedge_{i=1}^{n-1}(a_i + x_i v) \wedge (a_n + x_n v) \\
&= (\bigwedge_{i=1}^{n-1} a_i + \sum_{i=1}^{n-1} a_1 \wedge \cdots \wedge x_i v \wedge \cdots \wedge a_{n-1}) \wedge (a_n + x_n v) \\
&= \bigwedge_{i=1}^{n} a_i + \sum_{i=1}^{n-1} a_1 \wedge \cdots \wedge x_i v \wedge \cdots \wedge a_n + \bigwedge_{i=1}^{n-1} a_i \wedge x_n v \\
&= \bigwedge_{i=1}^{n} a_i + \sum_{i=1}^{n} a_1 \wedge \cdots \wedge x_i v \wedge \cdots \wedge a_n \qquad \blacksquare
\end{aligned}$$

In the following sections, we will obtain the following information. First we will calculate the characteristic polynomial for CTs in spaces of constant curvature. Using this, for ICTs we will calculate the transformation from the canonical coordinates they induce to Cartesian coordinates, and we will calculate the metric in canonical coordinates.

9.4.1 Central Concircular tensors

The following general lemma will be used to calculate the characteristic polynomial of central CTs.

Lemma 9.4.6 (Determinant of Central Concircular tensors)
Suppose $L = A + r \otimes r^\flat$ is a central Concircular tensor, where $r^i = x^i$. Then,

$$\det L = \bigwedge_{i=1}^{n} a_i + \sum_{i=1}^{n} a_1 \wedge \cdots \wedge x_i r \wedge \cdots \wedge a_n \tag{9.4.2}$$

Suppose U is a non-degenerate A-invariant subspace (hence $U^\perp$ is A-invariant), let $L_u = L|_U$ and $L_{u^\perp} = L|_{U^\perp}$, then:

$$\det L = \det L_u \det A_{u^\perp} + \det A_u(\det L_{u^\perp} - \det A_{u^\perp}) \tag{9.4.3}$$

$\square$

PROOF The first statement follows from Lemma 9.4.5 by taking $A \to A$, $r \to v$ and $r^\flat \to x$.

Now for the second part, let $k = \dim U$, then in a basis adapted to the decomposition $V = U \bigoplus U^\perp$, we have:

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where B is a $k \times k$ matrix and C is a $(n-k) \times (n-k)$ matrix. Furthermore $r = r_b + r_c$ where $r_b \in U$ and $r_c \in U^\perp$. The main fact we use is that for any square matrix, T, of the form:

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

we have $\det T = \det A \det C$. Thus:

$$\begin{aligned}
\det L &= \bigwedge_{i=1}^{n} a_i + \sum_{i=1}^{n} a_1 \wedge \cdots \wedge x_i r \wedge \cdots \wedge a_n \\
&= \bigwedge_{i=1}^{k} b_i \wedge \bigwedge_{i=1}^{n-k} c_i + (\sum_{i=1}^{k} b_1 \wedge \cdots \wedge x_i r_b \wedge \cdots \wedge b_k) \wedge \bigwedge_{i=1}^{n-k} c_i \\
&\quad + \bigwedge_{i=1}^{k} b_i \wedge (\sum_{i=1}^{n-k} c_1 \wedge \cdots \wedge x_i r_c \wedge \cdots \wedge c_{n-k}) \\
&= (\bigwedge_{i=1}^{k} b_i + \sum_{i=1}^{k} b_1 \wedge \cdots \wedge x_i r_b \wedge \cdots \wedge b_k) \wedge \bigwedge_{i=1}^{n-k} c_i \\
&\quad + \bigwedge_{i=1}^{k} b_i \wedge (\sum_{i=1}^{n-k} c_1 \wedge \cdots \wedge x_i r_c \wedge \cdots \wedge c_{n-k}) \\
&= \det L_u \det A_{u^\perp} + \det A_u (\det L_{u^\perp} - \det A_{u^\perp})
\end{aligned}$$

■

Now consider the simplest case where $A = \operatorname{diag}(\lambda_1, ..., \lambda_n)$. Then Eq. (9.4.2) can be used to get the characteristic polynomial of L, which is:

$$p(z) = \det(zI - L) = \prod_{i=1}^{n} (z - \lambda_i) - \sum_{i=1}^{n} x_i x^i \prod_{j \neq i} (z - \lambda_j) \tag{9.4.4}$$

Now suppose L is an ICT with eigenfunctions $(u^1, \dots, u^n)$, then from the above equation we have:

$$\prod_{j=1}^{n}(u^j - \lambda_i) = p(\lambda_i) = -\varepsilon_i (x^i)^2 \prod_{j \neq i}(\lambda_i - \lambda_j)$$

One can check that by assumption we must have $\lambda_i \neq \lambda_j$ if $i \neq j$. This will eventually be proven later. Thus we deduce the transformation from the coordinates $(u^1, \ldots, u^n)$ to Cartesian coordinates to be:

$$(x^i)^2 = \varepsilon_i \frac{\prod_{j=1}^{n}(u^j - \lambda_i)}{\prod_{j \neq i}(\lambda_j - \lambda_i)} \tag{9.4.5}$$

The derivation of the transformation to Cartesian coordinates follows that of [Cra03, section 5]. We will use this method for all other types of CTs as well. Now, it will be useful to write the characteristic polynomial in standard form:

Proposition 9.4.7

Suppose L is a central CT with parameter matrix $A = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ and arbitrary orthogonal metric. Write the characteristic polynomial of A as:

$$B(z) = \det(zI - A) = \sum_{l=0}^{n} a_l z^l$$

Then the characteristic polynomial of L is:

$$p(z) = \det(zI - L) = \sum_{l=0}^{n}(a_l - \sum_{j=0}^{n-1-l} a_{j+1+l} \langle r, A^j r \rangle) z^l \tag{9.4.6}$$

◇

Proof We will prove this formula by expanding Eq. (9.4.4). For the following calculations, if $a(z)$ is a polynomial in z, then $[z^l]a(z)$ is the coefficient of z^l in this polynomial. First observe that

$$\begin{aligned}
[z^l]\prod_{j}(z - \lambda_j) &= [z^l][z\prod_{j \neq i}(z - \lambda_j) - \lambda_i \prod_{j \neq i}(z - \lambda_j)] \\
&= [z^{l-1}]\prod_{j \neq i}(z - \lambda_j) - \lambda_i [z^l]\prod_{j \neq i}(z - \lambda_j) \\
\Rightarrow [z^{l-1}]\prod_{j \neq i}(z - \lambda_j) &= [z^l]\prod_{j}(z - \lambda_j) + \lambda_i [z^l]\prod_{j \neq i}(z - \lambda_j)
\end{aligned}$$

We also have

$$[z^{n-1}]\prod_{j\neq i}(z-\lambda_j)=1$$

We will prove inductively that

$$[z^l]\prod_{j\neq i}(z-\lambda_j)=\sum_{j=0}^{n-1-l}\lambda_i^j a_{j+1+l}$$

Then by inductive hypothesis, we have

$$\begin{aligned}[z^{l-1}]\prod_{j\neq i}(z-\lambda_j)&=a_l+\lambda_i\sum_{j=0}^{n-1-l}\lambda_i^j a_{j+1+l}\\&=a_l+\sum_{j=1}^{n-l}\lambda_i^j a_{j+l}\\&=\sum_{j=0}^{n-l}\lambda_i^j a_{j+l}\end{aligned}$$

Then

$$\begin{aligned}[z^l]\sum_{i=1}^n x_i x^i\prod_{j\neq i}(z-\lambda_j)&=\sum_{i=1}^n g_{ii}(x^i)^2[z^l]\prod_{j\neq i}(z-\lambda_j)\\&=\sum_{i=1}^n g_{ii}(x^i)^2\sum_{j=0}^{n-1-l}\lambda_i^j a_{j+1+l}\\&=\sum_{j=0}^{n-1-l}a_{j+1+l}\sum_{i=1}^n g_{ii}(x^i)^2\lambda_i^j\\&=\sum_{j=0}^{n-1-l}a_{j+1+l}\left\langle r,A^j r\right\rangle\end{aligned}$$

Which together with Eq. (9.4.4) proves the proposition. ■

In the following theorem we collect a useful limiting procedure for dealing with Jordan blocks. It has been proven by Kalnins, Miller, and Reid in [KMR84] for general dimensions. We have independently verified it only for dimensions less than three. The details of this verification are only partially included in the following proof, which can be omitted without loss of continuity.

Theorem 9.4.8 ([KMR84])
Let $A_0 := J_n^T(\lambda_1)$ and $g_0 := \varepsilon S_n$. For $n \leq 3$, there exists a sequence of diagonal matrices $A := \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, $g := \operatorname{diag}(a_1, \dots, a_n)$ and transformation matrices Λ such that

$$\Lambda^{-1} A \Lambda \to A_0 \qquad\qquad \Lambda^T g \Lambda \to g_0 \qquad \square$$

PROOF First consider the following definitions:

$$\Lambda^i{}_j := \epsilon^{j-1}_{i+1-j} = \prod_{l=2}^{j} (\epsilon^1_{i-1} - \epsilon^1_{l-2}) \qquad\qquad \epsilon^k_l := \begin{cases} 0 & \text{if } l \leq 0 \\ 1 & \text{if } k \leq 0 \end{cases}$$

$$a_i := \frac{\varepsilon}{\prod_{k \neq i} (\epsilon^1_{i-1} - \epsilon^1_{k-1})}$$

Note that ϵ^k_l is of order k if $k, l > 0$. Finally let $\lambda_i := \lambda_1 + \epsilon^1_{i-1}$. Then the conclusion follows by direct calculation if for each $i = 2, \dots, n$, $\epsilon^1_i \to 0$. ∎

Now suppose L is a central CT with parameter matrix $A = J_k^T(0)$. We will use the above theorem to obtain this CT as a limit of central CTs with parameter matrix $A = \operatorname{diag}(0, \lambda_2, \dots, \lambda_k)$. The characteristic polynomial of these CTs is given by Eq. (9.4.6). In order to obtain the characteristic polynomial for a CT with $A = J_k^T(0)$ we will use the fact that the characteristic polynomial of $J_k^T(0)$ is z^k. Then starting with $A = \operatorname{diag}(0, \lambda_2, \dots, \lambda_k)$, by Eq. (9.4.6) we have:

$$\begin{aligned} p(z) &= \sum_{l=0}^{k} (a_l - \sum_{j=0}^{k-1-l} a_{j+1+l} \langle r, A^j r \rangle) z^l \\ &\to z^k - \sum_{l=0}^{k-1} \langle r, A^{k-1-l} r \rangle z^l \\ &= z^k - \sum_{l=0}^{k-1} \langle r, A^{k-1-l} r \rangle z^l \\ &= z^k - \varepsilon \sum_{l=0}^{k-1} \sum_{i=1}^{l+1} x^i x^{l+2-i} z^l \end{aligned}$$

Thus we have proven part of the following:

Proposition 9.4.9
Suppose L is a central CT with parameter matrix $A = J_k^T(0)$ and metric $g = \varepsilon S_k$. Then

the characteristic polynomial of L is:

$$p(z) = \det(zI - L) = z^k - \varepsilon \sum_{l=0}^{k-1} \sum_{i=1}^{l+1} x^i x^{l+2-i} z^l$$

Furthermore the following are true:

- *L has no constant eigenfunctions.*
- *If $T(z) = \frac{p(z)}{B(z)}$ and $k \leq 3$, then $\langle dT, dT \rangle = 4\frac{d}{dz}T(z)$* $\diamondsuit$

Proof We first prove the case where A is a real Jordan block. To prove that L has no constant eigenfunctions, we differentiate an equation preceding this proposition to obtain:

$$\nabla p = -2 \sum_{l=0}^{k-1} z^l A^{k-1-l} r$$

from which we see that $\langle e_k, \nabla p \rangle = -2\varepsilon z^{k-1} x^1$. Thus L cannot have a constant eigenfunction. The equation for $\langle \mathrm{d}T, \mathrm{d}T \rangle$ is proven as follows. When $A = \mathrm{diag}(0, \lambda_2, \dots, \lambda_k)$ one can easily prove this formula using Eq. (9.4.4). Then the formula for $A = J_k^T(0)$ follows by applying the limiting technique in Theorem 9.4.8 used above. Finally, for the case of a complex Jordan block, i.e. $A = J_k^T(\lambda)$ where $\lambda \in \mathbb{C}$, note that these proofs hold by replacing $A \to A - \lambda I$ and $z \to z + \lambda$. ∎

Now one can use the second part of Lemma 9.4.6 to obtain the characteristic polynomial of any central CT in $\mathbb{E}^n_\nu$. Indeed, suppose L is a central CT with parameter matrix

$$A = J_k^T(0) \oplus \mathrm{diag}(\lambda_{k+1}, \dots, \lambda_n) \qquad g = \varepsilon_0 S_k \oplus \mathrm{diag}(\varepsilon_{k+1}, \dots, \varepsilon_n)$$

We can apply Lemma 9.4.6 with U equal to the subspace corresponding to $J_k^T(0)$, then

$$\begin{aligned} p(z) = \det(zI - L) = (\prod_{i=k+1}^{n} y_i) \left(z^k - \varepsilon_0 \sum_{l=0}^{k-1} \left(\sum_{i=1}^{l+1} x^i x^{l+2-i} \right) z^l \right) \\ - z^k (\sum_{i=k+1}^{n} x_i x^i \prod_{j=k+1, j \neq i}^{n} y_j) \end{aligned}$$

Now when L is an ICT, we can obtain a transformation from canonical coordinates to Cartesian coordinates. Our formula is motivated by one in [KMR84] and is given as follows:

$$\sum_{i=1}^{l+1} x^i x^{l+2-i} = \frac{-\varepsilon_0}{l!}(\frac{d}{dz})^l(\frac{p(z)}{B_{u^\perp}(z)})\big|_{z=0} \quad l = 0, \ldots, k-1 \tag{9.4.7a}$$

$$(x^i)^2 = -\varepsilon_i \frac{p(\lambda_i)}{B'(\lambda_i)} \qquad i = k+1, ..., n \tag{9.4.7b}$$

The following lemma will be used to obtain the metric in canonical coordinates adapted to an ICT defined in a space of constant curvature.

Lemma 9.4.10

Suppose L is a central CT with parameter matrix A. Let

$$T(z) = \frac{p(z)}{B(z)}$$

Then $\langle dT, dT \rangle = 4\frac{d}{dz}T(z)$. □

Proof We prove this by induction. The base cases are given by Proposition 9.4.9. Suppose U is a non-degenerate invariant subspace of A such that L_u has the form given by Proposition 9.4.9 and $U^\perp$ satisfies the induction hypothesis.

By Eq. (9.4.3) we can write:

$$p(z) = p_u(z)B_{u^\perp}(z) + B_u(z)(p_{u^\perp}(z) - B_{u^\perp}(z))$$

Then

$$\mathrm{d}p = B_{u^\perp}\mathrm{d}p_u + B_u\mathrm{d}p_{u^\perp}$$

Thus from the above equation, we have:

$$\begin{aligned}
\frac{\mathrm{d}p}{B} &= \frac{\mathrm{d}p_u}{B_u} + \frac{\mathrm{d}p_{u^\perp}}{B_{u^\perp}} \\
\Rightarrow \mathrm{d}T &= \mathrm{d}T_u + \mathrm{d}T_{u^\perp} \\
\Rightarrow \langle \mathrm{d}T, \mathrm{d}T \rangle &= \langle \mathrm{d}T_u, \mathrm{d}T_u \rangle + \langle \mathrm{d}T_{u^\perp}, \mathrm{d}T_{u^\perp} \rangle \\
&= 4\frac{d}{dz}T_u(z) + 4\frac{d}{dz}T_{u^\perp}(z) \\
&= 4\frac{d}{dz}T(z)
\end{aligned}$$ ■

Examples We end this section with some separable coordinate systems induced by central ICTs which can be analyzed fairly easily. These examples are a natural generalization of those presented in [Cra03, section 5] by Crampin.

Example 9.4.11 (Generalization of elliptic coordinates to $\mathbb{E}^n_\nu$)
Our first example is the central CT in $\mathbb{E}^n_\nu$ with parameter matrix $A = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ and orthogonal metric $g = (-1, \ldots, -1, 1, \ldots, 1)$. This CT is easiest to analyze if we assume $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Recall from Eq. (9.4.4), the characteristic polynomial of L is:

$$p(z) = \det(zI - L) = \prod_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i x^i \prod_{j \neq i} y_j$$

Using the above formula, one can show that L has no constant eigenfunctions (e.g. see the proof of Proposition 9.4.9). Then by Proposition 9.4.2, this CT is an ICT near any point where the eigenfunctions of L are simple. We will now show that L is an ICT in a dense subset of $\mathbb{E}^n_\nu$. First note that

$$p(\lambda_i) = -\varepsilon_i (x^i)^2 \prod_{j \neq i} (\lambda_i - \lambda_j) \tag{9.4.8}$$

Assume each $x^i \neq 0$, then from Equation 9.4.8, we find that $\operatorname{sgn} p(\lambda_i) = \varepsilon_i (-1)^{n+1-i}$. Also since the coefficient of leading degree of $p(z)$ is z^n, we find that $\lim\limits_{z \to \infty} p(z) = 1$ and $\lim\limits_{z \to -\infty} p(z) = (-1)^n$. Since by assumption we have that $\varepsilon_n = 1$, we can use the intermediate value theorem to deduce the following about the roots of $p(z)$. If $\nu = 0$ (i.e. in Euclidean space), there are n distinct roots $u^1, ..., u^n$ satisfying:

$$\lambda_1 < u^1 < \lambda_2 < u^2 \cdots < \lambda_n < u^n$$

If $\nu > 0$ then there are n distinct roots $u^1, ..., u^n$ satisfying:

$$u^1 < \lambda_1 < u^2 \cdots < u^\nu < \lambda_\nu < \lambda_{\nu+1} < u^{\nu+1} < \lambda_{\nu+2} < u^{\nu+2} \cdots < \lambda_n < u^n \tag{9.4.9}$$

Hence L is an IC-tensor on an open dense subset of $\mathbb{E}^n_\nu$; because of this property one could consider the induced separable coordinates to be a generalization of elliptic coordinates. Since $p(\lambda_i) = \prod\limits_{j=1}^{n} (\lambda_i - u^j)$, by Equation (9.4.8), we can obtain the Cartesian coordinates in terms of the separable coordinates $u^1, ..., u^n$

$$(x^i)^2 = \varepsilon_i \frac{\prod\limits_{j=1}^{n} (u^j - \lambda_i)}{\prod\limits_{j \neq i} (\lambda_j - \lambda_i)}$$

By using Eq. (9.4.9) and Proposition 9.4.15, one can check that in the separable coordinates $(u^1, \ldots, u^n)$, for $1 \leq i \leq \nu$, $\operatorname{sgn} g^{ii} = \frac{(-1)^{n-i+1}}{(-1)^{n-i}} = -1$. Hence $\partial_1, \ldots, \partial_\nu$ are timelike vector fields and the remaining ones are spacelike. □

We now show that if we relax the condition that $\lambda_1 < \cdots < \lambda_n$ in the above example then the coordinate system may no longer be defined on a dense subset of $\mathbb{E}^n_\nu$. Although one should note that the in $\mathbb{E}^n$ that condition was not restrictive. The simplest case occurs in $\mathbb{E}^2_1$.

Example 9.4.12
Consider a central CT L in $\mathbb{E}^2_1$ with parameter matrix $A = \mathrm{diag}(\lambda_1, \lambda_2)$ where $\lambda_1 > \lambda_2$ and orthogonal metric $g = \mathrm{diag}(-1, 1)$. Denote Cartesian coordinates by (t, x). In this case the characteristic polynomial of L, $p(z)$, given by Eq. (9.4.6) reduces to:

$$p(z) = z^2 + (2(t^2 - x^2) - \lambda_1 - \lambda_2)z - 2t^2\lambda_2 + 2x^2\lambda_1 + \lambda_1\lambda_2$$

One can calculate the discriminant of this polynomial to be:

$$4\left((t-x)^2 + \frac{\lambda_2 - \lambda_1}{2}\right)\left((t+x)^2 + \frac{\lambda_2 - \lambda_1}{2}\right)$$

If we define new Cartesian coordinates (y^1, y^2) by:

$$y^1 := \sqrt{2}(t - x) \qquad\qquad y^2 := \sqrt{2}(t + x)$$

and we let $e := \sqrt{\lambda_1 - \lambda_2}$, then L is a Benenti tensor on the following connected regions:

Region	(u^1, u^2)
N	$y^1 > e, y^2 < -e$
E	$y^1, y^2 > e$
S	$y^1 < -e, y^2 > e$
W	$y^1, y^2 > -e$
C	$\lvert y^1\rvert, \lvert y^2\rvert < e$

Hence the regions are separated by the lightlike lines $|y^i| = e$. Thus as claimed the associated separable coordinate systems aren't defined on a dense subset.

One can also find the coordinate domains as follows. Suppose L is an ICT with eigenfunctions $u^1 < u^2$. Then by requiring that the metric in these coordinates given by Proposition 9.4.15 to be Lorentzian, one finds the following constraints:

$$\begin{aligned}
&u^1 < u^2 < \lambda_2 < \lambda_1 \\
&\lambda_2 < \lambda_1 < u^1 < u^2 \\
&\lambda_2 < u^1 < u^2 < \lambda_1 \\
&u^1 < \lambda_2 < \lambda_1 < u^2
\end{aligned}$$

The above inequalities shown that in the subset where L is a Benenti tensor, if the eigenfunctions transition from one coordinate domain to another then one of the eigenfunctions must take the value λ_1 or λ_2. Hence the transition manifolds are solutions of $p(\lambda_i) = 0$, i.e. by Eq. (9.4.4) where $(x^i)^2 = 0$. In this case, the eigenfunctions of L can be readily calculated:

$$\begin{aligned}
t = 0 &\Rightarrow \lambda_1, \lambda_2 + x^2 \\
x = 0 &\Rightarrow \lambda_1 - t^2, \lambda_2
\end{aligned}$$

Using the values of the eigenfunctions on these subsets and their possible ranges given in Eq. (9.4.10) one can deduce the following:

(y^1, y^2)	(u^1, u^2)
E, W	$u^1 < u^2 < \lambda_2 < \lambda_1$
N,S	$\lambda_2 < \lambda_1 < u^1 < u^2$
C	$\lambda_2 < u^1 < u^2 < \lambda_1$

Together with Eq. (9.4.5), this completes the analysis of these coordinate systems. □

Even in three dimensions, the above analysis becomes much more difficult. This is because in three dimensions one can show that the discriminant is an eight degree polynomial in the coordinates with many terms. Although we note two simplifications that could be made for the general case. First by transferring to a geometrically equivalent CT, we could have assumed one of the eigenvalues of A were zero. Secondly since the characteristic polynomial of L, given by Eq. (9.4.4) only depends on the quantities $(x^i)^2$ and not x^i explicitly, one can restrict the analysis to the quadrant where each $x^i > 0$ while losing no generality. This symmetry is a consequence of the non-uniqueness of the chosen basis, in particular due to the fact that if v is an eigenvector of A then so is $-v$.

9.4.2 Axial Concircular tensors

Proposition 9.4.13

Let L be an axial CT with parameter matrix $A = J_k(0)^T$ and metric $g = \varepsilon S_k$. Then

$$p(z) = \det(zI - L) = z^k + \sum_{l=2}^{k} \sum_{i=1}^{l-1} x^{k+1+i-l} x^{k+1-i} z^{k-l} - 2\varepsilon \sum_{i=1}^{k} x^{k-i+1} z^{k-i} \tag{9.4.11}$$

Furthermore the following are true:

- *L has no constant eigenfunctions.*
- *If $k \leq 3$, then $\langle dp, dp \rangle = 4\varepsilon \frac{d}{dz} p(z)$.* ♢

Proof We first outline how one proves the above formula for $p(z)$. It is sufficient to calculate $\det L$ when L has the parameter matrix $A = J_k(\lambda)^T$. Let $\tilde{A} = [\tilde{a_1}, ..., \tilde{a_n}] := A + \varepsilon r \otimes e_k$. Then applying Lemma 9.4.5 to $L = \tilde{A} + e_1 \otimes r^\flat$ gives:

$$\det L = \bigwedge_{i=1}^{n} \tilde{a_i} + \sum_{i=1}^{n} \tilde{a_1} \wedge \cdots \wedge x_i e_1 \wedge \cdots \wedge \tilde{a_n}$$

After expanding r and e_1 in the basis $\{a_1, \ldots, a_k\}$ and simplifying, the result then follows by a straightforward but tedious calculation.

Suppose the above formula for $p(z)$ holds. We now show that L has no constant eigenfunctions. The constant term of dp is:

$$-2\varepsilon \sum_{i=1}^{k} z^{k-i} \mathrm{d}x^{k-i+1}$$

If $\lambda \in \mathbb{R}$ satisfies $p(\lambda) \equiv 0$, then the above form must be identically zero. A contradiction, hence L has no constant eigenfunctions.

The formula involving $\langle \mathrm{d}p, \mathrm{d}p \rangle$ can be checked manually for the cases $k \leq 3$. ■

The following proposition will reduce the calculation of the characteristic polynomial for general axial concircular tensors to cases already considered.

Proposition 9.4.14 (Determinant of Axial Concircular tensors)
Suppose L is an axial CT in canonical form given as follows:

$$L = A + e_1 \otimes r^\flat + r \otimes e_1^\flat$$
$$A = A_d \oplus A_c$$

where $A_d = J_k^T(\lambda)$. Then $p(z) = \det(zI - L)$ is given as follows:

$$p(z) = p_d(z)B(z) + \varepsilon(p_c(z) - B(z)) \tag{9.4.12}$$

♢

PROOF First note that it is sufficient to calculate $\det L$. Write $r = r_d + r_c$ adapted to the decomposition $\mathbb{E}^n_\nu = D \oplus D^\perp$ where D is the A-invariant subspace generated by e_1. Then

$$L = L_d + A_c + e_1 \otimes (r_c)^\flat + r_c \otimes e_1^\flat$$

where L_d is L restricted to D and A_c is A restricted to $D^\perp$. Let $\tilde{L} = L_d + A_c + e_1 \otimes (r_c)^\flat$, then applying Lemma 9.4.5 to $L = \tilde{L} + \varepsilon r_c \otimes e_k$ gives:

$$\det L = \det \tilde{L} + \varepsilon \tilde{L}_1 \wedge \cdots \wedge r_c \wedge \cdots \wedge \tilde{L}_n \tag{9.4.13}$$

where r_c appears in the kth spot. Now note that in block diagonal form

$$\tilde{L} = \begin{pmatrix} L_d & e_1 \otimes (r_c)^\flat \\ 0 & A_c \end{pmatrix}$$

Then after applying Lemma 9.4.5 once more, we get

$$\begin{aligned}
\tilde{L}_1 \wedge \cdots \wedge r_c \wedge \cdots \wedge \tilde{L}_n &= \bigwedge_{i=1}^{k-1} (L_d)_i \wedge r_c \wedge \Big(\sum_{i=k+1}^{n} a_{k+1} \wedge \cdots \wedge x_i e_1 \wedge \cdots \wedge a_n\Big) \\
&= -\bigwedge_{i=1}^{k-1} (L_d)_i \wedge e_1 \wedge \Big(\sum_{i=k+1}^{n} a_{k+1} \wedge \cdots \wedge x_i r_c \wedge \cdots \wedge a_n\Big) \\
&= -\bigwedge_{i=1}^{k-1} a_i \wedge e_1 \wedge \Big(\sum_{i=k+1}^{n} a_{k+1} \wedge \cdots \wedge x_i r_c \wedge \cdots \wedge a_n\Big) \\
&= (-1)^k e_1 \wedge \cdots \wedge e_k \wedge \Big(\sum_{i=k+1}^{n} a_{k+1} \wedge \cdots \wedge x_i r_c \wedge \cdots \wedge a_n\Big) \\
&= (-1)^k (\det(L_c) - \det(A_c))
\end{aligned}$$

where the second last equation follows by expanding e_1 in the basis $\{a_1, \ldots, a_k\}$. The result then follows by Eq. (9.4.13). ∎

Now one can use Proposition 9.4.14 to obtain the characteristic polynomial of any axial CT in $\mathbb{E}^n_\nu$. This is done as in the example in the discussion following Proposition 9.4.9. For example, we will calculate the Cartesian coordinates for a non-null axial CT (i.e. $k = 1$). Indeed, suppose L is a non-null axial CT and an ICT with eigenfunctions $(u^1, \ldots, u^n)$. Let $A_c = \operatorname{diag}(\lambda_2, \ldots, \lambda_n)$, then from Eq. (9.4.12) and Eq. (9.4.11), we see that

$$p(z) = \det(zI - L) = \Big(\prod_{i=2}^{n} y_i\Big)(z - 2\varepsilon x^1) - \varepsilon\Big(\sum_{i=2}^{n} x_i x^i \prod_{j=2, j\neq i}^{n} y_j\Big)$$

where $y_i = z - \lambda_i$. Since $p(z) = \prod_{i=1}^{n}(z - u^i)$, we can deduce the transformation from the coordinates $(u^1, \ldots, u^n)$ to Cartesian coordinates as follows. By evaluating $p(\lambda_i)$, we get

$$(x^i)^2 = -\varepsilon_i \varepsilon \frac{\prod_{j=1}^{n}(u^j - \lambda_i)}{\prod_{j \geq 2, j \neq i}(\lambda_j - \lambda_i)} \qquad i = 2, ..., n \tag{9.4.14}$$

By taking the coefficient of z^{n-1} of $p(z)$, we get:

$$x^1 = \frac{\varepsilon}{2}(u^1 + \cdots + u^n - \lambda_2 - \cdots - \lambda_n) \tag{9.4.15}$$

In conclusion, we note that this procedure can be generalized for $k \geq 2$.

Observe that Eq. (9.4.12) holds for a central CT if we define $p_d(z) \equiv 1$ in this case. We will use Eq. (9.4.12) and Lemma 9.4.10 to obtain the metric in canonical coordinates for some ICTs in $\mathbb{E}^n_\nu$. We have the following:

Proposition 9.4.15 (ICT metrics in $\mathbb{E}^n_\nu$)
Suppose L is an ICT in Euclidean or Minkowski space in canonical form with eigenfunctions $(u^1, \ldots, u^n)$. Then the metric in adapted coordinates is orthogonal and

$$g_{ii} = \frac{\varepsilon}{4} \frac{p'(u^i)}{B(u^i)} = \frac{\varepsilon}{4} \frac{\prod_{j \neq i}(u^i - u^j)}{\prod_{j=1}^{n-k}(u^i - \lambda_j)}$$

where ε is the sign associated with L and $\lambda_1, \ldots, \lambda_{n-k}$ are the roots of $B(z)$. □

Remark 9.4.16
The above formula likely holds in general (see [KMR84]) but we haven't verified it for null axial CTs when $k > 3$. □

Proof Let $T(z) := \frac{p(z)}{B(z)}$, $S(z) = p_d(z)$ and $\tilde{T}(z) := \frac{p_c(z)}{B(z)}$, then Eq. (9.4.12) implies:

$$\mathrm{d}T = \varepsilon \mathrm{d}\tilde{T} + \mathrm{d}S$$

Also recall that in these spaces, the index $k \leq 3$. Hence

$$\begin{aligned}
\langle \mathrm{d}T, \mathrm{d}T \rangle &= \mathrm{d}T(\nabla T) \\
&= \left\langle \mathrm{d}\tilde{T}, \mathrm{d}\tilde{T} \right\rangle + \langle \mathrm{d}S, \mathrm{d}S \rangle \\
&= 4\frac{d}{dz}\tilde{T}(z) + 4\varepsilon\frac{d}{dz}S(z) \quad \text{by Lemma 9.4.10 and Proposition 9.4.13} \\
&= 4\varepsilon\frac{d}{dz}(\varepsilon\tilde{T}(z) + S(z)) \\
&\overset{(9.4.12)}{=} 4\varepsilon\frac{d}{dz}\frac{p(z)}{B(z)}
\end{aligned}$$

Thus we have the following:

$$\begin{aligned}
\frac{\langle (\mathrm{d}p)|_{z=u^i}, (\mathrm{d}p)|_{z=u^i} \rangle}{B(u^i)^2} &= 4\varepsilon\frac{d}{dz}\frac{p(z)}{B(z)}\bigg|_{z=u^i} \\
&= 4\varepsilon\frac{p'(u^i)}{B(u^i)}
\end{aligned}$$

From Proposition 9.4.3 we have:

$$\begin{aligned}
g^{ii} &= \frac{\langle (\mathrm{d}p)|_{z=u^i}, (\mathrm{d}p)|_{z=u^i} \rangle}{p'(u^i)^2} \\
&= 4\varepsilon\frac{B(u^i)}{p'(u^i)} \\
&= 4\varepsilon\frac{\prod\limits_{j=k+1}^{n}(u^i - \lambda_j)}{\prod\limits_{j\neq i}(u^i - u^j)}
\end{aligned} \qquad \blacksquare$$

Remark 9.4.17

The above trick for calculating the metric is based on Moser's calculation of the metric for sphere-elliptic coordinates in [Mos11, P. 179-180]. □

Corollary 9.4.18

Suppose L is a non-degenerate CT in Euclidean or Minkowski space in canonical form. Then the points at which a real eigenvalue of A_c is an eigenvalue of L are singular, i.e. L cannot be an ICT in any neighborhood of these points. □

9.4.3 Concircular tensors in Spherical Submanifolds of pseudo-Euclidean space

In this section we treat the case of CTs defined on $\mathbb{E}^n_\nu(\kappa)$. We will be able to reduce most calculations to similar ones involving central CTs. The following proposition will allow us to do this.

Proposition 9.4.19 (Determinant of Spherical CTs)
Suppose $L = RL_cR^*$ is a CT in $\mathbb{E}^n_\nu(\frac{1}{r^2})$, the following holds:

$$p(z) = \det(zR - L + \frac{r \otimes r^\flat}{r^2}) = r^{-2}(B(z) - p_c(z)) \tag{9.4.16}$$

$\diamondsuit$

Proof It is sufficient to prove that:

$$\det(L + \frac{r \otimes r^\flat}{r^2}) = r^{-2}(\det L_c - \det A)$$

Observe that:

$$\begin{aligned} L + \frac{r \otimes r^\flat}{r^2} &= AR + \frac{[(r \cdot A \cdot r) + r^2]}{r^4} r \otimes r^\flat - \frac{1}{r^2} r \otimes r^\flat \cdot A \\ &= AR + r \otimes d \end{aligned}$$

for some vector d and

$$AR = A - \frac{1}{r^2} Ar \otimes r^\flat$$

Let b_i be the columns of AR, then by Lemma 9.4.5 we have

$$\det(L + \frac{r \otimes r^\flat}{r^2}) = \bigwedge_{i=1}^{n} b_i + \sum_{i=1}^{n} b_1 \wedge \cdots \wedge d_i r \wedge \cdots \wedge b_n$$

Now observe that

$$0 = \det L = \bigwedge_{i=1}^{n} b_i + \sum_{i=1}^{n} b_1 \wedge \cdots \wedge (d_i - \frac{x_i}{r^2}) r \wedge \cdots \wedge b_n$$

Thus

$$\det(L + \frac{r \otimes r^\flat}{r^2}) = \frac{1}{r^2} \sum_{i=1}^{n} b_1 \wedge \cdots \wedge x_i r \wedge \cdots \wedge b_n \tag{9.4.17}$$

Now, again using Lemma 9.4.5, we have:

$$\begin{aligned} b_1 \wedge \cdots \wedge r \wedge \cdots \wedge b_n &= (-1)^{i-1} r \wedge b_1 \wedge \cdots \wedge \hat{b}_i \wedge \cdots \wedge b_n \\ &= (-1)^{i-1} r \wedge (a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n - r^{-2} \sum_{j \neq i} a_1 \wedge \cdots \wedge x_j A r \wedge \cdots \wedge a_n) \end{aligned}$$

Note that the term $\hat{b}_i$, means b_i is missing from the product. Now note that for $i \neq j$

$$(-1)^{i-1} x_i r \wedge a_1 \wedge \cdots \wedge x_j A r \wedge \cdots \wedge a_n = -(-1)^{j-1} x_j r \wedge a_1 \wedge \cdots \wedge x_i A r \wedge \cdots \wedge a_n$$

Thus

$$\begin{aligned} \det(L + \frac{r \otimes r^\flat}{r^2}) &\overset{(9.4.17)}{=} r^{-2} \sum_{i=1}^{n} b_1 \wedge \cdots \wedge x_i r \wedge \cdots \wedge b_n \\ &= r^{-2} \sum_{i=1}^{n} a_1 \wedge \cdots \wedge x_i r \wedge \cdots \wedge a_n \\ &= r^{-2} (\det(A + r \otimes r^\flat) - \det A) \end{aligned}$$

∎

Using Eq. (9.4.16), for ICTs, the transformation from canonical coordinates to Cartesian coordinates can be calculated using the standard method. Indeed, if L is an ICT in $\mathbb{E}^n_\nu(\frac{1}{r^2})$ with parameter matrix:

$$A = J_k^T(0) \oplus \operatorname{diag}(\lambda_{k+1}, \ldots, \lambda_n) \qquad g = \varepsilon_0 S_k \oplus \operatorname{diag}(\varepsilon_{k+1}, \ldots, \varepsilon_n)$$

Then by a calculation almost identical to the one used to derive Eqs. (9.4.7a) and (9.4.7b), one obtains the following now using Eq. (9.4.16):

$$\sum_{i=1}^{l+1} x^i x^{l+2-i} = \frac{r^2 \varepsilon_0}{l!} (\frac{d}{dz})^l (\frac{p(z)}{B_{u^\perp}(z)})\Big|_{z=0} \quad l = 0, \ldots, k-1 \tag{9.4.18a}$$

$$(x^i)^2 = r^2 \varepsilon_i \frac{p(\lambda_i)}{B'(\lambda_i)} \qquad i = k+1, ..., n \tag{9.4.18b}$$

The transformation from canonical coordinates $(u^1, \ldots, u^{n-1})$ to Cartesian coordinates are obtained by noting that $p(z) = \prod_{i=1}^{n-1}(z - u^i)$.

Example 9.4.20 (Circular coordinates)
Let $M = \mathbb{E}^2_\nu(\kappa)$ where $\kappa = \pm 1$. Consider the CT in M with parameter matrix:

$$A = \mathrm{diag}(0, 1) \qquad g = \mathrm{diag}(\kappa_1, \varepsilon) \quad \kappa_1, \varepsilon \in \{-1, 1\}$$

Then by Eqs. (9.4.18a) and (9.4.18b), Cartesian coordinates (x, y) are given by:

$$\begin{aligned} x^2 &= \kappa\kappa_1 u \\ y^2 &= \kappa\varepsilon(1 - u) \end{aligned}$$

We now show how to obtain the standard parameterizations of these coordinates. First note that by the metric-Jordan canonical form theory, there are three isometrically inequivalent cases[4]:

Case 1 $\kappa_1 = \kappa$ and $\varepsilon = \kappa$, thus $g = \mathrm{diag}(\kappa, \kappa)$

If we take $u = \cos^2(t)$, then we obtain:

$$\begin{aligned} x^2 &= \cos^2(t) \\ y^2 &= \sin^2(t) \end{aligned}$$

Case 2 $\kappa_1 = \kappa$ and $\varepsilon = -\kappa$, thus $g = \mathrm{diag}(\kappa, -\kappa)$

If we take $u = \cosh^2(t)$, then we obtain:

$$\begin{aligned} x^2 &= \cosh^2(t) \\ y^2 &= \sinh^2(t) \end{aligned}$$

Case 3 $\kappa_1 = -\kappa$ and $\varepsilon = \kappa$, thus $g = \mathrm{diag}(-\kappa, \kappa)$

[4] Note that these cases additionally depend on ν.

If we take $u = -\sinh^2(t)$, then we obtain:

$$\begin{aligned} x^2 &= \sinh^2(t) \\ y^2 &= \cosh^2(t) \end{aligned}$$

Although the last two cases are geometrically equivalent, it will be useful to distinguish them when we move on to reducible CTs. □

Also using Eq. (9.4.16), one can obtain the metric in ICT induced coordinates.

Proposition 9.4.21 (ICT metrics in $\mathbb{E}^n_\nu(\kappa)$)
Suppose L is an ICT in $\mathbb{E}^n_\nu(\frac{1}{r^2})$ with eigenfunctions $(u^1, \ldots, u^{n-1})$. Then the metric in adapted coordinates is orthogonal and

$$g_{ii} = \frac{-r^2}{4}\frac{p'(u^i)}{B(u^i)} = \frac{-r^2}{4}\frac{\prod\limits_{j\neq i}(u^i - u^j)}{\prod\limits_{j=1}^{n}(u^i - \lambda_j)}$$

where $\lambda_1, \ldots, \lambda_n$ are the roots of $B(z)$. □

Proof We will reduce this calculation to the corresponding one for L_c using Eq. (9.4.16). We will assume that L is an ICT with eigenfunctions $(u^1, \ldots, u^{n-1})$ in some neighborhood in $\mathbb{E}^n_\nu(\frac{1}{r^2})$.

Now if we let $\tilde{d}$ denote the exterior derivative on the sphere, note that

$$\tilde{\mathrm{d}}p = R^*\mathrm{d}p$$

Now we make the following observation.

$$\left\langle \mathrm{d}p, r^\flat \right\rangle = \nabla_r p = 0$$

This can be proven, for example, by using Eq. (9.4.2) and the fact that r is a CV. Note that the above equation also implies that $\left\langle \mathrm{d}p_c, r^\flat \right\rangle = -2r^2 p$.

Hence we see that

$$\left\langle \tilde{\mathrm{d}}p, \tilde{\mathrm{d}}p \right\rangle = \langle \mathrm{d}p, \mathrm{d}p \rangle$$

Thus at a root $z = u^i$, we have

$$\left\langle \tilde{\mathrm{d}}p, \tilde{\mathrm{d}}p \right\rangle = r^{-4} \langle \mathrm{d}p_c, \mathrm{d}p_c \rangle$$

Then at $z = u^i$ we have

$$\begin{aligned}
\frac{\left\langle \tilde{\mathrm{d}}p, \tilde{\mathrm{d}}p \right\rangle}{B^2} &= \frac{r^{-4}\left\langle \mathrm{d}p_c, \mathrm{d}p_c \right\rangle}{B^2} \\
&\overset{9.4.10}{=} 4r^{-4}\frac{d}{dz}\frac{p_c(z)}{B(z)}\bigg|_{z=u^i} \\
&= -4r^{-2}\frac{d}{dz}\frac{p(z)}{B(z)}\bigg|_{z=u^i} \\
&= -4r^{-2}\frac{p'(u^i)}{B(u^i)}
\end{aligned}$$

Thus Proposition 9.4.21 follows from the above equation and Proposition 9.4.3. ∎

9.5 Classification of reducible concircular tensors

In this section, we will show how to find a warped product which "decomposes"[5] a given reducible OCT defined in a space of constant curvature. To do this, we will use the knowledge of warped product decompositions of these spaces summarized in Section 8.4 and Proposition 6.3.7 which gives us a method to construct reducible OCTs.

The following definition will be useful.

Definition 9.5.1

Suppose L is a CT in M and let $N = N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k$ be a local warped product decomposition of M passing through $\bar{p} \in N \subseteq M$. We say L is *decomposable* in this warped product if for each $p \in N$ and $i > 0$, T_pN_i is an invariant subspace for L. □

9.5.1 In pseudo-Euclidean space

Suppose $N = N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k$ is a warped product and $\tilde{L}$ is a CT in N_0. We say $\tilde{L}$ can be *extended to a CT in* N if $\tilde{L}$ satisfies Eq. (6.3.3) for each i with some $\lambda_i \in \mathbb{R}$. Assuming $\tilde{L}$ is an OCT, then Proposition 6.3.7 allows one to define a CT on N which restricts to $\tilde{L}$ on N_0. The following lemma will be our main tool for classifying reducible concircular tensors.

Lemma 9.5.2

Fix a proper warped product decomposition $(V_0 \oplus V_1; a)$ of $\mathbb{E}^n_\nu$ and let $L^i{}_j = A^i{}_j + mx^ix_j + w^ix_j + x^iw_j$ be a concircular tensor in N_0. Then L can be extended to concircular tensor in $\mathbb{E}^n_\nu$ decomposable in this warped product iff a is an eigenvector of A orthogonal to w. □

Proof First observe

[5] This amounts to partially diagonalizing these CTs.

$$\begin{aligned}
v^k \nabla_k \operatorname{tr}(L) &= v^k \nabla_k (m x_i x^i + 2x^i w_i) \\
&= m[(v^k \nabla_k x_i) x^i + x_i (v^k \nabla_k x^i)] + 2[(v^k \nabla_k w_i) x^i + w_i (v^k \nabla_k x^i)] \\
&= m(v_i x^i + x_i v^i) + 2 v^i w_i \\
&= 2 m v^i x_i + 2 v^i w_i
\end{aligned}$$

Hence $\nabla^i \operatorname{tr}(L) = 2(mx^i + w^i)$. Now let $\rho = a^i x_i = \langle a, x \rangle > 0$, then one can similarly show that

$$\nabla^i \log \rho = \frac{a^i}{\rho}$$

Then,

$$\begin{aligned}
L^i{}_j \nabla^j \log \rho - \frac{1}{2} \nabla^i \operatorname{tr}(L) &= \frac{1}{\rho}(A^i{}_j a^j + m x^i x_j a^j + w^i x_j a^j + x^i w_j a^j) - m x^i - w^i \\
&= \frac{1}{\rho}(A^i{}_j a^j + x^i w_j a^j) + \frac{1}{\rho}(m x^i \rho + w^i \rho) - m x^i - w^i \\
&= \frac{1}{\rho}(A^i{}_j a^j + x^i w_j a^j)
\end{aligned}$$

By definition, L can be extended to a CT decomposable in this warped product iff $L^i{}_j \nabla^j \log \rho - \frac{1}{2} \nabla^i \operatorname{tr}(L) \in \operatorname{span}\{\nabla^i \log \rho\}$. The above equation implies that this happens iff a is an eigenvector of A and $a \in w^\perp$. ■

We now use the above lemma to construct reducible CTs in $\mathbb{E}^n_\nu$.

Proposition 9.5.3 (Constructing Reducible CTs in $\mathbb{E}^n_\nu$)
Fix a proper warped product decomposition $(V_0 \ominus V_1; a)$ of $\mathbb{E}^n_\nu$ and let $\tilde{L} = \tilde{A} + m\tilde{r} \odot \tilde{r} + 2\tilde{r} \odot \tilde{w}$ be a concircular tensor in N_0 (in contravariant form) which can be extended to a concircular tensor L in $\mathbb{E}^n_\nu$ via the above lemma. Since $N_0 \subset V_0 \subset \mathbb{E}^n_\nu$, we can consider $\tilde{L}$ to be a tensor in $\mathbb{E}^n_\nu$. Then L is given as follows:

$$L = A + m r \odot r + 2 r \odot \tilde{w}$$

where as a linear operator, $A = \tilde{A} + \lambda I_{V_1}$, where λ is the eigenvalue of $\tilde{A}$ associated with a and I_{V_1} is the identity on V_1. ♢

Proof Throughout the proof, G is the contravariant metric for $\mathbb{E}^n_\nu$ and this metric adapted to the warped product is given as follows:

$$G = G' + \frac{1}{\rho^2} G_1$$

The non-null case: In this case $\kappa_1 := a^2 = \pm 1$. Let $m := \dim V_0$ and choose an orthonormal basis for V_0, $\{a_1, ..., a_m\}$ with $a_m = a$.

First note that for $p = (p_0, p_1) \in N_0 \times N_1$ and $v = (v_0, v_1) \in T_p(N_0 \times N_1)$, Eq. (8.4.3) implies that

$$\psi_* v = P_0 v_0 + \langle a, v_0 \rangle (p_1 - c) + \langle a, p_0 \rangle v_1$$

Hence we observe the following:

$$\begin{aligned} \psi_* p_0 &= P_0 p_0 + \langle a, p_0 \rangle (p_1 - c) \\ &= \psi(p_0, p_1) \end{aligned} \tag{9.5.1}$$

and

$$\psi_* a_i = a_i \quad \text{for } i = 1, ..., m-1 \tag{9.5.2}$$

Now let $\tilde{L} = \tilde{A} + m\tilde{r} \odot \tilde{r} + 2\tilde{w} \odot \tilde{r}$ be a concircular tensor in N_0 satisfying $\tilde{A}a = \lambda a$ for some λ and $\langle a, \tilde{w} \rangle = 0$. Then from Lemma 9.5.2 we know that $\psi_*(\tilde{L} + \frac{\lambda}{\rho^2} G_1)$ is a concircular tensor in $\mathbb{E}^n_\nu$. We now calculate $\psi_*(\tilde{L} + \frac{\lambda}{\rho^2} G_1)$ explicitly.

First note that

$$\tilde{A} = A_0 + \lambda \kappa_1 a \odot a$$

where $A_0 a = 0$ and so $\psi_* A_0 = A_0$ by Eq. (9.5.2). Let G be the contravariant metric for $\mathbb{E}^n_\nu$ and G_0 be the restriction of G to W_0, then

$$\begin{aligned} G &= G' + \frac{1}{\rho^2} G_1 \\ &= G_0 + \kappa_1 a \odot a + \frac{1}{\rho^2} G_1 \end{aligned}$$

Thus

$$\frac{1}{\rho^2} G_1 = G - G_0 - \kappa_1 a \odot a$$

Let G_{V_1} be the restriction of G to V_1, then

$$
\begin{aligned}
\psi_*(\tilde{A} + \frac{\lambda}{\rho^2}G_1) &= \psi_*(A_0 + \lambda\kappa_1 a \odot a + \lambda(G - G_0 - \kappa_1 a \odot a)) \\
&= \psi_*(A_0 + \lambda(G - G_0)) \\
&= A_0 + \lambda(G - G_0) \\
&= \tilde{A} + \lambda G_{V_1}
\end{aligned}
$$

where the second last equality follows from Eq. (9.5.2) and the fact that ψ is an isometry.

Eq. (9.5.1) implies that $\psi_*\tilde{r} = r$, also Eq. (9.5.2) together with the fact that $\langle a, \tilde{w}\rangle = 0$ implies that $\psi_*\tilde{w} = \tilde{w}$. Thus we conclude that

$$
\psi_*(\tilde{L} + \frac{\lambda}{\rho^2}G_1) = A + mr \odot r + 2r \odot \tilde{w}
$$

where as a linear operator, $A = \tilde{A} + \lambda I_{V_1}$ where I_{V_1} is the identity on V_1.

The null case: In this case a is a lightlike vector. Let $m := \dim V_0$ and choose a basis $\{a_1, ..., a_{m-2}, a, b\}$ for V_0 where $\{a_1, ..., a_{m-2}\}$ is an orthonormal basis for W_0 and a, b are as in the null warped product decomposition.

First note that for $p = (p_0, p_1) \in N_0 \times N_1$ and $v = (v_0, v_1) \in T_p(N_0 \times N_1)$, Eq. (8.4.4) implies that

$$
\begin{aligned}
\psi_* v = P_0 v_0 &+ (\langle b, v_0\rangle - \frac{1}{2}\langle a, v_0\rangle (P_1 p_1)^2 - \langle a, p_0\rangle \langle P_1 p_1, P_1 v_1\rangle)a + \langle a, v_0\rangle b \\
&+ \langle a, v_0\rangle P_1 p_1 + \langle a, p_0\rangle P_1 v_1
\end{aligned}
$$

Hence we observe the following:

$$
\begin{aligned}
\psi_* p_0 &= P_0 p_0 + (\langle b, p_0\rangle - \frac{1}{2}\langle a, p_0\rangle (P_1 p_1)^2)a + \langle a, p_0\rangle b + \langle a, p_0\rangle P_1 p_1 \\
&= \psi(p_0, p_1)
\end{aligned}
\tag{9.5.3}
$$

and

$$
\begin{aligned}
\psi_* a_i &= a_i \quad i = 1, ..., m-2 \\
\psi_* a &= a
\end{aligned}
\tag{9.5.4}
$$

Now let $\tilde{L} = \tilde{A} + m\tilde{r} \odot \tilde{r} + 2\tilde{w} \odot \tilde{r}$ be a concircular tensor on N_0 satisfying $\tilde{A}a = \lambda a$ for some λ and $\langle a, \tilde{w}\rangle = 0$. Then from Lemma 9.5.2 we know that $\psi_*(\tilde{L} + \frac{\lambda}{\rho^2}G_1)$ is a concircular tensor in $\mathbb{E}^n_\nu$. We now calculate $\psi_*(\tilde{L} + \frac{\lambda}{\rho^2}G_1)$ explicitly.

Since $\tilde{A}a = \lambda a$, $\tilde{A}$ can be decomposed in contravariant form as follows:

$$\tilde{A} = A_0 + 2\lambda a \odot b$$

where $A_0 a = 0$ and so $\psi_* A_0 = A_0$ by Eq. (9.5.4). Let G be the contravariant metric for $\mathbb{E}^n_\nu$ and G_0 be the restriction of G to W_0, then we see that

$$\frac{1}{\rho^2}G_1 = G - G_0 - 2a \odot b$$

Let G_{V_1} be the restriction of G to V_1, then

$$\begin{aligned}
\psi_*(\tilde{A} + \frac{\lambda}{\rho^2}G_1) &= \psi_*(A_0 + 2\lambda a \odot b + \lambda(G - G_0 - 2a \odot b)) \\
&= \psi_*(A_0 + \lambda(G - G_0)) \\
&= A_0 + \lambda(G - G_0) \\
&= A_0 + 2\lambda a \odot b + \lambda G_{V_1} \\
&= \tilde{A} + \lambda G_{V_1}
\end{aligned}$$

where the third equality follows from Eq. (9.5.4) and the fact that ψ is an isometry.

Eq. (9.5.3) implies that $\psi_* \tilde{r} = r$, also Eq. (9.5.4) together with the fact that $\langle a, \tilde{w} \rangle = 0$ implies that $\psi_* \tilde{w} = \tilde{w}$. Thus we conclude that

$$\psi_*(\tilde{L} + \frac{\lambda}{\rho^2}G_1) = A + mr \odot r + 2r \odot \tilde{w}$$

where as a linear operator, $A = \tilde{A} + \lambda I_{V_1}$ where I_{V_1} is the identity on V_1. ■

Remark 9.5.4

Note that even though the extended CT, L, can be naturally extended to all of $\mathbb{E}^n_\nu$. It is the extension of $\tilde{L}$ only for the subset $\mathrm{Im}(\psi)$ of $\mathbb{E}^n_\nu$ given by Theorem 8.4.5, which is in general not a dense subset of $\mathbb{E}^n_\nu$. □

The following corollary will be useful later on.

Corollary 9.5.5

Fix a proper warped product decomposition ψ determined by the data $(V_0 \ominus V_1; a)$ with $\kappa_1 := a^2 = \pm 1$. Let $\tilde{r} = P_1 r$ be the dilatational vector in W_1 and G_1 be the metric in W_1. Write the metric adapted to the warped product as $G = G' + \frac{1}{\rho^2}\tilde{G}$, then:

$$\psi_* \tilde{G} = \kappa_1 \tilde{r}^2 (G_1 - \frac{1}{\tilde{r}^2}\tilde{r} \odot \tilde{r}) \qquad \square$$

PROOF Let G be the contravariant metric for $\mathbb{E}^n_\nu$ and G_0 (resp. G_1) be the restriction of G to W_0 (resp. W_1), then recall that

$$\frac{1}{\rho^2}\tilde{G} = G - G_0 - \kappa_1 a \odot a$$

Hence the above equation together with Eq. (9.5.2) implies that

$$\begin{aligned}\psi_* \tilde{G} &= \rho^2 (G - G_0 - \kappa_1 \psi_*(a \odot a)) \\ &= \rho^2 (G_1 - \kappa_1 \psi_*(a \odot a))\end{aligned}$$

Let $\tilde{p}_1 = p_1 - c \in W_1(\kappa_1)$ then $\tilde{r} = P_1 r = \langle a, p_0 \rangle \tilde{p}_1$. Then by Eq. (9.5.1)

$$\begin{aligned}\psi_* a &= \kappa_1 \tilde{p}_1 \\ &= \kappa_1 \frac{\tilde{r}}{\langle a, p_0 \rangle} \\ &= \kappa_1 \frac{\tilde{r}}{\rho}\end{aligned}$$

Thus since $\tilde{r}^2 = \frac{\rho^2}{\kappa_1}$, we have:

$$\begin{aligned}\psi_* \tilde{G} &= \rho^2 (G_1 - \kappa_1 \psi_*(a \odot a)) \\ &= \rho^2 (G_1 - \kappa_1 \frac{1}{\rho^2} \tilde{r} \odot \tilde{r}) \\ &= \kappa_1 \tilde{r}^2 (G_1 - \frac{1}{\tilde{r}^2} \tilde{r} \odot \tilde{r})\end{aligned}$$

■

We now present some examples which show how to use the above proposition (Proposition 9.5.3) to construct warped products which decompose a given reducible CT.

Example 9.5.6

Let $M = \mathbb{E}^n_\nu$ where $n \geq 3$. Consider the central CT L with parameter matrix $A = \varepsilon e \odot e$ with $\varepsilon := e^2 = \pm 1$.

Let $W := e^\perp$ and P be the orthogonal projection onto W. Choose $\bar{p} \in \mathbb{E}^n_\nu$ such that $(P\bar{p})^2 \neq 0$, WLOG we assume $(P\bar{p})^2 = \pm 1$. We now construct a warped product passing through $\bar{p}$ which decomposes L.

Let $\kappa_1 := \operatorname{sgn}(P\bar{p})^2$ and take $a := \kappa_1 P\bar{p} \in W$. Let $V_1 = W \cap a^\perp$ and $V_0 = V_1^\perp = \mathbb{R}e \oplus \mathbb{R}a$. Note that a was chosen so that the initial data $(\bar{p}; V_0 \ominus V_1; a)$ is in canonical form and also note $\kappa_1 = a^2$. Let $\psi : N_0 \times_\rho N_1 \to \mathbb{E}^n_\nu$ be the warped product in Theorem 8.4.5 determined by this initial data.

Now let $\tilde{A} := \varepsilon e \odot e + 0a \odot a \in C^2_0(N_0)$, then by construction we have that:

$$A = \tilde{A} + 0I_{V_1}$$

Let $\tilde{L}$ be the central CT in N_0 with parameter matrix $\tilde{A}$ and suppose the contravariant metric in the warped product decomposes as $G = G' + \frac{1}{\rho^2}G_1$. The above proposition shows that:

$$\psi_*(\tilde{L} + 0\frac{1}{\rho^2}G_1) = L$$

for all points in the image of ψ, which includes $\bar{p}$. Hence this warped product decomposition decomposes L. Note that this warped product was constructed so that $\tilde{A}$ has simple eigenvalues and so $\tilde{L}$ is no longer reducible.

In the following we replace N_1 with $N_1 - c_1$ so that N_1 is a central hyperquadric. Then by Eq. (8.4.3), we have for $(p_0, p) = (\kappa_1 xa + ye, p) \in N_0 \times N_1$

$$\psi(p_0, p) = xp + ye \qquad \square$$

The above example will be applied to construct separable coordinates in Section 9.6.2, see Example 9.6.4. We now give a non-Euclidean variation of the above example.

Example 9.5.7
Let $M = \mathbb{E}^n_\nu$ where $n \geq 3$. Consider the central CT L with parameter matrix $A = a \odot a$ with $a^2 = 0$ and $a \neq 0$.

Let $W = a^\perp$. Choose $\bar{p} \notin W$, WLOG we assume $\langle \bar{p}, a\rangle = \pm 1$. We now construct a warped product passing through $\bar{p}$ which decomposes L.

If $\langle \bar{p}, a\rangle = -1$, then set $a := -a$, so we can assume $\langle \bar{p}, a\rangle = 1$. Define b as follows:

$$b := \bar{p} - \frac{\bar{p}^2}{2}a \tag{9.5.5}$$

Note that b is a lightlike vector satisfying $\langle a, b\rangle = 1$. Define $V_1 = a^\perp \cap b^\perp$ and $V_0 = \text{span}\{a, b\}$. Note that b was chosen so that the initial data $(\bar{p}; V_0 \ominus V_1; a)$ is in canonical form. Let $\psi : N_0 \times_\rho N_1 \to \mathbb{E}^n_\nu$ be the warped product in Theorem 8.4.5 determined by this initial data.

Note that $\{b, a\}$ forms a cycle of generalized eigenvectors for A and $A|_{V_1} = 0I_{V_1}$. Hence by the above proposition, $(\psi^{-1})_*L$ is decomposable in this warped product. Also by Theorem 8.4.5, $\bar{p} \in \text{Im}(\psi)$. Also, the restriction of $(\psi^{-1})_*L$ to N_0, $\tilde{L}$, is a central CT with 2D parameter matrix $a \odot a$.

In the following we replace N_1 with $P_1(N_1 - \bar{p})$ so that $N_1 = V_1$ is a vector space. Then by Eq. (8.4.4), we have for $(p_0, p) = (xb + ya, p) \in N_0 \times N_1$

$$\psi(p_0, p) = x(b + p - \frac{1}{2}p^2 a) + ya \qquad \square$$

General Construction We will now show how to use Proposition 9.5.3 to construct a warped product which decomposes an interesting class[6] of non-degenerate reducible CTs. This construction generalizes the above examples. First we need a preliminary definition. Suppose A is a linear operator on a vector space. We say that a vector v is a *proper generalized eigenvector of* A if $(A - \lambda I)^k v = 0$ for some $\lambda \in \mathbb{C}$ and $k > 1$.

Let $L = A + m r \odot r + 2r \odot w$ be a non-degenerate CT in $\mathbb{E}^n_\nu$ in the canonical form given by Theorem 9.1.8. We let the subspace D and the matrix A_c be as in the remarks following that theorem. We will assume that each real generalized eigenspace of A_c admits at most one proper generalized eigenvector. We lose no generality when working in Euclidean or Minkowski space (see Section 8.2.1).

Now let $W_1, \ldots, W_k$ be the multidimensional (real) eigenspaces of A_c with corresponding eigenvalues $\lambda_1, \ldots, \lambda_k$. The following construction is based on the metric-Jordan canonical form of A_c, see Theorem 8.2.5.

Case 1 W_i is a non-degenerate subspace

Choose a unit vector $a_i \in W_i$ and define $V_i := W_i \cap a_i^\perp$. The pair (V_i, a_i) determine a sphere.

Case 2 W_i is a degenerate subspace

Consider the metric-Jordan canonical form for A_c. By assumption there must be a single cycle $v_1, \ldots, v_r$ of generalized eigenvectors with $v_r \in W_i$ being a lightlike eigenvector. Let $a_i := v_r$ and $V_i := W_i \cap v_1^\perp$, note that V_i is non-degenerate.

Now let $V_0 := \cap_{i=1}^k V_i^\perp$ and $\tilde{A} := A|_{V_0}$. By construction, the data $(\bigoplus_{i=0}^k V_i; a_1, ..., a_k)$, determines a warped product decomposition $\psi : N_0 \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu$ in canonical form. By repeatedly applying Proposition 9.5.3 we see that L is decomposable in the warped product decomposition induced by ψ, with the following properties:

- $((\psi^{-1})_* L)|_{N_0} = \tilde{A} + m\tilde{r} \odot \tilde{r} + 2\tilde{r} \odot w$ where $\tilde{r}$ is the dilatational vector field in N_0
- $\tilde{A}|_{D^\perp}$ only has eigenspaces of dimension one, i.e. each Jordan block of $\tilde{A}|_{D^\perp}$ has a distinct eigenvalue.
- For each $i > 0$, TN_i is an eigenspace of $(\psi^{-1})_* L$ with constant eigenfunction λ_i

[6] This class includes all reducible OCTs in Euclidean and Minkowski space.

On Completeness We will end this section by showing that the above construction is complete, meaning that the restriction of $(\psi^{-1})_* L$ to the geodesic factor N_0 no longer has constant eigenfunctions.

We also note here that with an appropriate choice of $a_1, \ldots, a_k$ we can choose warped product decompositions to cover all of $\mathbb{E}^n_\nu$ except for a union of closed submanifolds with dimension strictly less than n. Examples 9.5.6 and 9.5.7 give more details on how to do this. In other words, for the non-degenerate CTs considered above, there exists a warped product decomposition $\psi : N_0 \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu$ such that $\mathrm{Im}(\psi)$ is a dense subset of $\mathbb{E}^n_\nu$. Although the cost of this is that the factors N_i may no longer be connected subsets.

The following lemma shows that the classification of reducible CTs given above is complete for central CTs.

Lemma 9.5.8 (Reducible central CTs)
Let L be a central CT with parameter matrix A. Suppose that each real generalized eigenspace of A has at most one proper generalized eigenvector. Then A has a real eigenspace $\tilde{E}_\lambda$ with dimension $m > 1$ iff L has a non-degenerate eigenspace E_λ (defined on a dense subset of $\mathbb{E}^n_\nu$) with constant eigenfunction λ and dimension $m - 1$. □

Proof It was proven above that under the hypothesis, if A has a real eigenspace with dimension $m > 1$ then L has a non-degenerate eigenspace E_λ with dimension $m - 1$. We will now prove the converse.

To prove the converse, we simply have to prove that if all real eigenspaces of A are at most one dimensional then L has no non-degenerate eigenspaces with constant eigenfunctions defined on open subsets of $\mathbb{E}^n_\nu$. It is sufficient to show that L has no constant eigenfunctions defined on open subsets of $\mathbb{E}^n_\nu$.

We prove this by induction. The base cases are given by Proposition 9.4.9. Suppose U is a non-degenerate invariant subspace of A such that L_u has the form given by Proposition 9.4.9 and $U^\perp$ satisfies the induction hypothesis. By Eq. (9.4.3) we can write:

$$p(z) = p_u(z) B_{u^\perp}(z) + B_u(z)(p_{u^\perp}(z) - B_{u^\perp}(z))$$

Then

$$\mathrm{d}p = B_{u^\perp} \mathrm{d}p_u + B_u \mathrm{d}p_{u^\perp}$$

By the induction hypothesis, $L_{u^\perp}$ has no constant eigenfunctions. Suppose λ is a constant eigenfunction of p, then by Proposition 9.4.9 and the above equation, it follows that

$$B_{u^\perp}(\lambda) = B_u(\lambda) = 0$$

If B_u has no real roots, we reach a contradiction. Otherwise, by construction A must have a real eigenspace with dimension $m > 1$, a contradiction. Hence we conclude that L has no constant eigenfunctions which proves the claim by induction. ∎

Since a multidimensional eigenspace of an OCT has a constant eigenfunction, the above proposition allows us to classify these eigenspaces when the CTs considered induce an OCT on some subset of $\mathbb{E}^n_\nu$. For completeness sake, we will now show that the hypothesis of the above proposition is the most general when it comes to classifying OCTs.

Proposition 9.5.9

Let L be a central CT with parameter matrix A. Suppose A has a real generalized eigenspace with multiple proper generalized eigenvectors, then L is not an OCT. ♢

Proof WLOG we can assume that that this generalized eigenspace of A is associated with the eigenvalue zero. First we have

$$
\begin{aligned}
L &= A + r \odot r \\
L^2 &= A^2 + Ar \odot r + r^2 r \odot r
\end{aligned}
$$

By hypothesis, $\dim N(L) \geq 1$. We also have that $\dim N(A^2) \geq 4$. The above equation shows that the range of L^2 is spanned by $\{r, Ar\}$ and the range of A^2 (on a dense subset of $\mathbb{E}^n_\nu$), hence we see that $\dim N(L^2) \geq 1 + \dim N(L)$. This implies that L is not point-wise diagonalizable on some dense subset of $\mathbb{E}^n_\nu$ (see for example [FIS03]). ∎

In fact one can show that if $A = J_2(0) \oplus J_2(0)$, then the associated central CT has a 2-cycle of generalized eigenvectors associated with eigenvalue zero.

The following lemma is the analogue of Lemma 9.5.8 for axial CTs. Its proof is also analogous and reduces to Lemma 9.5.8 with the help of Eq. (9.4.12) and Proposition 9.4.13.

Lemma 9.5.10 (Reducible axial CTs)

Let L be an axial CT with parameter matrix A. Suppose that each real generalized eigenspace of A_c has at most one proper generalized eigenvector. Then A_c has a real eigenspace $\tilde{E}_\lambda$ with dimension $m > 1$ iff L has a non-degenerate eigenspace E_λ (defined on a dense subset of $\mathbb{E}^n_\nu$) with constant eigenfunction λ and dimension $m - 1$. □

In conclusion we have the following theorem which summarizes our classification:

Theorem 9.5.11 (Classification of Reducible CTs in $\mathbb{E}^n_\nu$)

Let L be a non-degenerate CT in $\mathbb{E}^n_\nu$ such that each real generalized eigenspace of A_c has at most one proper generalized eigenvector. Then L is reducible iff A_c has a multidimensional real eigenspace. If L is reducible, then there exists an explicitly constructible warped product decomposition $\psi : N_0 \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu$ such that the following hold:

- *L is decomposable in the warped product $N_0 \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k$.*
- *The restriction of $(\psi^{-1})_* L$ to N_0 has no constant eigenfunctions.*
- $\mathrm{Im}(\psi)$ *is an open dense subset of $\mathbb{E}^n_\nu$.* ♢

9.5.2 In Spherical submanifolds of pseudo-Euclidean space

In this section we will show how the problem of classifying reducible CTs in $\mathbb{E}^n_\nu(\kappa)$ can be reduced to the same problem in $\mathbb{E}^n_\nu$. By using Theorem 8.4.7, we will show how to restrict a reducible CT in $\mathbb{E}^n_\nu$ to one in $\mathbb{E}^n_\nu(\kappa)$.

Proposition 9.5.12 (Restricting Reducible CTs to $\mathbb{E}^n_\nu(\kappa)$)
Let $\psi : N_0 \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu$ be a proper warped product decomposition in canonical form and let $\bar{p} \in \mathrm{Im}(\psi)$ as in Theorem 8.4.7. Suppose L_c is a reducible central CT in $\mathbb{E}^n_\nu$ satisfying

$$L_c = \psi_*(\tilde{L}_c + \sum_{i=1}^{k} \lambda_i G_i)$$

where G_i is the restriction of G to TN_i, $\lambda_i \in \mathbb{R}$ and $\tilde{L}_c$ is a CT in N_0. Let $\phi := \psi|_{N'}$ be the induced warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ as in Theorem 8.4.7. Then if we let L (resp. $\tilde{L}$) be the restriction of L_c (resp. $\tilde{L}_c$) to $\mathbb{E}^n_\nu(\kappa)$ (resp. $N_0(\kappa)$), then

$$L = \phi_*(\tilde{L} + \sum_{i=1}^{k} \lambda_i G_i) \qquad ♢$$

Proof Let $\tilde{r}$ (resp. r) be the dilatational vector field in N_0 (resp. $\mathbb{E}^n_\nu$). We will use the fact that $\psi_* \tilde{r} = r$; this can be deduced from the proof of Proposition 9.5.3 or Eq. (8.4.1). We let $R^* = I - \dfrac{r \otimes r^\flat}{r^2}$ be the orthogonal projection onto $T\mathbb{E}^n_\nu(\kappa)$ with a similar definition for $\tilde{R}^*$ with respect to $TN_0(\kappa)$. In the following, given $L \in S^2(\mathbb{E}^n_\nu)$, we denote by R^*L the restricted tensor given by $(R^*L)^{ij} = R^i{}_l L^{lk} R^j{}_k$.

Using the fact that ψ is an isometry and $\psi_* \tilde{r} = r$, one can show that $R^* \circ \psi_* = \psi_* \circ \tilde{R}^*$. Also note that $\tilde{R}^* G_i = G_i$. Thus

$$\begin{aligned} R^* L_c &= R^* \psi_*(\tilde{L}_c + \sum_{i=1}^{k} \lambda_i G_i) \\ &= \psi_*(\tilde{R}^* \tilde{L}_c + \sum_{i=1}^{k} \lambda_i \tilde{R}^* G_i) \\ &= \psi_*(\tilde{R}^* \tilde{L}_c + \sum_{i=1}^{k} \lambda_i G_i) \end{aligned}$$

By evaluating the above equation in $N_0(\kappa) \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k$, one obtains the desired result. ∎

Now we show how to apply the above results to obtain a warped product decomposition in which a given CT in $\mathbb{E}^n_\nu(\kappa)$ is decomposable. Let L be a CT in $\mathbb{E}^n_\nu(\kappa)$, then there is a unique central CT, L_c, such that $L = R^* L_c$. As described in the previous section, provided L_c is reducible, we can choose a warped product decomposition of $\mathbb{E}^n_\nu$, ψ, such that $L_c = \psi_*(\tilde{L}_c + \sum_{i=1}^k \lambda_i G_i)$ satisfying the hypothesis of the above proposition. Thus the above proposition gives a warped product decomposition ϕ which decomposes L, and is obtained by an appropriate restriction of ψ. We now give some examples of this procedure to obtain the standard spherical coordinates.

Example 9.5.13 (Spherical Coordinates I)
Let $M = \mathbb{E}^n_\nu(\kappa)$ where $\kappa = \pm 1$ and $n \geq 3$. Consider the CT L in $\mathbb{E}^n_\nu(\kappa)$ induced by $A = \varepsilon e \odot e$ with $\varepsilon := e^2 = \pm 1$. Let P be the orthogonal projector onto $e^\perp$ and choose $\bar{p} \in \mathbb{E}^n_\nu(\kappa)$ such that $(P\bar{p})^2 = \pm 1$. By Example 9.5.6 there is a warped product decomposition $\psi : N_0 \times_\rho N_1 \to \mathbb{E}^n_\nu$ passing through $\bar{p}$ which decomposes $L_c := A + r \odot r$. For $(p_0, p) = (x\kappa_1 a + ye, p) \in N_0 \times N_1$, we have

$$\psi(p_0, p) = xp + ye$$

To obtain a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$, by Theorem D.6.5 we need to restrict ψ to $N_0(\kappa) \times N_1$. Let ϕ be the induced warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$, then it follows by Proposition 9.5.12 that L is decomposable in this warped product. We now give the standard forms of this warped product by parameterizing (x, y) as in Example 9.4.20 while enforcing $x = \langle a, p_0 \rangle > 0$ and $N_0(\kappa)$ to be connected. We have three different cases:

Case 1 $\kappa_1 = \kappa$ and $\varepsilon = \kappa$

$$\phi : \begin{cases} (0, \pi) \times_{\sin} N_1 & \to \mathbb{E}^n_\nu(\kappa) \\ (t, p) & \mapsto \sin(t)p + \cos(t)e \end{cases}$$

Case 2 $\kappa_1 = \kappa$ and $\varepsilon = -\kappa$

$$\phi : \begin{cases} \mathbb{R} \times_{\cosh} N_1 & \to \mathbb{E}^n_\nu(\kappa) \\ (t, p) & \mapsto \cosh(t)p + \sinh(t)e \end{cases}$$

Case 3 $\kappa_1 = -\kappa$ and $\varepsilon = \kappa$

$$\phi : \begin{cases} \mathbb{R}^+ \times_{\sinh} N_1 & \to \mathbb{E}^n_\nu(\kappa) \\ (t,p) & \mapsto \sinh(t)p + \cosh(t)e \end{cases}$$

Note that even though there is only one inequivalent coordinate system on $\mathbb{E}^2_\nu(\kappa)$, the last two warped products are inequivalent. This is due to the fact that $a^2 = \kappa_1$ is different in those cases and $N_0 = \{p \in V_0 | \, \langle a, p\rangle > 0\}$. □

The following example is on spherical coordinates that only occur in non-Euclidean spheres.

Example 9.5.14 (Spherical Coordinates II)
Let $M = \mathbb{E}^n_\nu(\kappa)$ where $\kappa = \pm 1$ and $n \geq 3$. We now consider the CT L in $\mathbb{E}^n_\nu(\kappa)$ induced by $A = a \odot a$ with $a^2 = 0$ and $a \neq 0$. This example proceeds similarly to the first. Fix $\bar{p} \in \mathbb{E}^n_\nu(\kappa)$ such that $\langle a, \bar{p}\rangle = 1$. By Example 9.5.7 there is a warped product decomposition $\psi : N_0 \times_\rho N_1 \to \mathbb{E}^n_\nu$ passing through $\bar{p}$ which decomposes $L_c := A + r \odot r$. For $(p_0, p) = (xb + ya, p) \in N_0 \times N_1$, we have

$$\psi(p_0, p) = x(b + p - \frac{1}{2}p^2 a) + ya$$

Restricting ψ to $N_0(\kappa) \times N_1$ forces:

$$\kappa = p_0^2 = 2xy$$

Let ϕ be the warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ induced by ψ as in Theorem 8.4.7. Again, it follows by Proposition 9.5.12 that L is decomposable in this warped product. We now give ϕ with the standard parameterization of $N_0(\kappa)$, by enforcing $x = \langle a, p_0\rangle > 0$ and $N_0(\kappa)$ to be connected. These conditions are all satisfied if we take $x = \frac{1}{\sqrt{2}}\exp(t)$. Then we have the following:

$$\phi : \begin{cases} \mathbb{R} \times_{\frac{1}{\sqrt{2}}\exp} \mathbb{E}^{n-2}_{\nu-1} & \to \mathbb{E}^n_\nu(\kappa) \\ (t,p) & \mapsto \frac{1}{\sqrt{2}}\exp(t)(b + p - \frac{1}{2}p^2 a) + \frac{\kappa}{\sqrt{2}}\exp(-t)a \end{cases}$$

Also note that if $\nu = -\kappa = 1$, then ϕ is an isometry onto a connected component of $\mathbb{E}^n_1(-1) \simeq H^{n-1}$. □

In conclusion we have the following theorem which summarizes our classification:

Theorem 9.5.15 (Classification of Reducible CTs in $\mathbb{E}^n_\nu(\kappa)$)
Let L be a CT in $\mathbb{E}^n_\nu(\kappa)$ such that each real generalized eigenspace of A has at most one proper generalized eigenvector. Then L is reducible iff A has a multidimensional real eigenspace. If L is reducible, then there exists an explicitly constructible warped product decomposition $\psi : N_0 \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu(\kappa)$ such that the following hold:

1. *L is decomposable in the warped product $N_0 \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k$.*

2. *The restriction of $(\psi^{-1})_* L$ to N_0 has no constant eigenfunctions.*

3. *$\mathrm{Im}(\psi)$ is an open dense subset of $\mathbb{E}^n_\nu(\kappa)$.* $\diamondsuit$

PROOF We give the proof of Item 2. First suppose λ is a constant eigenfunction of L, then one can naturally lift λ to a constant function on $\mathbb{E}^n_\nu$. Let $p(z)$ be the characteristic polynomial of L having the form given by Eq. (9.4.16). Then since $\mathcal{L}_r p = 0$ (see the proof of Proposition 9.4.21), we must have $p(\lambda) = 0$ on some open subset of $\mathbb{E}^n_\nu$. Then the proof of Lemma 9.5.8 holds verbatim by Eq. (9.4.16), which proves the result.

Item 3 follows from the construction of ψ (see Proposition 9.5.12) and Theorem 9.5.11. ∎

9.6 Applications and Examples

In this section we will show how to apply the theory developed in this chapter to solve some of the motivating problems stated in the introduction. First, in Section 9.6.1 we will show how to enumerate the isometrically inequivalent separable coordinates in a given space of constant curvature. Then in Section 9.6.2 we will show how to construct separable coordinate systems by way of examples. Finally, in Section 10.3 we will show how to explicitly execute the BEKM separation algorithm in general. We also give the details of executing the BEKM separation algorithm for the Calogero-Moser system.

9.6.1 Enumerating inequivalent separable coordinates

In this section we show how one can use the theory developed in this chapter to enumerate the isometrically inequivalent separable coordinate systems on a given space of constant curvature. For dimensions greater than two, this problem is recursive as described in Section 6.5. This recursive nature was originally discovered by Kalnins et al. and is discussed more concretely in [Kal86]. So one will also have to enumerate the separable coordinate systems on spherical submanifolds of the underlying space and then construct the separable coordinates systems using warped products (see the beginning of Section 9.1.3 and also Section 6.5).

The main step is to enumerate the geometrically inequivalent CTs, so we will focus on this. To do this, one has to enumerate the canonical forms summarized in Section 9.1.3 together with the metric-Jordan canonical forms for A_c and take into account geometric equivalence. We illustrate this idea with a few examples.

Example 9.6.1 (Central CTs)
Let L be a central CT with parameter matrix A. In this case, we essentially have to enumerate the different metric-Jordan canonical forms for A. Fix $\lambda_1 < \cdots < \lambda_n \in \mathbb{R}$.

In Euclidean space there is only one central CT we can build from these parameters, it is given by the parameter matrix $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and it induces the well known elliptic coordinate system (see Example 9.4.11).

In Minkowski space there are n (geometrically inequivalent) central CTs we can build from these parameters, they are given as follows:

$$
\begin{aligned}
A &= J_{-1}(\lambda_1) \oplus J_1(\lambda_2) \oplus \cdots J_1(\lambda_n) \\
&\vdots \\
A &= J_1(\lambda_1) \oplus J_1(\lambda_2) \oplus \cdots J_{-1}(\lambda_n)
\end{aligned}
$$

They differ by the eigenvalue of A which is timelike. Similarly there are $n-1$ central CTs built only using $\lambda_2 < \cdots < \lambda_n$ with parameter matrix of the form:

$$A = J_{\pm 2}(\lambda_2) \oplus J_1(\lambda_3) \oplus \cdots J_1(\lambda_n)$$

Now consider the case where A has a two dimensional eigenspace, the rest being simple. Using $\lambda_2 < \cdots < \lambda_n$, in Euclidean space there are $n-1$ central CTs depending on which λ_i corresponds to the two dimensional eigenspace[7]. Each of these cases in Euclidean space induce $n-1$ different cases in Minkowski space depending on which λ_i becomes timelike, hence there are a total of $(n-1)^2$ cases in Minkowski space.

Finally we note that in Minkowski space A can have two complex conjugate eigenvalues, then since the corresponding real Jordan block is distinguishable from the other real eigenvalues of A, a similar analysis applies. In general one would have to order the complex eigenvalues (see Definition E.0.9). $\square$

Enumerating inequivalent axial CTs can largely be reduced to the same problem for central CTs. For example, in Euclidean space there is only one type of axial CT if all the eigenvalues of A_c are distinct. We end with CTs in spherical submanifolds of pseudo-Euclidean space as these are somewhat different.

Example 9.6.2 (CTs in $\mathbb{E}^n_\nu(\kappa)$)
Let L be the CT in $\mathbb{E}^n_\nu(\kappa)$ with parameter matrix A. Fix $\lambda_1 < \cdots < \lambda_n \in \mathbb{R}$. In this case there are sometimes less geometrically inequivalent CTs then isometrically inequivalent ones.

[7] When $n = 3$ the two different cases induce the oblate and prolate spheroidal coordinate systems.

In the Euclidean sphere there is only one CT we can build from these parameters, it is given by the parameter matrix $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and it induces the sphere-elliptic coordinate system.

Now suppose the ambient space is Minkowski space. Then we only need to consider $\lceil \frac{n}{2} \rceil$ cases given by (see Example 9.3.6):

$$\begin{aligned}
&A = J_{-1}(\lambda_1) \oplus J_1(\lambda_2) \oplus \cdots J_1(\lambda_n) \\
&\vdots \\
&A = J_1(\lambda_1) \oplus J_1(\lambda_2) \oplus \cdots \oplus J_{-1}(\lambda_{\lceil \frac{n}{2} \rceil}) \oplus \cdots J_1(\lambda_n)
\end{aligned}$$

Note that only the first $\lceil \frac{n}{2} \rceil$ eigenvalues of A are made timelike.

Most of the other cases can be deduced from the first example if one desires. Although we illustrate one difference with an example. For the Euclidean sphere $\mathbb{E}^3(1)$, fix $\lambda_1 < \lambda_2 \in \mathbb{R}$ and consider the CT induced by the following parameter matrices:

$$\begin{aligned}
A_1 &= \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2) \\
A_2 &= \operatorname{diag}(\lambda_1, \lambda_2, \lambda_2)
\end{aligned}$$

Note that $-A_2$ has the same form as A_1, specifically the smallest eigenvalue of $-A_2$ is repeated. Hence in considering parameter matrices with two dimensional eigenspaces, we only need to enumerate those with the form given by A_1, where the smaller eigenvalue is repeated. □

We have described how to enumerate the geometrically inequivalent CTs in spaces of constant curvature. One should note though, that in non-Euclidean spaces a given CT could induce different coordinate systems on disjoint connected subsets of the space (see Example 9.4.12). Hence in these cases, more work has to be done to enumerate the isometrically inequivalent separable coordinate systems.

9.6.2 Constructing separable coordinates

In a two dimensional Riemannian manifold, all non-trivial CTs are Benenti tensors. Hence in this case, one can enumerate all isometrically inequivalent separable coordinates simply by enumerating the geometrically inequivalent CTs. The latter problem can be solved in pseudo-Euclidean space using Theorem 9.1.8. In Table 9.1 we have done this for $\mathbb{E}^2$ and included the standard transformations from separable to Cartesian coordinates.

Table 9.1: Separable Coordinate Systems in $\mathbb{E}^2$

1. Cartesian coordinates	$L = d \odot d$	$x\, d + y\, e$
2. Polar coordinates	$L = r \odot r$	$\rho \cos\theta\, d + \rho \sin\theta\, e$
3. Elliptic coordinates	$L = d \odot d + a^{-2} r \odot r$	$a \cos\phi \cosh\eta\, d + a \sin\phi \sinh\eta\, e$
4. Parabolic coordinates	$L = 2r \odot d$	$\frac{1}{2}(\mu^2 - \nu^2)\, d + \mu\nu\, e$

The vectors d, e form an orthonormal basis for $\mathbb{E}^2$ and $a > 0$.

We now show how one obtains the coordinate formula in Table 9.1 from formulas we have already calculated. For elliptic coordinates, take Cartesian coordinates (x, y) on $\mathbb{E}^2$ and let L be the central CT with parameter matrix $A = \operatorname{diag}(\lambda_1, \lambda_2)$ where $\lambda_2 > \lambda_1$. Then the transformation from canonical coordinates (u^1, u^2) to Cartesian coordinates (x, y) read (see Eq. (9.4.5)):

$$x^2 = \frac{(\lambda_1 - u^1)(\lambda_1 - u^2)}{(\lambda_2 - \lambda_1)} \qquad y^2 = \frac{(\lambda_2 - u^1)(\lambda_2 - u^2)}{(\lambda_1 - \lambda_2)}$$

We can obtain the standard parameterization of elliptic coordinates as follows. Note that $L = \lambda_1 G + (\lambda_2 - \lambda_1)\tilde{L}$ where $\tilde{L} = e \odot e + (\lambda_2 - \lambda_1)^{-1} r \odot r$ is geometrically equivalent to L. The eigenfunctions of $\tilde{L}$, $(\tilde{u}^1, \tilde{u}^2)$, are related to those of L by $u^i = \lambda_1 + (\lambda_2 - \lambda_1)\tilde{u}^i$. Letting $a^2 := \lambda_2 - \lambda_1$ and substituting this expression for u^i in the above equation gives:

$$x^2 = a^2 \tilde{u}^1 \tilde{u}^2 \qquad y^2 = a^2 (1 - \tilde{u}^1)(\tilde{u}^2 - 1)$$

Then making the transformation $\tilde{u}^1 = \cos^2\phi$ and $\tilde{u}^2 = \cosh^2\eta$, we obtain the formula in Table 9.1.

The formula for parabolic coordinates follow similarly from Eqs. (9.4.14) and (9.4.15), after taking $u^1 = -\nu^2$ and $u^2 = \mu^2$ assuming $u^1 < u^2$.

We end with a few more examples to further illustrate the theory. The first example shows how to obtain coordinates which diagonalize a Benenti tensor which is not an ICT.

Example 9.6.3 (Spherical coordinates in $\mathbb{S}^2$)
Fix $d \in \mathbb{S}^2$ and let L be the CT induced in $\mathbb{S}^2$ by restricting $d \odot d$. As we observed earlier, L is necessarily a Benenti tensor. In Example 9.5.13 it was shown that a warped product which decomposes L is given by:

$$\psi(\phi, p) = \cos\phi\, d + \sin\phi\, p$$

where $p \in d^{\perp}(1)$, i.e. $p \in \mathbb{S}^2 \cap d^{\perp}$ and $\phi \in (0, \pi)$. Since $d^{\perp}(1)$ is the unit circle we obtain coordinates on it by taking $p = \cos\theta\, e + \sin\theta\, f$ where e, f is an orthonormal basis for $d^{\perp}$.

Then the above equation becomes:

$$\psi(\phi, p) = \cos\phi\, d + \sin\phi(\cos\theta\, e + \sin\theta\, f)$$

Furthermore, since ψ is a warped product decomposition with warping function $\sin\phi$, it follows from Example 9.5.13 that the metric is:

$$g = (\mathrm{d}\phi)^2 + \sin^2\phi(\mathrm{d}\theta)^2$$ □

Example 9.6.4 (Oblate/Prolate spheroidal coordinates in $\mathbb{E}^3$)
Fix a unit vector $d \in \mathbb{E}^n$, $c \neq 0$ and consider the following CT in $\mathbb{E}^n$:

$$L = c\, d \odot d + r \odot r \tag{9.6.1}$$

It follows from Example 9.5.6 that a warped product ψ which decomposes L is given as follows: Let $e \in d^\perp$ be a unit vector, then for $(p_0, p) = (xd + ye, p) \in N_0 \times N_1$

$$\psi(p_0, p) = xd + yp$$

Observe that $N_0 \simeq \mathbb{E}^2$ and L induces a Benenti tensor, $\tilde{L}$, on N_0 which has the form given by Eq. (9.6.1). If we let $a := \sqrt{|c|}$, then using Table 9.1 we can take coordinates on N_0 which diagonalize $\tilde{L}$ yielding the following maps.

$$\psi(p_0, p) = \begin{cases} c > 0 & a\cos\phi\cosh\eta\, d + a\sin\phi\sinh\eta\, p \\ c < 0 & a\sin\phi\sinh\eta\, d + a\cos\phi\cosh\eta\, p \end{cases}$$

Also N_1 is the unit sphere in $d^\perp$, hence $N_1 \simeq \mathbb{S}^{n-2}$. We can obtain separable coordinates for $\mathbb{E}^n$ by taking any separable coordinates for $\mathbb{S}^{n-2}$ on N_1 (see Section 6.5). For example, if $c > 0$ and $n = 3$, we obtain prolate spheroidal coordinates:

$$\psi(p_0, p) = a\cos\phi\cosh\eta\, d + a\sin\phi\sinh\eta\,(\cos\theta\, e + \sin\theta\, f)$$

where e, f is any orthonormal basis for $d^\perp$. Also note that using Proposition 9.4.15 and the fact that ψ is a warped product decomposition with warping function $a\sin\phi\sinh\eta$, one can obtain the following expression for the metric:

$$g = a^2(\sinh^2\eta + \sin^2\phi)((\mathrm{d}\phi)^2 + (\mathrm{d}\eta)^2) + a^2\sin^2\phi\sinh^2\eta(\mathrm{d}\theta)^2$$

Finally note that oblate spheroidal coordinates can be obtained by taking $c < 0$. □

Example 9.6.5 (Product coordinates in $\mathbb{E}^4$)
Consider the decomposition $\mathbb{E}^n = V \oplus W$ into non-trivial subspaces. Let $\tilde{G}$ denote the induced contravariant metric in V and consider the following CT in $\mathbb{E}^n$:

$$L = \tilde{G}$$

Observe that the warped product $\psi : V \times_1 W \to \mathbb{E}^n$ given by $(q, p) \to q + p$ is adapted to the eigenspaces of L. We can construct separable coordinates by parameterizing q (resp. p) with separable coordinates on V (resp. W). For example, if $\dim V = \dim W = 2$, by taking polar (resp. elliptic) coordinates on V (resp. W) from Table 9.1, we have the following separable coordinates on $\mathbb{E}^4$:

$$\psi(q, p) = \rho \cos\theta\, b + \rho \sin\theta\, c + a \cos\phi \cosh\eta\, d + a \sin\phi \sinh\eta\, e$$

where b, c (resp. d, e) is an orthonormal basis for V (resp. W). $\square$

In conclusion, as an exercise, we recommend the reader prove that there are eleven classes of isometrically inequivalent separable coordinate systems in $\mathbb{E}^3$.

9.7 Notes

A classification of CTs modulo the action of the isometry group in Euclidean space can be found in [Lun03] (cf. [Ben05]). A complete classification of these tensors for Euclidean space and the Euclidean sphere is implicit in [WW03].

Different parts of this problem have been solved for special cases by different researchers over the past few decades. A classification of separable coordinate systems in Riemannian spaces of constant curvature was originally done by Kalnins and Miller in [KM86; KM82], see also [Kal86] which is a book containing their results. Their solution primarily involves degenerating a general coordinate system (elliptic coordinates for $\mathbb{E}^n$) by taking limits of certain parameters appearing in the coordinate system. The insight provided by their classification was crucial for the development of the theory presented in this thesis. They have extended this work to spaces of constant curvature with arbitrary signature in [KMR84] to obtain a partial classification.

In [Kal75] orthogonal separable coordinates in two dimensional Minkowski space have been classified and those in three dimensional Minkowski space have been partially classified. A more detailed classification of a more general class of orthogonal separable coordinates in three dimensional Minkowski space has been given in [KM76]. This classification has been further refined in [Hin98] (cf. [HM08]). A classification of orthogonal separable coordinates for four dimensional Minkowski space has been given in [KM78] and references therein. Finally, building on results in [Kal86], a version of the BEKM separation algorithm has been given in [WW03] for Euclidean space and the Euclidean sphere.

Our approach to this problem has several advantages over previous approaches. First we gave a unified theory applicable to spaces of constant curvature with both Euclidean and Lorentzian signatures. This approach allows one to solve the different but related problems listed in the introduction. We gave a precise notion of in-equivalence for orthogonal separable coordinate systems in Minkowski space and thereby gave a clear, rigorous and complete classification in this space. The main drawback of our approach is that it is theoretical and not as easy to apply for those who wish to.

Chapter 10

Separation of Natural Hamiltonians

In this chapter we will use the theory developed so far to answer the following questions: Are there any special separable potentials one can construct for a given separable web, especially a KEM web? We will also answer the converse question for KEM webs (equivalently separable webs) in spaces of constant curvature, i.e. what are the separable webs (if any) in which a given potential separates? In other words, we will solve problems (2) and (3) (from the introduction) for spaces of constant curvature. Note that the answer to the first question is known in general (see Eq. (5.4.2)) in separable coordinates. The difference is that we give special potentials which have coordinate-independent formulas. The answer to the second question will involve working out the details of the BEKM separation algorithm which was introduced in Section 6.7.

An answer to the first question will be given in the first section. In the following section, we will answer the second question for a specific potential, namely the Calogero-Moser system. Finally, in the last section, we will answer the second question in the general case.

In order to execute the BEKM separation algorithm in $\mathbb{E}^n_\nu$, we will need the KBD equation in $\mathbb{E}^n_\nu$ and in $\mathbb{E}^n_\nu(\kappa)$. Fix a function $V \in \mathcal{F}(\mathbb{E}^n_\nu)$ and suppose $n > 1$. Then if L is the general CT in $\mathbb{E}^n_\nu$ given by Eq. (9.1.1) and $K_e := \operatorname{tr}(L)G - L$ is its KBDT, then the KBD equation in $\mathbb{E}^n_\nu$ is:

$$\mathrm{d}(K_e \mathrm{d}V) = 0$$

We will often refer to the above equation as just the *KBD equation*. It will be convenient to evaluate the KBD equation in $\mathbb{E}^n_\nu(\kappa)$ via its embedding in $\mathbb{E}^n_\nu$. Then if $\tilde{L}$ is the general CT in $\mathbb{E}^n_\nu(\kappa)$ given in $\mathbb{E}^n_\nu$ by Eq. (9.1.5), let $L := r^2\tilde{L}$ and $K_s := \operatorname{tr}(L)R - L$, then the KBD equation in $\mathbb{E}^n_\nu(\kappa)$ (embedded in $\mathbb{E}^n_\nu$) is:

$$\mathrm{d}(K_s \mathrm{d}V) = 0 \tag{10.0.1}$$

We will often refer to the above equation as the *spherical KBD equation*. We will show

how this equation is derived in Section 10.3.1.

We should also mention here that we carry out the BEKM separation algorithm slightly differently than described in Section 6.7. We construct warped products which decompose reducible OCTs such that the induced CT on the geodesic factor is an ICT as opposed to a Benenti tensor. This allows one to simultaneously construct separable coordinates while carrying out the algorithm, as illustrated by Section 10.2.

10.1 Known Separable Potentials

In this section we will use some notation introduced in Appendix A. Our first method for constructing separable potentials comes from a reinterpretation of Proposition 6.3.7 in terms of separating potentials. Suppose $L \in S^2(M)$ is an OCT and $M \times_\rho F$ is a warped product with $\kappa := \rho^{-2}$ and adpated contravariant metric $G = G' + \tilde{G}$. From Proposition 6.3.7, it follows that we can extend L to an OCT on $M \times_\rho F$ iff there exists $t \in \mathbb{R}$ such that

$$L(\mathrm{d}\kappa) + \kappa \mathrm{d}\,\mathrm{tr}(L) = \mathrm{d}(t\kappa)$$

It then follows from Proposition 5.3.10 that the L-sequence generated by L can be extended to KTs on $M \times_\rho F$. This has already been done in Proposition 6.6.3. It was shown that if K_a are the elements of the L-sequence generated by L, then they admit the following extensions

$$\bar{K}_a := K_a + (\sum_{i=0}^{a} (-t)^i \sigma_{a-i}) \tilde{G}$$

where $\sigma_a := (\wedge^n L^a)^{\wedge *}$ (see Appendix A). If we let $V_a := (\sum_{i=0}^{a} (-t)^i \sigma_{a-i})\kappa$, then it follows from Proposition 4.5.2 that they satisfy:

$$\mathrm{d}V_a = K_a \mathrm{d}\kappa$$

If L is a Benenti tensor, then the above equation implies that the potential κ is separable in the web associated with L. Generalizing these observations, we have the following:

Proposition 10.1.1 (Constructing separable potentials I)
Let $L \in S^2(M)$ is CT. Suppose $\kappa \in \mathcal{F}(M)$ satisfies the following equation

$$L(d\kappa) + \kappa d\,\mathrm{tr}(L) = d(t\kappa) \tag{10.1.1}$$

for some $t \in \mathcal{F}(M)$. Then the functions

$$V_a := (\sum_{i=0}^{a}(-t)^i\sigma_{a-i})\kappa$$

for $a = 0, \ldots, n-1$ *satisfy* $dV_a = K_a d\kappa$ *iff* t *satisfies*

$$L\,dt = t\,dt \qquad \diamondsuit$$

PROOF The proof is a straightforward generalization of Proposition 6.6.3 together with the above observations. ∎

We note that the left hand side of Eq. (10.1.1) is a closed form iff κ satisfies the KBD equation with L. Hence one can check if a given potential satisfies the hypothesis of the above proposition while executing the BEKM separation algorithm.

As a corollary of the proof, we observe the following [Ben04]:

Corollary 10.1.2

Suppose $L \in S^2(M)$ *is CT. If* K_a *are elements of the L-sequence for* $a = 0, \ldots, n-1$*, then the following hold*

$$d\sigma_{a+1} = K_a d\,\mathrm{tr}(L)$$

In particular, if L *is an OCT then the potential* $\mathrm{tr}(L)$ *is separable in any KEM web defined by* L*.* □

PROOF See Proposition B.1.2. ∎

We have another corollary, which is an application of the above proposition to spaces of constant curvature.

Corollary 10.1.3

Suppose $L = A + mr \odot r + 2w \odot r$ *is a CT in* $\mathbb{E}^n_\nu$ *and let* $\tilde{L}$ *be the restriction of* L *to* $\mathbb{E}^n_\nu(\kappa)$*. Let* a *be a covariantly constant vector and let* $V := \langle r, a\rangle^{-2}$*. If* a *is an eigenvector of* A *orthogonal to* w *then* V *satisfies the KBD equation with* L *in* $\mathbb{E}^n_\nu$*. If* a *is an eigenvector of* A *then the restriction of* V *to* $\mathbb{E}^n_\nu(\kappa)$ *satisfies the KBD equation with* $\tilde{L}$ *in* $\mathbb{E}^n_\nu(\kappa)$*.* $\diamondsuit$

PROOF We first consider the case in $\mathbb{E}^n_\nu$. Under these hypothesis it follows by Lemma 9.5.2 that if $\rho := |\langle r, a\rangle|$, then we have:

$$L(\mathrm{d}\log\rho) = \mathrm{d}(\lambda\log\rho + \frac{1}{2}\,\mathrm{tr}(L))$$

for some $\lambda \in \mathbb{R}$. The result then follows from Proposition 10.1.1. A similar proof holds for the case in $\mathbb{E}^n_\nu(\kappa)$, but now the above equation with $\tilde{L}$ follows either by restriction of the one in the ambient space or by Proposition 9.5.12 together with Eq. (6.3.3) from Proposition 6.3.7. ∎

One can naturally construct separable potentials from the above proposition, as the following example shows.

Example 10.1.4 (Constructing separable potentials I)
Suppose $a_1, \ldots, a_n$ is an orthonormal basis for $\mathbb{E}^n_\nu$, then the above corollary implies that the following potential is separable in generalized elliptic coordinates (see Example 9.4.11):

$$V = \sum_i k_i \langle r, a_i \rangle^{-2}$$

for some $k_i \in \mathbb{R}$. In fact this potential is clearly multi-separable. Furthermore we can also obtain a multi-separable potential on $\mathbb{E}^n_\nu(\kappa)$ by restriction. For these potentials the commuting first integrals guaranteed by the Jacobi theorem (see Corollary 5.4.2) can be explicitly calculated. Indeed, this follows by Theorem 6.6.2 together with Proposition 10.1.1. □

There are some additional separable potentials that can be constructed for KEM webs. These potentials arise from a different approach to the theory of separation, see [Bla05, Section 3.3] for the original derivation. They have also been derived using yet another approach in [Lun03]. See [Lun01] for a review.

Let L be a Benenti tensor. Fix a potential $V_1 \in \mathcal{F}(M)$. By Theorem 5.4.1 and Theorem 6.6.2, it follows that V_1 is separable in the web induced by L iff there exist functions $V_1, \ldots, V_n \in \mathcal{F}(M)$ satisfying

$$\mathrm{d}V_{a+1} = K_a \mathrm{d}V_1 \quad a = 0, \ldots, n-1 \tag{10.1.2}$$

Expanding K_a in terms of its recursive definition (see Eq. (A.0.2)), we get:

$$\begin{aligned} \mathrm{d}V_{a+1} &= K_a \mathrm{d}V_1 \\ &= \sigma_a \mathrm{d}V_1 - K_{a-1} L \mathrm{d}V_1 \\ &= \sigma_a \mathrm{d}V_1 - L K_{a-1} \mathrm{d}V_1 \\ &= \sigma_a \mathrm{d}V_1 - L \mathrm{d}V_a \end{aligned}$$

where we have used the fact that each K_a commutes with L since the K_a are polynomials in L. Hence we have that

$$\mathrm{d}V_{a+1} = \sigma_a \mathrm{d}V_1 - L \mathrm{d}V_a \quad a = 0, \ldots, n-1 \tag{10.1.3}$$

A straightforward calculation shows that the above equation is equivalent to Eq. (10.1.2). Now suppose we have functions $V_1, \ldots, V_n \in \mathcal{F}(M)$ satisfying the above equation, i.e. they form a *separable chain*. We are interested in creating new functions $\bar{V}_1, \ldots, \bar{V}_n \in \mathcal{F}(M)$ which form a separable chain. The following proposition does just this.

Proposition 10.1.5 (Constructing separable potentials II [Bla03; Bla05])
Suppose L is an OCT and V_a form a separable chain with respect to L, i.e. they satisfy Eq. (10.1.3). Then the chain $\bar{V}_a$ defined as follows:

$$\bar{V}_a := V_{a+1} - \sigma_a V_1 \tag{10.1.4}$$

is separable. Furthermore the following chain, which is an "inverse" of the first is also separable:

$$\bar{V}_a := V_{a-1} - \frac{\sigma_{a-1}}{\sigma_n} V_n \qquad \bar{V}_1 := -\frac{V_n}{\sigma_n} \tag{10.1.5}$$

where $\sigma_a := (\wedge^n L^a)^{\wedge *}$. □

Proof We will only prove that the first set of functions form a separable chain. Suppose V_a form a separable chain and define $\bar{V}_a$ by Eq. (10.1.4). Following Blaszak in [Bla03, section 4.3], we show that $\bar{V}_a$ form a separable chain:

$$\begin{aligned}
L\mathrm{d}\bar{V}_a - \sigma_a \mathrm{d}\bar{V}_1 &= L\mathrm{d}(V_{a+1} - \sigma_a V_1) - \sigma_a \mathrm{d}(V_2 - \sigma_1 V_1) \\
&= L\mathrm{d}V_{a+1} - \sigma_a L\mathrm{d}V_1 - V_1 L\mathrm{d}\sigma_a - \sigma_a \mathrm{d}V_2 + \sigma_a\sigma_1 \mathrm{d}V_1 + \sigma_a V_1 \mathrm{d}\sigma_1 \\
&= L\mathrm{d}V_{a+1} - \sigma_a L\mathrm{d}V_1 - V_1(L\mathrm{d}\sigma_a - \sigma_a \mathrm{d}\sigma_1) + \sigma_a(\sigma_1 \mathrm{d}V_1 - \mathrm{d}V_2) \\
&= L\mathrm{d}V_{a+1} - \sigma_a L\mathrm{d}V_1 + V_1 \mathrm{d}\sigma_{a+1} + \sigma_a L\mathrm{d}V_1 \\
&= L\mathrm{d}V_{a+1} + V_1 \mathrm{d}\sigma_{a+1} \\
&= -\mathrm{d}V_{a+2} + \sigma_{a+1}\mathrm{d}V_1 + V_1 \mathrm{d}\sigma_{a+1} \\
&= -\mathrm{d}(V_{a+2} - \sigma_{a+1}V_1) \\
&= -\mathrm{d}(\bar{V}_{a+1})
\end{aligned}$$

Which proves the result. ■

We give some applications of the above proposition to spaces of constant curvature in the following example.

Example 10.1.6 (Constructing separable potentials II)
Consider the central CT, $L = A + r \odot r$, in $\mathbb{E}^n_\nu$ where $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Fix Cartesian coordinates (x^i). Assuming each $\lambda_i \neq 0$, by Lemma 9.4.6, we see that

$$\det L = \sigma_n = \left(\prod_{i=1}^{n} \lambda_i\right)\left(1 + \sum_{i=1}^{n} \frac{x_i x^i}{\lambda_i}\right)$$

The second chain given by Eq. (10.1.5) applied to the constant potential gives the following separable potential:

$$V = \frac{1}{1 + \sum_{i=1}^{n} \frac{x_i x^i}{\lambda_i}}$$

Similarly if we let $\tilde{L}$ be the induced CT in $\mathbb{E}^n_\nu(\kappa)$, then by Eq. (9.4.16) we obtain the following separable potential:

$$V = \frac{1}{\sum_{i=1}^{n} \frac{x_i x^i}{\lambda_i}}$$

□

10.2 Example: Calogero-Moser system

We first present an example which separates in several different coordinate systems and hence provides a good example for the BEKM separation algorithm. Our example is the Calogero-Moser system, which will be defined shortly. Another advantage of this example is that its separability properties have been studied by several different authors [HMS05; WW05; WW03; BCR00; Cal69], hence it allows one to compare and contrast different methods. Finally we mention that we obtained this example from [WW03] where an algorithm equivalent to the BEKM separation algorithm was used to study this example.

The n-dimensional Calogero-Moser system is given by the following natural Hamiltonian [Cal08]:

$$H(p, q) = \frac{1}{2} \sum_{i=1}^{n} \left(p_i^2 + \omega^2 q_i^2\right) + \sum_{1 \leq i < j \leq n} \frac{g^2}{(q_i - q_j)^2} \tag{CM}$$

We will take $\omega = 0, g = 1$ for convenience. In this case this Hamiltonian models n point particles moving on a line acted on by forces depending on their relative distances. We can write the potential V as follows:

$$V = \sum_i \langle r, a_i \rangle^{-2}$$

where $a_i = e_k - e_l$ for some $k, l \in \{1, \ldots, n\}$ with $e_i := \partial_i$. Furthermore we let

$$d = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i$$

The following proposition gives the general solution to the KBD equation for the Calogero-Moser system.

Proposition 10.2.1
If V is the potential of the Calogero-Moser system given by Eq. (CM), then the general solution to the KBD equation is:

$$L = c\, d \odot d + 2w\, d \odot r + m\, r \odot r \tag{10.2.1}$$

where $c, w, m \in \mathbb{R}$. Furthermore the restriction of the above CT to $\mathbb{S}^{n-1}$ is a solution of the spherical KBD equation. ♢

PROOF We will first apply Corollary 10.1.3 to show that the above CT is in fact a solution to the KBD equation. Consider the vectors $b_i := e_1 - e_i$ for $i \neq 1$. We will construct the most general CT, $L = A + mr \odot r + 2w \odot r$, for which each vector b_i is an eigenvector of A and orthogonal to w. Observe that none of the b_i are pair-wise orthogonal, they span an $n-1$ dimensional subspace, and

$$\cap_i\, b_i^\perp = (\oplus_i \operatorname{span}\{b_i\})^\perp = \operatorname{span}\{d\}$$

Now suppose A is a self-adjoint operator such that each b_i is an eigenvector of A. Then it follows that A must have $d^\perp$ as an eigenspace, hence $A = kI + cd \odot d$ for some $k, c \in \mathbb{R}$. Thus up to equivalence the above form of L (Eq. (10.2.1)) satisfies our requirements, and it follows by Corollary 10.1.3 that L satisfies the KBD equation with V.

The second statement on the spherical KBD equation follows by a similar argument using Corollary 10.1.3.

We will now show that Eq. (10.2.1) is in fact the most general solution to the KBD equation. Suppose L is the general CT in $\mathbb{E}^n$ given by Eq. (9.1.1). For $\kappa := \rho^{-2}$ where $\rho := (\langle r, a\rangle)^{-2}$, we have from the proof of Lemma 9.5.2 that

$$L(\mathrm{d}\kappa) + \kappa \mathrm{d}\operatorname{tr}(L) = -2\rho^{-3}(Aa + \langle a, w\rangle\, r)$$

Also recall that:

$$\mathrm{d}\log\rho = \frac{a}{\rho}$$
$$\Rightarrow \mathrm{d}\rho^{-3} = -3\rho^{-4} a$$

Hence

$$\begin{aligned}\mathrm{d}(L(\mathrm{d}\kappa) + \kappa \mathrm{d}\,\mathrm{tr}(L)) &= -2\mathrm{d}\rho^{-3} \wedge (Aa + \langle a, w\rangle\, r) \\ &= 6\rho^{-4}[a \wedge (Aa + \langle a, w\rangle\, r)]\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{6}\sum_{i=1} c_i \mathrm{d}(L(\mathrm{d}\kappa_i) + \kappa_i \mathrm{d}\,\mathrm{tr}(L)) &= \sum_{i=1} c_i\rho_i^{-4}[a_i \wedge (Aa_i + \langle a_i, w\rangle\, r)] \\ &= \sum_{i=1} c_i\rho_i^{-4} a_i \wedge Aa_i + \sum_{i=1} c_i\rho_i^{-4} a_i \wedge \langle a_i, w\rangle\, r \\ &= \sum_{i=1} c_i\rho_i^{-4} a_i \wedge Aa_i + \sum_{i=1} (c_i\rho_i^{-4}\langle a_i, w\rangle\, a_i) \wedge r\end{aligned}$$

Denote the above 2-form by ω, then note that for $x \in \mathfrak{X}(\mathbb{E}^n)$ that $\mathcal{L}_x\omega = \mathrm{d}\omega(x)$. Let $\alpha = 6\rho^{-4}[a \wedge (Aa + \langle a, w\rangle\, r)]$, then

$$\alpha(x) = 6\rho^{-4}[\langle a, x\rangle\,(Aa + \langle a, w\rangle\, r) - \langle x, Aa + \langle a, w\rangle\, r\rangle\, a]$$

Suppose x is constant, then

$$\begin{aligned}\mathrm{d}\alpha(x) &= 6\,\langle a, x\rangle\, \mathrm{d}\rho^{-4} \wedge (Aa + \langle a, w\rangle\, r) - 6\rho^{-4}\langle a, w\rangle\, x \wedge a \\ &= 6\,\langle a, x\rangle\,(-4\rho^{-5}) a \wedge (Aa + \langle a, w\rangle\, r) - 6\rho^{-4}\langle a, w\rangle\, x \wedge a\end{aligned}$$

If we take $x = d$, we see that

$$\begin{aligned}\mathcal{L}_d\omega &= -6\sum_i c_i\rho_i^{-4}\langle a_i, w\rangle\, d \wedge a_i \\ &= 6(\sum_i c_i\rho_i^{-4}\langle a_i, w\rangle\, a_i) \wedge d\end{aligned}$$

Hence $\omega \equiv 0$ iff $(\sum_i c_i\rho_i^{-4}\langle a_i, w\rangle\, a_i) \in \mathrm{span}\{d\}$ iff

$$\sum_i c_i\rho_i^{-4}\langle a_i, w\rangle\, a_i = 0$$

If we differentiate the above equation with respect to to e_j, we get

$$\sum_i c_i \rho_i^{-5} \langle a_i, e_j \rangle \langle a_i, w \rangle a_i = 0$$

Since the a_i satisfying $\langle a_i, e_j \rangle \neq 0$ are linearly independent, the above equation implies that $\langle a_i, w \rangle = 0$ for each of these a_i. Thus we see that $\langle a_i, w \rangle = 0$ for each i is a necessary condition for separability. Now we are left to solve the following equation:

$$\sum_i c_i \rho_i^{-4} a_i \wedge A a_i = 0$$

By differentiating the above equation with respect to to e_j, we get

$$\sum_i c_i \rho_i^{-5} \langle a_i, e_j \rangle a_i \wedge A a_i = 0$$

Now differentiate the above with respect to to e_k to get the following:

$$\sum_i c_i \rho_i^{-6} \langle a_i, e_k \rangle \langle a_i, e_j \rangle a_i \wedge A a_i = 0$$

The above sum has precisely one term, it shows that $a_i \wedge A a_i = 0$. Thus a necessary condition for separability is that for each i, a_i is an eigenvector of A.

In conclusion we see that a necessary and sufficient condition for separability is that for each i, a_i is an eigenvector of A satisfying $\langle a_i, w \rangle = 0$. This confirms that Eq. (10.2.1) is in fact the most general solution of the KBD equation by the preceding calculations.■

Remark 10.2.2

When $n = 3$ one can check that the solution to the spherical KBD equation given in the above proposition is the most general. □

Canonical forms We now obtain the canonical forms according to Theorem 9.1.8 for the CTs given by Eq. (10.2.1). First the constants ω_i from Eq. (9.1.2) are given as follows:

$$\omega_0 = m$$
$$\omega_1 = w^2$$

Note that in Euclidean space, one only needs to calculate ω_0 and ω_1 to carry out the classification. We now break into the cases given by Theorem 9.1.8:

Case 1 Elliptic: $\omega_0 \neq 0$

By applying the translation given by Eq. (9.1.3) and changing to a geometrically equivalent CT one obtains:

$$L = cd \odot d + r \odot r \tag{10.2.2}$$

for some $c \in \mathbb{R}$.

Case 2 **Parabolic**: $\omega_0 = 0, \omega_1 \neq 0$

By applying the translation given by Eq. (9.1.4) and changing to a geometrically equivalent CT one obtains:

$$L = 2d \odot r \tag{10.2.3}$$

Case 3 **Cartesian**: $\omega_0 = 0, \omega_1 = 0, c \neq 0$

In this case after changing to a geometrically equivalent CT, we have:

$$L = d \odot d \tag{10.2.4}$$

Hence the three geometrically inequivalent solutions of the KBD equation for the Calogero-Moser potential are given by Eqs. (10.2.2) to (10.2.4). Note that we can obtain these CTs from Eq. (10.2.1) with an appropriate choice of parameters, hence there is no need to apply any isometries.

Determining Separability We now analyze these solutions further to find separable coordinates. We will obtain a compete analysis for the case $n \leq 3$ for convenience. For the following analysis, we fix unit vectors $a \in d^{\perp}$ and $e \in \mathrm{d}^{\perp} \cap a^{\perp}$.

We define N_1 to be the unit sphere in $d^{\perp}$:

$$N_1 = \{p \in d^{\perp} \mid p^2 = 1\}$$

Note if $d^{\perp} = \mathbb{R}a$, then we take $N_1 = \{a\}$. When $\dim N_1 = 1$, we take coordinates on it as follows:

$$\sigma(\theta) = \cos(\theta)a + \sin(\theta)e$$

Case 1 Elliptic with $c \neq 0$

When $n > 2$, this CT is reducible and a warped product decomposition ψ which decomposes this CT is given by Example 9.5.6. First define N_0 as follows:

$$N_0 = \{p \in \mathbb{R}d \oplus \mathbb{R}a \mid \langle a, p\rangle > 0\}$$

For $(p_0, p) = (xa + yd, p) \in N_0 \times N_1$, ψ is given as follows (see Example 9.5.6):

$$\psi(p_0, p) = xp + yd$$

Note that this equation also holds when $n = 2$, but in this case ψ is not a warped product decomposition. Now, to separate V, we have to apply the BEKM separation algorithm with V restricted to N_1 on N_1. Although it will be more convenient to use the spherical KBD equation in $d^\perp$, see the next section for more details.

When $n \leq 3$, no additional steps are needed since in this case $\dim N_1 \leq 1$. Indeed, by Example 9.4.11 L restricted to N_0 is an ICT (in a dense subset) hence L has simple eigenfunctions (locally), and so one obtains separable coordinates for V by taking elliptic coordinates on N_0 Section 6.7. When $c < 0$ we obtain oblate spheroidal coordinates and when $c > 0$ we obtain prolate spheroidal coordinates; see Example 9.6.4 for more details.

Case 2 Parabolic

When $n > 2$, then proceeding as in Example 9.5.6 (see also Section 9.5.1), one observes that the same warped product ψ as in the above case decomposes this CT. When $n \leq 3$, with similar arguments as in the above case, one finds that L locally has simple eigenfunctions, and one obtains separable coordinates for V by taking parabolic coordinates on N_0 Section 6.7. The resulting coordinate system is often called rotationally symmetric parabolic coordinates.

Case 3 Spherical: Elliptic with $c = 0$

In this case, one can check that the following warped product, ψ, decomposes L. For $(p_0, p) = (\rho a, p) \in \mathbb{R}^+ a \times \mathbb{S}^{n-1}$, ψ is given as follows:

$$\psi(p_0, p) = \rho p$$

Now observe that even when $n = 3$, L does not have simple eigenfunctions; in contrast with the previous two cases. To fill the multidimensional eigenspace of L corresponding to $r^\perp$, we have to solve the spherical KBD equation (see the next section for more details). Although when $n = 3$, we can fill this degeneracy by using the solution to the spherical KBD equation given by Proposition 10.2.1. Indeed, that proposition shows that the CT on $\mathbb{S}^{n-1}$ induced by $d \odot d$ is a solution of the spherical KBD equation. Hence by Example 9.6.3, this induced CT is diagonalized in spherical coordinates, and we see that V separates in the following coordinates Section 6.7.

$$\psi(\rho a, p) = \rho(\sin(\phi)(\cos(\theta)a + \sin(\theta)e) + \cos(\phi)d)$$

Case 4 Cartesian

In this case we obtain a product which decomposes L as follows. First let $N_0 = \mathbb{R}d$ and $N_1 = d^{\perp}$, then for $(p_0, p) = (xd, p) \in N_0 \times N_1$, we have:

$$\psi(p_0, p) = xd + p$$

As in the above case, even when $n = 3$, L does not have simple eigenfunctions. Hence we have to apply the BEKM separation algorithm with V restricted to N_1 on N_1. When $n = 3$ one finds that the general solution to the KBD equation is $\tilde{r} \odot \tilde{r}$ where $\tilde{r}$ is the dilatational vector field in N_1. Thus if we take polar coordinates in N_1, we obtain separable coordinates for V. For $(p_0, p) = (xd, y\sigma(\theta)) \in N_0 \times N_1$ with $y > 0$, we have:

$$\psi(p_0, y\sigma(\theta)) = xd + y(\cos(\theta)a + \sin(\theta)e)$$

We conclude with some remarks. First the analysis given above is complete when $n \leq 3$. Although when $n > 3$ the warped product decompositions obtained may allow for partial separation of the Hamilton-Jacobi equation. When $n = 4$ it was shown in [WW05] that no additional solutions to the (spherical) KBD equation could be obtained. Hence our analysis above is complete when $n = 4$.

Furthermore the above analysis holds verbatim for the weighted Calogero-Moser system with unequal masses, which can be modeled using the natural Hamiltonian in $\mathbb{E}^n$ associated with the following potential (see e.g. [WW05, Section 3.3]):

$$V = \sum_{1 \leq i < j \leq n} \frac{g_{ij}}{(m_i q_i - m_j q_j)^2}$$

The only difference is that in this case:

$$d = \frac{1}{\sqrt{M}} \sum_{i=1}^{n} \frac{e_i}{m_i}, \quad M = \sum_{i=1}^{n} \frac{1}{m_i^2}$$

More examples can be found in [WW03, section 7], where an algorithm equivalent to the BEKM separation algorithm is used to determine separability of some natural Hamiltonians defined in $\mathbb{E}^3$. See also [Ben93] where some Kepler type potentials are tested for separability in elliptic coordinates in $\mathbb{E}^2$.

10.3 The BEKM separation algorithm

In this section we show how to execute the BEKM separation algorithm (see Section 6.7 for the general theory) in spaces of constant curvature using the classification of CTs given in the previous chapter. This generalizes the example given in the previous section to arbitrary potentials.

10.3.1 Spherical KBD Equation

We first show how to derive the spherical KBD equation. Suppose $V \in \mathcal{F}(\mathbb{E}^n_\nu)$ is a potential in $\mathbb{E}^n_\nu$ which satisfies the KBD equation with $r \odot r$. Choose $a \in \mathbb{E}^n_\nu$ with $\kappa := a^2 = \pm 1$ and let $\rho := \langle a, r \rangle$. Then we can easily construct a warped product $\psi : \mathbb{R}^+ a \times_\rho \mathbb{E}^n_\nu(\kappa) \to \mathbb{E}^n_\nu$ which decomposes this CT. Let $\tau : \mathbb{E}^n_\nu(\kappa) \to \mathbb{E}^n_\nu$ be the standard embedding of this sphere. Hence to find separable coordinates for V, we have to apply the BEKM separation algorithm with $\tilde{V} := \tau^* V$ in $\mathbb{E}^n_\nu(\kappa)$.

If $\tilde{L}$ is the general CT in $\mathbb{E}^n_\nu(\kappa)$ and $\tilde{K} := \mathrm{tr}(\tilde{L})R - \tilde{L}$ is the KBDT where R is the metric in $\mathbb{E}^n_\nu(\kappa)$, then we have to solve the equation (see Section 6.7):

$$\mathrm{d}(\tilde{K}\mathrm{d}\tilde{V}) = 0$$

Now let K be the lift of $\tilde{K}$ (as a contravariant tensor) to $\mathbb{E}^n_\nu$ via the warped product ψ. Then Proposition 5.4.3 shows that the above equation is locally satisfied iff

$$\mathrm{d}(K\mathrm{d}V) = 0$$

Hence if we calculate this lift of K, we only need to solve the above equation in $\mathbb{E}^n_\nu$. We now proceed to calculate this lift. Note that it is sufficient to find a contravariant tensor in $\mathbb{E}^n_\nu$ which restricts to the KBDT in $\mathbb{E}^n_\nu(\kappa)$ and satisfies $\mathcal{L}_r K = 0$. It will be sufficient to do this for the CT then calculate the KBDT using its defining equation. Also noting that r is a CV, we will do the following calculations in a more general context just using this fact.

Let r be a non-null CV, since $r \odot r$ is an OCT, it follows that any integral manifold of $r^\perp$ is a spherical submanifold. Hence Proposition 9.3.1 shows that any CT on M induces one on any leaf of the foliation induced by $r^\perp$. The following proposition shows how to solve the problem described earlier in this more general context.

Proposition 10.3.1
Suppose L is a CT on M and r is a non-null CV. Let $E := r^\perp$, and $L_E := L|_E$. Then $\tilde{L} := r^2 L_E$ restricts to a CT on any integral manifold of E and it satisfies $\mathcal{L}_r \tilde{L} = 0$ on M where $\tilde{L}$ is in contravariant form. ♢

PROOF The proof of this fact is a straightforward calculation. We first note that since r is a CV with conformal factor ϕ, we have that

$$\nabla_{(i}r_{j)} = \phi g_{ij}$$

Suppose $u, v \in \Gamma(E)$, then

$$\begin{aligned}
(\mathcal{L}_r L_{ij})u^i v^j &= (\nabla_r L_{ij})u^i v^j + L_{ij}(\nabla_u r^i)v^j + L_{ij}(\nabla_v r^i)u^j \\
&= \alpha_{(i}r_{j)}u^i v^j + +2\phi L_{ij}u^i v^j \\
&= 2\phi L_{ij}u^i v^j
\end{aligned}$$

Thus

$$\begin{aligned}
(\mathcal{L}_r L^{ij})u_i v_j &= \mathcal{L}_r(G^{ik}L_{kj}G^{lj})u_i v_j \\
&= -2\phi L_{ij}u^i v^j + (\mathcal{L}_r L_{ij})u^i v^j - 2\phi L_{ij}u^i v^j \\
&= -2\phi L_{ij}u^i v^j
\end{aligned}$$

Finally

$$\begin{aligned}
(\mathcal{L}_r(r^2 L^{ij}))u_i v_j &= r^2(\mathcal{L}_r L^{ij})u_i v_j + (\nabla_r r^2)L^{ij}u_i v_j \\
&= -2r^2\phi L^{ij}u_i v_j + 2r^2\phi L^{ij}u_i v_j \\
&= 0
\end{aligned}$$

Thus since $r^\flat$ is closed, we conclude that $\mathcal{L}_r\tilde{L} = 0$. Also, as we noted earlier, Proposition 9.3.1 implies that $\tilde{L}$ induces a CT on any integral manifold of E. ■

Remark 10.3.2
The above ansatz for $\tilde{L}$ was deduced by studying results obtained by Benenti in [Ben08]. Although one can also obtain $\tilde{L}$ by solving a certain differential equation. □

Now back in $\mathbb{E}^n_\nu$, let r be the dilatational vector field and $L = r^2 L_E$ as in the above proposition. Note that L_E is given in general by Eq. (9.1.5). Let G be the metric of $\mathbb{E}^n_\nu$, then $R = G_E$ is the induced metric on $\mathbb{E}^n_\nu(\frac{1}{r^2})$ and the above proposition shows that $\mathcal{L}_r(r^2 R) = 0$. Hence $r^2 R$ is the r-lift of the metric of $\mathbb{E}^n_\nu(\kappa)$ (up to sign). Hence if $\operatorname{tr}(L)$ is obtained by using the metric of $\mathbb{E}^n_\nu$, the lifted KBDT is given as follows:

$$K_s = (\operatorname{tr}(L)\frac{1}{r^2})(r^2 R) - L = \operatorname{tr}(L)R - L$$

which is the KBDT in $\mathbb{E}^n_\nu(\kappa)$ embedded in $\mathbb{E}^n_\nu$. Also note that it follows from Proposition 4.5.2 that K_s is a KT in $\mathbb{E}^n_\nu$. Also, using Eq. (9.1.5), one can calculate K_s explicitly:

$$K_s = \operatorname{tr}(A)r^2R - \langle r, Ar\rangle\, G - r^2A + 2Ar \odot r$$

Note that since the term $\operatorname{tr}(A)r^2R$ is a multiple of the metric of $\mathbb{E}^n_\nu(\kappa)$, that term can be removed. We summarize our results in the following statement:

Proposition 10.3.3 (Spherical KBD equation)
Suppose $V \in \mathcal{F}(\mathbb{E}^n_\nu)$ is a potential in $\mathbb{E}^n_\nu$ which satisfies the KBD equation with $r \odot r$. Let L be a CT in $\mathbb{E}^n_\nu(\kappa)$ with parameter matrix A. Then V satisfies the KBD equation induced by L in $\mathbb{E}^n_\nu(\kappa)$ iff it satisfies the spherical KBD equation (Eq. (10.0.1)) with L in $\mathbb{E}^n_\nu$. $\diamondsuit$

10.3.2 In pseudo-Euclidean space

We show how to execute the BEKM separation algorithm in pseudo-Euclidean space. Fix a non-trivial solution L of the KBD equation in $\mathbb{E}^n_\nu$. First apply the classification given by Theorem 9.1.8 to L. We will now assume that L is in one of the canonical forms listed in that theorem. If L is a Cartesian CT then the analysis is straightforward, see Section 10.2 for example. So we now assume L is non-degenerate and each generalized eigenspace of A_c has at most one proper generalized eigenvector[1].

First if A_c has no multidimensional (real) eigenspaces, then it is not reducible by Theorem 9.5.11. Hence one obtains separable coordinates for the natural Hamiltonian on the subset where L is an ICT.

Now suppose A_c has multidimensional (real) eigenspaces $W_1, \dots, W_k$. It was shown in Section 9.5.1 that one can obtain data $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$ which determines a warped product decomposition $\psi : N_0 \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu$ in canonical form. Note that ψ decomposes the KBDT, K, associated with L. We now work with K.

We

case later. Suppose K is an orthogonal KT in $\mathbb{E}^n_\nu$ which is decomposed by the warped product ψ just constructed. Furthermore assume that each N_i corresponds to a distinct eigenspace of K. Now we show how to apply the BEKM separation algorithm on the spheres N_i by working only in a pseudo-Euclidean space.

Case 1 N_i is a non-null sphere, i.e. $a_i^2 \neq 0$

Let $W_{i\perp} := W_i^\perp$ and $c_i := \bar{p} - \frac{a_i}{\kappa_i}$. Define $\phi : W_{i\perp} \times W_i \to \mathbb{E}^n_\nu$ to be the standard product decomposition. Embed W_i in $\mathbb{E}^n_\nu$ as follows:

[1] It follows from the classification in Section 8.2.1 that we lose no generality with this assumption in Euclidean or Minkowski space.

$$\tau_i : \begin{cases} W_i & \to \mathbb{E}^n_\nu \\ p_i & \mapsto \phi(c_i, p_i) = c_i + p_i \end{cases}$$

Note that $N_i = W_i(\kappa_i)$ via the above affine embedding of W_i. Let r_i be the dilatational vector field in W_i. By Corollary 9.5.5 and Proposition 5.4.3, it follows that $\tau_i^* V$ satisfies the KBD equation with $r_i \odot r_i$. Hence by Proposition 10.3.3 it is necessary and sufficient to solve the spherical KBD equation on W_i with $\tau_i^* V$.

Case 2 N_i is a null sphere, i.e. $a_i^2 = 0$

Embed N_i in $\mathbb{E}^n_\nu$ as follows (see Eq. (8.4.1)):

$$\tau_i : \begin{cases} N_i & \to \mathbb{E}^n_\nu \\ p_i & \mapsto \psi(\bar{p}, \ldots, \bar{p}, p_i, \bar{p}, \ldots, \bar{p}) = p_i \end{cases}$$

In this case N_i is isometric to V_i which is a pseudo-Euclidean space. Hence the BEKM separation algorithm can be applied on V_i.

In the following section we will show how to apply the BEKM separation algorithm on $\mathbb{E}^n_\nu(\kappa)$.

10.3.3 In Spherical submanifolds of pseudo-Euclidean space

We show how to execute the BEKM separation algorithm in $\mathbb{E}^n_\nu(\kappa)$. First we show to change this to a problem in $\mathbb{E}^n_\nu$. Let $\tilde{V}$ be a potential in $\mathbb{E}^n_\nu(\kappa)$. Note that $\tilde{V}$ can be naturally lifted to a potential in $\mathbb{E}^n_\nu$ satisfying $\mathcal{L}_r \tilde{V} = 0$ using an appropriate coordinate system. Then, one can check that the potential

$$V := \frac{\tilde{V}}{\kappa r^2}$$

in $\mathbb{E}^n_\nu$ satisfies the KBD equation with $r \odot r$ in $\mathbb{E}^n_\nu$ and restricts to $\tilde{V}$ when restricted to $\mathbb{E}^n_\nu(\kappa)$. So we lose no generality in assuming $V \in \mathcal{F}(\mathbb{E}^n_\nu)$ and satisfies the KBD equation with $r \odot r$.

First note that by Proposition 10.3.3, we only need to consider solutions of the spherical KBD equation in $\mathbb{E}^n_\nu$. So let L be a non-trivial solution of the spherical KBD equation (Eq. (10.0.1)). As in the pseudo-Euclidean case, we assume each generalized eigenspace of A has at most one proper generalized eigenvector. In order to execute the BEKM separation algorithm in $\mathbb{E}^n_\nu$, we will need the following lemma:

Lemma 10.3.4
Let L_c be the central CT associated with L and $K_s = \mathrm{tr}(L)R - L$ be the KBDT associated with L. Suppose L_c is reducible and let $\psi : N_0 \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu$ be a warped product which decomposes L_c. Then ψ decomposes K_s. □

Proof This follows from the proof of Proposition 9.5.12. In that proof we obtained the following equation:

$$R^* L_c = \psi_*(\tilde{R}^* \tilde{L}_c + \sum_{i=1}^{k} \lambda_i G_i)$$

Then we have:

$$\begin{aligned} L = r^2 R^* L_c &= \psi_*(\tilde{r}^2 \tilde{R}^* \tilde{L}_c + \sum_{i=1}^{k} \lambda_i \tilde{r}^2 G_i) \\ R &= \psi_*(\tilde{R} + \sum_{i=1}^{k} G_i) \end{aligned}$$

Hence the result follows. ■

Now by Proposition 9.5.12 it follows that L is reducible iff L_c is reducible. Hence if L_c is not reducible, one obtains separable coordinates for the natural Hamiltonian on the subset (of $\mathbb{E}^n_\nu(\kappa)$) where L is an ICT.

If L_c is reducible, then by the above lemma, one can follow the arguments given in the previous section using the warped product decomposition induced by L_c which decomposes the KT K_s. We now give some crucial remarks. Let $\psi : N_0 \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu$ be a warped product decomposition which decomposes L_c and let $\phi : N_0(\kappa) \times_{\rho_1} N_1 \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu(\kappa)$ be an induced warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ as in Theorem 8.4.7. First note that the separable coordinates are constructed using the warped product ϕ. Also because the spherical factors N_i (where $i > 0$) are simultaneously spherical factors of ψ and ϕ (see Theorem D.6.5), there is no difference coming from working in the ambient space.

Chapter 11

Conclusion

In this thesis we have shown how to solve problems (1), (2), and (3) in spaces of constant curvature using concircular tensors. In doing so, we have given a covariant theory of a special class of separable coordinates called Kalnins-Eisenhart-Miller (KEM) coordinates. An overview of this theory can be found in Chapter 2.

In our solution, there is one important problem that has been unresolved. In Minkowski space, M^n, with $n \geq 3$, it is still computationally difficult to find the subset on which a given concircular tensor (CT) is a Benenti tensor. This implies that we still don't have a complete understanding of the separable coordinate systems for these spaces.

In Minkowski space it is well known that non-orthogonal separation can occur [Ben92b; KM79]. Hence it would be interesting to see if the present theory could be generalized to this case. A natural question is if the non-orthogonal separable coordinates in these spaces could be intrinsically characterized using concircular tensors, and conversely if non-orthogonal CTs could be used to define non-orthogonal separable coordinates. See Section 6.8 for more on this.

In Minkowski space it is also known that complex separation can occur [DR07]. Hence similar questions to those stated above for non-orthogonal separation also apply to this case.

One can also try to apply this theory to spaces with non-constant curvature. Some general results related to this idea are given in Section 6.3.3. Applications to general relativity have been studied in [Gro11].

APPENDICES

Appendix A

Cofactor Operators via Exterior Algebra

In this appendix we summarize some results from the book [Win10] by Winitzki. We mainly show how to obtain the cofactor operator (matrix) coordinate-independently and list some related results in this notation. In addition, we present some results on the derivative of a characteristic polynomial and projectors onto eigenspaces. These results are heavily used only in Section 6.6. Throughout this appendix V is an n-dimensional vector space.

Given $A \in \mathrm{End}(V)$ and a basis $\{v_1, \ldots, v_n\}$ for V, define $\wedge^k A^k \in \mathrm{End}(\wedge^k V)$ as follows:

$$\wedge^k A^k(v_1 \wedge \cdots \wedge v_k) = (Av_1 \wedge \cdots \wedge Av_k)$$

We define $\wedge^k A^1 \in \mathrm{End}(\wedge^k V)$ by

$$\wedge^k A^1(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k}(v_1 \wedge \cdots \wedge Av_i \wedge \cdots \wedge v_k)$$

For $1 \leq l \leq k$, the operator $\wedge^k A^l \in \mathrm{End}(\wedge^k V)$ is the most natural generalization of the above definitions (see [Win10, section 3.7]). Also if $l > k$ then $\wedge^k A^l := 0$.

Now for $A \in \mathrm{End}(\wedge^k V)$, we can define the exterior adjoint (transpose) $A^{\wedge *} \in \mathrm{End}(\wedge^{n-k} V)$ as follows:

$$A^{\wedge *} v \wedge w = v \wedge Aw$$

where $v \in \wedge^{n-k} V$ and $w \in \wedge^k V$. One can prove that $A^{\wedge *}$ is well defined either by direct calculation or using a proof analogous to the case of adjoints coming from inner products. The main use of the exterior adjoint is to give a basis independent definition of the determinant and the cofactor operator. Indeed, for any $A \in \mathrm{End}(V)$ we define $\det A \in \mathbb{R}$ by:

$$\det A := (\wedge^n A^n)^{\wedge *}$$

and the *cofactor operator*, $\operatorname{cof}(A) \in \operatorname{End}(V)$ by:

$$\operatorname{cof}(A) := (\wedge^{n-1} A^{n-1})^{\wedge *}$$

One can show that these definitions agree with the conventional ones [Win10]. Our first result is the following combinatorial lemma [Win10, lemma 1, P.138].

Lemma A.0.5
For $A \in \operatorname{End}(V)$, and $1 \leq j \leq k \leq n$ the following holds:

$$\wedge^k (A+I)^j = \sum_{i=0}^{j} \binom{k-i}{k-j} \wedge^k A^i \qquad \square$$

We will only make use of the cases $(k,j) = (n,n), (n-1,j)$ from the above lemma. We have the following corollary:

Corollary A.0.6
For $A \in \operatorname{End}(V)$, the following holds:

$$\det(zI - A) = (\wedge^n (zI - A)^n)^{\wedge *} = \sum_{i=0}^{n} z^{n-i} (-1)^i (\wedge^n A^i)^{\wedge *}$$
$$\operatorname{cof}(zI - A) = (\wedge^{n-1} (zI - A)^{n-1})^{\wedge *} = \sum_{i=0}^{n-1} z^{n-1-i} (-1)^i (\wedge^{n-1} A^i)^{\wedge *} \tag{A.0.1}$$
$\square$

Note that the above formula for the determinant implies that $(\wedge^n A^1)^{\wedge *} = \operatorname{Tr} A$. Another useful formula is the following [Win10, lemma 1, P.152]:

Proposition A.0.7
For $A \in \operatorname{End}(V)$ the following equation holds:

$$(\wedge^{n-1} A^{k-1})^{\wedge *} A + (\wedge^{n-1} A^k)^{\wedge *} = (\wedge^n A^k)^{\wedge *} I \tag{A.0.2}$$
$\square$

Proof First we consider the case $k = n$, then have to prove that:

$$(\wedge^{n-1} A^{n-1})^{\wedge *} A = (\wedge^n A^n)^{\wedge *} I$$

For $\omega = \omega_1 \wedge \cdots \wedge \omega_n$, let $\alpha = \omega_1 \wedge \cdots \wedge \omega_{n-1}$ and $\beta = \omega_n$, then

$$
\begin{aligned}
\wedge^n A^n \omega &= A\omega_1 \wedge \cdots \wedge A\omega_n \\
&= A\omega_1 \wedge \cdots \wedge A\omega_{n-1} \wedge A\omega_n \\
&= (\wedge^{n-1} A^{n-1})\alpha \wedge A\beta \\
&= \alpha \wedge (\wedge^{n-1} A^{n-1})^{\wedge *} A\beta
\end{aligned}
$$

which proves the result. The general case is a consequence of the identity [Win10]:

$$
(\wedge^n A^k)\alpha \wedge \beta = \wedge^{n-1} A^{k-1}\alpha \wedge A\beta + \wedge^{n-1} A^k \alpha \wedge \beta \qquad \blacksquare
$$

Note that the above proposition implies the following well known formula:

$$
\operatorname{cof}(A)A = \det(A)I \tag{A.0.3}
$$

As a corollary of the above proposition, we see that $(\wedge^{n-1}A^k)^{\wedge *}$ can be expressed as a polynomial in A [Win10, exercise 2, P.152].

Corollary A.0.8

For $A \in \operatorname{End}(V)$ the following equation holds:

$$
(\wedge^{n-1} A^k)^{\wedge *} = \sum_{i=0}^{k} (\wedge^n A^{k-i})(-1)^i A^i \tag{A.0.4}
$$

$\square$

The operators $(\wedge^{n-1}A^k)^{\wedge *}$ also admit a recursive formula [Win10, statement 3, P.159]:

Proposition A.0.9 (Leverrier sequence)

For $A \in \operatorname{End}(V)$, let $A_k := (\wedge^{n-1}A^k)^{\wedge *}$ where $1 \leq k \leq n-1$ and $A_0 := I$. Then the following formulas hold:

$$
A_k = \frac{1}{k}\operatorname{Tr}(A_{k-1}A)I - A_{k-1}A \qquad \square
$$

Proof Note that these formulas follow from Eq. (A.0.2) if we can prove that:

$$
\frac{1}{k}\operatorname{Tr}(A_{k-1}A) = (\wedge^n A^k)^{\wedge *}
$$

To prove this, we use the fact that $\operatorname{Tr} A_k = (n-k)(\wedge^n A^k)^{\wedge *}$ (see [Win10, statement 2, P.158]). Then taking the trace of Eq. (A.0.2) shows that:

$$
\begin{aligned}
\operatorname{Tr}((\wedge^{n-1}A^{k-1})^{\wedge *}A) &= n(\wedge^n A^k)^{\wedge *} - \operatorname{Tr}((\wedge^{n-1}A^k)^{\wedge *}) \\
&= n(\wedge^n A^k)^{\wedge *} - (n-k)(\wedge^n A^k)^{\wedge *} \\
&= k(\wedge^n A^k)^{\wedge *}
\end{aligned}
$$

which proves the result. ■

For convenience, we let $\sigma_k := (\wedge^n A^k)^{\wedge *}$ and $A_k := (\wedge^{n-1} A^k)^{\wedge *}$. The following is from [Win10, statement, P.179].

Proposition A.0.10 (Derivative of the Characteristic Polynomial)
Suppose the coefficients of A in some basis are a function t, then for $1 \leq k \leq n$ we have the following:

$$\frac{d\sigma_k}{dt} = \mathrm{Tr}(A_{k-1}\frac{dA}{dt}) \qquad \square$$

Remark A.0.11
When $k = n$ the above formula expressing the derivative of the determinant is called *Jacobi's formula.* □

Proof The proof is given on P. 179 in [Win10]. We will prove the special case when $k = n$ following the proof of lemma 4 in [Win10, P. 177]. Let $v_1, \ldots, v_n$ be a basis for V.

$$\begin{aligned}
\partial_t(\wedge^n A^n)v_1 \wedge \cdots \wedge v_n &= \partial_t((\wedge^n A^n)v_1 \wedge \cdots \wedge v_n) \\
&= \partial_t(Av_1 \wedge \cdots \wedge Av_n) \\
&= \sum_{k=1}^{n} Av_1 \wedge \cdots \wedge (\partial_t A)v_k \wedge \cdots \wedge Av_n \\
&= \sum_{k=1}^{n} v_1 \wedge \cdots \wedge (\wedge^{n-1}A^{n-1})^{\wedge *}(\partial_t A)v_k \wedge \cdots \wedge v_n \\
&= \wedge^n (A_{n-1}\partial_t A)^1 v_1 \wedge \cdots \wedge v_n \\
&= \mathrm{Tr}(A_{n-1}\partial_t A)v_1 \wedge \cdots \wedge v_n
\end{aligned}$$

The last equality follows from Corollary A.0.6. ■

The final fact we will need is the following [Win10, statement 2, P.195]:

Proposition A.0.12 (Projectors onto Eigenspaces)
Suppose $A \in \mathrm{End}(V)$ and λ is an eigenvalue of A with geometric and algebraic multiplicity k, then the operator

$$P_\lambda^k = \frac{(-1)^{n-k}}{\sigma_{n-k}}(\wedge^{n-1}(\lambda I - A)^{n-k})^{\wedge *}$$

is a projector onto the eigenspace corresponding to λ. □

Appendix B

Nijenhuis tensors, Haantjes tensors and Integrability of Eigenspaces

In this appendix M is a manifold of dimension n and T is an endomorphism of TM (i.e. a $\binom{1}{1}$-tensor). We assume the reader is familiar with Section 3.4. The theory presented in this appendix is motivated by the following question: Let $\mathcal{E} = (E_i)_{i=1}^k$ where E_i are the eigenspaces of T (which are assumed to be distributions), suppose $TM = \bigoplus_{i=1}^k E_i$. Then are there tensorial conditions depending only on T (and its derivatives) that characterize when $\mathcal{E}$ is an integrable net? Nijenhuis, Haantjes and colleagues have answered this question in the affirmative. They have provided conditions in terms of the Haantjes tensor (of T) which is defined in terms of the Nijenhuis tensor (of T). We shall see that the Nijenhuis tensor (which is a byproduct of the solution to this problem) is a useful tool in integrable systems theory in its own right.

We give original references throughout this appendix. For contemporary references mainly related to integrable systems theory see [GVY08; Bog06].

The primary definition is the following:

Definition B.0.13 (Nijenhuis tensor [Nij51])
If T is an endomorphism of TM, then the *Nijenhuis tensor (torsion)* of T is a $\binom{1}{2}$-tensor skew-symmetric in its covariant components, denoted by N_T and is defined as follows:

$$N_T(u, v) := T^2[u, v] + [Tu, Tv] - T([Tu, v] + [u, Tv])$$

Furthermore T is called *torsionless* if its Nijenhuis tensor vanishes. □

The following proposition will make it clear that N_T is actually a tensor. It gives equivalent definitions of the Nijenhuis tensor.

Proposition B.0.14 (Equivalent Definitions of The Nijenhuis tensor)
Suppose T is an endomorphism of TM, then the following are equivalent definitions of N_T:

1. $N_T(u, v) = (\mathcal{L}_{Tu}T - T\mathcal{L}_u T)v$ *for all* $u, v \in \mathfrak{X}(M)$ *where* $\mathcal{L}$ *is the Lie derivative*

2. $N_T(u, v) = (\nabla_{Tu}T)v - (\nabla_{Tv}T)u - T((\nabla_u T)v - (\nabla_v T)u)$ *for all* $u, v \in \mathfrak{X}(M)$ *where* ∇ *is a torsion-free connection*

3. $N_T = \frac{1}{2}[T, T]$ *where* $[\cdot, \cdot]$ *is the Frölicher-Nijenhuis bracket [FN56]* □

Proof We prove the first equation as follows

$$\begin{aligned} T^2[u, v] + [Tu, Tv] - T([Tu, v] + [u, Tv]) &= T^2[u, v] - \mathcal{L}_{Tv}Tu + T(\mathcal{L}_v Tu + \mathcal{L}_{Tv}u) \\ &= T^2[u, v] - T\mathcal{L}_{Tv}u - (\mathcal{L}_{Tv}T)u \\ &\quad + T(T\mathcal{L}_v u + (\mathcal{L}_v T)u + \mathcal{L}_{Tv}u) \\ &= -(\mathcal{L}_{Tv}T)u + T(\mathcal{L}_v T)u \\ &= (T(\mathcal{L}_v T) - (\mathcal{L}_{Tv}T))u \end{aligned}$$

Thus $N_T(u, v) = (T\mathcal{L}_v T - \mathcal{L}_{Tv}T)u$, hence $N_T(u, v) = (\mathcal{L}_{Tu}T - T\mathcal{L}_u T)v$.

Now suppose ∇ is a torsion-free connection, then $[u, v] = \nabla_u v - \nabla_v u$ for all $u, v \in \mathfrak{X}(M)$, thus

$$\begin{aligned} &T^2[u, v] + [Tu, Tv] - T([Tu, v] + [u, Tv]) \\ &= T^2[u, v] + [Tu, Tv] - T(\nabla_{Tu}v - \nabla_v Tu + \nabla_u Tv - \nabla_{Tv}u) \\ &= T^2[u, v] + [Tu, Tv] - T(\nabla_{Tu}v - (\nabla_v T)u - T\nabla_v u + (\nabla_u T)v + T\nabla_u v - \nabla_{Tv}u) \\ &= T^2[u, v] + T^2(\nabla_v u - \nabla_u v) + [Tu, Tv] - T(\nabla_{Tu}v - (\nabla_v T)u + (\nabla_u T)v - \nabla_{Tv}u) \\ &= \nabla_{Tu}Tv - \nabla_{Tv}Tu - T(\nabla_{Tu}v - (\nabla_v T)u + (\nabla_u T)v - \nabla_{Tv}u) \\ &= (\nabla_{Tu}T)v - (\nabla_{Tv}T)u - T(-(\nabla_v T)u + (\nabla_u T)v) \end{aligned}$$

The equation in terms of the Frölicher-Nijenhuis bracket is included for completeness, see for example [GVY08, Equation 13.37] for more details. ■

Remark B.0.15

Note in coordinates the equation for N_T in terms of a torsion-free connection ∇ can be written as follows[1]:

$$(N_T)^k{}_{ij} = 2(T^l{}_{[i}\nabla_{|l|}T^k{}_{j]} - T^k{}_l \nabla_{[i}T^l{}_{j]})$$

[1]The bar around the index l means that it's excluded from the anti-symmetrization.

Proof

$$\begin{aligned}
N_T(u,v) &= (\nabla_{Tu}T)v - (\nabla_{Tv}T)u - T((\nabla_u T)v - (\nabla_v T)u)\\
(N_T)^k{}_{ij}u^iv^j &= T^l{}_i u^i(\nabla_l T^k{}_j)v^j - T^l{}_j v^j(\nabla_l T^k{}_i)u^i - T^k{}_l(u^i(\nabla_i T^l{}_j)v^j - v^j(\nabla_j T^l{}_i)u^i)\\
&= (T^l{}_i\nabla_l T^k{}_j - T^l{}_j\nabla_l T^k{}_i - T^k{}_l(\nabla_i T^l{}_j - \nabla_j T^l{}_i))u^iv^j\\
&= 2(T^l{}_{[i}\nabla_{|l|}T^k{}_{j]} - T^k{}_l\nabla_{[i}T^l{}_{j]})u^iv^j
\end{aligned}$$

■

Remark B.0.16
Note that the first characterization implies that T is torsionless iff for every $v \in \mathfrak{X}(M)$ we have:

$$\mathcal{L}_{Tv}T = T\mathcal{L}_v T \tag{B.0.1}$$

We say that a vector field v is a symmetry of T if $\mathcal{L}_v T = 0$. The above equation shows the remarkable property that if T is torsionless, then T maps symmetries to symmetries.□

The Nijenhuis tensor is fundamental in this theory, but it alone is not enough to answer the question posed at the beginning of this appendix. For this we need to introduce the Haantjes tensor:

Definition B.0.17 (Haantjes tensor [Haa55])
If T is an endomorphism of TM, then the *Haantjes tensor (torsion)* of T is a $\binom{1}{2}$-tensor skew-symmetric in its covariant components, denoted by H_T and is defined as follows:

$$H_T(u,v) := T^2 N_T(u,v) + N_T(Tu,Tv) - T(N_T(Tu,v) + N_T(u,Tv))$$

□

The fact that H_T is a tensor follows from the fact that N_T is a tensor. Now we need the following lemma, the proof of which follows by a direct calculation.

Lemma B.0.18
Suppose T is an endomorphism of TM. Let X, Y be eigenvector fields of T with eigenfunctions λ, μ respectively. Then N_T satisfies the following:

$$N_T(X,Y) = (T-\lambda)(T-\mu)[X,Y] + (\lambda-\mu)((Y\lambda)X + (X\mu)Y) \tag{B.0.2}$$

□

Now we can prove the main theorem in this theory:

Theorem B.0.19 (Haantjes [Haa55; FN56])
Suppose T is an endomorphism of TM with eigenspaces $\mathcal{E} = (E_i)_{i=1}^k$. Assume each eigenspace is a distribution and together they satisfy:

$$TM = \bigoplus_{i=1}^k E_i$$

In other words T is point-wise diagonalizable. Then $\mathcal{E}$ is an integrable net iff the Haantjes tensor of T vanishes. □

PROOF We follow the proof given in [GVY08] which is originally from [FN56]. Suppose X, Y are eigenvector fields of T with eigenfunctions λ, μ respectively. Then using the above lemma one can calculate the following:

$$H_T(X,Y) = (N-\lambda)^2(N-\mu)^2[X,Y]$$

Now we note that if $\lambda \neq \mu$, since T is point-wise diagonalizable $H_T(X,Y) = 0$ iff

$$(N-\lambda)(N-\mu)[X,Y] = 0$$

This equation holds iff $[X,Y] \in E_\lambda \oplus E_\mu$; it holds for arbitrary X, Y iff $E_\lambda \oplus E_\mu$ is Frobenius integrable. Similarly if $\lambda = \mu$, then $H_T(X,Y) = 0$ iff

$$(N-\lambda)[X,Y] = 0$$

This equation holds iff $[X,Y] \in E_\lambda$; it holds for arbitrary X, Y iff E_λ is Frobenius integrable. The theorem then follows from Theorem 3.4.3. ■

Now we prove another important theorem which follows from the above lemma.

Theorem B.0.20 (Nijenhuis [Nij51])
Suppose T is an endomorphism of TM with eigenspaces $\mathcal{E} = (E_i)_{i=1}^k$ and corresponding eigenfunctions $(\lambda_i)_{i=1}^k$. Assume each eigenspace is a distribution and together they satisfy:

$$TM = \bigoplus_{i=1}^{k} E_i$$

In other words T is point-wise diagonalizable. Then $\mathcal{E}$ is an integrable net with each eigenfunction λ_i depending only on[2] E_i iff the Nijenhuis tensor of T vanishes. □

PROOF First assume that the Nijenhuis tensor of T vanishes. Then from the Haantjes theorem above, $\mathcal{E}$ is an integrable net. Furthermore by Eq. (B.0.2) in the above lemma it follows that each eigenfunction λ_i depends only on E_i.

Conversely if $\mathcal{E}$ is an integrable net and each eigenfunction λ_i depends only on E_i then by Eq. (B.0.2) in the above lemma it follows that the Nijenhuis tensor of T vanishes identically. ■

The following optional result is a straightforward consequence of the Nijenhuis theorem.

[2] By this we mean that $Y(\lambda_i) = 0$ for every $Y \in E_j$ and $j \neq i$.

Corollary B.0.21 (Integrability of almost product structures)
Suppose T is an endomorphism of TM defining an almost product structure, i.e. $T^2 = I$. Then the almost product structure is integrable iff $N_T = 0$. □

The above result can be generalized to almost complex structures (i.e. $T^2 = -I$) as well; this is known as the Newlander-Nirenberg theorem.

B.1 Properties of Torsionless Tensors

In this section we will list some identities satisfied by torsionless tensors which are used in the thesis. The contemporary references given earlier have more results on these tensors. We will make use of some notations and results from Appendix A; we will mainly be applying Proposition A.0.10.

Proposition B.1.1 (Jacobi's formula for Torsionless Tensors)
Suppose T is a torsionless tensor. Then for every $v \in \mathfrak{X}(M)$ we have:

$$\mathcal{L}_{Tv} \det T = \det T \mathcal{L}_v(\operatorname{Tr} T)$$

In terms of the adjoint T^*, this equation can be written:

$$T^* d(\det T) = \det T\, d(\operatorname{Tr} T)$$ □

Proof This is a consequence of Proposition A.0.10 when $k = 1$. Indeed that proposition implies that:

$$\mathcal{L}_v \det T = \operatorname{Tr}((\wedge^{n-1} T^{n-1})^{\wedge *} \mathcal{L}_v T)$$

for any $v \in \mathfrak{X}(M)$. Using the fact that T is torsionless, we have the following:

$$\begin{aligned} \mathcal{L}_{Tv} \det T &= \operatorname{Tr}((\wedge^{n-1} T^{n-1})^{\wedge *} \mathcal{L}_{Tv} T) \\ &\overset{(B.0.1)}{=} \operatorname{Tr}((\wedge^{n-1} T^{n-1})^{\wedge *} T \mathcal{L}_v T) \\ &= (\det T) \mathcal{L}_v(\operatorname{Tr} T) \end{aligned}$$

where in the last line we used the fact that $(\wedge^{n-1} T^{n-1})^{\wedge *} T = (\det T) I$. ■

More generally, we have the following formulas:

Proposition B.1.2
Suppose T is a torsionless tensor. For $1 \le k \le n$, let $\sigma_k := (\wedge^n T^k)^{\wedge *}$, then for every $v \in \mathfrak{X}(M)$ we have:

$$\mathcal{L}_{Tv}\sigma_k = \sigma_k \mathcal{L}_v \operatorname{Tr} T - \mathcal{L}_v \sigma_{k+1}$$

Equivalently, if for $1 \leq k \leq n-1$ *we let* $S_k := (\wedge^{n-1}T^k)^{\wedge *}$, *then for* $1 \leq k \leq n$

$$d\sigma_k = S^*_{k-1} d(\operatorname{Tr} T) \tag{B.1.1}$$

□

Proof Proposition A.0.10 implies that for $v \in \mathfrak{X}(M)$ we have:

$$\mathcal{L}_v \sigma_k = \operatorname{Tr}((\wedge^{n-1}T^{k-1})^{\wedge *}\mathcal{L}_v T)$$

Thus

$$\begin{aligned}
\mathcal{L}_{Tv}\sigma_k &= \operatorname{Tr}((\wedge^{n-1}T^{k-1})^{\wedge *}\mathcal{L}_{Tv} T) \\
&\overset{(B.0.1)}{=} \operatorname{Tr}((\wedge^{n-1}T^{k-1})^{\wedge *} T \mathcal{L}_v T) \\
&\overset{(A.0.2)}{=} \sigma_k \operatorname{Tr} \mathcal{L}_v T - \operatorname{Tr}((\wedge^{n-1}T^{k})^{\wedge *}\mathcal{L}_v T) \\
&= \sigma_k \operatorname{Tr} \mathcal{L}_v T - \mathcal{L}_v \sigma_{k+1}
\end{aligned}$$

If T^* denotes the adjoint of T, then we have:

$$\mathrm{d}\sigma_k = \sigma_{k-1}\mathrm{d}(\operatorname{Tr} T) - T^* \mathrm{d}\sigma_{k-1}$$

Thus Eq. (B.1.1) follows by induction on k from the above equation and the recursive equation Eq. (A.0.2) for S^*_k. ■

Appendix C

Self-adjoint operators in pseudo-Euclidean space

Self-adjoint operators are ubiquitous in pseudo-Riemannian geometry and hence in general relativity as well. Any linear operator metrically equivalent to a symmetric contravariant tensor is self-adjoint. The Ricci tensor, the Hessian of a smooth function, the shape operator associated with a pseudo-Riemannian hypersurface or an umbilical pseudo-Riemannian submanifold [O'N83, Definition 4.18], and Killing and conformal Killing tensors are all examples of such tensors. In general relativity the energy-momentum tensor is one as well. Hence their algebraic classification is an important problem.

In this section we call a pseudo-Euclidean space V a scalar product space, following [O'N83] (we will give more details shortly). Recall, a linear operator T on a scalar product space V is said to be self-adjoint if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in V$. Following O'Neil's solution given in exercises 18-19 in [O'N83, P. 260-261], we will obtain a canonical form for self-adjoint operators. Specifically, motivated by the Jordan canonical form, we develop an algorithm to find a Jordan canonical basis for a self-adjoint operator which also gives a canonical form for the scalar product. Our derivation has some advantages: it only depends on some results from Jordan form theory, we are able to prove existence and uniqueness of the canonical form independently of the corresponding results from Jordan form theory (i.e. ours results have few dependencies), and we obtain a simple algorithm to calculate the canonical forms for self-adjoint operators. The draw back is that our solution is less general than others (see for example [GL05]).

Finally, we note that the contents of this appendix are from [Raj14a]. They are included here for completeness.

C.1 Preliminaries on Scalar product spaces

We will assume the reader is familiar with Sections 1.4.1 and 8.1. In this appendix, we will also work with the complexification of a real vector space V denoted $V^{\mathbb{C}}$. We define the complexified bilinear form to be the symmetric bilinear form in $V^{\mathbb{C}}$ obtained from the real one by a linear extension. Note that the complexified bilinear form is symmetric, in contrast with the usual Hermitian form which is not symmetric. It follows immediately from the definition that the complexified bilinear form is non-degenerate iff the real bilinear form is non-degenerate. Thus a real scalar product space, V, can be canonically complexified to a complex scalar product space, hereafter denoted $V^{\mathbb{C}}$.

C.2 Preliminaries from Operator Theory

Given a complex scalar λ, a non-zero vector $x \in V$ is called a generalized eigenvector for T corresponding to λ if $(T - \lambda I)^p x = 0$ for some positive integer p.

Definition C.2.1 (Generalized Eigenspaces)
Let T be a linear operator on finite-dimensional complex vector space V and let λ be an eigenvalue of T. The generalized eigenspace (g-space) corresponding to λ, denoted K_λ, is the subset of V defined by:

$$K_\lambda = \{x \in V : (T - \lambda I)^p(x) = 0 \text{ for some positive integer p}\} \qquad \square$$

We say a set of distinct scalars $\lambda_1, ..., \lambda_k$ are the spectrum of T if they constitute all eigenvalues of T. Furthermore the kernel of an operator T is denoted by $\ker T$ or $N(T)$.

The following results concerning the g-spaces of a linear operator are proven in [FIS03, Section 7.1].

Theorem C.2.2
Let T be a linear operator on finite-dimensional complex vector space V. Suppose λ and μ are distinct eigenvalues of T, then the following statements are true:

1. *K_λ is a non-zero T invariant subspace of V*
2. *$K_\lambda \cap K_\mu = \{0\}$*
3. *Let $U = (T - \lambda I)|_{K_\mu}$, then K_μ is $(T - \lambda I)$-invariant and U is a bijection.*
4. *If m is the algebraic multiplicity of λ then $K_\lambda = N(T - \lambda I)^m$ and $\dim K_\lambda \leq m$.*
5. *If $\lambda_1, ..., \lambda_k$ is the spectrum of T, then $V = \bigoplus_{i=1}^{k} K_{\lambda_i}$* $\square$

Hence the above theorem implies T is block diagonal in a basis adapted to the g-spaces.

Definition C.2.3

Let T be a linear operator on finite-dimensional complex vector space E and let x be a generalized eigenvector of T corresponding to the eigenvalue λ. Suppose that p is the smallest positive integer such that $(T-\lambda I)^p(x) = 0$. Then the ordered set

$$\{(T-\lambda I)^{p-1}(x), (T-\lambda I)^{p-2}(x), \ldots, (T-\lambda I)(x), x\}$$

is called a cycle (p-cycle) of generalized eigenvectors for T with eigenvalue λ. $(T-\lambda I)^{p-1}(x)$ and x are called the initial vector and the end vector of the cycle, respectively. We say that the cycle has length p and x generates a cycle (p-cycle) of generalized eigenvectors. $\square$

We first note that the subspace spanned by a p-cycle has dimension p. We also observe that a given cycle of generalized eigenvectors generated by x with eigenvalue λ lie in K_λ. Also T restricted to this cycle has the following matrix representation:

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & \ddots & & 0 \\ & & \ddots & 1 & \\ & & & \lambda & 1 \\ & 0 & & & \lambda \end{pmatrix}$$

We denote $U_\lambda = T - \lambda I$ and if λ is fixed we remove the subscript and refer to $U_\lambda = U$.

Suppose T is a real linear operator and let λ be an eigenvalue with non-zero imaginary part. Suppose x generates a cycle of generalized eigenvectors of length p with eigenvalue λ whose end vector has linearly independent real and imaginary parts. Then it follows that $\overline{x}$ generates a cycle of generalized eigenvectors of length p with eigenvalue $\overline{\lambda}$ which is linearly independent of the cycle generated by x. We denote the real subspace generated by these vectors as $K_{\lambda\oplus\overline{\lambda}}$ and call this the *real subspace* spanned by the cycle generated by x. If $\lambda \in \mathbb{R}$, then this real subspace is just K_λ.

Knowledge of the Jordan canonical form is unnecessary for our derivation. Although for readers familiar with it, note that $K_\lambda \simeq N_1 \oplus \dfrac{N_2}{N_1} \oplus \ldots \oplus \dfrac{N_p}{N_{p-1}}$ where $N_i = \ker U_\lambda^i$. This shows the non-uniqueness of a given Jordan canonical basis. We will use this fact to find a Jordan canonical basis for a self-adjoint operator adapted to the scalar product.

In order to prove the uniqueness of the metric-Jordan canonical form of a self-adjoint operator we will need some theory on symmetric bilinear forms. First a *diagonal representation* of a symmetric bilinear form is a basis in which the matrix representation of the form is diagonal.

Theorem C.2.4 (Sylvester's Law of Inertia)
For any symmetric bilinear form defined over a real vector space, the number of positive diagonal entries and negative diagonal entries in a diagonal representation is independent of the diagonal representation.

For any symmetric bilinear form defined over a complex vector space, the number of non-zero diagonal entries in a diagonal representation is independent of the diagonal representation. □

Proof For the real case, see Theorem 6.38 in [FIS03] or Theorem 6.8 in [Jac12]. For the complex case, see Theorem 6.6 in [Jac12] ■

C.3 Existence of the metric-Jordan canonical form

In this section we will show how to obtain the canonical form, culminating in Theorem C.3.7. First we need some properties of self-adjoint operators.

Theorem C.3.1 (Fundamental Properties of Self-Adjoint Operators)
Suppose V is a scalar product space and T is a self-adjoint operator on V. Suppose $H \subseteq V$ is an invariant subspace of T. Then

1. $T(H^\perp) \subseteq H^\perp$, *i.e.* $H^\perp$ *is an invariant subspace of* T.
2. $(\ker T)^\perp = \operatorname{range} T$ *and* $V = \ker T \oplus \operatorname{range} T$ *iff either* $\ker T$ *or* $\operatorname{range} T$ *is a non-degenerate subspace*
3. *Any polynomial in* T *is self-adjoint.* □

Proof The proofs are immediate. ■

Remark C.3.2
The first statement of the above theorem also holds for unitary operators on V, as noted by O'Neil in [O'N83, Section 9.4]. □

The idea behind obtaining the canonical forms is as follows. First suppose that T is a self-adjoint operator on a scalar product space. When E is a Euclidean space, one can easily diagonalize T using property 1 and the fact that self-adjoint operators in Euclidean space have real eigenvalues. Indeed, after one finds a single eigenvector v, one can use property 1 to deduce that the subspace orthogonal to v must be T-invariant. Since in Euclidean space the subspace orthogonal to v must be complementary to v, one can repeat this procedure to find a basis of eigenvectors for T.

For general indefinite scalar products, our goal will be to find a cycle of generalized eigenvectors for T such that they span a non-degenerate subspace. Then as in the

Euclidean case, we can use property 1 to inductively build a Jordan canonical basis for T. We will now develop a series of lemmas to show that any self-adjoint operator admits a cycle of generalized eigenvectors whose span is a non-degenerate subspace. Then we will combine these lemmas in Theorem C.3.7 which shows how to obtain a Jordan canonical basis for T which also puts the scalar product in a canonical form.

The following theorem starts us off by showing that the g-spaces of a self-adjoint operator are always non-degenerate, in fact it says even more:

Lemma C.3.3

Suppose $V^{\mathbb{C}}$ is a scalar product space and T is a real self-adjoint operator on $V^{\mathbb{C}}$. Let λ and μ be distinct eigenvalues of T, then $K_\lambda \perp K_\mu$, hence if $\lambda_1, ..., \lambda_k$ is the spectrum of T then by Theorem C.2.2, $V^{\mathbb{C}} = \bigoplus_{i=1}^{k} K_{\lambda_i}$.

As an immediate corollary we find that each generalized eigenspace is a non-degenerate subspace. □

Proof Suppose $x \in K_\lambda$ and $y \in K_\mu$. Suppose $U_\lambda^p(x) = 0$, since $\mu \neq \lambda$ Theorem C.2.2 says that U_λ is a bijection when restricted to K_μ, hence there exists a $z \in K_\mu$ such that $y = U_\lambda^p(z)$. Since U_λ^p is self-adjoint, property 2 implies that $\langle x, y\rangle = 0$.

Thus $K_\lambda \perp K_\mu$. As a consequence of this and Theorem C.2.2 we see that $E = K_\lambda \oplus K_\lambda^\perp$, hence K_λ is non-degenerate. ■

Suppose V is a scalar product space and T is a self-adjoint operator on V. Suppose λ is an eigenvalue of T and $x \in K_\lambda$ generates a cycle of generalized eigenvectors of T of length p. Let $U = (T - \lambda I)$ and $v_i = U^{p-i}x$ for $i \in \{1, ..., p\}$. Then observe that

$$\begin{aligned} \langle v_i, v_j\rangle &= \left\langle U^{p-i}x, U^{p-j}x\right\rangle \\ &= \left\langle U^{2p-i-j}x, x\right\rangle \end{aligned} \tag{C.3.1}$$

If $i + j \leq p$ then by property 2 and the fact that $U^p x = 0$ the above equation implies that $\langle v_i, v_j\rangle = 0$. If $i + j > p$ then the above equation implies that $\langle v_i, v_j\rangle$ only depends on the sum $i + j$. Thus in a cycle of length p there are only p scalar products that are variable and the above equation shows us that we only need to deal with the products $\langle v_i, v_p\rangle$. The following lemma will show that for every g-space we can always find a generator of a cycle such that $\langle v_1, v_p\rangle \neq 0$.

Lemma C.3.4

Suppose V is a scalar product space and T is a self-adjoint operator on V. Fix an eigenvalue λ of T and let $U = (T - \lambda I)|_{K_\lambda}$. Suppose $k \geq 0$ satisfies $U^k \neq 0$, then there exists an $x \in K_\lambda$ such that $\left\langle U^k x, x\right\rangle \neq 0$. □

PROOF Suppose to the contrary that $\left\langle U^k(x), x\right\rangle = 0$ for all $x \in K_\lambda$. Define a bilinear form $[\cdot,\cdot] : K_\lambda \times K_\lambda \to \mathbb{F}$ by $[x,y] = \left\langle U^k(x), y\right\rangle$. Since U^k is self-adjoint, $[\cdot,\cdot]$ is a symmetric bilinear form. Thus by the polarization identity, it follows that for any $x, y \in K_\lambda$

$$0 = [x,y] = \left\langle U^k(x), y\right\rangle$$

Now, since $U^k \neq 0$, there exists an $x \in K_\lambda$ such that $U^k x \neq 0$. But by Lemma C.3.3 the scalar product is non-degenerate, hence the above equation implies that $U^k x = 0$, a contradiction. Hence the conclusion holds. ∎

Assuming $\langle v_1, v_p\rangle \neq 0$, the following proposition shows how to adapt the cycle so that any other remaining scalar products are zero.

Lemma C.3.5

Suppose the v_i are as defined as above for a cycle of generalized eigenvectors of T generated by $x \in K_\lambda$. Let $H \subseteq K_\lambda$ be the subspace corresponding to the cycle generated by x. If $\langle v_1, v_p\rangle \neq 0$, then we can choose an $x' \in H$ such that x' generates a cycle of generalized eigenvectors $v_i' = U^{p-i}x'$ of length p spanning H, such that $v_1, \ldots, v_p$ forms a skew-normal sequence of sign $\operatorname{sgn}\langle v_1, v_p\rangle$ if $\lambda \in \mathbb{R}$ or 1 if $\lambda \in \mathbb{C}\setminus\mathbb{R}$. □

PROOF Suppose first that $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then let $v_p' = \langle v_1, v_p\rangle^{-\frac{1}{2}} v_p$ where any square root is fine. Then observe that:

$$\begin{aligned}\left\langle v_1', v_p'\right\rangle &= \langle v_1, v_p\rangle^{-1}\langle v_1, v_p\rangle \\ &= 1\end{aligned}$$

If $\lambda \in \mathbb{R}$ then let $v_p' = |\langle v_1, v_p\rangle|^{-\frac{1}{2}} v_p$. Then observe that:

$$\begin{aligned}\left\langle v_1', v_p'\right\rangle &= |\langle v_1, v_p\rangle|^{-1}\langle v_1, v_p\rangle \\ &= \pm 1\end{aligned}$$

Thus we can assume that $|\langle v_1, v_p\rangle| = 1$. Inductively suppose that $|\langle v_1, v_p\rangle| = 1$ and that $\langle v_i, v_p\rangle = 0$ for $2 \leq i \leq k-1$ for some $k \geq 2$.

Let $v_p' = v_p + a v_{p-k+1}$ where a is to be determined. Now for $i \in \{1, ..., p\}$

$$\begin{aligned}v_i' &= U^{p-i} v_p' \\ &= U^{p-i} v_p + a U^{p-i} v_{p-k+1} \\ &= v_i + a v_{i-k+1}\end{aligned}$$

Observe that $v_i' = v_i$ if $i - k + 1 \leq 0$, i.e. $i \leq k - 1$. The above equation also shows that each $v_i' \in H$ and since $v_1' = v_1 \neq 0$ the cycle generated by v_p' has length p and thus forms a basis for H. Now using the fact that $\langle v_i, v_j \rangle$ only depends on $i + j$, we find that:

$$\begin{aligned}
\left\langle v_k', v_p' \right\rangle &= \langle v_k + av_1, v_p + av_{p-k+1} \rangle \\
&= \langle v_k, v_p \rangle + a \langle v_k, v_{p-k+1} \rangle + a \langle v_1, v_p \rangle + a^2 \langle v_1, v_{p-k+1} \rangle \\
&= \langle v_k, v_p \rangle + 2a \langle v_1, v_p \rangle
\end{aligned}$$

where $\langle v_1, v_{p-k+1} \rangle = 0$ since $p - k + 2 \leq p$. Thus let $a = -\dfrac{\langle v_k, v_p \rangle}{2 \langle v_1, v_p \rangle}$ which forces $\left\langle v_k', v_p' \right\rangle = 0$.

Now suppose $1 \leq i < k$, then note that $v_i' = v_i$, thus

$$\begin{aligned}
\left\langle v_i', v_p' \right\rangle &= \langle v_i, v_p + av_{p-k+1} \rangle \\
&= \langle v_i, v_p \rangle + a \langle v_i, v_{p-k+1} \rangle \\
&= \langle v_i, v_p \rangle
\end{aligned}$$

where $\langle v_i, v_{p-k+1} \rangle = 0$ follows from the induction hypothesis in conjunction with the fact that because $k \geq 2$, we have that $p + i - k + 1 \leq p + k - 1$ and $k \leq p$ implies $p + i - k + 1 \neq 1$. Thus v_p' satisfies the induction hypothesis and after relabeling v_p' as v_p we can apply the induction hypothesis again until $k = p$ in which case we will have proven the statement. ∎

Suppose x generates a cycle of generalized eigenvectors satisfying the conclusions of the above proposition and let $z_i = U^{p-i}x$. Then by Eq. (C.3.1) we find that the only non-zero scalar products are $\langle z_i, z_j \rangle = \langle z_1, z_p \rangle$ where $i + j = p + 1$. Thus we say a given cycle of generalized eigenvectors with eigenvalue λ for a self-adjoint operator are adapted to the scalar product, if they form a skew-normal sequence of sign ± 1 if $\lambda \in \mathbb{R}$ or sign 1 if $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If $\lambda \in \mathbb{R}$, then $\{z_1, \ldots, z_p\}$ form a real basis for the Jordan canonical form of T. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (WLOG we can assume $\mathrm{Im}(\lambda) > 0$), then we choose a canonical real basis $\{u_1, v_1, ..., u_p, v_p\}$ for T as follows. Let

$$u_i = \frac{1}{\sqrt{2}}(z_i + \overline{z_i}) \tag{C.3.2a}$$

$$v_i = \frac{1}{i\sqrt{2}}(z_i - \overline{z_i}) \tag{C.3.2b}$$

Note that

$$\begin{aligned}
\langle u_i, u_j \rangle &= \frac{1}{2}(\langle z_i, z_j \rangle + \langle \overline{z_i}, \overline{z_j} \rangle) \\
\langle v_i, v_j \rangle &= \frac{-1}{2}(\langle z_i, z_j \rangle + \langle \overline{z_i}, \overline{z_j} \rangle) \\
\langle u_i, v_j \rangle &= \frac{1}{2i}(\langle z_i, z_j \rangle - \langle \overline{z_i}, \overline{z_j} \rangle) = 0
\end{aligned}$$

It then follows that $\langle u_i, u_j \rangle = 1 = -\langle v_i, v_j \rangle$ if $i + j = p + 1$ with all other scalar products zero. Hence $\{u_i\}$ (resp. $\{v_i\}$) form a skew-normal sequence of sign 1 (resp. -1). Now if we set $u_{p+1} = v_{p+1} = 0$, T acts on this basis as follows:

$$\begin{aligned}
Tu_i &= \frac{1}{\sqrt{2}}(\lambda z_i + z_{i+1} + \overline{\lambda}\overline{z}_i + \overline{z}_{i+1}) \\
&= \frac{1}{\sqrt{2}}((a + ib)z_i + (a - ib)\overline{z}_i) + u_{i+1} \\
&= au_i + \frac{b}{i\sqrt{2}}(\overline{z}_i - z_i) + u_{i+1} \\
&= au_i - bv_i + u_{i+1}
\end{aligned}$$

Similarly

$$\begin{aligned}
Tv_i &= \frac{1}{i\sqrt{2}}(\lambda z_i + z_{i+1} - \overline{\lambda}\overline{z}_i - \overline{z}_{i+1}) \\
&= \frac{1}{i\sqrt{2}}((a + ib)z_i - (a - ib)\overline{z}_i) + v_{i+1} \\
&= av_i + \frac{b}{\sqrt{2}}(z_i + \overline{z}_i) + v_{i+1} \\
&= av_i + bu_i + v_{i+1}
\end{aligned}$$

In the following proposition we use these basis to show that the real subspace spanned by an adapted p-cycle is non-degenerate.

Lemma C.3.6

Suppose V is a real scalar product space and T is a self-adjoint operator on V. Let x be a generator for a p-cycle of generalized eigenvectors for T with eigenvalue λ adapted to the scalar product. Let $z_i = U^{p-i}x$, H be the real subspace spanned by this cycle and $\epsilon = \langle z_1, z_p \rangle = \pm 1$. Then H is non-degenerate.

Furthermore if $\lambda \in \mathbb{R}$, *then*

$$\dim H = p$$
$$\operatorname{ind} H = \begin{cases} \lfloor \frac{p+1}{2} \rfloor & \text{if } \epsilon = -1 \\ p - \lfloor \frac{p+1}{2} \rfloor & \text{if } \epsilon = 1 \end{cases}$$

If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *then*

$$\dim H = 2p$$
$$\operatorname{ind} H = p$$

□

Proof If $\lambda \in \mathbb{R}$, then the result follows by Lemma 8.1.1 applied to $z_1, \ldots, z_p$. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, consider the real vectors $\{u_1, v_1, ..., u_p, v_p\}$ defined in Eqs. (C.3.2a) and (C.3.2b). The result follows by Lemma 8.1.1 applied to the sequence $u_1, \ldots, u_p$ and then to $v_1, \ldots, v_p$. ■

The following theorem is from [O'N83, P. 260-261].

Theorem C.3.7 (Existence of the metric-Jordan canonical form [O'N83]) *A linear operator T on a scalar product space V is self adjoint if and only if $V = \bigoplus_{i=1}^{k} V_i$ (hence each V_i is non-degenerate) where each subspace V_i is T-invariant and $T|_{V_i}$ has one of the following forms:*

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & \ddots & & 0 \\ & & \ddots & 1 & \\ & & & \lambda & 1 \\ & 0 & & & \lambda \end{pmatrix}$$

relative to a skew-normal sequence $\{v_1, ..., v_p\}$ with all scalar products zero except $\langle v_i, v_j \rangle = \varepsilon = \pm 1$ when $i + j = p + 1$, or

$$\begin{pmatrix} a & b & 1 & 0 & & & & \\ -b & a & 0 & 1 & & & 0 & \\ & & & & & & & \\ & & & \ddots & & & & \\ & & & & a & b & 1 & 0 \\ & & & & -b & a & 0 & 1 \\ & 0 & & & & & a & b \\ & & & & & & -b & a \end{pmatrix}$$

relative to a basis $\{u_1, v_1, ..., u_p, v_p\}$ *with all scalar products zero except* $\langle u_i, u_j \rangle = 1 = -\langle v_i, v_j \rangle$ *if* $i + j = p + 1$.

The index and dimension of V_i *is determined by the blocks* $T|_{V_i}$ *due to Lemma C.3.6, hence we must have* $\operatorname{ind} V = \sum_{i=1}^{k} \operatorname{ind} V_i$ *and* $n = \sum_{i=1}^{k} \dim V_i$. □

PROOF We proceed by induction. If $n = 1$ then this result trivially holds. So suppose $n \geq 2$ and this result is true for all self-adjoint operators on scalar product spaces of dimension strictly less than n. Now we show that this holds when $\dim V = n$.

Fix an eigenvalue λ for T (which exists after complexification of V if necessary). Let $U = (T - \lambda I)$. Let p be the smallest integer such that $\dim N(U^p) = \dim N(U^{p+1})$, thus $K_\lambda = N(U^p)$. Then $\dim N(U^{p-1}) < \dim N(U^p)$, hence $U^{p-1}|_{K_\lambda} \neq 0$, thus by Lemma C.3.4 there exists an $x \in K_\lambda$ such that $\langle U^{p-1}x, x \rangle \neq 0$. Note that by construction p is the smallest integer such that $U^p x = 0$, hence x generates a p-cycle of generalized eigenvectors with eigenvalue λ.

Hence by Lemma C.3.5, the p-cycle of generalized eigenvectors generated by x can be modified into another such p-cycle spanning the same subspace as the original and adapted to the scalar product. Thus we now assume that the p-cycle of generalized eigenvectors generated by x is adapted to the scalar product. Note that it follows by Lemma 8.1.1 that the set of p vectors in this cycle are linearly independent. Let H be the real subspace spanned by the p-cycle(s) generated by x if $\lambda \in \mathbb{R}$ or by x and its conjugate if $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By Lemma C.3.6, H is non-degenerate and by construction H is T-invariant. If $H = V$ then we are done, so assume $H \subsetneq V$. Then by property 1, $H^\perp$ is an invariant subspace of T, and is complementary to H by non-degeneracy of H. Let $T' = T|_{H^\perp}$, then $H^\perp$ is a scalar product space with $0 < \dim H^\perp < n$ and T' is a self-adjoint operator on $H^\perp$. Hence the induction hypothesis applies to T', in which case we conclude that the result holds for T.

The converse is also easily checked. ■

C.4 Uniqueness of the metric-Jordan canonical form

In this section T is self adjoint operator on a scalar product space V. We will show in what sense each self-adjoint operator T admits a "unique" metric-Jordan canonical form. We will do this by showing that the parameters appearing in any two canonical forms derived by Theorem C.3.7 must be the same. Then we will show how this result can be used to determine if two self-adjoint operators are isometrically equivalent.

Lemma C.4.1
Let $U = (T - \lambda I)$ *for some eigenvalue* λ, *suppose* x *generates an adapted cycle of length* p *and sign* ε *and denote by* $v_i = U^{p-i}x$. *Also let* H *be the subspace spanned by this cycle.*

For any $0 \leq k \leq p-1$ define a symmetric bilinear form $[\cdot,\cdot]_k$ on H by

$$[x,y]_k = \left\langle U^k x, y \right\rangle$$

for $x, y \in H$. Then the number of zeros in any diagonal representation for $[\cdot,\cdot]_k$ is k. If the $\lambda \in \mathbb{R}$ then the number of negative entries in any diagonal representation for $[\cdot,\cdot]_k$ is

$$\begin{cases} \lfloor \frac{(p-k)+1}{2} \rfloor & \text{if } \epsilon = -1 \\ (p-k) - \lfloor \frac{(p-k)+1}{2} \rfloor & \text{if } \epsilon = 1 \end{cases}$$

In conclusion, we see that the of invariants of $[\cdot,\cdot]_k$ depends only on p, k, ε. □

Proof We prove this by exhibiting a diagonal representation for $[\cdot,\cdot]$ restricted to H.

First observe that for $i, j \in \{1, \ldots, p\}$

$$[v_i, v_j]_k = \left\langle U^k U^{p-i} x, U^{p-j} x \right\rangle = \left\langle U^{2p+k-i-j} x, x \right\rangle$$

The above equation is non-zero iff

$$\begin{aligned} 2p + k - i - j &= p - 1 \\ \Leftrightarrow p + k - i - j &= -1 \\ \Leftrightarrow i + j &= p + k + 1 \end{aligned}$$

It follows that if $i < k+1$, then $[v_i, v_j] = 0$ for any j. Now define vectors $v_i' = v_{i+k}$ for $i \in \{1, \ldots, p-k\}$. Then $\left\langle v_i', v_j' \right\rangle \neq 0$ iff

$$\begin{aligned} i + k + j + k &= p + k + 1 \\ \Leftrightarrow i + j &= p - k + 1 \end{aligned}$$

Hence $v_1', \ldots, v_{p-k}'$ (or equivalently $v_{k+1}, \ldots, v_p$) form a pseudo-orthonormal set of vectors with sign ε. Thus the formula for the number of negative entries when $\lambda \in \mathbb{R}$ follows from Lemma 8.1.1. Also observe that the number of zeros is k. Then by Sylvester's law of inertia it follows that the invariants of $[\cdot,\cdot]_k$ are given as above and hence depend only on p, k, ε. ■

For a real eigenvalue λ, an adapted cycle $x, Ux, \ldots, U^{p-1}x$ is called *positive* if $\langle U^{p-1}x, x\rangle = 1$ or negative if $\langle U^{p-1}x, x\rangle = -1$. By a metric-Jordan canonical basis, we mean one that is obtained from Theorem C.3.7.

Theorem C.4.2 (Uniqueness of the metric-Jordan canonical form)
Suppose λ is an eigenvalue of T. If $\lambda \in \mathbb{R}$, then the number of positive (negative) cycles in K_λ of a given length is independent of any metric-Jordan canonical basis. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then the number of cycles in K_λ of a given length is independent of any metric-Jordan canonical basis. □

PROOF Fix an eigenvalue λ of T and let $U = (T - \lambda I)$. Restrict the argument to the vector space K_λ, i.e set $V = K_\lambda$. Denote by $[\cdot,\cdot]_i$ the symmetric bilinear form given by:

$$[x, y]_i = \langle U^i x, y \rangle$$

for $x, y \in K_\lambda$. We will prove that the number of positive (negative) cycles of a given length depend only on the number of positive (negative) entries in a diagonal representation for $[\cdot,\cdot]_0, [\cdot,\cdot]_1, \ldots, [\cdot,\cdot]_n$. It is understood that the complex representations are chosen so that there are only positive or zero entries in it. It will follow by Sylvester's law of inertia that these signs are independent of any basis.

Fix a metric-Jordan canonical basis for $T|_{K_\lambda}$. It's known that $U^l = 0$ for any $l > n$, hence it follows that the number of cycles of length larger than n are determined by invariants of $[\cdot,\cdot]_l$ for $l > n$. Suppose inductively that the statement holds for all cycles of length strictly larger than p. We will now prove the statement for cycles of length p.

Denote by H the T-invariant non-degenerate (possibly zero) subspace spanned by all cycles of length strictly larger than p in this canonical basis. Observe that since H is T-invariant, it follows for any $l \geq 0$ that $[x, y]_l = 0$ for $x \in H$ and $y \in H^\perp$.

Case 1 There are no cycles of length p in this canonical basis.
Then note that $[x, y]_{p-1} \equiv 0$ for any $x, y \in H^\perp$ and if $H \neq 0$ the invariants of $[\cdot,\cdot]_{p-1}$ on H are uniquely determined by invariants of $[\cdot,\cdot]_l$ for $l \geq p$ by Lemma C.4.1. Also the invariants of $[\cdot,\cdot]_{p-1}$ over K_λ are determined by Sylvester's law of inertia, hence it follows that the number of cycles of length p are uniquely determined.

Case 2 Let $x_1, \ldots, x_m$ be generators for cycles of length p in this canonical basis.
For vectors from $H^\perp$ in this canonical basis the only non-zero diagonal entries of $[\cdot,\cdot]_{p-1}$ are

$$[x_i, x_i]_{p-1} = \langle U^{p-1} x_i, x_i \rangle = \pm 1 \quad i = 1, \ldots, m$$

Again, if $H \neq 0$ the invariants of $[\cdot,\cdot]_{p-1}$ on H are uniquely determined by invariants of $[\cdot,\cdot]_l$ for $l \geq p$ by Lemma C.4.1. The invariants of $[\cdot,\cdot]_{p-1}$ over K_λ are determined by Sylvester's law of inertia, hence it follows that the number of positive (and negative) cycles of length p are uniquely determined.

Thus the result follows by induction on p. ■

We can now state what we mean by "the" metric-Jordan canonical form:

Definition C.4.3

Let T be a self-adjoint operator on a scalar product space V. To each adapted p-cycle of sign ε with eigenvalue $\lambda \in \mathbb{C}$ we associate a 3-tuple $(\lambda, p, \varepsilon)$. A canonical form given by Theorem C.3.7 gives an un-ordered list of such 3-tuples counting multiplicities. We call this list *the metric-Jordan canonical form.* □

By the above theorem, it follows that the above definition is well defined, i.e. each self-adjoint operator T admits precisely one metric-Jordan canonical form. The following example shows that the signs appearing in these canonical forms add some subtleties:

Example C.4.4

Suppose V is Minkowski space equipped with the standard metric

$$g = \mathrm{diag}(-1, 1, \ldots, 1)$$

For $\lambda_1 < \ldots < \lambda_n \in \mathbb{R}$ define two self-adjoint operators T_1 and T_2 as follows:

$$\begin{aligned} T_1 &= \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \\ T_2 &= \mathrm{diag}(\lambda_2, \lambda_1, \lambda_3, \ldots, \lambda_n) \end{aligned}$$

Now observe that even though T_1 and T_2 have the same eigenvalues, they have different metric-Jordan canonical forms. We will show shortly that T_1 and T_2 are isometrically inequivalent, in the sense that there is no $R \in O(V)$ which relates T_1 and T_2 by a similarity transformation. □

Note that the above example is in sharp contrast with the Euclidean case where T_1 and T_2 as defined above would be isometrically equivalent.

Theorem C.4.5 (Isometric Equivalence of self-adjoint operators)

Suppose S and T are self-adjoint operators on a scalar product space V. Then S and T differ by an isometry $R \in O(V)$ iff they have the same metric-Jordan canonical form. □

Proof It's clear that if S and T have the same metric-Jordan canonical form then there is an isometry $R \in O(V)$ which relates the two operators, namely the transformation that relates a metric-Jordan canonical basis of S to a metric-Jordan canonical basis of T.

Suppose T is given as follows relative to S:

$$T = RSR^{-1}$$

Let $\beta = \{v_1, \ldots, v_n\}$ be a canonical basis for S. Then consider the basis $\tilde{\beta} = \{Rv_1, \ldots, Rv_n\}$ for T. Since R is an isometry, we have

$$g|_{\tilde{\beta}} = g|_{\beta}$$

The equation relating T to S implies that

$$T|_{\tilde{\beta}} = S|_{\beta}$$

Hence S and T have the same metric-Jordan canonical form. ∎

Appendix D

Warped products in Spaces of Constant Curvature

We will obtain the warped product decompositions of spaces of constant curvature (with arbitrary signature) in their natural models as subsets of pseudo-Euclidean space (see Section 8.3). This generalizes the corresponding result by Nolker in [Nol96] to arbitrary signatures, and has a similar level of detail. Although our derivation is complete in some sense, none is proven.

Our solution can fairly easily be deduced from that in [Nol96], and it is. Thus the goal of this appendix is to expose the results for reference purposes. We also note that the contents of this appendix are from [Raj14b].

Our primary motivation comes from Section 6.5, where it was shown that one can use the warped product decompositions of a given space to try to construct coordinates which separate the Hamilton-Jacobi equation. Thus these decompositions can be used to construct KEM coordinates.

Another motivation is because warped products are ubiquitous in applications of pseudo-Riemannian geometry, particularly in general relativity [DU05]. Hence it may be of some general interest to pursue this problem.

Our work is mainly self-contained, so it can be used as a reference. The material covered in Section 3.1 will be assumed throughout. We use the notation from Section 8.3 and assume knowledge of warped products from Section 3.5. We also use some results from the theory of pseudo-Riemannian submanifolds in [Che11], which is only necessary to understand certain proofs. We also assume the reader is familiar with [O'N83]. Familiarity with the article [Nol96] is useful but not necessary.

This appendix is organized as follows. In Appendices D.1 to D.3 we review preliminary theory on the spherical submanifolds and warped products in spaces of constant curvature. We give the warped product decompositions of pseudo-Euclidean space in Appendix D.4 and of spherical submanifolds of pseudo-Euclidean space in Appendix D.6. Appendix D.5

is an optional section which gives the isometry groups of spherical submanifolds of pseudo-Euclidean space, which builds on results from [O'N83].

D.1 Spherical Submanifolds of Spaces of Constant Curvature

In this section κ is allowed to be zero. The following optional proposition relates umbilical submanifolds to spherical ones in spaces of constant curvature.

Proposition D.1.1
Any umbilical submanifold of $\mathbb{E}^n_\nu(\kappa)$ with dimension greater than one is necessarily spherical. □

Proof This follows from Lemma 3.2 (a) in [Che11]. ■

Here we state some properties of spherical submanifolds in spaces of constant curvature.

Proposition D.1.2 (Spherical Submanifolds in Spaces of Constant Curvature)
Let $\phi : N \to \mathbb{E}^n_\nu(\kappa)^\circ$ be an isometric immersion of a pseudo-Riemannian manifold N. If N is a spherical submanifold, then

(a) $\langle H, H\rangle$ is constant.

(b) N is of constant curvature $\kappa + \langle H, H\rangle$

Proof Lemma 3.2 from [Che11]. ■

D.2 Standard spherical submanifolds of pseudo-Euclidean space

We collect some properties of $\mathbb{E}^n_\nu(\kappa)$ in the following proposition.

Proposition D.2.1
Let r denote the dilatational vector field and $r^2 = \langle r, r\rangle$. Fix $r^2 \in \mathbb{R}$, the following are true about $\mathbb{E}^n_\nu(\frac{1}{r^2})$

(a) It is a spherical submanifold with mean curvature normal

$$H = -\frac{r}{r^2} \tag{D.2.1}$$

(b) It has constant curvature $\dfrac{1}{r^2}$ and is geodesically complete. □

PROOF The first follows from [O'N83, Lemma 4.27]. When $\dim \mathbb{E}^n_\nu(\frac{1}{r^2}) > 1$, the first result together with Proposition D.1.1 shows that $\mathbb{E}^n_\nu(\frac{1}{r^2})$ is a spherical submanifold. In any case, it follows from Eq. (D.2.1) that $\mathbb{E}^n_\nu(\frac{1}{r^2})$ is a spherical submanifold. Hence the second result follows from Proposition D.1.2 (b). It follows from lemma 4.29 in [O'N83] that $\mathbb{E}^n_\nu(\frac{1}{r^2})$ is geodesically complete. ∎

We collect similar properties of $\mathbb{P}^n_\nu$.

Proposition D.2.2
The following are true about $\mathbb{P}^n_\nu$ with mean curvature vector $-a$:

(a) It is a spherical submanifold with mean curvature normal

$$H = -a$$

(b) It is globally isometric to $\mathbb{E}^n_\nu$. □

PROOF Consider the map ψ given by Eq. (8.3.1). It then follows that for $v \in TV$,

$$\psi_* v = v - \langle v, x\rangle\, a$$

The above equation shows that the induced metric at each point is the induced metric on V. Hence $\mathbb{P}^n_\nu \simeq \mathbb{E}^n_\nu$. Now to calculate the second fundamental form, we have for $w, v \in TV$:

$$\begin{aligned}\nabla_{\psi_* w}\psi_* v &= \nabla_w v - \langle \nabla_w v, x\rangle\, a - \langle v, w\rangle\, a \\ &= \psi_* \nabla_w v - \langle v, w\rangle\, a\end{aligned}$$

Hence it follows that $\mathbb{P}^n_\nu$ is umbilical with mean curvature vector $-a$. Since $-a$ is covariantly constant, it follows that $\mathbb{P}^n_\nu$ is spherical. ∎

D.3 Warped product decompositions of Spaces of Constant Curvature

In this section we study warped product decompositions of $\mathbb{E}^n_\nu(\kappa)$ where κ may equal zero. Recall that warped products were introduced in Section 3.5. A *warped product decomposition* of a given pseudo-Riemannian manifold M is a warped product which is (locally) isometric to M. Let $M = M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ be a warped product and $\psi : M \to \mathbb{E}^n_\nu(\kappa)$ a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$. Recall that M_0 is a geodesic submanifold and each M_i for $i > 0$ is a spherical submanifold (see Theorem 3.5.10). Fix $\bar{p} \in \psi(M)$. Let $H_i = -\nabla(\log \rho_i)$ be the mean curvature vector field associated to the canonical foliation L_i generated by M_i (see Proposition 3.5.9). Let $V_i := T_{\bar{p}_i} M_i$ for each i and $z_i := H_i|_{\bar{p}} \in V_0$ for $i > 0$. Then note that

$$T_{\overline{p}} M = \bigoplus_{i=0}^{k} V_i$$

Equation (3.5.4) implies that the mean curvature vectors satisfy the following equation for $i \neq j$:

$$\langle z_i, z_j \rangle = -\kappa \tag{D.3.1}$$

In this case we say that ψ is a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ associated with the initial data $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$ where $a_i := \kappa\bar{p} - z_i$.

Conversely, let $\overline{p} \in \mathbb{E}^n_\nu(\kappa)$ where $n \geq 2$ and consider the following decomposition of $T_{\overline{p}}\mathbb{E}^n_\nu(\kappa)$, $T_{\overline{p}}\mathbb{E}^n_\nu = \bigoplus_{i=0}^{k} V_i$ into non-trivial subspaces (hence non-degenerate) with $k \geq 1$. Suppose $z_1, ..., z_k \in V_0$ satisfy Eq. (D.3.1). Let $a_i := \kappa\bar{p} - z_i$ and assume additionally that the subset of non-zero a_i are linearly independent. In this case, we say that $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$ are *initial data* for a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$. We will show later on that in a space of constant curvature there always exists a warped product decomposition with a given initial data. It follows from Theorem 3.2.6 that in the category of Riemannian manifolds with $n > 2$, this property characterizes spaces of constant curvature.

The additional condition requiring the a_i to be linearly independent trivially holds in Euclidean space and in motivating applications. The reason we make this assumption will become more apparent later. Here is an optional lemma, which is given for completeness, and hints at why we make this assumption.

Lemma D.3.1
Suppose $a_1, \ldots, a_k$ are linearly independent pair-wise orthogonal lightlike vectors. Then there exist vectors $b_1, \ldots, b_k$ such that $\langle a_i, b_j \rangle = \delta_{ij}$ and $\langle b_i, b_j \rangle = 0$. □

Proof Suppose to the contrary that for any b_1 satisfying $\langle b_1, a_i \rangle = 0$ for $i > 1$ we have $\langle b_1, a_1 \rangle = 0$. Thus

$$\cap_{i=2}^{k} a_i^\perp \subseteq a_1^\perp$$

Define $T : V \to \mathbb{R}^k$ by:

$$T(v) = (\langle a_1, v\rangle, \ldots, \langle a_k, v\rangle)$$

By hypothesis we have $\dim \ker T \geq n - (k-1)$, hence $\dim \operatorname{Im} T \leq k-1$ by the rank-nullity theorem. Thus $a_1^\flat, \ldots, a_k^\flat$ are linearly dependent, a contradiction.

Thus there exists $b_1 \in \cap_{i=2}^k a_i^\perp$ with $\langle a_1, b_1\rangle = 1$. The result then follows by induction. Indeed the next step is to find b_2 by applying the above result to $\{a_2, \ldots, a_k\} \subset \operatorname{span}\{a_1, b_1\}^\perp$ making use of the fact that $\operatorname{span}\{a_1, b_1\}$ is non-degenerate by construction. ■

It has been shown by Nolker in [Nol96] that given any initial data for Riemannian spaces of constant curvature, there exists a unique warped product decomposition associated with the initial data. In this appendix we will show that given any initial data for a WP-decomposition of $\mathbb{E}^n_\nu(\kappa)$, there exists a WP-decomposition associated with the initial data. This WP-decomposition is probably uniquely determined but we don't use or prove this supposition.

Equation (3.5.3) implies that the Hessian H of each warping function ρ of a space of constant curvature satisfies the following equation on the geodesic factor:

$$H(X, Y) = -\kappa\rho \langle X, Y\rangle$$

This proves the following fact:

Lemma D.3.2
A space of constant non-zero curvature does not admit product decompositions. □

D.4 Warped product decompositions of pseudo-Euclidean space

D.4.1 Spherical submanifolds of pseudo-Euclidean space

We first describe the spherical submanifolds of pseudo-Euclidean space. The following theorem is a generalization of Lemma 5 in [Nol96] to pseudo-Euclidean space.

Theorem D.4.1 (Spherical submanifolds of $\mathbb{E}^n_\nu$)
Let $\overline{p} \in$ be arbitrary, $V \subseteq \mathbb{E}^n_\nu$ a non-degenerate subspace with $m := \dim V \geq 1$, $\mu := \operatorname{ind} V$ and $z \in V^\perp$. Let $\tilde{\kappa} := z^2$, $a := -z$ and $W = \mathbb{R}a \ominus V$. There is exactly one m-dimensional connected and geodesically complete spherical submanifold $\tilde{N}$ with $\overline{p} \in \tilde{N}$, $T_{\overline{p}}\tilde{N} = V$ and having mean curvature vector at $\overline{p}$, z. $\tilde{N}$ is an open submanifold of N; N is referred to as the spherical submanifold determined by $(\overline{p}, V, a)$ and is given as follows (where $\simeq$ means isometric to):

(a) *$a = 0$ iff N is geodesic, in this case $N \simeq \mathbb{E}^m_\mu$*

$$N = \overline{p} + V$$

(b) *a is timelike, then $\mu \le \nu - 1$ and $N \simeq H^m_\mu(\tilde{\kappa})$*

(c) *a is spacelike, then $N \simeq S^m_\mu(\tilde{\kappa})$*
For cases (b) and (c), let $c = \overline{p} - \frac{a}{\tilde{\kappa}}$ be the center of N, then N is given as follows:

$$N = c + \{p \in W \mid p^2 = \frac{1}{\tilde{\kappa}}\}$$

(d) *a is lightlike, then $\mu \le \nu - 1$ and $N \simeq E^m_\mu$*

$$N = \overline{p} + \{p - \frac{1}{2}p^2 a \mid p \in V\}$$ □

Remark D.4.2
$\tilde{N} = N$ except in the following two cases (which are anti-isometric): When $N \simeq H^m_0(\tilde{\kappa})$ or $N \simeq S^m_m(\tilde{\kappa})$, N is disconnected [O'N83, Section 4.6] and so $\tilde{N}$ is given as follows:

$$\tilde{N} = N \cap (c + \{p \in W \mid \langle a, p\rangle > 0\})$$ □

Proof First we note that it suffices to show that there exists a single connected and geodesically complete sphere satisfying the initial conditions. By Lemma 3.2.5, it must be unique.

Item (a) is clear. For Items (b) and (c), it follows from Proposition D.2.1 that N is a sphere and the initial conditions are easily checked. The connectedness properties follow from lemma 4.25 in [O'N83]. It follows from lemma 4.29 in [O'N83] that N is geodesically complete.

Item (d) follows from Proposition D.2.2. ■

Remark D.4.3
See [Che11] for a different proof. □

Since circles are one dimensional spherical submanifolds, we can use the above theorem to describe the circles in pseudo-Euclidean space.

Example D.4.4 (Proper Circles in pseudo-Euclidean space)
Suppose $(\bar{p}, \bar{V}, k\bar{Y})$ are initial conditions for a proper circle as in Lemma 3.2.1 with $\varepsilon_0 := \bar{V}^2 = \pm 1$, $\varepsilon_1 := \bar{Y}^2 = \pm 1$ and $\left\|k\bar{Y}\right\| \neq 0$. We now describe the circle determined by this data.

By Example 3.2.2 the proper circle determined by these initial conditions determine a spherical submanifold of $\mathbb{E}^n_\nu$ characterized by $(\bar{p}, \mathbb{R}\bar{V}, \varepsilon_0 k\bar{Y})$. Now let $H := \varepsilon_0 k\bar{Y}$, $\kappa := \langle H, H\rangle = \varepsilon_1 k^2$ and $c := \overline{p} + \frac{H}{\kappa} = \bar{p} - \frac{\varepsilon_0\varepsilon_1\bar{Y}}{k}$.

Case 1 Euclidean circle, $\gamma = \mathbb{S}^1$: $\varepsilon_0 = \varepsilon_1 = \pm 1$

$$\gamma(t) = c + \frac{1}{k}(\sin(kt)\bar{V} - \cos(kt)\bar{Y})$$

Case 2 Hyperbolic circle, $\gamma = H^1$: $\varepsilon_0 = 1, \varepsilon_1 = -1$

Case 3 de Sitter circle, $\gamma = \mathbb{S}^1_1$: $\varepsilon_0 = -1, \varepsilon_1 = 1$

In the last two cases (which are anti-isometric), γ is given as follows:

$$\gamma(t) = c + \frac{1}{k}(\sinh(kt)\bar{V} - \varepsilon_0\varepsilon_1\cosh(kt)\bar{Y}) \qquad \square$$

One can give a similar example for geodesics and null circles.

D.4.2 Warped product decompositions of pseudo-Euclidean space

Our classification of the warped product decompositions of $\mathbb{E}^n_\nu$ is based on the fact that a specification of the tangent spaces and mean curvature normals of the spherical foliations of a warped product at one point $\overline{p}$, uniquely determines a warped product decomposition in a neighborhood of $\overline{p}$. We now carry out this classification as follows. Suppose $\psi : N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu$ is a warped product decomposition of $\mathbb{E}^n_\nu$ associated with initial data $(\bar{p}; \bigoplus_{i=0}^{k} V_i; -z_1, ..., -z_k)$. By Eq. (D.3.1), the mean curvature vectors at $\bar{p}$ satisfy the following equation:

$$\langle z_i, z_j\rangle = 0 \quad i \neq j$$

We now only consider the case $\nu \leq 1$ as the other signatures are straightforward generalizations of these standard ones. In this case, we will use Theorem D.4.1 to classify N_i up to homothety as follows. Say $z_1, ..., z_l = 0$ and the remaining are non-zero, then for $i = 1, ..., l$ the N_i are pair-wise orthogonal planes passing through $\overline{p}$. We now consider the remaining possibilities:

Case 1 Since the z_i are orthogonal, there is at most one lightlike direction, say z_{l+1}. The remaining lightlike z_i are proportional to z_{l+1}, but since we assume the non-zero z_i are linearly independent, we will work with only one lightlike vector z_{l+1}. Then N_{l+1} a paraboloid isometric to Euclidean space. The orthogonality relations force the remaining z_i to be space-like and hence the remaining N_i are Euclidean spheres.

Case 2 Similarly, at most one of the z_i can be timelike, say z_{l+1}. Then N_{l+1} is isometric to hyperbolic space. The orthogonality relations force the remaining z_i to be space-like and hence the remaining N_i are Euclidean spheres.

Case 3 The remaining z_i are spacelike. If $\operatorname{ind} V_0 = 1$ or $\operatorname{ind} V_0 = 0$ in Euclidean space, then the remaining N_i are Euclidean spheres. If $\operatorname{ind} V_0 = 0$ in Minkowski space, then $\operatorname{ind} V_j = 1$ for precisely one $j \geq 1$, then N_j is de Sitter space while the remaining N_i are Euclidean spheres.

Case 4 All z_i are zero. Then each N_i is an affine plane and the warped product is a product of planes.

We summarize our findings in the following theorem.

Theorem D.4.5 (Warped products in $\mathbb{E}^n$ and M^n)
Suppose $N = N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k$ is a proper warped product decomposition of an open subset of $\mathbb{E}^n_\nu$. If at most one of the N_i are intrinsically flat, then N is isometric to one of the following warped products:

If $\mathbb{E}^n_\nu$ is Euclidean space:

$$\mathbb{E}^m \times_{\rho_1} S^{n_1} \times \cdots \times_{\rho_s} S^{n_s}$$

If $\mathbb{E}^n_\nu$ is Minkowski space:

$$\begin{aligned}
M^m \times_{\lambda_1} \mathbb{E}^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\
M^m \times_{\tau_1} H^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\
\mathbb{E}^m \times_{\rho_1} dS^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\
M^m \times_{\rho_1} S^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s}
\end{aligned}$$

where $\nabla\rho_i$,$\nabla\tau_i$,$\nabla\lambda_i$ is a spacelike,timelike, lightlike vector field respectively. □

The above theorem shows that there are at 1 and 4 distinct types of proper singly warped products in Euclidean and Minkowski space respectively. One can show that the multiply warped products can be built up from the singly warped products by iteratively decomposing the geodesic factor of the warped product into another warped product which is “compatible” with the original. Thus we only describe a special subset of warped products for simplicity.

The following theorem describes this interesting class of warped products. Its proof can be deduced from Theorem 7 in [Nol96]. It is a generalization of that theorem to pseudo-Euclidean space.

Theorem D.4.6 (Standard Warped Products in $\mathbb{E}^n_\nu$ [Nol96])
Fix $\overline{p} \in \mathbb{E}^n_\nu$ where $n \geq 2$ and the following decomposition of $T_{\overline{p}}\mathbb{E}^n_\nu$, $T_{\overline{p}}\mathbb{E}^n_\nu = \bigoplus_{i=0}^{k} V_i$ into non-trivial subspaces (hence non-degenerate) with $k \geq 1$. Suppose $a_1, ..., a_k \in V_0$ are pair-wise orthogonal. Let $\kappa_i := a_i^2$ and $\epsilon_i := \operatorname{sgn} \kappa_i$. We consider the following warped decompositions:

non-null warped decomposition *Let $\mu \geq 0$*

$$\kappa_1 \leq \cdots \leq \kappa_\mu < 0 < \kappa_{\mu+1} \leq \cdots \leq \kappa_k$$

In this case, let $c = \overline{p} - \sum_{i=1}^{k} \frac{a_i}{\kappa_i}$ and $c_i = \overline{p} - \frac{a_i}{\kappa_i}$ for each $i = 1, ..., k$.

null warped decomposition *$k = 1$, $a_1 := a$, $\kappa_1 = a^2 = 0$ but $a \neq 0$, i.e. a is lightlike.*

In this case, fix a lightlike vector $b \in V_0$ such that $\langle a, b\rangle = 1$ and let $c = \overline{p} - b$.

Now, define N_0 as follows:

$$N_0 := c + \{p \in V_0 |\, \langle a_i, p\rangle > 0 \text{ for all } i\ \}$$

Note that N_0 is an open subset of the plane determined by $(\overline{p}, V_0, 0)$. For $i = 1, ..., k$, let N_i be the spherical submanifold of $\mathbb{E}^n_\nu$ determined by $(\overline{p}, V_i, a_i)$. Define

$$\rho_i : \begin{cases} N_0 & \to \mathbb{R}_+ \\ p_0 & \mapsto \langle a_i, p_0 - c\rangle = 1 + \langle a_i, p_0 - \overline{p}\rangle \end{cases}$$

For $i = 1, ..., k$, let $W_i := \mathbb{R}a_i \ominus V_i$ and $P : \mathbb{E}^n_\nu \to W_i$ be the orthogonal projection. Then the map

$$\psi : \begin{cases} N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k & \to \mathbb{E}^n_\nu \\ (p_0, ..., p_k) & \mapsto p_0 + \sum_{i=1}^{k} \rho_i(p_0)(p_i - \overline{p}) \end{cases} \tag{D.4.1}$$

is an isometry onto the following set[1]:

$$\operatorname{Im}(\psi) := \begin{cases} c + \{p \in \mathbb{E}^n_\nu \mid \operatorname{sgn}(P_i(p))^2 = \epsilon_i, \text{ for each } i = 1, ..., k\} & \textit{non-null case} \\ c + \{p \in \mathbb{E}^n_\nu \mid \langle a, p\rangle > 0\} & \textit{null case} \end{cases}$$

$\operatorname{Im}(\psi)$ *is dense in $\mathbb{E}^n_\nu$ only for a non-null warped decomposition when each W_i for $i = 1, ..., k$ is Euclidean or anti-isometric to a Euclidean space.* □

[1] Note that $\operatorname{sgn} 0 = 0$, otherwise for $a \neq 0$, $\operatorname{sgn} a$ is the sign of a.

Remark D.4.7

Note that $\rho_i(\bar{p}) = 1$ for $i = 1, ..., k$. Also for each $p_i \in N_i$ we have $\psi(\bar{p}, \ldots, p_i, \ldots, \bar{p}) = p_i$, hence $\psi(\bar{p}, \ldots, \bar{p}) = \bar{p}$.

If the N_i are required to be connected, then $\mathrm{Im}(\psi)$ has to be modified slightly. For each N_i that is disconnected (see the remark following Theorem D.4.1), in addition to the restriction that $\mathrm{sgn}(P_i(p))^2 = \epsilon_i$ in the definition of $\mathrm{Im}(\psi)$, add the restriction that $\langle a_i, P_i(p)\rangle > 0$. ▫

Proof The idea of this proof is to assume Eq. (D.4.1) holds and then expand it by choosing an appropriate basis for V_0. In the expanded form we will be able to prove all the claims made in the theorem. We have the following two cases.

The non-null case: Let W_0 be the orthogonal complement of $\bigoplus_{i=1}^{k} \mathbb{R}a_i$ in V_0; which is well defined since $a_i^2 \neq 0$ for each i. Thus we have that

$$V_0 = W_0 \oplus \bigoplus_{i=1}^{k} \mathbb{R}a_i \tag{D.4.2}$$

which implies:

$$\begin{aligned}
\mathbb{E}^n_\nu &= \bigoplus_{i=0}^{k} V_i \\
&= W_0 \oplus \bigoplus_{i=1}^{k} \mathbb{R}a_i \oplus \bigoplus_{i=1}^{k} V_i \\
&= W_0 \oplus \bigoplus_{i=1}^{k} (\mathbb{R}a_i \oplus V_i) \\
&= W_0 \oplus \bigoplus_{i=1}^{k} W_i
\end{aligned}$$

Now let $P_i : \mathbb{E}^n_\nu \to W_i$ denote the orthogonal projection for $i = 0, ..., k$. Then from Eq. (D.4.2), we get the following orthogonal decomposition of V_0 which will be used extensively:

$$p = P_0 p + \sum_{i=1}^{k} \frac{1}{\kappa_i} \langle a_i, p\rangle\, a_i \qquad \text{for all } p \in V_0 \tag{D.4.3}$$

Now we use the above decomposition of $p \in V_0$ to write $\psi(p_0, ..., p_k)$ adapted to the following affine decomposition of $\mathbb{E}^n_\nu$

$$\mathbb{E}^n_\nu = c + \bigoplus_{i=0}^{k} W_i$$

We get the following for $(p_0, ..., p_k) \in N_0 \times \cdots \times N_k$

$$\psi(p_0, ..., p_k) = c + P_0(p_0 - c) + \sum_{i=1}^{k} \langle a_i, p_0 - c \rangle (p_i - c_i) \tag{D.4.4}$$

Now we prove that ψ is injective: Let $(p_0, ..., p_k), (q_0, ..., q_k) \in N_0 \times \cdots \times N_k$ and suppose that $\psi(p_0, ..., p_k) = \psi(q_0, ..., q_k)$. From Eq. (D.4.4), we deduce the following:

$$\begin{aligned} P_0(p_0 - c) &= P_0(q_0 - c) \\ \langle a_i, p_0 - c \rangle (p_i - c_i) &= \langle a_i, q_0 - c \rangle (q_i - c_i) \end{aligned}$$

Since for each $i = 1, ..., k$, $(p_i - c_i)^2 = (q_i - c_i)^2 = \frac{1}{\kappa_i}$ and $\langle a_i, p_0 - c \rangle, \langle a_i, q_0 - c \rangle \in \mathbb{R}^+$, we deduce that $p_i = q_i$. Then Eq. (D.4.3) shows $p_0 = q_0$.

Now for surjectivity: From Eq. (D.4.4) it's clear that $\psi(N_0 \times \cdots \times N_k) \subseteq \mathrm{Im}(\psi)$. Given $p \in \mathrm{Im}(\psi)$, using Eq. (D.4.4) in conjunction with Eq. (D.4.3) we can readily calculate the inverse $q = \psi^{-1}(p)$ given in components as follows:

$$\begin{aligned} q_0 &= c + P_0(p - c) + \sum_{i=1}^{k} \frac{\epsilon_i}{\sqrt{|\kappa_i|}} \|P_i(p - c)\| a_i \\ q_i &= c_i + \frac{1}{\sqrt{|\kappa_i|}} \frac{P_i(p - c)}{\|P_i(p - c)\|} \quad i = 1, ..., k \end{aligned}$$

Now we show that ψ is an isometry. Note first that for $p = (p_0, ..., p_k) \in N_0 \times \cdots \times N_k$ and $v = (v_0, ..., v_k) \in T_p(N_0 \times \cdots \times N_k)$, Eq. (D.4.4) implies that

$$\psi_* v = P_0 v_0 + \sum_{i=1}^{k} \langle a_i, v_0 \rangle (p_i - c_i) + \sum_{i=1}^{k} \langle a_i, p_0 - c \rangle v_i$$

Hence also using the fact that:

$$\langle p_i - c_i, v_i \rangle = 0 \text{ for } i = 1, ..., k$$

we get:

$$
\begin{aligned}
(\psi_* v)^2 &= (P_0 v_0)^2 + \sum_{i=1}^{k} (\langle a_i, v_0 \rangle (p_i - c_i))^2 + \sum_{i=1}^{k} (\langle a_i, p_0 - c \rangle v_i)^2 \\
&= (P_0 v_0)^2 + \sum_{i=1}^{k} \frac{\langle a_i, v_0 \rangle^2}{\kappa_i} + \sum_{i=1}^{k} \rho_i(p_0)^2 v_i^2 \\
&= (P_0 v_0 + \sum_{i=1}^{k} \frac{\langle a_i, v_0 \rangle}{\kappa_i} a_i)^2 + \sum_{i=1}^{k} \rho_i(p_0)^2 v_i^2 \\
&= v_0^2 + \sum_{i=1}^{k} \rho_i(p_0)^2 v_i^2
\end{aligned}
$$

where the last two lines follow from the fact that $v_0 \in V_0$ and Eq. (D.4.3).

The null case: We have the following decomposition of V_0:

$$
V_0 = W_0 \ominus \operatorname{span}\{a, b\}
$$

where W_0 is the orthogonal complement of $\operatorname{span}\{a, b\}$ relative to V_0. Let P_i denote the orthogonal projection onto W_0 for $i = 0$ and onto V_1 for $i = 1$. Then for $p \in \mathbb{E}^n_\nu$:

$$
p = P_0 p + \langle b, p \rangle a + \langle a, p \rangle b + P_1 p \tag{D.4.5}
$$

and

$$
p^2 = (P_0 p)^2 + 2 \langle b, p \rangle \langle a, p \rangle + (P_1 p)^2 \tag{D.4.6}
$$

Let $c = \overline{p} - b$, $\tilde{p}_0 = p_0 - c$ and $\tilde{p}_1 = p_1 - \overline{p}$, then for $(p_0, p_1) \in N_0 \times N_1$

$$
\begin{aligned}
\psi(p_0, p_1) = c &+ P_0(\tilde{p}_0) + (\langle b, \tilde{p}_0 \rangle - \frac{1}{2} \langle a, \tilde{p}_0 \rangle (P_1(\tilde{p}_1))^2) a + \langle a, \tilde{p}_0 \rangle b \\
&+ \langle a, \tilde{p}_0 \rangle P_1(\tilde{p}_1)
\end{aligned} \tag{D.4.7}
$$

where the last two lines follow from Eq. (D.4.6).

Injectivity of ψ follows readily from Eq. (D.4.7).

Now for surjectivity: From Eq. (D.4.7) it's clear that $\psi(N_0 \times N_1) \subseteq \operatorname{Im}(\psi)$. Given $p \in \operatorname{Im}(\psi)$, let $\tilde{p} = p - c$, then using Eq. (D.4.4) in conjunction with Eq. (D.4.3) we can readily calculate the inverse $q = \psi^{-1}(p)$ given in components as follows:

$$\begin{aligned} q_0 &= c + P_0(\tilde{p}) + (\langle b, \tilde{p}\rangle + \frac{1}{2\langle a, \tilde{p}\rangle}(P_1(\tilde{p}))^2)a + \langle a, \tilde{p}\rangle b \\ q_1 &= \overline{p} + \frac{1}{\langle a, \tilde{p}\rangle}P_1(\tilde{p}) - \frac{1}{2\langle a, \tilde{p}\rangle^2}(P_1(\tilde{p}))^2 a \end{aligned} \tag{D.4.8}$$

Now we show that ψ is an isometry. Note first that for $p = (p_0, p_1) \in N_0 \times N_1$ and $v = (v_0, v_1) \in T_p(N_0 \times N_1)$, Eq. (D.4.7) implies that

$$\begin{aligned} \psi_* v = P_0 v_0 &+ (\langle b, v_0\rangle - \frac{1}{2}\langle a, v_0\rangle (P_1(\tilde{p}_1))^2 - \langle a, \tilde{p}_0\rangle \langle P_1\tilde{p}_1, P_1 v_1\rangle)a + \langle a, v_0\rangle b \\ &+ \langle a, v_0\rangle P_1(\tilde{p}_1) + \langle a, \tilde{p}_0\rangle P_1(v_1) \end{aligned}$$

Hence we get that:

$$\begin{aligned} (\psi_* v)^2 &= (P_0 v_0)^2 + 2\langle b, v_0\rangle \langle a, v_0\rangle + \langle a, \tilde{p}_0\rangle^2 (P_1 v_1)^2 \\ &= (\langle b, v_0\rangle a + \langle a, v_0\rangle b + P_0 v_0)^2 + \rho(p_0)^2 (P_1 v_1)^2 \\ &= v_0^2 + \rho(p_0)^2 v_1^2 \end{aligned}$$

where the last two lines follow from the fact that $v_0 \in V_0$, Eq. (D.4.6) and since $P_1 : T_{p_1}N_1 \to V_1$ is an isometry for each $p_1 \in N_1$. ■

Definition D.4.8

We call ψ the warped product decomposition of $\mathbb{E}^n_\nu$ determined by $(\overline{p}; N_1, ..., N_k)$ or by $(\overline{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$ as in the hypothesis of the above theorem. □

Note that in the context of the above definition, the warped product decomposition is proper if each $a_i \neq 0$. For actual calculations we wish to work with canonical forms. The following definition will be particularly convenient.

Definition D.4.9 (Canonical form for Warped products of $\mathbb{E}^n_\nu$)

We say that a proper warped product decomposition of $\mathbb{E}^n_\nu$ determined by $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$ is in canonical form if: $\bar{p} \in V_0$ and $\langle \bar{p}, a_i\rangle = 1$. □

We note here that any proper warped product decomposition ψ of $\mathbb{E}^n_\nu$ can be brought into canonical form by the translation $\psi \to \psi - c$. This follows from the above theorem by observing that $\langle \bar{p} - c, a_i\rangle = 1$ for each $i > 0$. The following corollary gives the standard warped product decompositions of $\mathbb{E}^n_\nu$ in canonical form.

Corollary D.4.10 (Canonical form for Warped products of $\mathbb{E}^n_\nu$)
Let ψ be a proper warped product decomposition of $\mathbb{E}^n_\nu$ determined by $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$ which is in canonical form.

Then the conclusions of Theorem D.4.6 simplify as follows:

$$N_0 = \{p \in V_0 | \langle a_i, p\rangle > 0 \text{ for all } i \}$$

$$\rho_i = \langle a_i, p_0\rangle$$

$$\mathrm{Im}(\psi) = \begin{cases} \{p \in \mathbb{E}^n_\nu \mid \mathrm{sgn}(P_i(p))^2 = \epsilon_i, \ \text{for each } i = 1, ..., k\} & \text{non-null case} \\ \{p \in \mathbb{E}^n_\nu \mid \langle a, p\rangle > 0\} & \text{null case} \end{cases}$$

For $(p_0, ..., p_k) \in N_0 \times \cdots \times N_k$, ψ has the following form:

$$\psi(p_0, ..., p_k) = \begin{cases} P_0 p_0 + \sum_{i=1}^{k} \langle a_i, p_0\rangle (p_i - c_i) & \text{non-null case} \\ P_0 p_0 + (\langle b, p_0\rangle - \frac{1}{2}\langle a, p_0\rangle (P_1(p_1))^2) a + \langle a, p_0\rangle b + \langle a, p_0\rangle P_1 p_1 & \text{null case} \end{cases}$$

Furthermore, the following equation holds:

$$\psi(p_0, ..., p_k)^2 = p_0^2 \tag{D.4.9}$$

□

Proof First note that for the non-null case:

$$\begin{aligned} \langle a_i, c\rangle &= \left\langle a_i, \bar{p} - \frac{a_i}{\kappa_i}\right\rangle \\ &= 1 - \frac{\langle a_i, a_i\rangle}{\kappa_i} \\ &= 0 \end{aligned}$$

Similarly for the null-case:

$$\begin{aligned} \langle c, a\rangle &= \langle \bar{p} - b, a\rangle \\ &= 0 \end{aligned}$$

Thus we see that

$$\begin{aligned} N_0 &= c + \{p \in V_0 |\, \langle a_i, p\rangle > 0 \text{ for all i } \} \\ &= \{p \in V_0 |\, \langle a_i, p\rangle > 0 \text{ for all i } \} \end{aligned}$$

The formula for $\mathrm{Im}(\psi)$ follows similarly. Clearly $\rho_i(p_0) = \langle a_i, p_0 - c\rangle = \langle a_i, p_0\rangle$. Now we break into cases.

The non-null case:

Note that $c \in W_0$, so $P_0 c = c$, hence

$$\begin{aligned} \psi(p_0, ..., p_k) &= c + P_0(p_0 - c) + \sum_{i=1}^{k} \langle a_i, p_0 - c\rangle (p_i - c_i) \\ &= c + P_0(p_0 - c) + \sum_{i=1}^{k} \langle a_i, p_0\rangle (p_i - c_i) - \sum_{i=1}^{k} \langle a_i, c\rangle (p_i - c_i) \\ &= P_0 p_0 + \sum_{i=1}^{k} \langle a_i, p_0\rangle (p_i - c_i) \end{aligned}$$

It follows from the above equation that $\psi(p_0, ..., p_k)^2 = p_0^2$.

The null case:

By Eq. (D.4.5), c can be written as follows:

$$c = P_0 c + \langle b, c\rangle\, a$$

Thus Eq. (D.4.7) reduces to

$$\begin{aligned} \psi(p_0, p_1) = c &+ P_0(p_0) - P_0 c - \langle b, c\rangle\, a + (\langle b, p_0\rangle - \frac{1}{2}\langle a, p_0\rangle\, (P_1(\tilde{p}_1))^2)a + \langle a, p_0\rangle\, b \\ &+ \langle a, p_0\rangle\, P_1(\tilde{p}_1) \\ = P_0(p_0) &+ (\langle b, p_0\rangle - \frac{1}{2}\langle a, p_0\rangle\, (P_1(p_1))^2)a + \langle a, p_0\rangle\, b \\ &+ \langle a, p_0\rangle\, P_1(p_1) \end{aligned}$$

In the last equation we used the fact that $P_1 \tilde{p}_1 = P_1 p_1$ since $\bar{p} \in V_0$.

Finally, it follows from the above equation that $\psi(p_0, p_1)^2 = p_0^2$. ∎

D.5 Isometry groups of Spherical submanifolds of pseudo-Euclidean space*

Warped products of spaces of constant curvature are closely related to certain integrable subgroups of the isometry group due to the following fact [Zeg11]:

Proposition D.5.1 (Lifting isometries from Killing distributions)
Let $M = B \times_\rho F$ be a warped product and suppose $\tilde{f} : F \to F$ is an isometry of F. Then the lift f defined by

$$f(x, y) := (x, \tilde{f}(y)), \quad (x, y) \in B \times F$$

is an isometry of M. □

Theorem 5.1 in [Zeg11] shows conversely that given a certain integrable group action on a pseudo-Riemannian manifold M, one can obtain a warped product whose spherical foliation is invariant under the action of the group. Hence in spaces of constant curvature one can show that the above property characterizes warped products. In view of this, in this section we state the isometry groups which preserve the spherical submanifolds of pseudo-Euclidean space.

The isometry groups of H^n_ν and S^n_ν are well documented, see for example [O'N83, section 9.2]. In this section we will describe the isometry group of $\mathbb{P}^n_\nu$. This is given in [Nol96, lemma 6] for the case when $\nu = 0$; that proof should generalize easily. Although, we will give a different proof (motivated by Nolker's results) using our knowledge of warped product decompositions and Proposition D.5.1.

We denote the homogeneous isometry group (i.e. orthogonal group) of $\mathbb{E}^{n+2}_{\nu+1}$ by $O_{\nu+1}(n+2)$ (see [O'N83]). Then we have the following:

Proposition D.5.2
Let $-a$ be the mean curvature vector of $\mathbb{P}^n_\nu$. The isometry group of $\mathbb{P}^n_\nu$ is:

$$I(\mathbb{P}^n_\nu) = \{T \in O_{\nu+1}(n+2) \mid Ta = a\}$$

Furthermore suppose we fix an embedding of $\mathbb{E}^n_\nu$ by fixing a subspace $V \simeq \mathbb{E}^n_\nu$, then for $p \in V$ and $\tilde{p} \in V^\perp$ we have the following Lie group isomorphism:

$$\phi : \begin{cases} O(V) \ltimes V & \to I(\mathbb{P}^n_\nu) \\ (B, v) & \mapsto \phi(B, v) \end{cases}$$

where

$$\phi(B, v)(p + \tilde{p}) = \tilde{p} + Bp + \langle a, \tilde{p} \rangle v - (\langle Bp, v \rangle + \frac{1}{2} \langle a, \tilde{p} \rangle v^2))a$$ □

PROOF Consider the warped product decomposition:

$$\psi(p_0, p) = \langle a, p_0\rangle b + \langle a, p_0\rangle p + (\langle b, p_0\rangle - \frac{1}{2}\langle a, p_0\rangle p^2)a$$

for $p_0 \in N_0$ and $p \in V$. Note that

$$\psi(b, p) = b + p - \frac{1}{2}p^2 a$$

is a map onto $\mathbb{P}^n_\nu$. As in Eq. (D.4.8), one can deduce that the inverse of ψ is

$$\begin{aligned} q_0 &= (\langle b, p\rangle + \frac{1}{2\langle a, p\rangle}(Pp)^2)a + \langle a, p\rangle b \\ q_1 &= \frac{1}{\langle a, p\rangle}Pp \end{aligned}$$

Let $B \in O(V)$, $v \in V$ and define $Tp = Bp + v$ for $p \in V$. Now define $\hat{T}$ by:

$$\hat{T} : \begin{cases} \mathbb{E}^{n+2}_{\nu+1} & \to \mathbb{E}^{n+2}_{\nu+1} \\ p & \mapsto \psi(p_0, Tp_1) \end{cases}$$

Since ψ is a warped product decomposition, it follows by Proposition D.5.1 that $\hat{T}$ induces an isometry of some open subset of $\mathbb{E}^{n+2}_{\nu+1}$ onto itself. We will now calculate $\hat{T}$ explicitly.

For arbitrary $x \in \mathbb{E}^{n+2}_{\nu+1}$ write $x = p + \tilde{p}$ where $p \in V$ and $\tilde{p} \in V^\perp$.

$$\begin{aligned} (Tq_1)^2 &= \left\| \frac{1}{\langle a, x\rangle}Bp + v \right\| \\ &= (\frac{1}{\langle a, x\rangle^2}(Px)^2 + \frac{2}{\langle a, x\rangle}\langle Bp, v\rangle + v^2) \end{aligned}$$

$$\begin{aligned} \psi(q_0, Tq_1) &= \langle a, q_0\rangle b + \langle a, q_0\rangle Tq_1 + (\langle b, q_0\rangle - \frac{1}{2}\langle a, q_0\rangle (Tq_1)^2)a \\ &= \langle a, x\rangle b + \langle a, x\rangle Tq_1 + (\langle b, x\rangle + \frac{1}{2\langle a, x\rangle}(Px)^2 - \frac{1}{2}\langle a, x\rangle (Tq_1)^2)a \\ &= \langle a, x\rangle b + \langle a, x\rangle Tq_1 + (\langle b, x\rangle - (\langle Bp, v\rangle + \frac{1}{2}\langle a, x\rangle v^2))a \\ &= \langle b, x\rangle a + \langle a, x\rangle b + Bp + \langle a, x\rangle v - (\langle Bp, v\rangle + \frac{1}{2}\langle a, x\rangle v^2))a \\ &= \tilde{p} + Bp + \langle a, x\rangle v - (\langle Bp, v\rangle + \frac{1}{2}\langle a, x\rangle v^2))a \end{aligned}$$

Hence if $p := Px$ and $\tilde{p} := (I - P)x$ then

$$\hat{T}x = \tilde{p} + Bp + \langle a, x\rangle v - (\langle Bp, v\rangle + \frac{1}{2}\langle a, x\rangle v^2))a$$

Thus since $\hat{T}$ is a linear isometry of $\mathbb{E}^{n+2}_{\nu+1}$ it follows that $\hat{T} \in O(\mathbb{E}^{n+2}_{\nu+1})$. Also $\hat{T}$ clearly fixes a so $\hat{T} \in I(\mathbb{P}^n_\nu)$.

Let the map ϕ be as in the hypothesis. Note that $\phi(B, v) = \hat{T}$. ϕ is a Lie group homomorphism, since

$$\begin{aligned}
(\hat{TS})x &= \psi(x_0, TSx_1) \\
&= \psi((\hat{S}x)_0, T((\hat{S}x)_1) \\
&= \hat{T}\hat{S}x
\end{aligned}$$

By definition of $\hat{T}$ it follows that ϕ is injective.

To show that ϕ is surjective, fix $T \in I(\mathbb{P}^n_\nu)$. Consider the decomposition:

$$p = \langle a, p\rangle b + \langle b, p\rangle a + Pp, \quad p \in \mathbb{E}^n_\nu$$

Using the fact that $(Tp)^2 = p^2$ with the above decomposition we obtain the following equations:

$$\begin{aligned}
p^2 = (Tp)^2 &= 2\langle a, Tp\rangle \langle b, Tp\rangle + (PTp)^2 \\
p \in V \Rightarrow p^2 &= (PTp)^2 \\
p = b \Rightarrow 0 = b^2 &= 2\langle b, Tb\rangle + (PTb)^2 \\
p = \psi(b, \tilde{p}) \Rightarrow 0 = p^2 &= 2\langle b, T\tilde{p}\rangle + \langle PTb, PT\tilde{p}\rangle
\end{aligned}$$

The second equation implies that $PT \in O(V)$. We claim that $\phi(PT, PTb) = T$. This can be seen by decomposing the action of T with respect to the above decomposition and then using the last three equations and the fact that $T \in I(\mathbb{P}^n_\nu)$.

Hence ϕ is a Lie group isomorphism. ■

We also note that if $\psi : \mathbb{E}^n_\nu \to \mathbb{P}^n_\nu$ is the standard embedding from Eq. (8.3.1), then ψ is equivariant, i.e. in the notation of the proof $\psi \circ T(p) = \hat{T} \circ \psi(p)$.

D.6 Warped Product decompositions of Spherical submanifolds of Pseudo-Euclidean space

D.6.1 Spherical submanifolds of $\mathbb{E}^n_\nu(\kappa)$

In this section we will classify the spherical submanifolds of $\mathbb{E}^n_\nu(\kappa)$. In particular we will show that they all have the form $\mathbb{E}^n_\nu(\kappa) \cap (\bar{p} + W)$ for some $\bar{p} \in \mathbb{E}^n_\nu$ and some subspace W. Although not all spherical submanifolds will have this form since we are only considering the case of pseudo-Riemannian manifolds. We will see that all spherical submanifolds of $\mathbb{E}^n_\nu(\kappa)$ arise as restrictions of spherical submanifolds of $\mathbb{E}^n_\nu$.

The following lemma concerns a submanifold N of $\mathbb{E}^n_\nu(\kappa)$. We denote by H' the mean curvature normal of N in $\mathbb{E}^n_\nu(\kappa)$ and H the mean curvature normal of N in $\mathbb{E}^n_\nu$. Similar definitions hold for the second fundamental forms h' and h. As usual r denotes the dilatational vector field.

Lemma D.6.1

If N is a submanifold of $\mathbb{E}^n_\nu(\kappa)$ then the following equations hold:

$$h(X,Y) = h'(X,Y) - \langle X, Y \rangle \frac{r}{r^2}$$

$$H = H' - \frac{r}{r^2} \tag{D.6.1}$$

In particular, N is an umbilical submanifold of $\mathbb{E}^n_\nu(\kappa)$ iff it is an umbilical submanifold of $\mathbb{E}^n_\nu$. In fact, N is a spherical submanifold of $\mathbb{E}^n_\nu(\kappa)$ iff it is a spherical submanifold of $\mathbb{E}^n_\nu$. □

Proof These formulas follow from lemma 3.5 and corollary 3.1 in [Che11]. ■

Now we consider the problem of finding the sphere in $\mathbb{E}^n_\nu(\kappa)$ passing through a point $\overline{p}$ with tangent space V and mean curvature normal z at $\overline{p}$. We make this precise as follows.

Let $\overline{p} \in \mathbb{E}^n_\nu(\kappa)$ be arbitrary, $V \subset T_{\overline{p}}\mathbb{E}^n_\nu(\kappa)$ a non-degenerate subspace with $m := \dim V \geq 1$, $\mu := \operatorname{ind} V$ and $z \in V^\perp \cap T_{\overline{p}}\mathbb{E}^n_\nu(\kappa)$.

Now let $a := \kappa\overline{p} - z$. Then assuming this data defines a submanifold of $\mathbb{E}^n_\nu(\kappa)$, we use Eq. (D.6.1) to obtain the mean curvature normal in $\mathbb{E}^n_\nu$ at $\bar{p}$, which is given as follows:

$$z - \kappa\overline{p} = -a$$

Then this determines a sphere in $\mathbb{E}^n_\nu$ with initial data $(\overline{p}, V, a)$ by Theorem D.4.1. Note that $a \neq 0$. In the following theorem we will show that this sphere in $\mathbb{E}^n_\nu$ is in fact the sphere in $\mathbb{E}^n_\nu(\kappa)$ determined by $(\overline{p}, V, a)$. First let $W := \mathbb{R}a \oplus V$ and $\tilde{\kappa} := a^2$.

Theorem D.6.2 (Spherical submanifolds of $\mathbb{E}^n_\nu(\kappa)$)
There is exactly one m-dimensional connected and geodesically complete spherical submanifold $\tilde{N}$ of $\mathbb{E}^n_\nu(\kappa)$ with $\bar{p} \in \tilde{N}$, $T_{\bar{p}}\tilde{N} = V$ and having mean curvature vector at $\bar{p}$, z. $\tilde{N}$ is an open submanifold of N; $N = \mathbb{E}^n_\nu(\kappa) \cap (\bar{p} + W)$ is the spherical submanifold determined by $(\bar{p}, V, a)$ in $\mathbb{E}^n_\nu(\kappa)$ and $\mathbb{E}^n_\nu$. In fact, N can be given explicitly as follows (where $\simeq$ means isometric to):

(a) *a is timelike, then $\mu \leq \nu - 1$ and $N \simeq H^m_\mu(\tilde{\kappa})$*

(b) *a is spacelike, then $N \simeq S^m_\mu(\tilde{\kappa})$*
For cases (b) and (c), let $c = \bar{p} - \frac{a}{\tilde{\kappa}}$ be the center of N, then N is given as follows:

$$N = c + \{p \in W \mid p^2 = \frac{1}{\tilde{\kappa}}\}$$

(c) *a is lightlike, then $\mu \leq \nu - 1$ and $N \simeq \mathbb{E}^m_\mu$*

$$N = \bar{p} + \{p - \frac{1}{2}p^2 a \mid p \in V\}$$

□

Remark D.6.3
The relationship between $\tilde{N}$ and N follows from Remark D.4.2, since the above spheres are the spheres in $\mathbb{E}^n_\nu$ determined by $(\bar{p}, V, a)$ in Theorem D.4.1. □

Remark D.6.4
N is a geodesic submanifold of $\mathbb{E}^n_\nu(\kappa)$ iff $z = 0$ iff W intersects the origin. □

Proof First we note that it suffices to show that there exists a single connected and geodesically complete sphere satisfying the initial conditions. By Lemma 3.2.5, it must be unique.

The three definitions of N given above follow directly from Theorem D.4.1 with initial data $(\bar{p}, V, a)$. Hence the relevant intrinsic properties of N follow from Theorem D.4.1. For the remainder of the proof we will assume N is given by those definitions, and we will prove the following.

Claim D.6.4.1
$N = \mathbb{E}^n_\nu(\kappa) \cap (\bar{p} + W)$ □

Proof Note that the following equations are satisfied: $\langle \bar{p}, \bar{p} \rangle = \dfrac{1}{\kappa}$, $\langle a, \bar{p} \rangle = 1$

First we consider the case of Items (a) and (b). We can always write $p = c + \tilde{p}$ where $\tilde{p} \in W$. Also note that the following holds:

$$\begin{aligned}\langle c, c\rangle &= \langle \bar{p}, \bar{p}\rangle - 2\left\langle \bar{p}, \frac{a}{\tilde{\kappa}}\right\rangle + \frac{1}{\tilde{\kappa}^2}\langle a, a\rangle \\ &= \frac{1}{\kappa} - 2\frac{1}{\tilde{\kappa}} + \frac{1}{\tilde{\kappa}} \\ &= \frac{1}{\kappa} - \frac{1}{\tilde{\kappa}}\end{aligned}$$

Then since $\langle c, a\rangle = 0$, we have

$$\begin{aligned}\langle p, p\rangle &= c^2 - 2\langle c, \tilde{p}\rangle + \tilde{p}^2 \\ &= \frac{1}{\kappa} - \frac{1}{\tilde{\kappa}} + \tilde{p}^2\end{aligned}$$

The above equation shows that $p \in \mathbb{E}^n_\nu(\kappa)$ iff $\tilde{p} \in W(\tilde{\kappa})$, which proves the result.

Now for Item (c). We can always write $p = \bar{p} + v + wa$ where $v \in V$ and $w \in \mathbb{R}$. Hence

$$\langle p, p\rangle = \frac{1}{\kappa} + v^2 + 2w$$

The above equation shows that $p \in \mathbb{E}^n_\nu(\kappa)$ iff $w = -\frac{1}{2}v^2$, which proves the result. ■

Thus we have shown that N is a spherical submanifold of $\mathbb{E}^n_\nu$ contained in $\mathbb{E}^n_\nu(\kappa)$. It then follows from Lemma D.6.1 that N is a spherical submanifold of $\mathbb{E}^n_\nu(\kappa)$ with mean curvature normal z at $\bar{p}$. Furthermore by Proposition D.1.2 (b), this sphere is of constant curvature $\kappa + z^2 = a^2 = \tilde{\kappa}$. ■

Now we mention when we can restrict a sphere in $\mathbb{E}^n_\nu$ to one in $\mathbb{E}^n_\nu(\kappa)$. Suppose $(\bar{p}, V, -z)$ determines a sphere in $\mathbb{E}^n_\nu$ with $\bar{p} \in \mathbb{E}^n_\nu(\kappa)$ and $V \subset T_{\bar{p}}\mathbb{E}^n_\nu(\kappa)$. Then define z' as

$$z' := z + \kappa\bar{p} \in V^\perp$$

We know that $\bar{p} \in V^\perp$ and $z \in V^\perp$ by hypothesis. In order for $\langle z', \bar{p}\rangle = 0$, we must additionally assume $\langle z, \bar{p}\rangle = -1$. In this case, $(\bar{p}, V, -z)$ define initial data for a sphere in $\mathbb{E}^n_\nu(\kappa)$. It follows from the above theorem that this sphere is simultaneously the sphere in $\mathbb{E}^n_\nu$ and in $\mathbb{E}^n_\nu(\kappa)$ determined by $(\bar{p}, V, -z)$.

D.6.2 Warped Product decompositions of Spherical submanifolds of Pseudo-Euclidean space

Suppose $\psi : N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k \to \mathbb{E}^n_\nu(\kappa)$ is a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ associated with initial data $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$ where each $a_i = \kappa\bar{p} - z_i$. By Eq. (D.3.1), the mean curvature vectors at $\bar{p}$ satisfy the following equation:

$$\langle z_i, z_j \rangle = -\kappa \quad i \neq j$$

By Theorem D.6.2, $L_i(\bar{p})$ is a spherical submanifold of $\mathbb{E}^n_\nu$ determined by $(\bar{p}, V_i, a_i)$. Note that $a_i \neq 0$. Furthermore the above equation implies that

$$\langle a_i, a_j \rangle = 0 \quad i \neq j$$

Also recall that by assumption, the a_i are linearly independent. Thus the initial data $(\bar{p}; (\mathbb{R}\bar{p} \oplus V_0) \oplus V_1 \oplus \cdots \oplus V_k; a_1, \ldots, a_k)$ determines a proper warped product decomposition of the ambient space $\mathbb{E}^n_\nu$. Furthermore, we note that this warped product decomposition is in canonical form; the canonical form was specifically designed to have this property. We now consider the converse problem of restricting a warped product decomposition in $\mathbb{E}^n_\nu$ to $\mathbb{E}^n_\nu(\kappa)$. The following theorem shows that this is always possible when the warped product in $\mathbb{E}^n_\nu$ is proper and in canonical form:

Theorem D.6.5 (Restricting Warped products to $\mathbb{E}^n_\nu(\kappa)$)
Let ψ be a proper warped product decomposition of $\mathbb{E}^n_\nu$ associated with $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$ which is in canonical form. Suppose $\kappa^{-1} := \bar{p}^2 \neq 0$ and let $N' := N_0(\kappa) \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k$. Note that $N_0(\kappa)$ is an open subset of the sphere in $\mathbb{E}^n_\nu(\kappa)$ determined by $(\bar{p}, (\bar{p}^\perp \cap V_0), 0)$. Then $\phi : N' \to \mathbb{E}^n_\nu(\kappa)$ defined by $\phi := \psi|_{N'}$ is a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ determined by $(\bar{p}; (\bar{p}^\perp \cap V_0) \bigoplus_{i=1}^{k} V_i; a_1, ..., a_k)$.

Furthermore for any point $p \in \mathrm{Im}(\psi)$ with $p^2 \neq 0$, the leaf of the foliation induced by N_i, $L_i(p)$, is simultaneously a sphere in $\mathbb{E}^n_\nu$ and $\mathbb{E}^n_\nu(\frac{1}{p^2})$. Also ψ is in canonical form at every $p \in \mathrm{Im}(\psi)$. □

Proof By Eq. (D.4.9) in Corollary D.4.10 it follows that ϕ is a diffeomorphism onto $\phi(N') \subseteq \mathbb{E}^n_\nu(\kappa)$. Clearly the restriction of the metric on N to N' is still a warped product metric. Hence it follows that ϕ is a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$, i.e. an isometry from a warped product. Furthermore by Theorem D.6.2 it follows that for each $i > 0$, N_i is also the sphere in $\mathbb{E}^n_\nu(\kappa)$ determined by $(\bar{p}, V_i, z_i)$.

Now for the last point, fix $p \in \mathrm{Im}(\psi)$ with $p^2 \neq 0$. Let $\tilde{r}$ be the dilatational vector field in N_0 and $r := \psi_*\tilde{r}$. Can show that r is also the dilatational vector field in $\mathbb{E}^n_\nu$ (e.g.

see Eq. (D.4.1)). Now if $\rho_i = \langle \tilde{r}, a_i \rangle$, then it follows from Proposition 3.5.9 (2) that the mean curvature vector H_i is:

$$H_i = -\frac{a_i}{\rho_i}$$

Hence $\langle \tilde{r}, -H_i \rangle = 1$. Thus at p, by making the identification $r = p$, we see that TN_i is orthogonal to $p = \psi_* \tilde{p}$ and $\langle \tilde{p}, -H_i \rangle = 1$. It follows from the discussion following Theorem D.6.2 that $L_i(p)$ is also a sphere in $\mathbb{E}^n_\nu(\frac{1}{p^2})$. ∎

Remark D.6.6 (Connectedness)
Remark D.4.7 gives the appropriate modifications of $\mathrm{Im}(\psi)$ when each N_i for $i > 0$ are required to be connected. When $N_0(\kappa) \simeq H^m_0(\tilde{\kappa})$ or $N_0(\kappa) \simeq S^m_m(\tilde{\kappa})$, $N_0(\kappa)$ is disconnected [O'N83, Section 4.6] and so we modify $N_0(\kappa)$ as follows: By Theorem D.6.2 it follows that $N_0(\kappa)$ is an open subset of the sphere in $\mathbb{E}^n_\nu$ determined by $(\bar{p}, (\bar{p}^\perp \cap V_0), \kappa\bar{p})$. Thus to enforce connectedness, it follows by Remark D.4.2 that we must replace $N_0(\kappa)$ with

$$N_0(\kappa) \cap \{p \in V_0 \mid \langle \kappa\bar{p}, p \rangle > 0\}$$

Proof $a = \kappa\bar{p}$, $\tilde{\kappa} = a^2 = \kappa$

$$\begin{aligned} c &= \bar{p} - \frac{a}{\tilde{\kappa}} \\ &= 0 \end{aligned}$$

∎

Now we show the effect of this on $\phi(N')$ when $\nu = 1$.

Case 1 a_i is time-like for some i

$N_0(\kappa)$ is automatically connected since $N_0(\kappa) \subset \{p \in V_0 \mid \langle a_i, p \rangle > 0 \text{ for each i}\}$, then $N_0(\kappa) \subset \{p \in V_0 \mid \langle \kappa\bar{p}, p \rangle > 0\}$ since $\langle a_i, \kappa\bar{p} \rangle = \kappa < 0$ (see [O'N83, P. 143] and Nolker's proof of the hyperbolic case).

Case 2 null case, $a := a_1$ is light-like

$N_0(\kappa)$ is connected here as well. First observe that it follows from the equation for ψ in Corollary D.4.10 that

$$\langle a, \psi(p_0, p_1) \rangle = \langle a, p_0 \rangle > 0$$

Thus it follows that $N_0(\kappa)$ and $\phi(N')$ are in the time cone opposite to a (see remarks preceding Nolker's proof of the hyperbolic case). Thus it follows that $N_0(\kappa) \subset \{p \in V_0 \mid \langle \kappa\bar{p}, p \rangle > 0\}$, so $N_0(\kappa)$ and hence $\phi(N')$ are connected.

In this case $\phi(N')$ is the maximal connected component of $\mathbb{E}^n_\nu(\kappa)$ passing through $\bar{p}$.

Case 3 a_i is space-like for each i

First observe that it follows from the proof of Corollary D.4.10 that $c = P_0 c \in W_0$ and $p_i - c_i \in W_i$ for $i > 0$, hence

$$\begin{aligned}\langle c, \psi(p_0, \ldots, p_k)\rangle &= \left\langle c, P_0 p_0 + \sum_{i=1}^{k} \langle a_i, p_0\rangle (p_i - c_i)\right\rangle \\ &= \langle c, P_0 p_0\rangle \\ &= \langle c, p_0\rangle\end{aligned}$$

Also since since $\langle c, a_i\rangle = 0$, we have that

$$\begin{aligned}\langle c, c\rangle &= \langle c, \bar{p}\rangle \\ &= \left\langle \bar{p} - \sum_{i=1}^{k} \frac{a_i}{\kappa_i}, \bar{p}\right\rangle \\ &= \frac{1}{\kappa} - \sum_{i=1}^{k} \frac{1}{\kappa_i} \\ &< 0\end{aligned}$$

In other words, c is time-like. Also the above equation shows that $\langle c, \kappa\bar{p}\rangle > 0$, thus c and $\kappa\bar{p}$ are in opposite time cones (see [O'N83, P. 143]). Hence,

$$\{p \in V_0 \mid \langle \kappa\bar{p}, p\rangle > 0\} = \{p \in V_0 \mid \langle \kappa c, p\rangle > 0\}$$

Thus since $\langle c, \psi(p_0, \ldots, p_k)\rangle = \langle c, p_0\rangle$, we see that $\phi(N')$ becomes

$$\phi(N') \cap \{p \in \mathbb{E}^n_\nu \mid \langle \kappa c, p\rangle > 0\}$$ □

In the following corollary we show how to obtain any warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ by restricting an appropriate warped product decomposition of $\mathbb{E}^n_\nu$. The "appropriate" warped product product decomposition of $\mathbb{E}^n_\nu$ to restrict follows from the discussion preceding the above theorem. Thus together with the above theorem, we have the following corollary:

Corollary D.6.7 (Warped product decompositions of $\mathbb{E}^n_\nu(\kappa)$)
Suppose $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, \ldots, a_k)$ define initial data for a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$.

Let ϕ be the warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ given in the above theorem by restricting the warped product decomposition of $\mathbb{E}^n_\nu$ with initial data $(\bar{p}; (\mathbb{R}\bar{p} \ominus V_0) \bigoplus_{i=1}^{k} V_i; a_1, \ldots, a_k)$.

Then ϕ is a warped product decomposition of $\mathbb{E}^n_\nu(\kappa)$ determined by $(\bar{p}; \bigoplus_{i=0}^{k} V_i; a_1, ..., a_k)$. □

We now mention which warped product decompositions are possible in $\mathbb{E}^n_\nu(\kappa)$. We do this by finding out when it's possible to restrict a warped product on the ambient space. Given a warped product $(V_0 \oplus V_1 \oplus \cdots \oplus V_k; a_1, \ldots, a_k)$ passing through an arbitrary point in $\mathbb{E}^n_\nu$, in order to restrict it to $\mathbb{E}^n_\nu(\kappa)$, we need it to pass through a point $\bar{p} \in V_0$ with $\bar{p}^2 = \kappa$ satisfying $\langle \bar{p}, a_i \rangle = 1$. So for a fixed $\kappa \neq 0$, we enumerate the distinct warped products in $\mathbb{E}^n_\nu$, expand $\bar{p} \in V_0$ so that $\langle \bar{p}, a_i \rangle = 1$ and determine if it's possible for $\bar{p}^2 = \kappa$. By making use of Theorem D.4.5, we have the following results:

Theorem D.6.8 (Warped products in Spherical submanifolds of $\mathbb{E}^n$ and M^n) *Suppose $N = N_0 \times_{\rho_1} N_1 \times \cdots \times_{\rho_k} N_k$ is a warped product decomposition of an open subset of a spherical submanifold of $\mathbb{E}^n$ or M^n. This warped product is necessarily proper. If at most one of the N_i are intrinsically flat, then N is isometric to one of the following warped products:*

In S^n:

$$S^m \times_{\rho_1} S^{n_1} \times \cdots \times_{\rho_s} S^{n_s}$$

In dS^n:

$$\begin{gathered} dS^m \times_{\lambda_1} \mathbb{E}^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\ dS^m \times_{\tau_1} H^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\ S^m \times_{\rho_1} dS^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\ dS^m \times_{\rho_1} S^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \end{gathered}$$

In H^n:

$$\begin{gathered} H^m \times_{\lambda_1} \mathbb{E}^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\ H^m \times_{\tau_1} H^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \\ H^m \times_{\rho_1} S^{n_1} \times_{\rho_2} S^{n_2} \times \cdots \times_{\rho_s} S^{n_s} \end{gathered}$$

where $\nabla\rho_i$,$\nabla\tau_i$,$\nabla\lambda_i$ is a spacelike,timelike, lightlike vector field respectively. □

Proof For the proof that the warped products are proper, see Lemma D.3.2. ■

Appendix E

Lexicographic ordering of complex numbers

Complex numbers can be given a natural lexicographic ordering (as in dictionaries) by using their Cartesian product structure:

Definition E.0.9
Suppose $\lambda = a + ib$ and $\omega = c + id$ are complex numbers. We write $\lambda < \omega$ if: $b < d$ or $(b = d$ and $a < c)$ $\square$

In the following we use "xor" to mean exclusive or and "or" has its standard meaning. Suppose $\lambda, \omega, \nu \in \mathbb{C}$ and $a \in \mathbb{R}^+$, one can check that this ordering has the following properties:

trichotomy: $\lambda = \omega$ xor $\lambda < \omega$ xor $\omega < \lambda$

transitivity: If $\lambda < \omega$ and $\omega < \nu$ then $\lambda < \nu$

translation invariance: If $\lambda < \omega$ then $\lambda + \nu < \omega + \nu$

dilatation invariance: If $\lambda < \omega$ then $a\lambda < a\omega$

skew symmetry: If $\lambda < \omega$ then $-\omega < -\lambda$

Furthermore we note that if $\lambda, \omega \in \mathbb{R}$ then this ordering reduces to the natural ordering of real numbers.

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