Families of Thue Inequalities with Transitive Automorphisms

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

A family of parameterized Thue equations is defined as

$$F_{\{t,s,\ldots\}}(X,Y) = m, \qquad m \in \mathbb{Z}$$

where $F_{\{t,s,\ldots\}}(X,Y)$ is a form in X and Y with degree greater than or equal to 3 and integer coefficients that are parameterized by $t, s, \ldots \in \mathbb{Z}$. A variety of these families have been studied by different authors.

In this thesis, we study the following families of Thue inequalities

$$\begin{aligned}
|sx^3 - tx^2y - (t+3s)xy^2 - sy^3| &\le 2t + 3s, \\
|sx^4 - tx^3y - 6sx^2y^2 + txy^3 + sy^4| &\le 6t + 7s, \\
|sx^6 - 2tx^5y - (5t+15s)x^4y^2 - 20sx^3y^3 + 5tx^2y^4 & \\
+(2t+6s)xy^5 + sy^6| &\le 120t + 323s,
\end{aligned}$$

where s and t are integers. The forms in question are “simple”, in the sense that the roots of the underlying polynomials can be permuted transitively by automorphisms.

With this nice property and the hypergeometric functions, we construct sequences of good approximations to the roots of the underlying polynomials. We can then prove that under certain conditions on s and t there are upper bounds for the number of integer solutions to the above Thue inequalities.

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Dedication

This dissertation is lovingly dedicated to my parents, Xilian Chen and Ziyi An. Their support, encouragement, and constant love have sustained me throughout my life.

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Chapter 1

Introduction

A Diophantine equation is a polynomial equation over rationals in two or more unknowns such that only the integer solutions are searched or studied. It has been a subject of investigation for over 1800 years.

The word *Diophantine* refers to the Hellenistic mathematician of the 3rd century, *Diophantus of Alexandria*, who made a study of such equations and was one of the first mathematicians to introduce symbolism into algebra. The mathematical study of Diophantine problems that Diophantus initiated is now called Diophantine analysis.

The reason people are interested in studying Diophantine equations includes the following:

- Its a fun challenge.
- It gives justication for other studying subjects, *e.g.*, algebraic number theory or algebraic geometry.
- It leads to other interesting questions. For example Pell equations, $x^2 - dy^2 = 1$, lead to questions about continued fractions and fundamental units. Ljunggrens equation $A^4 - 2B^2 = 8$ is related to approximations of π. Fermats Last Theorem $x^n + y^n = z^n$ lead to questions about unique factorization domains, cyclotomic fields, elliptic curves and modular forms.

We start with the simplest linear Diophantine equation in two variables

$$ax + by = c,$$

where $a, b, c \in \mathbb{Z}$. This equation has solutions if and only if $\gcd(a, b)|c$, in which case the solution can be found by a reverse process of Euclidean algorithm.

With the next step up in complexity, let's look at Pell's equation

$$x^2 - dy^2 = 1,$$

where d is a positive square-free integer. The non-trivial solutions (other than $(\pm 1, 0)$) are related to the fundamental unit for the ring $\mathbb{Z}[\sqrt{d}]$ and can be found by the rational approximation to $\sqrt{d}$. More precisely, the above Pell's equation can be written as

$$\left(\frac{x}{y}\right)^2 = d + \frac{1}{y^2}.$$

As $1/y^2$ can be arbitrarily small with big enough y, a solution (x, y) gives a rational approximation x/y to $\sqrt{d}$. In fact, the solutions can be found by performing the continued fraction expansion of $\sqrt{d}$ and testing each successive convergent until a solution to Pell's equation is found.

P. Fermat, J. Wallis, L. Euler, J.L. Lagrange, and C.F. Gauss in the early 19th century mainly studied Diophantine equations of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

where $a, b, c, d, e,$ and f are integers, *i.e.*, general inhomogeneous equations of the second degree with two unknowns. Lagrange used continued fractions in his study of general inhomogeneous Diophantine equations of the second degree with two unknowns. Gauss developed the general theory of quadratic forms, which is the basis of solving certain types of Diophantine equations.

In studies on Diophantine equations of degrees higher than two significant success was attained only in the 20th century. It was established by A. Thue. Let $F \in \mathbb{Z}[X, Y]$ be a homogeneous polynomial of degree $n \geq 3$ which is irreducible over the rationals and m be an integer. Then the diophantine equation

$$F(x, y) = m \tag{1.1}$$

is called a *Thue equation*. In 1909, Thue proved his famous result about this equation:

Theorem (Thue). *(1.1) has only finitely many solutions* $(x, y) \in \mathbb{Z}^2$.

Thue's proof is based on his approximation theorem: Let α be an algebraic number of degree $n \geq 2$ and $\epsilon > 0$. Then there exists a positive number $c(\alpha, \epsilon)$, such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha, \epsilon)}{q^{n/2+1+\epsilon}}.$$

The constant $c(\alpha, \epsilon)$ is not effective in that given α and ϵ the proof does not give a means of calculating $c(\alpha, \epsilon)$. Since his approximation is not effective, Thue's theorem is not effective, meaning that it does not give an upper bound for the sizes of the solutions. It does, however, lead to an upper bound for the number of solutions.

Since then, many authors studied the Thue equation in various forms and by different methods. In this chapter, we will give a brief survey of these results.

1.1 Solution of Single Thue Equations

In 1968, after his great work on linear forms in logarithms of algebraic numbers, A. Baker [7] could give an effective upper bound for the solutions of any given Thue equation (1.1):

Theorem (Baker). *Let $\kappa > n + 1$ and $(x, y) \in \mathbb{Z}^2$ be a solution of (1.1). Then*

$$\max\{|x|, |y|\} < Ce^{\log^{\kappa} m},$$

where $C = C(n, \kappa, F)$ is an effectively computable number.

These bounds have been improved since that time. For example, Bugeaud and Győry [10] proved the following:

Theorem (Bugeaud-Győry). *Let $B \geq \max\{|m|, e\}$, α be a root of $F(X, 1)$, $K := \mathbb{Q}(\alpha)$, $R := R_K$ the regulator of K and r the unit rank of K. Let $H \geq 3$ be an upper bound for the absolute values of the coefficients of F.*

Then all solutions $(x, y) \in \mathbb{Z}^2$ of (1.1) satisfy

$$\max\{|x|, |y|\} < \exp\left(c \cdot R \cdot \max\{\log R, 1\} \cdot (R + \log(HB))\right)$$

and

$$\max\{|x|, |y|\} < \exp\left(c' \cdot H^{2n-2} \cdot \log^{2n-1} H \cdot \log B\right),$$

where $c = 3^{r+27}(r + 1)^{7r+19}n^{2n+6r+14}$ and $c' = 3^{3(n+9)}n^{18(n+1)}$.

The bounds for the solutions obtained by Baker's method are rather large, thus the solutions cannot be found practically by enumeration. For a similar problem Baker and Davenport [6] proposed a method to reduce drastically the bound by using continued fraction reduction. Pethő and Schulenberg [31] replaced the continued fraction reduction by the LLL-algorithm and gave a general method to solve (1.1) for the totally real case with $m = 1$ and arbitrary degree n. Tzanakis and de Weger [45] described the general case. Finally, Bilu and Hanrot [8] were able to replace the LLL-algorithm by the much faster continued fraction method and solve Thue equations up to degree 1000.

1.2 Number of Solutions

We define a solution (x, y) to the Thue equation $F(x, y) = m$ to be primitive, if x and y are coprime integers. The problem of giving upper bounds (depending on m and the degree n) for the number of primitive solutions goes back to Siegel. Such a bound has been given by Evertse [15] in 1983:

Theorem (Evertse)**.** *Let $F(x, y)$ be an irreducible binary form with rational integral coefficients, of degree $n \geq 3$. Let m be a positive integer.*

Then the number of primitive solutions to

$$F(x, y) = m$$

does not exceed

$$7^{15\left(\binom{n}{3}+1\right)^2} + 6 \times 7^{2\binom{n}{3}(t+1)}$$

where t is the number of prime factors of the constant term m.

The above theorem is actually a special case of Evertse's work, in which he also treated equations in number fields. In 1987, an improved version was given by Bombieri and Schmidt [9]:

Theorem (Bombieri-Schmidt)**.** *Let m be a positive number and let $F(x, y)$ be an irreducible binary form of degree $n \geq 3$, with rational integral coefficients. Then the number of primitive solutions of the equation*

$$|F(x, y)| = m$$

does not exceed

$$cn^{1+t}$$

where c is an absolute constant and t is the number of distinct prime factors of m. When n is sufficiently large, the number of primitive solutions (with (x, y) and $(-x, -y)$ regarded as the same) does not exceed

$$215n^{1+t}.$$

This result is best possible (up to the constant 215), at least for $m = 1$, since the equation

$$|X^n + (X - Y)(2X - Y) \dots (nX - Y)| = 1$$

has at least $2(n + 1)$ solutions: $\pm\{(1, 1), \dots, (1, n), (0, 1)\}$.

In 1991, my supervisor, Stewart [34] showed the following:

Theorem. *Let F be a binary form with integer coefficients of degree $n \geq 3$, content 1, and nonzero discriminant D. Let m be a nonzero integer and let ϵ be a positive real number. Let g be any divisor of m with $g \geq |m|^{2/n+\epsilon}$. If $|m| \geq (\gcd(D, g^2))^{1/\epsilon}$, then the number of pairs of coprime integers (x, y) for which $F(x, y) = m$ is at most*

$$2800 \left(1 + \frac{1}{4\epsilon r}\right) n^{1+\omega(g)},$$

where $\omega(g)$ denotes the number of distinct prime factors of g.

Sharper bounds have been obtained for special classes of Thue equations. If only k coefficients of $F(x, y)$ are nonzero, the number of solutions depends on k and m only (and not on n). In 1987, Mueller and Schmidt [28] proved the following:

Theorem (Mueller-Schmidt)**.** *Let F be an irreducible binary form of degree n, with integral coefficients. If F has precisely 3 nonzero coefficients and $n \geq 9$, then the inequality*

$$|F(x, y)| \leq m$$

has at most $O(m^{2/n})$ solutions $(x, y) \in \mathbb{Z}^2$.

Shortly after that, they extended their result to the general case [29]:

Theorem (Mueller-Schmidt)**.** *Let F be an irreducible binary form of degree $n \geq 3$, with integral coefficients. If F has no more than k with $k \geq 3$ nonzero coefficients, then the inequality*

$$|F(x, y)| \leq m$$

has at most $O(k^2 m^{2/n}(1 + \log m^{1/n}))$ solutions $(x, y) \in \mathbb{Z}^2$.

In 2000, Thomas [38] gave absolute upper bounds for the number of solutions for $m = 1$ and $k = 3$:

Theorem (Thomas)**.** *Let F be an irreducible binary form of degree $n \geq 3$, with integral coefficients. Further suppose that F has precisely three nonzero coefficients. If $n \geq 38$, then the equation*

$$|F(x, y)| = 1$$

has at most 20 solutions $(x, y) \in \mathbb{Z}^2$ with $|xy| \geq 2$ ((x, y) and $(-x, -y)$ regarded as the same).

If only 2 coefficients of $F(x, y)$ are nonzero, the special case $ax^n - by^n = \pm 1$ with $ab \neq 0, x > 0, y > 0$ has been studied by many authors. In 2001, Bennett [7] proved there is at most one solution to this equation.

1.3 Families of Thue Equations

A family of parameterized Thue equations is a Thue equation with coefficients which are integer polynomials in one or more parameters. For example, a one-parameter family of Thue equations is the following:

$$F_t(X, Y) = m, \qquad m \in \mathbb{Z} \tag{1.2}$$

where $F_t \in \mathbb{Z}[t][X, Y]$ is an irreducible binary form of degree of at least 3 with coefficients that are integer polynomials in t.

In 1990, Thomas [35] investigated for the first time a parametrized family of cubic Thue equations. Since then, different families of Thue equations have been studied. Thomas proved

Theorem (Thomas)**.** *Let $t \in \mathbb{Z}$ and $t \geq 1.365 \times 10^7$. Then the equation*

$$x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \pm 1 \tag{1.3}$$

has only the trivial solutions: $(x, y) \in \pm\{(0, 1), (1, 0), (1, -1)\}$.

Mignotte [23] filled the gap $4 \leq t \leq 1.365 \times 10^7$ in 1993, proving that the only solution to (1.3) for these values of t are trivial ones (for $t = 0, 1, 2, 3$, (1.3) had been solved earlier).

The same family has been studied by Mignotte, Pethő and Lemmermeyer [25]. In 1996, they proved the following:

Theorem (Mignotte-Pethő-Lemmermeyer). *(1). Let $n \geq 1650$, k be positive integers. If*

$$|x^3 - (n-1)x^2y - (n+2)xy^2 - y^3| = k$$

for some $x, y \in \mathbb{Z}$, then

$$\log|y| < c_1 \log^2(n+2) + c_2 \log n \log k,$$

where

$$c_1 = 700 + 476.4\left(1 - \frac{1432.1}{n}\right)^{-1}\left(1.501 - \frac{1902}{n}\right) < 1956.4,$$

$$c_2 = 29.82 + \left(1 - \frac{1432.1}{n}\right)^{-1}\frac{1432}{n\log n} < 30.71.$$

(2). Let n be a nonnegative integer. If $(x, y) \in \mathbb{Z}^2$ is a solution of

$$|x^3 - (n-1)x^2y - (n+2)xy^2 - y^3| \leq 2n+1, \tag{1.4}$$

then either $(x, y) = t(u, v)$ with an integer t of absolute value $\leq \sqrt[3]{2n+1}$ and $\pm(u, v) \in \{(1,0), (0,1), (-1,1)\}$, or

$$\pm(x, y) \in \{(-1,1), (-1,2), (-1,n+1), (-n,-1), (n+1,-n), (2,-1)\},$$

except when $n = 2$, in which case (1.4) has the extra solutions

$$\pm(x, y) \in \{(-4,3), (8,3), (1,-4), (3,1), (3,-1)\}.$$

In 1991, Mignotte and Tzanakis [27] studied a family of cubic Thue equations that is similar to Thomas'. They proved

Theorem (Mignotte-Tzanakis). *Let $n \in \mathbb{Z}$ and $n \geq 3.67 \times 10^{32}$. Then the equation*

$$x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$$

has only the following solutions:

$$(x, y) \in \{(1,0), (0,-1), (1,-1), (-n-1,-1), (1,-n)\}.$$

Mignotte [24] could prove the same result for all $n \geq 3$ in 2000.

In 1991, Pethő [30] studied by using Thomas' method the two classes of Thue equations in the following theorem:

Theorem (Pethő). *Let $n \in \mathbb{Z}$. Put*

$$F_1(x,y) = x^4 - nx^3y - x^2y^2 + nxy^3 + y^4$$

and

$$F_2(x,y) = x^4 - nx^3y - 3x^2y^2 + nxy^3 + y^4$$

If $|n| \geq 9.9 \times 10^{27}$, then

(1) the only solutions to the equation

$$F_1(x,y) = 1$$

are $(x,y) \in \{(0,\pm1),(\pm1,0),(\pm1,\pm1),(\mp1,\pm1),(\pm n,\pm1),(\pm1,\mp n)\}$;

(2) the only solutions to the equation

$$|F_2(x,y)| = 1$$

are $(x,y) \in \{(0,\pm1),(\pm1,0),(\pm1,\pm1),(\pm1,\mp1)\}$.

The first result in the above theorem was improved by Mignotte, Pethő and Roth [26] in 1996. They solved this equation completely.

Theorem (Mignotte-Pethő-Roth). *Let $n \in \mathbb{Z}$. Then the only solutions to the equation*

$$x^4 - nx^3y - x^2y^2 + nxy^3 + y^4 = \pm1$$

are $\pm\{(0,1),(1,0),(1,1),(1,-1),(n,1),(1,-n)\}$ for $|n| \notin \{2,4\}$.

If $|n| = 2$, the family is reducible. If $|n| = 4$, four more solutions exist, they are

$$\pm(x,y) = \begin{cases} (8,7),(7,-8) & \text{if } n = 4 \\ (8,-7),(7,8) & \text{if } n = -4. \end{cases}$$

In 1993, Thomas [36] investigated the family of equation

$$\Phi_n(x,y) = x(x - a(n)y)(x - b(n)y) + uy^3,$$

where $n \in \mathbb{Z}, a(t), b(t) \in \mathbb{Z}[t]$ are monic polynomials of degree a and b respectively and $u \in \{\pm1\}$. Under a technical assumption on $a(n)$ and $b(n)$, he could prove that all solutions $(x,y) \in \mathbb{Z}^2$ to the equation $\Phi_n(x,y) = 1$ are given by $(1,0),(0,u),(a(n)u,u),(b(n)u,u)$, if n is greater than an effectively computable constant N. In particular, if $a(t), b(t)$ are monomials, Thomas' result gives:

Theorem (Thomas). *Let a and b be integers such that $0 < a < b$. Define a real number $N(a,b)$ by*

$$N(a,b) = \left(2 \cdot 10^6 \cdot (a+2b)\right)^{4.86/(b-a)}.$$

If $n \geq N(a,b)$, then the equation

$$x(x-n^a y)(x-n^b y) + uy^3 = 1, \qquad u = \pm 1$$

has only the four solutions $(1,0), (0,u), (n^a u, u), (n^b u, u)$.

In the same year, Thomas also published a paper about a two-parameter family of cubic Thue equations [37]. He proved

Theorem (Thomas). *Let b, c be nonzero integers such that $\Delta = 4c - b^2 > 0$, the discriminant of $t^3 - bt^2 + ct - 1$ is negative, and $c \geq \min\{4.2 \times 10^{41} \times |b|^{2.32}, 3.6 \times 10^{41} \times \Delta^{1.1582}\}$. Then the equation*

$$x^3 - bx^2y + cxy^2 - y^3 = 1$$

has only the trivial solutions, namely $(x,y) = (1,0), (0,-1)$.

In 1995, the family of quartic Thue equation $F_n(x,y) = x^4 - ax^3y - 6x^2y^2 + axy^3 + y^4 = c$ with $c \in \{\pm 1, \pm 4\}$ was completely solved by Lettl and Pethő [21]. They proved

Theorem (Lettl-Pethő). *Let $a \in \mathbb{Z}$ and $c \in \{\pm 1, \pm 4\}$. If $a \notin \{\pm 1, \pm 4\}$, the equation*

$$x^4 - ax^3y - 6x^2y^2 + axy^3 + y^4 = c$$

only has the trivial integral solutions in x, y, namely,

$$(x,y) \in \{(\pm 1, 0), (0, \pm 1), (\pm 1, 1), (\pm 1, -1).$$

Chen and Voutier [11] solved the equation $x^4 - ax^3y - 6x^2y^2 + axy^3 + y^4 = \pm 1$ independently in 1995.

The family of quartics $x^4 - a^2x^2y^2 + y^4$ was studied by Wakabayashi [46] in 1997. He proved

Theorem (Wakabayashi). *Let a be an integer. For $a \geq 8$, the only primitive solutions to the Thue inequality*

$$|x^4 - a^2x^2y^2 + y^4| \leq a^2 - 2$$

are $(x,y) = (0,0), (\pm 1, 0), (0, \pm 1), (\pm a, \pm 1), (\pm 1, \pm a), (\pm 1, \pm 1)$, with mixed signs.

Later in 2000, Wakabayashi [48] generalized this paper to the family of Thue inequalities of the form $|x^4 - a^2x^2y^2 - by^4| \leq a^2 + b - 1$. He found all solutions to this inequality when a is sufficiently large relative to b.

Theorem (Wakabayashi)**.** *Let $a, b \in \mathbb{N}$. Then the only primitive solutions to the Thue inequality*

$$|x^4 - a^2x^2y^2 - by^4| \leq a^2 + b - 1$$

are $(x, y) = (0, 0), (\pm 1, 0), (0, \pm 1), (\pm a, \pm 1), (\pm 1, \pm 1)$ with mixed signs, provided that

$$a \geq 5.3 \times 10^{10} b^{6.22},$$

or

$$b \in \{1, 2\}, \quad a \geq 1.$$

In 1997, Heuberger, Pethő and Tichy [19] completely solved the one-parameter family of quartic Thue equations

$$F_a(x, y) = x(x - y)(x - ay)(x - (a + 1)y) - y^4 = \pm 1,$$

where a is an integer. More precisely, they proved

Theorem (Heuberger-Pethő-Tichy)**.** *Let a be an integer. Put*

$$F_a(x, y) = x(x - y)(x - ay)(x - (a + 1)y) - y^4.$$

Then

$$|F_a(x, y)| = 1$$

only has the trivial solutions

$$(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1), (\pm a, \pm 1), (\pm(a + 1), \pm 1).$$

The more general form $x(x - y)(x - ay)(x - by) - y^4$ was studied by Pethő and Tichy [32]. They proved

Theorem (Pethő-Tichy)**.** *Let a, b be integers. Assume that*

$$10^{2 \cdot 10^{28}} < a + 1 < b \leq a \left(1 + \frac{1}{\log^4 a}\right).$$

Put

$$F_{a,b}(x,y) = x(x-y)(x-ay)(x-by) - y^4.$$

Then

$$F_{a,b}(x,y) = \pm 1$$

has only the trivial solutions

$$(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1), (\pm a, \pm 1), (\pm b, \pm 1).$$

In 1999, Lettl, Pethő and Voutier [22] published a paper about the simple families of Thue inequalities. The particular forms under their consideration are distinguished by being "simple" forms. They proved the following:

Theorem (Lettl-Pethő-Voutier)**.** *Let a be an integer. Put*

$$F_a^{(3)}(x,y) = x^3 - ax^2y - (a+3)xy^2 - y^3$$

$$F_a^{(4)}(x,y) = x^4 - ax^3y - 6x^2y^2 + axy^3 + y^4$$

$$F_a^{(6)}(x,y) = x^6 - 2ax^5y - (5a+15)x^4y^2 - 20x^3y^3 + 5ax^2y^4 + (2a+6)xy^5 + y^6$$

(1) For $a \geq 89$, the only primitive solutions $(x,y) \in \mathbb{Z}^2$ to the inequality

$$|F_a^{(6)}(x,y)| \leq 120a + 323$$

with $-\frac{y}{2} < x \leq y$ are $(0,1), (1,1), (1,2), (-1,3)$;

(2) For $a \geq 58$, the only primitive solutions $(x,y) \in \mathbb{Z}^2$ to the inequality

$$|F_a^{(4)}(x,y)| \leq 6a + 7$$

with $|x| \leq y$ are $(0,1), (\pm 1, 1), (\pm 1, 2)$;

(3) For $a \geq 30$, let $(x,y) \in \mathbb{Z}^2$ be a primitive solution to

$$|F_a^{(3)}(x,y)| \leq k(a)$$

with $\frac{8k(a)}{2a+3} \leq y$ and $-\frac{y}{2} < x \leq y$. Then

$$|y| < 0.4(120k(a))^{1+\epsilon(a)}, \qquad \text{with } \epsilon(a) = \frac{2.14}{\log(a+1.5) - 3.44}.$$

These forms have been studied by different authors. For example, $|F_a^{(4)}(x,y)| = 1$ has been solved completely by Lettl and Pethő [21], and by Chen and Voutier [11] independently.

This type of form is the focus of this thesis. Let F be a binary form. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$$

and define the binary form F^A by

$$F^A(x,y) = F(ax+by, cx+dy).$$

This defines an action of $GL_2(\mathbb{Q})$ on $\mathbb{Q}[x,y]$.

Definition Two forms $F, G \in \mathbb{Q}[x,y]$ are called *equivalent* if there exists some $A \in GL_2(\mathbb{Q})$ and $r \in \mathbb{Q}^*$ such that $rG = F^A$, where $\mathbb{Q}^* = \mathbb{Q}\backslash\{0\}$.

Definition Let $F \in \mathbb{Q}[x,y]$ be a form. We call $A \in GL_2(\mathbb{Z})$ an automorphism of F if $F^A = F$.

Definition A form $F \in \mathbb{Q}[x,y]$ is called *simple* if F is irreducible over $\mathbb{Q}$ with degree ≥ 3 and there exists some non-trivial $A \in GL_2(\mathbb{Q})/\mathbb{Q}^* I_2$ such that $\phi_A : z \to Az := \frac{az+b}{cz+d}$ permutes the zeros of the underlying polynomial $F(x,1)$ transitively; here I_2 is the identity matrix of order 2 and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$.

One can see that if a form F is simple, then it is close to having non-trivial automorphism. The three forms in the previous theorem are all simple, since we have

$$F_a^{(3)}(y, -x-y) = F_a^{(3)}(x,y),$$

$$F_a^{(4)}(x-y, x+y) = -4F_a^{(4)}(x,y),$$
$$F_a^{(6)}(x-y, x+2y) = -27F_a^{(6)}(x,y).$$

We'll consider the same forms but with two parameters.

In 1999, Wakabayashi [47] [49] proved

Theorem (Wakabayashi)**.** *Let a, b be integers.*

(1) *Suppose that* $a \geq 360b^4$. *Then the only primitive solutions with* $y \geq 0$ *of the Thue inequality*

$$|x^3 + axy^2 + by^3| \leq a + |b| + 1$$

are $(0,0), (\pm 1, 0), (0,1), (\pm 1, 1), (-b/d, a/d)$, *where* $d = \gcd(a,b)$.

(2) *Suppose that* $|b| = 1$ *or* $|b| = 2$. *Then for all* $a \geq 1$ *the only primitive solutions to*

$$|x^3 + axy^2 + by^3| \leq a + |b| + 1$$

are $(0,0), (\pm 1, 0), (0,1), (\pm 1, 1), (-b/d, a/d)$, *where* $d = \gcd(a,b)$, *except the cases* $|b| = 1$, $1 \leq a \leq 3$ *and* $|b| = 2$, $1 \leq a \leq 7$. *Further, all solutions in the exceptional cases can be listed.*

A family of quintic Thue equations had been investigated by Gaál and Lettl [16]. In 2000, they proved

Theorem (Gaál-Lettl)**.** *Let* $t \in \mathbb{Z}$. *If* $|t| \geq 3.28 \times 10^{15}$, *then the only integral solutions* (x,y) *to the equation*

$$\begin{aligned} F_t(x,y) = x^5 + (t-1)x^4y - (2t^3 + 4t + 4)x^3y^2 + \\ (t^4 + t^3 + 2t^2 + 4t - 3)x^2y^2 + (t^3 + t^2 + 5t + 3)xy^4 + y^5 = \pm 1 \end{aligned}$$

are $(\pm 1, 0), (0, \pm 1)$.

In 2000, Togbé [39] proved

Theorem (Togbé)**.** *Let* n *be an integer such that* n, $n+2$ *and* n^2+4 *are square-free. Then the equation*

$$x^4 - n^2x^3y - (n^3 + 2n^2 + 4n + 2)x^2y^2 - n^2xy^3 + y^4 = 1$$

has only the trivial solutions $(\pm 1, 0), (0, \pm 1)$ *for* $n \leq 5 \times 10^6$ *or for* $n \geq 1.191 \times 10^{19}$.

Tobgé̋ [42] improved his result in 2006. He showed that

Theorem (Togbé)**.** *Let* n *be an integer such that* n, $n+2$ *and* n^2+4 *are square-free. Then the equation*

$$x^4 - n^2x^3y - (n^3 + 2n^2 + 4n + 2)x^2y^2 - n^2xy^3 + y^4 = 1$$

has only the trivial solutions $(\pm 1, 0), (0, \pm 1)$ *for* $n \geq 2$. *In the case of* $n = 1$, *there exists an extra solution* $\pm(1, -1)$ *besides* $\pm(0, 1)$ *and* $\pm(1, 0)$.

In 2002, Dujella and Jadrijević [13] solved another family of quartic Thue equations. Later in 2004 they extended the same family of quartic Thue equation to the inequality case [14].

Theorem (Dujella-Jadrijević)**.** *Let a be an integer.*

(2002) If $a \geq 3$, then the equation

$$x^4 - 4ax^3y + (6a+2)x^2y^2 + 4axy^3 + y^4 = 1$$

has only the trivial solutions $(x, y) \in \{(\pm 1, 0), (0, \pm 1)\}$.

(2004) If $a \geq 4$, then the inequality

$$|x^4 - 4ax^3y + (6a+2)x^2y^2 + 4axy^3 + y^4| \leq 6a + 4$$

has only the following solutions (x, y) in integers:

$$(\pm 1, 0), (0, \pm 1), (1, \pm 1), (-1, \pm 1), (\pm 1, \mp 2), (\pm 2, \pm 1).$$

In 2003, Wakabayashi proved

Theorem (Wakabayashi)**.** *Let $a \in \mathbb{Z}$. If $a \geq 1.35 \times 10^{14}$, then the equation*

$$x^3 - a^2xy^2 + y^3 = 1$$

has only the trivial integral solutions:

$$(x, y) \in \{(0, 1), (1, 0), (1, a^2), (a, 1), (-a, 1)\}.$$

In 2004, Togbé [40] proved

Theorem (Togbé)**.** *Let $n \geq 1$ be an integer. The equation*

$$x^3 - (n^3 - 2n^2 + 3n - 3)x^2y - n^2xy^2 - y^3 = \pm 1$$

has only the trivial integral solutions:

$$\pm\{(1, 0), (0, 1)\},$$

except for the case $n = 2$, when there are seven more pairs of solutions:

$$\pm\{(9, -13), (5, -14), (4, 1), (2, -3), (1, -1), (1, -3), (1, -2), (0, 1), (1, 0)\}.$$

In 2005, Jadrijević [20] proved

Theorem (Jadrijević). *Let $m, n \in \mathbb{Z}$ and $m > 0, n > 0$. Then there are no solutions to the equation*

$$x^4 - 2mnx^3y + 2(m^2 - n^2 + 1)x^2y^2 + 2mnxy^3 + y^4 = 1$$

satisfying the additional conditions $\gcd(xy, mn) = 1$ *and* $xy \neq 0$.

In 2006, Togbé proved in [41]

Theorem (Togbé). *Let n be a nonnegative integer. Put*

$$\begin{aligned}\Phi_n(x, y) = x^3 &+ (n^8 + 2n^6 - 3n^5 + 3n^4 - 4n^3 + 5n^2 - 3n + 3)x^2y \\ &- (n^3 - 2)n^2xy^2 - y^3\end{aligned}$$

Then the solutions in integers x, y to the equation

$$\Phi_n(x, y) = \pm 1$$

are

$$\{\pm(1, 0), \pm(0, 1)\}, \qquad \text{if } n \geq 2,$$

and

$$\begin{cases} \{\pm(1, 0), \pm(0, 1), \pm(1, -1)\} \text{ if } n = 1, \\ \{\pm(1, 0), \pm(0, 1), \pm(1, -1), \pm(1, 2), \pm(2, -3), \pm(3, -1)\} \text{ if } n = 0. \end{cases}$$

and in [43]

Theorem (Togbé). *Let $a \in \mathbb{N}$. Put*

$$\begin{aligned}\Phi_a(x, y) = x^6 &- (a - 2)x^5y - (a^2 + a + 6)x^4y^2 \\ &+ (a^3 - 2a^2 + 6a - 10)x^3y^3 + (a^3 + 5a + 3)x^2y^4 \\ &+ (a^2 - a + 4)xy^5 - y^6\end{aligned}$$

If $a > 1.078 \times 10^{12}$, then the equation

$$\Phi_a(x, y) = \pm 1$$

only has the integral solutions $(x, y) = (0, \pm 1), (\pm 1, 0), (\pm 1, 1)$.

In the same year, Ziegler [53] investigated a family of quartic Thue equations with three parameters. He showed

Theorem (Ziegler)**.** *Let (x, y) be a solution to Thue equation*

$$x^4 - 4sx^3y - (2ab + 4s(a+b))x^2y^2 - 4absxy^3 + a^2b^2y^4 = \mu$$

with $s \in \mathbb{Z}, a, b \in \frac{1}{4}\mathbb{Z}, |a| \geq |b|$ and $0 \neq ab \in \mathbb{Z}$ and suppose $s > 7.23 \times 10^{10}|a|^{\frac{29+\sqrt{241}}{2}}$. Then necessarily $\mu = 1$. Furthermore, the only solutions are $(x, y) = (\pm 1, 0), (0, \pm 1)$ if $ab = \pm 1$ or those listed as follows:

$$\begin{aligned}(a, b, x, y) \in \{&(-17/4, -4, \pm 4, \pm 1), (17/4, 4, \pm 4, \mp 1)\\ &(-5/2, -2, \pm 2, \pm 1), (5/2, 2, \pm 2, \mp 1),\\ &(-2, -1, \pm 1, \pm 1), (2, 1, \pm 1, \mp 1),\\ &(-4, -15/4, \pm 4, \pm 1), (4, 15/4, \pm 4, \mp 1),\\ &(-2, -3/2, \pm 2, \pm 1), (2, 3/2, \pm 2, \mp 1)\}\end{aligned} \tag{1.5}$$

In 2007, Wakabayashi [50] studied cubic Thue equations with nontrivial automorphisms. He proved

Theorem (Wakabayashi)**.** *Let F be an irreducible cubic form with integer coefficients. Suppose that the discriminant of F is positive and F has non-trivial automorphism.*

Let $a, b \in \mathbb{Z}$. Then the number of integer solutions to the Thue equation

$$F(x, y) = bx^3 - ax^2y - (a + 3b)xy^2 - by^3 = 1$$

is three or zero, except for the following case, where the number of solutions is N_F,

$$\begin{aligned} &F \sim x^3 + x^2y - 2xy^2 - y^3, && N_F = 9,\\ &F \sim x^3 - 3xy^2 - y^3, && N_F = 6,\\ &F \sim x^3 - 2x^2y - 5xy^2 - y^3, && N_F = 6.\end{aligned}$$

For two forms $F, G \in \mathbb{Z}[x, y]$, "$F \sim G$" means there exists a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$$

such that

$$F^A(x, y) = F(ax + by, cx + dy) = G(x, y).$$

In [51], Wakabayashi extended Lettl, Pethő and Voutier's work [22] to two-parameter families of Thue inequalities. He obtained the following results:

Theorem (Wakabayashi)**.** *Let $s, t \in \mathbb{Z}$. Put*

$$F_{s,t}^{(3)}(x, y) = sx^3 - tx^2y - (t + 3s)xy^2 - sy^3,$$

$$F_{s,t}^{(4)}(x, y) = sx^4 - tx^3y - 6sx^2y^2 + txy^3 + sy^4,$$

$$F_{s,t}^{(6)}(x, y) = sx^6 - 2tx^5y - (5t + 15s)x^4y^2 - 20sx^3y^3 + 5tx^2y^4 + (2t + 6s)xy^5 + sy^6.$$

(1) If $s \geq 1$ and $t \geq 97.3s^{48/19}$, then the only primitive solutions $(x, y) \in \mathbb{Z}^2$ to the Thue inequality

$$|F_{s,t}^{(6)}(x, y)| \leq 120t + 323s$$

with $y \geq 0$ are

$$(\pm 1, 0), (0, 1), (\pm 1, 1), (\pm 2, 1), (-3, 1), (\pm 1, 2), (-3, 2), (-1, 3), (-2, 3)$$

(2) If $s \geq 1$ and $t \geq 70s^{28/9}$, then the only primitive solutions $(x, y) \in \mathbb{Z}^2$ to the Thue inequality

$$|F_{s,t}^{(4)}(x, y)| \leq 6t + 7s$$

with $y \geq 0$ are

$$(\pm 1, 0), (0, 1), (\pm 1, 1), (\pm 2, 1), (\pm 1, 2)$$

(3) Let $s \geq 1$ and $t \geq 64s^{9/2}$, then the only primitive solutions $(x, y) \in \mathbb{Z}^2$ to the Thue inequality

$$|F_{s,t}^{(3)}(x, y)| \leq 2t + 3s$$

with $-1/2 < x/y \leq 1$ and $y > 0$ are

$$\begin{cases} (0, 1), (1, 1), (-1, t + 2) & \text{if } s = 1, \\ (0, 1), (1, 1) & \text{if } s \geq 2. \end{cases}$$

Further, the only primitive solutions $(x, y) \in \mathbb{Z}^2$ with $y \geq 0$ are

$$\begin{cases} (1, 0), (0, 1), (\pm 1, 1), (-2, 1), (-1, 2), & \\ (-1, t + 2), (-t - 2, t + 1), (t + 1, 1) & \text{if } s = 1, \\ (1, 0), (0, 1), (\pm 1, 1), (-2, 1), (-1, 2) & \text{if } s \geq 2. \end{cases}$$

In 2008, Togbé [44] completely solved another family of cubic Thue equations.

Theorem (Togbé)**.** *Let $n \in \mathbb{Z}$ be nonnegative. Then the integer solutions to the equation*

$$x^3 - n(n^2+n+3)(n^2+2)x^2y - (n^3+2n^2+3n+3)xy^2 - y^3 = \pm 1$$

are

$$\begin{cases} \{(\pm 1, 0), (0, \pm 1)\}, & \text{if } n > 0; \\ \{\pm(-3,2), \pm(-1,1), \pm(-1,3), \pm(0,1), \pm(1,0), \pm(2,1)\}, & \text{if } n = 0. \end{cases}$$

In 2009, He, Jadrijević and Togbé [17] proved

Theorem (He-Jadrijević-Togbé)**.** *Let $c \geq 1$ be an integer. Then for all $c \geq 1$, the Thue inequality*

$$|x^4 - 4x^3y - (2c-2)x^2y^2 + (4c+4)xy^3 - (2c-1)y^4| \leq \max\left\{\frac{c}{4}, 4\right\}$$

has primitive solutions of the form $(x,y) = (\pm 1, 0), \pm(1,1)$. These solutions are the only primitive solutions if $c \neq 2n^2 - 2, n \in \mathbb{N}, n > 1$ and $c \neq 1, 2$. The additional primitive solutions are given by:

(i) $(x,y) = \pm(n+1, n), \pm(n-1, n), \pm(2n+1, 1), \pm(2n-1, -1)$ *for* $c = 2n^2 - 2, n \in \mathbb{N}, n > 1$*;*

(ii) $(x,y) = \pm(0,1), \pm(2,1)$ *for* $c = 2$*;*

(iii) $(x,y) = (0, \pm 1), \pm(2,1), \pm(3,1)$ *for* $c = 1$.

Also in 2009, Akhtari [1] studied general cubic forms with big discriminant. She proved:

Theorem (Akhtari)**.** *Let F be a binary cubic form of degree with integer coefficients. If its discriminant $D_F > 1.4 \times 10^{57}$, then the equation*

$$|F(x,y)| = 1$$

has at most 7 integer solutions.

If F is equivalent to a reduced form which is not monic and has discriminant $D > 9 \times 10^{58}$, then the equation

$$F(x,y) = 1$$

has at most 6 integer solutions.

Akhtari and Okazaki proved a similar result for quartic Thue equations in 2010.

Theorem (Akhtari-Okazaki). *Let F be an irreducible quartic form with integer coefficients and D_F be its discriminant. If $D_F > 10^{500}$, then the equation*

$$|F(x,y)| = 1$$

has at most 61 integer solutions, counting (x,y) and $(-x,-y)$ only once.

For a special quartic form with vanishing J-invariant, Akhtari [2] proved the following

Theorem (Akhtari). *Let*

$$F(x,y) = a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4$$

be an irreducible binary form with integer coefficients and positive discriminant that splits in $\mathbb{R}$. Let

$$I_F = a_2^2 - 3a_1a_3 + 12a_0a_4$$

and

$$J_F = 2a_2^3 - 9a_1a_2a_3 + 27a_1^2a_4 - 72a_0a_2a_4 + 27a_0a_3^2.$$

If $J_F = 0$, then the equation

$$|F(x,y)| = 1$$

has at most 12 solutions in integers x and y (with (x,y) and $(-x,-y)$ regarded as the same); and the inequality

$$|F(x,y)| \leq h$$

has at most 12 primitive solutions (x,y), with $|y| \geq \frac{h^{3/4}}{(3I_F)^{1/8}}$.

In 2011, Dujella, Ibrahimpašić and Jadrijević [12] solved the following family of quartic Thue inequalities:

Theorem (Dujella-Ibrahimpašić-Jadrijević). *Let $n \geq 3$ be an integer. Then all the primitive solutions to the inequality*

$$|x^4 + 2(1-n^2)x^2y^2 + y^4| \leq 2n + 3$$

are $(0, \pm 1), (\pm 1, 0), (\pm 1, \pm\sqrt{2(n^2-1)}), (\pm\sqrt{2(n^2-1)}, \pm 1)$, where the latter two solutions are only valid if $2(n^2-1)$ is a perfect square.

In the same year, He, Kihel, and Togbé [18] proved

Theorem (He-Kihel-Togbé)**.** *Let $c \geq 3$ be an integer. Suppose $n = c^2 + c - 5$ and $0 < |\mu| \leq c + 2$. Then the equation*

$$x^4 - (n+1)x^3y - nx^2y^2 + 2xy^3 + y^4 = \mu$$

has integer solutions (x, y) if and only if $\mu = 1$. In this case, all primitive solutions are given by $(x, y) = (0, \pm 1), (\pm 1, 0), \pm(1, -1)$.

In 2012, Akhtari [3] improved the result of Okazaki and herself by showing:

Theorem (Akhtari)**.** *Let F be an irreducible binary quartic form with integer coefficients. If the discriminant of F is greater than an explicitly computable constant D_0, then the equation*

$$|F(x, y)| = 1$$

has at most U_F integer solutions, counting (x, y) and $(-x, -y)$ only once, where $U_F = 6$ if $F(x, 1) = 0$ has no real root, $U_F = 14$ if $F(x, 1) = 0$ has two real and one pair of complex conjugate roots and $U_F = 26$ if $F(x, 1) = 0$ has four real roots.

Wakabayashi extended his work on cubic Thue equation with automorphisms to the quartic case in 2012. He proved [52]

Theorem (Wakabayashi)**.** *Let $a, b \in \mathbb{Z}$. Then the equation*

$$|bx^4 - ax^3y - 6bx^2y^2 + axy^3 + by^4| = 1$$

has 0 or 4 integer solutions, except for the cases $b = 1, a = \pm 1, \pm 4$ when there are 8 solutions.

Again, Put

$$\begin{aligned}
F_{s,t}^{(3)}(x, y) &= sx^3 - tx^2y - (t + 3s)xy^2 - sy^3, \\
F_{s,t}^{(4)}(x, y) &= sx^4 - tx^3y - 6sx^2y^2 + txy^3 + sy^4, \\
F_{s,t}^{(6)}(x, y) &= sx^6 - 2tx^5y - (5t + 15s)x^4y^2 \\
&\qquad - 20sx^3y^3 + 5tx^2y^4 + (2t + 6s)xy^5 + sy^6.
\end{aligned}$$

Consider the Thue inequalities:

$$|F_{s,t}^{(3)}(x, y)| \leq 2t + 3s, \tag{1.6}$$

$$|F_{s,t}^{(4)}(x, y)| \leq 6t + 7s, \tag{1.7}$$

$$|F_{s,t}^{(6)}(x, y)| \leq 120t + 323s. \tag{1.8}$$

Lettl, Pethő and Voutier [22] had completely solved these inequalities for $s = 1$ and t greater than a determined positive number. Wakabayashi [51] extended their work and completely solved the inequalities with the following conditions:

$$\begin{aligned} &s \geq 1, \quad t \geq 64 s^{9/2}, \quad \text{for (1.6)}, \\ &s \geq 1, \quad t \geq 70 s^{28/9}, \quad \text{for (1.7)}, \\ &s \geq 1, \quad t \geq 97.3 s^{48/19}, \quad \text{for (1.8)}. \end{aligned} \tag{1.9}$$

In this thesis, we'll prove the following:

Theorem 1.1. *Let τ be an integer with $\tau \geq 5$ and let s, t be positive integers such that the form*

$$F^{(3)}_{s,t}(x, y) = sx^3 - tx^2y - (t + 3s)xy^2 - sy^3$$

is irreducible over $\mathbb{Q}$. Suppose that $s \geq 1$ and $t \geq 1.2 \cdot 10^6 s^{3+21/2^\tau}$. Then other than the trivial solutions

$$\begin{aligned} \pm\{&(0, 1), (1, -1), (-1, 0), (1, 1), (1, -2), (-2, 1), \\ &(-1, t + 2), (-t - 2, t + 1), (t + 1, 1)\} \text{ if } s = 1, \\ \pm\{&(0, 1), (1, -1), (-1, 0), (1, 1), (1, -2), (-2, 1)\} \text{ if } s \geq 2, \end{aligned}$$

there are at most 6τ primitive integer solutions to the Thue inequality

$$|F^{(3)}_{s,t}(x, y)| \leq 2t + 3s.$$

Theorem 1.2. *Let τ be an integer with $\tau \geq 2$ and let s, t be positive integers such that the form*

$$F^{(4)}_{s,t}(x, y) = sX^4 - tX^3Y - 6sX^2Y^2 + tXY^3 + sY^4$$

is irreducible over $\mathbb{Q}$. Suppose that $s \geq 1$ and $t \geq 1200 s^{2+4/3^\tau}$. Then other than the trivial solutions

$$\begin{aligned} \pm\{&(1, 0), (0, 1), (1, 1), (1, -1), \\ &(1, 2), (2, -1), (2, 1), (1, -2)\} \end{aligned}$$

there are at most 8τ primitive integer solutions to the Thue inequality

$$|F^{(4)}_{s,t}(x, y)| \leq 7s + 6t.$$

Theorem 1.3. *Let τ be an integer with $\tau \geq 1$ and let s, t be positive integers such that the form*

$$F_{s,t}^{(6)}(x,y) = sx^6 - 2tx^5y - (5t+15s)x^4y^2 \\ - 20sx^3y^3 + 5tx^2y^4 + (2t+6s)xy^5 + sy^6$$

is irreducible over $\mathbb{Q}$. Suppose that $s \geq 1$ and $t \geq 200s^{12/7+1/5^\tau}$. Then other than the trivial solutions

$$\pm\{(0,1),(1,0),(1,1),(-1,2),(-1,1),(-2,1), \\ (2,1),(-1,3),(-3,2),(1,2),(-2,3),(-3,1)\}$$

there are at most 12τ integer solutions to the Thue inequality

$$|F_{s,t}^{(6)}(x,y)| \leq 120t + 323s.$$

Compared with the results of Wakabayashi [51], we extend the range of the parameters s and t, but with the cost of weakened results. More precisely, with the condition in (1.9), Wakabayashi proved that the inequalities in Theorem 1.1, 1.2 and 1.3 have only trivial solutions. We loosen the condition by considering a wider range of s and t. In this case we are not able to explicitly solve the inequalities but instead, we have to assume a possible solution. Thus our results are ineffective. The following tables sketch the comparison:

	Wakabayashi [51]	This thesis
Cubic case	$t \geq 64s^{4.5}$	$t \geq 1.2 \cdot 10^6 s^{3.66}$
Quartic case	$t \geq 70s^{3.111}$	$t \geq 1200s^{2.45}$
Sextic case	$t \geq 97.3s^{2.526}$	$t \geq 200s^{1.92}$

Table 1.1: Comparison of the conditions, assume $s \geq 1$ in all cases

	Wakabayashi [51]	This thesis
Cubic case	0	at most 30
Quartic case	0	at most 16
Sextic case	0	at most 12

Table 1.2: Comparison of the results: the number of solutions other than the trivial ones

Chapter 2

Hypergeometric Method and Gap Principle

In this chapter, we prepare some results that will be needed in the later chapters. Throughout this chapter, μ denotes either $3, 4$ or 6.

2.1 Contour integrals and the hypergeometric method

We're going to follow the arguments of Rickert [33] and Wakabayashi [49] to prove some preliminary results that will be used to obtain the irrationality measures of certain algebraic numbers. The idea here is by finding the Padé approximation of the function

$$\frac{\sqrt[\mu]{1+x}}{\sqrt[\mu]{1-x}}$$

one can construct a sequence of "good" approximations to some algebraic number related to it and further deduce an irrationality measure of this number.

For integers $n \geq 1, l = 0, 1$ and $j = 1, 2$, define integrals

$$I_{ln} = \frac{1}{2\pi i} \int_{\Gamma} \frac{z^l (1 + xz)^{n+\frac{1}{\mu}}}{(z^2 - 1)^{n+1}} dz \tag{2.1}$$

and

$$I_{ljn} = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{z^l (1 + xz)^{n+\frac{1}{\mu}}}{(z^2 - 1)^{n+1}} dz, \tag{2.2}$$

where Γ is a simple closed counter-clockwise curve enclosing both the point 1 and -1, and Γ_1 (Γ_2) is a simple closed counter clockwise curve enclosing 1 (-1) and not enclosing -1 (1). These integrals are well-defined for $|x| < 1$ if we take Γ and Γ_j so that they do not enclose $-1/x$.

Lemma 2.1. *For $n \geq 1$, $l = 0, 1$*

$$I_{l1n}(x) = p_{ln}(x)\sqrt[\mu]{1+x} \tag{2.3}$$

$$I_{l2n}(x) = (-1)^{l+1} p_{ln}(-x)\sqrt[\mu]{1-x} \tag{2.4}$$

$$I_{ln}(x) = p_{ln}(x)\sqrt[\mu]{1+x} - (-1)^{l} p_{ln}(-x)\sqrt[\mu]{1-x}, \tag{2.5}$$

where $p_{ln}(x)$ are polynomials of degree at most n with rational coefficients given by

$$p_{0n}(x) = \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{\mu}}{h} \binom{2n-h}{n-h} \frac{x^h(1+x)^{n-h}}{2^{2n+1-h}} \tag{2.6}$$

and

$$p_{1n}(x) = \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{\mu}}{h} \left(\binom{2n-h}{n-h} \frac{1}{2^{2n+1-h}} - \binom{2n-h-1}{n-h-1} \frac{1}{2^{2n-h}} \right) x^h (1+x)^{n-h}. \tag{2.7}$$

Proof. Obviously,

$$I_{ln}(x) = I_{l1n}(x) + I_{l2n}(x)$$

for $l = 0, 1$.

By residue theory, we have

$$
\begin{aligned}
I_{01n}(x) &= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(1+xz)^{n+\frac{1}{\mu}}}{(z^2-1)^{n+1}}dz \\
&= \frac{1}{n!}\lim_{z\to 1}\frac{d^n}{dz^n}\left((z-1)^{n+1}\cdot\frac{(1+xz)^{n+\frac{1}{\mu}}}{(z^2-1)^{n+1}}\right) \\
&= \frac{1}{n!}\lim_{z\to 1}\frac{d^n}{dz^n}\left((1+xz)^{n+\frac{1}{\mu}}(z+1)^{-(n+1)}\right) \\
&= \frac{1}{n!}\lim_{z\to 1}\sum_{h=0}^{n}\binom{n}{h}\frac{d^h}{dz^h}\left((1+xz)^{n+\frac{1}{\mu}}\right)\cdot\frac{d^{n-h}}{dz^{n-h}}\left((z+1)^{-(n+1)}\right) \\
&= \frac{1}{n!}\lim_{z\to 1}\sum_{h=0}^{n}\binom{n}{h}k!\binom{n+\frac{1}{\mu}}{h}x^h(1+xz)^{n-h+\frac{1}{\mu}} \\
&\qquad\qquad \cdot(-1)^{n-h}(n-h)!\binom{2n-h}{n-h}(z+1)^{-(2n+1-h)} \\
&= \lim_{z\to 1}\sum_{h=0}^{n}\binom{n}{h}\frac{h!(n-h)!}{n!}(-1)^{n-h}\binom{n+\frac{1}{\mu}}{h}\binom{2n-h}{n-h} \\
&\qquad\qquad \cdot x^h(1+xz)^{n-h+\frac{1}{\mu}}(z+1)^{-(2n+1-h)} \\
&= \sum_{h=0}^{n}(-1)^{n-h}\binom{n+\frac{1}{\mu}}{h}\binom{2n-h}{n-h}\frac{x^h(1+x)^{n-h+\frac{1}{\mu}}}{2^{2n+1-h}} \\
&= p_{0n}(x)\sqrt[\mu]{1+x}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_{11n}(x) &= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{z(1+xz)^{n+\frac{1}{\mu}}}{(z^2-1)^{n+1}}dz \\
&= \frac{1}{n!}\lim_{z\to 1}\frac{d^n}{dz^n}\left(z(1+xz)^{n+\frac{1}{\mu}}(z+1)^{-(n+1)}\right) \\
&= \frac{1}{n!}\lim_{z\to 1}\sum_{h=0}^{n}\binom{n}{h}\frac{d^h}{dz^h}(1+xz)^{n+\frac{1}{\mu}}\cdot\frac{d^{n-h}}{dz^{n-h}}\left(z(z^2-1)\right)^{-(n+1)}) \\
&= \frac{1}{n!}\lim_{z\to 1}\sum_{h=0}^{n}\binom{n}{h}h!\binom{n+\frac{1}{\mu}}{h}x^h(1+xz)^{n-h+\frac{1}{\mu}} \\
&\qquad\cdot\left(\binom{n-h}{0}z\frac{d^{n-h}}{dz^{n-h}}(z+1)^{-(n+1)}+\binom{n-h}{1}\frac{d^{n-h-1}}{dz^{n-h-1}}(z+1)^{-(n+1)}\right) \\
&= \frac{1}{n!}\lim_{z\to 1}\sum_{h=0}^{n}\binom{n}{h}h!\binom{n+\frac{1}{\mu}}{h}x^h(1+xz)^{n-h+\frac{1}{\mu}}\left((-1)^{n-h}(n-h)!\binom{2n-h}{n-h}\right. \\
&\qquad \left. z(z+1)^{-(2n+1-h)}+(-1)^{n-h-1}(n-h)!\binom{2n-h-1}{n-h-1}(z+1)^{-(2n-h)}\right) \\
&= \sum_{h=0}^{n}(-1)^{n-h}\binom{n+\frac{1}{\mu}}{h}\left(\binom{2n-h}{n-h}\frac{1}{2^{2n+1-h}}-\binom{2n-h-1}{n-h-1}\frac{1}{2^{2n-h}}\right) \\
&\qquad\cdot x^h(1+x)^{n-h+\frac{1}{\mu}} \\
&= p_{1n}(x)\sqrt[\mu]{1+x}.
\end{aligned}
$$

By a change of variables, $z' = -z$, we see that, for $l = 0, 1$,

$$
\begin{aligned}
I_{l2n}(x) &= (-1)^{l+1}I_{l1n}(-x) \\
&= (-1)^{l+1}p_{ln}(-x)\sqrt[\mu]{1-x}.
\end{aligned}
$$

This completes the proof of the lemma. □

Put

$$
J_h = \frac{1}{2\pi i}\int_{\Gamma}\frac{z^h}{(z^2-1)^{n+1}}dz,
$$

and define the generating function

$$
J(x) = \sum_{h=0}^{\infty} J_h x^h.
$$

Lemma 2.2. *$J_h = 0$ for $0 \le h \le 2n$, and $J_{2n+1} = 1$. Further, for $|x| < 1$,*

$$J(x) = \frac{x^{2n+1}}{(1-x^2)^{n+1}}. \tag{2.8}$$

Proof. By the residue theory, it is well-known that the integrand is a rational function $P(z)/Q(z)$ with $\deg(Q) > 1 + \deg(P)$ and the integral over any closed contour containing all the zeros of Q, is equal to zero (This can be shown by a combination of partial fraction decomposition and residue calculation). Thus $J_h = 0$ for $0 \le h \le 2n$.

For $h = 2n + 1$, suppose that

$$\frac{z^{2n+1}}{(z^2-1)^{n+1}} = \sum_{j=1}^{n+1} \frac{a_j}{(z-1)^j} + \sum_{j=1}^{n+1} \frac{b_j}{(z+1)^j}.$$

Then we get

$$z^{2n+1} = (z+1)^{n+1} \sum_{j=1}^{n+1} a_j (z-1)^{n+1-j} + (z-1)^{n+1} \sum_{j=1}^{n+1} b_j (z+1)^{n+1-j}.$$

Comparing the coefficients before z^{2n+1} on both sides of the above equation, we get that

$$a_1 + b_1 = 1.$$

On the other hand, by the relation between residue and Laurent series expansion, we see that $J_{2n+1} = a_1 + b_1 = 1$. One can also prove this by a change of variables $z = 1/w$ together with residue calculus.

For $|x| < 1$,carefully choose Γ so that $|xz| < 1$. Then $\sum_{h=0}^{\infty} x^h z^h$ converges to $(1-xz)^{-1}$ on Γ. Thus,

$$\begin{aligned}
J(x) &= \sum_{h=0}^{\infty} J_h x^h \\
&= \sum_{h=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma} \frac{(xz)^h}{(z^2-1)^{n+1}} dz \\
&= \frac{1}{2\pi i} \int_{\Gamma} \frac{\sum_{h=0}^{\infty} (xz)^h}{(z^2-1)^{n+1}} dz \\
&= \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{(1-xz)(z^2-1)^{n+1}} dz.
\end{aligned}$$

Let $z = 1/w$. Then we have

$$\begin{aligned} J(x) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{(1-xz)(z^2-1)^{n+1}} dz \\ &= \frac{1}{2\pi i} \int_{\Gamma'} \frac{1}{\left(1-\frac{x}{w}\right)\left(\frac{1}{w^2}-1\right)^{n+1}} \left(-\frac{1}{w^2}\right) dw \\ &= \frac{1}{2\pi i} \int_{-\Gamma'} \frac{w^{2n+1}}{(w-x)(1-w^2)^{n+1}} dw, \end{aligned}$$

where $-\Gamma'$ is a counterclock-wise curve containing x but not 1 or -1. Thus,

$$J(x) = \lim_{w \to x} \frac{w^{2n+1}}{(1-w^2)^{n+1}} = \frac{x^{2n+1}}{(1-x^2)^{n+1}}.$$

□

Lemma 2.3. *The function $I_{0n}(x)$ has a zero of order $2n+1$ at $x = 0$, and the function $I_{1n}(x)$ has a zero of order $2n$ at $x = 0$.*

Proof. By Taylor expansion,

$$(1+xz)^{n+\frac{1}{\mu}} = \sum_{h=0}^{\infty} \binom{n+\frac{1}{\mu}}{h} x^h z^h$$

Then for $l = 0, 1$,

$$I_{ln}(x) = \sum_{h=0}^{\infty} \binom{n+\frac{1}{\mu}}{h} J_{h+l} x^h = \sum_{h=2n+1-l}^{\infty} \binom{n+\frac{1}{\mu}}{h} J_{h+l} x^h$$

by Lemma 2.2. This proves the lemma.

□

Lemma 2.4.

$$\Delta := \begin{vmatrix} p_{0n}(x) & -p_{0n}(-x) \\ p_{1n}(x) & p_{1n}(-x) \end{vmatrix} = c_{2n} x^{2n} \tag{2.9}$$

with

$$c_{2n} = \frac{(-1)^n}{2^{2n+1}} \binom{2n}{n} \binom{n+\frac{1}{\mu}}{2n}.$$

Proof. By Lemma 2.1, the degree of $\Delta(x)$ is at most $2n$. Also, by the definitions from Lemma 2.1, we see that

$$\begin{aligned}\Delta(x)\sqrt[\mu]{1-x} &= \begin{vmatrix} p_{0n}(x) & -p_{0n}(-x)\sqrt[\mu]{1-x} \\ p_{1n}(x) & p_{1n}(-x)\sqrt[\mu]{1-x} \end{vmatrix} \\ &= \begin{vmatrix} p_{0n}(x) & p_{0n}(x)\sqrt[\mu]{1+x} - p_{0n}(-x)\sqrt[\mu]{1-x} \\ p_{1n}(x) & p_{1n}(x)\sqrt[\mu]{1+x} + p_{1n}(-x)\sqrt[\mu]{1-x} \end{vmatrix} \\ &= \begin{vmatrix} p_{0n}(x) & I_{0n}(x) \\ p_{1n}(x) & I_{1n}(x) \end{vmatrix}.\end{aligned}$$

Then by Lemma 2.3, $\Delta(x)\sqrt[\mu]{1-x} = p_{0n}(x)I_{1n}(x) - p_{1n}(x)I_{0n}(x)$ has a Taylor expansion

$$p_{0n}(0)\binom{n+\frac{1}{\mu}}{2n}J_{2n+1}x^{2n} + \cdots .$$

Notice that the constant term in the Taylor expansion of $\sqrt[\mu]{1-x}$ is 1 and $p_{0n}(0) = \frac{(-1)^n}{2^{2n+1}}\binom{2n}{n}$, $J_{2n+1} = 1$. Therefore, we have

$$\Delta(x) = \frac{(-1)^n}{2^{2n+1}}\binom{2n}{n}\binom{n+\frac{1}{\mu}}{2n}x^{2n}.$$

□

Lemma 2.5. *Let ξ be a non-zero real number. Suppose that there are positive numbers ρ, P, l, L, d, Δ with $L/\Delta > 1$, and for each integer $n \geq 1$, two linear forms*

$$p_{jn} + q_{jn}\xi = l_{jn} \qquad j = 0, 1$$

in ξ with rational coefficients p_{jn} and q_{jn} satisfying the following conditions:

(i) $\begin{vmatrix} p_{0n} & q_{0n} \\ p_{1n} & q_{1n} \end{vmatrix} \neq 0$

(ii) $|q_{jn}| \leq \varrho P^n$

(iii) $|l_{jn}| \leq lL^{-n}$

(iv) *p_{jn} and q_{jn}, $j = 0, 1$ have a common denominator $\Delta_n \leq d\Delta^n$.*

Then for any integers p and q with $q > 0$, we have

$$\left|\xi - \frac{p}{q}\right| > \frac{1}{Cq^{\lambda}},$$

where

$$\lambda = 1 + \frac{\log(\Delta P)}{\log(L/\Delta)},$$

$$C = 2\varrho d\Delta P \left(\max\{2dl, 1\}\right)^{\frac{\log(\Delta P)}{\log(L/\Delta)}}.$$

Proof. Let p, q be integers with $q > 0$. Put

$$\delta = \left|\xi - \frac{p}{q}\right|.$$

For any $n \geq 1$, $j = 0, 1$, let $\eta_{jn} = q_{jn}p + qp_{jn}$. Note that

$$ql_{jn} - \eta_{jn} = q(q_{jn}\xi + p_{jn}) - (q_{jn}p + qp_{jn}) = qq_{jn}\left(\xi - \frac{p}{q}\right).$$

It follows that, for $j = 0, 1$,

$$\begin{aligned} |\eta_{jn}| &\leq \left|qq_{jn}\left(\xi - \frac{p}{q}\right)\right| + |ql_{jn}| \\ &\leq q\varrho P^n\delta + qlL^{-n}. \end{aligned}$$

By condition (i), for any n, we can fix a j so that $|\eta_{jn}| \neq 0$. This is a rational number with denominator Δ_n. Thus, by condition (iv), we have

$$|\eta_{jn}| \geq \frac{1}{\Delta_n} \geq \frac{1}{d\Delta^n}.$$

By assumption, $L/\Delta > 1$. Put

$$n = 1 + \left[\frac{\log(Cq)}{\log(L/\Delta)}\right],$$

where $C = \max\{2dl, 1\}$. This implies that

$$qlL^{-n} \leq \frac{1}{2d\Delta^n}.$$

Therefore, we have

$$q\varrho P^n \delta > \frac{1}{2d\Delta^n}.$$

It follows that

$$\delta > \frac{1}{2d\varrho q(P\Delta)^n} \geq \frac{1}{2d\varrho q(P\Delta)^{1+\frac{\log(Cq)}{\log(L/\Delta)}}} = \frac{1}{2d\varrho P\Delta C^{\frac{\log(P\Delta)}{\log(L/\Delta)}} q^{1+\frac{\log(P\Delta)}{\log(L/\Delta)}}}.$$

□

2.2 Gap principle

Lemma 2.6. *Let B, μ and ξ be real numbers with B and μ positive. Suppose that (x_1, y_1) and (x_2, y_2) are two pairs of integers with $x_1/y_1 \neq x_2/y_2$ satisfying*

$$\left|\xi - \frac{x_i}{y_i}\right| \leq \frac{1}{2By_i^{\mu}}, \qquad i = 1, 2. \tag{2.10}$$

Further suppose that $y_2 \geq y_1 > 0$. Then

$$y_2 \geq By_1^{\mu-1}. \tag{2.11}$$

Proof. By assumption, we have

$$\frac{x_1}{y_1} \neq \frac{x_2}{y_2}, \qquad y_2 \geq y_1 > 0,$$

that is,

$$x_1 y_2 - x_2 y_1 \neq 0.$$

Then

$$
\begin{aligned}
1 &\leq |x_1 y_2 - x_2 y_1| \\
&= |x_1 y_2 - y_1 y_2 \xi + y_1 y_2 \xi - x_2 y_1| \\
&= \left| y_1 y_2 \left(\frac{x_1}{y_1} - \xi \right) + y_1 y_2 \left(\xi - \frac{x_2}{y_2} \right) \right| \\
&\leq y_1 y_2 \left| \frac{x_1}{y_1} - \xi \right| + y_1 y_2 \left| \xi - \frac{x_2}{y_2} \right| \\
&\leq y_1 y_2 \left(\frac{1}{2By_1^{\mu}} + \frac{1}{2By_2^{\mu}} \right) \\
&\leq y_1 y_2 \cdot \frac{1}{2By_1^{\mu}} \cdot 2 \\
&= \frac{y_2}{By_1^{\mu-1}}.
\end{aligned}
$$

This gives

$$
y_2 \geq By_1^{\mu-1}. \tag{2.12}
$$

□

Chapter 3

Cubic Simple Form

In this chapter, we'll study the following inequality

$$|sx^3 - tx^2y - (t+3s)xy^2 - sy^3| \leq k, \tag{3.1}$$

where s, t are integers and $k = k(t,s)$ is linear in t and s. Let

$$F(x,y) = sx^3 - tx^2y - (t+3s)xy^2 - sy^3.$$

Suppose that s and t are positive integers such that F is irreducible over $\mathbb{Q}$. We have that F is a simple form since

$$F(y, -x-y) = F(x,y) \tag{3.2}$$

and the map

$$z \to -\frac{1}{z+1} \tag{3.3}$$

permutes the roots of $F(x,1)$ transitively. As discussed in the first chapter, Wakabayashi [51] completely solved (3.1) for $s \geq 1, t \geq 64s^{9/2}$ and $k = 2t+3s$. For the same k, we'll prove the following result:

Theorem 3.1. *Let s, t be positive integers such that*

$$sx^3 - tx^2y - (t+3s)xy^2 - sy^3$$

is irreducible over $\mathbb{Q}$ and let τ be an integer with $\tau \geq 5$. Suppose that $s \geq 1$ and $t \geq 1.2 \cdot 10^6 s^{3+21/2^\tau}$. Then other than the trivial solutions

$$\begin{aligned} &\pm\{(0,1),(1,-1),(-1,0),(1,1),(1,-2),(-2,1),\\ &\qquad (-1,t+2),(-t-2,t+1),(t+1,1)\} \text{ if } s=1,\\ &\pm\{(0,1),(1,-1),(-1,0),(1,1),(1,-2),(-2,1)\} \text{ if } s \geq 2, \end{aligned}$$

there are at most 6τ integer solutions to the Thue inequality

$$|sx^3 - tx^2y - (t+3s)xy^2 - sy^3| \le 2t + 3s. \tag{3.4}$$

Since the case when $s = 1$ had been explicitly solved by Lettl, Pethő and Voutier [22], in the following proof we always assume $s \ge 2$.

The main proof is based on the observation that the root of the underlying polynomial $F(x, 1)$ can be expressed in terms of cubic roots of algebraic numbers, due to the special shape of the simple form F. With hypergeometric functions, rational approximations to the (quotient of) cubic roots of algebraic numbers can be constructed, which, in turn, will give us a good rational approximation to the root of the underlying polynomial. This leads to an irrationality measure for the root. Then we use a routine argument to derive the upper bound for the size of the solutions from this measure. Together with a gap principle, we prove the bound for the number of solutions.

3.1 Elementary properties

From the relation (3.2), it is easy to see that if (x, y) is a solution to inequality (3.1), then

$$(y, -x-y), (-x-y, x), (-x, -y), (-y, x+y), (x+y, -x)$$

are also solutions to (3.1). Notice that the map (3.3) permutes the intervals

$$\left(-\frac{1}{2}, 1\right], \left(-2, -\frac{1}{2}\right], (-\infty, -2] \cup (1, +\infty).$$

If there exists an integer solution (x, y) to (3.1), we can always choose it from the above set of solutions to satisfy the following condition:

$$-\frac{1}{2} < \frac{x}{y} \le 1, \qquad \gcd(x, y) = 1, \qquad y \ge 0. \tag{3.5}$$

In the following proof, we'll always assume (x, y) satisfies (3.5) if it is a solution to (3.1). Let

$$f(x) = s^{-1}F(x, 1) = x^3 - wx^2 - (w+3)x - 1, \tag{3.6}$$

where $w = t/s$. Then we have

Lemma 3.2. *For $w \geq 4$, f has three real roots θ_{-1}, θ_0 and θ_w that satisfy the following:*

$$\begin{aligned} -1 - \frac{1}{w} + \frac{1}{w^2} < \theta_{-1} < -1 - \frac{1}{w} + \frac{2}{w^2}, \\ -\frac{1}{w+2} < \theta_0 < -\frac{1}{w+2} + \frac{4}{(w+2)w^2}, \\ w + 1 + \frac{2}{w} - \frac{3}{w^2} < \theta_w < w + 1 + \frac{2}{w}. \end{aligned}$$

Proof. For $w \geq 4$, direct computation gives

$$\begin{aligned} f\left(-1 - \frac{1}{w} + \frac{1}{w^2}\right) &= -\frac{w^4 - 4w^3 + 3w - 1}{w^6} < 0, \\ f\left(-1 - \frac{1}{w} + \frac{2}{w^2}\right) &= \frac{w^5 + w^4 + 7w^3 - 6w^2 - 12w + 8}{w^6} > 0, \\ f\left(-\frac{1}{w+2}\right) &= \frac{2w+3}{(w+2)^3} > 0, \\ f\left(-\frac{1}{w+2} + \frac{4}{(w+2)w^2}\right) &= -\frac{2w^4 + 5w^3 - 6w^2 + 12w - 8}{w^6} < 0, \\ f\left(w + 1 + \frac{2}{w} - \frac{3}{w^2}\right) &= -\frac{w^5 + 12w^4 + 10w^3 + 9w^2 - 54w + 27}{w^6} < 0, \\ f\left(w + 1 + \frac{2}{w}\right) &= \frac{3w^3 + 8w^2 + 12w + 8}{w^3} > 0. \end{aligned}$$

Then the lemma follows.

□

Suppose (x, y) is an integer solution to (3.1) that satisfies (3.5). From Lemma 3.2, we can see that $\frac{x}{y}$ is bounded away from θ_{-1} and θ_w, and it is close to θ_0 for $w \geq 4$. We then denote θ_0 by θ in the rest of this chapter. We now define the interval

$$\mathbb{I} = \left(-\frac{1}{w+2}, -\frac{1}{w+2} + \frac{4}{(w+2)w^2}\right).$$

We divide all integer solutions (x, y) with $y \geq 2$ of (3.1) that satisfy (3.5) into two groups.

Definition We call (x, y) an integer solution to (3.1) of type I if $\gcd(x, y) = 1, y \geq 2$ and

$$\frac{x}{y} \in \left(-\frac{1}{2}, -\frac{1}{w+2}\right] \cup \left[-\frac{1}{w+2} + \frac{4}{(w+2)w^2}, 1\right];$$

(x, y) is of type II if $\gcd(x, y) = 1, y \geq 2$ and

$$\frac{x}{y} \in \mathbb{I} = \left(-\frac{1}{w+2}, -\frac{1}{w+2} + \frac{4}{(w+2)w^2}\right).$$

Lemma 3.3. *Let (x, y) be an integer solution to (3.1) of type II. For $w \geq 1000$, we have*

$$\left|\theta - \frac{x}{y}\right| \leq \frac{1}{By^3}, \tag{3.7}$$

where

$$B = \frac{0.999t}{k}.$$

Proof. From Lemma 3.2, we have for $w \geq 4$

$$\theta_{-1} < -1, \qquad \theta_w > w + 1.$$

Since (x, y) is of type II,

$$-0.001 < \frac{x}{y} \leq 0.$$

We have

$$\left|\frac{x}{y} - \theta_{-1}\right| > 0.999, \qquad \left|\frac{x}{y} - \theta_w\right| > w, \tag{3.8}$$

for $w \geq 1000$. On the other hand, (x, y) satisfies

$$|F(x, y)| \leq k.$$

This is equivalent to

$$\left|sy^3\left(\frac{x}{y} - \theta_{-1}\right)\left(\frac{x}{y} - \theta\right)\left(\frac{x}{y} - \theta_w\right)\right| \leq k, \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\left|\theta - \frac{x}{y}\right| \leq \frac{k}{0.999swy^3} = \frac{1}{(0.999t/k)y^3}.$$

□

3.2 Irrationality of the root of f

Suppose that (x_0, y_0) is an integer solution to (3.1) that satisfies (3.5). In this section, we'll calculate a measure of irrationality of θ in terms of this solution. The idea is that one can rewrite θ in terms of $\frac{\sqrt[3]{1+\gamma}}{\sqrt[3]{1-\gamma}}$ for some algebraic number γ, thanks to the special form of F. Then we can apply the hypergeometric method discussed in the previous chapter to construct a sequence of "good" approximations to θ, from which the irrationality measure can be deduced.

For any complex number λ, let $\overline{\lambda}$ denote the complex conjugate of λ.

Lemma 3.4. *The form F can be rewritten as*

$$\begin{aligned} F(x,y) &= sx^3 - tx^2y - (t+3s)xy^2 - sy^3 \\ &= \frac{1}{2}\left(\eta(x-\rho y)^3 + \overline{\eta}(x-\overline{\rho}y)^3\right), \end{aligned}$$

where

$$\eta = s - \frac{(2t+3s)\sqrt{3}i}{9}, \qquad \rho = \frac{-1+\sqrt{3}i}{2},$$

and $i = \sqrt{-1}$.

Proof. By direct calculation, we have

$$\begin{aligned} (x-\rho y)^3 &= \left(x - \frac{-1+\sqrt{3}i}{2}y\right)^3 \\ &= \left(x^3 + \frac{3}{2}x^2y - \frac{3}{2}xy^2 - y^3\right) - \left(\frac{3\sqrt{3}}{2}x^2y + \frac{3\sqrt{3}}{2}xy^2\right)i. \end{aligned}$$

We need only to verify that the real part of $\eta(x-\rho y)^3$ is equal to $F(x,y)$. That is,

$$\begin{aligned} &\frac{1}{2}\left(\eta(x-\rho y)^3 + \overline{\eta}(x-\overline{\rho}y)^3\right) \\ =& s\left(x^3 + \frac{3}{2}x^2y - \frac{3}{2}xy^2 - y^3\right) - \frac{(2t+3s)\sqrt{3}}{9}\left(\frac{3\sqrt{3}}{2}x^2y + \frac{3\sqrt{3}}{2}xy^2\right) \\ =& sx^3 - tx^2y - (t+3s)xy^2 - sy^3 \\ =& F(x,y). \end{aligned}$$

□

Recall from last section that θ is a root of $f(x) = F(x,1)/s$. Then by Lemma 3.4, we have

$$\eta(\theta-\rho)^3 + \overline{\eta}(\theta-\overline{\rho})^3 = 0. \tag{3.10}$$

This gives

$$\frac{\eta}{\overline{\eta}} = -\frac{(\theta-\overline{\rho})^3}{(\theta-\rho)^3}. \tag{3.11}$$

On the other hand, since (x_0, y_0) is a solution to (3.1), we can then put

$$F(x_0, y_0) = m, \tag{3.12}$$

for some integer m with $|m| \leq k$. Again by Lemma 3.4, we have

$$\frac{1}{2}\left(\eta(x_0-\rho y_0)^3 + \overline{\eta}(x_0-\overline{\rho}y_0)^3\right) = m. \tag{3.13}$$

Then we can write

$$\eta(x_0-\rho y_0)^3 = m + Ai \tag{3.14}$$

with

$$A = -\frac{\sqrt{3}}{9}H, \tag{3.15}$$

where

$$H = (2t+3s)x_0^3 + (3t+18s)x_0^2y_0 - (3t-9s)x_0y_0^2 - (2t+3s)y_0^3 \in \mathbb{Z}. \tag{3.16}$$

Since $A \in \mathbb{R}$, we have

$$\frac{\eta(x_0-\rho y_0)^3}{\overline{\eta}(x_0-\overline{\rho}y_0)^3} = \frac{m+Ai}{m-Ai}. \tag{3.17}$$

Combining (3.11) and (3.17), we have

$$-\frac{(\theta-\overline{\rho})^3(x_0-\rho y_0)^3}{(\theta-\rho)^3(x_0-\overline{\rho}y_0)^3} = \frac{m+Ai}{m-Ai}. \tag{3.18}$$

Simplify this equation and write

$$\gamma = \frac{m}{Ai} = \frac{3\sqrt{3}mi}{H}. \tag{3.19}$$

It follows that

$$\frac{(\theta-\overline{\rho})^3(x_0-\rho y_0)^3}{(\theta-\rho)^3(x_0-\overline{\rho}y_0)^3} = \frac{1+\gamma}{1-\gamma}. \tag{3.20}$$

Taking cubic root on both sides, we obtain

$$\frac{(\theta-\overline{\rho})(x_0-\rho y_0)}{(\theta-\rho)(x_0-\overline{\rho}y_0)}=\frac{\sqrt[3]{1+\gamma}}{\sqrt[3]{1-\gamma}}, \tag{3.21}$$

where we choose the cubic roots so that their arguments lie in the interval $(-\pi/6,\pi/6)$ since from the last section x_0/y_0 is close to θ and so the left side is close to 1.

Now we can apply Lemma 2.1 from Chapter 2 with $\mu=3$ and $x=\gamma$. It follows that for any integer $n\geq 1$, we have relations

$$I_{0n}(\gamma)=p_{0n}(\gamma)\sqrt[3]{1+\gamma}-p_{0n}(-\gamma)\sqrt[3]{1-\gamma} \tag{3.22}$$

and

$$I_{1n}(\gamma)=p_{1n}(\gamma)\sqrt[3]{1+\gamma}+p_{1n}(-\gamma)\sqrt[3]{1-\gamma}, \tag{3.23}$$

where

$$p_{0n}(\gamma)=\sum_{h=0}^{n}(-1)^{n-h}\binom{n+\frac{1}{3}}{h}\binom{2n-h}{n-h}\frac{\gamma^h(1+\gamma)^{n-h}}{2^{2n+1-h}}, \tag{3.24}$$

$$\begin{aligned}p_{1n}(\gamma)=\sum_{h=0}^{n}(-1)^{n-h}\binom{n+\frac{1}{3}}{h}&\left(\binom{2n-h}{n-h}\frac{1}{2^{2n+1-h}}\right.\\&\left.-\binom{2n-h-1}{n-h-1}\frac{1}{2^{2n-h}}\right)\gamma^h(1+\gamma)^{n-h},\end{aligned} \tag{3.25}$$

and

$$I_{ln}(\gamma)=\frac{1}{2\pi i}\int_{\Gamma}\frac{z^l(1+\gamma z)^{n+\frac{1}{3}}}{(z^2-1)^{n+1}}dz, \tag{3.26}$$

for $j=0,1$. Dividing both sides of (3.22) and (3.23) by $\sqrt[3]{1-\gamma}$ and then substituting (3.21) and multiplying both sides by $(\theta-\rho)(x_0-\overline{\rho}y_0)$, we obtain

$$q'_{0n}\theta+p'_{0n}=l'_{0n} \tag{3.27}$$

with

$$\begin{aligned}q'_{0n}&=p_{0n}(\gamma)(x_0-\rho y_0)-p_{0n}(-\gamma)(x_0-\overline{\rho}y_0),\\p'_{0n}&=-\overline{\rho}p_{0n}(\gamma)(x_0-\rho y_0)+\rho p_{0n}(-\gamma)(x_0-\overline{\rho}y_0),\\l'_{0n}&=\frac{I_{0n}(\gamma)(\theta-\rho)(x_0-\overline{\rho}y_0)}{\sqrt[3]{1-\gamma}}\end{aligned}$$

and

$$q'_{1n}\theta + p'_{1n} = l'_{1n} \tag{3.28}$$

with

$$\begin{aligned}
q'_{1n} &= p_{1n}(\gamma)(x_0 - \rho y_0) + p_{1n}(-\gamma)(x_0 - \overline{\rho} y_0),\\
p'_{1n} &= -\overline{\rho} p_{1n}(\gamma)(x_0 - \rho y_0) - \rho p_{1n}(-\gamma)(x_0 - \overline{\rho} y_0),\\
l'_{1n} &= \frac{I_{1n}(\gamma)(\theta - \rho)(x_0 - \overline{\rho} y_0)}{\sqrt[3]{1-\gamma}}.
\end{aligned}$$

Put

$$M_j = \begin{cases} 2^{2n}H^n/\sqrt{3}i & \text{if } j = 0,\\ 2^{2n}H^n & \text{if } j = 1.\end{cases} \tag{3.29}$$

Then we have the following:

Lemma 3.5. *With the above notation, for $n \geq 1$, $j = 0, 1$, we have*

$$M_j q'_{jn} \in \mathbb{Z}, \qquad M_j p'_{jn} \in \mathbb{Z}.$$

Proof. First we have, for all integers n, h with $n \geq 1$, $h \leq n$,

$$3^{\left[\frac{3h}{2}\right]} \cdot \binom{n + \frac{1}{3}}{h} \in \mathbb{Z}, \tag{3.30}$$

where $\left[\frac{3h}{2}\right]$ denotes the greatest integer that is less than or equal to $\frac{3h}{2}$. To show this, note that

$$\begin{aligned}
3^{\left[\frac{3h}{2}\right]} \cdot \binom{n + \frac{1}{3}}{h} &= 3^{\left[\frac{h}{2}\right]+h} \cdot \frac{\left(n + \frac{1}{3}\right)\left(n - 1 + \frac{1}{3}\right)\dots\left((n-h+1) + \frac{1}{3}\right)}{h!}\\
&= 3^{\left[\frac{h}{2}\right]} \cdot \frac{(3n+1)(3(n-1)+1)\dots(3(n-h+1)+1)}{h!}.
\end{aligned}$$

The number of 3-factors in $h!$ is at most

$$\left[\frac{h}{3}\right] + \left[\frac{h}{3^2}\right] + \ldots \leq \sum_{j=1}^{\infty} \frac{h}{3^j} = \frac{h}{2}.$$

Now we consider the other prime factors of $h!$ that are not 3. Suppose p is a prime such that $p|h!$ with $p \neq 3$ and a is a positive integer such that $p^a|h!$ but $p^{a+1} \nmid h!$. First notice that $p \leq h$. Then consider the natural integer sequence modulo p:

$$1, 2, 3, \ldots, p-1, 0, 1, 2, \ldots \tag{3.31}$$

The exponent of p-factor in $h!$, a, depends on the number of times 0 appears in the first h elements in the above sequence:

$$(1, 2, \ldots, h) \mod p. \tag{3.32}$$

In other words, a depends on how many complete residual sets $(1, 2, \ldots, p-1, 0)$ (3.32) contains mod p. Notice that $(n-h+1, n-h+2, \ldots, n)$ is a sequence of h consecutive integers. We have that modulo p, it contains the same number of complete residual sets as (3.32). Since $\gcd(3, p) = 1$, it follows that

$$(3(n-h+1)+1, 3(n-h+2)+1, \ldots, 3n+h) \mod p \tag{3.33}$$

contains the same number of complete residual sets $(1, 2, \ldots, p-1, 0)$ as well. This implies that

$$p^a \mid \prod_{j=0}^{h-1}(3(n-j)+1).$$

Therefore, (3.30) holds. It follows that

$$(3\sqrt{3})^h \cdot \binom{n+\frac{1}{3}}{h} \in \mathbb{Z}[\sqrt{3}], \tag{3.34}$$

since if h is even, then $(3\sqrt{3})^h = 3^{[3h/2]}$ and thus

$$(3\sqrt{3})^h \cdot \binom{n+\frac{1}{3}}{h} \in \mathbb{Z};$$

and if h is odd, then $(3\sqrt{3})^h = 3^{[3h/2]}\sqrt{3}$ and thus

$$(3\sqrt{3})^h \cdot \binom{n+\frac{1}{3}}{h} \in \mathbb{Z}[\sqrt{3}].$$

Recall that

$$\gamma = \frac{3\sqrt{3}mi}{H},$$

which is a purely imaginary number. By the definition of q'_{0n}, p'_{0n}, we have

$$\begin{aligned} q'_{0n} &= 2i\Im(p_{0n}(\gamma)(x_0 - \rho y_0)), \\ p'_{0n} &= -2i\Im(p_{0n}(\gamma)(x_0 - \rho y_0)\overline{\rho}), \end{aligned}$$

where

$$p_{0n}(\gamma) = \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{3}}{h} \binom{2n-h}{n-h} \frac{\gamma^h (1+\gamma)^{n-h}}{2^{2n+1-h}},$$

and

$$\rho = \frac{-1+\sqrt{3}i}{2}.$$

It follows that

$$\begin{aligned} M_0 q'_{0n} &= M_0 \cdot 2i\Im \left((\Re(p_{0n}(\gamma)) + i\Im(p_{0n}(\gamma))) \cdot \left(x_0 + \frac{y_0}{2} - \frac{\sqrt{3}y_0}{2} i \right) \right) \\ &= i(2x_0 + y_0)\Im(p_{0n}(\gamma))M_0 - i\sqrt{3}y_0 \Re(p_{0n}(\gamma))M_0 \end{aligned}$$

and

$$\begin{aligned} M_0 p'_{0n} &= -M_0 \cdot 2i\Im \left((\Re(p_{0n}(\gamma)) + i\Im(p_{0n}(\gamma))) \cdot \left(x_0 + \frac{y_0}{2} - \frac{\sqrt{3}y_0}{2} i \right) \right. \\ &\qquad \left. \cdot \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \right) \\ &= i(x_0 + 2y_0)\Im(p_{0n}(\gamma))M_0 + i\sqrt{3}x_0 \Re(p_{0n}(\gamma))M_0. \end{aligned}$$

Thus to show $M_0 q'_{0n} \in \mathbb{Z}, M_0 p'_{0n} \in \mathbb{Z}$, it suffices to show that

$$\Im(p_{0n}(\gamma)) \cdot (iM_0) \in \mathbb{Z}, \qquad \sqrt{3}\Re(p_{0n}(\gamma)) \cdot (iM_0) \in \mathbb{Z}.$$

We have

$$\begin{aligned} iM_0 p_{0n}(\gamma) &= \frac{2^{2n} H^n}{\sqrt{3}} \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{3}}{h} \binom{2n-h}{n-h} \frac{\gamma^h (1+\gamma)^{n-h}}{2^{2n+1-h}} \\ &= \frac{1}{\sqrt{3}} \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{3}}{h} \binom{2n-h}{n-h} \frac{(H\gamma)^h (H + H\gamma)^{n-h}}{2^{1-h}} \\ &= \frac{1}{\sqrt{3}} \sum_{h=0}^{n} (-1)^{n-h} \left(\binom{n+\frac{1}{3}}{h} (3\sqrt{3}mi)^h \right) \left(\binom{2n-h}{n-h} \frac{1}{2^{1-h}} \right) \\ &\qquad \cdot (H + 3\sqrt{3}mi)^{n-h}. \end{aligned}$$

Notice that for $h = 0, 1 \ldots, n$

$$\binom{2n-h}{n-h}\frac{1}{2^{1-h}} \in \mathbb{Z}$$

and by (3.34)

$$\binom{n+\frac{1}{3}}{h}(3\sqrt{3}mi)^h \in \mathbb{Z}[\sqrt{3}].$$

It follows that

$$i\sqrt{3}M_0 p_{0n}(\gamma) \in \mathbb{Z}[\sqrt{3}].$$

Since

$$\begin{aligned} i\sqrt{3}M_0 p_{0n}(\gamma) &= \sum_{h=0}^{n}(-1)^{n-h}\binom{n+\frac{1}{3}}{h}\binom{2n-h}{n-h}\frac{(H\gamma)^h(H+H\gamma)^{n-h}}{2^{1-h}} \\ &= \sum_{h=0}^{n}\frac{(-1)^{n-h}}{2^{1-h}}\binom{n+\frac{1}{3}}{h}\binom{2n-h}{n-h}\sum_{l=0}^{n-h}H^{n-h-l}(3\sqrt{3}mi)^{h+l} \end{aligned}$$

we see that each term in the real part of $i\sqrt{3}M_0 p_{0n}(\gamma)$ is in $\mathbb{Z}$ and each term in the imaginary part of $i\sqrt{3}M_0 p_{0n}(\gamma)$ is of the form of an integer multiplied by $\sqrt{3}$. Therefore $iM_0 p_{0n}(\gamma)$ can be written as

$$\frac{a}{\sqrt{3}} + bi,$$

where $a, b \in \mathbb{Z}$. It follows that

$$\Im(p_{0n}(\gamma)) \cdot (iM_0) \in \mathbb{Z}, \qquad \sqrt{3}\Re(p_{0n}(\gamma)) \cdot (iM_0) \in \mathbb{Z}.$$

since $iM_0 \in \mathbb{R}$. This proves $M_0 q'_{0n} \in \mathbb{Z}, M_0 p'_{0n} \in \mathbb{Z}$.

Similarly, we have

$$\begin{aligned} q'_{1n} &= 2\Re(p_{1n}(\gamma)(x_0 - \rho y_0)), \\ p'_{1n} &= -2\Re(p_{1n}(\gamma)(x_0 - \rho y_0)\overline{\rho}), \end{aligned}$$

where

$$p_{1n}(\gamma) = \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{3}}{h} \left(\binom{2n-h}{n-h} \frac{1}{2^{2n+1-h}} - \binom{2n-h-1}{n-h-1} \frac{1}{2^{2n-h}} \right) \gamma^h (1+\gamma)^{n-h}.$$

Since

$$\begin{aligned} M_1 q'_{1n} &= M_1 \cdot 2\Re \left((\Re(p_{1n}(\gamma)) + i\Im(p_{1n}(\gamma))) \cdot \left(x_0 + \frac{y_0}{2} - \frac{\sqrt{3}y_0}{2} i \right) \right) \\ &= (2x_0 + y_0)\Re(p_{1n}(\gamma))M_1 + \sqrt{3}y_0 \Im(p_{1n}(\gamma))M_1 \end{aligned}$$

and

$$\begin{aligned} M_1 p'_{1n} &= -M_1 \cdot 2\Re \left((\Re(p_{1n}(\gamma)) + i\Im(p_{1n}(\gamma))) \cdot \left(x_0 + \frac{y_0}{2} - \frac{\sqrt{3}y_0}{2} i \right) \right. \\ &\qquad \left. \cdot \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \right) \\ &= (x_0 + 2y_0)\Re(p_{1n}(\gamma))M_1 - \sqrt{3}x_0 \Im(p_{1n}(\gamma))M_1, \end{aligned}$$

it suffices to show that

$$\Re(p_{1n}(\gamma))M_1 \in \mathbb{Z}, \qquad \sqrt{3}\Im(p_{1n}(\gamma))M_1 \in \mathbb{Z}.$$

We have

$$\begin{aligned} M_1 p_{1n}(\gamma) &= 2^{2n} H^n \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{3}}{h} \left(\binom{2n-h}{n-h} \frac{1}{2^{2n+1-h}} \right. \\ &\qquad \left. - \binom{2n-h-1}{n-h-1} \frac{1}{2^{2n-h}} \right) \gamma^h (1+\gamma)^{n-h} \\ &= \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{3}}{h} \left(\binom{2n-h}{n-h} \frac{1}{2^{1-h}} \right. \\ &\qquad \left. - \binom{2n-h-1}{n-h-1} \frac{1}{2^{-h}} \right) (H\gamma)^h (H + H\gamma)^{n-h}. \end{aligned}$$

Using the same argument as for $j = 0$, we see that $M_1 p_{1n}(\gamma)$ can be written as

$$a + b\sqrt{3}i,$$

for some integers a, b. It then follows that

$$\Re(p_{1n}(\gamma))M_1 \in \mathbb{Z}, \qquad \sqrt{3}\Im(p_{1n}(\gamma))M_1 \in \mathbb{Z}.$$

which implies that $M_1 q'_{1n} \in \mathbb{Z}, M_1 p_{1n} \in \mathbb{Z}$.

□

Put

$$q_{jn} = M_j q'_{jn}, \quad p_{jn} = M_j p'_{jn}, \quad l_{jn} = M_j l_{jn}, \tag{3.35}$$

for $j = 0, 1$ and $n \geq 1$. Summarizing the discussion in this section, we obtain

Lemma 3.6. *For $n \geq 1$, put*

$$\begin{aligned}
q_{0n} &= (p_{0n}(\gamma)(x_0 - \rho y_0) - p_{0n}(-\gamma)(x_0 - \overline{\rho} y_0))2^{2n}H^n/\sqrt{3}i, \\
p_{0n} &= (-\overline{\rho} p_{0n}(\gamma)(x_0 - \rho y_0) + \rho p_{0n}(-\gamma)(x_0 - \overline{\rho} y_0))2^{2n}H^n/\sqrt{3}i, \\
q_{1n} &= (p_{1n}(\gamma)(x_0 - \rho y_0) + p_{1n}(-\gamma)(x_0 - \overline{\rho} y_0))2^{2n}H^n, \\
p_{1n} &= (-\overline{\rho} p_{1n}(\gamma)(x_0 - \rho y_0) - \rho p_{1n}(-\gamma)(x_0 - \overline{\rho} y_0))2^{2n}H^n.
\end{aligned}$$

Then q_{0n}, p_{0n}, q_{1n} and p_{1n} are rational integers satisfying the following relations:

$$\begin{aligned}
q_{0n}\theta + p_{0n} =& l_{0n}, \\
q_{1n}\theta + p_{1n} =& l_{1n},
\end{aligned}$$

where

$$\begin{aligned}
l_{0n} &= \frac{I_{0n}(\gamma)(\theta - \rho)(x_0 - \overline{\rho} y_0)2^{2n}H^n}{\sqrt[3]{1-\gamma}\sqrt{3}i}, \\
l_{1n} &= \frac{I_{1n}(\gamma)(\theta - \rho)(x_0 - \overline{\rho} y_0)2^{2n}H^n}{\sqrt[3]{1-\gamma}}.
\end{aligned}$$

To apply Lemma 2.5, we need the following condition and estimates.

Lemma 3.7. *Let p_{jn}, q_{jn} defined as in Lemma 3.6 for $j = 0, 1$, we have*

$$\begin{vmatrix} p_{0n} & q_{0n} \\ p_{1n} & q_{1n} \end{vmatrix} \neq 0,$$

for any $n \geq 1$.

Proof. By Lemma 2.4 and $\gamma \neq 0$, we have

$$\Delta(\gamma) = \begin{vmatrix} p_{0n}(\gamma) & -p_{0n}(-\gamma) \\ p_{1n}(\gamma) & p_{1n}(-\gamma) \end{vmatrix} = c_{2n}\gamma^{2n} \neq 0. \tag{3.36}$$

Put $A_j = p_{jn}(\gamma)(x_0 - \rho y_0)$ and $B_j = p_{jn}(-\gamma)(x_0 - \overline{\rho} y_0)$ for $j = 0, 1$. Then

$$\begin{aligned} q_{0n} &= (A_0 - B_0)M_0, \quad p_{0n} = (-\overline{\rho} A_0 + \rho B_0)M_0 \\ q_{1n} &= (A_1 + B_1)M_1 \quad p_{1n} = -(\overline{\rho} A_1 + \rho B_1)M_1. \end{aligned}$$

It follows that

$$\begin{aligned} \begin{vmatrix} p_{0n} & q_{0n} \\ p_{1n} & q_{1n} \end{vmatrix} &= \begin{vmatrix} (-\overline{\rho} A_0 + \rho B_0)M_0 & (A_0 - B_0)M_0 \\ -(\overline{\rho} A_1 + \rho B_1)M_1 & (A_1 + B_1)M_1 \end{vmatrix} \\ &= M_0 M_1 \begin{vmatrix} (-\overline{\rho} A_0 + \rho B_0) & (A_0 - B_0) \\ -(\overline{\rho} A_1 + \rho B_1) & (A_1 + B_1) \end{vmatrix} \\ &= M_0 M_1 \begin{vmatrix} (\rho - \overline{\rho}) A_0 & (A_0 - B_0) \\ (\rho - \overline{\rho}) A_1 & (A_1 + B_1) \end{vmatrix} \\ &= M_0 M_1 (\rho - \overline{\rho}) \begin{vmatrix} A_0 & -B_0 \\ A_1 & B_1 \end{vmatrix} \\ &= M_0 M_1 (\rho - \overline{\rho}) \begin{vmatrix} p_{0n}(\gamma)(x_0 - \rho y_0) & -p_{0n}(-\gamma)(x_0 - \overline{\rho} y_0) \\ p_{0n}(\gamma)(x_0 - \rho y_0) & p_{0n}(-\gamma)(x_0 - \overline{\rho} y_0) \end{vmatrix} \\ &= M_0 M_1 (\rho - \overline{\rho}) |x_0 - \rho y_0|^2 \begin{vmatrix} p_{0n}(\gamma) & -p_{0n}(-\gamma) \\ p_{1n}(\gamma) & p_{1n}(-\gamma) \end{vmatrix} \\ &\neq 0. \end{aligned}$$

□

Lemma 3.8. *Suppose that $|\gamma| < 1/\sqrt{2}$. For $n \geq 1, j = 0, 1$,*

$$|q_{jn}| < \varrho P^n,$$

where

$$\varrho = \frac{1.67|x_0 - \rho y_0|\sqrt[3]{1+\sqrt{2}|\gamma|}}{\sqrt[3]{1-|\gamma|}}$$

and

$$P = 4|H|(1+\sqrt{2}|\gamma|).$$

Proof. From the proof of Lemma 3.5, we see that

$$|q_{jn}| \leq 2|p_{jn}(\gamma)(x_0 - \rho y_0)M_j| = 2|p_{jn}(\gamma)| \cdot |x_0 - \rho y_0| \cdot |M_j|, \tag{3.37}$$

for $j = 0, 1, n \geq 1$. By the definition of M_j, we have

$$|M_j| \leq 2^{2n}|H|^n. \tag{3.38}$$

By Lemma 2.1, we have, for $j = 0, 1$,

$$p_{jn}(x) = \frac{I_{j1n}(x)}{\sqrt[3]{1+x}}, \tag{3.39}$$

where

$$I_{j1n}(x) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{z^j(1+xz)^{n+\frac{1}{3}}}{(z^2-1)^{n+1}}dz.$$

Consider the curve $\Gamma : \{z \in \mathbb{C} : |z^2 - 1| = 1\}$. It consists of two closed curves. Let Γ_1 be the one enclosing the point 1. Notice that $|z|^2 = |z^2| \leq |z^2 - 1| + 1 = 2$ and $\sqrt{2} \in \Gamma_1$. Hence $\max_{z\in\Gamma_1} |z| = \sqrt{2}$. Then for $|x| < 1/\sqrt{2}$, we have

$$\begin{aligned} |I_{j1n}(x)| &= \left|\frac{1}{2\pi i}\int_{\Gamma_1} \frac{z^j(1+xz)^{n+\frac{1}{3}}}{(z^2-1)^{n+1}}dz\right| \\ &\leq \frac{|\Gamma_1|}{2\pi} \cdot \max_{z\in\Gamma_1}\left|\frac{z^j(1+xz)^{n+\frac{1}{3}}}{(z^2-1)^{n+1}}\right| \\ &= \frac{|\Gamma_1|\sqrt{2}(1+\sqrt{2}|x|)^{n+\frac{1}{3}}}{2\pi}, \end{aligned} \tag{3.40}$$

for $j = 0, 1$, where $|\Gamma_1|$ denotes the length of Γ_1.

Write $z = a + bi$. By the definition of Γ, we can obtain the equation of Γ_1 on ab-plane:

$$(a^2 + b^2)^2 - 2(a^2 - b^2) = 0, \qquad 0 \le a \le \sqrt{2}.$$

We can find the length of Γ_1 by an integral along the above curve. Numerical integration by Maple gives

$$|\Gamma_1| = 3.70814935\ldots < 3.709. \tag{3.41}$$

Notice that

$$|\sqrt[3]{1+x}| = \sqrt[3]{|1+x|} \ge \sqrt[3]{1-|x|}, \tag{3.42}$$

for $|x| < 1/\sqrt{2}$. Then combining (3.37), (3.38), (3.39), (3.40), (3.41), and (3.42), we obtain

$$\begin{aligned} |q_{jn}| &= 2|x_0 - \rho y_0| \cdot |M_j| \cdot |p_{jn}(\gamma)| \\ &\le 2|x_0 - \rho y_0| 2^{2n} |H|^n \frac{3.709 \cdot \sqrt{2}(1+\sqrt{2}|\gamma|)^{n+\frac{1}{3}}}{2\pi\sqrt[3]{1-|\gamma|}} \\ &< \frac{1.67|x_0 - \rho y_0|\sqrt[3]{1+\sqrt{2}|\gamma|}}{\sqrt[3]{1-|\gamma|}} \left(4|H|(1+\sqrt{2}|\gamma|)\right)^n. \end{aligned}$$

□

Lemma 3.9. *Suppose that $w \ge 4$ and $|\gamma| < 1/\sqrt{2}$. For $n \ge 1, j = 0, 1$,*

$$|l_{jn}| \le l L^{-n},$$

where

$$l = \frac{4|x_0 - \overline{\rho} y_0|}{9\sqrt[3]{1-|\gamma|}(1-|\gamma|^2)}$$

and

$$L = \frac{1-|\gamma|^2}{|H||\gamma|^2}.$$

Proof. By the definition of l_{jn} from Lemma 3.6, we have

$$|l_{jn}| \leq \frac{2^{2n}|H|^n|\theta - \rho||x_0 - \overline{\rho}y_0||I_{jn}(\gamma)|}{|\sqrt[3]{1-\gamma}|}, \tag{3.43}$$

for $j = 0, 1$ and $n \geq 1$. By Lemma 3.2, we have

$$-\frac{1}{w+2} < \theta < -\frac{1}{w+2} + \frac{4}{(w+2)w^2} < 0,$$

for $w \geq 4$. Thus

$$|\theta - \rho| < \sqrt{\left(-\frac{1}{2} - 0\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1. \tag{3.44}$$

By Lemma 2.3 and the definition of I_{jn}, we have

$$I_{jn}(x) = \sum_{h=2n+1-j}^{\infty} \binom{n+\frac{1}{3}}{h} J_{h+j} x^h,$$

where

$$J_h = \frac{1}{2\pi i} \int_{\Gamma} \frac{z^h}{(z^2-1)^{n+1}} dz.$$

As a consequence of Lemma 2.2, $J_h \geq 0$, since

$$\sum_{h=0}^{\infty} J_h x^h = \frac{x^{2n+1}}{(1-x^2)^{n+1}}$$

and the Taylor expansion at 0 of the right hand side of the above equation obviously has non-negative coefficients. Notice that for $h \geq 2n$, we have

$$\frac{\left|\binom{n+\frac{1}{3}}{h}\right|}{\left|\binom{n+\frac{1}{3}}{h+1}\right|} = \left|\frac{h+1}{n-h-\frac{1}{3}}\right| = \frac{h+1}{h-n-\frac{1}{3}} > 1.$$

Thus $\left|\binom{n+\frac{1}{3}}{h}\right|$ decreases as h increases. It follows that

$$
\begin{aligned}
|I_{jn}(x)| &\leq \left|\binom{n+\frac{1}{3}}{2n}\right| \sum_{2n+1-j}^{\infty} |J_{h+j}||x|^h \\
&= \left|\binom{n+\frac{1}{3}}{2n}\right| |x|^{-j} \sum_{h=2n+1}^{\infty} J_h |x|^h \\
&= \left|\binom{n+\frac{1}{3}}{2n}\right| |x|^{-j} \frac{|x|^{2n+1}}{(1-|x|^2)^{n+1}},
\end{aligned}
$$

for $|x| < 1$ by Lemma 2.2. By induction, we can have the following estimate:

$$\left|\binom{n+\frac{1}{3}}{2n}\right| \leq \frac{4}{9}\left(\frac{1}{4}\right)^n,$$

since for $n = 1$,

$$\left|\binom{n+\frac{1}{3}}{2n}\right| = \frac{2}{9} = \frac{4}{9}\left(\frac{1}{4}\right)^1,$$

and for $n \geq 2$,

$$
\begin{aligned}
\frac{\left|\binom{n+\frac{1}{3}}{2n}\right|}{\left|\binom{n+1+\frac{1}{3}}{2(n+1)}\right|} &= \left|\frac{(2n+1)(2n+2)}{\left(n+1+\frac{1}{3}\right)\left(-n+\frac{1}{3}\right)}\right| \\
&= 4 \cdot \frac{n^2+\frac{3}{2}n+\frac{1}{2}}{n^2+n-\frac{4}{9}} \\
&> 4.
\end{aligned}
$$

Therefore, we obtain

$$|I_{jn}(x)| \leq \frac{4}{9}\left(\frac{1}{4}\right)^n |x|^{-j} \frac{|x|^{2n+1}}{(1-|x|^2)^{n+1}}. \tag{3.45}$$

Notice that for $|\gamma| < 1/\sqrt{2}$,

$$|\sqrt[3]{1-\gamma}| = \sqrt[3]{|1-\gamma|} \geq \sqrt[3]{1-|\gamma|}. \tag{3.46}$$

Then we combine (3.43), (3.44), (3.45) and (3.46). It follows that

$$|l_{jn}| \leq \frac{2^{2n}|H|^n|x_0 - \overline{\rho}y_0|}{\sqrt[3]{1-|\gamma|}} \cdot \frac{4}{9}\left(\frac{1}{4}\right)^n |\gamma|^{-j} \frac{|\gamma|^{2n+1}}{(1-|\gamma|^2)^{n+1}}$$
$$= \frac{4|x_0 - \overline{\rho}y_0|}{9\sqrt[3]{1-|\gamma|}(1-|\gamma|^2)} \cdot \left(\frac{|H||\gamma|^2}{1-|\gamma|^2}\right)^n$$
$$= \frac{4|x_0 - \overline{\rho}y_0|}{9\sqrt[3]{1-|\gamma|}(1-|\gamma|^2)} \cdot \left(\frac{1-|\gamma|^2}{|H||\gamma|^2}\right)^{-n}.$$

□

Lemma 3.10. *Suppose that $|H| \geq 54|m|^2$, $|\gamma| < 1/\sqrt{2}$ and $w \geq 4$. With the notations as above, we have, for any integers p and q with $q > 0$,*

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{Cq^\lambda},$$

where

$$\lambda = 1 + \frac{\log\left(4|H|(1+\sqrt{2}|\gamma|)\right)}{\log\left(\frac{1-|\gamma|^2}{|H||\gamma|^2}\right)}$$

and

$$C = \frac{13.36|x_0 - \rho y_0|\sqrt[3]{1+\sqrt{2}|\gamma|}|H|(1+\sqrt{2}|\gamma|)}{\sqrt[3]{1-|\gamma|}} \cdot \max\left(\frac{8|x_0-\overline{\rho}y_0|}{9\sqrt[3]{1-|\gamma|}(1-|\gamma|^2)}, 1\right)^{\lambda-1}.$$

Proof. Recall that

$$\gamma = \frac{3\sqrt{3}mi}{H}.$$

Then by the assumption that $|H| \geq 54|m|^2$, we have that

$$(|H|+1)\cdot|\gamma|^2 < 2|H|\cdot|\gamma|^2 = \frac{54|m|^2}{|H|} \leq 1.$$

It follows that

$$L = \frac{1 - |\gamma|^2}{|H||\gamma|^2} - 1 = \frac{1 - (|H| + 1)|\gamma|^2}{|H||\gamma|^2} \geq 0.$$

Then we can apply Lemma 2.5, together with Lemmas 3.5, 3.7, 3.8 and 3.9. We have that for any integers p and q with $q > 0$,

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{Cq^{\lambda}},$$

where

$$\begin{aligned} \lambda &= 1 + \frac{\log P}{\log L} \\ &= 1 + \frac{\log\left(4|H|(1 + \sqrt{2}|\gamma|)\right)}{\log\left(\frac{1-|\gamma|^2}{|H||\gamma|^2}\right)} \end{aligned}$$

and

$$\begin{aligned} C &= 2\varrho P \max(2l, 1)^{\log P / \log L} \\ &= 2 \cdot \frac{1.67|x_0 - \rho y_0| \sqrt[3]{1 + \sqrt{2}|\gamma|}}{\sqrt[3]{1 - |\gamma|}} \cdot 4|H|(1 + \sqrt{2}|\gamma|) \\ &\quad \cdot \max\left(\frac{2 \cdot 4|x_0 - \overline{\rho} y_0|}{9\sqrt[3]{1 - |\gamma|}(1 - |\gamma|^2)}, 1\right)^{\lambda - 1} \\ &= \frac{13.36|x_0 - \rho y_0| \sqrt[3]{1 + \sqrt{2}|\gamma|}|H|(1 + \sqrt{2}|\gamma|)}{\sqrt[3]{1 - |\gamma|}} \\ &\quad \cdot \max\left(\frac{8|x_0 - \overline{\rho} y_0|}{9\sqrt[3]{1 - |\gamma|}(1 - |\gamma|^2)}, 1\right)^{\lambda - 1}. \end{aligned}$$

□

3.3 Upper bounds for the solutions

Lemma 3.11. *Let $\epsilon \in (0, 1)$ and let λ be defined as in the last section. Suppose that $w \geq 4$ and*

$$|H| \geq 2^{1+4/\epsilon} 3^{2+2/\epsilon} k^{2+2/\epsilon}.$$

Then we have

$$\lambda < 2 + \epsilon.$$

Proof. By the assumption on H and the definition of γ, it is easy to see that

$$|\gamma| = \frac{3\sqrt{3}|m|}{|H|} \leq \frac{3\sqrt{3}k}{|H|} < \frac{1}{\sqrt{2}}.$$

Thus the conditions in Lemma 3.10 are satisfied. Then we have P, L, λ defined as in Lemma 3.8, 3.9 and 3.10. With $0 < |\gamma| < 1/\sqrt{2}$, we have

$$P = 4|H|(1 + \sqrt{2}|\gamma|) < 8|H|, \tag{3.47}$$

and

$$L = \frac{1 - |\gamma|^2}{|H||\gamma|^2} = (1 - |\gamma|^2)\frac{|H|}{(|H||\gamma|)^2} > \frac{|H|}{18|m|^2} \geq \frac{|H|}{18k^2}, \tag{3.48}$$

since $\gamma = \frac{3\sqrt{3}mi}{H}$ and $|m| \leq k$. From the assumption on H, it follows that

$$\frac{\left(\frac{|H|}{18k^2}\right)^{\epsilon+1}}{8|H|} = \frac{|H|^{\epsilon}}{8 \cdot 18^{\epsilon+1} \cdot k^{2\epsilon+2}} \geq \frac{2^{\epsilon+4} \cdot 3^{2\epsilon+2} \cdot k^{2\epsilon+2}}{8 \cdot 18^{\epsilon+1} \cdot k^{2\epsilon+2}} = 1.$$

Then combining (4.32) and (4.33), we have

$$L^{\epsilon+1} > \left(\frac{|H|}{18k^2}\right)^{\epsilon+1} \geq 8|H| > P.$$

Taking logarithms, we obtain

$$(\epsilon + 1)\log L > \log P.$$

Therefore,

$$\lambda = 1 + \frac{\log P}{\log L} < 1 + (\epsilon + 1) = 2 + \epsilon.$$

$\square$

Lemma 3.12. *With the same notation as before, suppose that* (x_0, y_0), (x, y) *are solutions to (3.1) of type II. Let* H *be defined as in (3.16). Assume that as in Lemma 3.11* $|H|$ *is sufficiently large so that* $\lambda < 3$. *Then we have for* $w \geq 1000$

$$y < \left(\frac{C}{B}\right)^{\frac{1}{3-\lambda}}$$

where

$$B = \frac{0.999t}{k}$$

and

$$C = \frac{13.36|x_0 - \rho y_0| \sqrt[3]{1+\sqrt{2}|\gamma|}|H|(1+\sqrt{2}|\gamma|)}{\sqrt[3]{1-|\gamma|}} \cdot \max\left(\frac{8|x_0 - \overline{\rho} y_0|}{9\sqrt[3]{1-|\gamma|}(1-|\gamma|^2)}, 1\right)^{\lambda-1}.$$

Proof. Notice that if $w \geq 4$ and $|H|$ is large enough as in Lemma 3.11, the assumptions in Lemmas 3.3 and 3.10 can be easily satisfied. It then follows directly from these two lemmas that if (x, y) is a solution to (3.1) then

$$\frac{1}{Cy^{\lambda}} < \left|\theta - \frac{x}{y}\right| \leq \frac{1}{By^3}$$

which gives

$$y < \left(\frac{C}{B}\right)^{\frac{1}{3-\lambda}}$$

with B, C defined as in the statement of the lemma.

□

3.4 Proof of Theorem 3.1

Let $k = 2t + 3s$. It is obvious that

$$|F(0,1)| = s \leq k,$$
$$|F(1,1)| = 2t + 3s \leq k.$$

Then $(0,1),(1,1)$ are the only solutions (x,y) to inequality (3.1) with $y = 1$ that satisfy (3.5). We now suppose $y \geq 2$ in the rest of this chapter.

Lemma 3.13. *If (x,y) is an integer solution to inequality (3.1) with $k = 2t + 3s$ of type I, then*

$$y < w + 2.$$

Proof. Recall that

$$f(x) = x^3 - wx^2 - (w+3)x - 1.$$

Then we have

$$f'(x) = 3x^2 - 2wx - (w+3).$$

Since

$$f'\left(-\frac{1}{2}\right) = -\frac{9}{4} < 0,$$
$$f'(1) = -3w < 0,$$

we see that $f'(x) < 0$ on interval $(-1/2, 1]$ and hence $f(x)$ decreases on $(-1/2, 1]$. On the other hand, we have

$$f\left(-\frac{1}{w+2}\right) = \frac{2w+3}{(w+2)^3} > 0,$$
$$f\left(-\frac{1}{w+2} + \frac{4}{(w+2)w^2}\right) = -\frac{2w^4 + 5w^3 - 6w^2 + 12w - 8}{w^6} < 0$$

and

$$\frac{2w+3}{(w+2)^3} < \frac{2w^4 + 5w^3 - 6w^2 + 12w - 8}{w^6},$$

for $w \geq 1$. It follows that if (x, y) is of type I, then

$$\left| f\left(\frac{x}{y}\right) \right| \geq \frac{2w+3}{(w+2)^3}.$$

Then we have

$$k = 2t + 3s \geq |F(x, y)| = sy^3 \left| f\left(\frac{x}{y}\right) \right| \geq \frac{(2t+3s)y^3}{(w+2)^3}.$$

This gives

$$y \leq w + 2.$$

Notice that $\gcd(s, t) = 1$ and $y \in \mathbb{N}$. Thus $y \neq w + 2$. Then we obtain

$$y < w + 2.$$

$\square$

Lemma 3.14. *There is no integer solution (x, y) to (3.1) of type I, where in (3.1) $k = 2t + 3s$.*

Proof. Suppose that (x, y) is a solution to (3.1) of type I. By assumption, $x/y \neq 0$ since $y \geq 2$ and $\gcd(x, y) = 1$. We then have

$$\frac{x}{y} \notin \left(-\frac{1}{y}, \frac{1}{y}\right),$$

since otherwise

$$\frac{1}{y} > \left|\frac{x}{y} - 0\right| = \frac{|x|}{y} \geq \frac{1}{y},$$

which is a contradiction. Put

$$h(y) = (-y^3 + (w+3)y^2 - wy - 1) - (2w + 3).$$

Since

$$h(1) = -2 - 2w \leq 0, \qquad h(2) = h(w+2) = 0,$$

we have that $h(y) \geq 0$ on interval $[2, w+2)$. By Lemma 3.13, we have

$$y < w + 2.$$

Thus, we see that if (x, y) is of type I, then

$$-y^3 + (w+3)y^2 - wy - 1 \geq 2w + 3. \tag{3.49}$$

Notice that (3.49) takes the equality sign only when $y = 2$. If $y = 2$, then by condition (3.5), the only choice for x is 1. However, $(1, 2)$ is not a solution since $|F(1, 2)| = 6t + 19s$. Thus we can remove the equality sign in (3.49) and get

$$-y^3 + (w+3)y^2 - wy - 1 > 2w + 3. \tag{3.50}$$

From the proof of Lemma 3.13, we know that function $f(x)$ is decreasing on interval $(-1/2, 1]$. By (3.50), we have

$$f\left(-\frac{1}{y}\right) = \frac{-y^3 + (w+3)y^2 - wy - 1}{y^3} > \frac{2w+3}{y^3}$$
$$f\left(\frac{1}{y}\right) = -\frac{(y^2+y)w + (y^3 + 3y^2 - 1)}{y^3} < -\frac{2w+3}{y^3}$$

Therefore

$$\left|f\left(\frac{x}{y}\right)\right| > \frac{2w+3}{y^3},$$

if (x, y) is of type I. It follows that

$$k = 2t + 3s \geq |F(x, y)| = sy^3 \left|f\left(\frac{x}{y}\right)\right| > 2t + 3s,$$

which is a contradiction.

□

Lemma 3.15. *If (x, y) is an integer solution to (3.1) of type II, then*

$$y > \frac{t^2}{4s^3}.$$

Proof. By the definition of type II, if (x, y) is of this type, then

$$-\frac{1}{w+2} < \frac{x}{y} < -\frac{1}{w+2} + \frac{4}{(w+2)w^2}.$$

Multiply the above inequality by $(w+2)sy$. We have

$$-sy < x(w+2)s = x(t+2s) < -sy + \frac{4sy}{w^2}.$$

Since both $-sy$ and $x(t+2s)$ are integers, it follows that

$$\frac{4sy}{w^2} > 1.$$

That is

$$y > \frac{w^2}{4s} = \frac{t^2}{4s^3}.$$

$\square$

Lemma 3.16. *With the assumption in Theorem 3.1, there are at most τ integer solutions to (3.1) of type II.*

Proof. Suppose that (x_0, y_0) is a non-trivial integer solution to (3.1) of type II. Then

$$\frac{x_0}{y_0} \in \left(-\frac{1}{w+2}, -\frac{1}{w+2} + \frac{4}{(w+2)w^2}\right). \tag{3.51}$$

Recall the definition of H:

$$H = (2t+3s)x_0^3 + (3t+18s)x_0^2y_0 - (3t-9s)x_0y_0^2 - (2t+3s)y_0^3.$$

Put

$$g(x) = (2w+3)x^3 + (3w+18)x^2 - (3w-9)x - (2w+3).$$

Then

$$g'(x) = (6w+9)x^2 + (6w+36)x - 3w + 9.$$

Since

$$g'(-1) = -3w - 18 < 0,$$
$$g'(0) = -3w + 9 < 0,$$

for $w > 3$, we have that $g(x)$ is decreasing on the interval $(-1, 0)$. Then if x is in the interval given by (3.51), then

$$g\left(-\frac{1}{w+2} + \frac{4}{(w+2)w^2}\right) < g(x) < g\left(-\frac{1}{w+2}\right).$$

We have

$$g\left(-\frac{1}{w+2}\right) = -\frac{2w^4 + 12w^3 + 36w^2 + 54w + 27}{(w+2)^3} < 0,$$
$$g\left(-\frac{1}{w+2} + \frac{4}{(w+2)w^2}\right) = -\frac{2w^7 + 12w^5 - 22w^4 + 51w^3 - 66w^2 + 20w - 24}{w^6} < 0.$$

It follows that

$$\frac{2w^4 + 12w^3 + 36w^2 + 54w + 27}{(w+2)^3} < |g(x)| < \frac{2w^7 + 12w^5 - 22w^4 + 51w^3 - 66w^2 + 20w - 24}{w^6}.$$

For $w \geq 1000$, we have

$$\frac{2w^4 + 12w^3 + 36w^2 + 54w + 27}{(w+2)^3} \geq 2w,$$
$$\frac{2w^7 + 12w^5 - 22w^4 + 51w^3 - 66w^2 + 20w - 24}{w^6} \leq 2.000012w.$$

Thus

$$2w < |g(x)| < 2.000012w.$$

By the definition of $g(x)$ and H, together with (3.51), we obtain

$$2ty_0^3 < |H| < 2.000012ty_0^3. \tag{3.52}$$

By Lemma 3.15, we have that if (x, y) is a solution to (3.1) of type II then

$$y > \frac{t^2}{4s^3}. \tag{3.53}$$

By the assumption of the theorem, we have that

$$t \geq 1.2 \cdot 10^6 s^{3+\chi}, \tag{3.54}$$

where

$$\chi = \frac{21}{2^\tau}, \qquad \text{for } \tau \geq 5. \tag{3.55}$$

Then we have

$$0 < \chi \leq \frac{21}{32}. \tag{3.56}$$

(3.54) implies that

$$s^{-1} \geq (1.2 \cdot 10^6)^{\frac{1}{3+\chi}} t^{-\frac{1}{3+\chi}}. \tag{3.57}$$

From (3.53) and (3.57), we have

$$y > \frac{t^2}{4} \cdot \left((1.2 \cdot 10^6)^{\frac{1}{3+\chi}} t^{-\frac{1}{3+\chi}}\right)^3 \geq 24321 t^{\frac{3+2\chi}{3+\chi}}. \tag{3.58}$$

The last inequality holds since the coefficient takes minimal value when χ takes maximal value 21/32. (3.58) is equivalent to

$$t < 24321^{-\frac{3+\chi}{3+2\chi}} y^{\frac{3+\chi}{3+2\chi}} < 5230^{-1} y^{\frac{3+\chi}{3+2\chi}}, \tag{3.59}$$

since

$$24321^{\frac{3+\chi}{3+2\chi}} \geq 24321^{\frac{39}{46}} > 5230.$$

In particular, since (x_0, y_0) is a solution of type II, we have

$$t < 5230^{-1} y_0^{\frac{3+\chi}{3+2\chi}} \tag{3.60}$$

From (3.52), together with (3.58), we have

$$\begin{aligned}|H| &> 2ty_0^3\\ &> 2t\cdot\left(24321t^{\frac{3+2\chi}{3+\chi}}\right)^3\\ &= 2\cdot 24321^3\cdot t^{1+3\cdot\frac{3+2\chi}{3+\chi}}\\ &> 2.8\cdot 10^{13}t^{\frac{12+7\chi}{3+\chi}}. \end{aligned}\tag{3.61}$$

Put

$$\epsilon=\frac{2(3+\chi)}{6+5\chi}.\tag{3.62}$$

It is easy to see that $\epsilon\in(0,1)$. For $w\geq 500$,

$$k=2t+3s\leq 2.006t.$$

Then we have

$$\begin{aligned}2^{1+4/\epsilon}3^{2+2/\epsilon}k^{2+2/\epsilon} &\leq 2^{-3}\left(4\cdot 3\cdot 2.006\right)^{2+2/\epsilon}t^{2+2/\epsilon}\\ &= 2^{-3}\cdot 24.072^{\frac{12+7\chi}{3+\chi}}t^{\frac{12+7\chi}{3+\chi}}\\ &< 2.33\cdot 10^5 t^{\frac{12+7\chi}{3+\chi}}.\end{aligned}\tag{3.63}$$

The last inequality holds since $\frac{12+7\chi}{3+\chi}$ takes maximal value when χ takes its maximal value 21/32. By (3.61) and (3.63), we see that the condition in Lemma 3.11 is satisfied and thus

$$|H|\geq 2^{1+4/\epsilon}3^{2+2/\epsilon}k^{2+2/\epsilon}.\tag{3.64}$$

Hence we can apply Lemma 3.11, which gives $\lambda<2+\epsilon<3$. By Lemma 3.12, we have that if (x,y) is a solution to (3.1) of type II then

$$y<\left(\frac{C}{B}\right)^{\frac{1}{3-\lambda}},\tag{3.65}$$

where

$$B=\frac{0.999t}{k}$$

and

$$C = \frac{13.36|x_0 - \rho y_0| \sqrt[3]{1+\sqrt{2}|\gamma|}|H|(1+\sqrt{2}|\gamma|)}{\sqrt[3]{1-|\gamma|}} \cdot \max\left(\frac{8|x_0 - \overline{\rho} y_0|}{9\sqrt[3]{1-|\gamma|}(1-|\gamma|^2)}, 1\right)^{\lambda-1}.$$

We now estimate B and C in the current case. Since by assumption, we have

$$w = t/s \geq 1.2 \cdot 10^6 s^{2+\chi} \geq 1.2 \cdot 10^6 \cdot 2^2 = 4.8 \cdot 10^6.$$

Thus

$$B = \frac{0.999t}{k} = \frac{0.999t}{2t+3s} = \frac{0.999}{2+3/w} > 0.499499. \tag{3.66}$$

Notice that (x_0, y_0) is of type II. Then

$$\frac{x_0}{y_0} \in \left(-\frac{1}{w+2}, -\frac{1}{w+2} + \frac{4}{(w+2)w^2}\right) \subset \left(-\frac{1}{4.8 \cdot 10^6}, 0\right).$$

We have

$$|x_0 - \rho y_0| = y_0 \left|\frac{x_0}{y_0} - \rho\right| < \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = 1,$$

$$|x_0 - \overline{\rho} y_0| = y_0 \left|\frac{x_0}{y_0} - \overline{\rho}\right| < \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1.$$

From condition (3.64) and $k = 2t + 3s = s(2w+3) \geq 2(2 \cdot 4.8 \cdot 10^6 + 3) > 10^7$, it follows

$$\begin{aligned} |\gamma| &= \frac{3\sqrt{3}|m|}{|H|} \\ &\leq \frac{3\sqrt{3}k}{|H|} \\ &\leq \frac{3\sqrt{3}k}{2^{1+4/\epsilon}3^{2+2/\epsilon}k^{2+2/\epsilon}} \\ &< \frac{3\sqrt{3}}{2^5 \cdot 3^4 \cdot (10^7)^3} \\ &< 3 \cdot 10^{-24}. \end{aligned}$$

Therefore,

$$\begin{aligned}C &= \frac{13.36|x_0-\rho y_0|\sqrt[3]{1+\sqrt{2}|\gamma|}|H|(1+\sqrt{2}|\gamma|)}{\sqrt[3]{1-|\gamma|}} \\ &\quad \cdot \max\left(\frac{8|x_0-\overline{\rho}y_0|}{9\sqrt[3]{1-|\gamma|}(1-|\gamma|^2)},1\right)^{\lambda-1} \\ &< \frac{13.36y_0\sqrt[3]{1+\sqrt{2}\cdot 3\cdot 10^{-24}}|H|(1+\sqrt{2}\cdot 3\cdot 10^{-24})}{\sqrt[3]{1-3\cdot 10^{-24}}} \\ &\quad \cdot \max\left(\frac{8y_0}{9\sqrt[3]{1-3\cdot 10^{-24}}(1-(3\cdot 10^{-24})^2)},1\right)^{\lambda-1} \\ &< 13.361\cdot 0.8889^{\lambda-1}|H|y_0^{\lambda} \\ &< 10.5572|H|y_0^{\lambda},\end{aligned}$$

since $y_0 \geq 2$ and $\lambda < 3$. Thus (3.65) implies

$$y < \left(21.14|H|y_0^{\lambda}\right)^{\frac{1}{3-\lambda}}. \tag{3.67}$$

Recall the right hand side of (3.52),

$$|H| < 2.000012ty_0^3.$$

Together with (3.60) and (3.62), we obtain

$$\begin{aligned}y &< \left(21.14|H|y_0^{\lambda}\right)^{\frac{1}{3-\lambda}} \\ &< \left(21.14\cdot 2.000012ty_0^3y_0^{\lambda}\right)^{\frac{1}{3-\lambda}} \\ &< \left(\frac{42.2803}{5230}\cdot y_0^{\frac{3+\chi}{3+2\chi}}y_0^{3+\lambda}\right)^{\frac{1}{3-\lambda}} \\ &< (0.0080842)^{\frac{6+5\chi}{3\chi}}y_0^{\frac{126+174\chi+59\chi^2}{3(3+2\chi)\chi}}.\end{aligned} \tag{3.68}$$

The last inequality holds because $\frac{3+\lambda}{3-\lambda}$ is increasing in $\lambda \in (0,3)$ and $\lambda < 2+\epsilon < 3$. Assume that there are $\tau+1$ solutions $(x_0,y_0),(x_1,y_1),\ldots,(x_\tau,y_\tau)$ to (3.1) with condition (3.5). Further assume that

$$y_0 \leq y_1 \leq \ldots \leq y_\tau.$$

Then by Lemmas 2.6 and 3.3, we have by induction

$$\begin{aligned} y_\tau &\geq \left(\frac{B}{2}\right) y_{\tau-1}^2 \\ &\geq \left(\frac{B}{2}\right)\left(\left(\frac{B}{2}\right) y_{\tau-2}^2\right)^2 \\ &\geq \cdots \\ &\geq \left(\frac{B}{2}\right)^{2^\tau - 1} y_0^{2^\tau}. \end{aligned}$$

Together with the above estimation for B, we get

$$y_\tau > 0.2497495^{2^\tau-1} y_0^{2^\tau}. \tag{3.69}$$

By Lemma 3.14, we know that (x_τ, y_τ) has to be type II as well. Then (3.68) applies to y_τ. Together with (3.69) we have

$$0.2497495^{2^\tau-1} y_0^{2^\tau} < (0.0080842)^{\frac{6+5\chi}{3\chi}} y_0^{\frac{126+174\chi+59\chi^2}{3(3+2\chi)\chi}}.$$

Notice that $\chi = 21/2^\tau$. Then $2^\tau = 21/\chi$. It follows that

$$0.2497495^{\frac{21}{\chi}-1} y_0^{\frac{21}{\chi}} < (0.0080842)^{\frac{6+5\chi}{3\chi}} y_0^{\frac{126+174\chi+59\chi^2}{3(3+2\chi)\chi}}.$$

That is,

$$y_0^{21-\frac{126+174\chi+59\chi^2}{3(3+2\chi)}} < 0.2497495^{-21+\chi} \cdot (0.0080842)^{\frac{6+5\chi}{3}}. \tag{3.70}$$

Put

$$\phi = 21 - \frac{126+174\chi+59\chi^2}{3(3+2\chi)}.$$

Then

$$\phi = -\frac{59\chi^2+48\chi-63}{3(3+2\chi)}.$$

Since $59x^2 + 48x - 63 = 0$ has two roots at $-1.5173\ldots$ and $0.7037\ldots$, we have $\phi > 0$ for $\chi \in \left(0, \frac{21}{32}\right]$ from (3.56). Therefore, (3.70) gives

$$\begin{aligned} y_0 &< \left(0.2497495^{-21+\chi} \cdot (0.0080842)^{\frac{6+5\chi}{3}}\right)^{\frac{1}{\phi}} \\ &= \exp\left(\frac{1}{\phi} \cdot \left(\log(0.2497495)(-21+\chi) + \log(0.0080842)\frac{6+5\chi}{3}\right)\right) \\ &< \exp\left(\frac{3(3+2\chi)(19.5-9.4\chi)}{63-48\chi-59\chi^2}\right) \\ &< \exp(28.32) \\ &< 2 \cdot 10^{12}. \end{aligned} \tag{3.71}$$

The second last inequality holds since $\frac{3(3+2\chi)(19.5-9.4\chi)}{63-48\chi-59\chi^2}$ takes its maximal value at $\chi = 21/32$. On the other hand, from (3.53) and (3.54), we have

$$\begin{aligned} y_0 &> \frac{t^2}{4s^3} \\ &\geq \frac{(1.2 \cdot 10^6)^2 s^{3+2\chi}}{4} \\ &\geq \frac{(1.2 \cdot 10^6)^2 2^3}{4} \\ &= 2.88 \cdot 10^{12}. \end{aligned}$$

This is a contradiction to (3.71). Thus it follows that there are at most τ solutions to (3.1) of type II.

□

Combining Lemmas 3.14 and 3.16, we have that there are at most τ integer solutions (x, y) to (3.1) that satisfies (3.5) and $y \geq 2$. Then by (3.2), we conclude that for $s \geq 2, t \geq 3 \cdot 10^5 s^{3+21/2^\tau}$ with $\tau \geq 5$, other than the trivial solutions

$$\begin{aligned} \pm\{(0,1),(1,-1),(-1,0),(1,1),(1,-2),(-2,1),& \\ (-1,t+2),(-t-2,t+1),(t+1,1)\}& \text{ if } s = 1, \\ \pm\{(0,1),(1,-1),(-1,0),(1,1),(1,-2),(-2,1)\}& \text{ if } s \geq 2, \end{aligned}$$

there are at most 6τ integer solutions to the Thue inequality

$$|sx^3 - tx^2y - (t+3s)xy^2 - sy^3| \leq 2t + 3s.$$

Chapter 4

Quartic Simple Form

Let $s, t \in \mathbb{N}$ and

$$F(X,Y) = sx^4 - tx^3y - 6sx^2y^2 + txy^3 + sy^4$$

be an irreducible quartic form. F is a simple form by the definition in Chapter 1 since direct calculation implies

$$F(x-y, x+y) = -4F(x,y) \tag{4.1}$$

and accordingly the map

$$\psi : z \mapsto \frac{z-1}{z+1} \tag{4.2}$$

permutes the roots of $F(x,1)$ transitively. In this chapter, we mainly focus on the inequality:

$$|F(X,Y)| = |sx^4 - tx^3y - 6sx^2y^2 + txy^3 + sy^4| \leq k, \tag{4.3}$$

where k is a linear form in s and t. As discussed in Chapter 1, Wakabayashi [51] completely solved (4.3) for $s \geq 1, t \geq 70s^{28/9}$ and $k = 7s + 6t$. We shall prove the following result:

Theorem 4.1. *Let τ be an integer with $\tau \geq 2$ and let s, t be positive integers such that*

$$sx^4 - tx^3y - 6sx^2y^2 + txy^3 + sy^4 \tag{4.4}$$

is irreducible over $\mathbb{Q}$. Suppose that $s \geq 1$ and $t \geq 1200s^{2+4/3^\tau}$. Then other than the trivial solutions

$$\pm\{(1,0),(0,1),(1,1),(1,-1),$$
$$(1,2),(2,-1),(2,1),(1,-2)\}$$

there are at most 8τ *primitive integer solutions to the Thue inequality*

$$|sx^4 - tx^3y - 6sx^2y^2 + txy^3 + sy^4| \leq 7s + 6t. \tag{4.5}$$

Since the case when $s = 1$ had been explicitly solved by Lettl, Pethő and Voutier [22], in the following proof we always assume $s \geq 2$. The proof is similar to the cubic case.

4.1 Elementary properties

In this section, we shall study some elementary properties of the underlying polynomial

$$F(x, y) = sx^4 - tx^3y - 6sx^2y^2 + txy^3 + sy^4.$$

From (4.1), we can see that if (x, y) is a solution to (4.3), then any element in

$$\pm\{(x, y), (x - y, x + y), (y, -x), (x + y, -x + y)\} \tag{4.6}$$

is a solution to the following inequality

$$|F(x, y)| \leq 4k. \tag{4.7}$$

Notice that the map (4.2) permutes the intervals

$$[1/2, 3), [-1/3, 1/2), [-2, -1/3), (-\infty, -2) \cup [3, +\infty).$$

If there exists an integer solution (x, y) to (4.3), we can always choose one from the set of solutions (4.6) to the inequality (4.7) that satisfies the following condition

$$\frac{1}{2} \leq \frac{x}{y} < 3, \qquad \gcd(x, y) = 1, \qquad y \geq 0. \tag{4.8}$$

Let

$$f(x) = \frac{1}{s}F(x, 1) = x^4 - \left(\frac{t}{s}\right)x^3 - 6x^2 + \left(\frac{t}{s}\right)x + 1.$$

Put $w = t/s$. We have

$$f(x) = x^4 - wx^3 - 6x^2 + wx + 1. \tag{4.9}$$

Lemma 4.2. *$f(x) = 0$ has four real roots. Denote these zeros by $\theta_{-1}, \theta_0, \theta_1, \theta_2$. Further, if $w \geq 3000$, then the following holds:*

$$-1 - \frac{2}{w-1} < \theta_{-1} < -1 - \frac{2}{w}$$

$$-\frac{1}{w} < \theta_0 < -\frac{1}{w+1}$$

$$1 - \frac{2}{w+1} < \theta_1 < 1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}$$

$$w < \theta_2 < w + \frac{5}{w}$$

Proof. Since $w = t/s > 0$, we have

$$f(-3)f(-1) = -112 - 96w < 0$$

$$f(-1)f(0) = -4 < 0$$

$$f(0)f(1) = -4 < 0$$

$$f(1) = -4 \ \& \ f(+\infty) = +\infty$$

It follows that f has four real roots.

Further, if $w \geq 3000$, then we have $1 - 5w^2 < 0$ and $w^3 - 2w^2 - 8w - 4 = w((w-1)^2 - 9) - 4 > 0$. We have the following:

$$f\left(-1 - \frac{2}{w-1}\right) = \frac{4(5w^2-1)}{(w-1)^4} > 0, \ f\left(-1 - \frac{2}{w}\right) = -\frac{4(w^3 - 2w^2 - 8w - 4)}{w^4} < 0$$

$$f\left(-\frac{1}{w}\right) = -\frac{5w^2-1}{w^4} < 0, \ f\left(-\frac{1}{w+1}\right) = \frac{w^3 - 2w^2 - 8w - 4}{(w+1)^4} > 0$$

$$f\left(1 - \frac{2}{w+1}\right) = \frac{4(5w^2-1)}{(w+1)^4} > 0$$

$$\begin{aligned} & f\left(1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}\right) \\ &= -\frac{1}{(w+1)^4(w^2+w+5)^4}\big(2w^{10} + 8w^9 + 144w^8 + 529w^7 + 2022w^6 \\ &\qquad +5538w^5 + 11021w^4 + 18736w^3 + 22176w^2 + 14034w + 3479\big) \\ &< 0 \end{aligned}$$

$$f(w) = 1 - 5w^2 < 0,\ f\left(w + \frac{5}{w}\right) = \frac{21w^4 + 225w^2 + 625}{w^4} > 0$$

Then the second assertion follows.

□

Suppose that (x, y) is an integer solution to (4.7) that satisfies (4.8). From the above lemma, we see that x/y is bounded away from all roots of f except θ_1 for w large enough. In the rest of the proof in chapter, let $\theta = \theta_1$. We thus define the following interval that contains this root:

$$\mathbb{I} = \left(1 - \frac{2}{w+1}, 1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}\right) \tag{4.10}$$

and further we have the following defintion:

Definition For $w \geq 4$, we call (x, y) an integer solution to (4.7) of type I if (x, y) satisfies (4.7) with $\gcd(x, y) = 1, y \geq 4$ and

$$\frac{x}{y} \in \left[\frac{1}{2}, 1 - \frac{2}{w+1}\right] \cup \left[1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}, 3\right);$$

(x, y) an integer solution to (4.7) of type II if (x, y) satisfies (4.7) with $\gcd(x, y) = 1, y \geq 4$ and

$$\frac{x}{y} \in \mathbb{I}.$$

Lemma 4.3. *Suppose that (x, y) is a primitive solution to (4.7) of type II and $w \geq 3000$. Then we have*

$$\left|\theta - \frac{x}{y}\right| \leq \frac{1}{By^4},$$

where

$$B = \frac{0.499t}{k}.$$

Proof. Put

$$\delta_i = \left|\theta_i - \frac{x}{y}\right|,$$

for $i = -1, 0, 1, 2$. By the definition of type II solutions and Lemma 4.2, we have, for $w \geq 3000$,

$$\frac{x}{y} \in \left(1 - \frac{2}{w+1}, 1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}\right) \subset (0.999, 1)$$

and then

$$\delta_{-1} > 1.999, \quad \delta_0 > 0.999, \quad \delta_2 > w - 1.$$

Since

$$F(x,y) = sy^4 \left(\frac{x}{y} - \theta_{-1}\right)\left(\frac{x}{y} - \theta_0\right)\left(\frac{x}{y} - \theta\right)\left(\frac{x}{y} - \theta_2\right)$$

and

$$|F(x,y)| \leq 4k,$$

we have

$$\begin{aligned} \left|\theta - \frac{x}{y}\right| &= \frac{|F(x,y)|}{sy^4\delta_{-1}\delta_0\delta_2} \\ &\leq \frac{4k}{sy^4\delta_{-1}\delta_0\delta_2} \\ &< \frac{4k}{sy^4 1.999 \cdot 0.999(w-1)} \\ &< \frac{k}{0.499ty^4} \\ &= \frac{1}{\left(\frac{0.499t}{k}\right) y^4}, \end{aligned}$$

for $w \geq 2000$.

□

4.2 Irrationality of the root of f

Suppose that (x_0, y_0) is an integer solution to (4.7) that satisfies (4.8). We'll derive an irrationality measure for θ in term of (x_0, y_0). Let $\overline{\lambda}$ denote the complex conjugate for any complex number λ.

Lemma 4.4. *The form F can be written as*

$$\begin{aligned} F(x,y) &= sx^4 - tx^3y - 6sx^2y^2 + txy^3 + sy^4 \\ &= \frac{1}{2}\left(\eta(x-\rho y)^4 + \overline{\eta}(x-\overline{\rho}y)^4\right) \end{aligned}$$

where

$$\eta = s - \frac{t}{4}i, \qquad \rho = i$$

and $i = \sqrt{-1}$.

Proof. We only need to verify that F is equal to the real part of $\eta(x-\rho y)^4$. Direct calculation gives

$$(x-iy)^4 = (x^4 - 6x^2y^2 + y^4) - (4x^3y - 4xy^3)i.$$

Then the real part of $\eta(x-\rho y)^4$ is

$$\begin{aligned} \Re(\eta(x-\rho)^4) &= s(x^4 - 6x^2y^2 + y^4) - \frac{t}{4}(4x^3y - 4xy^3) \\ &= sx^4 - tx^3y - 6sx^2y^2 + txy^3 + sy^4 \\ &= F(x,y). \end{aligned}$$

□

Recall from last section that θ is a root of $f(x) = F(x,1)/s$. Then by Lemma 4.4, we have

$$\eta(\theta - i)^4 + \overline{\eta}(\theta + i)^4 = 0.$$

This gives

$$\frac{\eta}{\overline{\eta}} = -\frac{(\theta+i)^4}{(\theta-i)^4}. \tag{4.11}$$

On the other hand, since (x_0, y_0) is a solution to (4.7), we can put

$$F(x_0, y_0) = m,$$

for some fixed integer m with $|m| \le 4k$. Then by Lemma 4.4 again,

$$\frac{1}{2}(\eta(x_0 - iy_0)^4 + \overline{\eta}(x_0 + iy_0)^4) = m.$$

Thus, we can put

$$\eta(x_0 - iy_0)^4 = m + Ai, \tag{4.12}$$

where

$$A = -(t/4)x_0^4 - 4sx_0^3y_0 + (3t/2)x_0^2y_0^2 + 4sx_0y_0^3 - (t/4)y_0^4. \tag{4.13}$$

Hence

$$\frac{\eta(x_0 - iy_0)^4}{\overline{\eta}(x_0 + iy_0)^4} = \frac{m + Ai}{m - Ai}.$$

Together with (4.11), it follows that

$$\frac{(\theta + i)^4}{(\theta - i)^4} \cdot \frac{(x_0 - iy_0)^4}{(x_0 + iy_0)^4} = \frac{Ai + m}{Ai - m} = \frac{1 + \gamma}{1 - \gamma},$$

where $\gamma = m/Ai$. Take a 4th root on both sides of the above equation. We have

$$\frac{\theta + i}{\theta - i} \cdot \frac{x_0 - iy_0}{x_0 + iy_0} = \frac{\sqrt[4]{1 + \gamma}}{\sqrt[4]{1 - \gamma}}, \tag{4.14}$$

where we choose the quartic roots so that their arguments lie in the interval $(-\pi/6, \pi/6)$ since from last section x_0/y_0 is close to θ and so the left side is close to 1.

Let $I_{jn}(x), p_{jn}(x)$, $j = 0, 1$, be defined as in Lemma 2.1. Then by applying Lemma 2.1 with $\mu = 4$ and $x = \gamma$, for any $n \ge 1$, $j = 0, 1$, we have

$$I_{jn}(\gamma) = p_{jn}(\gamma)\sqrt[4]{1 + \gamma} - (-1)^j p_{jn}(-\gamma)\sqrt[4]{1 - \gamma}, \tag{4.15}$$

where

$$p_{0n}(\gamma) = \sum_{h=0}^{n} (-1)^{n-h} \binom{n + \frac{1}{4}}{h} \binom{2n - h}{n - h} \frac{\gamma^h (1 + \gamma)^{n-h}}{2^{2n+1-h}} \tag{4.16}$$

$$p_{1n}(\gamma) = \sum_{h=0}^{n} (-1)^{n-h} \binom{n + \frac{1}{4}}{h} \left(\binom{2n - h}{n - h} \frac{1}{2^{2n+1-h}} - \binom{2n - h - 1}{n - h - 1} \frac{1}{2^{2n-h}} \right) \gamma^h (1 + \gamma)^{n-h}, \tag{4.17}$$

and

$$I_{jn}(\gamma) = \frac{1}{2\pi i}\int_{\Gamma} \frac{z^j(1+\gamma z)^{n+\frac{1}{4}}}{(z^2-1)^{n+1}}dz, \tag{4.18}$$

for $j = 0, 1$. Dividing both sides of (4.15) by $\sqrt[4]{1-\gamma}$, we have

$$\frac{I_{jn}(\gamma)}{\sqrt[4]{1-\gamma}} = p_{jn}(\gamma)\frac{\sqrt[4]{1+\gamma}}{\sqrt[4]{1-\gamma}} - (-1)^j p_{jn}(-\gamma).$$

Substituting (4.14), we get

$$\frac{I_{jn}(\gamma)}{\sqrt[4]{1-\gamma}} = p_{jn}(\gamma)\frac{(\theta+i)(x_0-iy_0)}{(\theta-i)(x_0+iy_0)} - (-1)^j p_{jn}(-\gamma).$$

It follows that

$$\begin{aligned}\frac{(\theta-i)(x_0+iy_0)I_{jn}(\gamma)}{\sqrt[4]{1-\gamma}} &= \theta\left(p_{jn}(\gamma)(x_0-iy_0) - (-1)^j p_{jn}(-\gamma)(x_0+iy_0)\right) \\ &\quad + \left(p_{jn}(\gamma)(x_0-iy_0)i + (-1)^j p_{jn}(-\gamma)(x_0+iy_0)i\right).\end{aligned}$$

Put, for $n \geq 1$,

$$\begin{aligned} p_{0n} &= -(p_{0n}(\gamma)(x_0-iy_0) + p_{0n}(-\gamma)(x_0+iy_0)) \\ q_{0n} &= i(p_{0n}(\gamma)(x_0-iy_0) - p_{0n}(-\gamma)(x_0+iy_0)) \\ l_{0n} &= \frac{i(\theta-i)(x_0+iy_0)I_{0n}(\gamma)}{\sqrt[4]{1-\gamma}} \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} p_{1n} &= i(p_{1n}(\gamma)(x_0-iy_0) - p_{1n}(-\gamma)(x_0+iy_0)) \\ q_{1n} &= p_{1n}(\gamma)(x_0-iy_0) + p_{1n}(-\gamma)(x_0+iy_0) \\ l_{1n} &= \frac{(\theta-i)(x_0+iy_0)I_{1n}(\gamma)}{\sqrt[4]{1-\gamma}}. \end{aligned} \tag{4.20}$$

Then we have, for $n \geq 1, j = 0, 1$,

$$\theta q_{jn} + p_{jn} = l_{jn}. \tag{4.21}$$

Since γ is a purely imaginary number, it is easy to see from the definition of p_{jn} and q_{jn} that

$$p_{jn}, q_{jn} \in \mathbb{Q}, \qquad \text{for } j = 0, 1, n \geq 1.$$

Lemma 4.5. *With p_{jn}, q_{jn} defined in (4.19) and (4.20) for $j = 0, 1$, we have*

$$\begin{vmatrix} p_{0n} & q_{0n} \\ p_{1n} & q_{1n} \end{vmatrix} \neq 0,$$

for any $n \geq 1$.

Proof. By Lemma 2.4, we have

$$\Delta(\gamma) = \begin{vmatrix} p_{0n}(\gamma) & -p_{0n}(-\gamma) \\ p_{1n}(\gamma) & p_{1n}(-\gamma) \end{vmatrix} = c_{2n}\gamma^{2n} \neq 0. \tag{4.22}$$

Put $A_j = p_{jn}(\gamma)(x_0 - iy_0),\ B_j = p_{jn}(-\gamma)(x_0 + iy_0)$ for $j = 0, 1$. Then by (4.19) and (4.20), we have

$$p_{0n} = -(A_0 + B_0),\ p_{1n} = i(A_1 - B_1),$$
$$q_{0n} = i(A_0 - B_0),\ q_{1n} = (A_1 + B_1).$$

It follows that

$$\begin{aligned}
\begin{vmatrix} p_{0n} & q_{0n} \\ p_{1n} & q_{1n} \end{vmatrix} &= \begin{vmatrix} -(A_0 + B_0) & i(A_0 - B_0) \\ i(A_1 - B_1) & (A_1 + B_1) \end{vmatrix} \\
&= \begin{vmatrix} -2A_0 & i(A_0 - B_0) \\ 2iA_1 & (A_1 + B_1) \end{vmatrix} \\
&= 2\begin{vmatrix} -A_0 & -iB_0 \\ iA_1 & B_1 \end{vmatrix} \\
&= -2\begin{vmatrix} A_0 & -B_0 \\ A_1 & B_1 \end{vmatrix} \\
&= -2\begin{vmatrix} p_{0n}(\gamma)(x_0 - iy_0) & -p_{0n}(-\gamma)(x_0 + iy_0) \\ p_{1n}(\gamma)(x_0 - iy_0) & p_{1n}(-\gamma)(x_0 + iy_0) \end{vmatrix} \\
&= -2(x_0^2 + y_0^2)\begin{vmatrix} p_{0n}(\gamma) & -p_{0n}(-\gamma) \\ p_{1n}(\gamma) & p_{1n}(-\gamma) \end{vmatrix} \\
&= -2(x_0^2 + y_0^2)\Delta(\gamma) \\
&\neq 0
\end{aligned}$$

by (4.22).

□

Lemma 4.6. *Suppose that $|\gamma| < 1/\sqrt{2}$. For $n \geq 1$, $j = 0, 1$,*

$$|q_{jn}| < \rho P^n,$$

where

$$\rho = 1.67\sqrt{x_0^2 + y_0^2}\left(\frac{1+\sqrt{2}|\gamma|}{1-|\gamma|}\right)^{\frac{1}{4}}$$

and

$$P = 1 + \sqrt{2}|\gamma|.$$

Proof. By the definition of q_{jn}, we have

$$|q_{jn}| \leq 2|p_{jn}(\gamma)(x_0 - iy_0)| \tag{4.23}$$

By Lemma 2.1,

$$p_{jn}(x) = \frac{I_{j1n}(x)}{\sqrt[4]{1+x}}, \tag{4.24}$$

where

$$I_{j1n}(x) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{z^j(1+xz)^{n+\frac{1}{4}}}{(z^2-1)^{n+1}}dz.$$

To estimate $I_{j1n}(x)$, let's consider curve $\Gamma : \{z \in \mathbb{C} : |z^2 - 1| = 1\}$. It consists of two closed curves. Let Γ_1 be the one enclosing the point 1. Notice that $|z|^2 \leq |z^2 - 1| + 1 = 2$ and $\sqrt{2} \in \Gamma_1$. Hence $\max_{z\in\Gamma_1} |z| = \sqrt{2}$. Then for $|x| < 1/\sqrt{2}$, we have

$$\begin{aligned}
|I_{j1n}(x)| &= \left|\frac{1}{2\pi i}\int_{\Gamma_1} \frac{z^j(1+xz)^{n+\frac{1}{4}}}{(z^2-1)^{n+1}}dz\right| \\
&\leq \frac{|\Gamma_1|}{2\pi} \cdot \max_{z\in\Gamma_1}\left|\frac{z^j(1+xz)^{n+\frac{1}{4}}}{(z^2-1)^{n+1}}\right| \\
&= \frac{|\Gamma_1|\sqrt{2}^j(1+\sqrt{2}|x|)^{n+\frac{1}{4}}}{2\pi},
\end{aligned} \tag{4.25}$$

where $|\Gamma_1|$ denotes the length of Γ_1.

Write $z = a + bi$. By the definition of Γ, we can obtain the equation of Γ_1 on ab-plane:

$$(a^2 + b^2)^2 - 2(a^2 - b^2) = 0, \qquad 0 \leq a \leq \sqrt{2}.$$

We can find the length of Γ_1 by an integral along the above curve. Numerical integration gives the estimate

$$|\Gamma_1| = 3.70814935\ldots < 3.709. \tag{4.26}$$

Also, $|\sqrt[4]{1-x}| = \sqrt[4]{|1-x|} \geq \sqrt[4]{1-|x|}$. Together with (4.23), (4.24), (4.25), we get

$$\begin{aligned}
|q_{jn}| &\leq 2|x_0 - iy_0||p_{jn}(\gamma)| \\
&\leq \frac{2\sqrt{x_0^2+y_0^2}|\Gamma_1|\sqrt{2}^j(1+\sqrt{2}|\gamma|)^{n+\frac{1}{4}}}{2\pi\sqrt[4]{1-|\gamma|}} \\
&\leq \frac{1.67\sqrt{x_0^2+y_0^2}\sqrt[4]{1+\sqrt{2}|\gamma|}}{\sqrt[4]{1-|\gamma|}} \cdot (1+\sqrt{2}|\gamma|)^n.
\end{aligned}$$

□

Lemma 4.7. *Suppose that* $|\gamma| < 1/\sqrt{2}$. *For* $n \geq 1$, $j = 0, 1$,

$$|l_{jn}| \leq lL^{-n},$$

where

$$l = \frac{0.625\sqrt{\theta^2+1}\sqrt{x_0^2+y_0^2}}{\sqrt[4]{1-|\gamma|}(1-|\gamma|^2)}$$

and

$$L = \frac{4(1-|\gamma|^2)}{|\gamma|^2}.$$

Proof. By the definition of l_{jn}, we have that

$$|l_{jn}| = \frac{|\theta - i||x_0+iy_0||I_{jn}(\gamma)|}{|\sqrt[4]{1-\gamma}|}. \tag{4.27}$$

By Lemma 2.2 and Lemma 2.3, we have

$$I_{jn}(x) = \sum_{h=2n+1-j}^{\infty} \binom{n+\frac{1}{4}}{h} J_{h+j} x^h,$$

where

$$J_h = \frac{1}{2\pi i}\int_\Gamma \frac{z^h}{(z^2-1)^{n+1}}dz.$$

As a consequence of Lemma 2.2, $J_h \geq 0$. Notice that for $h \geq 2n$, $\left|\binom{n+\frac{1}{4}}{h}\right|$ decreases as h increases. It follows that

$$\begin{aligned}
|I_{jn}(x)| &\leq \left|\binom{n+\frac{1}{4}}{2n}\right| \sum_{h=2n+1-j}^{\infty} |J_{h+j}||x|^h \\
&= \left|\binom{n+\frac{1}{4}}{2n}\right| |x|^{-j} \sum_{h=2n+1}^{\infty} J_h |x|^h \\
&= \left|\binom{n+\frac{1}{4}}{2n}\right| |x|^{-j} \frac{|x|^{2n+1}}{(1-|x|^2)^{n+1}}
\end{aligned}$$

by Lemma 2.2. Moreover, one can show by induction that

$$\left|\binom{n+\frac{1}{4}}{2n}\right| \leq \frac{5}{8}\left(\frac{1}{4}\right)^n.$$

Therefore, together with (4.27), we get

$$\begin{aligned}
|l_{jn}| &\leq \frac{5|\theta-i||x_0+iy_0|}{8|\sqrt[4]{1-\gamma}|}\left(\frac{1}{4}\right)^n \frac{|\gamma|^{2n}}{(1-|\gamma|^2)^{n+1}} \\
&\leq \frac{0.625\sqrt{\theta^2+1}\sqrt{x_0^2+y_0^2}}{\sqrt[4]{1-|\gamma|}(1-|\gamma|^2)}\left(\frac{|\gamma|^2}{4(1-|\gamma|^2)}\right)^n.
\end{aligned}$$

□

Lemma 4.8. *Let Δ_n be the common denominator of p_{jn}, q_{jn} for $n \geq 1$. Then*

$$\Delta_n \leq d\Delta^n,$$

where $d = 0.25$, $\Delta = 128|A|$.

Proof. From the definition of p_{jn}, q_{jn} and the fact $x_0, y_0 \in \mathbb{Z}$, we see that Δ_n divides the common denominator of the coefficients of $p_{jn}(\gamma)$. Notice that the number of 2-factors in $h!$ is at most $h-1$. Hence we have

$$\begin{aligned}
4^h 2^{h-1}\binom{n+\frac{1}{4}}{h} &= 4^h 2^{h-1}\frac{(n+1/4)(n-1+1/4)\cdots(n-h+1+1/4)}{h!} \\
&= 2^{h-1}\frac{(4n+1)(4(n-1)+1)\cdots(4(n-h+1)+1)}{h!}.
\end{aligned}$$

Suppose p is a prime such that $p|h!$ with $p \neq 2$ and a is a positive integer such that $p^a|h!$ but $p^{a+1} \nmid h!$. First notice that $p \leq h$. Then consider the natural integer sequence modulo p:

$$1, 2, 3, \ldots, p-1, 0, 1, 2, \ldots \tag{4.28}$$

The exponent of p-factor in $h!$, a, depends on the number of times 0 appears in the first h elements in the above sequence:

$$(1, 2, \ldots, h) \mod p. \tag{4.29}$$

In other words, a depends on how many complete residual sets $(1, 2, \ldots, p-1, 0)$ (4.29) contains mod p. Notice that $(n-h+1, n-h+1, \ldots, n)$ is a sequence of h consecutive integers. We have that modulo p, it contains the same number of complete residual sets as (4.29). Since $\gcd(2, p) = 1$, it follows that

$$(4(n-h+1)+1, 4(n-h+2)+1, \ldots, 4n+1) \mod p \tag{4.30}$$

contains the same number of complete residual sets $(1, 2, \ldots, p-1, 0)$ as well. This implies that

$$p^a \mid \prod_{j=0}^{h-1}(4(n-j)+1).$$

Hence we have

$$4^h 2^{h-1}\binom{n+\frac{1}{4}}{h} \in \mathbb{Z}.$$

Also notice that for $n \geq 1$, $\binom{2n}{n}$ has at least one 2-factor and

$$4A = 4\left(-\frac{t}{4}x_0^4 - 4sx_0^3y_0 + \frac{3t}{2}x_0^2y_0^2 + 4sx_0y_0^3 - \frac{t}{4}y_0^4\right) \in \mathbb{Z}.$$

Then by the definition of $p_{jn}(\gamma)$ in (4.16) and (4.17), we have

$$2^{2n}2^{n-1}4^n(4A)^n p_{jn}(\gamma) \in \mathbb{Z}[i].$$

Therefore, by the definition of p_{jn}, q_{jn}, we have

$$2^{-1}2^{2n}2^{n-1}4^n(4A)^n p_{jn} \in \mathbb{Z}, \qquad 2^{-1}2^{2n}2^{n-1}4^n(4A)^n q_{jn} \in \mathbb{Z}.$$

It follows that

$$\Delta_n \leq 0.25(128|A|)^n.$$

□

Lemma 4.9. *Suppose that $|A| \geq 32m^2 + |m|$. With the notation as above, we have, for any integers p and q with $q > 0$,*

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{Cq^{\lambda}},$$

where

$$\lambda = 1 + \frac{\log(128|A|(1+\sqrt{2}|\gamma|))}{\log\left(\frac{1-|\gamma|^2}{32|A||\gamma|^2}\right)}$$

and

$$C = 106.88\sqrt{x_0^2 + y_0^2}\left(\frac{1+\sqrt{2}|\gamma|}{1-|\gamma|}\right)^{\frac{1}{4}} |A|(1+\sqrt{2}|\gamma|)$$
$$\cdot \left(\max\left\{\frac{0.3125\sqrt{\theta^2+1}\sqrt{x_0^2+y_0^2}}{\sqrt[4]{1-|\gamma|}(1-|\gamma|^2)}, 1\right\}\right)^{\lambda-1}.$$

Proof. By assumption $|A| \geq 32m^2 + |m|$. Then we have

$$\frac{L}{\Delta} = \frac{4(1-|\gamma|^2)}{128|A||\gamma|^2} = \frac{1+|m|/|A|}{32}\left(\frac{|A|}{m^2} - \frac{1}{|m|}\right) > 1.$$

Apply Lemma 2.5 on (4.21), with the conditions shown by Lemmas 4.5, 4.6, 4.7, 4.8. It follows that

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{Cq^{\lambda}},$$

for any $p, q \in \mathbb{Z}$, $q > 0$, where

$$\begin{aligned}\lambda &= 1 + \frac{\log(\Delta P)}{\log(L/\Delta)} \\ &= 1 + \frac{\log(128|A|(1+\sqrt{2}|\gamma|))}{\log\left(\frac{1-|\gamma|^2}{32|A||\gamma|^2}\right)}\end{aligned}$$

and

$$
\begin{aligned}
C &= 2\rho d\Delta P\left(\max\{2dl,1\}\right)^{\lambda-1}\\
&= 2\cdot 1.67\sqrt{x_0^2+y_0^2}\left(\frac{1+\sqrt{2}|\gamma|}{1-|\gamma|}\right)^{\frac{1}{4}}\cdot 0.25\cdot 128|A|(1+\sqrt{2}|\gamma|)\\
&\quad\cdot\left(\max\left\{\frac{2\cdot 0.25\cdot 0.625\sqrt{\theta^2+1}\sqrt{x_0^2+y_0^2}}{\sqrt[4]{1-|\gamma|}(1-|\gamma|^2)},1\right\}\right)^{\lambda-1}\\
&= 106.88\sqrt{x_0^2+y_0^2}\left(\frac{1+\sqrt{2}|\gamma|}{1-|\gamma|}\right)^{\frac{1}{4}}|A|(1+\sqrt{2}|\gamma|)\\
&\quad\cdot\left(\max\left\{\frac{0.3125\sqrt{\theta^2+1}\sqrt{x_0^2+y_0^2}}{\sqrt[4]{1-|\gamma|}(1-|\gamma|^2)},1\right\}\right)^{\lambda-1}.
\end{aligned}
$$

□

4.3 Upper bounds for the solutions

Lemma 4.10. *Let $\epsilon\in(0,2)$. If*

$$|A|\geq 128.1^{1/\epsilon}\cdot 512.1^{1+1/\epsilon}k^{2+2/\epsilon}, \tag{4.31}$$

then $\lambda<2+\epsilon$.

Proof. By the assumption on A, we have

$$|A|>128^{1/\epsilon}\cdot 512^{1+1/\epsilon}k^{2+2/\epsilon}>2^6(4k)^2>32m^2+|m|$$

then the condition in Lemma 4.9 is satisfied. By the definition of γ,

$$
\begin{aligned}
|\gamma| &= \frac{|m|}{|A|}\\
&\leq \frac{4k}{|A|}\\
&\leq \frac{4k}{128.1^{1/\epsilon}\cdot 512.1^{1+1/\epsilon}k^{2+2/\epsilon}}\\
&< \frac{4}{128^{1/2}\cdot 512^{3/2}}\\
&= 2^{-17}.
\end{aligned}
$$

Then we have P, L, Δ, λ defined as in the previous lemmas. It follows that

$$\Delta P = 128|A|(1+\sqrt{2}|\gamma|) < 128.1|A|, \tag{4.32}$$

and

$$\begin{aligned}
\frac{L}{\Delta} &= \frac{1-|\gamma|^2}{32|A||\gamma|^2} \\
&= (1-|\gamma|^2)\frac{|A|}{32(|A||\gamma|)^2} \\
&= (1-|\gamma|^2)\frac{|A|}{32|m|^2} \\
&\geq (1-|\gamma|^2)\frac{|A|}{32\cdot 16k^2} \\
&\geq \frac{|A|}{512.1k^2},
\end{aligned} \tag{4.33}$$

since $\gamma = \frac{m}{Ai}$ and $|m| \leq 4k$. From the assumption on A, it follows that

$$\frac{\left(\frac{|A|}{512.1k^2}\right)^{\epsilon+1}}{128.1|A|} = \frac{|A|^{\epsilon}}{128.1 \cdot 512.1^{\epsilon+1} \cdot k^{2\epsilon+2}} \geq \frac{128.1 \cdot 512.1^{\epsilon+1}k^{2\epsilon+2}}{128.1 \cdot 512.1^{\epsilon+1} \cdot k^{2\epsilon+2}} = 1.$$

Then combining (4.32) and (4.33), we have

$$\left(\frac{L}{\Delta}\right)^{\epsilon+1} > \left(\frac{|A|}{512.1k^2}\right)^{\epsilon+1} \geq 128.1|A| > \Delta P.$$

Taking logarithms, we obtain

$$(\epsilon+1)\log(L/\Delta) > \log(\Delta P).$$

Therefore,

$$\lambda = 1 + \frac{\log(\Delta P)}{\log(L/\Delta)} < 1 + (\epsilon+1) = 2+\epsilon.$$

□

Lemma 4.11. *With the same notation as before, suppose that (x_0, y_0), (x, y) are solutions to (4.7) of type II. Let A be defined as in (4.13). Assume that as in Lemma 4.10 $|A|$ is sufficiently large so that $\lambda < 4$. Then we have for $w \geq 2000$,*

$$y < \left(\frac{C}{B}\right)^{\frac{1}{4-\lambda}},$$

where

$$B = \frac{0.499t}{k}$$

and

$$C = 106.88\sqrt{x_0^2 + y_0^2}\left(\frac{1+\sqrt{2}|\gamma|}{1-|\gamma|}\right)^{\frac{1}{4}}|A|(1+\sqrt{2}|\gamma|)$$
$$\cdot\left(\max\left\{\frac{0.3125\sqrt{\theta^2+1}\sqrt{x_0^2+y_0^2}}{\sqrt[4]{1-|\gamma|}(1-|\gamma|^2)}, 1\right\}\right)^{\lambda-1}.$$

Proof. Notice that if $w \geq 4$ and $|A|$ is large enough as in Lemma 4.10, the assumptions in Lemmas 4.3 and 4.9 can be easily satisfied. It then follows directly from these two lemmas that if (x, y) is a solution to (4.7) then

$$\frac{1}{Cy^{\lambda}} < \left|\theta - \frac{x}{y}\right| \leq \frac{1}{By^4},$$

which gives

$$y < \left(\frac{C}{B}\right)^{\frac{1}{4-\lambda}}$$

with B, C defined as in the statement of the lemma.

□

4.4 Proof of Theorem 4.1

Let $k = 6t + 7s$. For $1 \le y \le 3$, the following pairs satisfy assumption (4.8):

$$(1,1), (2,1), (1,2), (3,2), (5,2), (2,3), (4,3), (5,3), (7,3), (8,3).$$

Substituting these integer pairs to (4.7), we see that only

$$(1,1), (2,1), (1,2)$$

are integer solutions to (4.7) with condition (4.8). Then by (4.1) we can get all the other trivial solutions to (4.3) as stated in the theorem. In the following proof, we assume that $y \ge 4$.

Lemma 4.12. *Let $(x,y) \in \mathbb{Z}^2$ be a solution to (4.3) of type I. Then*

$$y < \frac{w}{2},$$

for $w \ge 400$.

Proof. Consider

$$f(x) = \frac{1}{s} F(x,1) = x^4 - wx^3 - 6x^2 + wx + 1.$$

For $w \ge 3000$, from the shape of $f(x)$, we see that if (x,y) is of type I, then

$$\left| f\left(\frac{x}{y}\right) \right| \ge \min \left\{ \left| f\left(1 - \frac{2}{w+1}\right) \right|, \left| f\left(1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}\right) \right| \right\}.$$

Similar to that in the proof of Lemma 4.2,

$$f\left(1 - \frac{2}{w+1}\right) = \frac{4(5w^2-1)}{(w+1)^4} > \frac{19}{w^2}$$

and

$$\begin{aligned} & f\left(1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}\right) \\ &= -\frac{1}{(w+1)^4(w^2+w+5)^4} \big(2w^{10} + 8w^9 + 144w^8 + 529w^7 + 2022w^6 \\ &\quad +5538w^5 + 11021w^4 + 18736w^3 + 22176w^2 + 14034w + 3479\big) \\ &< -\frac{1}{w^2}. \end{aligned}$$

Therefore, if (x, y) is of type I, then

$$\left| f\left(\frac{x}{y}\right) \right| > \frac{1}{w^2}.$$

Then

$$4(7s + 6t) \geq |F(x, y)| = sy^4 \left| f\left(\frac{x}{y}\right) \right| > \frac{sy^4}{w^2}.$$

Thus

$$y^4 < 4(6w + 7)w^2 < \left(\frac{1}{2}\right)^4 w^4,$$

for $w \geq 400$. That is,

$$y < \frac{w}{2}.$$

□

Lemma 4.13. *Let $(x, y) \in \mathbb{Z}^2$ be a solution to (4.3) of type II. Then*

$$y > \frac{t^2}{11s^3}.$$

Proof. If (x, y) is of type II, then

$$1 - \frac{2}{w+1} < \frac{x}{y} < 1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}. \tag{4.34}$$

Multiplying (4.34) by $s(w + 1)y$, we obtain

$$y(t + s) - 2ys < x(t + s) < y(t + s) - 2ys + \frac{11ys}{w^2 + w + 5}$$

Since both $x(t + s)$ and $y(t + s) - 2ys$ are integers, we have

$$\frac{11ys}{w^2 + w + 5} > 1.$$

Thus

$$y > \frac{w^2 + w + 5}{11s} > \frac{w^2}{11s} = \frac{t^2}{11s^3}.$$

□

Then the proof of Theorem 4.1 will be completed by the following two claims:

Claim I. There is no primitive solution (x, y) to (4.3) of type I with $y \geq 4$.

Proof. Suppose that (x, y) is a primitive solution to (4.3) of type I with $y \geq 4$. We then divide the interval of interest $[1/2, 3)$ into two intervals

$$[1/2, 3) = [1/2, 3/5] \cup (3/5, 3).$$

For $x/y \in [1/2, 3/5]$, from the shape of $f(x)$ and Lemma 4.2, we have

$$\left| f\left(\frac{x}{y}\right) \right| = f\left(\frac{x}{y}\right) \geq \min\left\{ f\left(\frac{1}{2}\right), f\left(\frac{3}{5}\right) \right\}.$$

Since

$$f\left(\frac{1}{2}\right) = \frac{3}{8}w - \frac{7}{16} > \frac{1}{4}w$$

and

$$f\left(\frac{3}{5}\right) = \frac{48}{125}w - \frac{644}{625} > \frac{1}{4}w,$$

for $w \geq 8$. Thus, for $x/y \in [1/2, 3/5]$, we have

$$\left| f\left(\frac{x}{y}\right) \right| > \frac{1}{4}w = \frac{y^4}{4}\frac{w}{y^4} \geq 4^3\frac{w}{y^4} > \frac{25w}{y^4}. \tag{4.35}$$

Then we treat the case when $x/y \in (3/5, 3)$. Since $\gcd(x, y) = 1$ and $y \geq 4$, we have $x \neq y$. Therefore,

$$\frac{x}{y} \notin \left(1 - \frac{1}{y}, 1 + \frac{1}{y}\right)$$

since otherwise,

$$\frac{1}{y} > \left|\frac{x}{y} - 1\right| = \frac{|x - y|}{y} \geq \frac{1}{y},$$

a contradiction. We have

$$f\left(1 + \frac{1}{y}\right) = -\frac{(2y^3 + 3y^2 + y)w}{y^4} - \frac{4y^4 + 8y^3 - 4y - 1}{y^4} < -\frac{25w}{y^4} < 0,$$

for $y \geq 4$; and

$$f\left(1 - \frac{1}{y}\right) = \frac{(2y^3 - 3y^2 + y)w + (-4y^4 + 8y^3 - 4y + 1)}{y^4}.$$

Put

$$h(y) = (2y^3 - 3y^2 + y)w + (-4y^4 + 8y^3 - 4y + 1) - 25w.$$

Then

$$h(y) = -4y^4 + (2w + 8)y^3 - 3wy^2 + (-4 + w)y + 1 - 25w.$$

The discriminant of h is

$$-269996w^6 + 4800w^5 - 6719808w^4 + 256153600w^3 - 38396928w^2 + 1228800w + 16384,$$

which is negative, for $w \geq 10$. This means that h has 2 real roots and a pair of complex conjugate roots. Since we have

$$h(2) = -7 - 19w < 0, \qquad h(3) = -119 + 5w > 0,$$

for $w \geq 20$ and

$$h\left(\frac{w}{2}\right) = \frac{1}{4}w^3 - 27w + \frac{1}{2}w^2 + 1 > 0,$$
$$h\left(\frac{w}{2} + 1\right) = -\frac{1}{4}w^3 + \frac{1}{2}w^2 - 23w + 1 < 0,$$

for $w \geq 10$. Then, for w big enough, h has a real root between 2 and 3, another real root between $w/2$ and $w/2 + 1$. From the shape of h, we conclude that if $y \in (3, w/2)$, then

$$h(y) > 0,$$

that is

$$(2y^3 - 3y^2 + y)w + (-4y^4 + 8y^3 - 4y + 1) - 25w > 0.$$

It follows that under this condition

$$f\left(1 - \frac{1}{y}\right) = \frac{(2y^3 - 3y^2 + y)w + (-4y^4 + 8y^3 - 4y + 1)}{y^4} > \frac{25w}{y^4} > 0.$$

Notice that $f(x)$ is a decreasing function over the interval $(3/5, 3)$, since

$$f'(x) < 0, \qquad \text{for all } x \in (3/5, 3).$$

By Lemma 4.12, if (x, y) is of type I, then

$$y < w/2,$$

for $w \geq 400$. So if (x, y) is of type I and $x/y \in (3/5, 3)$, then

$$\left| f\left(\frac{x}{y}\right) \right| = f\left(\frac{x}{y}\right) \geq f\left(1 - \frac{1}{y}\right) > \frac{25w}{y^4} \tag{4.36}$$

if $x/y \in (3/5, 1 - 1/y]$ and

$$\left| f\left(\frac{x}{y}\right) \right| = -f\left(\frac{x}{y}\right) \geq -f\left(1 + \frac{1}{y}\right) > \frac{25w}{y^4} \tag{4.37}$$

if $x/y \in [1 + 1/y, 3)$. Combining (4.35), (4.36) and (4.37), we obtain

$$\left| f\left(\frac{x}{y}\right) \right| > \frac{25w}{y^4}$$

if (x, y) is a primitive solution to (4.3) with $y \geq 4$. It follows that, for $w \geq 30$,

$$25t > 4(6t + 7s) \geq |F(x, y)| = sy^4 \left| f\left(\frac{x}{y}\right) \right| > sy^4 \frac{25w}{y^4} = 25t,$$

which is a contradiction. This completes the proof of the claim.

□

Claim II. There are at most τ primitive solutions to (4.7) of type II.

Proof. Recall that we've assumed that (x_0, y_0) is a non-trivial solution to (4.7) satisfying condition (4.8). From Claim I, we know that (x_0, y_0) is a solution of type II and thus

$$\frac{x_0}{y_0} \in \left(1 - \frac{2}{w+1}, 1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}\right).$$

By the assumption of the theorem, we have

$$w = \frac{t}{s} \geq 1200 s^{1+\chi} \geq 1200 \cdot 2 = 2400. \tag{4.38}$$

We've also defined

$$A = -\frac{t}{4}x_0^4 - 4sx_0^3y_0 + \frac{3t}{2}x_0^2y_0^2 + 4sx_0y_0^3 - \frac{t}{4}y_0^4.$$

Put

$$g(x) = -\frac{w}{4}x^4 - 4x^3 + \frac{3w}{2}x^2 + 4x - \frac{w}{4}.$$

We have the following:

$$g(0) = -\frac{1}{4}w < 0, \qquad g\left(\frac{1}{2}\right) = \frac{7}{64}w + \frac{3}{2} > 0$$

$$g\left(\frac{3}{2}\right) = \frac{119}{64}w - \frac{15}{2} > 0, \qquad g(3) = -7w - 96 < 0$$

$$g'(-1) = -2w - 8 < 0, \qquad g'\left(\frac{1}{2}\right) = \frac{11}{8}w + 1$$

$$g'\left(\frac{3}{2}\right) = \frac{9}{8}w - 23 > 0, \qquad g'(2) = -2w - 44 < 0$$

for $w \geq 30$. Then on the interval $(1/2, 3/2)$, the function $g(x)$ is increasing as x increases and $g(x) > 0$ for all $x \in (1/2, 3/2)$. Obviously, for $w \geq 2000$,

$$\frac{x_0}{y_0} \in \left(1 - \frac{2}{w+1}, 1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}\right) \subset \left(\frac{1}{2}, \frac{3}{2}\right)$$

Thus we obtain

$$sy_0^4 g\left(1 - \frac{2}{w+1}\right) < sy_0^4 g\left(\frac{x_0}{y_0}\right) = |A| < sy_0^4 g\left(1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}\right). \tag{4.39}$$

Since

$$g\left(1 - \frac{2}{w+1}\right) = \frac{w^4 + 10w^2 - 15}{(w+1)^4} \cdot w > 0.998w \tag{4.40}$$

and

$$\begin{aligned} & g\left(1 - \frac{2}{w+1} + \frac{11}{(w+1)(w^2+w+5)}\right) \\ &= \frac{1}{4(w+1)^4(w^2+w+5)^4} \cdot \big(4w^{13} + 16w^{12} + 144w^1 1 + 504w^1 0 + 2000w^9 \\ &\quad + 5712w^8 + 13968w^7 + 30056w^6 + 49632w^5 + 68692w^4 + 71888w^3 \\ &\qquad +37320w^2 - 617w - 5280\big) \\ &< w, \end{aligned} \tag{4.41}$$

for $w \geq 2000$. Combining (4.39), (4.40) and (4.41), we get

$$0.998ty_0^4 < |A| < ty_0^4. \tag{4.42}$$

By Lemma 4.13, if (x, y) is a solution of type II, then

$$y > \frac{t^2}{11s^3}. \tag{4.43}$$

Since

$$t \geq 1200s^{2+\chi}, \tag{4.44}$$

where $\chi = 4/3^\tau$ with $\tau \geq 2$, we have

$$s^{-1} \geq \left(\frac{t}{1200}\right)^{-\frac{1}{2+\chi}}.$$

Combining with (4.43), we obtain

$$y > \frac{t^2}{11}\left(\frac{t}{1200}\right)^{-\frac{3}{2+\chi}} = \frac{1200^{\frac{3}{2+\chi}}}{11} \cdot t^{\frac{1+2\chi}{2+\chi}} > 546t^{\frac{1+2\chi}{2+\chi}}. \tag{4.45}$$

The last inequality holds since by the definition of χ, we have

$$0 < \chi \leq \frac{4}{9} \tag{4.46}$$

and $\frac{3}{2+\chi}$ takes it minimal value when χ takes its maximal value. Note that (4.45) is equivalent to

$$t < \left(\frac{y}{546}\right)^{\frac{2+\chi}{1+2\chi}} < 3485^{-1}y^{\frac{2+\chi}{1+2\chi}}. \tag{4.47}$$

The last inequality holds since

$$546^{\frac{2+\chi}{1+2\chi}} > 3485,$$

for $0 < \chi < 4/9$. In particular, since (x_0, y_0) is a solution of type II, we have

$$t < 3485^{-1}y_0^{\frac{2+\chi}{1+2\chi}}. \tag{4.48}$$

Then from the left side of (4.42), we have

$$\begin{aligned}|A| &> 0.998ty_0^4\\ &> 0.998t\left(546t^{\frac{1+2\chi}{2+\chi}}\right)^4\\ &= 0.998\cdot 546^4 t^{\frac{6+9\chi}{2+\chi}}\\ &> 8.8\cdot 10^{10}t^{\frac{6+9\chi}{2+\chi}}. \end{aligned} \tag{4.49}$$

Put

$$\epsilon = \frac{4+2\chi}{2+7\chi}. \tag{4.50}$$

Obviously, $\epsilon \in (0,2)$. Then we have for $w \geq 2000$,

$$\begin{aligned}128.1^{1/\epsilon}\cdot 512.1^{1+1/\epsilon}k^{2+2/\epsilon} &= 128.1^{1/\epsilon}\cdot 512.1^{1+1/\epsilon}((6t+7s))^{2+2/\epsilon}\\ &\leq 128.1^{1/\epsilon}\cdot 512.1^{1+1/\epsilon}\cdot 6.0035^{2+2/\epsilon}t^{2+2/\epsilon}\\ &= 128.1^{\frac{2+7\chi}{4+2\chi}}\cdot 512.1^{\frac{6+9\chi}{4+2\chi}}\cdot 6.0035^{\frac{6+9\chi}{2+\chi}}t^{\frac{6+9\chi}{2+\chi}}\\ &\leq 128.1^{\frac{23}{22}}\cdot 512.1^{\frac{45}{22}}\cdot 6.0035^{\frac{45}{11}}t^{\frac{6+9\chi}{2+\chi}}\\ &< 8.51\cdot 10^{10}t^{\frac{6+9\chi}{2+\chi}}. \end{aligned}$$

Combining the above inequality with (4.49), we have

$$|A| \geq 128.1^{1/\epsilon}\cdot 512.1^{1+1/\epsilon}k^{2+2/\epsilon}.$$

Then by Lemma 4.10, we get $\lambda < 2+\epsilon < 4$. By Lemma 4.11, we have an upper bound for y if (x,y) is a solution to (4.7) of type II:

$$y < \left(\frac{C}{B}\right)^{\frac{1}{4-\lambda}}, \tag{4.51}$$

where

$$B = \frac{0.499t}{k}$$

and

$$\begin{aligned}C = 106.88\sqrt{x_0^2+y_0^2}&\left(\frac{1+\sqrt{2}|\gamma|}{1-|\gamma|}\right)^{\frac{1}{4}}|A|(1+\sqrt{2}|\gamma|)\\ &\cdot\left(\max\left\{\frac{0.3125\sqrt{\theta^2+1}\sqrt{x_0^2+y_0^2}}{\sqrt[4]{1-|\gamma|}(1-|\gamma|^2)},1\right\}\right)^{\lambda-1}.\end{aligned}$$

We now estimate C and B under the assumption of the theorem. By (4.38), we have

$$B = \frac{0.499t}{k} = \frac{0.499t}{6t+7s} = \frac{0.499}{6+7/w} > 0.08312. \tag{4.52}$$

Since (x_0, y_0) is a type II solution,

$$\theta, \frac{x_0}{y_0} \in \mathbb{I} \subset (0.999, 1).$$

Then we have

$$\begin{aligned}
\sqrt{\theta^2+1} &< \sqrt{2}, \\
\sqrt{x_0^2+y_0^2} &= y_0\sqrt{\left(\frac{x_0}{y_0}\right)^2+1} < \sqrt{2}y, \\
|\gamma| = \frac{|m|}{|A|} &\le \frac{4k}{128.1^{1/2}\cdot 512.1^{3/2}k^3} \\
&\le \frac{4}{128.1^{1/2}\cdot 512.1^{3/2}(2(6\cdot 2400+7))^2} = 4\cdot 10^{-14}.
\end{aligned}$$

It follows that

$$\begin{aligned}
C &= 106.88\sqrt{x_0^2+y_0^2}\left(\frac{1+\sqrt{2}|\gamma|}{1-|\gamma|}\right)^{\frac{1}{4}}|A|(1+\sqrt{2}|\gamma|) \\
&\quad\cdot\left(\max\left\{\frac{0.3125\sqrt{\theta^2+1}\sqrt{x_0^2+y_0^2}}{\sqrt[4]{1-|\gamma|}(1-|\gamma|^2)}, 1\right\}\right)^{\lambda-1} \\
&< 106.88\sqrt{2}y_0\left(\frac{1+\sqrt{2}\cdot 4\cdot 10^{-14}}{1-4\cdot 10^{-14}}\right)^{\frac{1}{4}}|A|(1+\sqrt{2}\cdot 4\cdot 10^{-14}) \\
&\quad\cdot\left(\max\left\{\frac{0.3125\sqrt{2}\sqrt{2}y_0}{\sqrt[4]{1-4\cdot 10^{-14}}(1-(4\cdot 10^{-14})^2)}, 1\right\}\right)^{\lambda-1} \\
&< 151.2y_0|A|\cdot(0.6251y_0)^{\lambda-1} \\
&< 151.2\cdot 0.6251|A|y_0^{\lambda} \\
&< 94.52|A|y_0^{\lambda}.
\end{aligned}$$

Together with (4.52), (4.42), (4.48), (4.50) and (4.51), we have

$$\begin{aligned} y &< \left(\frac{C}{B}\right)^{\frac{1}{4-\lambda}} \\ &< \left(\frac{94.52|A|y_0^{\lambda}}{0.08312}\right)^{\frac{1}{4-\lambda}} \\ &< (1137.2ty_0^{4+\lambda})^{\frac{1}{4-\lambda}} \\ &< \left(\frac{1137.2}{3485}y_0^{\frac{2+\chi}{1+2\chi}+4+\lambda}\right)^{\frac{1}{4-\lambda}} \\ &< 0.3264^{\frac{2+7\chi}{12\chi}}y_0^{\frac{20+92\chi+95\chi^2}{12\chi(1+2\chi)}}. \end{aligned} \tag{4.53}$$

Assume that there are $\tau+1$ solutions $(x_0, y_0), (x_1, y_1), \ldots, (x_\tau, y_\tau)$ to (4.7) of type II. Further assume that

$$y_0 \le y_1 \le \ldots \le y_\tau.$$

Then by Lemmas 4.3 and 2.6, together with (4.52), we have

$$y_\tau \ge 0.04156y_{\tau-1}^3 \ge \ldots \ge 0.04156^{1+3+\ldots+3^{\tau-1}}y_0^{3^\tau} = 0.04156^{\frac{3^\tau-1}{2}}y_0^{3^\tau}. \tag{4.54}$$

Combining (4.53), (4.54) and noting $3^\tau = \frac{4}{\chi}$, we get

$$0.04156^{\frac{4-\chi}{2\chi}}y_0^{\frac{4}{\chi}} < 0.3264^{\frac{7\chi+2}{12\chi}} \cdot y_0^{\frac{20+92\chi+95\chi^2}{12\chi(1+2\chi)}}.$$

Simplify this inequality and we can get

$$y_0^{4-\frac{20+92\chi+95\chi^2}{12(1+2\chi)}} < 0.04156^{-\frac{4-\chi}{2}} \cdot 0.3264^{\frac{7\chi+2}{12}}. \tag{4.55}$$

Put

$$\phi = 4 - \frac{20+92\chi+95\chi^2}{12(1+2\chi)} = \frac{28+4\chi-95\chi^2}{12(1+2\chi)}.$$

Since $28+4x-95x^2=0$ has two roots at $-0.52225\ldots$ and $0.56435\ldots$, we have that, for

$\chi \in (0, 4/9)$, $\phi > 0$. Thus (4.55) implies

$$\begin{aligned} y_0 &< \left(0.04156^{-\frac{4-\chi}{2}} \cdot 0.3264^{\frac{7\chi+2}{12}}\right)^{\frac{1}{\phi}} \\ &= \exp\left(\frac{1}{\phi}\left(\left(-\frac{4-\chi}{2}\right)\log(0.04156) + \left(\frac{7\chi+2}{12}\right)\log(0.3264)\right)\right) \\ &< \exp\left(\frac{(74.1-26.9\chi)(1+2\chi)}{28+4\chi-95\chi^2}\right) \\ &< \exp(10.7) \\ &< 4.5 \cdot 10^4. \end{aligned} \tag{4.56}$$

On the other hand, from (4.44) and (4.43), we have

$$y_0 > \frac{t^2}{11s^3} \geq \frac{1200^2(s^{2+\chi})^2}{11s^3} \geq \frac{1200^2 \cdot 2}{11} > 2.6 \cdot 10^5. \tag{4.57}$$

This contradicts (4.56). It then follows that there are at most τ solutions to (4.7) of type II.

□

Chapter 5

Sextic Simple Form

In this chapter, we'll study the following inequality

$$\begin{aligned} &|sx^6 - 2tx^5y - (5t+15s)x^4y^2 \\ &\quad - 20sx^3y^3 + 5tx^2y^4 + (2t+6s)xy^5 + sy^6| \leq k \end{aligned} \tag{5.1}$$

where s, t are integers and $k = k(t, s)$ is linear in t and s. Let

$$\begin{aligned} F(x,y) = {} & sx^6 - 2tx^5y - (5t+15s)x^4y^2 \\ & - 20sx^3y^3 + 5tx^2y^4 + (2t+6s)xy^5 + sy^6. \end{aligned}$$

We have that F is a simple form since F is irreducible over $\mathbb{Q}$ and

$$F(x-y, x+2y) = -27F(x,y) \tag{5.2}$$

and the map

$$z \to -\frac{z-1}{z+2} \tag{5.3}$$

permutes the roots of $F(x,1)$ transitively. As discussed in the first chapter, Wakabayashi [51] completely solved (5.1) for $s \geq 1, t \geq 97.3s^{48/19}$ and $k = 120t + 323s$. For the same k, we'll prove the following result:

Theorem 5.1. *Let s, t be positive integers such that*

$$\begin{aligned} & sx^6 - 2tx^5y - (5t+15s)x^4y^2 \\ & \quad - 20sx^3y^3 + 5tx^2y^4 + (2t+6s)xy^5 + sy^6 \end{aligned}$$

is irreducible over $\mathbb{Q}$. *Suppose that* $s \geq 1$ *and* $t \geq 200 s^{12/7+1/5^\tau}$ *with* $\tau \geq 1$. *Then other than the trivial solutions*

$$\pm\{(0,1),(1,0),(1,1),(-1,2),(-1,1),(-2,1),\\ (2,1),(-1,3),(-3,2),(1,2),(-2,3),(-3,1)\}$$

there are at most 12τ *integer solutions to the Thue inequality*

$$|sx^6 - 2tx^5y - (5t+15s)x^4y^2 \\ - 20sx^3y^3 + 5tx^2y^4 + (2t+6s)xy^5 + sy^6| \leq 120t + 323s. \tag{5.4}$$

Since the case when $s = 1$ has been explicitly solved by Lettl, Pethő and Voutier, in the following proof we always assume $s \geq 2$.

5.1 Elementary properties

From the relation (5.2), it is easy to see that if (x, y) is a solution to (5.1), then any element in

$$\pm\{(x,y),(x-y,x+2y),(-y,x+y),\\ (-x-2y,2x+y),(-x-y,x),(-2x-y,x-y)\} \tag{5.5}$$

is a solution to the following inequality:

$$|F(x,y)| \leq 27k, \tag{5.6}$$

since

$$\begin{aligned} &F(x-y,x+2y) = -27F(x,y),\\ &F(-y,x+y) = F(x,y),\\ &F(-x-2y,2x+y) = -27F(x,y),\\ &F(-x-y,x) = F(x,y),\\ &F(-2x-y,x-y) = -27F(x,y). \end{aligned}$$

Notice that the map (5.3) permutes the intervals

$$\left(\frac{1}{2},4\right], \left(-\frac{1}{5},\frac{1}{2}\right], \left(-\frac{2}{3},-\frac{1}{5}\right], \left(-\frac{5}{4},-\frac{2}{3}\right] \\ \left(-3,-\frac{5}{4}\right], (-\infty,-3)\cup(4,\infty).$$

If there exists an integer solution (x, y) to (5.1), we can always choose it from the above set of solutions (5.5) of the inequality (5.6) to satisfy the following condition:

$$\frac{1}{2} < \frac{x}{y} \leq 4, \qquad \gcd(x, y) = 1, \qquad y \geq 0. \tag{5.7}$$

In the following proof, we'll consider integer solutions (x, y) to (5.6) that satisfy (5.7). Let

$$\begin{aligned} f(x) &= s^{-1}F(x, 1) \\ &= x^6 - 2wx^5 - (5w + 15)x^4 - 20x^3 + 5wx^2 + (2w + 6)x + 1, \end{aligned}$$

where $w = t/s$. Then we have

Lemma 5.2. *For $w \geq 6$, f has six real roots $\theta_{-2}, \theta_{-1}, \theta_{-1/2}, \theta_0, \theta_1, \theta_{2w}$ that satisfy the following:*

$$\begin{aligned} -2 - \frac{3}{2w} + \frac{9}{8w^2} <&\theta_{-2} < -2 - \frac{3}{2w} + \frac{4}{3w^2}, \\ -1 - \frac{1}{2w} + \frac{3}{5w^2} <&\theta_{-1} < -1 - \frac{1}{2w} + \frac{11}{16w^2}, \\ -\frac{1}{2} - \frac{3}{8w} + \frac{1}{2w^2} <&\theta_{-1/2} < -\frac{1}{2} - \frac{3}{8w} + \frac{9}{16w^2}, \\ -\frac{1}{2w} + \frac{7}{8w^2} - \frac{1}{w^3} <&\theta_0 < -\frac{1}{2w} + \frac{7}{8w^2}, \\ 1 - \frac{6}{4w + 9} <&\theta_1 < 1 - \frac{6}{4w + 9} + \frac{14}{(4w + 9)w^2}, \\ 2w + \frac{5}{2} + \frac{10}{3w} <&\theta_{2w} < 2w + \frac{5}{2} + \frac{35}{8w}. \end{aligned}$$

Proof. For $w \geq 6$, direct computations give

$$\begin{aligned} f\left(-2 - \frac{3}{2w} + \frac{9}{8w^2}\right) = \frac{27}{262144w^{12}} \big(&425984w^{10} + 307200w^9 \\ &- 2064384w^8 - 1050624w^7 + 3013632w^6 \\ &+ 357696w^5 - 1866240w^4 + 431568w^3 \\ &+314928w^2 - 157464w + 19683\big) > 0, \end{aligned}$$

$$f\left(-2-\frac{3}{2w}+\frac{4}{3w^2}\right)=-\frac{1}{46656w^{12}}\big(174960w^{11}-1259712w^{10}$$
$$+72900w^9+11524032w^8+1921752w^7$$
$$-21034161w^6+1866672w^5+14713920w^4$$
$$-5412864w^3-2617344w^2+1769472w$$
$$-262144)<0,$$

$$f\left(-1-\frac{1}{2w}+\frac{3}{5w^2}\right)=-\frac{1}{1000000w^{12}}\big(50000w^{11}+750000w^{10}$$
$$-5012500w^9-2162500w^8+8287500w^7$$
$$-1950625w^6-2155500w^5+2254500w^4$$
$$-1248480w^3-19440w^2+233280w$$
$$-46656)<0,$$

$$f\left(-1-\frac{1}{2w}+\frac{11}{16w^2}\right)=\frac{1}{16777216w^{12}}\big(2097152w^{11}-5242880w^{10}$$
$$+96665600w^9+26279936w^8-182484992w^7$$
$$+52977664w^6+48792832w^5-55601920w^4$$
$$+37438368w^3-1405536w^2-7730448w$$
$$+1771561)>0,$$

$$f\left(-\frac{1}{2}-\frac{3}{8w}+\frac{1}{2w^2}\right)=\frac{1}{262144w^{12}}\big(18432w^{11}+20736w^{10}$$
$$-656640w^9+544608w^8-98168w^7$$
$$-345351w^6+558648w^5-241680w^4$$
$$+41216w^3+9984w^2-18432w+4096)>0,$$

$$f\left(-\frac{1}{2}-\frac{3}{8w}+\frac{9}{16w^2}\right)=-\frac{27}{16777216w^{12}}\big(16384w^{10}+1720320w^9$$
$$-1529856w^8+331776w^7+931392w^6$$
$$-1850688w^5+874800w^4-163296w^3$$
$$-26244w^2+78732w-19683\big)<0,$$

$$f\left(-\frac{1}{2w}+\frac{7}{8w^2}-\frac{1}{w^3}\right) = -\frac{1}{262144w^{18}}\big(294912w^{16} - 1314816w^{15} + 5390336w^{14} - 10680320w^{13} + 17890304w^{12} - 24673472w^{11} + 24963968w^{10} - 19690128w^9 + 10144400w^8 + 405928w^7 - 8986449w^6 + 11809616w^5 - 9833408w^4 + 6952960w^3 - 3796992w^2 + 1376256w - 262144\big) < 0,$$

$$f\left(-\frac{1}{2w}+\frac{7}{8w^2}\right) = \frac{1}{262144w^{12}}\big(229376w^{10} + 1576960w^9 - 3096576w^8 + 6092800w^7 - 5766144w^6 + 3579072w^5 - 1348480w^4 - 707952w^3 + 576240w^2 - 403368w + 117649\big) > 0,$$

$$f\left(1-\frac{6}{w+9}\right) = \frac{27}{(4w+9)^6}\big(8960w^4 + 53760w^3 + 118944w^2 + 114912w + 40851\big) > 0,$$

$$f\left(1-\frac{6}{4w+9}+\frac{14}{(4w+9)w^2}\right) = -\frac{1}{(4w+9)^6w^{12}}\big(16128w^{16} + 1064448w^{15} + 7916832w^{14} + 24919776w^{13} + 47487663w^{12} + 81370548w^{11} + 131743472w^{10} + 153994260w^9 + 152929980w^8 + 163322880w^7 + 110802720w^6 + 53590320w^5 + 45791872w^4 - 3226944w^3 - 9680832 * w^2 - 7529536\big) < 0,$$

$$f\left(2w+\frac{5}{2}+\frac{10}{3w}\right) = -\frac{1}{46656w^6}\big(1555200w^{10} + 1866240w^9 - 20080224w^8 - 123070752w^7 - 360032661w^6 - 696642120w^5 - 951030000w^4 - 937440000w^3 - 645600000w^2 - 288000000w - 64000000) < 0,$$

$$f\left(2w+\frac{5}{2}+\frac{35}{8w}\right)=\frac{1}{262144w^6}\left(55050240w^9+395902976w^8\right.$$
$$+1628110848w^7+4447576064w^6+8882657280w^5$$
$$+13194720000w^4+14652960000w^3+11764900000w^2$$
$$\left.+6302625000w+1838265625\right)>0.$$

Then the lemma follows.

□

Suppose (x, y) is an integer solution to (5.6) that satisfies (5.7). From Lemma 5.2, we can see that $\frac{x}{y}$ is bounded away from all roots of f except θ_1, for $w \geq 6$. In the following proof, we let $\theta = \theta_1$. Since this root is our focus, we define the interval

$$\mathbb{I}=\left(1-\frac{6}{4w+9}, 1-\frac{6}{4w+9}+\frac{14}{(4w+9)w^2}\right). \tag{5.8}$$

and further we have the following defintion:

Definition We call (x, y) an integer solution to (5.6) of type I if $\gcd(x, y) = 1, y \geq 4$ and

$$\frac{x}{y} \in \mathbb{I};$$

(x, y) is of type II if $\gcd(x, y) = 1, y \geq 4$ and

$$\frac{x}{y} \in \left(\frac{1}{2}, 1-\frac{6}{4w+9}\right] \cup \left[1-\frac{6}{4w+9}+\frac{14}{(4w+9)w^2}, 4\right].$$

Lemma 5.3. *Let (x, y) be an integer solution to (5.6) of type I. For $w \geq 300$, we have*

$$\left|\theta-\frac{x}{y}\right| \leq \frac{1}{By^6}, \tag{5.9}$$

where

$$B=\frac{0.65t}{k}.$$

Proof. From Lemma 5.2, we have for $w \geq 6$

$$\theta_{-2} < -2, \qquad \theta_{-1} < -1, \qquad \theta_{-1/2} < -\frac{1}{2},$$
$$\theta_0 < 0, \qquad \theta_{2w} > 2w + \frac{5}{2}.$$

For $w \geq 300$,

$$\frac{x}{y} \in \mathbb{I} = \left(1 - \frac{6}{4w+9}, 1 - \frac{6}{4w+9} + \frac{14}{(4w+9)w^2}\right) \subset (0.995, 1).$$

We then have

$$\left|\frac{x}{y} - \theta_{-2}\right| > 2.995, \quad \left|\frac{x}{y} - \theta_{-1}\right| > 1.995, \quad \left|\frac{x}{y} - \theta_{-1/2}\right| > 1.495,$$
$$\left|\frac{x}{y} - \theta_0\right| > 0.995, \quad \left|\frac{x}{y} - \theta_{2w}\right| > 2w,$$

and hence

$$\left|\prod_{j \neq 1}\left(\frac{x}{y} - \theta_j\right)\right| > 17.77w.$$

On the other hand, (x, y) satisfies

$$|F(x, y)| \leq 27k.$$

This is equivalent to

$$\left|sy^6 \prod_{j}\left(\frac{x}{y} - \theta_j\right)\right| \leq 27k. \tag{5.10}$$

It then follows that

$$\left|\theta - \frac{x}{y}\right| \leq \frac{27k}{17.77swy^6} = \frac{1}{(0.65t/k)y^3}.$$

□

5.2 Irrationality of the root of f

Suppose that (x_0, y_0) is an integer solution to (5.6) that satisfies (5.7). Similar to the previous two chapters, we'll obtain an irrationality measure of θ in terms of (x_0, y_0).

Lemma 5.4. *The form F can be rewritten as*

$$\begin{aligned} F(x,y) &= sx^6 - 2tx^5y - (5t+15s)x^4y^2 \\ &\qquad - 20sx^3y^3 + 5tx^2y^4 + (2t+6s)xy^5 + sy^6 \\ &= \frac{1}{2}\left(\eta(x-\rho y)^6 + \overline{\eta}(x-\overline{\rho}y)^6\right) \end{aligned}$$

where

$$\eta = s - \frac{(2t+3s)\sqrt{3}i}{9}, \qquad \rho = \frac{-1+\sqrt{3}i}{2}.$$

and $i = \sqrt{-1}$.

Proof. By direct calculation, we have

$$\begin{aligned} (x-\rho y)^6 &= \left(x - \frac{-1+\sqrt{3}i}{2}y\right)^6 \\ &= \left(x^6 + 3x^5y - \frac{15}{2}x^4y^2 - 20x^3y^3 - \frac{15}{2}x^2y^4 + 3xy^5 + y^6\right) \\ &\qquad - \frac{3\sqrt{3}xy}{2}\left(2x^4 + 5x^3y - 5xy^3 - 2y^4\right) i. \end{aligned}$$

We need only to verify that the real part of $\eta(x-\rho y)^6$ is equal to $F(x,y)$. That is,

$$\begin{aligned} &\frac{1}{2}\left(\eta(x-\rho y)^6 + \overline{\eta}(x-\overline{\rho}y)^6\right) \\ =& s\left(x^6 + 3x^5y - \frac{15}{2}x^4y^2 - 20x^3y^3 - \frac{15}{2}x^2y^4 + 3xy^5 + y^6\right) \\ &\qquad - \frac{(2t+3s)\sqrt{3}}{9}\left(\frac{3\sqrt{3}xy}{2}\left(2x^4 + 5x^3y - 5xy^3 - 2y^4\right)\right) \\ =& sx^6 - 2tx^5y - (5t+15s)x^4y^2 - 20sx^3y^3 + 5tx^2y^4 \\ &\qquad + (2t+6s)xy^5 + sy^6 \\ =& F(x,y). \end{aligned}$$

□

Recall from last section that θ is a root of $f(x) = F(x,1)/s$. Then by Lemma 5.4, we have

$$\eta(\theta - \rho)^6 + \overline{\eta}(\theta - \overline{\rho})^6 = 0. \tag{5.11}$$

This gives

$$\frac{\eta}{\overline{\eta}} = -\frac{(\theta - \overline{\rho})^6}{(\theta - \rho)^6}. \tag{5.12}$$

On the other hand, since (x_0, y_0) is a solution to (5.6), we can then put

$$F(x_0, y_0) = m, \tag{5.13}$$

for some integer m with $|m| \leq 27k$. Again by Lemma 5.4, we have

$$\frac{1}{2}\left(\eta(x_0 - \rho y_0)^6 + \overline{\eta}(x_0 - \overline{\rho} y_0)^6\right) = m. \tag{5.14}$$

Then we can write

$$\eta(x_0 - \rho y_0)^6 = m + Ai \tag{5.15}$$

with

$$A = -\frac{\sqrt{3}}{9}H, \tag{5.16}$$

where

$$\begin{aligned} H = {} & (2t + 3s)x_0^6 + (6t + 36s)x_0^5 y_0 + (45s - 15t)x_0^4 y_0^2 - (60s + 40t)x_0^3 y_0^3 \\ & - (90s + 15t)x_0^2 y_0^4 + (6t - 18s)x_0 y_0^5 + (2t + 3s)y_0^6 \in \mathbb{Z}. \end{aligned} \tag{5.17}$$

Since $A \in \mathbb{R}$, we have

$$\frac{\eta(x_0 - \rho y_0)^6}{\overline{\eta}(x_0 - \overline{\rho} y_0)^6} = \frac{m + Ai}{m - Ai}. \tag{5.18}$$

Combining (5.12) and (5.18), we have

$$-\frac{(\theta - \overline{\rho})^6 (x_0 - \rho y_0)^6}{(\theta - \rho)^6 (x_0 - \overline{\rho} y_0)^6} = \frac{m + Ai}{m - Ai}. \tag{5.19}$$

Simplify this equation and write

$$\gamma = \frac{m}{Ai} = \frac{3\sqrt{3}mi}{H}. \tag{5.20}$$

It follows that

$$\frac{(\theta - \overline{\rho})^6 (x_0 - \rho y_0)^6}{(\theta - \rho)^6 (x_0 - \overline{\rho} y_0)^6} = \frac{1 + \gamma}{1 - \gamma}. \tag{5.21}$$

Taking sixth roots on both sides, we obtain

$$\frac{(\theta-\overline{\rho})(x_0-\rho y_0)}{(\theta-\rho)(x_0-\overline{\rho} y_0)}=\frac{\sqrt[6]{1+\gamma}}{\sqrt[6]{1-\gamma}}, \tag{5.22}$$

where we choose the 6th roots so that their arguments lie in the interval $(-\pi/6, \pi/6)$ since from last section x_0/y_0 is close to θ and so the left side is close to 1.

Now we can apply Lemma 2.1 from Chapter 2 with $\mu = 6$ and $x = \gamma$. It follows that, for any integer $n \geq 1$, we have relations

$$I_{0n}(\gamma) = p_{0n}(\gamma)\sqrt[6]{1+\gamma} - p_{0n}(-\gamma)\sqrt[6]{1-\gamma} \tag{5.23}$$

and

$$I_{1n}(\gamma) = p_{1n}(\gamma)\sqrt[6]{1+\gamma} + p_{1n}(-\gamma)\sqrt[6]{1-\gamma}, \tag{5.24}$$

where

$$p_{0n}(\gamma) = \sum_{h=0}^{n}(-1)^{n-h}\binom{n+\frac{1}{6}}{h}\binom{2n-h}{n-h}\frac{\gamma^h(1+\gamma)^{n-h}}{2^{2n+1-h}} \tag{5.25}$$

and

$$p_{1n}(\gamma) = \sum_{h=0}^{n}(-1)^{n-h}\binom{n+\frac{1}{6}}{h}\left(\binom{2n-h}{n-h}\frac{1}{2^{2n+1-h}} - \binom{2n-h-1}{n-h-1}\frac{1}{2^{2n-h}}\right)\gamma^h(1+\gamma)^{n-h} \tag{5.26}$$

and

$$I_{jn}(\gamma) = \frac{1}{2\pi i}\int_{\Gamma}\frac{z^j(1+\gamma z)^{n+\frac{1}{6}}}{(z^2-1)^{n+1}}dz, \tag{5.27}$$

for $j = 0, 1$. Dividing both sides of (5.23) and (5.24) by $\sqrt[6]{1-\gamma}$ and then substituting (5.22) and multiplying both sides by $(\theta-\rho)(x_0-\overline{\rho}y_0)$, we obtain

$$q'_{0n}\theta + p'_{0n} = l'_{0n} \tag{5.28}$$

with

$$\begin{aligned}
q'_{0n} &= p_{0n}(\gamma)(x_0-\rho y_0) - p_{0n}(-\gamma)(x_0-\overline{\rho}y_0),\\
p'_{0n} &= -\overline{\rho}p_{0n}(\gamma)(x_0-\rho y_0) + \rho p_{0n}(-\gamma)(x_0-\overline{\rho}y_0),\\
l'_{0n} &= \frac{I_{0n}(\gamma)(\theta-\rho)(x_0-\overline{\rho}y_0)}{\sqrt[6]{1-\gamma}}
\end{aligned}$$

and

$$q'_{1n}\theta + p'_{1n} = l'_{1n} \tag{5.29}$$

with

$$\begin{aligned} q'_{1n} &= p_{1n}(\gamma)(x_0 - \rho y_0) + p_{1n}(-\gamma)(x_0 - \overline{\rho} y_0), \\ p'_{1n} &= -\overline{\rho} p_{1n}(\gamma)(x_0 - \rho y_0) - \rho p_{1n}(-\gamma)(x_0 - \overline{\rho} y_0), \\ l'_{1n} &= \frac{I_{1n}(\gamma)(\theta - \rho)(x_0 - \overline{\rho} y_0)}{\sqrt[6]{1-\gamma}}. \end{aligned}$$

Put

$$M_j = \begin{cases} 2^{3n} H^n / \sqrt{3} i & \text{if } j = 0, \\ 2^{3n} H^n & \text{if } j = 1. \end{cases} \tag{5.30}$$

Then we have the following:

Lemma 5.5. *With the above notation, for $n \geq 1$, $j = 0, 1$, we have*

$$M_j q'_{jn} \in \mathbb{Z}, \qquad M_j p'_{jn} \in \mathbb{Z}.$$

Proof. First we have, for all integers n, h with $n \geq 1$, $1 \leq h \leq n$,

$$2^{2h-1} \cdot 3^{\left[\frac{3h}{2}\right]} \cdot \binom{n + \frac{1}{6}}{h} \in \mathbb{Z}, \tag{5.31}$$

where $\left[\frac{3h}{2}\right]$ denotes the greatest integer that is less than or equal to $\frac{3h}{2}$. To show this, note that

$$\begin{aligned} 2^{2h-1} \cdot 3^{\left[\frac{3h}{2}\right]} \cdot \binom{n + \frac{1}{6}}{h} &= 2^{h-1} \cdot 3^{\left[\frac{h}{2}\right]} \cdot 6^h \cdot \frac{\left(n + \frac{1}{6}\right) \dots \left((n-h+1) + \frac{1}{6}\right)}{h!} \\ &= 2^{h-1} \cdot 3^{\left[\frac{h}{2}\right]} \cdot \frac{(6n+1) \dots (6(n-h+1)+1)}{h!}. \end{aligned}$$

The number of 2-factors in $h!$ is at most

$$\left[\frac{h}{2}\right] + \left[\frac{h}{2^2}\right] + \dots < \sum_{j=1}^{\infty} \frac{h}{2^j} = h.$$

The number of 3-factors in $h!$ is at most

$$\left[\frac{h}{3}\right] + \left[\frac{h}{3^2}\right] + \dots < \sum_{j=1}^{\infty} \frac{h}{3^j} = \frac{h}{2}.$$

Suppose p is a prime such that $p|h!$ with $p \neq 2, 3$ and a is a positive integer such that $p^a|h!$ but $p^{a+1} \nmid h!$. First notice that $p \leq h$. Then consider the natural integer sequence modulo p:

$$1, 2, 3, \ldots, p-1, 0, 1, 2, \ldots \tag{5.32}$$

The exponent a of p-factor in $h!$ depends on the number of times 0 appears in the first h elements in the above sequence:

$$(1, 2, \ldots, h) \mod p. \tag{5.33}$$

In other words, a depends on how many complete residual sets $(1, 2, \ldots, p-1, 0)$ (5.33) contains mod p. Notice that $(n-h+1, n-h+1, \ldots, n)$ is a sequence of h consecutive integers. We have that modulo p, it contain the same number of complete residual sets as (5.33). Since $\gcd(6, p) = 1$, it follows that

$$(6(n-h+1)+1, 6(n-h+2)+1, \ldots, 6n+1) \mod p \tag{5.34}$$

contains the same number of complete residual sets $(1, 2, \ldots, p-1, 0)$ as well. This implies that

$$p^a \mid \prod_{j=0}^{h-1}(6(n-j)+1).$$

Therefore, (5.31) holds. This implies that

$$2^{n+(h-1)} \cdot (3\sqrt{3})^h \cdot \binom{n+\frac{1}{6}}{h} \in \mathbb{Z}[\sqrt{3}], \tag{5.35}$$

since if h is even, then $(3\sqrt{3})^h = 3^{[3h/2]}$ and thus

$$2^{n+(h-1)} \cdot (3\sqrt{3})^h \cdot \binom{n+\frac{1}{6}}{h} \in \mathbb{Z};$$

and if h is odd, then $(3\sqrt{3})^h = 3^{[3h/2]}\sqrt{3}$ and thus

$$2^{n+(h-1)} \cdot (3\sqrt{3})^h \cdot \binom{n+\frac{1}{6}}{h} \in \mathbb{Z}[\sqrt{3}].$$

Recall that

$$\gamma = \frac{3\sqrt{3}mi}{H},$$

which is a purely imaginary number. By the definition of q'_{0n}, p'_{0n}, we have

$$q'_{0n} = 2i\Im(p_{0n}(\gamma)(x_0 - \rho y_0)),$$
$$p'_{0n} = -2i\Im(p_{0n}(\gamma)(x_0 - \rho y_0)\overline{\rho}),$$

where

$$p_{0n}(\gamma) = \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{3}}{h} \binom{2n-h}{n-h} \frac{\gamma^h (1+\gamma)^{n-h}}{2^{2n+1-h}},$$

and

$$\rho = \frac{-1+\sqrt{3}i}{2}.$$

It follows that

$$\begin{aligned} M_0 q'_{0n} &= M_0 \cdot 2i\Im\left((\Re(p_{0n}(\gamma)) + i\Im(p_{0n}(\gamma))) \cdot \left(x_0 + \frac{y_0}{2} - \frac{\sqrt{3}y_0}{2} i \right) \right) \\ &= i(2x_0 + y_0)\Im(p_{0n}(\gamma))M_0 - i\sqrt{3}y_0 \Re(p_{0n}(\gamma))M_0 \end{aligned}$$

and

$$\begin{aligned} M_0 p'_{0n} &= -M_0 \cdot 2i\Im\left((\Re(p_{0n}(\gamma)) + i\Im(p_{0n}(\gamma))) \cdot \left(x_0 + \frac{y_0}{2} - \frac{\sqrt{3}y_0}{2} i \right) \right. \\ &\qquad \left. \cdot \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \right) \\ &= i(x_0 + 2y_0)\Im(p_{0n}(\gamma))M_0 + i\sqrt{3}x_0 \Re(p_{0n}(\gamma))M_0. \end{aligned}$$

Thus to show $M_0 q'_{0n} \in \mathbb{Z}, M_0 p'_{0n} \in \mathbb{Z}$, it suffices to show that

$$\Im(p_{0n}(\gamma)) \cdot (iM_0) \in \mathbb{Z}, \qquad \sqrt{3}\Re(p_{0n}(\gamma)) \cdot (iM_0) \in \mathbb{Z}.$$

We have

$$\begin{aligned} iM_0 p_{0n}(\gamma) &= \frac{2^{3n} H^n}{\sqrt{3}} \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{6}}{h} \binom{2n-h}{n-h} \frac{\gamma^h (1+\gamma)^{n-h}}{2^{2n+1-h}} \\ &= \frac{1}{\sqrt{3}} \sum_{h=0}^{n} (-1)^{n-h} 2^{n+(h-1)} \binom{n+\frac{1}{6}}{h} \binom{2n-h}{n-h} (H\gamma)^h (H + H\gamma)^{n-h} \\ &= \frac{1}{\sqrt{3}} \sum_{h=0}^{n} (-1)^{n-h} \left(\binom{n+\frac{1}{6}}{h} 2^{n+(h-1)} (3\sqrt{3}mi)^h \right) \binom{2n-h}{n-h} \\ &\qquad \cdot (H + 3\sqrt{3}mi)^{n-h}. \end{aligned}$$

By (5.35)

$$\binom{n+\frac{1}{6}}{h} \cdot 2^{n+(h-1)} \cdot (3\sqrt{3}mi)^h \in \mathbb{Z}[\sqrt{3}].$$

It follows that

$$i\sqrt{3}M_0 p_{0n}(\gamma) \in \mathbb{Z}[\sqrt{3}].$$

Since

$$\begin{aligned} i\sqrt{3}M_0 p_{0n}(\gamma) &= \sum_{h=0}^{n} (-1)^{n-h} 2^{n+(h-1)} \binom{n+\frac{1}{6}}{h} \binom{2n-h}{n-h} (H\gamma)^h (H+H\gamma)^{n-h} \\ &= \sum_{h=0}^{n} (-1)^{n-h} 2^{n+(h-1)} \binom{n+\frac{1}{6}}{h} \binom{2n-h}{n-h} \\ &\qquad \cdot \sum_{l=0}^{n-h} H^{n-h-l} (3\sqrt{3}mi)^{h+l}, \end{aligned}$$

we see that each term in the real part of $i\sqrt{3}M_0 p_{0n}(\gamma)$ is in $\mathbb{Z}$ and each term in the imaginary part of $i\sqrt{3}M_0 p_{0n}(\gamma)$ is of the form of an integer multiplied by $\sqrt{3}$. Therefore, $iM_0 p_{0n}(\gamma)$ can be written as

$$\frac{a}{\sqrt{3}} + bi,$$

where $a, b \in \mathbb{Z}$. It follows that

$$\Im(p_{0n}(\gamma)) \cdot (iM_0) \in \mathbb{Z}, \qquad \sqrt{3}\Re(p_{0n}(\gamma)) \cdot (iM_0) \in \mathbb{Z},$$

since $iM_0 \in \mathbb{R}$. This proves $M_0 q'_{0n} \in \mathbb{Z}$, $M_0 p'_{0n} \in \mathbb{Z}$.

Similarly, we have

$$\begin{aligned} q'_{1n} &= 2\Re(p_{1n}(\gamma)(x_0 - \rho y_0)), \\ p'_{1n} &= -2\Re(p_{1n}(\gamma)(x_0 - \rho y_0)\overline{\rho}), \end{aligned}$$

where

$$\begin{aligned} p_{1n}(\gamma) = \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{6}}{h} \Bigg(&\binom{2n-h}{n-h} \frac{1}{2^{2n+1-h}} \\ &- \binom{2n-h-1}{n-h-1} \frac{1}{2^{2n-h}} \Bigg) \gamma^h (1+\gamma)^{n-h}. \end{aligned}$$

Since

$$\begin{aligned}M_1 q'_{1n} &= M_1 \cdot 2\Re\left((\Re(p_{1n}(\gamma)) + i\Im(p_{1n}(\gamma))) \cdot \left(x_0 + \frac{y_0}{2} - \frac{\sqrt{3}y_0}{2} i \right) \right) \\ &= (2x_0 + y_0)\Re(p_{1n}(\gamma))M_1 + \sqrt{3}y_0\Im(p_{1n}(\gamma))M_1\end{aligned}$$

and

$$\begin{aligned}M_1 p'_{1n} &= -M_1 \cdot 2\Re\left((\Re(p_{1n}(\gamma)) + i\Im(p_{1n}(\gamma))) \cdot \left(x_0 + \frac{y_0}{2} - \frac{\sqrt{3}y_0}{2} i \right) \right. \\ &\qquad \left. \cdot \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \right) \\ &= (x_0 + 2y_0)\Re(p_{1n}(\gamma))M_1 - \sqrt{3}x_0\Im(p_{1n}(\gamma))M_1,\end{aligned}$$

it suffices to show that

$$\Re(p_{1n}(\gamma))M_1 \in \mathbb{Z}, \qquad \sqrt{3}\Im(p_{1n}(\gamma))M_1 \in \mathbb{Z}.$$

We have

$$\begin{aligned}M_1 p_{1n}(\gamma) &= 2^{3n} H^n \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{6}}{h} \left(\binom{2n-h}{n-h} \frac{1}{2^{2n+1-h}} \right. \\ &\qquad \left. - \binom{2n-h-1}{n-h-1} \frac{1}{2^{2n-h}} \right) \gamma^h (1+\gamma)^{n-h} \\ &= \sum_{h=0}^{n} (-1)^{n-h} \binom{n+\frac{1}{6}}{h} \left(\binom{2n-h}{n-h} 2^{n+(h-1)} \right. \\ &\qquad \left. - \binom{2n-h-1}{n-h-1} 2^{n+h} \right) (H\gamma)^h (H+H\gamma)^{n-h} \\ &= \sum_{h=0}^{n} (-1)^{n-h} \cdot 2^{n+(h-1)} \binom{n+\frac{1}{6}}{h} \left(\binom{2n-h}{n-h} \right. \\ &\qquad \left. - 2\binom{2n-h-1}{n-h-1} \right) (H\gamma)^h (H+H\gamma)^{n-h}.\end{aligned}$$

Using the same argument as for $j = 0$, we see that $M_1 p_{1n}(\gamma)$ can be written as

$$a + b\sqrt{3}i,$$

for some integer a, b. It then follows that

$$\Re(p_{1n}(\gamma))M_1 \in \mathbb{Z}, \qquad \sqrt{3}\Im(p_{1n}(\gamma))M_1 \in \mathbb{Z},$$

which implies that $M_1 q'_{1n} \in \mathbb{Z}, M_1 p_{1n} \in \mathbb{Z}$.

□

Put

$$q_{jn} = M_j q'_{jn}, \quad p_{jn} = M_j p'_{jn}, \quad l_{jn} = M_j l_{jn}, \tag{5.36}$$

for $j = 0, 1$ and $n \geq 1$. Summarizing the discussion in this section, we obtain:

Lemma 5.6. *For $n \geq 1$, put*

$$\begin{aligned}
q_{0n} &= (p_{0n}(\gamma)(x_0 - \rho y_0) - p_{0n}(-\gamma)(x_0 - \overline{\rho} y_0))2^{3n} H^n / \sqrt{3} i, \\
p_{0n} &= (-\overline{\rho} p_{0n}(\gamma)(x_0 - \rho y_0) + \rho p_{0n}(-\gamma)(x_0 - \overline{\rho} y_0))2^{3n} H^n / \sqrt{3} i, \\
q_{1n} &= (p_{1n}(\gamma)(x_0 - \rho y_0) + p_{1n}(-\gamma)(x_0 - \overline{\rho} y_0))2^{3n} H^n, \\
p_{1n} &= (-\overline{\rho} p_{1n}(\gamma)(x_0 - \rho y_0) - \rho p_{1n}(-\gamma)(x_0 - \overline{\rho} y_0))2^{3n} H^n.
\end{aligned}$$

Then q_{0n}, p_{0n}, q_{1n} and p_{1n} are rational integers satisfying the following relations:

$$\begin{aligned}
q_{0n}\theta + p_{0n} =& l_{0n}, \\
q_{1n}\theta + p_{1n} =& l_{1n},
\end{aligned}$$

where

$$\begin{aligned}
l_{0n} &= \frac{I_{0n}(\gamma)(\theta - \rho)(x_0 - \overline{\rho} y_0)2^{3n} H^n}{\sqrt[6]{1-\gamma}\sqrt{3} i}, \\
l_{1n} &= \frac{I_{1n}(\gamma)(\theta - \rho)(x_0 - \overline{\rho} y_0)2^{3n} H^n}{\sqrt[6]{1-\gamma}}.
\end{aligned}$$

To apply Lemma 2.5, we need the following condition and estimates.

Lemma 5.7. *Let p_{jn}, q_{jn} be defined as in Lemma 5.6 for $j = 0, 1$. We have*

$$\begin{vmatrix} p_{0n} & q_{0n} \\ p_{1n} & q_{1n} \end{vmatrix} \neq 0$$

for any $n \geq 1$.

Proof. By Lemma 2.4 and $\gamma \neq 0$, we have

$$\Delta(\gamma) = \begin{vmatrix} p_{0n}(\gamma) & -p_{0n}(-\gamma) \\ p_{1n}(\gamma) & p_{1n}(-\gamma) \end{vmatrix} = c_{2n}\gamma^{2n} \neq 0. \tag{5.37}$$

Put $A_j = p_{jn}(\gamma)(x_0 - \rho y_0)$ and $B_j = p_{jn}(-\gamma)(x_0 - \overline{\rho} y_0)$ for $j = 0, 1$. Then

$$\begin{aligned} q_{0n} &= (A_0 - B_0)M_0, \quad p_{0n} = (-\overline{\rho}A_0 + \rho B_0)M_0 \\ q_{1n} &= (A_1 + B_1)M_1 \quad p_{1n} = -(\overline{\rho}A_1 + \rho B_1)M_1. \end{aligned}$$

It follows that

$$\begin{aligned} \begin{vmatrix} p_{0n} & q_{0n} \\ p_{1n} & q_{1n} \end{vmatrix} &= \begin{vmatrix} (-\overline{\rho}A_0 + \rho B_0)M_0 & (A_0 - B_0)M_0 \\ -(\overline{\rho}A_1 + \rho B_1)M_1 & (A_1 + B_1)M_1 \end{vmatrix} \\ &= M_0M_1 \begin{vmatrix} (-\overline{\rho}A_0 + \rho B_0) & (A_0 - B_0) \\ -(\overline{\rho}A_1 + \rho B_1) & (A_1 + B_1) \end{vmatrix} \\ &= M_0M_1 \begin{vmatrix} (\rho - \overline{\rho})A_0 & (A_0 - B_0) \\ (\rho - \overline{\rho})A_1 & (A_1 + B_1) \end{vmatrix} \\ &= M_0M_1(\rho - \overline{\rho}) \begin{vmatrix} A_0 & -B_0 \\ A_1 & B_1 \end{vmatrix} \\ &= M_0M_1(\rho - \overline{\rho}) \begin{vmatrix} p_{0n}(\gamma)(x_0 - \rho y_0) & -p_{0n}(-\gamma)(x_0 - \overline{\rho} y_0) \\ p_{0n}(\gamma)(x_0 - \rho y_0) & p_{0n}(-\gamma)(x_0 - \overline{\rho} y_0) \end{vmatrix} \\ &= M_0M_1(\rho - \overline{\rho})|x_0 - \rho y_0|^2 \begin{vmatrix} p_{0n}(\gamma) & -p_{0n}(-\gamma) \\ p_{1n}(\gamma) & p_{1n}(-\gamma) \end{vmatrix} \\ &\neq 0. \end{aligned}$$

□

Lemma 5.8. *Suppose that* $|\gamma| < 1/\sqrt{2}$. *For* $n \geq 1, j = 0, 1$,

$$|q_{jn}| < \varrho P^n,$$

where

$$\varrho = \frac{1.67|x_0 - \rho y_0| \sqrt[6]{1 + \sqrt{2}|\gamma|}}{\sqrt[6]{1 - |\gamma|}},$$

and

$$P = 8|H|(1 + \sqrt{2}|\gamma|).$$

Proof. From the proof of Lemma 5.5, we see that

$$|q_{jn}| \leq 2|p_{jn}(\gamma)(x_0 - \rho y_0)M_j| = 2|p_{jn}(\gamma)| \cdot |x_0 - \rho y_0| \cdot |M_j|, \tag{5.38}$$

for $j = 0, 1, n \geq 1$. By the definition of M_j, we have

$$|M_j| \leq 2^{3n}|H|^n. \tag{5.39}$$

By Lemma 2.1, we have, for $j = 0, 1$,

$$p_{jn}(x) = \frac{I_{j1n}(x)}{\sqrt[6]{1+x}}, \tag{5.40}$$

where

$$I_{j1n}(x) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{z^j(1+xz)^{n+\frac{1}{6}}}{(z^2-1)^{n+1}}dz.$$

Consider the curve $\Gamma : \{z \in \mathbb{C} : |z^2 - 1| = 1\}$. It consists of two closed curves. Let Γ_1 be the one enclosing the point 1. Notice that $|z|^2 = |z^2| \leq |z^2 - 1| + 1 = 2$ and $\sqrt{2} \in \Gamma_1$. Hence $\max_{z\in\Gamma_1}|z| = \sqrt{2}$. Then, for $|x| < 1/\sqrt{2}$, we have

$$\begin{aligned} |I_{j1n}(x)| &= \left|\frac{1}{2\pi i}\int_{\Gamma_1} \frac{z^j(1+xz)^{n+\frac{1}{6}}}{(z^2-1)^{n+1}}dz\right| \\ &\leq \frac{|\Gamma_1|}{2\pi} \cdot \max_{z\in\Gamma_1}\left|\frac{z^j(1+xz)^{n+\frac{1}{6}}}{(z^2-1)^{n+1}}\right| \\ &= \frac{|\Gamma_1|\sqrt{2}(1+\sqrt{2}|x|)^{n+\frac{1}{6}}}{2\pi}, \end{aligned} \tag{5.41}$$

for $j = 0, 1$, where $|\Gamma_1|$ denotes the length of Γ_1.

Write $z = a + bi$. By the definition of Γ, we can obtain the equation of Γ_1 on ab-plane:

$$(a^2 + b^2)^2 - 2(a^2 - b^2) = 0, \qquad 0 \leq a \leq \sqrt{2}.$$

We can find the length of Γ_1 by an integral along the above curve. Numerical integration gives

$$|\Gamma_1| = 3.70814935\ldots < 3.709. \tag{5.42}$$

Notice that

$$|\sqrt[6]{1+x}| = \sqrt[6]{|1+x|} \geq \sqrt[6]{1-|x|}, \tag{5.43}$$

for $|x| < 1/\sqrt{2}$. Then combining (5.38), (5.39), (5.40), (5.41), (5.42), and (5.43), we obtain

$$\begin{aligned} |q_{jn}| &= 2|x_0 - \rho y_0| \cdot |M_j| \cdot |p_{jn}(\gamma)| \\ &\leq 2|x_0 - \rho y_0| 2^{3n} |H|^n \frac{3.709 \cdot \sqrt{2}(1+\sqrt{2}|\gamma|)^{n+\frac{1}{6}}}{2\pi \sqrt[6]{1-|\gamma|}} \\ &< \frac{1.67|x_0 - \rho y_0| \sqrt[6]{1+\sqrt{2}|\gamma|}}{\sqrt[6]{1-|\gamma|}} \left(8|H|(1+\sqrt{2}|\gamma|)\right)^n . \end{aligned}$$

□

Lemma 5.9. *Suppose that $w \geq 4$ and $|\gamma| < 1/\sqrt{2}$. For $n \geq 1, j = 0, 1,$*

$$|l_{jn}| \leq l L^{-n},$$

where

$$l = \frac{4|x_0 - \overline{\rho} y_0|}{9\sqrt[6]{1-|\gamma|}(1-|\gamma|^2)}$$

and

$$L = \frac{1-|\gamma|^2}{|H||\gamma|^2}.$$

Proof. By the definition of l_{jn} from Lemma 5.6, we have

$$|l_{jn}| \leq \frac{2^{3n}|H|^n |\theta - \rho| |x_0 - \overline{\rho} y_0| |I_{jn}(\gamma)|}{|\sqrt[6]{1-\gamma}|}, \tag{5.44}$$

for $j = 0, 1$ and $n \geq 1$. By Lemma 5.2, we have

$$1 - \frac{6}{4w+9} < \theta < 1 - \frac{6}{4w+9} + \frac{25}{(4w+9)w^2} < 1,$$

for $w \geq 6$. Thus

$$|\theta - \rho| < \sqrt{\left(-\frac{1}{2} - 1\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3}. \tag{5.45}$$

By Lemma 2.3 and the definition of I_{jn}, we have

$$I_{jn}(x) = \sum_{h=2n+1-j}^{\infty} \binom{n+\frac{1}{6}}{h} J_{h+j} x^h,$$

where

$$J_h = \frac{1}{2\pi i} \int_\Gamma \frac{z^h}{(z^2-1)^{n+1}} dz.$$

As a consequence of Lemma 2.2, $J_h \geq 0$, since

$$\sum_{h=0}^{\infty} J_h x^h = \frac{x^{2n+1}}{(1-x^2)^{n+1}}$$

and the Taylor expansion at 0 of the right hand side of the above equation obviously has non-negative coefficients. Notice that for $h \geq 2n$, we have

$$\frac{\left|\binom{n+\frac{1}{6}}{h}\right|}{\left|\binom{n+\frac{1}{6}}{h+1}\right|} = \left|\frac{h+1}{n-h-\frac{1}{6}}\right| = \frac{h+1}{h-n+\frac{1}{6}} > 1.$$

Thus $\left|\binom{n+\frac{1}{6}}{h}\right|$ decreases as h increases. It follows that

$$\begin{aligned}|I_{jn}(x)| &\leq \left|\binom{n+\frac{1}{6}}{2n}\right| \sum_{2n+1-j}^{\infty} |J_{h+j}||x|^h \\ &= \left|\binom{n+\frac{1}{6}}{2n}\right| |x|^{-j} \sum_{h=2n+1}^{\infty} J_h |x|^h \\ &= \left|\binom{n+\frac{1}{6}}{2n}\right| |x|^{-j} \frac{|x|^{2n+1}}{(1-|x|^2)^{n+1}},\end{aligned}$$

for $|x| < 1$ by Lemma 2.2. By induction, we can have the following estimate:

$$\left|\binom{n+\frac{1}{6}}{2n}\right| \leq \frac{7}{18}\left(\frac{1}{4}\right)^n,$$

since for $n = 1$,

$$\left|\binom{n+\frac{1}{6}}{2n}\right| = \frac{7}{72} = \frac{7}{18}\left(\frac{1}{4}\right)^1,$$

and for $n \geq 2$,

$$\begin{aligned}
\frac{\left|\binom{n+\frac{1}{6}}{2n}\right|}{\left|\binom{n+1+\frac{1}{6}}{2(n+1)}\right|} &= \left|\frac{(2n+1)(2n+2)}{\left(n+1+\frac{1}{6}\right)\left(-n+\frac{1}{6}\right)}\right| \\
&= 4 \cdot \frac{n^2 + \frac{3}{2}n + \frac{1}{2}}{n^2 + n - \frac{7}{36}} \\
&> 4.
\end{aligned}$$

Therefore, we obtain

$$|I_{jn}(x)| \leq \frac{7}{18}\left(\frac{1}{4}\right)^n |x|^{-j} \frac{|x|^{2n+1}}{(1-|x|^2)^{n+1}}. \tag{5.46}$$

Notice that for $|\gamma| < 1/\sqrt{2}$,

$$|\sqrt[6]{1-\gamma}| = \sqrt[6]{|1-\gamma|} \geq \sqrt[6]{1-|\gamma|}. \tag{5.47}$$

Then we combine (5.44), (5.45), (5.46) and (5.47). It follows that

$$\begin{aligned}
|l_{jn}| &\leq \frac{2^{3n}|H|^n|x_0 - \overline{\rho}y_0|}{\sqrt[6]{1-|\gamma|}} \cdot \frac{7}{18}\left(\frac{1}{4}\right)^n |\gamma|^{-j} \frac{|\gamma|^{2n+1}}{(1-|\gamma|^2)^{n+1}} \\
&= \frac{7|x_0 - \overline{\rho}y_0|}{18\sqrt[6]{1-|\gamma|}(1-|\gamma|^2)} \cdot \left(\frac{2|H||\gamma|^2}{1-|\gamma|^2}\right)^n \\
&= \frac{7|x_0 - \overline{\rho}y_0|}{18\sqrt[6]{1-|\gamma|}(1-|\gamma|^2)} \cdot \left(\frac{1-|\gamma|^2}{2|H||\gamma|^2}\right)^{-n}.
\end{aligned}$$

□

Lemma 5.10. *Suppose that $|H| \geq 81|m|^2$, $|\gamma| < 1/\sqrt{2}$ and $w \geq 4$. With the notations as above, we have, for any integers p and q with $q > 0$,*

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{Cq^\lambda},$$

where

$$\lambda = 1 + \frac{\log\left(8|H|(1+\sqrt{2}|\gamma|)\right)}{\log\left(\frac{1-|\gamma|^2}{2|H||\gamma|^2}\right)}$$

and

$$C = \frac{26.72|x_0 - \rho y_0| \sqrt[6]{1+\sqrt{2}|\gamma|}|H|(1+\sqrt{2}|\gamma|)}{\sqrt[6]{1-|\gamma|}} \cdot \max\left(\frac{7|x_0 - \overline{\rho} y_0|}{18\sqrt[6]{1-|\gamma|}(1-|\gamma|^2)}, 1\right)^{\lambda-1}.$$

Proof. Recall that

$$\gamma = \frac{3\sqrt{3}mi}{H}.$$

Then by the assumption that $|H| \geq 81|m|^2$, we have that

$$(2|H|+1)\cdot|\gamma|^2 < 3|H|\cdot|\gamma|^2 = \frac{81|m|^2}{|H|} \leq 1.$$

It follows that

$$L = \frac{1-|\gamma|^2}{2|H||\gamma|^2} - 1 = \frac{1-(2|H|+1)|\gamma|^2}{2|H||\gamma|^2} \geq 0.$$

Then we can apply Lemma 2.5, together with Lemmas 5.5, 5.7, 5.8 and 5.9. We have that, for any integers p and q with $q > 0$,

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{Cq^\lambda},$$

where

$$\lambda = 1 + \frac{\log P}{\log L}$$
$$= 1 + \frac{\log\left(8|H|(1+\sqrt{2}|\gamma|)\right)}{\log\left(\frac{1-|\gamma|^2}{2|H||\gamma|^2}\right)}$$

and

$$C = 2\varrho P \max(2l, 1)^{\log P/\log L}$$
$$= 2 \cdot \frac{1.67|x_0 - \rho y_0|\sqrt[6]{1+\sqrt{2}|\gamma|}}{\sqrt[6]{1-|\gamma|}} \cdot 8|H|(1+\sqrt{2}|\gamma|)$$
$$\cdot \max\left(\frac{2\cdot 7|x_0 - \overline{\rho} y_0|}{18\sqrt[6]{1-|\gamma|}(1-|\gamma|^2)}, 1\right)^{\lambda-1}$$
$$= \frac{26.72|x_0 - \rho y_0|\sqrt[6]{1+\sqrt{2}|\gamma|}|H|(1+\sqrt{2}|\gamma|)}{\sqrt[6]{1-|\gamma|}}$$
$$\cdot \max\left(\frac{7|x_0 - \overline{\rho} y_0|}{9\sqrt[6]{1-|\gamma|}(1-|\gamma|^2)}, 1\right)^{\lambda-1}.$$

□

5.3 Upper bounds for the solutions

Lemma 5.11. *Let $\epsilon \in (0,4)$ and let λ be defined as in the last section. Suppose that $w \geq 4$ and*

$$|H| \geq 36 \cdot 576^{1/\epsilon} \cdot (27k)^{2+2/\epsilon}.$$

Then we have

$$\lambda < 2 + \epsilon.$$

Proof. By the assumption on H and the definition of γ, it is easy to see that

$$|\gamma| = \frac{3\sqrt{3}|m|}{|H|} \leq \frac{3\sqrt{3}(27k)}{|H|} < \frac{1}{\sqrt{2}}.$$

Thus the conditions in Lemma 5.10 are satisfied. Then we have P, L, λ defined as in Lemmas 5.8, 5.9 and 5.10. With $0 < |\gamma| < 1/\sqrt{2}$, we have

$$P = 8|H|(1 + \sqrt{2}|\gamma|) < 16|H|, \tag{5.48}$$

and

$$L = \frac{1 - |\gamma|^2}{2|H||\gamma|^2} = (1 - |\gamma|^2)\frac{|H|}{2(|H||\gamma|)^2} > \frac{|H|}{36|m|^2} \geq \frac{|H|}{36(27k)^2}, \tag{5.49}$$

since $\gamma = \frac{3\sqrt{3}mi}{H}$ and $|m| \leq 27k$. From the assumption on H, it follows that

$$\frac{\left(\frac{|H|}{36k^2}\right)^{\epsilon+1}}{16|H|} = \frac{|H|^\epsilon}{16 \cdot 36^{\epsilon+1} \cdot (27k)^{2\epsilon+2}} \geq \frac{36^\epsilon \cdot 576 \cdot (27k)^{2\epsilon+2}}{576 \cdot 36^\epsilon \cdot (27k)^{2\epsilon+2}} = 1.$$

Then combining (5.48) and (5.49), we have

$$L^{\epsilon+1} > \left(\frac{|H|}{36(27k)^2}\right)^{\epsilon+1} \geq 16|H| > P.$$

Taking logarithms, we obtain

$$(\epsilon + 1)\log L > \log P.$$

Therefore,

$$\lambda = 1 + \frac{\log P}{\log L} < 1 + (\epsilon + 1) = 2 + \epsilon.$$

□

Lemma 5.12. *With the same notations as before, suppose that (x_0, y_0), (x, y) are solutions to (5.6) of type I. Let H be defined as in (5.17). Assume that as in Lemma 5.11 $|H|$ is sufficiently large so that $\lambda < 6$. Then we have for $w \geq 300$,*

$$y < \left(\frac{C}{B}\right)^{\frac{1}{6-\lambda}}$$

where

$$B = \frac{0.65t}{k}$$

and

$$C = \frac{26.72|x_0 - \rho y_0| \sqrt[6]{1+\sqrt{2}|\gamma|}|H|(1+\sqrt{2}|\gamma|)}{\sqrt[6]{1-|\gamma|}} \cdot \max\left(\frac{7|x_0 - \overline{\rho} y_0|}{9\sqrt[6]{1-|\gamma|}(1-|\gamma|^2)}, 1\right)^{\lambda-1}.$$

Proof. Notice that if $w \geq 4$ and $|H|$ is large enough as in Lemma 5.11, the assumptions in Lemmas 5.3 and 5.10 can be easily satisfied. It then follows directly from these two lemmas that if (x, y) is a solution to (5.1) then

$$\frac{1}{Cy^{\lambda}} < \left|\theta - \frac{x}{y}\right| \leq \frac{1}{By^6}$$

which gives

$$y < \left(\frac{C}{B}\right)^{\frac{1}{6-\lambda}}$$

with B, C defined as in the statement of the lemma.

□

5.4 Proof of Theorem 5.1

Let $k = 120t + 323s$. Then

$$27k = 3240t + 8721s.$$

It is obvious that

$$\begin{aligned}
|F(1,1)| &= 27s \le k,\\
|F(2,1)| &= 120t + 323s \le k,\\
|F(3,1)| &= 840t + 1007s \le 27k,\\
|F(4,1)| &= 3240t + 999s \le 27k,\\
|F(3,2)| &= 1680t + 7811s \le 27k,\\
|F(2,3)| &= 1680t - 2771s \le 27k.
\end{aligned}$$

We can verify that $(1,1),(2,1),(3,1),(4,1),(3,2)$ and $(2,3)$ are the only solutions (x,y) to (5.6) with $y \le 3$ that satisfy (5.7). Now we only need to focus on the solutions with $y \ge 4$. Recall that we have divided integer solutions to (5.6) with condition (5.7) and $y \ge 4$ into two types. Then we have the following:

Lemma 5.13. *If (x,y) is an integer solution to (5.6) with $k = 120t + 323$ of type II, then*

$$y < \frac{2w}{3},$$

for $w \ge 300$.

Proof. Recall that

$$f(x) = x^6 - 2wx^5 - (5w+15)x^4 - 20x^3 + 5wx^2 + (2w+6)x + 1.$$

Then we have

$$f'(x) = 6x^5 - 10wx^4 - (20w+60)x^3 - 60x^2 + 10wx + (2w+6).$$

Since $f'(x)$ has three negative roots and one positive root in

$$\left(\frac{1}{2}, 1 - \frac{6}{4w+9}\right] \tag{5.50}$$

and the other positive root greater than w, we see that $f(x)$ increases and then decreases in interval (5.50) and it decreases in

$$\left[1 - \frac{6}{4w+9} + \frac{14}{(4w+9)w^2}, 4\right].$$

On the other hand, we have

$$\begin{aligned}
f\left(\frac{1}{2}\right) &= \frac{15w}{8} + \frac{37}{64}, \\
f\left(1 - \frac{6}{4w+9}\right) &= \frac{27}{(4w+9)^6}\big(8960w^4 + 53760w^3 \\
&\qquad +118944w^2 + 114912w + 40851\big) \\
&> \frac{57}{w^2}, \\
f\left(1 - \frac{6}{4w+9} + \frac{14}{(4w+9)w^2}\right) &= -\frac{1}{(4w+9)^6 w^{12}}\big(16128w^{16} \\
&\quad + 1064448w^{15} + 7916832w^{14} + 24919776w^{13} \\
&\quad + 47487663w^{12} + 81370548w^{11} + 131743472w^{10} \\
&\quad + 153994260w^9 + 152929980w^8 + 163322880w^7 \\
&\quad + 110802720w^6 + 53590320w^5 + 45791872w^4 \\
&\quad -3226944w^3 - 9680832w^2 - 7529536\big) \\
&< -\frac{3.9375}{w^2}.
\end{aligned}$$

for $w \geq 500$. It follows that if (x, y) is of type II, then

$$\left| f\left(\frac{x}{y}\right) \right| \geq \frac{3.9375}{w^2}.$$

Then we have

$$27k = 27(120w + 323)s \geq |F(x,y)| = sy^6 \left| f\left(\frac{x}{y}\right) \right| \geq \frac{3.9375sy^6}{w^2}.$$

For $w \geq 300$, we have

$$120w + 323 \leq 121.077w$$

Then

$$y^6 \leq \frac{27 \cdot 121.077w^3}{3.9375}.$$

It follows that

$$y \leq 3.1w^{1/2} < \frac{2w}{3},$$

for $w \geq 300$.

$\square$

Lemma 5.14. *Suppose $w \geq 300$. Then there is no integer solution (x, y) to (5.6) of type II, where in (5.6) $k = 120t + 323s$.*

Proof. Suppose that (x, y) is a solution to (5.6) of type II. By assumption, $x/y \neq 1$ since $y \geq 4$ and $\gcd(x, y) = 1$. We then have

$$\frac{x}{y} \notin \left(1 - \frac{1}{y}, 1 + \frac{1}{y}\right),$$

since otherwise

$$\frac{1}{y} > \left|\frac{x}{y} - 1\right| = \frac{|x - y|}{y} \geq \frac{1}{y},$$

which is a contradiction. Put

$$\begin{aligned} h(y) = &-27y^6 + (18w + 108)y^5 - (45w + 135)y^4 \\ &+ (40w + 60)y^3 - 15wy^2 + (2w - 6)y + 1 - 27 \cdot 121w. \end{aligned}$$

Since

$$h'(0) > 0, \quad h'\left(\frac{1}{5}\right) < 0, \quad h'\left(\frac{1}{2}\right) > 0,$$
$$h'\left(\frac{2}{3}\right) < 0, \quad h'(1) > 0, \quad h'\left(\frac{2w}{3}\right) < 0,$$

for $w \geq 300$, we see the distribution of the roots of $h'(y)$. Furthermore,

$$h(3) = -2771 - 1587w < 0, \quad h(4) = 5973w - 30743 > 0,$$
$$h\left(\frac{2w}{3}\right) = \frac{16}{3}w^5 - \frac{400}{27}w^4 + \frac{100}{9}w^3 + \frac{4}{3}w^2 - 3271w + 1 > 0$$

we have that $h(y) > 0$ on interval $[4, 2w/3)$. By Lemma 5.13, we have that if (x, y) is of type II, then

$$4 \leq y < \frac{2w}{3}.$$

Thus by the definition of h,

$$\begin{aligned} &-27y^6 + (18w+108)y^5 - (45w+135)y^4 \\ &\quad + (40w+60)y^3 - 15wy^2 + (2w-6)y + 1 > 27 \cdot 121w. \end{aligned} \tag{5.51}$$

Then we have

$$\begin{aligned} f\left(1 - \frac{1}{y}\right) &= \frac{1}{y^6}\big(-27y^6 + (18w+108)y^5 - (45w+135)y^4 \\ &\qquad +(40w+60)y^3 - 15wy^2 + (2w-6)y + 1\big) \\ &> \frac{27 \cdot 121w}{y^6}, \end{aligned}$$

and

$$\begin{aligned} f\left(1 + \frac{1}{y}\right) &= -\frac{1}{y^6}\big((18y^5 + 15y^2 + 45y^4 + 2y + 40y^3)w \\ &\qquad -1 - 6y + 108y^5 + 135y^4 + 27y^6 + 60y^3\big) \\ &< -\frac{27 \cdot 121w}{y^6}. \end{aligned}$$

From the shape of f, we have that if (x, y) is a solution to (5.6) of type II, then

$$\left| f\left(\frac{x}{y}\right) \right| > \frac{27 \cdot 121w}{y^6}.$$

It follows that

$$27k = 27(120t + 323s) \geq |F(x,y)| = sy^6 \left| f\left(\frac{x}{y}\right) \right| > 27 \cdot 121t.$$

This is a contradiction when $w \geq 300$.

□

Lemma 5.15. *If (x, y) is an integer solution to (5.6) of type I, then*

$$y > \frac{t^2}{14s^3}.$$

Proof. By the definition of solutions of type I, if (x, y) is of this type, then

$$1 - \frac{6}{4w+9} < \frac{x}{y} < 1 - \frac{6}{4w+9} + \frac{14}{(4w+9)w^2}.$$

Multiply the above inequality by $(4w+9)sy$. We have

$$(4w+9)sy - 6sy < x(4w+9)s = x(4t+9s) < (4w+9)sy - 6sy + \frac{14sy}{w^2}.$$

Since both $(4w+9)sy - 6sy$ and $x(4t+9s)$ are integers, it follows that

$$\frac{14sy}{w^2} > 1.$$

That is

$$y > \frac{w^2}{14s} = \frac{t^2}{14s^3}.$$

□

Lemma 5.16. *With the assumption in Theorem 5.1, there are at most τ integer solutions to (5.1) of type I.*

Proof. Recall that we've assumed that (x_0, y_0) is a non-trivial integer solution to (5.6) satisfying condition (5.7). From Lemma 5.14, we know that (x_0, y_0) has to be a solution of type I and thus

$$\frac{x_0}{y_0} \in \mathbb{I} = \left(1 - \frac{6}{4w+9}, 1 - \frac{6}{4w+9} + \frac{14}{(4w+9)w^2}\right). \tag{5.52}$$

Recall the definition of H from (5.17):

$$\begin{aligned} H = {} & (2t+3s)x_0^6 + (6t+36s)x_0^5y_0 + (45s-15t)x_0^4y_0^2 - (60s+40t)x_0^3y_0^3 \\ & - (90s+15t)x_0^2y_0^4 + (6t-18s)x_0y_0^5 + (2t+3s)y_0^6. \end{aligned}$$

Put

$$\begin{aligned} g(x) = {} & (2w+3)x^6 + (6w+36)x^5 - (15w-45)x^4 \\ & - (40w+60)x^3 - (15w+90)x^2 + (6w-18)x + 2w + 3. \end{aligned}$$

Since for $w \geq 300$ $g'(x)$ has three negative roots and one positive root less than 0.5 and the other positive root greater than 2, we have that $g(x)$ is decreasing on the interval $\mathbb{I}$. Then

$$g\left(1-\frac{6}{4w+9}+\frac{14}{(4w+9)w^2}\right) < g(x) < g\left(1-\frac{6}{4w+9}\right).$$

We have

$$\begin{aligned} g\left(1-\frac{6}{4w+9}\right) &= -\frac{27}{(4w+9)^6}\big(8192w^7+86016w^6 \\ &\quad +419328w^5+1209600w^4+2147040w^3 \\ &\quad +2258928w^2+1267434w+285687\big) \\ &< -53w, \\ g\left(1-\frac{6}{4w+9}+\frac{14}{(4w+9)w^2}\right) &= -\frac{1}{(4w+9)^6w^{12}}\big(221184w^{19}+2322432w^{18} \\ &\quad +11321856w^{17}+34981632w^{16}+78291360w^{15} \\ &\quad +143727696w^{14}+242392878w^{13}+342697149w^{12} \\ &\quad +345677220w^{11}+298409832w^{10}+227159100w^9 \\ &\quad -76919220w^8-297339840w^7-291906720w^6 \\ &\quad -492685200w^5-381009888w^4-164574144w^3 \\ &\quad -203297472w^2-15059072w-22588608\big) \\ &> -54w, \end{aligned}$$

for $w \geq 300$. It follows that

$$53w < |g(x)| < 54w.$$

By the definition of $g(x)$ and H, together with (5.52), we obtain

$$53ty_0^6 < |H| < 54ty_0^6. \tag{5.53}$$

By Lemma 5.15, we have

$$y > \frac{t^2}{14s^3}. \tag{5.54}$$

By the assumption of the theorem, we have

$$t \geq 200s^{12/7+\chi}, \tag{5.55}$$

where

$$\chi = \frac{1}{5^\tau}, \qquad \text{for } \tau \geq 1. \tag{5.56}$$

This implies that

$$w = \frac{t}{s} \geq 200 s^{5/7} \geq 200 \cdot 2^{5/7} > 300. \tag{5.57}$$

and

$$0 < \chi \leq \frac{1}{5}. \tag{5.58}$$

Then

$$s^{-1} \geq 200^{\frac{7}{12+7\chi}} t^{-\frac{7}{12+7\chi}}. \tag{5.59}$$

From (5.54) and (5.59), we have

$$y > \frac{t^2}{14} \cdot \left(200^{\frac{7}{12+7\chi}} t^{-\frac{7}{12+7\chi}}\right)^3 > 288 t^{\frac{3+14\chi}{12+7\chi}}. \tag{5.60}$$

The last inequality holds since $\frac{7}{12+7\chi}$ takes it minimal value when χ takes its maximal value in (5.58). This is equivalent to

$$t < 288^{-\frac{12+7\chi}{3+14\chi}} y^{\frac{12+7\chi}{3+14\chi}} < 2.1 \cdot 10^{-6} y^{\frac{12+7\chi}{3+14\chi}}. \tag{5.61}$$

In particular, since (x_0, y_0) is a solution of type I, we have

$$t < 2.1 \cdot 10^{-6} y_0^{\frac{12+7\chi}{3+14\chi}}. \tag{5.62}$$

From (5.53), together with (5.60), we have

$$\begin{aligned} |H| &> 53 t y_0^6 \\ &> 53 t \cdot \left(288 t^{\frac{3+14\chi}{12+7\chi}}\right)^6 \\ &= 53 \cdot 288^6 t^{1+6\cdot\frac{3+14\chi}{12+7\chi}} \\ &> 3 \cdot 10^{16} t^{\frac{30+91\chi}{12+7\chi}}. \end{aligned} \tag{5.63}$$

Put

$$\epsilon = \frac{2(12+7\chi)}{6+77\chi}. \tag{5.64}$$

It is easy to see that $\epsilon \in (0,4)$. For $w \geq 300$,

$$k = 120t + 323s \leq 121.077t.$$

Then we have

$$\begin{aligned}
36 \cdot 576^{1/\epsilon}(27k)^{2+2/\epsilon} &\leq 36 \cdot 576^{\frac{6+77\chi}{24+14\chi}} \cdot (27 \cdot 121.077t)^{\frac{30+91\chi}{12+7\chi}} \\
&= 36 \cdot 576^{\frac{6+77\chi}{24+14\chi}} \cdot 3269.079^{\frac{30+91\chi}{12+7\chi}} t^{\frac{30+91\chi}{12+7\chi}} \\
&< 2.6 \cdot 10^{16} t^{\frac{30+91\chi}{12+7\chi}}.
\end{aligned} \tag{5.65}$$

By (5.63) and (5.65), the condition in Lemma 5.11 is satisfied and thus

$$|H| \geq 36 \cdot 576^{1/\epsilon}(27k)^{2+2/\epsilon}. \tag{5.66}$$

Hence we can apply Lemma 5.11, which gives $\lambda < 2 + \epsilon < 6$. By Lemma 5.12, we have that if (x, y) is a solution to (5.1) then

$$y < \left(\frac{C}{B}\right)^{\frac{1}{6-\lambda}}, \tag{5.67}$$

where

$$B = \frac{0.65t}{k}$$

and

$$C = \frac{26.72|x_0 - \rho y_0|\sqrt[6]{1+\sqrt{2}|\gamma|}|H|(1+\sqrt{2}|\gamma|)}{\sqrt[6]{1-|\gamma|}} \cdot \max\left(\frac{7|x_0 - \overline{\rho} y_0|}{9\sqrt[6]{1-|\gamma|}(1-|\gamma|^2)}, 1\right)^{\lambda-1}.$$

We now estimate B and C in the current case. By (5.57), we have

$$B = \frac{0.65t}{k} = \frac{0.65t}{120t + 323s} > 0.005368. \tag{5.68}$$

Notice that (x_0, y_0) is of type I. Then

$$\frac{x_0}{y_0} \in \mathbb{I} \subset \left(\frac{401}{403}, 1\right).$$

We have

$$|x_0 - \rho y_0| = y_0 \left|\frac{x_0}{y_0} - \rho\right| < \sqrt{\left(1+\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3}y_0,$$
$$|x_0 - \overline{\rho} y_0| = y_0 \left|\frac{x_0}{y_0} - \overline{\rho}\right| < \sqrt{\left(1+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3}y_0.$$

From condition (5.66) and $k = 120t + 323s = s(120w + 323) \geq 2(120 \cdot 300 + 323) = 72646$, it follows that

$$\begin{aligned} |\gamma| &= \frac{3\sqrt{3}|m|}{|H|} \\ &\leq \frac{3\sqrt{3}(27k)}{|H|} \\ &\leq \frac{3\sqrt{3}(27k)}{36 \cdot 576^{1/\epsilon}(27k)^{2+2/\epsilon}} \\ &< \frac{3\sqrt{3}}{36 \cdot 576^{1/4} \cdot (27 \cdot 72646)^{3/2}} \\ &< 2 \cdot 10^{-11}. \end{aligned}$$

Therefore

$$
\begin{aligned}
C &= \frac{26.72|x_0-\rho y_0|\sqrt[6]{1+\sqrt{2}|\gamma|}|H|(1+\sqrt{2}|\gamma|)}{\sqrt[6]{1-|\gamma|}} \\
&\quad \cdot \max\left(\frac{7|x_0-\overline{\rho} y_0|}{9\sqrt[6]{1-|\gamma|}(1-|\gamma|^2)}, 1\right)^{\lambda-1} \\
&< \frac{26.72\sqrt{3}y_0\sqrt[6]{1+\sqrt{2}\cdot 2\cdot 10^{-11}}|H|(1+\sqrt{2}\cdot 2\cdot 10^{-11})}{\sqrt[6]{1-2\cdot 10^{-11}}} \\
&\quad \cdot \max\left(\frac{7\sqrt{3}y_0}{9\sqrt[6]{1-2\cdot 10^{-11}}(1-(2\cdot 10^{-11})^2)}, 1\right)^{\lambda-1} \\
&< 46.2804\cdot 1.34716^{\lambda-1}|H|y_0^{\lambda} \\
&< 205.35|H|y_0^{\lambda},
\end{aligned}
$$

since $\lambda < 6$. Thus (5.67) implies

$$
y < \left(38255|H|y_0^{\lambda}\right)^{\frac{1}{6-\lambda}}. \tag{5.69}
$$

Recall the right hand side of (5.53),

$$
|H| < 54ty_0^6.
$$

Together with (5.62) and (5.64), we obtain

$$
\begin{aligned}
y &< \left(38255|H|y_0^{\lambda}\right)^{\frac{1}{6-\lambda}} \\
&< \left(38255\cdot 54ty_0^6y_0^{\lambda}\right)^{\frac{1}{6-\lambda}} \\
&< \left(38255\cdot 54\cdot 2.1\cdot 10^{-6}\cdot y_0^{\frac{12+7\chi}{3+14\chi}}y_0^{6+\lambda}\right)^{\frac{1}{6-\lambda}} \\
&< (4.34)^{\frac{6+77\chi}{294\chi}}y_0^{\frac{288+3864\chi+9359\chi^2}{294(3+14\chi)\chi}}.
\end{aligned} \tag{5.70}
$$

The last inequality holds because $\frac{6+\lambda}{6-\lambda}$ is increasing in $\lambda \in (0,6)$ and $\lambda < 2+\epsilon < 6$. Assume that there are $\tau+1$ solutions $(x_0,y_0),(x_1,y_1),\ldots,(x_\tau,y_\tau)$ to (5.6) of type I. Further assume that

$$
y_0 \le y_1 \le \ldots \le y_\tau.
$$

Then by Lemmas 2.6 and 5.3, we have by induction

$$\begin{aligned} y_\tau &\geq \left(\frac{B}{2}\right) y_{\tau-1}^5 \\ &\geq \left(\frac{B}{2}\right)\left(\left(\frac{B}{2}\right) y_{\tau-2}^5\right)^5 \\ &\geq \cdots \\ &\geq \left(\frac{B}{2}\right)^{\frac{5^\tau-1}{4}} y_0^{5^\tau}. \end{aligned}$$

Together with the above estimation for B, we get

$$y_\tau > 0.002684^{\frac{5^\tau-1}{4}} y_0^{5^\tau}. \tag{5.71}$$

By Lemma 5.14, we know that (x_τ, y_τ) has to be type I as well. Then (5.70) applies to y_τ. Together with (5.71) we have

$$0.002684^{\frac{5^\tau-1}{4}} y_0^{5^\tau} < (4.34)^{\frac{6+77\chi}{294\chi}} y_0^{\frac{288+3864\chi+9359\chi^2}{294(3+14\chi)\chi}}.$$

Notice that $\chi = 1/5^\tau$. Then $5^\tau = 1/\chi$. It follows that

$$0.002684^{\frac{1/\chi-1}{4}} y_0^{1/\chi} < (4.34)^{\frac{6+77\chi}{294\chi}} y_0^{\frac{288+3864\chi+9359\chi^2}{294(3+14\chi)\chi}}.$$

That is,

$$y_0^{1-\frac{288+3864\chi+9359\chi^2}{294(3+14\chi)}} < 0.0027435^{-\frac{1-\chi}{4}} \cdot (4.34)^{\frac{6+77\chi}{294}}. \tag{5.72}$$

Put

$$\phi = 1 - \frac{288+3864\chi+9359\chi^2}{294(3+14\chi)}.$$

Then

$$\phi = -\frac{9359\chi^2 - 252\chi - 594}{294(3+14\chi)}.$$

Since $9359\chi^2 - 252\chi - 594 = 0$ has two roots at $-0.23882\ldots$ and $0.2657\ldots$, we have $\phi > 0$ for $\chi \in \left(0, \frac{1}{5}\right]$ from (5.58). Therefore, (5.72) gives

$$\begin{aligned}
y_0 &< \left(0.002684^{-\frac{1-\chi}{4}} \cdot (4.34)^{\frac{6+77\chi}{294}}\right)^{\frac{1}{\phi}} \\
&= \exp\left(\frac{1}{\phi}\left(\log(0.002684)\left(-\frac{1-\chi}{4}\right) + \log(4.34)\left(\frac{6+77\chi}{294}\right)\right)\right) \\
&< \exp\left(\frac{294(3+14\chi)(1.52-1.095\chi)}{594+252\chi-9359\chi^2}\right) \\
&< \exp(8.2154) \\
&< 3698. \qquad (5.73)
\end{aligned}$$

The second last inequality holds since

$$\frac{294(3+14\chi)(1.52-1.095\chi)}{594+252\chi-9359\chi^2}$$

takes its maximal value when χ takes its maximal value in (5.58). On the other hand, from (5.54) and (5.55), we have

$$\begin{aligned}
y_0 &> \frac{t^2}{14s^3} \\
&\geq \frac{200^2 s^{3/7+2\chi}}{14} \\
&\geq \frac{200^2 2^{3/7}}{14} \\
&> 3845.
\end{aligned}$$

This contradicts (5.73). It then follows that there are at most τ solutions to (5.1) of type I.

□

Combining Lemmas 5.14 and 5.16, we have that there are at most τ integer solutions (x, y) to (5.1) that satisfy (5.7) and $y \geq 4$. Then by the (5.2), we conclude that for $s \geq 2, t \geq 200s^{12/7+1/5^\tau}$ with $\tau \geq 1$, other than the trivial solutions

$$\pm\{(0,1),(1,0),(1,1),(-1,2),(-1,1),(-2,1),$$
$$(2,1),(-1,3),(-3,2),(1,2),(-2,3),(-3,1)\}$$

there are at most 12τ integer solutions to the Thue inequality

$$\begin{aligned}&|sx^6 - 2tx^5y - (5t+15s)x^4y^2 \\ &\quad - 20sx^3y^3 + 5tx^2y^4 + (2t+6s)xy^5 + sy^6| \leq 120t + 323s.\end{aligned}$$

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