Mode Invisibility Technique – A Quantum Non-Demolition Measurement for Light

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Science
in
Physics

Waterloo, Ontario, Canada, 2014

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The act of measuring an observable of a quantum system introduces backaction noise to the measured system. In the most extreme case, the system gets destroyed with its state left unavailable for future measurement. An example is the photodetection process where photons are destroyed upon detection. It is desirable to consider schemes that avoid the back action effect induced on detected observable due to the measurement. Such measurement scheme in which one is able to repeat a measurement on a physical observable, with the result of each measurement being the same for all measurement times is known as quantum non-demolition (QND) measurement. With the QND measurement idea, it has become possible to detect photons without destroying its state. In this thesis, we present an optical QND measurement scheme known as the Mode invisibility technique to measure a quantum state of light. Unlike other optical QND measurement scheme for photon detection, we show in this thesis that we can take advantage of the spatial distribution of cavity modes to minimize the measurement back action inherent during an atom-light field interaction. Our method is robust and applicable to detect the parameters that characterize a family of Gaussian state. We show that we obtain a very good resolution to measure photon population differences between two given Fock states by means of atomic interferometry.

Acknowledgements

I would like to acknowledge the university of Waterloo for hosting this research and extend my profound gratitude to the following persons who have made the completion of this thesis possible

My supervisor, Prof. R. B. Mann, of the department of physics and astronomy, university of Waterloo for his vital encouragement and support. Dr. Eduardo Martin-Martinez, for the helpful comments.

I am grateful to my family for their fervent prayers and support. To my love Oscar O. M. Iheshiulor, you are indeed my joy complete, thank you for being a part of me.

Most especially to my creator and advocate, the almighty God through whom all things were made possible, I am grateful.

Dedication

This thesis is dedicated to God.

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Chapter 1

Introduction

The quantum theory of measurement [6, 65] describes a measurement process as being irreversible, so that it is impossible to reverse the measurement after information has been acquired. It is a non-deterministic evolution in which before the measurement is completed, the measurement result and post-measurement states are unpredictable. In a more practical sense, a quantum measurement on a system to acquire information about the system must change the state of the system thereby adding excessive backaction noise effects on the measured system and making it difficult to repeat a measurement on the same measured observable. A simple example is the measurement of the photon distribution of an electromagnetic (EM) field using a photodetector. Photodetectors absorb incoming photons subjected to them, converting their energy into an electrical signal and destroying the field state in the process. Such excess destruction is not fundamental. Through a careful selection of measurement device, preparation of quantum states, and interaction model, one can indeed measure the photon number of an EM-field without destroying the field state. This is known as quantum non-demolition (QND) measurement [8, 25, 43, 44, 45, 53].

In an optical QND measurement, a probe system (measuring device) and the object system (system to be measured) interact weakly via a selected Hamiltonian and a correlation is established between the two systems. The weak interaction ensures that the 'object' system remains unperturbed during and after the interaction process. A direct measurement on the readout observable is performed to gain information about the system. Although a QND measurement scheme leaves the state of a quantum system unchanged, and accessible for future measurement; information acquired from the quantum state itself is reduced due to a weak interaction between the probe and measurement object. This leads to the questions: *1*) "How weak can we make the interaction between the probe and system, such that minimization of the system's perturbation is achieved without canceling the information gained in the quantum system?" In the context of optical QND measurement, this question has been discussed in [44]. The mode invisibility measurement technique emerged, which utilizes the spatial symmetry of the cavity modes to suppress the perturbation of light field due to an interaction with a beam of atom (the probe). *2*) "What kind of measurement scheme do we have in mind?" We could consider any sort of atomic interferometer [38, 40, 13, 64] that yields information about a quantum system with high sensitivity. The use of atomic interferometry promises impressive advances in metrology, from the measurement of space-time curvature of the Earth (see for instance [52] for a quick review) to proposals for improving the detectability of the Unruh effect by reducing required accelerations [40], to the implementation of a highly sensitive quantum thermometer [38]. In this thesis, we report a QND measurement technique useful for detecting some features of a given state of light [44] via atomic interferometry.

With the production of cavities with high quality factor that are able to sustain photons for a long damping time, it is possible to detect in a non-demolition way, single photons without destroying the field state [43]. A remarkable QND measurement scheme realized in an optical system of light field is discussed in [12, 29, 28]. In this scheme, a QND measurement was adopted to the counting of photons (light quanta) trapped in a high quality factor (high-Q) microwave cavity. The detector (probe) in the form of a beam of highly excited Rydberg atom evolves into a superposition of two distinct atomic states and was made to interact non-resonantly with the relevant field mode. The non-resonant interaction ensures a non-demolition of the field state. However a shift in the atomic gap proportional to the photon number is observed that is measurable using a Ramsey atomic interferometer. The idea behind the success of this scheme is the ability to trap single photons.

We pursue the same objective of this scheme to improve the amount of information obtainable from the system. Our method involves establishing a resonant interaction between the detector and light field. Although a resonant interaction results in a larger phase shift gain, however, it increases the perturbation experienced by the field system, thereby jeopardizing the idea of a QND measurement. Fortunately one can take advantage of the spatial distribution of the field modes in order to minimize the effect of the resonant mode on the transition probability and still have a strong contribution to the phase [54, 9, 56]. To this end, we proposed to let the atom interact only with an even mode of the cavity [44]. This had the advantage that the effect of resonant interaction was minimized while preserving the phase shift gain. Another QND measurement technique for light energy is reported in [51]. Unlike the previous technique [11] that detects single microwave photons, the later scheme measured single photons of visible light, a spectrum that is more useful in quantum optics [58]. The advantage of the scheme [44] is that it is applicable both at the optical regime and microwave regime provided that the ratio of cavity length to coupling strength is kept constant.

The name "coupling" suggests an interaction between a measurement object (observable) and measurement device (probe) via an interaction Hamiltonian. Virtually the QND measurement techniques described above are based on the principle of weak interaction with very small coupling strength such that the system's evolution is treated based on time dependent perturbation theory [28] on the coupling strength. We consider in this thesis, the Unruh De-Witt interaction Hamiltonian [38, 40] for our measurement model. Given an interaction Hamiltonian for the quantum system, it is possible to describe the pre- measurement and post measurement processes. In chapter 2, we comment on these interaction processes between an atom-light field system and discuss possible approximations for our QND scheme to be successful.

The outline of this thesis is as follows. In the remainder of this chapter, we give an overview of quantum measurement and its different kinds – an ideal quantum measurement and a generalized quantum measurement. Although the ideal quantum measurement assumes the measurement process to be noise free and void of any kind of measurement error, a generalized and practical measurement is far from ideal. This leads to the quantum measurement problem where one cannot measure a quantum system without perturbing the state of the quantum system. We will discuss how a QND scheme minimizes this measurement problem and leads to the actual realization of the ideal measurement in quantum mechanics [65].

The objective of this thesis is to see ways in which we can realize a QND technique that measures a state of light. It is important to look at the definition and properties of some light state of interest – Fock state, coherent state and squeezed coherent state. This we will see in chapter 2 and give a literature review of some successful QND measurement technique on light state [63]. Chapter 3 presents the 'mode invisibility' measurement technique. Finally applications and future directions are discussed in chapter 4. Before we discuss QND measurement theory, let us now briefly examine the history of quantum measurement.

1.1 Quantum Measurement - An Overview

1.1.1 Ideal (projective) Quantum Measurement

Given a quantum system prepared initially in a state ρ_{in} and let $|q\rangle$ be the normalized ket vector we wish to measure in this quantum system. The first description of what happens during a quantum measurement is [65]

1. The result of a measurement of an observable $\hat{q}$ will be nothing else but one of its eigenvalues
$$\hat{q}|q_n\rangle = q_n|q_n\rangle, \quad (n = 1, 2, \cdots) \tag{1.1}$$

2. The probability of obtaining the specific measurement results q_n (assumed for simplicity to be non-degenerate) is given by
$$P(q_n) = \mathrm{Tr}[|q_n\rangle\langle q_n|\hat{\rho}_{in}] \tag{1.2}$$
where $|q_n\rangle\langle q_n|$ is a set of projection operators that projects any initial state onto one of the basis states.

3. After such a measurement with the results q_n, the system's density operator jumps to a new (post-measurement) state given by
$$\hat{\rho}_f = \frac{1}{P(q_n)}|q_n\rangle\langle q_n|\hat{\rho}_{in}|q_n\rangle\langle q_n| \tag{1.3}$$

Here the eigenstate $|q_n\rangle$ of the observable $\hat{q}$ forms a complete orthonormal set i.e.
$$\sum_{q_n} |q_n\rangle\langle q_n| = \hat{I}.$$
The state jump describes the reduction (collapse) of the state vector $|q\rangle$ to $|q_n\rangle$. These postulates are known as the projection postulate. They define what is know as an ideal (exact) measurement because it is noise free and without any measurement error. However a practical measurement is far from an ideal measurement. In order to describe a more realistic measurement, the concept of measurement has to be analyzed such that it takes account of how the measurement is performed, describes the measurement apparatus used, and distinguishes measurements from other possible types of interactions.

1.1.2 Generalized quantum measurement

A generalized quantum measurement introduces the concept of measuring device (probe) which induces a back action effect on the measured observable in a way that violates Heisenberg uncertainty principle (HUP). Consider a target system (the quantum system we wish to measure) and a probe system (measurement device) we will call the probe. Each of these systems (probe and target systems) are each characterized by an observable and its conjugate.

Heisenberg uncertainty principle

To understand how the uncertainty principle play a major role in quantum measurement process, given two non-commuting observables P and Q i.e.

$$[P, Q] = PQ - QP \neq 0,$$

which in the quantum formalism are examples of physical observables (example is position x and momentum p of a quantum particle), the uncertainty principle postulates that there exists a lower bound in their measurement namely

$$\Delta P \Delta Q \geq \frac{1}{2}|\langle [P, Q] \rangle|. \tag{1.4}$$

ΔP and ΔQ are dispersions in the measurement of P and Q respectively. In theory, when a very precise measurement on P is made results in a large fluctuation in Q. Although the precision of measurement of P is not affected, however, the large fluctuation in Q may eventually couple back to P therefore perturbing it (measurement back-action). Example is discussed in section 1.1.3 shortly. A measurement back-action will cause a second (repeated) measurement on the same observable to yield a measurement outcome different from that of the first measurement. [1]

From equation (1.4), if the commutator is a constant, we can drop the expectation values. The result then becomes the Heisenberg uncertainty principle. Non-commuting observables obey a Heisenberg uncertainty relation; For example, given the position x and its conjugate, the momentum p of a particle, we have the relation

$$\Delta p \Delta x \geq \frac{\mathrm{i}}{2}\langle [\hat{p}, \hat{x}] \rangle = \frac{\hbar}{2}$$

For photodetection, the uncertainty produced in the EM-fields implies that we cannot simultaneously measure the field phase ϕ and photon number n.

$$\Delta N \Delta \phi \geq \frac{\hbar}{2}.$$

1.1.3 A quantum measurement problem

To have a clearer picture of how the HUP affects a measurement of an observable, let us take for instance the measurement of the position $\hat{x}$ of a particle. According to HUP, a

[1] A measurement back-action will cause a second (repeated) measurement on the same observable to yield a measurement outcome different from that of the first measurement.

precise measurement of x will have introduced an uncertainty in p such that

$$\Delta p \geq \frac{\hbar}{2\Delta x} \tag{1.5}$$

If we assume that the particle is a free particle described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m},$$

then in the Heisenberg picture [65], we have the subsequent evolution of $\hat{x}$ to be given by

$$\frac{d}{dt}\hat{x} = \frac{\mathrm{i}}{\hbar}[\hat{H},\ \hat{x}] = \frac{\hat{p}}{m}, \tag{1.6}$$

so that at time t, the position of the particle would be given as

$$\hat{x}(t) = \hat{x}(0) + \frac{\hat{p}(0)t}{m}. \tag{1.7}$$

Therefore the uncertainty in the second measurement of the particles position $\Delta x(t)$ would be

$$\langle \Delta x(t)^2 \rangle = \langle \Delta x(0)^2 \rangle + \frac{t^2}{m^2}\langle \Delta p(0)^2 \rangle + \frac{t}{m}\langle \Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0) \rangle \tag{1.8}$$

Given that the quantities $\Delta p(0)$ and $\Delta x(0)$ are independent, the last term drops and we have

$$\langle \Delta x(t)^2 \rangle \geq \langle \Delta x(0)^2 \rangle + \frac{\hbar t^2}{2m}\langle \Delta x(0)^2 \rangle \tag{1.9}$$

We observe that due to the first measurement we made, the dispersion or uncertainty in any subsequent measurement will be increased. As a result of this property of non-commuting observables, the final state of a measured system is different from the initial state; therefore sequential measurements of an observable may not give the same result. This describes a quantum measurement problem–an attempt to measure a quantum system perturbs the system, thereby introducing inevitable backaction effect to the measured system.

For an ideal measurement in quantum mechanics 1.1.1, one needs to realize some scheme that measures a quantum system without causing a measurement back action. In essence we would like to measure a quantum system such that subsequent measurements give results that can be completely determined based on the first result. Such a measurement is called a quantum nondemolition (QND) measurement.

1.1.4 Indirect quantum measurement

Any indirect measurement scheme introduces the concept of a measurement device (probe) which has its uncertainty and produces excess back action (measurement uncertainty) on the measured system. It is known that due to the uncertainty relation, such indirect measurement schemes can never exceed the standard quantum limit [42, 57, 70] which is imposed by the Heisenberg uncertainty principle (HUP). In this section, we will discuss how such measurement backaction is introduced on the system to be measured.

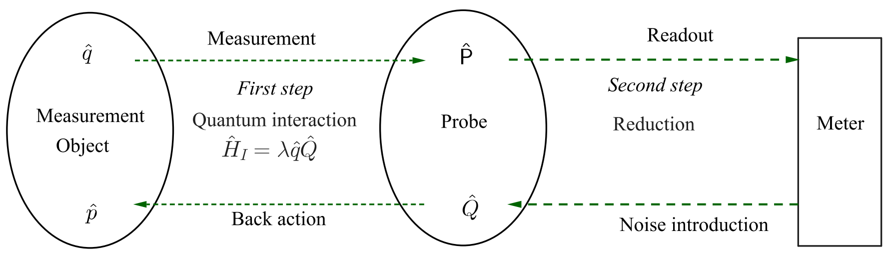


Figure 1.1: An indirect quantum measurement process. Here, measurement consists of a system to be measured, the measuring device (probe) that interacts with the target system and a meter that reads the probe observable.

In an indirect measurement process, we have a measuring device (probe or detector) initially prepared in a given quantum state, let's say $\hat{\rho}_p$. The probe is chosen optimally to copy information of the measured observable for the quantum object (system to be measured) which we also consider to be initially prepared in state $\hat{\rho}_s$. Two steps emerge from an indirect quantum measurement:

1. Quantum interaction between probe system and object-system governed by a unitary time evolution,
2. measurement of the readout observable (state reduction).

In the first step, which involves a quantum interaction between the probe and object system, an interaction is induced via an interaction Hamiltonian H_I generally given as (setting $\hbar = 1$)

$$H_I = \lambda\hat{q}\hat{Q}, \tag{1.10}$$

where λ is the coupling constant, $\hat{Q}$ is the conjugate observable to the probe observable $\hat{P}$ just like the position observable $\hat{x}$ of a particle has a conjugate in the particle's momentum $\hat{p}$. $\hat{q}$ with its conjugate $\hat{p}$ are observables of the object system. The nature of the interaction Hamiltonian (1.10) is such that it copies the information of the measured observable $\hat{q}$ of the object system onto the readout observable $\hat{P}$ of the probe. We note that there is an absence of the state reduction process (see discussion in 1.1.1) in this first step; however, a quantum correlation is established between the object observable $\hat{q}$ and readout observable $\hat{P}$ of the probe and a backaction noise is injected on $\hat{p}$ by the quantum noise of $\hat{Q}$. We can thus consider the combined 'object + probe' system as a closed system.

The total Hamiltonian describing such a system is given as

$$\hat{H}_T = \hat{H}_s + \hat{H}_p + \hat{H}_I, \tag{1.11}$$

where H_s and H_p are the free Hamiltonian for the system and probe respectively. H_I is the appropriate interaction Hamiltonian. Assuming that the HUP principle (1.4) holds for the non-commuting observables

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar, \qquad [\hat{P}, \hat{Q}] = \mathrm{i}\hbar,$$

then the evolution of the observables $\hat{p}$ and $\hat{P}$ is described by the Heisenberg equations of motion

$$\frac{d}{dt}\hat{P} = \frac{\mathrm{i}}{\hbar}[\hat{H}_I,\ \hat{P}] = \lambda\hat{q} \qquad \text{measurement,} \tag{1.12a}$$

$$\frac{d}{dt}\hat{p} = \frac{\mathrm{i}}{\hbar}[\hat{H}_I,\ \hat{p}] = \lambda\hat{Q} \qquad \text{backaction.} \tag{1.12b}$$

Here we assume that the interaction part of the equation of motion is fast compared to the free evolution. We briefly describe these equations. Equation (1.12a) implies that the information we gain from the combined 'object+probe' system depends on the measured observable of the system $\hat{q}$ and similarly, in equation (1.12b), we see that the measurement backaction on a quantum system is imposed by the probes conjugate observable $\hat{Q}$. A measurement back-action inherent during a quantum measurement process arises due to the quantum uncertainty of $\hat{Q}$ imposed on the conjugate observable $\hat{p}$ of a measured system.

The back action transferred from the probe to the object produces an irreversible and inevitable change in the object system. To acquire information of what changes have been made or not on the object-system requires a subsequent measurement. This leads to the second step in the indirect measurement process - the readout observable $\hat{P}$ of the probe.

In the initial stage of measurement, the combined probe-object system is defined as a product state in form a density operator

$$\hat{\rho}_{\mathrm{i}} = \hat{\rho}_s \otimes \hat{\rho}_p \tag{1.13}$$

The appropriate interaction Hamiltonian H_I introduces a unitary evolution defined by a unitary operator $\hat{U}$ for this combined object and probe Hilbert space $H_s \otimes H_p$. Such an interaction brings the object system and probe into a joint-correlated quantum state given by the density operator

$$\hat{\rho}_f = \hat{U}\hat{\rho}_s \otimes \hat{\rho}_p\hat{U}^{\dagger}. \tag{1.14}$$

The corresponding reduced density operators of the object and the probe alone are:

$$\hat{\rho'}_s = \mathrm{Tr}_p\left[\hat{U}\hat{\rho}_s \otimes \hat{\rho}_p\hat{U}^{\dagger}\right], \quad \hat{\rho'}_p = \mathrm{Tr}_s\left[\hat{U}\hat{\rho}_s \otimes \hat{\rho}_p\hat{U}^{\dagger}\right]$$

respectively. Tr_s and Tr_p denote a partial trace over the system and probe respectively and the new states (i.e the reduced states $\hat{\rho'}_s$ and $\hat{\rho'}_p$) are states on the Hilbert spaces H_s and H_p of the system and probe respectively. Since we have a composite system of probe and object and we wish to know what the measurement information about the object system is by looking at the readout observable of the probe, we disregard the object system and take a measurement on the probe. This "disregarding" is reflected by the operation of a partial trace.

Although an indirect measurement process offers the advantage that the measurement process does not perturb the object-system significantly, but the back action transferred from the probe to the object still produces an irreversible and inevitable change in the object system. Such measurement backactions are not fundamental; so it is possible to realize a measurement technique that could exceed the standard quantum limit. A non-standard measurement technique that exceeds the standard quantum limit (SQL) would involve a carefully chosen target system (signal beam), carefully prepared probe (probe beam) and making the interaction between the probe and target weak enough such that the perturbation on the target system is minimized, with the target state preserved after measurement.

1.2 Quantum Non-Demolition Measurement

1.2.1 Basic Concepts

We have already discussed in section 1.4 how a first measurement of a given observable of a quantum system will inevitably induce huge uncertainties in its conjugate observable, with the uncertainty coupling back (backaction) afterwards to the measured observable as the system evolves. Hence the result of any subsequent measurement of same observable is uncertain [7, 8].

A quantum non-demolition measurement scheme [8, 53, 25, 46, 4, 18, 45] tries to obtain the highest amount of information possible about a quantum system whilst minimizing the measurement back action. A complete history of the QND measurement scheme can be found in [8]. In general, the original motivation for performing QND measurements was the effort to minimize the measurement back action arising during a gravitational wave detection. A domain in which a QND measurement scheme has been greatly realized is quantum optics. This is due to advancement in laser, nonlinear optics and atomic physics technologies. A QND measurement of optical fields was first demonstrated [12]. Since its conceptual introduction, there has been a lot of theoretical [15] and experimental [23, 26, 43, 5] progress on this field.

1.2.2 Fundamentals of Quantum Non-Demolition Measurement

A key feature of QND measurements is repeatability. In such a measurement, one monitors a single observable $\hat{q}$ of the system we wish to measure. This observable must be a 'QND observable'– i.e. an observable that can be repeatedly measured. Let us discuss the general condition for making a QND measurement.

In the Heisenberg picture, the evolution of the measured observable $\hat{q}$ of the object system and the readout observable $\hat{P}$ of the probe is described by the equations

$$i\hbar\frac{d}{dt}\hat{q} = -[\hat{H}_s, \hat{q}] - [\hat{H}_I, \hat{q}], \qquad \text{(measurement)} \tag{1.15a}$$

$$i\hbar\frac{d}{dt}\hat{P} = -[\hat{H}_m, \hat{P}] - [\hat{H}_I, \hat{P}], \qquad \text{(backaction)} \tag{1.15b}$$

Here we consider the free Hamiltonians H_s and H_p of the object system and probe respectively for our discussion. The first commutators represent the free evolution of $\hat{q}$ and $\hat{P}$, while the second commutation represents the contribution of the interaction. As discussed earlier, a measurement of $\hat{q}$ introduces a measurement back action on its conjugate variable $\hat{p}$ caused by the inevitable uncertainty in $\hat{Q}$. We analyze here the two ways in which the measurement observable $\hat{q}$ can be changed.

1. A direct perturbation arising from its free evolution.
2. An indirect perturbation arising from the measurement back-action injected on its conjugate observable during the measurement process.

1.3 Criteria of Quantum non-Demolition Measurement

As we have discussed, in a general quantum measurement process, the measurement device becomes coupled under a unitary evolution operator to a quantum system which is carefully prepared in some initial state. The quantum system is left in the corresponding output state. Some measurement criteria for a QND measurement scheme which prevents changes on the measured observable have been stated [58]. Three distinct criteria thus emerge in the context of optical QND measurements:

1. The object system should not be excessively perturbed by the measurement process, so that repeated measurements give the same result. In other words, the QND measurement should be a good quantum state preparation (QSP) device. This is the quantum non-demolition measurement idea.

2. The probe observable should carry some information about the object-systems observable $\hat{q}$ so that a measurement is actually performed. This criteria describes the efficiency of the QND measurement scheme.

3. The probe observable $\hat{P}$ should be quantum correlated with the object-system observable $\hat{q}$ so that the readout of the probe does give some information about the outgoing signal (output quantum correlation)

With these criteria stated, we may now look at equations (1.15a) and (1.15b) and see how each term could be prepared to yield the QND measurement criteria. It follows that

1. A direct perturbation on the object observable $\hat{q}$ is due to its free evolution $[\hat{H}_I, \hat{q}]$. Therefore to protect $\hat{q}$ from the effect of interaction with the probe and against the measurement back action induced on object system, we expect that the interaction Hamiltonian H_I commutes with $\hat{q}$

$$[\hat{H}_I, \hat{q}] = 0 \tag{1.16}$$

   That is $\hat{H}_I$ must be a function of $\hat{q}$ so that a measurement of $\hat{q}$ would give an information about $\hat{q}$.

   An interaction Hamiltonian satisfying the condition (1.16) is termed "back action evading type" [7].

2. The free Hamiltonian of the object system $\hat{H}_s$ must not be a function of the conjugate observable $\hat{p}$ of the object system.

$$[\hat{H}_s, \hat{q}] = 0 \tag{1.17}$$

This condition ensures that the uncertainty introduced in $\hat{p}$ which gets to perturb $\hat{q}$ indirect, is eliminated. In this way, we get rid of the back action inherent during a measurement process.

An observable $\hat{q}$ satisfying equation (1.17) is called a "QND observable" [7].

3. The probe observable $\hat{P}$ cannot be a constant of the motion, and so

$$[\hat{H}_I, \hat{P}] \neq 0 \tag{1.18}$$

If it were, we could not use it to measure anything since we could not a detect a change in it.

If conditions (1.16) and (1.17) are satisfied, we can measure a system's observable $\hat{q}$ repeatedly with an infinitesimally small measurement error without introducing backaction noise on $\hat{q}$. Such a measurement is called a quantum nondemoiltion (QND) measurement.

Examples of QND observables

We have seen that to perform a QND measurement, the quantum observable to be measured must be a QND observable. Here we will present the example of such QND observables namely the momentum of a free particle, and the photon number operator $\hat{n}$.

Momentum of a free particle–Consider the measurement of the physical observables of a free particle namely, its position $\hat{x}$ and momentum $\hat{p}$ respectively. The free Hamiltonian of the particle is given by

$$\hat{H}_s = \frac{\hat{p}^2}{2m}, \tag{1.19}$$

It is evident that the free Hamiltonian operator commutes with the momentum operator $\hat{p}$ as required in QND condition (1.17). Since $\hat{p}$ has its conjugate variable $\hat{x}$, it is certain from (1.4) that the measurement of $\hat{p}$ perturbs the particles position $\hat{x}$ but the positions uncertainty does not feed back to the particles momentum since H_s is independent of $\hat{x}$. However, when it comes to the measurement of $\hat{x}$, the free Hamiltonian (1.19) does not commute with $\hat{x}$ and as such, measuring $\hat{x}$ will inevitably perturb the particle's momentum, thus an uncertainty is introduced in the particle's position arising from the momentum's uncertainty. [14]

Photon number operator-The free Hamiltonian of the harmonic oscillator with quantum number $\hat{n}$ is given by

$$H_s = \hbar\omega\left(\hat{n} + \frac{1}{2}\right) \tag{1.20}$$

Obviously, this quantity commutes with $\hat{n}$

Motivation

The act of measuring a quantum system to acquire information about it must necessarily disturb the system. An example is the photodetection process which creates or annihilate photons and projects the photon state into the vacuum. We have also seen that by preparing a careful measurement scheme, we can indeed detect single photons without destroying the photon state [43]. To ensure that such photon states are not destroyed, we need to measure a significant information about the quantum system after the interaction process.

Motivated by the QND measurement principle and a remarkable work as cited in [43], this thesis addresses a QND measurement technique that measures light not only in a non-demolition manner, but acquires information about the state of the measured observable. Seeing that photons which are quanta of light are examples of a QND observable, one can indeed realize a QND measurement of light by careful measurement technique.

Chapter 2

Background

The discussion of the previous chapter is exclusively concerned with a quantum measurement problem, which states that the act of measuring a quantum system to acquire information about its physical observables must necessarily disturb the system. This results in the inability of one repeating a measurement on the same measured observable. On the other hand, a quantum nondemolition (QND) measurement, that is, the measurement that minimizes the backaction effect inherent in a quantum measurement process, offers an approach by which one can realize measurement techniques that are noise free, repeatable and yield higher precision of measured observables. QND measurement schemes [25, 51, 44, 63, 14, 8] treat the interaction between a signal beam coupled to the beam.of a measuring device (probe).

The purpose of the present chapter is the QND treatment of the coupling of single atoms with the quantized electromagnetic field stored a cavity. Several theoretical and experimental attempts on QND measurement of a quantum system have been realized. However we present here a remarkable QND measurement scheme that detects photons without destroying the state of the measured photon [63]. The chapter is split into parts that separately discuss quantum theory of light (2.1), the atom-field interaction model (2.2), a QND measurement of light field (2.3).

2.1 Quantization of the Electromagnetic Field

Light is an important subject of study which is central to human knowledge. Physically, we understand that light is a propagating electromagnetic wave. It is thus described by one of its observables–an electric field $E(r,t)$ and a magnetic field $B(r,t)$. The fields are solutions of the Maxwell's equations which mathematically are linear partial differential equations. We can in general find a complete set of solutions of these equations, $E(r,t)$ and $B(r,t)$ which we call *modes of light*.

Suppose we have an EM-field confined to a cavity. We can call this configuration a quantum system. The major objective of this thesis is to propose a QND measurement of such a quantum system. In other words, what physical properties of an EM field can we obtain non-destructively? The first step to take is to consider the quantization of the EM-field. Since it is of little relevance to our study – how an electromagnetic field can be quantized – we refer the reader to the book [58] for an excellent literature review on the quantization of electromagnetic field.

2.1.1 Maxwell's Equations

With the objective of quantizing the EM-field in free space, it is convenient to begin with the classical description of the field based on Maxwell's equations. These equations describe the combined electric and magnetic response of a medium. For light (source free region), the Maxwell's equations are

$$\nabla \cdot E = 0 = \nabla \cdot B, \tag{2.1a}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{2.1b}$$

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}. \tag{2.1c}$$

Here, ϵ_0 and μ_0 are the free space permitivity and permeability respectively and $\mu_0\epsilon_0 = 1/c^2$, where c is the speed of light in vacuum. In terms of scalar potential $\phi(r,t)$ and vector potentials $A(r,t)$ respectively, EM-fields can be written as

$$E = -\nabla\Phi - \frac{\partial A}{\partial t} \tag{2.2}$$

$$B = \nabla \times A \tag{2.3}$$

so that under the invariant Lorentz gauge transformation, the vector potential satisfies the wave equation

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0. \tag{2.4}$$

2.1.2 Mode expansion of the electromagnetic field

Since the fields in cavities differ from that in free space, the interaction of an atom with an electromagnetic field in a cavity depends on the cavity boundary conditions. The solution to the equation (2.4) is given in terms of expansion in the normal cavity modes

$$\hat{A}(r,t) = -\mathrm{i}\sum_{\gamma}\sqrt{\frac{\hbar}{2\epsilon_0\omega_\gamma}}\big(a_\gamma(t)u_\gamma(r) + a_\gamma^\dagger(t)u_\gamma^*(r)\big), \qquad \gamma = 1, 2, \cdots \tag{2.5}$$

Here a normalization condition is conveniently chosen so that the amplitudes $a_\gamma(t)$ and $a_\gamma^\dagger$ are dimensionless. $u_\gamma(r)$ is the normal mode function having solutions for an appropriate boundary condition. Substituting equation (2.5) into the vector potential equation (2.4), we get

$$\left(\nabla^2 + \frac{\omega_\gamma^2}{c^2}\right)u_\gamma(r) = 0, \qquad \left(\frac{\partial^2}{\partial t^2} + \omega_\gamma^2\right)a_\gamma(t) = 0$$

with $a_\gamma(t) = a_\gamma e^{-\mathrm{i}\omega_\gamma t}$ and $a_\gamma^\dagger(t) = a_\gamma^\dagger e^{\mathrm{i}\omega_\gamma t}$. A boundary condition for a standing wave solution such as in an optical cavity in a simple 1D model would be

$$u_\gamma(r=0) = u_\gamma(r=L) = 0$$

This gives a mode function of

$$u_\gamma(r) \sim \sin(k_\gamma r)\hat{\epsilon}, \qquad k_\gamma = \frac{\gamma\pi}{L} \tag{2.6}$$

with cavity length L and polarization vector (dipole moment per unit volume) $\hat{\epsilon}$ which is required to be perpendicular to γ

$$\hat{\epsilon}\cdot\gamma = 0$$

Substituting equation (2.6) into equation (2.5), the quantized field for a standing wave is therefore given as

$$\hat{A}(r,t) = -\mathrm{i}\sum_{\gamma}\sqrt{\frac{\hbar}{2\epsilon_0\omega_\gamma}}\left(a_\gamma e^{-\mathrm{i}\omega_\gamma t} + a_\gamma^\dagger e^{\mathrm{i}\omega_\gamma t}\right)\sin(k_\gamma r(t)) \tag{2.7}$$

We will now discuss in the next section, some of the relevant EM-field states.

2.1.3 Important states of the Electromagnetic Field

The quantum operators $a_\gamma^\dagger$ and a_γ in equation (2.7) create and annihilate photons in mode γ and obeys the commutation relation

$$[a_\gamma, a_\beta^\dagger] = \delta_{\gamma,\beta}, \quad [a_\gamma, a_\beta] = [a_\gamma^\dagger, a_\beta^\dagger] = 0 \tag{2.8}$$

We will see why these operators are called the creation and annihilation operators later in chapter 3. The quantized free EM- field (2.7) is an infinite collection of uncoupled harmonic oscillators obeying the relation (2.8) and given by the Hamiltonian

$$\hat{H} = \sum_\gamma \left(a_\gamma^\dagger a_\gamma + \frac{1}{2} \right) \hbar\omega_\gamma. \tag{2.9}$$

Equation (2.9) represents the sum of the number of photons $a_\gamma^\dagger a_\gamma = \hat{n}_\gamma$ in each mode γ multiplied by the energy of a photon $\hbar\omega_\gamma$ in that mode, plus $\frac{1}{2}\hbar\omega_\gamma$ representing the energy of the vacuum fluctuations in each mode. ω_γ is the frequency of the field mode γ and $\hbar$ the reduced Planck's constant. Because all oscillators are uncoupled, it is sufficient to study a single mode γ. Consider a single mode γ, we define the vacuum state by a Dirac ket vector $|0\rangle$ satisfying the relation

$$a_\gamma|0\rangle = 0, \quad a_\gamma^\dagger|0\rangle = |1_\gamma\rangle \ \forall \ \gamma$$

We shall now consider three possible representations of the electromagnetic eld.

Fock or number states

Fock states or number states are defined as the eigenstates of the number operator $\hat{n}_\gamma$:

$$\hat{a}_\gamma^\dagger \hat{a}_\gamma |n_\gamma\rangle = n_\gamma |n_\gamma\rangle,$$

and the free EM-field Hamiltonian (2.9)

$$H|n_\gamma\rangle = E|n_\gamma\rangle \qquad (n = 1, 2, 3 \cdots \infty), \tag{2.10}$$

where the state $|n_\gamma\rangle$ consists of n_γ elementary field excitations (photons), in the field mode γ and contains a total number n of photons, i.e. n_γ counts the number of excitations (photons) in a mode γ. The creation operator $\hat{a}_\gamma^\dagger$ and annihilation operator $\hat{a}_\gamma$ each creates and destroys photons in a Fock state according to the following rule

$$\hat{a}_\gamma|n_\gamma\rangle = \sqrt{n_\gamma}|n_\gamma - 1\rangle$$
$$\hat{a}_\gamma^\dagger|n_\gamma\rangle = \sqrt{n_\gamma + 1}|n_\gamma + 1\rangle$$

The nth Fock state, equation (2.11) is created by the successive application of the creation operator on ground state $|0\rangle$ of the harmonic oscillator.

$$|n\rangle = \frac{1}{n!}(a_0^\dagger)^n|0\rangle \tag{2.11}$$

Some properties of Fock states

1. Fock states, being the eigenstates of the number operator are mutually orthogonal, hence for a single mode γ,

$$\langle m_\gamma|n_\gamma\rangle = \delta_{mn}$$

   and they form a complete set of orthonormal vectors

$$\sum_{\gamma=0}^{\infty}|n_\gamma\rangle\langle n_\gamma| = \hat{I},$$

   where $\hat{I}$ is the identity operator in the Hilbert space of the single-mode system.

2. For various modes of a Fock state $|n\rangle$:

$$\langle m|n\rangle = \delta_{m,n} \qquad \sum_{\gamma=0}^{\infty}|n_{\gamma 1}, n_{\gamma 2}, \cdots\rangle\langle n_{\gamma 1}, n_{\gamma 2}, \cdots| = \hat{I},$$

Although Fock states are well known in quantum mechanics and useful for calculations, because of the nature of the hamonic oscillator, transitions between adjacent Fock states are degenerate, making it impossible to generate and detect with classical linear techniques [67]. It is possible however to produce a single photon state which might be useful for quantum information processing [43, 67]

Coherent states

A special class of light states are the coherent states [21]. Coherent states are produced by classical light sources. The single-mode coherent state, symbolized by the state vector $|\alpha\rangle$, may be defined as the eigenstate of the annihilation operator with eigenvalue α namely

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{2.12}$$

with α being a complex number, which may be written as

$$\alpha = |\alpha|e^{\mathrm{i}\theta} \tag{2.13}$$

where the real numbers $|\alpha|$ and θ are the amplitude and phase of the coherent state respectively with $0 \leq |\alpha| < \infty$ and $0 < \theta \leq 2\pi$

In the number basis, $\{|n\rangle\}$, a single-mode coherent state $|\alpha\rangle$ takes the form

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{2.14}$$

which is obviously well normalized $\langle\alpha|\alpha\rangle = 1$. It is straightforward to check that the expectation value of the photon number on a coherent state is $\langle\hat{n}\rangle = |\alpha|^2$. Additionally we can define the displacement operator [21]

$$D(\alpha) = \exp\Big(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\Big), \tag{2.15}$$

since each a and $a^\dagger$ is non-commuting with the operator $\alpha a^\dagger - \alpha^* a$. It can be shown easily that the displacement operator (2.15) provides the following operator relations

$$D(\alpha)^\dagger \hat{a} D(\alpha) = a + \alpha, \tag{2.16a}$$

$$D(\alpha)^\dagger a^\dagger D(\alpha) = a^\dagger + \alpha^* \tag{2.16b}$$

This implies that when the displacement operator acts on a state it changes all moments of a and $a^\dagger$. A coherent state is also given in terms of the displacement operator namely

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{2.17}$$

These kind of states have been used to build a strong description of the EM field [21, 22]. We are going to study the effect of an interaction of a detector (single atom) with a single mode of the coherent state. We will show that it is possible to measure some properties of the coherent state in a non-destructive way [43, 44].

Squeezed states

A third quantum state of light we will study is the squeezed coherent state. These are states that minimize the uncertainty relationships with phase-sensitive fluctuations (where the uncertainty in the x and p quadratures is not necessarily the same). The single-mode squeeze operator is defined by

$$S(\zeta) = \exp\left(\frac{1}{2}\zeta^* \hat{a}\hat{a} - \frac{1}{2}\zeta \hat{a}^\dagger \hat{a}^\dagger\right), \qquad \zeta = re^{i\phi} \tag{2.18}$$

A representation of a squeezed coherent state ($|\zeta, \alpha\rangle$) can be obtained by applying the single mode-squeeze operator (2.18) on the coherent state, i.e.

$$|\zeta, \alpha\rangle \equiv S(\zeta)|\alpha\rangle. \tag{2.19}$$

$S(\zeta)$ is a unitary operator ($\hat{S}(\zeta)\hat{S}^\dagger(\zeta) = \hat{S}^\dagger(\zeta)\hat{S}(\zeta) = 1$) with magnitude $|S(\zeta)| = r$ ($0 \leq r < \infty$) and phase ϕ ($0 \leq \phi \leq 2\pi$) respectively [36]. The annihilation and creation operators transform as

$$S(\zeta)^\dagger_\kappa a_\kappa S(\zeta)_\kappa = a_\kappa \cosh(r) - a^\dagger_\kappa e^{i\phi} \sinh(r) \tag{2.20a}$$

$$S(\zeta)^\dagger_\kappa a^\dagger_\kappa S(\zeta)_\kappa| = a^\dagger_\kappa \cosh(r) - a_\kappa e^{-i\phi} \sinh(r). \tag{2.20b}$$

In general, squeezed states of light [58] describe very well the outcome of many non-linear quantum optics light sources (e.g. parametric down conversion) [59] and they have been widely studied in the past years as they have been proven useful for optical communications [51, 36] and for quantum metrology [52, 13] among other disciplines.

We will see how these states of light could be measured in a non-destructive way in our later discussion.

2.2 Interaction of Atom with Electromagnetic Field

Having reminded ourselves of the relevant properties of EM-field and optical cavities, we can now start a discussion of an interaction between light suspended in a cavity and an atom schematically in Figure 2.1. An atom in the presence of light field will respond to the electrical component of the EM-field. We assume that the atom is fired into the cavity in such a way that it can cause photon absorption and emission inside the cavity. We are particularly interested in the case where the transition frequency of the atom coincides with one of the resonant modes of the cavity. In these circumstances, we can expect that the interaction between the atom and the light field will be strongly affected, since the atom and cavity can exchange photons in a resonant way. This has the advantage that the atom is able to gain much information about the state of light however in a destructive way. We will show in chapter 3 that we can minimize the perturbation inherent during such resonant interaction while leaving the information gained untouched.

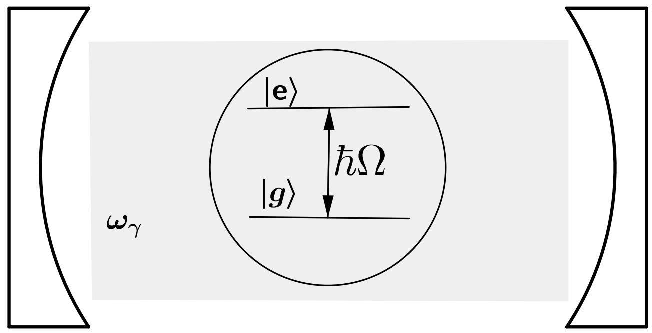


Figure 2.1: A schematic representation of two-level system in a cavity with light field modes suspended in it

2.2.1 Two-level atom approximation

Modelling the single atom (detector) as a two-level system (TLS) implies that there are only two allowed states for the detector's excitation, the ground state $|g\rangle$ with energy E_g and the excited state $|e\rangle$ with energy E_e respectively.

$$|g\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |e\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.21}$$

The Hamiltonian of the two-level atom is then given as

$$\hat{H}_{TLS} = E_g|g\rangle\langle g| + E_e|e\rangle\langle e| \tag{2.22}$$

$$= \begin{pmatrix} E_g & 0 \\ 0 & Ee \end{pmatrix} \tag{2.23}$$

We can simplify the Hamiltonian

$$\begin{aligned} \hat{H}_{TLS} =& \frac{1}{2}\begin{pmatrix} E_e + E_g & 0 \\ 0 & E_e + E_g \end{pmatrix} + \frac{1}{2}\begin{pmatrix} E_e - E_g & 0 \\ 0 & E_g - E_e \end{pmatrix} \\ =& \frac{1}{2}(E_e + E_g)\hat{\mathbf{I}} + \frac{1}{2}\Delta E\hat{\sigma}_z \end{aligned}$$

The Hamiltonian for a two level system is thus given as

$$H_{TLP} = \frac{\hbar\Omega}{2}\sigma_z \tag{2.24}$$

The difference in energy between the ground eigenstate and excited eigenstate of the TLS is thus related to the system's transition frequency Ω by

$$\Delta E = E_e - E_g = \hbar\Omega$$

One can define the lowering operator σ^- and raising operator σ^+ for the TLS respectively;

$$\sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.25}$$

which are capable of raising the ground state to an excited state and lowering the excited state to a ground state respectively. Mathematically, we have

$$\sigma^+|e\rangle = 0, \quad \sigma^-|g\rangle = 0, \quad \sigma^+|g\rangle = |e\rangle, \quad \sigma^-|e\rangle = |g\rangle$$

The TLS is used as a basic tool to act on EM-field, to manipulate them, to control their various degrees of freedom and more importantly as a source of information on them.

2.2.2 Atom-field interaction Hamiltonian

Like any physical interaction in quantum mechanics, our starting point is to write out an appropriate Hamiltonian for a TLS coupled to an electromagnetic field. The joint atom-field Hamiltonian $\hat{H}_T$ is given by

$$\hat{H}_T = \hat{H}_p + \hat{H}_F + \hat{H}_I. \tag{2.26}$$

Here, $\hat{H}_p$ is a free Hamiltonian for the atom (probe) as defined in equation (2.22). Similaly, $\hat{H}_F$ is the free Hamiltonian (2.9) of the electromagnetic field

$$H_F = \omega_\gamma\big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\big). \tag{2.27}$$

$\hat{H}_I$ is the interaction Hamiltonian-the Hamiltonian that describes the joint atom-field system which is given as

$$H_{p-F} = -\,\overrightarrow{\hat{D}} \cdot \overrightarrow{E}(x,t). \tag{2.28}$$

The dipole moment $\overrightarrow{D}$, is an operator for the system under consideration. Since we are interested in a TLS, the component of $\overrightarrow{D}$ are 2×2 Hermitian matrices which in essence could be represented as a linear superposition of Pauli spin matrices, we may therefore define the dipole moment for a TLS as

$$\overrightarrow{D} = \mu(\sigma^+ + \sigma^-),$$

μ is a quantity having the dimension of the dipole operator. It is known that when the interaction between the atom and field mode is relatively strong, then $\omega_\gamma \approx \Omega$. That is frequency of the EM-field is equal to the atomic transition frequency. We will see the effect of such strong interaction later in chapter 3. We remind the reader that the purpose of this thesis is to discuss possible techniques that non destructively measure a state of an EM-field. To satisfy the QND condition 1.16, we need an interaction Hamiltonian that is capable of imprinting the information of the EM-field on the measurement device.

2.2.3 The Unruh-DeWitt Hamiltonian

In this section, we wish to define a back action evading Hamiltonian necessary for our QND condition (1.16) to be satisfied. We consider a model known as the Unruh deWitt Hamiltonian [40] which is given by the expression

$$\hat{H}_I = \lambda\hat{\mu}(t)\hat{\phi}[x(t)]. \tag{2.29}$$

Here, $\phi[x(t)]$ is a scalar quantum field analogous to the vector field $\mathbf{A}(x)$ discussed in section 2.1.1

$$\phi[x(t)] = \sum_{\gamma=1}^{\infty}(a_\gamma^\dagger e^{\mathrm{i}\omega_\gamma t} + a_\gamma e^{-\mathrm{i}\omega_\gamma t})\sin[k_\gamma x(t)]. \tag{2.30}$$

However, a condition to the Unruh DeWitt Hamiltonian is that there is no exchange of angular momentum for the atom-cavity interaction [40]. The atomic monopole moment μ is defined as

$$\hat{\mu} = \sigma^+ e^{\mathrm{i}\Omega t} + \sigma^- e^{-\mathrm{i}\Omega t} \tag{2.31}$$

so that the interaction Hamiltonian becomes

$$H_I = \sum_{\gamma=1} \frac{\lambda \sin(k_\gamma x(t)).}{\sqrt{k_\gamma L}} \left(\sigma^+ a_\gamma^\dagger e^{\mathrm{i}(\Omega+\omega_\gamma)t} + \sigma^+ a_\gamma e^{-\mathrm{i}(-\Omega+\omega_\gamma)t} + \sigma^- a_\gamma^\dagger e^{\mathrm{i}(-\Omega+\omega_\gamma)t} + \sigma^- a_\gamma e^{-\mathrm{i}(\Omega+\omega_\gamma)t}\right). \tag{2.32}$$

Here ω_γ is the frequency of the field mode γ, $x(t)$ is the position of the detector in the cavity of length L at time t. λ is the coupling strength for the interaction of the atomic monopole moment with the field.

2.2.4 Resonant and Non-resonant interaction

Let us discuss the possible interactions that could occur in the cavity modes.

Non-resonant interaction

At non-resonance, the transition frequency of the atom is detuned from the cavity mode frequency. When this detuning becomes larger than the cause of atom-field interaction i.e the coupling, then energy exchange between the atom and relevant cavity modes becomes impossible [58]. Instead, the cavity modes induce a phase shift in the atomic state. Since the phase shift which is detectable by Ramsey interferometry is proportional to the photon number in the cavity, it is possible to perform a QND measurement of the photon number.

Resonant interaction

A typical resonant condition suggests that the atomic transition frequency equals the frequency of the cavity mode. As a result, strongest interaction between the atom and light field is achieved and the atomic transition frequency is maximum.

Although a resonant interaction between the atom and cavity mode would cause a high perturbation in the cavity modes, it is important however to learn more information about the field state. Our QND measurement model discussed shortly assumes a resonant interaction between the combined atom-light field system. We show how we are able to control the effect of the resonant interaction [44] so that the cavity mode is not perturbed significantly and the atomic phase shift is unperturbed and remains significantly measurable by atom interferometery. This is known as the "mode invisibility" technique.

The Hamiltonian (2.32) contains both the counter- rotating or rapidly oscillating component $(\omega_\gamma + \Omega)$ and the rotating or slowly oscillating component $(\omega_\gamma - \Omega)$.

1. The slowly varying term accumulates the atom-field interaction
2. The rapidly varying term cancels it out due to destructive interference.

2.3 Quantum Non-Demolition Measurement in Optics

Consider a QND measurement in the context of quantum optics. An example of a destructive measurement is the photodetection process. Suppose we want to measure the number of photons in a given light field (beam). The photodetection process simply use a photon counter to do this. But suppose we wanted to repeat a measurement on the same light beam whose photon number distribution we have acquired knowledge about, this implies that the state of light has to be preserved after a first measurement. A photon counter destroys the photon it counts leaving the state of light it measured inaccessible for future measurement as shown in figure 2.2. We can apply the principle of QND measurement to the measurement of light state.

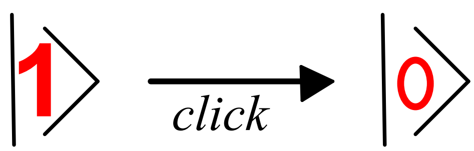


Figure 2.2: A schematic representation of a photodetection process. Detecting a photon destroys the photon thereby changing the state of light.

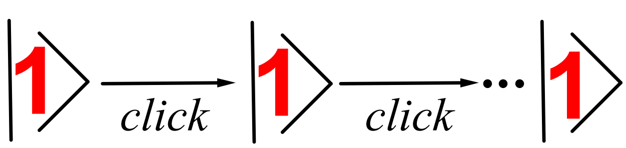


Figure 2.3: Schematic representation of QND measurement of single photons.Photons upon detection are not destroyed and repeated measurement yield the same result for all times.

Figure 2.2 is a sketch of what happens during a photodetection process. Detectors (photon counters) annihilate or absorb photons and convert the energy of these photons into an electrical signal (click). The detected photon gets destroyed in the process and the post-measurement state is always a vacuum instead of Fock state associated to the measurement result. This is not good for an ideal quantum measurement (see section 1.1.1). Can we make a repetitive measurement of the form in figure 2.3 where measurement of a photon state leaves the photon content unchanged and repeated measurement yields the same result? Such measurement is known as quantum non-demolition measurement (QND).

QND measurement schemes [7, 53, 25] allow one to project a system onto an eigenstate of the measured observable so that repetition of measurement yields the same outcome. Such a property is required for quantum error correction [61], one-way quantum computing [43, 50] and state preparation measurement, detection of gravitation waves and weak forces [13, 64]. To perform a QND measurement of photons requires:

1. The ability to trap and control these photons.
2. Photons should survive the effect of an interaction with a detector at any time.
3. Detectors need to be transparent, i.e. they need to pick up information from the field without exchanging energy with it.

With development in the field of atomic physics, solid state physics, laser physics, and optical physics, a very reflective cavity with high quality factor as high as 4×10^{10}, leading to a photon lifetime in the cavity of $0.3s$ have been realized [27]. For microwave frequencies, detection of confined photons in high Q-cavities has been proposed and experimentally demonstrated [43]. They all exploit strong interaction between photons and atoms on the single photon level. Detection scheme for traveling photons have also been proposed, but in those proposals, the photon is absorbed by the detector and therefore the measurement is destructive. Proposals for detecting itinerant photons using coupled cavities have also been suggested, but they are limited by the trade-off between interaction strength and signal loss due to reflection [46].

2.3.1 Seeing a single photon without destroying it

Here, we present a novel approach to detect single microwave photons trapped in a high Q cavity, which is sensitive to only one or two number of photons [11]. The measurement

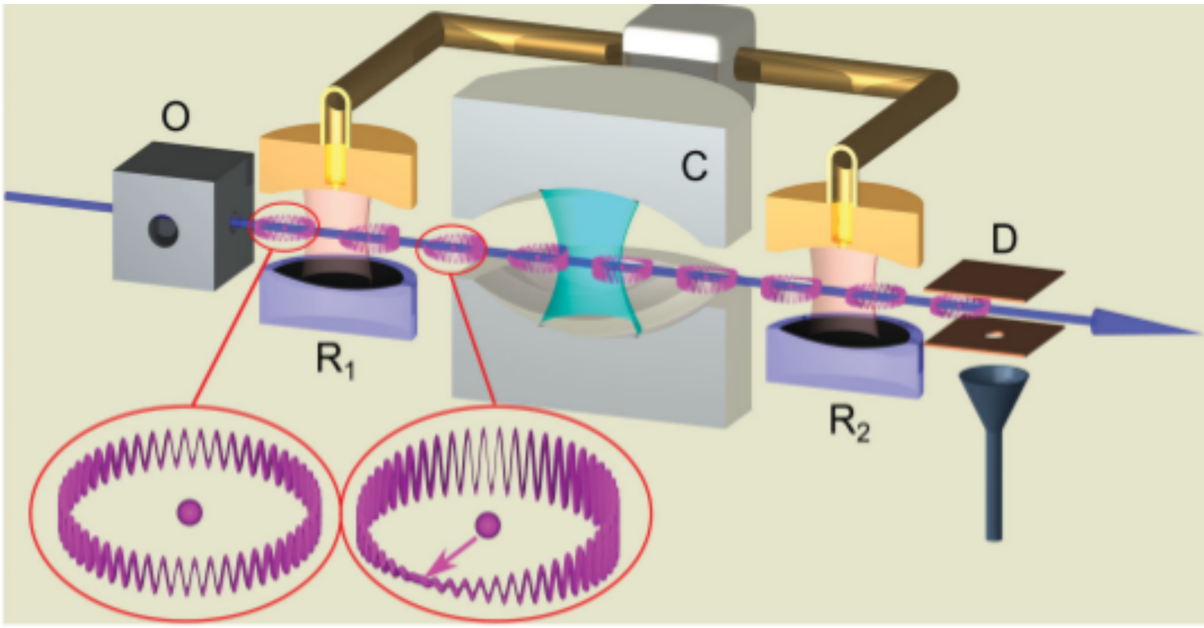


Figure 2.4: Experimental arrangement for detecting photons of a microwave cavity without destroying the field state [29].

scheme is shown in figure 2.4. Highly excited Rydberg atoms (probe) [32] interact with a single mode of a microwave cavity (measurement object). The experiment is designed such that the Rydberg atom is non-resonant with the microwave photons; i.e. there is a difference between the atomic transition frequency and the frequency of microwave photons. Since the atoms are non-resonant with the microwave photon, it implies that no photon is exchanged between them and the cavity modes, thus the measured field state is not destroyed while the atoms successively cross the cavity one by one. This shows that the measurement is indeed a QND one. The QND criteria property is enhanced when the atom-field coupling is adiabatic, that is the interaction causes the field to rise and lower slowly during the atom's trajectory through the cavity.

Although there is no exchange of photon during the interaction process, there is an observable phase shift in the atomic state which is proportional to the photon number in the cavity and measurable by the sensitive Ramsey method of interferometer [49].

2.3.2 The Mode Invisibility Technique

Method to improve the sensitivity of the QND scheme as discussed in subsection 2.3.1 have been proposed [43]. To achieve such sensitivity, atoms should be made to interact on resonance with the cavity modes; but at resonance, strongest interaction is achieved between an atom and the field mode. However this results in a large alteration of the cavity mode thereby jeopardizing the QND measurement criteria. We show how, by careful resonant interactions between two-level atoms and a single mode of the EM-field in a cavity, one can achieve a QND measurement scheme for light field with high sensitivity. The mode invisibility technique [44] described shortly, minimizes the effect of resonant interaction while leaving the information acquired from the field state unchanged.

Understanding the mode invisibility technique

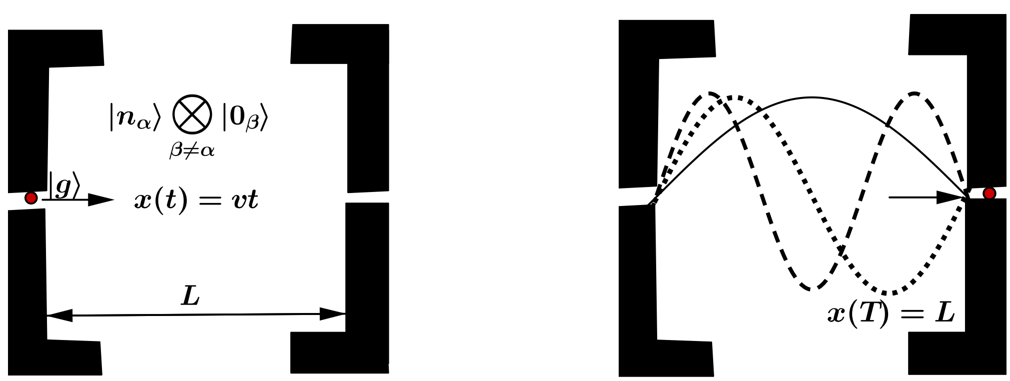


Figure 2.5: Scheme showing the motion of an atom through cavity modes.

When we subject single atoms to a motion at constant speed v (see figure 2.5) and let them interact at resonance with single even light mode trapped in a cavity of length L, it interacts basically in a non-destructive way with the relevant field mode [44]. The cavity mode function in equation (2.6) for standing waves

$$\mu_\gamma(x) \sim \sin(k_\gamma x) \quad k_\gamma = \frac{\pi\gamma}{L}, \tag{2.33}$$

has the property that its even harmonics ($\gamma = 2, 4, \cdots$) have odd parity and the odd harmonics ($\gamma = 3, 5, \cdots$) have even parity. Therefore, an atom flying through an even mode of the cavity (see dotted line in figure 2.5) will interact with light in that mode during the first half of its motion $x \in [0, L/2]$. However, whatever effect the atom's interaction had on the field mode would be canceled out during the second half of the atom's motion through the cavity $x \in [L/2, L]$. In this way, we could say the cavity modes are made invisible to the detector during the interaction process and thus leaves the field state and atomic state unchanged but for a phase shift on the atomic state. This is possible because the effective sign of the coupling to the cavity (λ times the spatial distribution of the mode) reverses half way through the flight path of the atom.

Definition 2.3.3 *The mode invisibility technique is a QND measurement technique that minimizes the effect of a resonant interaction between an atom and light field by canceling the largest contribution to the transition probability* [1].

Taking advantage of the mode invisibility technique, we proposed a QND measurement scheme that differentiates between two given light state in a nondestructive way [54, 9, 56]. We will discuss fully in chapter 3 how we have employed the mode invisibility technique to acquire information about Fock states, coherent states and squeezed state. More so we will see that it is possible to differentiate between a Fock state of parameter n from another Fock state of parameter $n + m$

2.4 Atomic Interferometery

An optical QND measurement scheme for light field assumes that the detector leaves the state of light it measures unperturbed but only acquires a phase shift due to its interaction with the light mode. The phase shift acquired by a detector interacting with a field state has been studied and observed to encode information about the field it measured. For instance, the phase shift was observed to encode information about the acceleration of a moving field [40], the temperature of a thermal field [38], and the intensity of a light field [11, 44] respectively. Two questions therefore arise:

[1] A detector traveling at constant speed through the cavity and made to probe only the even cavity modes would undo whatever effect it had on the field modes before it leaves the cavity.

1. How possible is it to measure this global phase shift?
2. How sensitive is the phase shift to the information of the field state?

The central goal of interferometry is to estimate phase shifts with the highest sensitivity [13, 41, 52]. There are several possible interferometric setup depending on what they are to be used for. Use of an atomic interferometer to measure the phase shift in a detector's state has been achieved [44, 40, 38]. In a measurement setting (see figure 3.1) to detect the Unruh effect at lower acceleration, the interferometric sensitivity was so strong, and it was shown that the measurement of the phase can in principle be used to detect the Unruh effect at a more feasible acceleration than the previous proposals [40]; similarly the same setup was proposed in principle to produce a high precision quantum thermometer [38]. We will give a full description of this measurement setup in chapter three and see how useful it is to measure the phase shift acquired by a detector (atom) interacting with a field state [44] during a QND measurement. Our measurement setup pursues the same purpose as discussed in [11, 43, 29], however we stand to improve the sensitivity of the phase acquired in the interferometeric measurement.

Chapter 3

Methodology

The discussion in chapter one encompasses the quantum measurement problem. A measurement process of an observable introduces a back action effect so that successive measurement of the same observable yields different results. In the most extreme case, the measured system gets destroyed. On the other hand, a measurement procedure which allows one to monitor a quantum system so that it is possible to perform a measurement on repeatable observables was also discussed. This measurement scheme is known as quantum non-demolition (QND) measurement. With a QND measurement scheme, it is possible to detect photons in a non-destructive way [43] and acquire measurable information about the light state in which the measured photon existed [44]. Such QND measurement involves an interaction between a two level atom and an electromagnetic field. In chapter 2, we discussed extensively the processes involved in atom-light field interaction. Unlike the QND measurement scheme in the reference [43] which incorporates a non-resonant interaction between the atom and light field so that the field is not detsroyed in the process, our measurement scheme [44] assumes a resonant interaction between the atom and light field so that we can acquire measurable information from the field state, however the field gets perturbed in the process jeopardizing the QND measurement scheme. We showed that by letting the detector (single atom) probe only the even cavity modes, it is possible to overcome the perturbation induced on the field due to its interaction with the detector, and still keep the information about the field state measurable. Such QND measurement scheme which allows one to measure only the even cavity modes of a light field so that the field is left unchanged while information is acquired is known as 'mode invisibility' measurement technique. Here in this chapter, we will give an application of the mode invisibility technique to the measurement of Fock states, coherent states, squeezed coherent states and squeezed vacuum states.

3.1 Mode Invisibility and Single Photon Detection

The first quantum system we will consider is an interaction between an atom, a two-level system and a Fock state trapped in a highly reflective cavity. Our measurement scheme is the atomic interferometer 3.1.

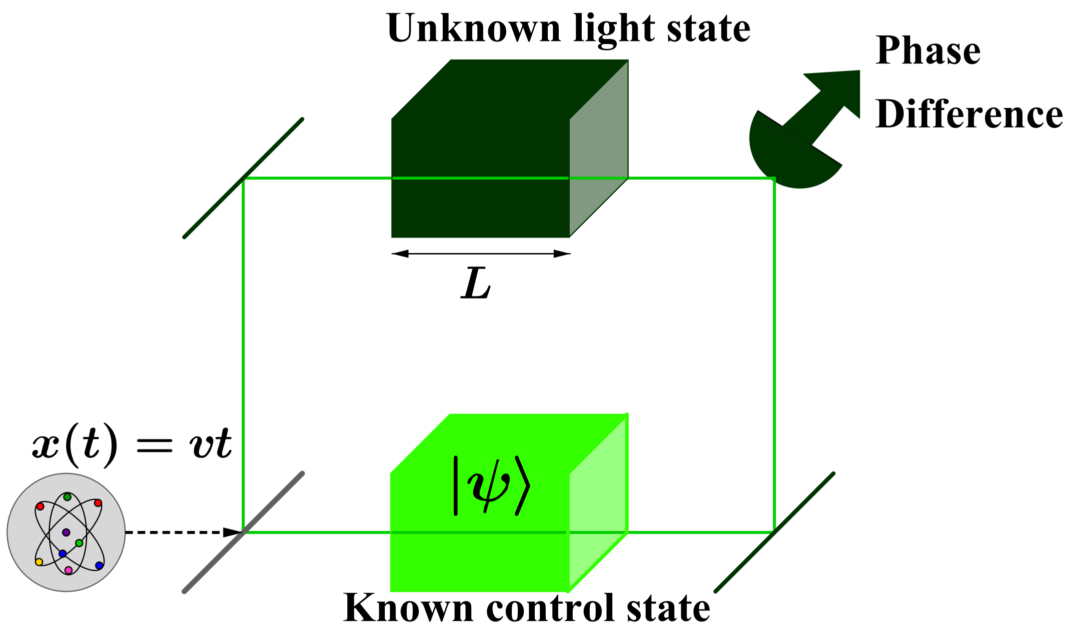


Figure 3.1: Schematic set-up for our QND scheme. An atom initially in its ground state is fired into the interferometer and made to interact with even cavity modes. The atom exits the cavity after an interaction time T leaving the field state unperturbed but for a phase shift.

Figure 3.1 shows the set-up for our measurement scheme. It is an atomic interferometer comprising of two optical cavities each of length L placed along the branches of the interferometer. In a mode of one cavity is stored an unknown Fock state (target) state while in a single mode of the second cavity is stored a known Fock (control) state. The aim of this measurement setup is to acquire information about the unknown Fock state using single atom as a probe. This kind of set-up has been considered previously as a way to detect the Unruh effect [40] and to measure, with great precision, the temperature of a hot source relative to some reference source [38]. In either case, the phase acquired by the detector in the two cavities differed and by looking at the phase difference after an interaction time T, we obtain information about the unknown field state. We propose to use this phase shift to perform a QND measurement of the unknown Fock state.

Single atoms initially in their ground state are sent into the atomic interferometer at constant speed. On entering the interferometer, this single atom gets split by the beam splitter into two partial beams and each partial beam travel independently along the branches of the interferometer. During their motion through the arms of the interferometer, each partial beam encounters a cavity with light field sustained in it. In one cavity is trapped an unknown Fock state which we want to probe and the second cavity contains a known field state which serve as a reference state. If our QND measurement technique is successful, the interaction would leave the field state unperturbed while the single atoms exit the cavity one by one but with a global phase factor. We use the atomic interferometer to compare the phase difference between the target state and control state. This phase difference reveals the information about the quantum state of light.

Modeling the measurement process

The joint atom-field system is given by the initial product state

$$|\psi(0)\rangle = |g\rangle \otimes |n_\alpha\rangle \bigotimes_{\beta\neq\alpha} |0_\beta\rangle^{1}, \tag{3.1}$$

where we consider the detector (single atom) to be a two level system (TLS) initially in the ground state $|g\rangle$. The cavity mode γ in which we want to probe is populated while we assume the rest cavity modes are less populated or vacuum. In terms of density operator, the state (3.1) reduces to the tensor product of the detector's density operator ρ_d and the field's density operator ρ_F respectively.

$$\rho(0) = \rho_d \otimes \rho_F = |g\rangle\langle g| \otimes |n_\alpha\rangle\langle n_\alpha|. \tag{3.2}$$

We bear in mind that the rest modes are vacuum state and photons could be created under the action of the field creation operator $\hat{a}_\kappa^\dagger$. That is, $\hat{a}_\kappa^\dagger|0_\kappa\rangle = |1_\kappa\rangle$. The appropraite interaction Hamiltonian we consider is the Unruh DeWitt interaction Hamiltonian [39, 41]

$$H_I = \sum_\kappa \frac{\lambda \sin(k_\kappa x(t))}{\sqrt{k_\kappa L}} \left(\sigma^+ a_\kappa^\dagger e^{\mathrm{i}(\Omega+\omega_\kappa)t} + \sigma^+ a_\kappa e^{-\mathrm{i}(\omega_\kappa-\Omega)t} + \sigma^- a_\kappa^\dagger e^{\mathrm{i}(\omega_\kappa-\Omega)t} + \sigma^- a_\kappa e^{-\mathrm{i}(\Omega+\omega_\kappa)t}\right). \tag{3.3}$$

We work in perturbation theory in the coupling strength λ where we have assumed that λ is a small parameter. Under the interaction Hamiltonian (3.3), the evolution of our quantum system from a time $t = 0$ to a time $t = T$ is given by the unitary operator

$$U(T,0) = \mathcal{T}\exp\left[\frac{1}{\mathrm{i}}\int_0^T dt H_I(t)\right]. \tag{3.4}$$

The different order contributions to (3.4) will be

$$\begin{aligned} U(0,T) = & \underbrace{(-\mathrm{i})\int_0^T dt_1 H_I(t_1)}_{U^{(1)}} \\ & + \underbrace{(-\mathrm{i})^2\int_0^T dt_1 \int_0^{t_1} dt_2\, H_I(t_1)H_I(t_2)}_{U^{(2)}} + \ldots \\ & + \underbrace{(-\mathrm{i})^n\int_0^T dt_1 \ldots \int_0^{t_{(n-1)}} dt_n\, H_I(t_1)\ldots H_I(t_n)}_{U^{(n)}} \end{aligned} \tag{3.5}$$

Therefore the joint quantum system evolve after an interaction time T to a final state given as

$$|\psi(T)\rangle = U(0,T)|\psi(0)\rangle, \tag{3.6}$$

where to different orders in the perturbation theory, equation (3.6) gives

$$|\psi(T)\rangle = |\psi(0)\rangle + \left|\psi_T^{(1)}\right\rangle + \left|\psi_T^{(2)}\right\rangle + \mathcal{O}(\lambda^3),$$

with

$$|\psi^n\rangle = U^{(n)}|\psi(0)\rangle. \tag{3.7}$$

Weak Adiabatic Approximation

For a QND measurement, we require that the joint atom-field state (3.6) remains in the same state after the the measurement and information has been acquired from the system. This implies that

- The probability that the whole system remains in the same state is approximately unity, i.e,

$$\left|\langle\psi(0)|U(0,T)|\psi(0)\rangle\right|^2 \approx 1. \tag{3.8}$$

- Under this assumption the final state of the system would be very approximately equal to the initial state except for a global dynamical phase

$$|\psi(T)\rangle = U(0,T)|\psi(0)\rangle \approx e^{i\gamma}|\psi(0)\rangle, \tag{3.9}$$

where γ is the phase factor to be determined. In particular the state of the measuring device (the single atom) remains the same before and after it exits the cavity except for a dynamical phase. In the regime where the coupling between atom and field mode is made weak such that the state of the field is non-significantly perturbed, it is expected that the atom holds information about the field state. We call this condition the 'weak adiabatic assumption'.

We will show that our mode invisibility technique yields a significant measurement of the relevant field mode and leaves the Fock state unperturbed. To ensure that this is the case, we will compute its transition probability into a state different from the original configuration, and require this quantity to be very small. This is equivalent to demanding that the probability that the system remains in the same initial state after the atom crosses the cavity is approximately unity.

Transition probability

The initial joint atom-field system evolves according to the equation

$$\rho^T = \mathrm{U(0,T)}\,\rho(0)\,\mathrm{U(0,T)}^\dagger \tag{3.10}$$

We will drop the $(0,T)$. The reduced density operator for the atom state after an interaction time is

$$\rho_p^T = \mathrm{Tr}_F[\mathrm{U}^{(1)}\,\rho(0)\,\mathrm{U}^{(1)\dagger}] \tag{3.11}$$

where we have considered the the first order term to the evolution operator (3.6) and we perform a trace on the composite system over the field state. From equation (3.5), we define the first order term to the evolution operator to be

$$U^{(1)} = -\mathrm{i}\lambda \int_0^T \mathrm{dt} \sum_\kappa \Big[\sigma^+ a_\kappa^\dagger e^{\mathrm{i}(\Omega+\omega_\kappa)t} + \sigma^+ a_\kappa e^{-\mathrm{i}(-\Omega+\omega_\kappa)t} + \sigma^- a_\kappa^\dagger e^{\mathrm{i}(-\Omega+\omega_\kappa)t} + \sigma^- a_\kappa e^{-\mathrm{i}(\Omega+\omega_\kappa)t}\Big] \times \frac{\sin(k_\kappa x(t)).}{\sqrt{k_\kappa L}} \tag{3.12}$$

For notational convenience, we rewrite $U^{(1)}$ as

$$\mathrm{U}^{(1)} = -\mathrm{i}\lambda \sum_\kappa \Big(\sigma^+ a_\kappa^\dagger X_{+,\kappa} + \sigma^- a_\kappa X_{+,\kappa}^* + \sigma^- a_\kappa^\dagger X_{-,\kappa} + \sigma^+ a_\kappa X_{-,\kappa}^*\Big), \tag{3.13}$$

where

$$X_{\pm,\beta} = \frac{1}{\sqrt{k_\beta L}} \int_0^{L/v} \mathrm{d}t\, e^{\mathrm{i}(\pm\Omega+\omega_\beta)t} \sin[k_\beta x(t)]. \tag{3.14}$$

We need to evaluate the operator

$$\rho^T = \mathrm{U}^{(1)\dagger}\, \rho(0)\, \mathrm{U}^{(1)} \equiv \underbrace{\Big(U^{(1)}|\psi(0)\rangle\Big)}_{A}\underbrace{\Big(\langle\psi(0)|U^{(1)\dagger}\Big)}_{B} \tag{3.15}$$

Inserting equation (3.13) and noting that $\sigma^-|g\rangle = 0$ and $\langle e|\sigma^+ = 0$,

$$\begin{aligned} A =& -\mathrm{i}\lambda \sum_\kappa \Big(\sigma^+ a_\kappa^\dagger X_{+,\kappa} + \sigma^+ a_\kappa X_{-,\kappa}^*\Big)|g\rangle \otimes |n_\alpha\rangle, \\ B =& \mathrm{i}\lambda \sum_\beta \langle g| \otimes \langle n_\alpha| \Big(\sigma^- a_\beta X_{+,\beta}^* + \sigma^- a_\beta^\dagger X_{-,\beta}\Big). \end{aligned}$$

There are two ways to evaluate equation (3.15). Its either we evaluate A first and apply the operators in B on the corresponding result, or we do vice versa. We will first evaluate A by taking each term explicitly. Two cases arise, when $\kappa = \alpha$ and $\kappa \neq \alpha$

$$\begin{aligned} \sum_\kappa X_{+,\kappa}\sigma^+ a_\kappa^\dagger |g\rangle \otimes |n_\alpha\rangle =& \sum_{\kappa\neq\alpha} X_\kappa \sigma^+ a_\kappa^\dagger |g\rangle \otimes |n_\alpha\rangle + X_\alpha \sigma^+ a_\alpha^\dagger |g\rangle \otimes |n_\alpha\rangle \\ &= |e\rangle \Big(\sum_{\kappa\neq\alpha} X_{+,\kappa} |n_\alpha 1_\kappa\rangle + X_\alpha \sqrt{n+1}|n+1_\alpha\rangle\Big) \\ \sum_\kappa X_{-,\kappa}^* \sigma^+ a_\kappa |g\rangle \otimes |n_\alpha\rangle =& |e\rangle \Big(X_{-,\alpha}^* \sqrt{n}|n-1_\alpha\rangle\Big) \end{aligned}$$

and we get

$$A = -\mathrm{i}\lambda|e\rangle\Big(\sum_{\kappa\neq\alpha} X_{+,\kappa}|n_\alpha 1_\kappa\rangle + X_\alpha\sqrt{n+1}|n+1_\alpha\rangle + X^*_{-,\alpha}\sqrt{n}|n-1_\alpha\rangle\Big).$$

This reduces to

$$\begin{aligned}\rho^T =& \lambda^2|e\rangle\langle g| \otimes \Bigg[\sum_{\kappa\neq\alpha} X_{+,\kappa}|n_\alpha 1_\kappa\rangle\langle n_\alpha 0_\kappa| + X_{+,\alpha}\sqrt{n+1}|(n+1)_\alpha\rangle\langle n_\alpha| \\ &+ X^*_{-,\alpha}\sqrt{n}|(n-1)_\alpha\rangle\langle n_\alpha|\Bigg] \times \sum_\beta \left(\sigma^- a_\beta X^*_{+,\beta} + \sigma^- a^\dagger_\beta X_{-,\beta}\right) \\ \rho^T =& \rho' T + \rho''^T,\end{aligned}$$

where

$$\begin{aligned}\rho'^T =& \lambda^2|e\rangle\langle g| \otimes \Bigg[\sum_{\kappa\neq\alpha} X_{+,\kappa}|n_\alpha 1_\kappa\rangle\langle n_\alpha\ 0_\kappa| + X_{+,\alpha}\sqrt{n+1}|(n+1)_\alpha\rangle\langle n_\alpha| \\ &+ X^*_{-,\alpha}\sqrt{n}|(n-1)_\alpha\rangle\langle n_\alpha|\Bigg] \sum_\beta X^*_{+,\beta}\sigma^- a_\beta \\ \rho''^T =& \lambda^2|e\rangle\langle g|\Bigg[\sum_{\gamma\neq\alpha} X_{+,\gamma}|n_\alpha 1_\gamma\rangle\langle n_\alpha\ 0_\gamma| + X_{+,\alpha}\sqrt{n+1}|(n+1)_\alpha\rangle\langle n_\alpha| \\ &+ X^*_{-,\alpha}\sqrt{n}|(n-1)_\alpha\rangle\langle n_\alpha|\Bigg] \sum_\beta X_{-,\beta}\sigma^- a^\dagger_\beta.\end{aligned}$$

Now we apply the operators from the right. The evaluation becomes complicated here. Taking each term explicitly and considering terms when $(\beta \neq \kappa, \alpha)$, $(\beta = \kappa \neq \alpha), (\beta =$

$\alpha, \beta \neq \kappa$).

$$\rho'^T = \lambda^2 |e\rangle\langle g| \otimes \left[\sum_{\kappa\neq\alpha} X_{+,\kappa} |n_\alpha 1_\kappa\rangle\langle n_\alpha\ 0_\kappa| + X_{+,\alpha}\sqrt{n+1}|(n+1)_\alpha\rangle\langle n_\alpha| \right.$$
$$\left. + X^*_{-,\alpha}\sqrt{n}|(n-1)_\alpha\rangle\langle n_\alpha| \right] \sum_\beta X^*_{+,\beta}\sigma^- a_\beta$$
$$= \lambda^2 |e\rangle\langle e| \left[\sum_{\beta\neq\kappa,\alpha}\sum_{\kappa\neq\alpha} X_{+,\kappa}X^*_{+,\beta}|n_\alpha\ 1_\kappa\ 0_\beta\rangle\langle n_\alpha\ 0_\kappa\ 1_\beta| + \sum_{\kappa\neq\alpha}|X_{+,\kappa}|^2|n_\alpha\ 1_\kappa\rangle\langle n_\alpha 1_\kappa| \right.$$
$$+ \sum_{\kappa\neq\alpha} X_{+,\kappa}X^*_{+,\alpha}\sqrt{n+1}|n_\alpha\ 1_\kappa\rangle\langle (n+1)_\alpha\ 0_\kappa| + \sum_{\beta\neq\alpha} X_{+,\alpha}X^*_{+,\beta}\sqrt{n+1}|(n+1)_\alpha\ 0_\beta\rangle\langle n_\alpha\ 1_\beta|$$
$$+ |X_{+,\alpha}|^2(n+1)|(n+1)_\alpha\rangle\langle (n+1)_\alpha| + \sum_{\beta\neq\alpha} X^*_{-,\alpha}X^*_{+,\beta}\sqrt{n}|(n-1)_\alpha\ 0_\beta\rangle\langle n_\alpha\ 1_\beta|$$
$$\left. + X^*_{-,\alpha}X^*_{+,\alpha}\sqrt{n(n+1)}|(n-1)_\alpha\rangle\langle (n+1)_\alpha| \right].$$

Similarly for terms in $\sigma^- a^\dagger_\beta X_{-,\beta}$, we have

$$\rho''^T = \lambda^2 |e\rangle\langle g| \left[\sum_{\gamma\neq\alpha} X_{+,\gamma}|n_\alpha 1_\gamma\rangle\langle n_\alpha\ 0_\gamma| + X_{+,\alpha}\sqrt{n+1}|(n+1)_\alpha\rangle\langle n_\alpha| \right.$$
$$\left. + X^*_{-,\alpha}\sqrt{n}|(n-1)_\alpha\rangle\langle n_\alpha| \right] \sum_\beta X_{-,\beta}\sigma^- a^\dagger_\beta$$
$$= \lambda^2 |e\rangle\langle e| \left[\sum_{\gamma\neq\alpha} X_{+,\gamma}X_{-\alpha}\sqrt{n}|n_\alpha\ 1_\gamma\rangle\langle (n-1)_\alpha\ 0_\gamma| + X_{+,\alpha}X_{-\ \alpha}\sqrt{n(n+1)}|(n+1)_\alpha\rangle\langle (n-1)_\alpha| \right.$$
$$\left. + |X_{-\alpha}|^2 n|(n-1)_\alpha\rangle\langle (n-1)_\alpha| \right].$$

Therefore back to equation (3.11),

$$\rho^T_d = \mathrm{Tr}_F[\rho^T] = \mathrm{Tr}_F[\rho'^T] + \mathrm{Tr}_F[\rho''^T]$$
$$= \lambda^2 |e\rangle\langle e| \otimes \left[|X_{-,\alpha}|^2 n + |X_{+,\alpha}|^2 n + \sum_\beta |X_{+,\beta}|^2 \right]$$

We are interested in finding the excitation transition probability of the atom system after the interaction time T, this is given by

$$P_{|e\rangle}(T) = \langle e| \,\mathrm{Tr}_F[\mathrm{U}^{(1)\dagger}\,\rho(0)\,\mathrm{U}^{(1)}]|e\rangle$$
$$P_{|e\rangle}(T) = \lambda^2 \left[|X_{-,\alpha}|^2 n + |X_{+,\alpha}|^2 n + \sum_{\beta} |X_{+,\beta}|^2 \right] \tag{3.16}$$

We will briefly describe each contributions to the excitation probability (3.16). The first term is the typical rotating wave contribution that comes from exciting the single atom (detector absorbing a photon from the field mode α) and destroying a single photon from the field. The second corresponds to the atom getting excited and emitting a photon to the mode α. This is the typical counter-rotating contribution. The third term corresponds to the vacuum fluctuations due to the rest of the modes (see for instance [58, 33]).

If we send an atom through a cavity of length L at a constant speed v, the time it spends inside the cavity is $T = L/v$. Taking into account that $k_\beta = \beta\pi/L$, the integrals in equation (3.16) results in

$$X_{\pm,\beta} = \frac{1}{\sqrt{\pi\beta}} \int_0^{L/v} \mathrm{d}t\, e^{\mathrm{i}(\pm\Omega+\omega_\beta)t} \sin\left(\frac{\beta\,\pi}{L} vt\right), \tag{3.17}$$

and can be readily solved, giving

$$X_{\pm,\beta} = \frac{\left[e^{\mathrm{i}\frac{L}{v}(\omega_\beta \pm \Omega)}(-1)^\beta - 1\right] Lv\sqrt{\beta\,\pi}}{(\beta\,\pi v)^2 - L^2(\omega_\beta \pm \Omega)^2} \tag{3.18}$$

We remind the reader that detectors are made to interact with the cavity mode at resonance, that is $\omega_\alpha = \Omega$ where the frequency of the mode coincides with the atomic gap. Evident in this expression (3.18), it is easily seen that the term which is probable to yield the largest contribution to the excitation probability (3.16) thereby jeopardizing the hypothesis (3.8) is the first term having the typical rotating wave term $X_{-,\alpha}$. All the counter-rotating contributions $X_{+,\beta}$ for all β are damped by the sum of frequency of the atomic gap and that of the cavity modes; and the off-resonant rotating-wave contributions $X_{-,\beta}$ for all $\omega_\beta \neq \Omega$ are damped by the difference between frequency of the atomic gap and the frequency of the cavity modes respectively. To achieve an excitation probability that is approximately zero i.e. $P_{|e\rangle}(T) \ll 1$, the rotating wave term coming from a resonant interaction between the atom and field mode has to be suppressed; however it is necessary we keep the resonant interaction if we must achieve our research goal– improving the phase shift acquired in the atomic state.

The mode invisibility technique eliminates such setbacks. From our interaction model (**??**), it is easily seen that the multiples of even field modes (even harmonics $\beta = 2, 4, \dots$) have the property that their spatial function is of odd parity. For example, for $X_{-,\alpha}$ (3.18) (corresponding to the resonance: $\omega_\alpha = \Omega$) we get

$$X_{-,\alpha} = \frac{\left[(-1)^\alpha - 1\right] L}{(\alpha\,\pi)^{3/2}\, v}. \tag{3.19}$$

.

- For $\alpha = 1, 3, \dots$ (odd harmonic of the relevant cavity mode), equation (3.19) would have a high value which violates our assumption 3.1.
- For $\alpha = 2, 4, \dots$ (even harmonics of the relevant cavity mode), then the contribution $X_{-,\alpha} = 0$.

Hence provided the highly excited state we wish to probe is prepared in one of those even modes, the mode is invisible to the atom (at leading order in perturbation theory) and therefore will not perturb it – this is the mode invisibility technique. Employing the mode invisibility technique, we achieved for a set of parameters we considered, an excitation transition probability of approximately 10^{-20}. This value becomes smaller for any other set of parameters considered.

Provided $P_{|e\rangle}(T) \ll 1$, we are certain that the atom does not alter the state of field mode it probes. The atom-field system evolve to almost the same state but for a phase factor according to the relation

$$|\psi(T)\rangle = \mathrm{U}(0, \mathrm{T})\,|\psi(0)\rangle \approx e^{\mathrm{i}\eta}|\psi(0)\rangle, \tag{3.20}$$

The next step therefore is to evaluate the phase shift η on the atomic state due to this atom-field interaction. Before we proceed, we need to evaluate the leading order to the phase value. The first contribution to the phase would come from $|\psi_T^{(2)}\rangle$, given by

$$|\psi_T^{(2)}\rangle = U^{(2)}|\psi(0)\rangle \tag{3.21}$$

where

$$\begin{aligned}\mathrm{U}^{(2)} = -\lambda^2 \sum_\kappa \sum_\beta \int_0^{\frac{L}{v}} dt \int_0^t dt' \Bigg(&\sigma^-\sigma^+ a_\kappa^\dagger a_\beta^\dagger e^{\mathrm{i}(\omega_\kappa - \Omega)t} e^{\mathrm{i}(\omega_\beta + \Omega)t'} \\ + \sigma^-\sigma^+ a_\kappa^\dagger a_\beta e^{\mathrm{i}(\omega_\kappa - \Omega)t} e^{-\mathrm{i}(\omega_\beta - \Omega)t'} &+ \sigma^-\sigma^+ a_\kappa a_\beta e^{-\mathrm{i}(\omega_\kappa + \Omega)t} e^{-\mathrm{i}(\omega_\beta - \Omega)t'} \\ + \sigma^-\sigma^+ a_\kappa a_\beta^\dagger e^{-\mathrm{i}(\omega_\kappa + \Omega)t} e^{\mathrm{i}(\omega_\beta + \Omega)t'} \Bigg) &\frac{\sin(k_\kappa x(t))}{\sqrt{k_\kappa L}} \frac{\sin(k_\beta x(t'))}{\sqrt{k_\beta L}}.\end{aligned} \tag{3.22}$$

Therefore, following the same procedure as before,

$$
\begin{aligned}
&U^{(2)}|\psi(0)\rangle = -\lambda^2 \int_0^{\frac{L}{v}} dt \int_0^t dt' |g\rangle \\
&\Bigg[\sum_{\kappa\neq\beta,\alpha} \sum_{\beta\neq\alpha} \frac{\sin(k_\kappa x(t))}{\sqrt{k_\kappa L}} \frac{\sin(k_\beta x(t'))}{\sqrt{k_\beta L}} e^{\mathrm{i}(\omega_\kappa-\Omega)t} e^{\mathrm{i}(\omega_\beta+\Omega)t'} |n_\alpha, 1_\beta, 1_\kappa\rangle \\
&+ \frac{\sin(k_\alpha x(t))\sin(k_\alpha x(t'))}{k_\alpha L} e^{-\mathrm{i}(\omega_\alpha+\Omega)(t+t')} \sqrt{n(n-1)} |(n-2)_\alpha\rangle \\
&+ \sum_{\beta\neq\alpha} \frac{\sin(k_\alpha x(t))}{\sqrt{k_\alpha L}} \frac{\sin(k_\beta x(t'))}{\sqrt{k_\beta L}} e^{\mathrm{i}(\omega_\beta-\Omega)t} e^{\mathrm{i}(\omega_\alpha+\Omega)t'} \sqrt{n+1} |(n+1)_\alpha \ 1_\beta\rangle \\
&+ \sum_{\beta\neq\alpha} \frac{\sin(k_\beta x(t))}{\sqrt{k_\beta L}} \frac{\sin(k_\alpha x(t'))}{\sqrt{k_\alpha L}} e^{\mathrm{i}(\omega_\beta-\Omega)t} e^{-\mathrm{i}(\omega_\alpha-\Omega)t'} \sqrt{n} |(n-1)_\alpha 1_\beta\rangle \\
&+ \sum_{\beta\neq\alpha} \frac{\sin(k_\alpha x(t))\sin(k_\alpha x(t'))}{k_\alpha L} e^{\mathrm{i}(\omega_\alpha-\Omega)t} e^{\mathrm{i}(\omega_\alpha+\Omega)t'} \sqrt{(n+1)(n+2)} |(n+2)_\alpha\rangle \\
&+ \sum_{\beta\neq\alpha} \frac{\sin(k_\alpha x(t))}{\sqrt{k_\alpha L}} \frac{\sin(k_\beta x(t'))}{\sqrt{k_\beta L}} e^{-\mathrm{i}(\omega_\beta-\Omega)t} e^{-\mathrm{i}(\omega_\alpha-\Omega)t'} \sqrt{n} |(n-1)_\alpha \ 1_\beta\rangle \\
&+ \sum_{\beta\neq\alpha} \frac{\sin(k_\beta x(t))\sin(k_\beta x(t'))}{k_\beta L} e^{\mathrm{i}(\omega_\beta-\Omega)t} e^{i(\omega_\beta+\Omega)t'} \sqrt{2} |n_\alpha \ 2_\beta\rangle \\
&+ \frac{\sin(k_\alpha x(t))\sin(k_\alpha x(t'))}{k_\alpha L} e^{-\mathrm{i}(\omega_\alpha+\Omega)(t-t')} (n+1) |n_\alpha\rangle \\
&+ \frac{\sin(k_\alpha x(t))\sin(k_\alpha x(t'))}{k_\alpha L} e^{\mathrm{i}(\omega_\alpha-\Omega)(t-t')} n |n_\alpha\rangle \\
&+ \sum_{\beta\neq\alpha} \frac{\sin(k_\beta x(t))\sin(k_\beta x(t'))}{k_\beta L} e^{-\mathrm{i}(\omega_\beta+\Omega)(t-t')} |n_\alpha\rangle \Bigg].
\end{aligned}
$$

By grouping these terms into the two including terms that are proportional to the initial state and terms that are orthogonal, we get

$$
|\psi^{(2)}(T)\rangle = \frac{-\lambda^2}{k_\alpha L} \left(C^*_{+,\alpha}(n+1) + C_{-,\alpha} n + \sum_{\beta\neq\alpha} C^*_{+,\beta} \right) |\psi(0)\rangle + |\psi(T)\rangle_\perp, \tag{3.23}
$$

where for notational convenience,

$$C_{\pm,\beta} = \int_0^T \mathrm{d}t \int_0^t \mathrm{d}t' \; e^{\mathrm{i}(\omega_\beta \pm \Omega)(t-t')} \sin[k_\beta x(t)] \sin[k_\beta x(t')].$$

$|\psi(T)\rangle_\perp$ in (3.23) is the second order contribution that is orthogonal to the initial state and which is irrelevant to the computation of the phase calculation and small enough for all our assumptions to hold. Its magnitude have an impact on the visibility of the fringes in the interferometric experiment as we will discuss later in the chapter 3; however, we show that we are able to compute the effect of this term and keep it under control.

The phase shift on atomic state

Having obtained the leading order to the phase factor, the next thing to do is estimate the phase acquired by the atom which we will obtain from the expression (3.9)

$$|\psi(T)\rangle = U(0,T)|\psi(0)\rangle \approx e^{i\gamma}|\psi(0)\rangle,$$

We can obtain an expression for η by multiplying the left and right hand side of the equation and taking their natural logarithm

$$\langle\psi(0)|e^{\mathrm{i}\eta}|\psi(0)\rangle \approx \langle\psi(0)|\,\mathrm{U}(0,\mathrm{T})\,|\psi(0)\rangle$$

We can easily extract the phase η by taking the natural logarithm of the two sides of the equation

$$\eta = -i\ln\langle\psi(0)|\,\mathrm{U}(0,\mathrm{T})\,|\psi(0)\rangle \tag{3.24}$$

To the leading order in phase factor,

$$\mathrm{U}(0,\mathrm{T})\,|\psi(0)\rangle = |\psi(0)\rangle + U^{(2)}|\psi(0)\rangle$$

where we have neglected higher terms.

$$\eta = -i\ln\Big[1 - \langle\psi(0)|\,\mathrm{U}^{(2)}\,|\psi(0)\rangle\Big] \tag{3.25}$$

Therefore substituting (3.23), we have the phase acquired by an atom crossing a cavity with Fock state sustained in it to be

$$\eta = -i\ln\left(1 - \lambda^2\left[n\frac{C_{-,\alpha}}{k_\alpha L} + \sum_{\beta\neq\alpha}\frac{C^*_{+,\beta}}{k_\beta L} + (n+1)\frac{C^*_{+,\alpha}}{k_\alpha L}\right]\right). \tag{3.26}$$

Note that η is not a real number (so strictly speaking, it is not technically a phase) because the second order correction has a contribution orthogonal to the initial state, as seen in (3.23). The phase factor γ to be determined is given by the real part of η

$$\gamma = \mathrm{Re}(\eta) \tag{3.27}$$

To obtain the leading-order approximation for the phase (3.27), let us first assume that we work in a regime where $|\eta| \ll 1$, which is consistent with our approximations. If that is the case we can expand the exponential term in (3.20), yielding

$$e^{i\eta} \sim 1 + \mathrm{i}\eta = 1 - \lambda^2 \left[n \frac{C_{-,\alpha}}{k_\alpha L} + \sum_{\beta \neq \alpha} \frac{C^*_{+,\beta}}{k_\beta L} + (n+1) \frac{C^*_{+,\alpha}}{k_\alpha L}, \right] \tag{3.28}$$

and so the phase $\gamma = \mathrm{Re}(\eta)$ will be given by

$$\gamma \simeq -\mathrm{Im}\left(\lambda^2 \left[n \frac{C_{-,\alpha}}{k_\alpha L} + \sum_{\beta \neq \alpha} \frac{C^*_{+,\beta}}{k_\beta L} + (n+1) \frac{C^*_{+,\alpha}}{k_\alpha L} \right] \right). \tag{3.29}$$

The term $C_{\pm,\beta}$ can be easily evaluated. We obtained a simplified analytical solution:

$$C_{\pm,\beta} = -\frac{\mathrm{i}L(\pm\Omega + \pi\beta c/L)}{2v\Big[(\pi\beta v/L)^2 - (\pm\Omega + \pi\beta c/L)^2\Big]} + \frac{(\pi\beta v/L)^2 \Big[(-1)^\beta e^{\mathrm{i}(\pm\Omega + \pi\beta c/L)L/v} - 1\Big]}{\Big[(\pi\beta v/L)^2 - (\pm\Omega + \pi\beta c/L)^2\Big]^2}.$$

The mode invisibility technique assumes a resonant interaction between atom and the relevant field mode. If the transition frequency of the atom is prepared in the α-th mode of the cavity ($\Omega = \frac{\alpha\pi}{L}c$), therefore, $C_{\pm,\beta}$ can be expressed as

$$C_{\pm,\beta} = L^2 \left(\frac{(\beta\pi v)^2 \left[(-1)^\beta e^{\mathrm{i}[\pi\frac{c}{v}(\beta\pm\alpha)]} - 1\right]}{\Big[(\beta\pi v)^2 - \pi^2 c^2 (\beta \pm \alpha)^2\Big]^2} - \frac{\mathrm{i}\pi c(\beta \pm \alpha)}{2v\Big[(\beta\pi v)^2 - \pi^2 c^2 (\beta \pm \alpha)^2\Big]} \right). \tag{3.30}$$

At resonance, $\beta = \alpha = 2, 4, \cdots$ where β is the field mode that was probed. Therefore it is easy to conclude from that $C_{-,\alpha} = 0$ while

$$C_{+,\alpha} = \frac{L^2}{\pi^2\alpha^2 v^2} \frac{e^{i(2\pi\alpha\frac{c}{v})} - 1}{\left(1 - 4\frac{c^2}{v^2}\right)^2} - \frac{\mathrm{i}c}{\pi\alpha v} \frac{L^2}{v^2 - 4c^2}. \tag{3.31}$$

From equation (3.27), we see that in the non-relativistic regime $v \ll c$, if we keep the ratio λ/Ω constant, the measured phase shift γ will be invariant under changes of length L of the cavity. To present result for a particular case, we consider a cavity length $L \sim 1$ μm and λ in the range $(10^{-6} - 10^{-4})\Omega$ as in a typical quantum optical settings [28]. For our measurement setting, the atomic gap is set on resonance with the lower even harmonic of the cavity. We trap an unknown Fock state we wish to measure in the relevant cavity mode and a known Fock state in an additional cavity mode as a reference and setup an atomic interferometer as shown in figure 3.1.

The first measurement approximation (3.9) will hold for any measurement to be carried out in the system provided that $P_{|e\rangle}(T)$ in equation (3.16) is approximately zero. To achieve this, we send the probe into the interferometer at a constant speed $v = 10^3 m/s$ through the cavity. Due to the mode invisibility effect, we find that the transition probability – even for a relatively strong coupling $\lambda = 10^{-4}\Omega$, remains below 10^{-20} for our choice of parameters, consistent with our perturbation approach and the assumption (3.9).[2] Hence the detector does not significantly modify the state of the field during interaction (QND requirement) but a phase shift is observed in the atomic state. We can proceed then to evaluate the phase shift on atomic state (see equation (3.27)).

A good question may arise thus: what happens when there is a slight detuning from resonance (i.e. if $\omega_\alpha - \Omega = \delta$)? From (3.18) it is easy to see that the largest contribution to the amplitude of transition probability for small δ when $\beta = 2n$ becomes

$$X_{-,\beta} \simeq \frac{\mathrm{i}L^2\delta\sqrt{2\pi n}}{(2\pi n v)^2 - L^2\delta^2} \simeq \frac{\mathrm{i}L^2\delta}{v^2\sqrt{(2\pi n)^3}} + \mathcal{O}(\delta^2), \tag{3.32}$$

We require the square of the absolute value to $X_{-,\beta}$ for the transition probability

$$|X_{-,\beta}|^2 \simeq \frac{L^4\delta^2}{v^4(2\pi n)^3} \simeq 0, \tag{3.33}$$

We see that the factor that contributes more to the transition probability also cancels out for small detuning. We may then conclude that the mode invisibility technique is robust against a slight detuning from resonance.

In figure 3.2, we show a plot of the phase shift γ as a function of number of photons contained in the Fock state. As it is expected [58, 43], we see that the phase shift is proportional to the photon number n. For very few photons, the phase response is linear with

[2]This is similar to the rate of response in the case of vacuum fluctuations, since the mode that is populated has been made invisible to the detector, as per the technique spelled out in section 2.3.2.

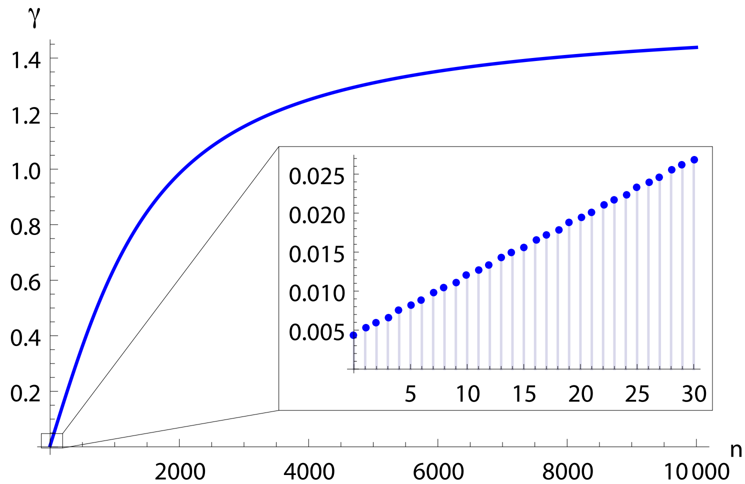


Figure 3.2: Quantum nondemolition measurement of Fock state of light by atomic interferometry. We see that the phase shift on atomic state depends linearly on the photon number in the Fock state of light and we are able to measure Fock states of very few photons.

the number of photons. The curve deviates from linearity for large numbers of photons, reducing the resolution of the setting. This implies that our setting is better suited to distinguish between states whose photon numbers differ by large amounts; it is more difficult to distinguish a state containing $n+1$ photons from a state containing n photons.

We remind the reader that our calculations are made in the context of perturbation theory in the coupling strength λ. We define the perturbation estimator as

$$\left(\frac{L}{v}\right)\left(\frac{\lambda}{\Omega}\right)n \ll 1 \tag{3.34}$$

when this criterion is satisfied, transition probabilities are small. For microwave cavities with $\lambda = 10^{-3}\Omega$, this perturbation estimator gives a value $10^{-9}n \ll 1$; for optical cavities the bound is even tighter. We therefore expect that the phase value γ could extend to states of more photons without major problems. The visibility factor, however, would decrease substantially outside of the perturbation regime so there will always be a trade-off imposed by the interferometric sensibility for cases where we want to probe a high enough number of photons.

Phase Resolution

This is the smallest signal that can be resolved in an atomic interferometer. Since the phase shift on atomic state reveals information about the number of photons in a cavity mode, a next step to consider is to differentiate between two Fock states having n and $n+m$ photon numbers respectively. This can be achieved by looking at the difference in

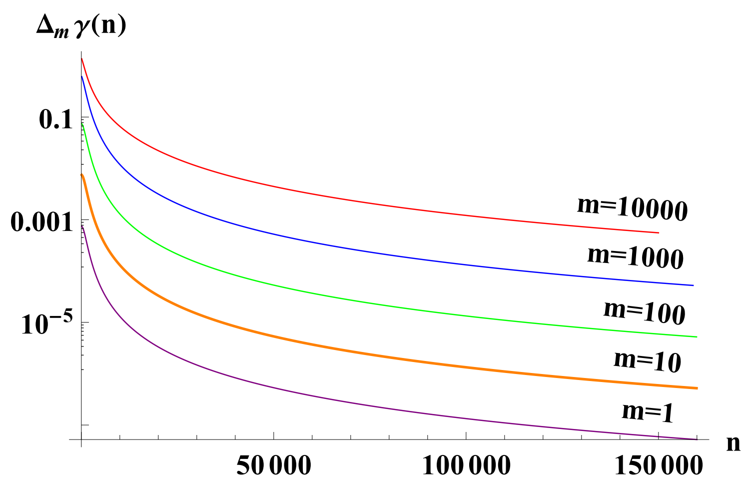


Figure 3.3: Phase resolution required to distinguish a Fock state containing $n+m$ photons from a state containing n photons. To show the trend in the behavior of this magnitude it is interesting to plot it for n above the threshold where the visibility would make the measurement experimentally challenging ($\sim 10^4$ photons: see figure 3.4)

phase shift acquired by a detector crossing a cavity with Fock state $|n\rangle$ and another cavity with Fock state $|n+m\rangle$

$$\Delta_m\gamma(n) = \gamma(n+m) - \gamma(n), \tag{3.35}$$

From equation (3.26) where $\gamma(n) = \mathrm{Re}[\eta(n)]$, we have

$$\Delta_m\gamma(n) = \mathrm{Re}\left[\mathrm{i}\ln\left(\frac{1-\lambda^2\left[\sum_{\beta\neq\alpha}\frac{C^*_{+,\beta}}{k_\beta L}+\frac{(n+m+1)C^*_{+,\alpha}}{k_\alpha L}\right]}{1-\lambda^2\left[\sum_{\beta\neq\alpha}\frac{C^*_{+,\beta}}{k_\beta L}+\frac{(n+1)C^*_{+,\alpha}}{k_\alpha L}\right]}\right)\right] \tag{3.36}$$

where we have assumed that $C_{-,\alpha} = 0$ from the mode invisibility approximation. In the regime $|\eta| \ll 1$, an approximate expression for $\Delta_m\gamma$ from equation (3.28) is

$$\Delta_m\gamma \simeq \frac{\lambda^2 m}{k_\alpha L}\mathrm{Im}\left[C^*_{+,\alpha}\right] \tag{3.37}$$

We see that while phase difference (3.36) computed from the complete expression (3.24) does depend on n, an approximate solution in the regime $|\eta| \ll 1$ is independent of n. For non-relativistic regime $v/c \ll 1$, the integral $\mathrm{Im}[C_{+,\alpha}]$ from (3.31) yields

$$\mathrm{Im}[C^*_{+,\alpha}] \simeq \frac{-c}{\pi\alpha v}\frac{L^2}{v^2-4c^2} \simeq \frac{L^2}{4\pi\alpha c v},$$

which upon substituting in (3.37) and taking into account that $k_\alpha L = \pi\alpha$ yields

$$\Delta_m\gamma \simeq \frac{\lambda^2 L^2}{4\pi^2\alpha^2 c v} m. \tag{3.38}$$

Equation (3.38) shows that in the very-few-photon regime the phase difference between two Fock states with n and $n+m$ photon numbers respectively, is a function of the difference in the photon number $n-(n+m)=m$ contained in these states. Figure 3.3 shows a plot of the phase difference as a function of n. Our ability to distinguish between two Fock states depends on the magnitude of the resolution of the interferometric experiment. From figure 3.3, we see that for small number of photons, we have more than enough resolution to tell apart states that differ by only one photon having in mind that typical resolutions in atomic interferometry are of the order of fractions of milliradians [28]. However, the resolution in our interferometric experiment rapidly decreases with n.

Although we might not be able to tell apart a state with a million photons from a state of a million and one photons, we can still obtain information about the photon content of the unknown state. The difference in phase between two states containing respectively n and $n+m$ photons is thus the required resolution of the interferometric experiment in order to distinguish states that differ by m photons as shown in figure 3.3 for several values of m.

Visibility Factor

As an atomic interferometer resolves a detector source thereby obtaining useful information about its structure, its visibility is reduced. For instance, if an atom within a two-path interferometer interacts with its environment (creates or annihilates a photon) the atom and the environment become entangled. Since we do not observe the field state, we found the final quantum state of the atom by averaging over all possible environment states. This is done by taking the trace of the atom-field density matrix over the degrees of freedom of the environment (see equation (3.16)). Applied to the atom interferometer, this procedure results in a reduction of contrast (visibility) and a phase shift given by the complex factor η for every interaction process (3.26).

In equation (3.9), we have a contribution $\exp[-|\text{Im}(\eta)|]$ in the component of $|\psi(T)\rangle$ proportional to $|\psi(0)\rangle$. In practical terms, this translates into a loss of visibility in the interference pattern. The desirable regime is therefore $\text{Im}(\eta) \ll 1$. Figure 3.4 shows the loss of contrast in the atom-field system with variable n–the number of photons in the field system.

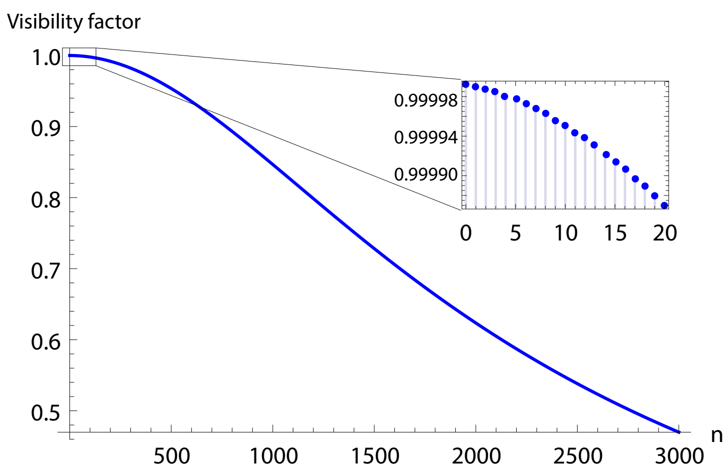


Figure 3.4: Plot of the visibility factor $e^{|\mathrm{Im}\eta|}$ showing that the interferometry will not be significantly disturbed by the second order effects that take us out of the approximation (3.8). Here $v = 1000$ m/s as in the previous plots.

3.2 Mode Invisibility – A QND Measurement for Coherent States

The mode invisibility technique for single photon detection in section 3.1 was presented to achieve a quantum non-demolition measurement of Fock states of light taking advantage of the spatial symmetry of the modes of the field in an optical cavity. Using this technique we showed that an atomic probe, on resonance with the target field mode we want to measure, can be sent through a cavity in a way in which the state of light in that mode is not altered, but at the same time the atom acquires a non-negligible phase easily appreciable in an atomic interferometry experiment. We exploited the 'mode invisibility' technique to suggest that a setting of two optical cavities – one containing a known state of light and another one containing the unknown state of light that we want to probe – allows for the effective distinction of Fock states containing very few photons.

However, how this 'mode invisibility' technique could be used to characterize, in a non-demolition way, some features of the Wigner function of different states of light (photon number expectations, phase space distribution first and second moments, etc) remained unexplored. In this section, we show how the mode-invisibility technique could be applied to the non-demolition measurement of coherent states of light [22, 21] and squeezed coherent states [36] of light respectively.

Furthermore, in chapter 2, to measure the phase shift on an atomic state (i.e. the amount of information gained in the system of light field), an interferometric scheme require a comparison of this phase shift with that obtained in another known field state which would serve as a reference. In so doing, we required a known Fock state of light as a reference state in order to probe the unknown Fock state of light. Since realizing a Fock state in the laboratory could be a challenging task and might limit the mode invisibility technique, we suggest in this section the use of a more realizable light state – 'the coherent state'. We will show that it is indeed possible to obtain information about an unknown state of light without employing a similar state as a reference. This adds an advantage to the mode invisibility technique.

Squeezed coherent state

A squeezed coherent state (SCS) $|\alpha, \zeta\rangle$ is obtained by first acting the displacement operator on the vacuum followed by the squeezed operator $\hat{S}(\zeta)$, i.e.

$$|\alpha, \zeta\rangle = \hat{S}(\zeta)\hat{D}(\alpha)|0\rangle, \tag{3.39}$$

with its density operator given as

$$\tilde{\rho} = |\alpha, \zeta\rangle\langle\alpha, \zeta|. \tag{3.40}$$

Modelling the measurement process

In this section, we study the effect of an atomic interaction on a (SCS). As before, our interaction Hamiltonian is the Unruh deWitt model (**??**) and the atom is prepared in its ground state $|g\rangle$. Consider the coupling between our detector and a (SC) trapped in an optical cavity.

Let the SCS be defined by the ket operator $|\zeta, \alpha\rangle_\kappa$ in a cavity mode κ of frequency $\omega_\kappa = \kappa\pi/L$ while all the rest of the modes are prepared in very low-populated states. The combined atom-field state before interaction is

$$|\psi(0)\rangle = |g\rangle \otimes |\zeta, \alpha\rangle_\kappa \bigotimes_{\gamma\neq\kappa} |0_\gamma\rangle. \tag{3.41}$$

In density operator notation,

$$\rho_0 = |g\rangle\langle g| \otimes |\zeta, \alpha\rangle_\kappa\langle\zeta, \alpha|_\kappa \tag{3.42}$$

Using the evolution operator (3.4), we will estimate the excitation transition probability of the atom system

$$P_{|e\rangle}(T) = \langle e| \operatorname{Tr}_F[U^{(1)}\rho_0 U^{(1)\dagger}]|e\rangle \tag{3.43}$$

Since we started our detector at ground state, the only contributing terms in $U^{(1)}$ as before will be

$$U^{(1)} = -\mathrm{i}\lambda\Big[\sum_{\gamma}\sigma^{+}a_{\gamma}^{\dagger}X_{+,\gamma} + \sum_{\gamma}\sigma^{+}a_{\gamma}X_{-\gamma}^{*}\Big] \tag{3.44}$$

Upon substitution of various terms, we get $U^{(1)}\rho_0 U^{\dagger(1)}$ to be given as

$$U^{(1)}\rho_0 U^{\dagger(1)} = \lambda^2\times$$

$$\Big[\sum_{\gamma}\sigma^{+}a_{\gamma}^{\dagger}X_{+,\gamma} + \sum_{\gamma}\sigma^{+}a_{\gamma}X_{-\gamma}^{*}\Big]\Big(|g\rangle\langle g| \otimes |\zeta,\alpha\rangle_{\kappa}\langle\zeta,\alpha|_{\kappa}\Big)\Big[\sum_{\beta}\sigma^{-}a_{\beta}X_{+,\kappa}^{*} + \sum_{\beta}\sigma^{-}a_{\beta}^{\dagger}X_{-,\beta}\Big]$$

Therefore,

$$\mathrm{Tr}[U^{(1)}\rho_0 U^{\dagger(1)}] = \lambda^2\times$$

$$\Big[\mathrm{Tr}_F\left[|e\rangle\langle e|\sum_{\beta,\gamma}X_{+,\beta}^{*}X_{+,\gamma}\langle\zeta,\alpha|_{\kappa}a_{\beta}a_{\gamma}^{\dagger}|\zeta,\alpha\rangle_{\kappa}\right] + \mathrm{Tr}_F\left[|e\rangle\langle e|\sum_{\beta,\gamma}X_{-,\beta}X_{-,\gamma}^{*}\langle\zeta,\alpha|_{\kappa}a_{\beta}^{\dagger}a_{\gamma}|\zeta,\alpha\rangle_{\kappa}\right]$$

$$+\,\mathrm{Tr}_F\left[|e\rangle\langle e|\sum_{\beta,\gamma}X_{-,\beta}X_{+,\gamma}\langle\zeta,\alpha|_{\kappa}a_{\beta}^{\dagger}a_{\gamma}^{\dagger}|\zeta,\alpha\rangle_{\kappa}\right] + \mathrm{Tr}_F\left[|e\rangle\langle e|\sum_{\beta,\gamma}X_{+,\beta}^{*}X_{-,\gamma}^{*}\langle\zeta,\alpha|_{\kappa}a_{\beta}a_{\gamma}|\zeta,\alpha\rangle_{\kappa}\right]\Big]$$

The trace expressions follow from equation (2.18) since we are interested in the trace over the field of the composite system. It is easy to check that applying the mode invisibility technique that ensures the detector to probe the even relevant field modes, the integrals $X_{-,\beta}X_{+,\gamma}$ and its conjugate $X_{-,\beta}^{*}X_{+,\gamma}^{*}$ vanish for non-relativistic terms $v/c \ll 1$. Therefore we are left with the first two terms. Taking note of the canonical relationship $a_{\beta}a_{\gamma}^{\dagger} = \delta_{\gamma,\beta} + a_{\gamma}^{\dagger}a_{\beta}$, we rewrite the expression

$$\mathrm{Tr}_F[U^{(1)}\rho U^{(1)\dagger}] = \lambda^2\underbrace{\Big[\mathrm{Tr}_F\left[|e\rangle\langle e|\sum_{\beta,\gamma}X_{+,\beta}^{*}X_{+,\gamma}\langle\zeta,\alpha|_{\kappa}\delta_{\gamma,\beta}|\zeta,\alpha\rangle_{\kappa}\right]}_{X}$$

$$+\lambda^2\underbrace{\mathrm{Tr}_F\left[|e\rangle\langle e|\sum_{\beta,\gamma}M_{\beta,\gamma}\langle\zeta,\alpha|_{\kappa}a_{\beta}^{\dagger}a_{\gamma}|\zeta,\alpha\rangle_{\kappa}\right]\Big]}_{Y} \tag{3.45}$$

where for notational convenience we have written

$$M_{\beta,\gamma} = X_{-,\beta}X_{-,\gamma}^{*} + X_{+,\beta}^{*}X_{+,\beta}. \tag{3.46}$$

The first term in equation X gives

$$X = \mathrm{Tr}_F \left[|e\rangle\langle e| \sum_{\beta,\gamma} X^*_{+,\beta} X_{+,\gamma} \langle \zeta, \alpha|_\kappa \delta_{\gamma,\beta} |\zeta, \alpha\rangle_\kappa \right] = |e\rangle\langle e| X^*_{+,\beta} X_{+,\beta}] \tag{3.47}$$

We will evaluate the sum in the second term Y. By writing the state $|\zeta, \alpha\rangle_\kappa = S(\zeta)_\kappa D(\alpha)_\kappa |0\rangle$, we need to evaluate

$$Y = \mathrm{Tr}_F \left[|e\rangle\langle e| \underbrace{\sum_{\beta,\gamma} M_{\beta,\gamma} \langle 0| D^\dagger_\kappa(\alpha) S(\zeta)^\dagger_\kappa | a^\dagger_\beta a_\gamma S(\zeta)_\kappa D(\alpha)_\kappa |0\rangle}_{Z} \right]$$

Since we have sums over two variables β and γ respectively with a fixed variable κ, we will consider the following steps in evaluating Z

(1) The sum over the variable β is kept constant, then we consider two cases when $\gamma = \kappa$ and $\gamma \neq \kappa$). This gives

$$\begin{aligned} \sum_{\beta,\gamma} M_{\beta,\gamma} \langle 0| D^\dagger_\kappa(\alpha) S(\zeta)^\dagger_\kappa | a^\dagger_\beta a_\gamma S(\zeta)_\kappa D(\alpha)_\kappa |0\rangle &= \sum_{\beta,\gamma\neq\kappa} M_{\beta,\gamma} \langle 0| D^\dagger_\kappa(\alpha) S^\dagger_\kappa(\zeta) a^\dagger_\beta a_\gamma S_\kappa(\zeta) D_\kappa(\alpha) |0\rangle \\ &+ \sum_{\beta,\gamma=\kappa} M_{\beta,\gamma} \langle 0| D^\dagger_\kappa(\alpha) S^\dagger_\kappa(\zeta) a^\dagger_\beta a_\gamma S_\kappa(\zeta) D_\kappa(\alpha) |0\rangle \end{aligned}$$

The first term on the right hand side vanishes since for $\gamma \neq \kappa$

$$a_\gamma S_\kappa(\zeta) D_\kappa(\alpha) |0\rangle = S_\kappa(\zeta) D_\kappa(\alpha) a_\gamma |0\rangle = 0$$

Therefore we have

$$\sum_{\beta,\gamma} M_{\beta,\gamma} \langle 0| D^\dagger_\kappa(\alpha) S(\zeta)^\dagger_\kappa | a^\dagger_\beta a_\gamma S(\zeta)_\kappa D(\alpha)_\kappa |0\rangle = \sum_{\beta} M_{\beta,\kappa} \langle 0| D^\dagger_\kappa(\alpha) S^\dagger_\kappa(\zeta) a^\dagger_\beta a_\kappa S_\kappa(\zeta) D_\kappa(\alpha) |0\rangle$$

We have remove the sum over γ when $\gamma = \kappa$ since κ is a fixed variable.

(2) The second step therefore is to consider the sum over β when $\beta = \kappa$ and $\beta \neq \kappa$ respectively.

$$\sum_{\beta} M_{\beta,\kappa}\langle 0|D_\kappa^\dagger(\alpha)S(\zeta)_\kappa^\dagger|a_\beta^\dagger a_\gamma S(\zeta)_\kappa D(\alpha)_\kappa|0\rangle = \underbrace{\sum_{\beta\neq\kappa} M_{\beta,\kappa}\langle 0|D_\kappa^\dagger(\alpha)S_\kappa^\dagger(\zeta)a_\beta^\dagger a_\kappa S_\kappa(\zeta)D_\kappa(\alpha)|0\rangle}_{Z'} + \underbrace{\sum_{\beta=\kappa} M_{\beta,\kappa}\langle 0|D_\kappa^\dagger(\alpha)S_\kappa^\dagger(\zeta)a_\beta^\dagger a_\kappa S_\kappa(\zeta)D_\kappa(\alpha)|0\rangle}_{Z''}$$

Where we have separated the term $Z = Z' + Z''$. We will treat Y' and Y'' separately.

$$Z' = \sum_{\beta\neq\kappa} M_{\beta,\kappa}\langle 0|D_\kappa^\dagger(\alpha)S_\kappa^\dagger(\zeta)a_\beta^\dagger a_\kappa S_\kappa(\zeta)D_\kappa(\alpha)|0\rangle$$

For $\beta \neq \kappa$,

$$a_\beta^\dagger a_\kappa S_\kappa(\zeta)D_\kappa(\alpha)|0\rangle = a_\kappa S_\kappa(\zeta)D_\kappa(\alpha)a_\beta^\dagger|0\rangle = a_\kappa S_\kappa(\zeta)D_\kappa(\alpha)|1_\beta\rangle,$$

so that

$$\sum_{\beta\neq\kappa} M_{\beta,\kappa}\langle 0|D_\kappa^\dagger S_\kappa^\dagger a_\beta^\dagger a_\kappa S_\kappa D_\kappa|0\rangle = \sum_{\beta\neq\kappa} M_{\beta,\kappa}\langle 0|D_\kappa^\dagger S_\kappa^\dagger a_\kappa S_\kappa D_\kappa|1_\beta\rangle \tag{3.48}$$

Recall from equations (2.20a)

$$S(\zeta)_\kappa^\dagger a_\kappa S(\zeta)_\kappa = a_\kappa \cosh(r) - a_\kappa^\dagger e^{i\phi}\sinh(r)$$

The right hand side of equation (3.48) gives,

$$\sum_{\beta\neq\kappa} M_{\beta,\kappa}\langle 0|D_\kappa^\dagger S_\kappa^\dagger(\zeta)a_\kappa S_\kappa(\zeta)D_\kappa|1_\beta\rangle = \sum_{\beta\neq\kappa} M_{\beta,\kappa}\langle 0|D_\kappa^\dagger\Big(a_\kappa C(r) - a_\kappa^\dagger e^{\mathrm{i}\phi}S(r)\Big)D_\kappa|1_\beta\rangle$$

Where we have re-written $\cosh(r) = C(r)$ and $\sinh(r) = S(r)$. We thus have

$$\sum_{\beta\neq\kappa} M_{\beta,\kappa}\langle 0|D_\kappa^\dagger(\alpha)S_\kappa^\dagger(\zeta)a_\kappa S_\kappa(\zeta)D_\kappa(\alpha)|0\rangle = \sum_{\beta\neq\kappa} M_{\beta,\kappa}[C(r)\langle 0|D_\kappa^\dagger(\alpha)a_\kappa D_\kappa(\alpha)|1_\beta\rangle - \sum_{\beta\neq\kappa} M_{\beta,\kappa}[e^{\mathrm{i}\phi}S(r)\langle 0|D_\kappa^\dagger(\alpha)a_\kappa^\dagger D_\kappa(\alpha)|1_\beta\rangle$$

Also recall from equations (2.16a) and (2.16b),

$$D_\kappa^\dagger(\alpha)a_\kappa D_\kappa(\alpha) = a_\kappa + \alpha, \qquad D_\kappa^\dagger(\alpha)a_\kappa^\dagger D_\kappa(\alpha) = a_\kappa^\dagger + \alpha^*$$

Therefore,

$$\sum_{\beta\neq\kappa} M_{\beta,\kappa}\langle 0|D^\dagger_\kappa S^\dagger_\kappa a_\kappa S_\kappa(\zeta) D_\kappa|1_\beta\rangle = \sum_{\beta\neq\kappa}\sum_{\beta\neq\kappa} M_{\beta,\kappa}\alpha C(r)\langle 0|1_\beta\rangle + \sum_{\beta\neq\kappa} M_{\beta,\kappa} C(r)\langle 0|a_\kappa|1_\beta\rangle$$
$$- \sum_{\beta\neq\kappa}\sum_{\beta\neq\kappa} M_{\beta,\kappa}\alpha^* e^{\mathrm{i}\phi} S(r)\langle 0|1_\beta\rangle - \sum_{\beta\neq\kappa} M_{\beta,\kappa}\alpha^* e^{\mathrm{i}\phi} S(r)\langle 0|a^\dagger_\kappa|1_\beta\rangle = 0$$

Therefore we have $Z' = 0$. We go ahead to evaluate Y'' given as

$$Z'' = \sum_{\beta=\kappa} M_{\beta,\kappa}\langle 0|D^\dagger_\kappa(\alpha) S^\dagger_\kappa(\zeta) a^\dagger_\beta a_\kappa S_\kappa(\zeta) D_\kappa(\alpha)|0\rangle$$

We recall that κ is a fixed variable, so we will drop the summation term to have

$$Z'' = M_{\kappa,\kappa}\langle 0|\underbrace{D^\dagger_\kappa(\alpha) S^\dagger_\kappa(\zeta) a^\dagger_\kappa S(\zeta)_\kappa S^\dagger_\kappa(\zeta) a_\kappa S_\kappa(\zeta) D_\kappa(\alpha)}_{H}|0\rangle \tag{3.49}$$

where we have made use of the unitarity of the squeeze operator $S(\zeta)_\kappa S^\dagger_\kappa(\zeta) = 1$. We will go ahead to evaluate H. First we note that from equations (2.20a),

$$S^\dagger_\kappa(\zeta) a^\dagger_\kappa S(\zeta)_\kappa S^\dagger_\kappa(\zeta) a_\kappa S_\kappa(\zeta) = C^2(r) a^\dagger_\kappa a_\kappa - e^{\mathrm{i}\phi} C(r)S(r) a^\dagger_\kappa a^\dagger_\kappa - e^{-\mathrm{i}\phi} C(r)S(r) a_\kappa a_\kappa + S^2(r) a_\kappa a^\dagger_\kappa$$

so that

$$\begin{aligned} H = {} & C^2(r)\alpha a^\dagger_\kappa + C^2(r)\alpha^* a_\kappa + C^2(r) a^\dagger_\kappa a_\kappa - e^{\mathrm{i}\phi} C(r)S(r) a^\dagger_\kappa a^\dagger_\kappa + S^2(r)\alpha^* a_\kappa \\ & - 2\alpha^* e^{\mathrm{i}\phi} C(r)S(r) a^\dagger_\kappa - e^{-\mathrm{i}\phi} C(r)S(r) a_\kappa a_\kappa - 2\alpha e^{-\mathrm{i}\phi} C(r)S(r) a_\kappa + \alpha S^2(r) a^\dagger_\kappa \\ & + S^2(r) a_\kappa a^\dagger_\kappa + |\alpha|^2 S^2(r) - (\alpha^*)^2 e^{\mathrm{i}\phi} C(r)S(r) - \alpha^2 e^{-2\mathrm{i}\phi} C(r)S(r) + C^2(r)|\alpha|^2 \end{aligned}$$

When we substitute H back in equation (3.49), the only non-zero terms will the terms at the last line namely

$$S^2(r) a_\kappa a^\dagger_\kappa + |\alpha|^2 S^2(r) - (\alpha^*)^2 e^{\mathrm{i}\phi} C(r)S(r) - \alpha^2 e^{-2\mathrm{i}\phi} C(r)S(r)$$

Therefore writing

$$-(\alpha^*)^2 e^{\mathrm{i}\phi} C(r)S(r) - \alpha^2 e^{-2\mathrm{i}\phi} C(r)S(r) = -2\,\mathrm{Re}[(\alpha^*)^2 e^{\mathrm{i}\phi}] C(r)S(r)$$

we have

$$Z'' = M_{\kappa,\kappa}\langle 0|S^2(r) a_\kappa a^\dagger_\kappa + |\alpha|^2 C^2(r) + |\alpha|^2 S^2(r) - 2\,\mathrm{Re}[(\alpha^*)^2 e^{\mathrm{i}\phi}] C(r)S(r)|0\rangle.$$

Upon substitution,

$$
\begin{aligned}
Y &= \mathrm{Tr}_F \left[|e\rangle\langle e| \underbrace{\sum_{\beta,\gamma} M_{\beta,\gamma}\langle 0|D^\dagger_\kappa(\alpha)S(\zeta)^\dagger_\kappa|a^\dagger_\beta a_\gamma S(\zeta)_\kappa D(\alpha)_\kappa|0\rangle}_{Z=Z'+Z''}\right] \\
&= \mathrm{Tr}_F \left[|e\rangle\langle e M_{\kappa,\kappa}\langle 0|S^2(r) + |\alpha|^2 C^2(r) + |\alpha|^2 S^2(r) - 2\,\mathrm{Re}[(\alpha^*)^2 e^{\mathrm{i}\phi}]C(r)S(r)|0\rangle\right].
\end{aligned}
$$

Equation (3.45) reads

$$
\mathrm{Tr}_F[U^{(1)}\rho_0 U^{(1)\dagger}] = X + Y,
$$

so that the reduced density operator for the atomic state $\mathrm{Tr}_F[U^{(1)}\rho_0 U^{(1)\dagger}]$ is given as

$$
\left[M_{\kappa,\kappa}\left(S^2(r) + |\alpha|^2 C^2(r) + |\alpha|^2 S^2(r) - 2\,\mathrm{Re}[(\alpha^*)^2 e^{\mathrm{i}\phi}]C(r)S(r)\right) + \sum_\gamma |X_{+,\gamma}|^2\right]|e\rangle\langle e|
$$

Recall from (3.46),

$$
M_{\kappa,\kappa} = |X_{-,\kappa}|^2 + |X_{+,\kappa}|^2.
$$

Thus the excitation transition probability (see expression (3.43)) for the atomic state is

$$
\begin{aligned}
P^{\alpha,r}_{|e\rangle} =& \langle e|\,\mathrm{Tr}_F[U^{(1)}\rho_0 U^{(1)\dagger}]|e\rangle \\
P^{\alpha,r}_{|e\rangle} =& \lambda^2 \frac{|\alpha|^2}{k_\kappa L}\left(|X_{-,\kappa}|^2 + |X_{+,\kappa}|^2\right)\left(C^2(r) + S^2(r)\right) + \left(|X_{-,\kappa}|^2 + |X_{+,\kappa}|^2\right)\lambda^2 \frac{S^2(r)}{k_\kappa L} \\
&+ \sum_\gamma \frac{\lambda^2}{k_\gamma L}|X_{+,\gamma}|^2 - 2\frac{\lambda^2}{k_\kappa L}\left(|X_{-,\kappa}|^2 + |X_{+,\kappa}|^2\right)S(r)C(r)\,\mathrm{Re}\left[|\alpha|^2 e^{\mathrm{i}(2\theta-\phi)}\right] \qquad (3.50)
\end{aligned}
$$

If we note from the definition of the squeezed coherent state $|\zeta,\alpha\rangle = S(\zeta)D(\alpha)|0\rangle$ where the unitary operators $D(\alpha)$ and $S(\zeta)$ have been defined in (2.15) and (2.18) respectively, the squeezed vacuum state $S(\zeta)|0\rangle$ is obtained for $\alpha = 0$ and the coherent state $D(\alpha)|0\rangle$ is obtained for $r = 0$. Therefore (3.50) gives also the correct transition probabilities for the squeezed vacuum and the coherent state as particular cases. Namely, for a coherent state,

$$
P^\alpha_{|e\rangle} = \lambda^2\left[\frac{|\alpha|^2}{k_\kappa L}|X_{-,\kappa}|^2 + \frac{|\alpha|^2}{k_\kappa L}|X_{+,\kappa}|^2 + \sum_\gamma \frac{|X_{+,\gamma}|^2}{k_\gamma L}\right], \qquad (3.51)
$$

and for the squeezed vacuum,

$$P^r_{|e\rangle} = \lambda^2 \left[\frac{S^2(r)}{k_\kappa L} |X_{-,\kappa}|^2 + |X_{+,\kappa}|^2 \frac{S^2(r)}{k_\kappa L} + \sum_\gamma \frac{|X_{+,\gamma}|^2}{k_\gamma L} \right]. \tag{3.52}$$

These transition probabilities $P^r_{|e\rangle}, P^\alpha_{|e\rangle}$ and $P^{\alpha,r}_{|e\rangle}$ each depend on the integrals $|X_{\pm,\gamma}|^2$ with $X_{\pm,\gamma}$ defined as in equation (3.17). Now, as discussed in chapter 2, the rotating-wave resonant term $|X_{-,\kappa}|^2$ gives, by far, the largest contribution to the probability of transition of the system. By means of the mode invisibility technique 2.3.2 we are able to cancel the contribution of this leading term to the transition probability.

Once we have achieved an excitation transition probability (3.50), (3.51) and (3.52) which satisfies our approximation 3.1, we proceed to calculate the phase acquired by the atom upon interaction with each light field. Before we continue, we will compute the leading order contribution to the phase as we did in chapter 2, which is given by the expression

$$|\psi^{(2)}(T)\rangle = U^{(2)}|\psi(0)\rangle, \tag{3.53}$$

where $U^{(2)}$ is defined in 3.22

$$\begin{aligned} \mathrm{U}^{(2)} = -\lambda^2 \sum_\gamma \sum_\beta \int_0^{\frac{L}{v}} dt \int_0^t dt' \Big(& \sigma^-\sigma^+ a^\dagger_\gamma a^\dagger_\beta e^{\mathrm{i}(\omega_\gamma - \Omega)t} e^{\mathrm{i}(\omega_\beta + \Omega)t'} + \sigma^-\sigma^+ a^\dagger_\gamma a_\beta e^{\mathrm{i}(\omega_\gamma - \Omega)t} e^{-\mathrm{i}(\omega_\beta - \Omega)t'} \\ & + \sigma^-\sigma^+ a_\gamma a_\beta e^{-\mathrm{i}(\omega_\gamma + \Omega)t} e^{-\mathrm{i}(\omega_\beta - \Omega)t'} + \sigma^-\sigma^+ a_\gamma a^\dagger_\beta e^{-\mathrm{i}(\omega_\gamma + \Omega)t} e^{\mathrm{i}(\omega_\beta + \Omega)t'} \Big) \frac{\sin(k_\gamma x(t))}{\sqrt{k_\gamma L}} \frac{\sin(k_\beta x(t'))}{\sqrt{k_\beta L}}. \end{aligned}$$

We have already defined all it takes to evaluate equation (3.53), therefore

$$\begin{aligned} |\psi^{(2)}\rangle = -\lambda^2 \Bigg[& \frac{\left(C^*_{+,\kappa} + C_{-,\kappa}\right)}{k_\kappa L} \Big(C^2(r) + S^2(r)\Big) |\alpha|^2 + \sum_\gamma \frac{C^*_{+,\gamma}}{k_\gamma L} + \Big(C^*_{+,\kappa} + C_{-,\kappa}\Big) \frac{\lambda^2 S^2(r)}{k_\kappa L} \\ & - \frac{2\lambda^2}{k_\kappa L} S(r) C(r) \,\mathrm{Re}\left[|\alpha|^2 e^{\mathrm{i}(2\theta - \phi)}\right] \Bigg] |\psi(0)\rangle + |\psi(T)\rangle_\perp \end{aligned} \tag{3.54}$$

where

$$C_{\pm,\kappa} = \int_0^{L/v} dt \int_0^t dt' e^{\mathrm{i}(\omega_\kappa \pm \Omega)(t-t')} \sin(k_\kappa v t) \sin(k_\kappa v t'). \tag{3.55}$$

$|\psi(0)\rangle$ is the initial state of the joint atom-squeezed coherent system and $|\psi(T)\rangle_\perp$ is the second order correction that is orthogonal to the initial state. For $r = 0$ we recover the case of a coherent state in the cavity mode we want to probe. For that particular case the expression simplifies to

$$|\psi(T)^{(2)}\rangle_\alpha = -\lambda^2 \left[\frac{\left(C^*_{+,\kappa} + C_{-,\kappa}\right)}{k_\kappa L} |\alpha|^2 + \sum_\gamma \frac{C^*_{+,\gamma}}{k_\gamma L} \right] |\psi(0)\rangle + |\psi(T)\rangle_\perp$$

Similarly for the case $\alpha = 0$ we recover the squeezed vacuum. In this case the leading order contribution is

$$|\psi(T)^{(2)}\rangle_r = -\lambda^2 \left[\frac{\left(C_{-,\kappa} + C^*_{+,\kappa}\right)}{k_\kappa L} \sinh^2(r) + \sum_\gamma \frac{C^*_{+,\gamma}}{k_\gamma L} \right] |\psi(0)\rangle + |\psi(T)\rangle_\perp$$

Evaluating the phase factor

Having found the leading order to the phase factor, we now go ahead to evaluate the phase value itself given by the equation (3.25)

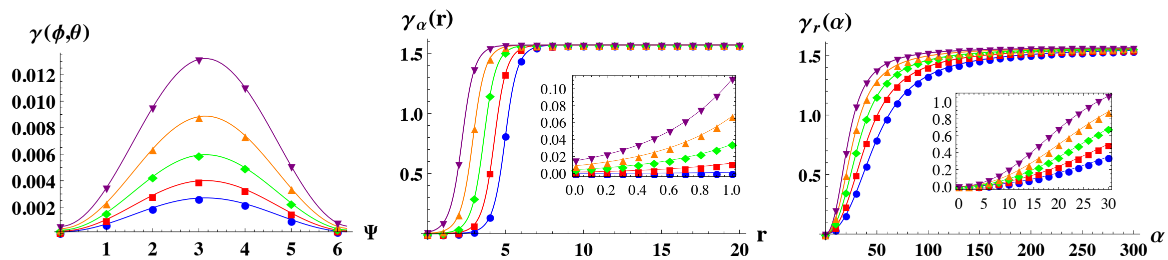
$$\eta = -\mathrm{i} \ln \left[1 - \langle \psi(0)| U(0,T)^{(2)} |\psi(T)\rangle \right], \tag{3.56}$$

Multiplying equation (3.54) by $\langle\psi(0)|$, we have

$$\langle\psi(0)|U^{(2)}(T)|\psi(0)\rangle = \sum_\gamma \frac{\lambda^2 C^*_{+,\gamma}}{k_\gamma L} +$$
$$\frac{\lambda^2}{k_\kappa L}(C^*_{+,\kappa} + C_{-,\kappa}) \left(S^2(r) + \big(C^2(r) + S^2(r)\big)|\alpha|^2 - 2S(r)C(r)\,\mathrm{Re}\left[|\alpha|^2 e^{\mathrm{i}(2\theta-\phi)}\right] \right)$$

Upon substitution, the phase acquired by a detector after it had crossed a cavity sustaining a squeezed coherent state is given by

$$\eta(r,\alpha) = -\mathrm{i}\ln \Bigg[1 - \Bigg(\sum_\gamma \frac{\lambda^2 C^*_{+,\gamma}}{k_\gamma L} + \frac{\lambda^2}{k_\kappa L}(C^*_{+,\kappa} + C_{-,\kappa})\big(C^2(r) + S^2(r)\big)|\alpha|^2 +$$
$$\frac{\lambda^2}{k_\kappa L}(C^*_{+,\kappa} + C_{-,\kappa}) \left(S^2(r) - 2S(r)C(r)\,\mathrm{Re}\left[|\alpha|^2 e^{\mathrm{i}(2\theta-\phi)}\right] \right) \Bigg) \Bigg] \tag{3.57}$$



(a) (b) (c)

Figure 3.5: Measuring a squeezed coherent state in a non-destructive manner. **(a)** Phase measured as a function of Ψ. **(b)** Phase as a function of amplitude r of the squeeze parameter **(c)** Phase measured as a function of the magnitude of the displacement parameter $|\alpha|$

For the particular cases when $r = 0$ and $|\alpha| = 0$, we have the phase acquired by an atom crossing a cavity with coherent state or squeezed vacuum state sustained in it to be respectively given as

$$\eta(r) = -\mathrm{i}\ln\left[1 - \lambda^2\left(\sum_{\gamma}\frac{C^*_{+,\gamma}}{k_\gamma L} + \frac{S^2(r)C_{-,\kappa}}{k_\kappa L} + \frac{S^2(r)C^*_{+,\kappa}}{k_\kappa L}\right)\right] \tag{3.58}$$

and

$$\eta(\alpha) = -\mathrm{i}\ln\left[1 - \lambda^2\left(\sum_{\gamma}\frac{C^*_{+,\gamma}}{k_\gamma L} + \frac{|\alpha|^2C_{-,\kappa}}{k_\kappa L} + \frac{|\alpha|^2C^*_{+,\kappa}}{k_\kappa L}\right)\right] \tag{3.59}$$

We note that with the mode invisibility technique, $C_{-,\gamma} = 0$ as shown in equation (3.30). From equation (3.57) we see that the global phase acquired by the atom crossing a cavity with a squeezed coherent state sustained in it depends on the following features:

1. the magnitude of the displacement parameter $|\alpha|$
2. the squeezing parameter r
3. the relative angle $2\theta - \phi$ between the squeezing and displacement operations in phase space

This means for our measurement setting, if we are able to measure the phase (see equation (3.57)) acquired by the detector during its interaction with a squeezed coherent

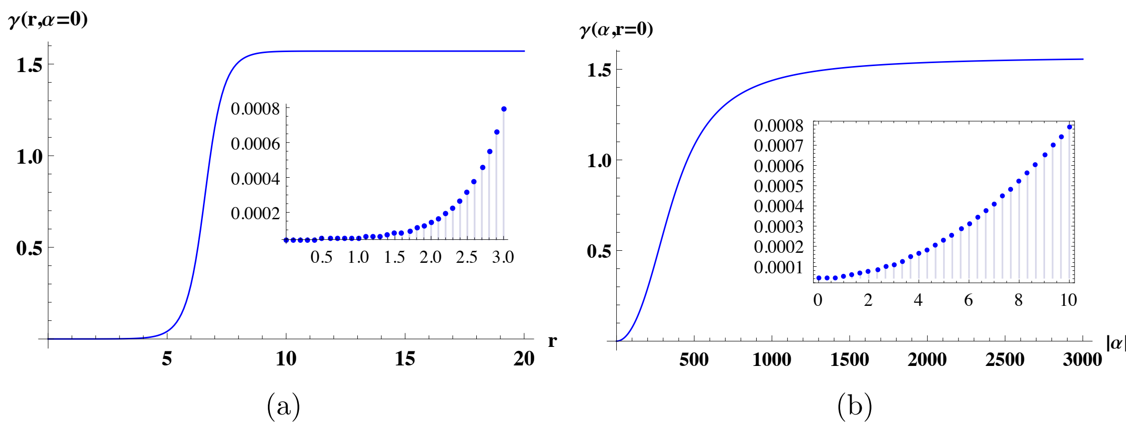


Figure 3.6: Measuring a squeezed coherent state in a non-destructive manner. **(a)** Phase measured as a function of Ψ. **(b)** Phase as a function of amplitude r of the squeeze parameter

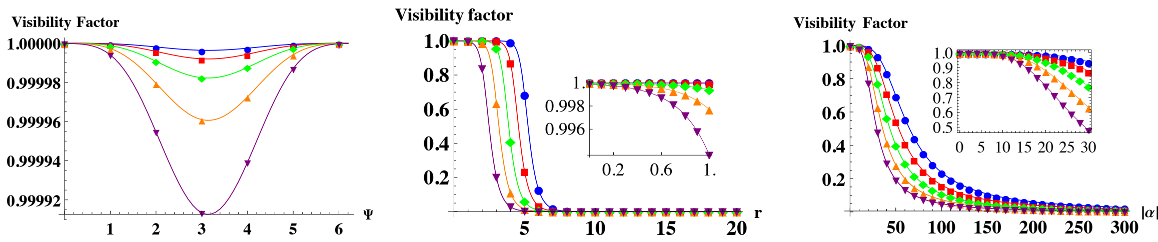
state, then we can obtain, in a non-destructive way, information about the amplitude of the squeezed amplitude r, the average number of photons and the relative phase ($\Psi = 2\phi - \theta$) between the squeeze and displacement operators respectively. These features characterizes the Wigner function [68] of a squeezed coherent state. We can therefore use this phase to characterize a coherent state, a squeezed vacuum state or to measure the direction of squeezing in phase space relative to the direction of displacement in a coherent squeezed state.

Figure 3.5 shows different plots of γ (the real value of $\eta(r, \alpha)$) versus its dependent variables α, r and of course the relative phase difference $\Psi = 2\theta - \phi$. The figure shows how the dynamical phase is sensitive to these three parameters. We summarize the relationship here

(a) Phase plot as a function of the squeeze operator amplitude r. We observe a maximum value of the squeezing at $\gamma = 1.5$. We see that for small squeezing parameter $0 = r \leq 1$, there is a clear difference between the various curves; however when the value of r increases, the curves get to a peak at $\gamma = 1.5$

(b) Phase plot as a function of the displacement amplitude $|\alpha|$. We also observe a maximum value of the displacement when $\gamma = 1.5$ and we can no longer obtain any information from the system.

(c) Phase plot as a function of the relative displacement $\Psi = 2\theta - \phi$. For the relative phase between the squeezing and the displacement, we see that there is indeed an



(a) (b) (c)

Figure 3.7: Visibility factor **(a)**as a function of Ψ. **(b)**as a function of amplitude r of the squeeze parameter **(c)**as a function of the magnitude of the displacement parameter $|\alpha|$

appreciable phase difference that increases when the photon number expectation of the mode increases. This increase is extremely sensitive to the values of the squeezing parameter and the displacement parameter for a range of values (although it loses sensitivity when the expected photon population increases). This suggests that the measurement of this phase would be an extremely good method for probing, in a non-demolition way, states that are very scarcely populated, losing sensitivity as the photon population increases.

The key of this technique is that whilst the probability that the system evolves to some different state is made negligible, the global phase factor (which can be determined interferometrically by comparison with a known state) is not negligible.

Visibility factor

Figure 3.7 shows the different visibility factors that would impact an interferometric experiment where we compare the phase acquired by an atom going through a cavity with the target state of light and some other reference phase. Consistently with the previous results, we see the method is much better off probing states with a low photon number expectation.

3.3 Coherent State – A Reference State for Interferometric Measurement

We have determined how the global phase acquired by an atomic probe behaves as a function of the parameters of the probed states of light in a QND measurement setting and how this phase can be measured by means of atomic interferometry [43].

Using an interferometer, we need a target state (state to be probed) and a reference state. When it comes to choosing a reference state, we would like it to be a state that is easy to prepare and control in the laboratory. As discussed in section 2, in order to probe an unknown Fock state, we used a known Fock state in another cavity as a reference state. However this is less than ideal since the preparation and control of Fock states of light is a rather challenging enterprise in quantum optics [30]. The constraint therefore could be removed from the mode invisibility QND measurement scheme. A natural candidate for a reference state is a coherent state. These states are among the easiest to prepare and control in quantum optic laboratories [58] and as we have shown, the mode invisibility technique is also applicable to coherent states of light. The rest of this chapter would therefore characterize the amount of information gained in different states of light using a coherent state as a reference.

Interferometric measurement of a squeezed coherent state using another coherent state as reference

The phase acquired by an atom crossing a cavity sustaining a squeezed coherent state of light is given in equation (3.57). Using a coherent state with amplitude α_R as the reference state in an atomic interferometry scheme, the interferometric phase measured would be given as

$$\Delta\gamma(r,\alpha,\Psi) = \mathrm{Re}[-\mathrm{i}\ln\chi(\alpha,r,\Psi)]$$

where

$$\chi(\alpha,r,\Psi) = \Bigg[1-\lambda^2\bigg(\frac{C^*_{+,\kappa}}{k_\kappa L}S^2(r)+\sum_\gamma\frac{C^*_{+,\gamma}}{k_\gamma L}+\frac{C^*_{+,\kappa}}{k_\kappa L}(C^2(r)+S^2(r))|\alpha|^2 - \frac{2}{k_\kappa L}C^*_{+,\kappa}S(r)C(r)\,\mathrm{Re}\left[|\alpha|^2 e^{\mathrm{i}(2\theta-\phi)}\right]\bigg)\Bigg] \times\left[1-\lambda^2\left(\frac{C^*_{+,\kappa}}{k_\kappa L}|\alpha_R|^2+\sum_\gamma\frac{C^*_{+,\gamma}}{k_\gamma L}\right)\right]^{-1} \tag{3.60}$$

For a case when $r=0$ we have the phase acquired by an atom crossing a cavity with a coherent state sustained in its mode when we use another coherent state with known

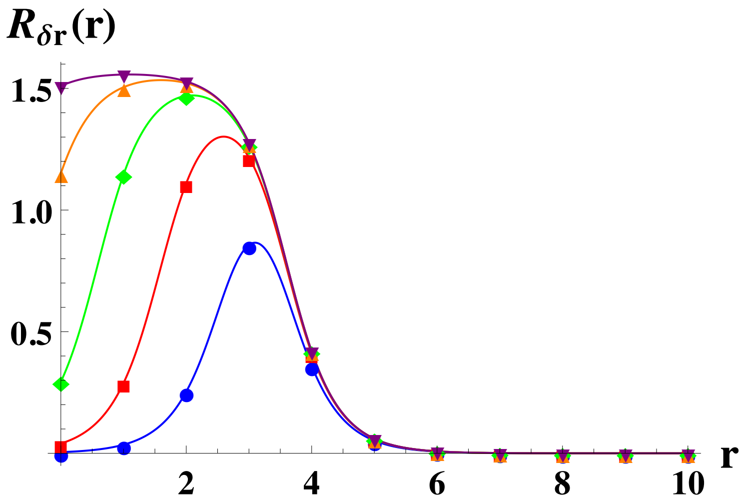


Figure 3.8: Phase resolution required to distinguish between a squeezed vacuum state with amplitude r using a coherent state of amplitude $|\alpha|^R$ as reference

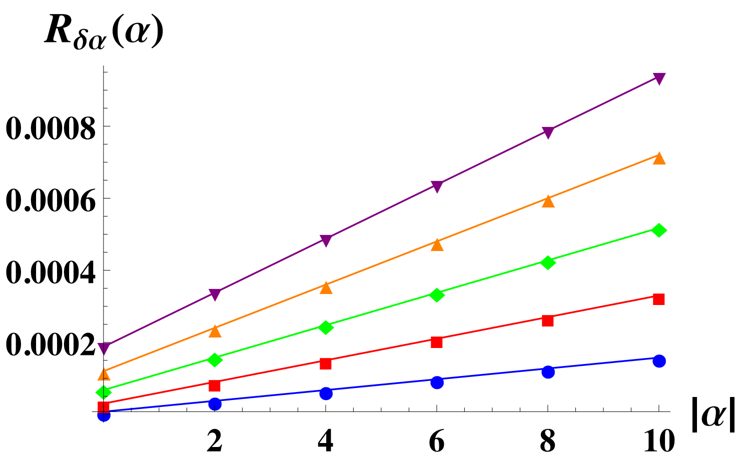


Figure 3.9: Phase resolution required to distinguish between a coherent state with parameter $|\alpha|$ from another coherent state with parameter $|\alpha + \delta\alpha|$.

amplitude $|\alpha_R|$ as reference. In the non-relativistic case $v/c \ll 1$, the expression is given as

$$\Delta\gamma_{\alpha_R}^{(\alpha)} = \frac{\lambda^2}{k_\kappa L}\mathrm{Im}[C_{+,\kappa}^*](|\alpha|^2 - |\alpha_R|^2)$$

The dependence on the value of $|\alpha|$ will be similar and has the same form and magnitude as the one shown in figure 3.5. One can compute the phase resolution $\mathcal{R}_{\delta\alpha}(\alpha)$ – the interferometric phase difference between two coherent states of amplitudes $|\alpha|$ and $|\alpha| + \delta\alpha$ – and plot the result in figure 3.9, showing our ability to distinguish two different coherent states as the amplitude of the state changes.

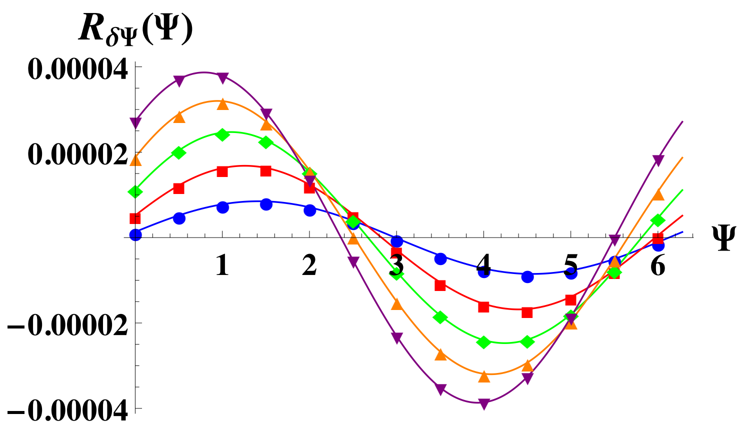


Figure 3.10: Phase resolution required to distinguish between the compound phase $\Psi = 2\theta - \phi$ of a squeezed coherent state from the compound state $\Psi + \delta\Psi$ of another squeezed coherent state

Similarly, for $\alpha = 0$ in equation (3.60), we obtain the interferometric phase observed in a setting with an atom flying through a cavity sustaining a squeezed vacuum light state and a reference cavity with a reference coherent state $|\alpha_R|$. In the non-relativistic limit, this expression is given as

$$\Delta\gamma_n^{(r)} = \frac{\lambda^2}{k_\kappa L}\mathrm{Im}[C^*_{+,\kappa}][\sinh^2(r) - |\alpha_R|^2]$$

The phase resolution to tell apart different values of r is shown in figure 3.8. Additionally, we also show in figure 3.10 the interferometric phase resolution to measure the relative direction of squeezing with respect to the displacement.

QND measurement of Fock states using coherent states as reference

In our earlier work [44], we discussed how it is possible to distinguish between a Fock state containing n photons from another Fock state containing $n + m$ photons using the 'mode invisibility' technique. The phase acquired by an atom crossing a cavity containing a Fock state of light is [44]:

$$\gamma(n) = \mathrm{Re}\left[-i\ln\left[1 - \lambda^2\left(\frac{C^*_{+,\kappa}}{k_\kappa L}n + \sum_\beta \frac{C^*_{+,\beta}}{k_\beta L}\right)\right]\right]$$

On the other hand, the phase acquired by an atom crossing a cavity sustaining a coherent state of light is given in equation (3.59). If we prepare an experimental setup as illustrated in Figure 3.1, and make the known state a coherent state, the difference between phases for the two given states is given by the expression

$$\Delta\gamma_n^{(\alpha)} = \gamma(n) - \gamma(\alpha),$$

which yields

$$\Delta\gamma_n^{(\alpha)} = \mathrm{Re}\left[-\mathrm{i}\ln\left(\frac{1-\lambda^2\left(\frac{C^*_{+,\kappa}}{k_\kappa L}n + \sum_\beta \frac{C^*_{+,\beta}}{k_\beta L}\right)}{1-\lambda^2\left(\frac{C^*_{+,\kappa}}{k_\kappa L}|\alpha_R|^2 + \sum_\beta \frac{C^*_{+,\beta}}{k_\beta L}\right)} \right)\right]. \tag{3.61}$$

In the non-relativistic case, if we assume $\gamma \ll 1$, then this expression is given as

$$\Delta\gamma_n^{(\alpha)} = \frac{\lambda^2}{k_\kappa L}\mathrm{Im}[C^*_{+,\kappa}](n - |\alpha|^2)$$

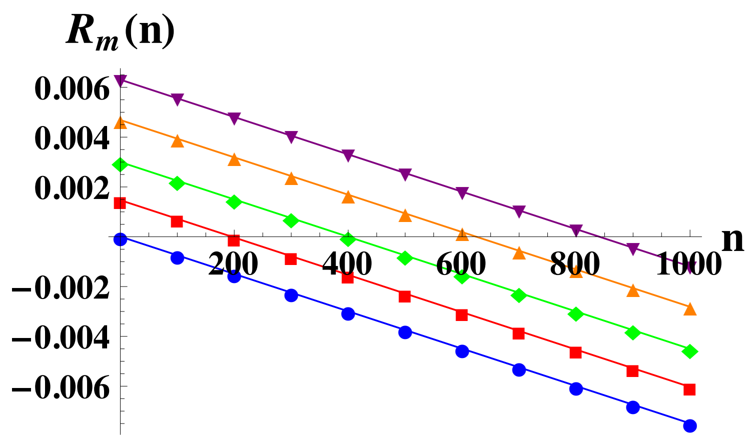


Figure 3.11: Phase resolution required to distinguish between a Fock state of unknown number n of photons from another with $n+m$ photons, using a coherent state as a reference.

We can compare the phase resolution in this case (the interferometric phase difference between a Fock state of n photons from a Fock state of $n+m$ photons) with the high phase resolution obtained using Fock states instead of coherent states in [44]. We defined the phase resolution of the interferometric experiment as the difference in the observed interferometric phase between states with n and $n+m$ photons. In section 3.1 we employed a known Fock state as a reference, yielding

$$\mathcal{R}_m(n) = \Delta\gamma(m+n) - \Delta\gamma(n),$$

where the respective interferometric phases are respectively $\Delta\gamma(m+n)$ and $\Delta\gamma(n)$ for $m+n$ and n photons. This phase resolution was found to respond linearly for small number of photons in the target state. However as the number of photons increases, the slope of the curve decreases logarithmically, worsening the resolution of the interferometric experiment [44]. This is not surprising: it is challenging to distinguish between a state with a million photons from one with a million and one photons, unlike distinguishing between single photon and two-photon states.

A significant challenge in the method 3.1 is to prepare a coherent state of light in the reference cavity. The discussion above indicates that coherent states can be used as a reference for the interferometric phase determination of an unknown Fock state.

In this new scenario, the interferometric phase between the reference coherent state and the unknown Fock state that we want to identify is given by (3.61). The phase resolution required to distinguish between a Fock state containing n photons and $n+m$ photons, in this new scheme where the reference is a known coherent-state, is

$$\Delta_m^{(\alpha)}\gamma(n) = \Delta\gamma_n^{(\alpha)} - \Delta\gamma_{n+m}^{(\alpha)}$$

which becomes, upon using equation (3.61)

$$\Delta_m^{(\alpha)}\gamma(n) = \mathrm{Re}\left[\mathrm{i}\ln\left(\frac{1-\lambda^2\left(\frac{nC^*_{+,\kappa}}{k_\kappa L}+\sum_\beta \frac{C^*_{+,\beta}}{k_\beta L}\right)}{1-\lambda^2\left(\frac{(n+m)C^*_{+,\kappa}}{k_\kappa L}+\sum_\beta \frac{C^*_{+,\beta}}{k_\beta L}\right)}\right)\right.$$

yielding an expression independent of the amplitude of the coherent state, one similar to that of a reference cavity sustaining a Fock state of known photon number m. figure 3.11 shows the phase resolution plotted against the unknown number of photons n in the Fock state that we want to probe. We see that the phase resolution is linearly dependent on the number of photons. Given that the phase resolution required for an interferometric experiment is of the order of fractions of milirands, we see from the plot that we have more than enough phase resolution to distinguish between Fock states differing only in one photon using a coherent state reference.

Chapter 4

Conclusion

4.1 General Disscusion

We have discussed the properties of a practical quantum measurement. Unlike an ideal measurement scheme [65], a realistic quantum measurement process such as photodetection process usually perturbs the quantity being measured by adding noise in the system under study. This has the disadvantage in that it changes the state of the quantum system and makes it inaccessible for future measurement.The purpose of a QND measurement is to control this backaction noise. Since its introduction, many theoretical and experimental work has been done to realize a QND measurement in quantum optics – a suitable domain for realizing QND measurements [45, 58].

We have achieved a kind of optical QND measurement technique – the 'mode invisibility' technique. With the objective to detect single photons, our measurement set up figure 3.1, consists of a meter beam coupled resonantly to a single mode of a highly reflective cavity where we trap a signal beam. The resonant interaction ensures a maximum correlation between the meter and signal beam however this yields highest alteration of the signal beam which is not ideal for a QND measurement scheme. The mode invisibility technique (see section 2.3.2) allows us to cancel the largest contribution to the transition probability of the meter beam while keeping the information gained in the cavity about the signal beam sensibly high. In this way it is possible to obtain information about the quantum state of light without perturbing the system very much via an atomic interferometric experiment. This is in satisfaction of the criteria that characterizes any QND measurement scheme [18, 45] which requires that the measured system should not be significantly perturbed and that the probe should pick information about the signal so that a measurement is actually defined.

We have used this mode invisibility technique in a combined system of single atoms and Fock state of light to measure the relative difference between two given Fock states of light [44]. We showed that any Fock state could be significantly measured without degrading of the field states excessively (non-demolition property). For realistic values of the parameters (microwave cavities) there is enough resolution to distinguish states that differ only by one photon and whose photon population can be of the order of 10^3. This opens up the possibility of constructing extremely sensitive measurement schemes with the ability to detect and identify states of light containing only a few photons with small measurement error.

We further extended the mode invisibility technique to build more general settings where coherent states and squeezed states of light can be distinguished. Our observation reveals an ability to acquire information about the Wigner function of more general states of light in a non-destructive way. In the context of a QND measurement of squeezed states, information about the amplitude of the squeezing parameter r, the coherent state parameter $|\alpha|$ and, even more, about the relative phase difference $(2\theta - \phi)$, (the direction of the squeezing in phase space relative to the direction of displacement) were revealed in the measured phase acquired by the probe. Since the method is non-demolition, we could employ successive measurements (which will not alter the state of light significantly) to characterize more than one parameter of the state of light.

Since to measure the phase factor which carries information about the system we have probed, we use an atomic interferometer which requires us to setup a reference cavity containing a known state of light, we also showed in our measurement technique for single photon detection that instead of preparing a Fock state in a reference cavity in order to probe a target Fock state, we could instead prepare a coherent state in this reference cavity. The use of coherent state sustained in a mode of a reference cavity during a single photon detection process yields the same phase resolution as when a Fock state is used in the reference cavity instead. This eliminates the setbacks of having to control and maintain a reference Fock state to probe another one [67].

4.2 Future Prospects

To date, QND measurement remains the only way that we can extract information from a quantum system without perturbing its state and then allowing a feedback and control of the same measured state. Its realization could be very useful for the development of quantum information technologies which require complete control of quantum measurement such as systems for quantum simulation and quantum computation, harnessing quantum mechanics toward the goal of practical application. Everything that could be known to realize a QND measurement for technology purposes be studied and investigated.

Although we have presented the ‘mode invisibility’ technique to be useful for the measurement of families of light state, many unusual aspects of QND measurement such as extracting information about the Wigner function of a given light state have not yet been studied and leave many exciting possibilities for future research. Mode invisibility measurement technique might even in particular be useful for quantum state preparation. We propose to measure the higher degrees of freedom such as the variance and standard deviation of light field by which it is possible to measure the impression in our measurement scheme.

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