Generalized Complex Structures on Kodaira Surfaces

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we study generalized complex structures on Kodaira surfaces, which are non-Kähler surfaces that admit holomorphic symplectic structures. We show, in particular, that the moduli space of even-type generalized complex structures on a Kodaira surface is smooth of complex dimension four. Furthermore, we give an explicit description of this moduli space using deformation theory. We also obtain a Global Torelli Theorem for Kodaira surfaces in the generalized setting, which is an analogue of Huybrechts' result for generalized K3 surfaces. Finally, we study generalized holomorphic bundles with respect to the even-type generalized complex structures previously obtained.

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Dedication

This thesis is dedicated to my wife, Parisa.

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Chapter 1

Introduction

Generalized complex geometry is a relatively new branch of mathematics first introduced by Hitchin in [17] (2001) and further developed by Gualtieri in his PhD thesis [11] and subsequent papers ([15], [12], [5], and [14] to name a few). In the generalized setting, instead of studying structures on the tangent and cotangent bundles of a manifold, one instead examines structures on their direct sum $T \oplus T^*$. This theory contains symplectic geometry and complex geometry as special cases, and it provides the proper framework for many questions arising in geometry and physics; it has become a very active research area. The goal of this thesis is to examine generalized complex structures and generalized holomorphic bundles on Kodaira surfaces, which are non-Kähler compact complex surfaces with trivial canonical bundle. By the Enriques-Kodaira classification, the only compact complex surfaces with trivial canonical bundle are K3 surfaces, complex 2-tori, and Kodaira surfaces. We also consider these structures and bundles on complex 2-tori.

In Chapter 2, we present general results pertaining to generalized complex geometry. We start with an examination of generalized complex structures in the linear setting and then extend them to manifolds; throughout Sections 2.1, 2.2, and 2.3, we follow Gualtieri's thesis [11]. Additionally, in Section 2.4, we introduce a special class of generalized complex structures, namely generalized Calabi-Yau structures, and prove that on a complex surface every generalized Calabi-Yau structure of even-type is the B-field transform of a complex structure or a symplectic structure (Theorem 2.51). We end the chapter by discussing generalized Kähler structures and note, in particular, that Kodaira surfaces do not admit them (Corollary 2.54), by invoking a result of Apostolov on biHermitian structures [1].

In Chapter 3, we present generalized versions of the classical Global Torelli Theorems for K3 surfaces, complex 2-tori, and Kodaira surfaces. The classical theorems state that cohomologous complex structures are isomorphic. One can, however, prove that cohomologous generalized Calabi-Yau structures are isomorphic in certain cases. For K3 surfaces and complex 2-tori, one requires a hyperKählerity assumption. The result was first obtained by Huybrechts in [21] for K3 surfaces, but extends naturally to complex 2-tori (see Theorem 3.14). Although Kodaira surfaces do not admit hyperKähler-type structures, we nonetheless obtain an analogue of the Global Torelli Theorem in Section 3.2 (Theorem 3.17). In the last section, we discover that while Kodaira surfaces are non-Kähler they do admit so called Kähler-type generalized Calabi-Yau structures. We construct an explicit family of such structures.

In Chapter 4, we present some general results about deformations of generalized complex structures and compute explicit deformations over Kodaira surfaces and complex 2-tori. We begin by outlining the deformation theory for generalized complex structures developed in [11] and [15], and prove that the deformation spaces of symplectic-type structures are always smooth (Theorem 4.15). We also prove that B-field transforms preserve the smoothness of deformation spaces (Theorem 4.17). Next, we compute explicit deformations of even-type structures on Kodaira surfaces and complex 2-tori in Sections 4.2 and 4.3, respectively. We, in particular, show that starting at a complex structure or a symplectic structure yields a smooth family of deformations (Theorems 4.21 and 4.30). We then use these results to give an explicit description of moduli spaces of even-type generalized complex structures on Kodaira surfaces and complex 2-tori in Section 4.4. In Section 4.5, we prove directly that generalized complex structures fail to give rise to positive definite metrics and hence generalized Kähler structures, in many cases. Finally, in Section 4.6, we construct an odd-type generalized complex structure on a Kodaira surface.

In Chapter 5, we study generalized holomorphic bundles, which are analogues of holomorphic vector bundles on complex manifolds in the generalized setting. More precisely, generalized holomorphic bundles are flat Lie algebroid connections for which the Lie algebroid is a generalized complex structure. In Section 5.1, we examine Lie algebroid connections and prove that flat connections are preserved under Lie algebroid isomorphisms (Proposition 5.2). We then restrict ourselves to generalized holomorphic bundles. If the base generalized complex structure comes from a complex structure, then generalized holomorphic bundles are co-Higgs bundles (see Section 5.1.1). Hitchin studies co-Higgs bundles in [19] and Rayan describes their moduli spaces over $\mathbb{CP}^1$ in [26] and over $\mathbb{CP}^2$ in [27]. We introduce the notion of twisted co-Higgs bundles in Section 5.1.2 and show that flatness is preserved by B-field transforms (Proposition 5.11). In Sections 5.1.3 and 5.1.4 we we study two special classes of generalized holomorphic bundles that correspond to Poisson modules and flat bundles, respectively. We end the chapter by giving an explicit description of generalized holomorphic bundles over Kodaira surfaces and complex 2-tori with respect to the families of generalized complex structures obtained in Sections 4.2 and 4.3.

Finally, in Chapter 6, we outline some (as of yet) unanswered questions regarding the above topics. This includes plans for future research as well as some classical open problems.

Chapter 2

Generalized Complex Structures

In this chapter, we introduce generalized complex geometry by first examining generalized complex structures in the linear setting (Section 2.1) and then extending our study to general manifolds (Section 2.3). We also summarize the necessary results concerning spinors and how they can be used to represent generalized complex structures in Section 2.2. In addition, we provide an exposition on generalized Calabi-Yau structures in Section 2.4. Lastly, generalized Kähler structures are presented in Section 2.5 and we consider the trivial example. This chapter follows Marco Gualtieri's PhD thesis [11] closely in the treatment of the subject. We focus on results that are important for the remainder of the thesis but the reader is encouraged to reference [11] and [15] for a more thorough examination of the subject and [17] and [21] for generalized Calabi-Yau structures.

2.1 Linear Algebra

Throughout this section we will use V to denote a real, finite dimensional vector space and $X + \alpha$ to denote a typical element of $V \oplus V^*$. We can define a non-degenerate bilinear form on $V \oplus V^*$; let $X, Y \in V$ and $\alpha, \beta \in V^*$ then for $X + \alpha, Y + \beta \in V \oplus V^*$ we define

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X))$$

which is symmetric. We call it the **inner product** on $V \oplus V^*$, and denote it by $\langle \cdot, \cdot \rangle$.

If we fix a basis $\{e_1, \ldots, e_n\}$ of V and consider the dual basis $\{e^1, \ldots, e^n\}$ for V^*, then with respect to the basis $\{e_i, e^j\}_{1\leq i,j\leq n}$ of $V \oplus V^*$ the matrix corresponding to $\langle \cdot, \cdot \rangle$ is

$$\frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

and so $\langle \cdot, \cdot \rangle$ has n positive and n negative eigenvalues, that is, has signature (n, n).

Definition 2.1. We call a subspace $L \leq V \oplus V^*$ **isotropic** if $\langle X + \alpha, Y + \beta \rangle = 0$ for any $X + \alpha, Y + \beta \in L$.

We wish to discuss isotropic subspaces with the largest possible dimension, and so we make use of the following claim:

Claim 2.2. *The maximal dimension of an isotropic subspace is* $\frac{1}{2}\dim(V \oplus V^*)$.

Proof. Let L be an isotropic subspace. First we define the orthogonal space for L:

$$L^{\perp} := \{v \in V \oplus V^* : \langle v, w \rangle = 0 \text{ for all } w \in L\}.$$

Then since L is isotropic, it is clear that $L \subseteq L^{\perp}$.

Next, we show that $\dim(L) + \dim(L^{\perp}) = \dim(V \oplus V^*)$. Define a map $h : V \oplus V^* \to L^*$ by $v \mapsto \langle v, - \rangle\,|_L$. It is clear that the kernel of this map is $L^{\perp}$. Then, the non-degeneracy of $\langle \cdot, \cdot \rangle$ implies the map $X \mapsto \langle X, - \rangle : V \oplus V^* \to V \oplus V^*$ is a linear isomorphism. Hence, h is surjective because the restriction map is the dual of the inclusion map (which is injective). In other words, $\operatorname{im}(h) = L^*$. Therefore, by the rank-nullity theorem,

$$\dim(V \oplus V^*) = \dim(\operatorname{im}(h)) + \dim(\ker(h)) = \dim(L^*) + \dim(L^{\perp}) = \dim(L) + \dim(L^{\perp}).$$

Finally, we note that if $\dim(L) > \frac{1}{2}\dim(V \oplus V^*)$ then since $L \subseteq L^{\perp}$, we obtain $\dim(V \oplus V^*) = \dim(L) + \dim(L^{\perp}) > \dim(V \oplus V^*)$, which is a contradiction. Hence the dimension of such a subspace can be no larger than $\frac{1}{2}\dim(V \oplus V^*)$. But V is clearly an isotropic subspace of $V \oplus V^*$, with dimension $\frac{1}{2}\dim(V \oplus V^*)$. Therefore the maximal dimension of an isotropic subspace is precisely $\frac{1}{2}\dim(V \oplus V^*)$. $\square$

Definition 2.3. A **maximal isotropic** subspace $L \leq V \oplus V^*$ is an isotropic subspace of maximal dimension.

Maximal isotropic subspaces will be of particular importance when we begin our discussion of generalized complex geometry. Let us examine some examples.

Example 2.4. The spaces V and V^* are both maximal isotropics.

Example 2.5. Let $E \leq V$ be a subspace, and define $\mathrm{Ann}(E)$ to be the annihilator of E in V^*. Then the subspace

$$E \oplus \mathrm{Ann}(E) \leq V \oplus V^*$$

is isotropic. It is also maximal since $\dim(\mathrm{Ann}(E)) = \dim(V) - \dim(E)$, and so $\dim(E \oplus \mathrm{Ann}(E)) = \dim(E) + \dim(\mathrm{Ann}(E)) = \dim(V) = \frac{1}{2}\dim(V \oplus V^*)$ as desired.

Example 2.6. Again, let $E \leq V$ be any subspace. Let $\epsilon \in \wedge^2 E^*$ be a 2-form. We can think of ϵ as a skew-symmetric map from V to V^* ($\epsilon : V \to V^*$) via $\epsilon : X \mapsto \epsilon(X, -)$. Now define a subspace $L(E, \epsilon) \leq V \oplus V^*$:

$$L(E, \epsilon) = \{X + \alpha \in E \oplus V^* : \alpha|_E = \epsilon(X, -)\}$$

which satisfies

$$\dim(L(E, \epsilon)) = \frac{1}{2}\dim(V \oplus V^*).$$

So we only need to check that $L(E, \epsilon)$ is isotropic. For this, fix $X + \alpha, Y + \beta \in L(E, \epsilon)$. Then,

$$\begin{aligned}
\langle X + \alpha, Y + \beta \rangle &= \frac{1}{2}(\beta(X) + \alpha(Y)) \\
&= \frac{1}{2}(\epsilon(Y, X) + \epsilon(X, Y)) \\
&= \frac{1}{2}(\epsilon(Y, X) - \epsilon(Y, X)) = 0,
\end{aligned}$$

which shows that $L(E, \epsilon)$ is isotropic.

It turns out that every maximal isotropic subspace of $V \oplus V^*$ is of this form as is illustrated in the following proposition.

Proposition 2.7 (Gualtieri, [11], Proposition 2.6)**.** *Every maximal isotropic of $V \oplus V^*$ is of the form $L(E, \epsilon)$.*

To relate this to Examples 2.4 and 2.5, we can see that: $V = L(V, 0)$, $E \oplus \mathrm{Ann}(E) = L(E, 0)$, and $V^* = L(\{0\}, 0)$.

We are now ready to define a generalized almost complex structure on V, which is an extension of both a complex structure and a symplectic structure. Recall that a complex structure on V is an endomorphism $J : V \to V$ such that $J^2 = -1$. Moreover, a symplectic structure ω on V is a nondegenerate 2-form $\omega \in \wedge^2 V^*$ which we can view as a map $\omega : V \to V^*$ via $\omega(X) = \iota_X \omega$. Using this interpretation of ω, we can see $\omega^* = -\omega : (V^*)^* = V \to V^*$, or in other words that ω is skew.

When we extend this notion to $V \oplus V^*$, we make a natural identification between $(V \oplus V^*)^*$ and $V \oplus V^*$ via the inner product by defining the action

$$(X+\alpha)(Y+\beta) =< X+\alpha, Y+\beta > .$$

Definition 2.8. A **generalized complex structure** on V is an endomorphism $\mathbb{J} : V \oplus V^* \to V \oplus V^*$ that is both complex ($\mathbb{J}^2 = -1$) and orthogonal with respect to $< \cdot, \cdot >$ (that is, $\mathbb{J}^*\mathbb{J} = 1$).

Another way to define a generalized complex structure on V is to define it as a complex structure on $V \oplus V^*$ that is "symplectic": $\mathbb{J}^* = -\mathbb{J}$. These definitions are equivalent, as demonstrated in the next proposition.

Proposition 2.9 (Gualtieri, [11], Proposition 4.2)**.** *A complex structure $\mathbb{J}$ on $V \oplus V^*$ satisfies $\mathbb{J}^*\mathbb{J} = 1$ if and only if $\mathbb{J}^* = -\mathbb{J}$.*

Proof. Using the fact that $\mathbb{J}^2 = -1$, we can see that $\mathbb{J}^{-1} = -\mathbb{J}$. Therefore, if $\mathbb{J}^* = -\mathbb{J}$, then $\mathbb{J}^* = \mathbb{J}^{-1}$, which means $\mathbb{J}^*\mathbb{J} = 1$. Conversely, if $\mathbb{J}^*\mathbb{J} = 1$ then $\mathbb{J}^* = \mathbb{J}^{-1}$, which implies $\mathbb{J}^* = -\mathbb{J}$. $\square$

To illustrate why such a structure is called *generalized*, we note that if we start with a complex or symplectic structure, then we get a generalized complex structure.

Example 2.10. (The Complex Case) Suppose we have a complex structure J on V, then we define

$$\mathbb{J}_J := \begin{bmatrix} -J & 0 \\ 0 & J^* \end{bmatrix}.$$

It is straightforward to verify $\mathbb{J}_J^2 = -1$, and $\mathbb{J}_J^*\mathbb{J}_J = 1$.

Example 2.11. (The Symplectic Case) If we have a symplectic structure ω on V, then we can define

$$\mathbb{J}_\omega := \begin{bmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{bmatrix}.$$

Once again, the conditions for $\mathbb{J}_\omega$ to be a generalized complex structure are easily verified.

In order to understand all generalized complex structures on V, we simply need to understand maximal isotropic subspaces as the next proposition illustrates.

Proposition 2.12 (Gualtieri, [11], Proposition 4.3)**.** *A generalized complex structure on V is equivalent to the specification of a complex maximal isotropic subspace $L \leq (V \oplus V^*) \otimes \mathbb{C}$ such that $L \cap \overline{L} = \{0\}$ (L has real index zero). Here, we extend the inner product to the complexification by complex linearity in each argument.*

Proof. First suppose $\mathbb{J}$ is a generalized complex structure on V. Define L to be the $+i$-eigenspace of $\mathbb{J}$ in $(V \oplus V^*) \otimes \mathbb{C}$. Then if $X, Y \in L$ we get

$$\begin{aligned} \langle X, Y \rangle &= \langle \mathbb{J}(X), \mathbb{J}(Y) \rangle \\ &= \langle iX, iY \rangle \\ &= -\langle X, Y \rangle \end{aligned}$$

which implies $\langle X, Y \rangle = 0$. Therefore L is isotropic and half-dimensional, which implies L is a maximal isotropic subspace. Furthermore, $\overline{L}$ is the $-i$-eigenspace, so $L \cap \overline{L} = \{0\}$.

Converesely, suppose L is a maximal isotropic subspace such that $L \cap \overline{L} = \{0\}$. This implies that $(V \oplus V^*) \otimes \mathbb{C} = L \oplus \overline{L}$. Hence we can define $\mathbb{J}$ by specifying its action on the spaces L and $\overline{L}$. So define $\mathbb{J}$ to be multiplication by i on L and multiplication by $-i$ on $\overline{L}$. Then $\mathbb{J}$ is a generalized complex structure on V. $\square$

Definition 2.13. The **type** of a generalized complex structure is the (complex) codimension of $\pi_{V_\mathbb{C}} L$ in $V_\mathbb{C}$ where $\pi_{V_\mathbb{C}}$ is the projection map.

Example 2.14. For the complex example above, $\mathbb{J}_J$ has corresponding maximal isotropic $L_J = V_{0,1} + V^*_{1,0}$ and has type $n = \dim(V_\mathbb{C})$. For the symplectic example let us compute L_ω, the maximal isotropic corresponding to $\mathbb{J}_\omega$. If $X + \alpha \in L_\omega$, then

$$\begin{bmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{bmatrix} \begin{bmatrix} X \\ \alpha \end{bmatrix} = \begin{bmatrix} iX \\ i\alpha \end{bmatrix}.$$

Therefore, we can see that $-\omega^{-1}\alpha = iX$, or $\alpha = -i\omega(X)$. Thus, we have shown that the $+i$-eigenspace has the form $L_\omega = \{X - i\omega(X) : X \in V \otimes \mathbb{C}\}$ and clearly has type zero.

The following proposition provides an easy way to determine if a vector space admits a generalized complex structure.

Proposition 2.15 (Gualtieri, [11], Proposition 4.5)**.** *A vector space V admits a generalized complex structure if and only if it is even-dimensional.*

Proof. First we note that if V is even dimensional, then it admits an actual complex structure, which we have proven induces a generalized complex structure.

For the converse, fix a generalized complex structure $\mathbb{J}$ on V. Then since $\langle\cdot,\cdot\rangle$ is indefinite, there must exist a vector $\gamma \in V \oplus V^*$ such that $\langle\gamma,\gamma\rangle = 0$. But since $\mathbb{J}$ is orthogonal with respect to the inner product, $\mathbb{J}\gamma$ is orthogonal to γ. Indeed,

$$\langle \mathbb{J}\gamma, \gamma\rangle = \left\langle \mathbb{J}^2\gamma, \mathbb{J}\gamma\right\rangle = -\langle \gamma, \mathbb{J}\gamma\rangle = -\langle \mathbb{J}\gamma, \gamma\rangle$$

which implies $\langle\mathbb{J}\gamma,\gamma\rangle = 0$. Similarly we may check $\langle\mathbb{J}\gamma,\mathbb{J}\gamma\rangle = 0$. Therefore, $\{\gamma, \mathbb{J}\gamma\}$ spans an isotropic subspace of $V \oplus V^*$, say P.

If P is not maximal then we can find another vector $\gamma' \notin P$ such that $\langle\gamma',\gamma'\rangle = 0$, and enlarge P by adding in γ' and $\mathbb{J}\gamma'$. The new subspace P is still isotropic. Continue in this manner (of adding in a vector η orthogonal to P satisfying $\langle\eta,\eta\rangle = 0$ together with $\mathbb{J}\eta$ to the spanning set) until a maximally isotropic subspace is obtained. By definition, a maximally isotropic subspace has dimension $\dim(V)$, but by construction this space is even dimensional, as we increase its dimension by two each step. It follows that V must be even dimensional. $\square$

There are two operations we can perform on elements of $V \oplus V^*$ that will be useful to us throughout the thesis: B- and β-transforms.

Definition 2.16. (**B-transform**) Let $B \in \wedge^2 V^*$. We define an action, e^B, on $V \oplus V^*$ as follows for $X + \alpha \in V \oplus V^*$,

$$X + \alpha \mapsto X + \alpha + \iota_X B.$$

It follows directly from the definition of a B-transform that they do not affect projections onto V and therefore preserve type. So we may transform a complex-type or symplectic-type structure by a B-field and obtain another complex-type or symplectic-type structure respectively. This action can also translate maximal isotropic subspaces of the form $L(E,\epsilon)$ from Example 2.6:

$$e^B L(E,\epsilon) = L(E, \epsilon + \iota^* B),$$

where $\iota : E \to V$ is the inclusion. Note that we can obtain any maximal isotropic $L(E,\epsilon)$ as a B-transform of $L(E,0)$ by choosing a 2-form B such that $\iota^*(B) = \epsilon$.

Definition 2.17. (**β-transform**) Let $\beta \in \wedge^2 V$ and define an action on $V \oplus V^*$, e^β, by

$$e^\beta(X + \alpha) := X + \iota_\alpha \beta + \alpha$$

for $X + \alpha \in V \oplus V^*$.

2.2 Spinors on $V \oplus V^*$

Before we discuss these structures on manifolds we introduce one more way of defining them via spinors. Recall that V is an n-dimensional real vector space. For $V \oplus V^*$, we the exterior algebra of V^*, namely, $\wedge^* V^*$; these are the spinors. For any non-zero spinor ϕ, its *null space* is defined as

$$L_\phi := \{X + \alpha \in V \oplus V^* : X \lrcorner \phi + \alpha \wedge \phi = 0\}.$$

It is straightforward to check that L_ϕ is isotropic. We consider spinors of even-type and odd-type where even-type spinors are elements of $\wedge^{2k} V^*$ and odd-type spinors are elements of $\wedge^{2k+1} V^*$. We now define a pairing on even- or odd-type spinors.

Definition 2.18. The **Mukai pairing** on spinors is defined differently if the arguments are even or odd. For two even spinors ϕ and ψ,

$$< \phi, \psi >= \sum_m (-1)^m \phi_{2m} \wedge \psi_{n-2m},$$

and for two odd spinors

$$< \phi, \psi >= \sum_m (-1)^m \phi_{2m+1} \wedge \psi_{n-2m-1}.$$

Definition 2.19. A spinor ϕ is called **pure** if the corresponding null space, L_ϕ, is maximally isotropic.

There is a simple expression for the purity condition on 4-dimensional vector spaces using the Mukai pairing from [17].

Proposition 2.20 (Hitchin, [17], Section 4.4)**.** *A spinor ϕ on a 4-dimensional vector space is pure if and only if $< \phi, \phi >= 0$.*

We now summarize some important results regarding pure spinors. See [11] or [7] for more details on spinors. Firstly, we note that *all* maximally isotropic subspaces of $V \oplus V^*$ are represented by a unique spinor (up to multiplication by a scalar) in $\wedge^* V^*$. Secondly, we must consider which maximal isotropics have real index zero ($L \cap \overline{L} = 0$) and for that we require the following proposition.

Proposition 2.21 (Chevalley, [7], Proposition III.2.4)**.** *Two maximal isotropic spaces L and L' satisfy $L \cap L' = \{0\}$ if and only if their respective pure spinor representatives ϕ and ϕ' satisfy $< \phi, \phi' > \neq 0$.*

Lastly, we wish to the identify the spinor line corresponding to the maximal isotropic subspace $L(E, \epsilon)$.

Proposition 2.22 (Gualtieri, [11], Proposition 2.24)**.** *Let $L(E, \epsilon)$ be any maximal isotropic. Then the pure spinor defining it is given by*

$$ce^B \theta_1 \wedge \ldots \wedge \theta_k$$

for any $c \neq 0$, where $B \in \wedge^2 V^*$ is a 2-form that satisfies $\iota^* B = \epsilon$ and $(\theta_1, \ldots, \theta_k)$ is a basis for $\mathrm{Ann}(E)$.

The above proposition shows that the spinor corresponding to $L(E, \epsilon)$ is unique up to multiplication by a non-zero constant. Therefore it is natural to speak of the *spinor line* corresponding to a maximal isotropic. Also, the next proposition shows that all of the above results carry over if we consider the complexified space $(V \oplus V^*) \otimes \mathbb{C}$.

Proposition 2.23 (Gualtieri, [11], Proposition 2.25)**.** *Any maximal isotropic subspace $L \leq (V \oplus V^*) \otimes \mathbb{C}$ of type k corresponds to a pure spinor line generated by*

$$\phi_L = e^{B + i\omega} \Omega,$$

where B and ω are real 2-forms and $\Omega = \theta_1 \wedge \ldots \wedge \theta_k$ for some linearly independent complex 1-forms θ_i. The integer k is the type of the generalized complex structure. This corresponds to the definition of type we gave in the previous section. Furthermore, L is of real index zero if and only if

$$\omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0.$$

Again, our two main examples stem from a complex structure and a symplectic structure, see [11].

Example 2.24. (The Complex Case) For $\mathbb{J}_J$ (from Example 2.10), the spinor line is generated by $\phi = \Omega^{n,0}$, where $\Omega^{n,0}$ is a generator of the $(n, 0)$-forms on V with respect to the complex structure J.

Example 2.25. (The Symplectic Case) For $\mathbb{J}_\omega$ (from Example 2.11), the spinor line is generated by $\phi = e^{i\omega}$ where ω is the given symplectic structure.

2.3 Generalized Complex Structures

We wish to translate the ideas presented in the previous sections to manifolds. The two main ideas are to replace the tangent bundle T with the direct sum of the tangent and cotangent bundles $T \oplus T^*$ and the Lie bracket with the *Courant bracket*, which we define in Definition 2.32. Recall that an *almost complex structure* on a manifold M with tangent bundle T is a linear endomorphism $J : T \to T$ such that $J^2 = -1$. Such a structure is *integrable* if and only if its i-eigenbundle, $L \subseteq T \otimes \mathbb{C}$, is closed under the Lie bracket.

Given Proposition 2.15, we restrict our attention to even-dimensional manifolds. We begin with a discussion of generalized almost complex structures and then proceed to an integrability condition, which is the same process that is used when transporting the notion of complex structures on vector spaces to manifolds.

Definition 2.26. A **generalized almost complex structure** on a real $2n$-dimensional manifold M is given by any of the following three equivalent sets of data:

1. An almost complex structure on $T \oplus T^*$ which is orthogonal with respect to $\langle \cdot, \cdot \rangle$,
2. A maximal isotropic subbundle $L \leq (T \oplus T^*) \otimes \mathbb{C}$ such that $L \cap \overline{L} = \{0\}$.
3. A pure spinor line bundle $U \leq \wedge^* T^* \otimes \mathbb{C}$ such that, for any generator ϕ of U,
$$< \phi, \overline{\phi} > \neq 0$$
at each point $x \in M$.

Definition 2.27. The **type** of a generalized complex structure is the (complex) codimension of the projection of L onto $T \otimes \mathbb{C} = T_{\mathbb{C}}$. Equivalently, if ϕ is a generator for the corresponding pure spinor line bundle, say locally that $\phi = e^{B+i\omega}\Omega$ where $\Omega \in \wedge^k T^*$, then the integer k is the type of the structure. If k is even then we say the generalized complex structure is of **even-type** and if it is odd the structure is of **odd-type**. On a real $2n$-dimensional manifold, a structure of type zero is called a **symplectic-type** structure while a structure of type n is called a **complex-type** structure. The type may not be the same at each point on the manifold, it is an integer-valued function on the manifold.

Both Examples 2.10 and 2.11 give examples of generalized almost complex structures. We examine these more closely after we understand the integrability condition. As in the linear setting, one can obtain an analogue of Proposition 2.20 in the manifold setting which we exhibit next.

Proposition 2.28 (Hitchin, [17], Section 4.4). *A spinor ϕ on a 4-dimensional manifold is pure if and only if $< \phi, \phi >= 0$.*

The first step is to discuss when a generalized almost complex structure exists on an even-dimensional manifold. Ideally we would like to relate it to the existence of almost complex structures which are well understood. We have already shown in the previous section how to obtain a generalized almost complex structure from an almost complex structure. Thus, any manifold admitting an almost complex structure also admits a generalized complex structure. The converse is also true: given a generalized almost complex structure on a manifold, we can construct an almost complex structure on the manifold.

Proposition 2.29 (Gualtieri, [11], Proposition 4.15). *For an even dimensional manifold M, the obstruction to the existence of a generalized almost complex structure is the same as the obstruction to the existence of an almost complex structure.*

Proof. Let $\mathbb{J}$ be a generalized almost complex structure. We can choose a positive definite subbundle, say $C_+ \leq T \oplus T^*$, that is complex with respect to $\mathbb{J}$ (that is, $\mathbb{J}(C_+) \subseteq C_+$). In fact, all we need is a subbundle of maximal (half) dimension for, if $\langle X, X \rangle > 0$, then $\langle \mathbb{J}X, \mathbb{J}X \rangle = \langle X, X \rangle > 0$. An example of such a subbundle is the the set of diagonal elements $span\{\frac{\partial}{\partial x_i} + dx_i\}$. Therefore, the orthogonal complement $C_- := C_+^\perp$ is negative definite as well as complex with respect to $\mathbb{J}$.

Since $C_\pm$ are both definite subbundles, while T^* is null ($\langle X, X \rangle = 0$ for all $X \in T$), the projection $\pi : C_\pm \to T$ is an isomorphism. Indeed, if $x \in C_+$ and $\pi(x) = 0$ then x only has a cotangent component which means $\langle x, x \rangle = 0$, which forces $x = 0$. Therefore, since $\mathbb{J}|_{C_\pm}$ are almost complex structures on $C_\pm$ and $C_\pm \cong T$, the two almost complex structures $\mathbb{J}|_{C_\pm}$ induce two almost complex structures $J_\pm$ on T. Hence, a generalized almost complex structure gives rise to an almost complex structure as desired. $\square$

Remark 2.30. We should remark that sufficient conditions for an almost complex structure to exist are not entirely known. In fact, they are only known in dimension ≤ 10. One necessary condition for the existence of an almost complex structure (and hence generalized almost complex structure) on an even-dimensional manifold is for all its odd Stiefel-Whitney classes to be zero, but there are others. For a discussion and more references, see [11].

We now extend the definition of a B-field transform (from Definition 2.16) to the manifold setting. The additional property we will require is that B is closed.

Definition 2.31. (B-Field Transforms) For a generalized almost complex structure L and a real *closed* 2-form B, define the B**-field transform** of L as $L_B := e^B(L)$ where $e^B(X+\alpha) = X+\alpha+\iota_X B$. Define **complex** B**-field transforms** analogously for complex closed 2-forms B.

Next, we proceed to the question of integrability, and for this we will need an extension of the Lie bracket, namely the Courant bracket.

Definition 2.32. The **Courant Bracket** is defined as

$$[X+\alpha, Y+\beta]_C = [X,Y] + \mathcal{L}_X(\beta) - \mathcal{L}_Y(\alpha) - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha),$$

for $X+\alpha, Y+\beta$ smooth sections of $T \oplus T^*$.

Note that this bracket is skew. It is not, however, a Lie bracket since it does not satisfy the Jacobi identity. However, if the bracket is restricted to an isotropic subbundle L, then the Jacobi identity is satisfied (so $[\cdot,\cdot]_C$ is a Lie bracket on L).

By definition B-field transforms do not alter projections onto the tangent bundle and therefore preserve type. The reason we want B to be closed is because we want the B-transform e^B to preserve the Courant bracket. This is illustrated in the next proposition.

Proposition 2.33 (Gualtieri, [11], Proposition 3.23)**.** *Let B be a 2-form, then for $x, y \in C^\infty(T \oplus T^*)$, we have $[e^B(x), e^B(y)]_C = e^B([x,y]_C)$ if and only if B is closed.*

Proof. Fix $X+\alpha, Y+\beta \in C^\infty(T\oplus T^*)$. A direct computation shows that

$$[e^B(X+\alpha), e^B(Y+\beta)]_C = e^B([X+\alpha, Y+\beta]_C) + \iota_X\iota_Y dB.$$

Therefore, $[e^B(X+\alpha), e^B(Y+\beta)]_C = e^B([X+\alpha, Y+\beta]_C)$ if and only if $\iota_X\iota_Y dB = 0$ for all X, Y, which is true only when $dB = 0$. □

Definition 2.34. A generalized almost complex structure $\mathbb{J}$ is said to be **integrable** (to a generalized complex structure) when its $+i$-eigenbundle $L \leq (T\oplus T^*)\otimes\mathbb{C}$ is closed under the Courant bracket.

In the next two examples we will show that classical complex and symplectic structures give rise to generalized complex structures and discuss some properties of these generalized complex structures.

Example 2.35. (The Complex Case) In this example we prove that a complex structure gives rise to a generalized complex structure. Let us begin with a complex structure J on T. Then, as we did in Example 2.10, we define

$$\mathbb{J}_J := \begin{bmatrix} -J & 0 \\ 0 & J^* \end{bmatrix},$$

which is a generalized almost complex structure. All that remains is to identify the $+i$-eigenbundle and show it is closed under the Courant bracket.

From the definition of $\mathbb{J}_J$, it is clear that its $+i$-eigenbundle is $L_J = T_{0,1} \oplus T^*_{1,0}$. Then, if J is an integrable complex structure, we know that $T_{0,1}$ is closed under the Lie bracket. Moreover, another consequence of integrability is that the derivative of a $(1,0)$-form is the sum of a $(2,0)$-form and a $(1,1)$-form. Now, we compute for $X+\alpha, Y+\beta \in C^\infty(L_J)$,

$$\begin{aligned} [X+\alpha, Y+\beta]_C &= [X,Y] + \mathcal{L}_X(\beta) - \mathcal{L}_Y(\alpha) - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha) \\ &= [X,Y] + \iota_X d\beta + d(\iota_X\beta) - \iota_Y d\alpha - d(\iota_Y\alpha) - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha). \end{aligned}$$

But since $X, Y \in C^\infty(T_{0,1})$ and $\alpha, \beta \in C^\infty(T^*_{1,0})$, we get that $\alpha(Y) = \beta(X) = 0$. Thus, we obtain

$$[X+\alpha, Y+\beta]_C = [X,Y] + \iota_X d\beta - \iota_Y d\alpha.$$

However, since $d\alpha$ and $d\beta$ have only $(2,0)$ and $(1,1)$ components we get that both $\iota_X d\beta$ and $\iota_Y d\alpha$ are of type $(1,0)$ for all $X, Y \in C^\infty T_{0,1}$. Hence,

$$[X+\alpha, Y+\beta]_C = ([X,Y] + \iota_X d\beta - \iota_Y d\alpha) \in C^\infty(L_J),$$

as desired. Therefore, $\mathbb{J}_J$ is an (integrable) generalized complex structure.

The converse is also true, the integrability of $\mathbb{J}_J$ implies the integrability of J since, if $\mathbb{J}_J$ is integrable, then

$$([X,Y] + \mathcal{L}_X(\beta) - \mathcal{L}_Y(\alpha) - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha)) \in C^\infty(L_J),$$

which gives us that $[X,Y] \in C^\infty T_{0,1}$ for any $X, Y \in C^\infty T_{0,1}$. Therefore, $\mathbb{J}_J$ is integrable if and only if J is.

Recall that the pure spinor line corresponding to a structure of this type is generated by $\Omega^{n,0}$, where $\Omega^{n,0}$ is a generator for the $(n,0)$-forms. Since $\Omega^{n,0}$ is an n-form the type of these structures is n, so this structure is of even- or odd-type depending on the parity of n.

Example 2.36. (The Symplectic Case) We show that the generalized almost complex structure associated to a symplectic structure is integrable. A *symplectic structure* on M is a real closed non-degenerate 2-form ω. Recall from Example 2.11 that we defined

$$\mathbb{J}_\omega = \begin{bmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{bmatrix},$$

which is a generalized almost complex structure. Recall from Example 2.14 that the $+i$-eigenbundle for this structure is $L_\omega = \{X - i\omega(X) : X \in T \otimes \mathbb{C}\}$.

Let us prove that this bundle is closed under the Courant bracket. Fix $X, Y \in C^\infty(T \otimes \mathbb{C})$ and compute

$$\begin{aligned} [X - i\omega(X), Y - i\omega(Y)]_C &= [X,Y] + \mathcal{L}_X(-i\omega(Y)) - \mathcal{L}_Y(-i\omega(X)) \\ &\quad -\frac{1}{2}d(\iota_X(-i\omega(Y)) - \iota_Y(-i\omega(X))). \end{aligned}$$

As as aside let us compute $[X,Y] - i\omega([X,Y])$, note that $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$.

$$\begin{aligned} [X,Y] - i\omega([X,Y]) &= [X,Y] + \mathcal{L}_X(-i\omega(Y)) - \iota_Y \mathcal{L}_X(-i\omega) \\ &= [X,Y] + \mathcal{L}_X(-i\omega(Y)) - \iota_Y(d(-i\omega(X)) + \iota_X d(-i\omega)) \\ &= [X,Y] + \mathcal{L}_X(-i\omega(Y)) - \iota_Y d(-i\omega(X)) + i(\iota_Y(\iota_X d(\omega))). \end{aligned}$$

Now, we return to our original computation,

$$\begin{aligned} [X - i\omega(X), Y - i\omega(Y)]_C &= [X,Y] + \mathcal{L}_X(-i\omega(Y)) - \iota_Y d(-i\omega(X)) + d(i\omega(X,Y)) \\ &\quad -\frac{1}{2}d(2i\omega(X,Y)) \\ &= [X,Y] + \mathcal{L}_X(-i\omega(Y)) - \iota_Y d(-i\omega(X)) \\ &= [X,Y] - i\omega[X,Y] - i(\iota_Y(\iota_X d(\omega))) \end{aligned}$$

from the aside above. Hence, we obtain that, if ω is a symplectic structure, then the $+i$-eigenbundle is closed under the Courant bracket since $d\omega = 0$.

Once again, we can see that the converse is true as well. If the generalized almost complex structure $\mathbb{J}_\omega$ is integrable, then $d\omega = 0$. Hence, $\mathbb{J}_\omega$ is integrable if and only if ω is closed. Since $L_\omega = \{X - i\omega(X) : X \in T \otimes \mathbb{C}\}$ we can see that L_ω has full-rank projection onto $T \otimes \mathbb{C}$. Therefore, the type of these generalized complex structures is 0 so these are always even-type structures. The pure spinor line is generated by $\phi = e^{i\omega}$ in this case.

These two examples will be of special importance for the remainder of the thesis. The following proposition fully classifies generalized complex structures of type 0 and type n.

Proposition 2.37 (Gualtieri, [11], Proposition 4.22). *On a 2n-dimensional manifold, a generalized complex structure of type zero (symplectic-type) is B-symplectic, that is, a B-field transform of a symplectic structure. On the other hand, any generalized complex structure of type n (complex-type) has the form $e^{\epsilon}\mathbb{J}_J$ where ϵ is a ∂-closed $(2,0)$-form.*

Remark 2.38. Since ϵ is not necessarily closed nor real, the above proposition shows that not all type n structures are B-field transforms of complex structures (by definition, a B-field transform is a closed, real 2-form). However, in section 2.4 we will see that on a Kodaira surface or complex 2-torus that every constant type n structure is a B-field transform of a complex structure.

Definition 2.39. The **Schouten-Nijenhuis bracket** (or Schouten bracket) is a natural extension of the Lie bracket and the Courant bracket. In this thesis, we will only need it for sections of $\wedge^2(T)$ or $\wedge^2(L)$. For $x, y, w, z \in C^\infty(T)$ (respectively $C^\infty(L)$):

$$[x \wedge y, w \wedge z]_S = [x, w] \wedge y \wedge z - [x, z] \wedge y \wedge w - [y, w] \wedge x \wedge z + [y, z] \wedge x \wedge w$$

where $[\cdot, \cdot]$ is the Lie bracket (respectively Courant bracket).

A special class of type zero generalized complex structures consists of structures that come from holomorphic Poisson structures as illustrated in the following example.

Example 2.40. Let $\beta \in C^\infty(\wedge^2 T_{1,0})$ be a holomorphic Poisson bivector, so that $[\beta, \beta]_S = 0$ where $[\cdot, \cdot]_S$ is the Schouten-Nijenhuis bracket. Write $\beta = P + iQ$ where P and Q are the real and imaginary parts of β, respectively. Since β is a $(2,0)$-bivector, it follows that P and Q are of type $(2,0)$ and $(0,2)$ (neither P nor Q can have a $(1,1)$-component or else β will also have one). If we write $Q = Q^{2,0} + Q^{0,2}$, then

$$QJ^* = iQ^{2,0} - iQ^{0,2},$$

where $QJ^*(\alpha) = Q(J^*\alpha)$ for any 1-form α.

We may also write $P = \frac{1}{2}(\beta + \bar{\beta})$ and $Q = -\frac{i}{2}(\beta - \bar{\beta})$, which means $Q^{2,0} = -\frac{i}{2}\beta$ and $Q^{0,2} = \frac{i}{2}\bar{\beta}$, and

$$QJ^* = \frac{1}{2}\beta + \frac{1}{2}\bar{\beta} = P.$$

The resulting deformed generalized complex structure has the form

$$\mathbb{J}_\beta = \begin{bmatrix} -J & Q \\ 0 & J^* \end{bmatrix}$$

and, using $P = QJ^*$, one can readily compute that this structure has $+i$-eigenbundle

$$L_\beta = T_{0,1} \oplus \Gamma_\beta,$$

where $\Gamma_\beta = \{\eta + \beta(\eta) : \eta \in T^*_{1,0}\}$. Note that integrability of this structure is equivalent to β being a Poisson structure. Indeed, one may show that $[\beta, \beta]_S = 0$ if and only if L_β is closed under the Courant bracket.

2.4 Generalized Calabi-Yau Structures

A generalized Calabi-Yau structure is a special kind of generalized complex structure. Instead of looking at a generalized complex structure as a map $\mathbb{J} : T \oplus T^* \to T \oplus T^*$, we may consider its associated spinor line bundle, which is generated by a pure spinor $\phi \in C^\infty(\wedge^* T^* \otimes \mathbb{C})$ as described in Section 2.2. It will be clear when we use $< \cdot, \cdot >$ on spinors that we are referring to the Mukai pairing.

Definition 2.41. A **generalized Calabi-Yau structure of even-type** on a smooth manifold M of dimension $2m$ is a complex spinor $\phi \in C^\infty(\wedge^{\text{even}} T^* \otimes \mathbb{C})$ on $T \oplus T^*$ satisfying:

- ϕ is pure,
- $< \phi, \overline{\phi} > \neq 0$,
- $d\phi = 0$.

Two generalized Calabi-Yau structures ϕ and ϕ' are **isomorphic** if there exists an exact B-field B and a diffeomorphism f of M such that

$$\phi = e^B f^* \phi'.$$

The third condition is the only difference between a generalized Calabi-Yau structure and a generalized complex structure. Indeed, the first condition guarantees the annihilator

$$L_\phi = \{X + \eta \in T \oplus T^* | (X + \eta) \cdot \phi := \iota_X \phi + \eta \wedge \phi = 0\}$$

is maximally isotropic. The second condition together with proposition 2.21 gives us that $L_\phi \cap \overline{L_\phi} = \{0\}$, because $\overline{L_\phi} = L_{\overline{\phi}}$. All that remains is to show that L_ϕ is closed under the Courant bracket. A proof of this can be found in [17].

Proposition 2.42 (Hitchin, [17], Proposition 1)**.** *If (M, ϕ) is a generalized Calabi-Yau manifold then the annihilator L_ϕ defines a generalized complex structure on M.*

Let us summarize the relationships between generalized complex structures and generalized Calabi-Yau structures:

$$\begin{aligned} \phi \text{ is pure} &\iff L_\phi \text{ is maximal isotropic (Definition 2.19)},\\ <\phi, \overline{\phi}> \neq 0 &\iff L_\phi \cap \overline{L_\phi} = \{0\} \text{ (Proposition 2.21)},\\ d\phi = 0 &\implies L_\phi \text{ is closed under the Courant bracket (Proposition 2.42)}. \end{aligned}$$

Remark 2.43. In general, the third condition is not an equivalence because some generalized complex structures correspond to pure spinor line bundles that are generated by a non-closed ϕ. In other words, generalized Calabi-Yau structures are a special class of generalized complex structures. Nonetheless, on some manifolds, generalized Calabi-Yau structures are equivalent to generalized complex structures as we will see in Theorem 2.46.

Let us look at two important examples of generalized Calabi-Yau structures. For more details regarding these examples, see [17], Section 4.1.

Example 2.44. (The Complex Case) Let J be a complex structure on M (a real $2m$-dimensional manifold) with trivial canonical bundle. Let Ω be a non-vanishing $(m, 0)$-form on M with respect to J. Then, Ω is a generalized Calabi-Yau structure.

Example 2.45. (The Symplectic Case) If ω is any real symplectic form on M, then $\phi_\omega := e^{i\omega} = (1 + i\omega - \frac{\omega\wedge\omega}{2} + \cdots)$ is a generalized Calabi-Yau structure.

Our goal in this thesis is to study left-invariant generalized complex structures on Kodaira surfaces and complex 2-tori, that is, generalized complex structures that descend from left-invariant generalized complex structures on $\mathbb{C}^2$. Both of these surfaces are complex nilmanifolds and the next theorem tells us that, on a complex nilmanifold, left-invariant generalized complex structures are in one-to-one correspondence with generalized Calabi-Yau structures.

Theorem 2.46 (Cavalcanti-Gualtieri, [6], Theorem 3.1)**.** *Any left-invariant generalized complex structure on a nilmanifold must be generalized Calabi-Yau.*

This means that studying left-invariant generalized complex structures on a Kodaira surface or complex torus is the same as studying generalized Calabi-Yau structures. If we are dealing with a nilmanifold we will use the term generalized complex structure to always mean a left-invariant generalized complex structure. The results for generalized Calabi-Yau structures we obtain in Sections 3.1 and 3.2 will be applied to better understand the moduli space of generalized complex structures in Section 4.4.

Remark 2.47. We will only focus on even-type generalized Calabi-Yau structures in this thesis but one can analogously define generalized Calabi-Yau structures of *odd-type*.

In Definition 2.31 we saw how to transform generalized complex structures by B-fields. The following lemma shows that we may transform a generalized Calabi-Yau structure ϕ by a B-field,

$$e^B\phi = \left(1 + B + \frac{B \wedge B}{2} + \cdots\right) \wedge \phi,$$

which is itself a generalized Calabi-Yau structure.

Lemma 2.48 (Hitchin, [17], Section 4.2)**.** *For a generalized Calabi-Yau structures ϕ and a B-field B, $e^B\phi$ is again generalized Calabi-Yau structure.*

We end this section with some results about even-type generalized Calabi-Yau structures on complex surfaces. We start with an important result regarding B-fields and the complex structures from Example 2.44.

Lemma 2.49. *Let M be a complex surface and σ a non-vanishing holomorphic $(2,0)$-form. Then, if τ is any 4-form, there exists a real closed 2-form B (that is, a B-field) such that $\tau = B \wedge \sigma$. Moreover, if τ is exact, then so is B.*

Proof. First, we note that, by consideration of type, we have $d\sigma = 0$, $\sigma \wedge \sigma = 0$ and $\sigma \wedge \overline{\sigma} \neq 0$. Before proceeding with the proof, we will show

$$H^{2,2}_{\overline{\partial}}(M, \mathbb{C}) \cong H^4_{DR}(M, \mathbb{C}).$$

We note, by the Dolbeault theorem,

$$H^{2,2}_{\overline{\partial}}(M, \mathbb{C}) \cong H^2(M, \Omega^2_M)$$

where Ω^2_M is the sheaf of holomorphic 2-forms on M. But we are on a surface, so Ω^2_M is the canonical bundle K_M. Hence, using Serre duality, we obtain

$$H^{2,2}_{\overline{\partial}}(M, \mathbb{C}) \cong H^2(M, K_M) \cong H^0(M, K_M \otimes K_M^*)^* \cong H^0(M, \mathcal{O})^* \cong \mathbb{C}.$$

Also,

$$H^4_{DR}(M, \mathbb{C}) \cong \mathbb{C},$$

so these two spaces have the same dimension. Let us define a map

$$\Psi : H^{2,2}_{\overline{\partial}}(M, \mathbb{C}) \to H^4_{DR}(M, \mathbb{C})$$

as

$$\Psi([\alpha]_{\overline{\partial}}) = [\alpha]_d.$$

This map is well-defined. Indeed, if $[\alpha]_{\overline{\partial}} \in H^{2,2}_{\overline{\partial}}(M, \mathbb{C})$, then α is a closed 4-form and so it makes sense to say $[\alpha]_d$. Further, if $[\alpha_1]_{\overline{\partial}} = [\alpha_2]_{\overline{\partial}}$, then $\alpha_1 - \alpha_2 = \overline{\partial}\gamma = (\partial + \overline{\partial})\gamma = d\gamma$ since γ is a $(2,1)$-form. So $[\alpha_1]_d = [\alpha_2]_d$, which means $\Psi([\alpha_1]_{\overline{\partial}}) = \Psi([\alpha_2]_{\overline{\partial}})$. Lastly, we verify this map is surjective. Then, since the dimensions agree, we will be done. So let $[\beta] \in H^4_{DR}(M, \mathbb{C})$ be given. Therefore, β is a closed 4-form, which means $\overline{\partial}\beta = 0$, and so

$$\Psi([\beta]_{\overline{\partial}}) = [\beta]_d,$$

which says that Ψ is surjective. Hence,

$$H^{2,2}_{\overline{\partial}}(M, \mathbb{C}) \cong H^4_{DR}(M, \mathbb{C}).$$

Now, we proceed with proof of the lemma. Let us first assume that τ is exact. Then, by our above argument,

$$[\tau]_d = 0 = [\tau]_{\overline{\partial}},$$

which means we can write $\tau = \overline{\partial}\gamma$ for some $(2,1)$-form γ. Furthermore, σ is non-vanishing and thus a global generator for the $(2,0)$-forms, we may find a $(0,1)$-form δ such that $\sigma \wedge \delta = \gamma$. Then, $\overline{\delta}$ is a $(1,0)$-form, which means $\sigma \wedge \overline{\delta} = 0$. Define $B := d(\delta + \overline{\delta})$, which is a real exact 2-form and we have

$$B \wedge \sigma = d(\delta + \overline{\delta}) \wedge \sigma = d(\delta \wedge \sigma) + d(\overline{\delta} \wedge \sigma) = d\gamma + 0 = \tau,$$

where the second equality follows from the fact that σ is closed.

Next, if τ is not exact, then we may write

$$\tau = (\tau - c\sigma \wedge \overline{\sigma}) + c\sigma \wedge \overline{\sigma},$$

where $\tau - c\sigma \wedge \overline{\sigma}$ is exact ($c \in \mathbb{C}$). This is because $\sigma \wedge \overline{\sigma}$ is a generator for $H^4_{DR}(M, \mathbb{C})$ which is a 1-dimensional complex vector space. So by our above argument we may find a real exact 2-form B' such that $B' \wedge \sigma = \tau - c\sigma \wedge \overline{\sigma}$. So define $B := B' + c\overline{\sigma} + \overline{c}\sigma$, a closed real 2-form. Thus,

$$\begin{aligned} B \wedge \sigma &= (B' + c\overline{\sigma} + \overline{c}\sigma) \wedge \sigma \\ &= B' \wedge \sigma + c\overline{\sigma} \wedge \sigma + 0 \\ &= \tau - c\sigma \wedge \overline{\sigma} + c\overline{\sigma} \wedge \sigma \\ &= \tau. \end{aligned}$$

$\square$

Remark 2.50. We remark that Hitchin discussed a more general version of the above lemma in [17] (Section 4.4). Moreover, Huybrechts has a version of it in [21] for Kähler surfaces (Proposition 1.5). But we have proven that it holds in general for any compact complex surface with trivial canonical bundle. This will be of fundamental importance for the remainder of this thesis.

Let us examine the general form of a generalized Calabi-Yau structure of even-type, say

$$\phi = \phi_0 + \phi_2 + \phi_4,$$

where ϕ_0 is a constant, ϕ_2 is a closed 2-form and ϕ_4 is a 4-form. The class of generalized Calabi-Yau structures is quite rigid, as the next theorem illustrates.

Theorem 2.51. *Let $\phi = \phi_0 + \phi_2 + \phi_4$ (where ϕ_i is an i-form) be a generalized Calabi-Yau structure of even-type on a compact complex surface with trivial canonical bundle X. Then:*

- *If $\phi_0 = 0$, we get $\phi = e^B\phi_2$ where B is a real closed 2-form and ϕ_2 is a generator for the space of holomorphic 2-forms corresponding to some complex structure on X, and is therefore of complex-type. Hence, complex-type generalized Calabi-Yau structures are B-field transforms of complex structures.*
- *If $\phi_0 \neq 0$, then $\phi = e^{B+i\omega}$ where B is a real closed 2-form and ω is a real symplectic form, which implies ϕ has type zero. Hence, symplectic-type generalized complex structures are B-field transforms of symplectic structures.*

Proof. Let $\phi = \phi_0 + \phi_2 + \phi_4$ be a generalized Calabi-Yau structure of even-type on a complex surface. The purity condition on a complex surface is equivalent to $< \phi, \phi >= 0$ by Proposition 2.28. This gives

$$2\phi_0\phi_4 - \phi_2 \wedge \phi_2 \;\; = \;\; 0, \tag{2.1}$$

and $< \phi, \overline{\phi} > \neq 0$ gives

$$\phi_0\overline{\phi_4} - \phi_2 \wedge \overline{\phi_2} + \overline{\phi_0}\phi_4 \neq 0. \tag{2.2}$$

Now let us consider some cases:

If $\phi_0 \neq 0$, we may solve equation (2.1) to give $\phi_4 = \frac{\phi_2 \wedge \phi_2}{2\phi_0}$ and, substituting this into (2.2), we get

$$\frac{\phi_0\overline{\phi_0}}{2}\left(\frac{(\overline{\phi_2})}{(\overline{\phi_0})} - \frac{\phi_2}{\phi_0}\right)^2 \neq 0.$$

Let us define $\frac{\phi_2}{\phi_0} := B + i\omega$. Then, the above equation implies that ω is non-degenerate. By definition, ω is closed and therefore ω is a real symplectic form. One can check that

$$\phi = e^{B+i\omega}$$

(up to scale) where B is a real closed 2-form, that is, a B-field transform. Therefore, in this case, we obtain B-field transforms of generalized Calabi-Yau structures that come from symplectic structures.

On the other hand, if $\phi_0 = 0$, then the three conditions give us $d\phi_2 = 0$, $\phi_2 \wedge \phi_2 = 0$, and $\phi_2 \wedge \overline{\phi_2} \neq 0$. Hence, we may view ϕ_2 as a non-vanishing holomorphic $(2,0)$-form, implying that it corresponds to a unique complex structure. Then, by Lemma 2.49, we get $\phi_4 = B \wedge \phi_2$ for a real closed 2-form B. Therefore, $\phi = e^B \phi_2$. $\square$

2.5 Generalized Kähler Structures

On a complex manifold, one often considers extra structures such as Hermitian metrics. A special class of such structures is given by the Kähler metrics. Recall that a Hermitian metric g on a complex manifold (M, J) is said to be *Kähler* if its fundamental form ω is d-closed; the choice of a Kähler metric g on (M, J) is called a *Kähler structure* on (M, J). Examples of manifolds that admit Kähler metrics are K3 surfaces and complex tori. Note, however, that not every complex manifold admits Kähler metrics, in which case we say they are *non-Kähler*; for example, Kodaira and Hopf surfaces are non-Kähler (because their first Betti numbers are odd). There also exists a notion of Kähler structure in the generalized setting.

Definition 2.52. A **generalized Kähler structure** is a pair of commuting generalized complex structures $(\mathbb{J}_1, \mathbb{J}_2)$ such that $G := -\mathbb{J}_1\mathbb{J}_2$ induces a positive definite metric $\widetilde{G}$ on $T \oplus T^*$ (where $\widetilde{G}(x,x) := \langle Gx, x\rangle = \langle \mathbb{J}_1 x, \mathbb{J}_2 x\rangle$ for all $x \in C^\infty(T \oplus T^*)$).

A trivial example of such a structure arises from an actual Kähler structure.

Example 2.53. (The Trivial Generalized Kähler Structure) If we start with a Kähler structure on M, say (g, J, ω), then we may consider $g : T \to T^*$ via $X \mapsto g(X, -)$. The properties of a Kähler structure give us that the diagram

$$\begin{array}{ccc} T & \xrightarrow{g} & T^* \\ J\uparrow & & \uparrow\omega \\ T & \xrightarrow{id} & T \end{array}$$

commutes. Then we can show $J^*\omega J = \omega$ by computing for $X, Y \in C^\infty(T)$:

$$\begin{aligned}(J^*\omega J(X))(Y) &= \omega(J(X))(J(Y)) \\ &= \omega(J(X), J(Y)) \\ &= g(J^2(X), J(Y)) \\ &= g(J(X), Y) \\ &= \omega(X, Y),\end{aligned}$$

so in particular

$$J^*\omega = \omega J^{-1} = -\omega J.$$

We will use the natural generalized complex structures, namely,

$$\mathbb{J}_J := \begin{bmatrix} J & 0 \\ 0 & -J^* \end{bmatrix} \text{ and } \mathbb{J}_\omega := \begin{bmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{bmatrix}.$$

We are using the negative version of Example 2.10 for $\mathbb{J}_J$, but this is still a generalized complex structure. To check they define a generalized Kähler structure we must show that they commute and that $G = -\mathbb{J}_J\mathbb{J}_\omega$ induces a positive definite metric.

Notice that

$$\mathbb{J}_J\mathbb{J}_\omega = \begin{bmatrix} 0 & -J\omega^{-1} \\ -J^*\omega & 0 \end{bmatrix}$$

and

$$\mathbb{J}_\omega\mathbb{J}_J = \begin{bmatrix} 0 & \omega^{-1}J^* \\ \omega J & 0 \end{bmatrix}.$$

Then we notice that $J^*\omega = -\omega J$ from the above computations, and inverting both sides of this gives $\omega^{-1}(J^*)^{-1} = -J^{-1}\omega^{-1}$ and hence $-\omega^{-1}J^* = J\omega^{-1}$. Therefore, $\mathbb{J}_J\mathbb{J}_\omega = \mathbb{J}_\omega\mathbb{J}_J$.

Next, observe that

$$G = -\mathbb{J}_J\mathbb{J}_\omega = -\begin{bmatrix} 0 & -J\omega^{-1} \\ -J^*\omega & 0 \end{bmatrix} = \begin{bmatrix} 0 & g^{-1} \\ g & 0 \end{bmatrix}.$$

Then, a direct computation with the identification $(T \oplus T^*)^* = T \oplus T^*$ yields

$$\langle G(X + \alpha), X + \alpha \rangle = g(X, X) + g(g^{-1}(\alpha), g^{-1}(\alpha)) > 0,$$

which implies that G induces a positive definite metric. Hence, we have shown that a Kähler structure gives rise to a generalized Kähler structure.

We have seen in Example 2.53, that any Kähler manifold admits a trivial generalized Kähler structure. For example, K3 surfaces and complex tori both admit trivial generalized Kähler structures. Nevertheless, Kähler manifolds may admit generalized Kähler structures other than the trivial ones; generalized Kähler structures and their deformations have been extensively studied by Goto in [10]. Non-Kähler manifolds can also admit generalized Kähler structures; for example, Hopf surfaces admit many (twisted) generalized Kähler structures (see [11] and [14] and references therein). Nonetheless, not every manifold admits generalized Kähler structures, as is the case for Kodaira surfaces.

In [11], it was proven that a generalized Kähler structure gives rise to a biHermitian structure. However, Apostolov proved the following in [1].

Proposition 2.54 (Apostolov, [1], Corollary 2)**.** *Any compact biHermitian surface with odd first Betti number has Kodaira dimension* $-\infty$*.*

Corollary 2.55. *Kodaira surfaces do not admit generalized Kähler structures.*

Proof. This follows from the fact that Kodaira surfaces have Kodaira dimension 0. □

While Apostolov's result gives us the answer via biHermitian geometry, we may check directly why certain pairs $(\mathbb{J}_1, \mathbb{J}_2)$ of generalized complex structures fail to form a generalized Kähler pair. We will see, in particular, that $G = -\mathbb{J}_1\mathbb{J}_2$ is not positive definite for those pairs. This will be done in Section 4.5.

Remark 2.56. Although the structures are not positive definite they are still non-degenerate and so give rise to *generalized pseudo-Kähler* structures.

Chapter 3

Global Torelli Theorems

In this chapter, we examine some generalizations of classical global Torelli theorems for complex surfaces. We work on compact holomorphic symplectic surfaces, which are K3 surfaces, complex 2-tori, and Kodaira surfaces. While K3 surfaces and complex tori are Kähler and, in fact, hyperKähler, Kodaira surfaces are non-Kähler. Sawon and Glover examine families of generalized complex structures on hyperKähler manifolds in [30] and construct a generalized *twistor space*. In [21] Huybrechts examines the moduli space of generalized Calabi-Yau structures on K3 surfaces. A generalized Global Torelli Theorem for K3 surfaces was presented by Huybrechts ([21], Proposition 2.11) by using the notions of *Kähler* and *hyperKähler* generalized Calabi-Yau structures. We will focus on extending this Global Torelli Theorem to complex 2-tori and Kodaira surfaces. In fact, the results he obtained can be extended directly to complex 2-tori for even-type generalized complex structures. We present a summary of these results in Section 3.1. Then, in Section 3.2, we use a similar technique to prove a global Torelli theorem for Kodaira surfaces. However, the assumptions required on a Kodaira surface are different than those for a Kähler surface. Finally, in Section 3.3, we explicitly construct a family of generalized Calabi-Yau structures of *Kähler-type* on a Kodaira surface, despite such surfaces being non-Kähler.

For reference, let us provide statements of the classical global Torelli theorems for K3 surfaces, complex 2-tori, and Kodaira surfaces. The versions we discuss give conditions for when two complex structures are isomorphic. Recall every complex structure J on a compact complex surface with trivial canonical bundle gives rise to a holomorphic non-vanishing $(2,0)$-form. On the other hand, a 2-form σ satisfying $d\sigma = 0$, $\sigma \wedge \sigma = 0$ and $\sigma \wedge \overline{\sigma} \neq 0$ defines a unique complex structure.

Theorem 3.1. ***(Global Torelli for K3 and 2-Tori)*** *Let M be a K3 surface or a* *complex 2-torus. Then, the complex structure defined by a 2-form σ depends (up to isomorphism) exclusively on the cohomology class $[\sigma] \in H^2_{DR}(M, \mathbb{C})$.*

Theorem 3.2 (Borcea, [2], Theorem 3)**.** ***(Global Torelli for Kodaira Surfaces)*** *Let N be a Kodaira surface. Then, the complex structure defined by a 2-form σ depends (up to isomorphism) exclusively on the cohomology class $[\sigma] \in H^2_{DR}(N, \mathbb{C})$.*

Theorem 3.1 is not the usual statement of the Global Torelli Theorem but rather an implication that we will use in the thesis. For more information see Huybrechts' paper on different versions of the Global Torelli Theorem in [20] (Remark 1.6, in particular). In what follows we will focus on only even-type generalized Calabi-Yau structures. An odd-type global Torelli theorem is a possible direction of future research.

3.1 Global Torelli on K3 Surfaces and Complex Tori

We now specialize to a K3 surface or a complex torus M. We only be considering generalized Calabi-Yau structures of even-type here. Although even-type structures are the theme of most of this thesis, odd-type structures cannot exist on a K3 surface because their first Betti number is zero (see [21]). We also note that given any complex structure J on M there exists a Kähler form corresponding to J. In fact, since we have the Calabi-Yau theorem on K3 surfaces and complex 2-tori, we can say that any complex structure on M admits a unique Ricci-flat Kähler structure in any Kähler class. However, on a K3 surface or complex 2-torus, Ricci-flat Kähler structures are precisely hyperKähler structures. Therefore we have established the following result.

Theorem 3.3 (Calabi-Yau)**.** *A complex structure and Kähler class on a K3 surface or complex 2-torus determine a unique hyperKähler metric.*

Associate to any generalized Calabi-Yau structure ϕ a real vector space $P_\phi \subseteq \mathcal{A}^*(M)$ which is spanned by the real and imaginary parts of ϕ. Similarly, $P_{[\phi]} \subseteq H^*(M, \mathbb{R})$ is generated by the real and imaginary parts of $[\phi] \in H^*(M, \mathbb{C})$. To motivate our definitions, consider a symplectic form ω. Then, it is of type $(1,1)$ with respect to a complex structure σ (a $(2,0)$-form) if and only if $\sigma \wedge \omega = 0$. In this case, ω (or $-\omega$) is a Kähler form for J. This implies that the vector spaces P_σ and $P_{e^{i\omega}}$ defined above are pointwise orthogonal with respect to the Mukai pairing. Mimicking this, we make the following definition.

Definition 3.4. Let ϕ be a generalized Calabi-Yau structure on M. Then ϕ is **Kähler** (or of **Kähler-type**) if there exists another generalized Calabi-Yau structure ϕ' such that P_ϕ and $P_{\phi'}$ are pointwise orthogonal. We then say that ϕ' **is a Kähler structure for ϕ.**

Remark 3.5. One can check that if ϕ' is a Kähler structure for ϕ and B is a B-field, then $e^B\phi'$ is a Kähler structure for $e^B\phi$.

Let us look at our two main examples.

Example 3.6. Let $\phi = \sigma$ be a complex structure. It can then be shown that any Kähler structure ϕ' for ϕ must be of symplectic-type, that is, $\phi'_0 \neq 0$ (see [21], Example 2.2). Theorem 2.51 says that there exists a B-field B and a symplectic structure ω such that $\phi' = e^{B+i\omega}$. A direct computation then shows that the orthogonality condition is equivalent to

$$\sigma \wedge \omega = 0 = \sigma \wedge B.$$

As we mentioned above, this means that ω (or $-\omega$) is a Kähler structure for σ; moreover, B is a closed real $(1,1)$-form with respect to σ. In particular, this implies that a complex structure σ is Kähler in the classical sense if and only if it is Kähler in the generalized Calabi-Yau sense. For this reason we will not be considering these structures when we move onto Kodaira surfaces.

Example 3.7. Suppose $\phi = e^{i\omega}$ is a generalized Calabi-Yau structure coming from a symplectic structure ω. If ϕ' is a Kähler structure for ϕ, then there are two cases: $\phi'_0 = 0$ or $\phi'_0 \neq 0$. If $\phi'_0 = 0$, then Theorem 2.51 tells us that $\phi' = e^B\sigma$ for a B-field B and a complex structure σ. By Remark 3.5, σ is a Kähler structure for $e^{-B}\phi = e^{-B+i\omega}$ and therefore we are in the situation of example 3.6. On the other hand, if $\phi'_0 \neq 0$, we may write $\phi' = e^{B'+i\omega'}$, and the orthogonality conditions are

$$B' \wedge \omega = B' \wedge \omega' = \omega \wedge \omega' = 0$$

and

$$B' \wedge B' = \omega \wedge \omega + \omega' \wedge \omega'.$$

The last condition implies that B' is also symplectic because $B'^2 = \omega^2 + \omega'^2 \neq 0$.

Remark 3.8. It is interesting to note that generalized Calabi-Yau structures of Kähler-type do, in fact, exist on Kodaira surfaces, despite those surfaces being non-Kähler. We will see this in Section 3.3.

Next, we wish to discuss an analogue for hyperKählerity for generalized Calabi-Yau structures. Classically, a symplectic form ω is hyperKähler with respect to the complex structure σ if $\sigma \wedge \overline{\sigma} = \lambda\omega \wedge \omega$ for some scalar $\lambda \in \mathbb{C}$.

Definition 3.9. A generalized Calabi-Yau structure ϕ is of **hyperKähler-type** if there exists another generalized Calabi-Yau structure ϕ' such that ϕ' is a Kähler structure for ϕ and $< \phi, \overline{\phi} >=< \phi', \overline{\phi'} >$ In this case, we call ϕ' a **hyperKahler structure for** ϕ.

Remark 3.10. As was the case for Kähler-type structures, hyperKähler structures are also preserved under B-transforms. That is, if ϕ' is a (hyper)Kähler structure for ϕ then $e^B\phi'$ is a (hyper)Kähler structure for $e^B\phi$.

Let us look again at our two examples.

Example 3.11. If $\phi = \sigma$ is a complex structure, then a Kähler structure for ϕ is $\phi' = \lambda e^{B+i\omega}$ where B is a closed real $(1,1)$-form and $\pm\omega$ is a Kähler form for σ. Then, ϕ' is a hyperKähler structure for ϕ if

$$2|\lambda|^2\omega \wedge \omega = \sigma \wedge \overline{\sigma},$$

which is equivalent to the condition $< \phi, \overline{\phi} >=< \phi', \overline{\phi'} >$. In other words, $\pm\omega$ is a hyperKähler form.

Example 3.12. If $\phi = e^{i\omega}$, then once again we have two cases. If the corresponding hyperKähler structure ϕ' is of complex-type, then $\phi' = e^B\sigma$ for a B-field B and a complex structure σ. Remark 3.10 tells us that σ is a hyperKähler structure for $e^{-B}\phi = e^{-B+i\omega}$, so we are in the same situation as Example 3.11.

On the other hand, if $\phi' = e^{(B'+i\omega')}$ then we have the same conditions we had from example 3.7 with the additional condition

$$\omega \wedge \omega = \omega' \wedge \omega'.$$

Consider the 2-form

$$\sigma := \frac{1}{\sqrt{2}}B' + i\omega'.$$

One readily checks that σ defines a complex structure for which $\pm\omega$ is a hyperKähler form.

Remark 3.13. In both of the above examples, we can see generalized Calabi-Yau structures of hyperKähler-type give rise to classical Kähler structures. So it is impossible to construct such structures on non-Kähler surfaces such as Kodaira surfaces.

We are now in a position to discuss the Global Torelli Theorem for a K3 surface or a complex 2-torus. This extends the result of Huybrechts, [21], Proposition 2.11.

Theorem 3.14. ***(Global Torelli)*** *Let ϕ and ψ be two generalized Calabi-Yau structures on a K3 surface or complex 2-torus M and suppose ϕ and ψ are both of hyperKähler-type. If $P_{[\phi]} = P_{[\psi]}$ then there exists a real exact B-field B and a diffeomorphism f such that, up to rescaling, $\phi = e^B f^*\psi$. That is, ϕ and ψ are isomorphic generalized Calabi-Yau structures.*

Proof. Suppose that ϕ and ψ are of complex-type, that is, $\phi = \phi_2 + \phi_4$ and $\psi = \psi_2 + \psi_4$, where ϕ_2 and ψ_2 correspond to complex structures on M. Since $P_{[\phi]} = P_{[\psi]}$, we get $[\phi_2] = [\psi_2]$ and the classical Torelli Theorem (3.1) provides a diffeomorphism f of M such that $f^*\psi_2 = \phi_2$. Thus, without loss of generality, let us then assume $\psi_2 = \phi_2$. So we have $\phi = \phi_2 + \phi_4$ and $\psi = \phi_2 + \psi_4$. We also know $[\phi_4] = [\psi_4]$ implying that $\phi_4 - \psi_4$ is exact. By Lemma 2.49, we can find an exact B-field B such that $\phi_4 - \psi_4 = B \wedge \phi_2$. Therefore, $\phi = \phi_2 + \psi_4 + B \wedge \phi_2 = e^B\psi$. Hence, ϕ and ψ are isomorphic as generalized Calabi-Yau structures.

On the other hand, if $\phi_0 \neq 0$, then $\psi_0 \neq 0$ as well since $[\phi_0] = [\psi_0]$. Let us rescale to make $\phi_0 = \psi_0 = 1$. Then, $\phi = e^{B+i\omega}$ and $\psi = e^{B'+i\omega'}$, for cohomologous symplectic structures ω and ω' and cohomologous real closed 2-forms B and B'. So B and B' differ by an exact 2-form. Without loss of generality, we may assume $\phi = e^{i\omega}$ and $\psi = e^{i\omega'}$ with $[\omega] = [\omega']$. Our goal is to find an isomorphism f such that $f^*\omega = \omega'$.

This is the point of the proof where we require the hyperKähler assumption. By Example 3.12, we can find complex structures σ and σ' such that ω and ω' are hyperKähler with respect to σ and σ', respectively. The fact that $[\omega] = [\omega']$ allows us to choose σ and σ' such that $[\sigma] = [\sigma']$ (see [21], Proposition 2.11). Once again, we invoke the classical Global Torelli Theorem (3.1) to obtain an isomorphism f such that $f^*\sigma = \sigma'$. This implies that both $f^*\omega$ and ω' are hyperKähler with respect to $f^*\sigma$; but the Calabi-Yau theorem (Theorem 3.3) tells us that hyperKähler forms are unique, so $f^*\omega = \omega'$. □

We have shown that generalized Calabi-Yau structures of hyperKähler-type depend, up to isomorphism, only on their cohomology class. Proving an analogous result on Kodaira surfaces is our next goal.

Remark 3.15. An important remark before we proceed to Kodaira surfaces is that the hyperKählerity assumption in the Global Torelli Theorem was only required when both structures were of symplectic-type. We cannot make the same assumption on Kodaira surfaces as they are non-Kähler. In that case we will need a different assumption to provide enough structure to prove the theorem. It should be stressed that it is not known whether or not the hyperKählerity assumption is required. On a K3 surface, Kodaira surface, and complex 2-torus it is unknown if $[\omega] = [\omega']$ implies the existence of a diffeomorphism f of M such that $f^*\omega = \omega'$. Counterexamples to the existence of such f are known for smooth manifolds of dimension 6, for example see [24] and [23].

3.2 Global Torelli on Kodaira Surfaces

As in the case of K3 surfaces and complex 2-tori, one has a classical Global Torelli Theorem for complex structures on Kodaira surfaces, Theorem 3.2. We will apply this theorem to obtain a global Torelli theorem for generalized complex structures. Denote by N a Kodaira surface, which is a complex nilmanifold. Proposition 2.46 says that generalized Calabi-Yau structures are equivalent to generalized complex structures on nilmanifolds. So let us work with generalized Calabi-Yau structures in this section and use these results to comment on the moduli space of generalized complex structures in Chapter 4.

Before we state the main theorem of this section, we recall a useful version of Moser's theorem (see [8]).

Proposition 3.16 (Moser). *Let S_c denote the set of symplectic forms ω on a manifold M such that $[\omega] = c$ for some fixed $c \in H^2_{DR}(M, \mathbb{R})$. If M is compact, all symplectic forms in the same path connected component of S_c are symplectomorphic.*

We know that a Kodaira surface N is non-Kähler, which means hyperKähler-type generalized complex structures do not exist on N (see remark 3.15). Instead, for the symplectic-type setting, we make sufficient assumptions to apply Moser's result.

Theorem 3.17. ***(Global Torelli)*** *Let ϕ be a generalized Calabi-Yau structure of even-type on a Kodaira surface N, then:*

1. *If ϕ is a generalized Calabi-Yau structure of complex-type, then ϕ depends, up to isomorphism, exclusively on its cohomology class*

$$[\phi] \in H^*_{DR}(N, \mathbb{C}).$$

2. *If ϕ is a generalized Calabi-Yau structure of symplectic-type (say $\phi = e^{B+i\omega}$), then up to isomorphism ϕ depends on its cohomology class*

$$[\phi] \in H^*_{DR}(N, \mathbb{C})$$

and the path connected component of

$$S_{[\omega]} \subseteq H^2_{DR}(N, \mathbb{R}).$$

Proof. Let $\phi = \phi_0 + \phi_2 + \phi_4$ and $\phi' = \phi'_0 + \phi'_2 + \phi'_4$ be two generalized Calabi-Yau structures of even-type on N and suppose $[\phi] = [\phi']$. Then, $[\phi_0] = [\phi'_0]$, so either both ϕ_0 and ϕ'_0 are zero or both are non-zero.

First, suppose $\phi_0 = \phi'_0 = 0$. Then, $\phi = \phi_2 + \phi_4$ and $\phi' = \phi'_2 + \phi'_4$, where ϕ_2 and ϕ'_2 correspond to complex structures on N. Moreover, $[\phi] = [\phi']$ implies $[\phi_2] = [\phi'_2]$. Theorem 3.2 guarantees the existence of a diffeomorphism f of N such that $f^*\phi'_2 = \phi_2$. Therefore, without loss of generality, we may assume $\phi_2 = \phi'_2$. So we have $\phi = \phi_2 + \phi_4$ and $\phi' = \phi_2 + \phi'_4$ where $[\phi_4] = [\phi'_4]$, that is, $\phi_4 - \phi'_4$ is exact. Lemma 2.49 gives an exact B-field B such that $\phi_4 - \phi'_4 = B \wedge \phi_2$. Hence, $\phi = \phi_2 + \phi'_4 + B \wedge \phi_2 = e^B \phi'$. Therefore ϕ and ϕ' are isomorphic generalized complex structures.

Next, suppose ϕ_0 and ϕ'_0 are non-zero. Then we may write $\phi = e^{B+i\omega}$ and $\phi' = e^{B'+i\omega'}$. Furthermore, suppose there is a path from ω' to ω in $S_{[\omega]}$. In this case, $[\phi] = [\phi']$ implies B and B' are cohomologous, which means $B - B'$ is an exact B-field. Without loss of generality, we may reduce to the case where $\phi = e^{i\omega}$ and $\phi' = e^{i\omega'}$. Then, we are in a situation where $[\omega] = [\omega']$ and, by assumption, there is a path from ω' to ω in $S_{[\omega]}$. Moser's result, Proposition 3.16, gives us a diffeomorphism f of M such that $f^*\omega = \omega'$. Then, $f^*\phi = \phi'$, as desired. □

Remark 3.18. Note that it is not clear that we require the path-connected assumption in the second part of Theorem 3.17. However, omitting it would require us to prove that any two symplectic forms ω and ω' satisfying $[\omega] = [\omega']$ are symplectomorphic. This is generally not known on complex surfaces.

3.3 Generalized Calabi-Yau Structures of Kähler-Type on Kodaira Surfaces

In this section, we construct a family of pairs (ϕ, ϕ') of generalized Calabi-Yau structures of Kähler-type such that ϕ and ϕ' are both of symplectic-type (as examined in Example 3.7). Recall that a generalized Calabi-Yau structure ϕ is of Kähler-type if there exists another generalized Calabi-Yau structure ϕ' with an orthogonality condition (see Definition 3.4). In the complex-type setting, the orthogonality condition implies that ϕ gives rise to a usual Kähler structure, which do not exist on Kodaira surfaces. However, in the symplectic-type setting, $\phi = e^{i\omega}$, one can explicitly find a second generalized Calabi-Yau structure $\phi' = e^{B'+i\omega'}$ of symplectic-type that makes (ϕ, ϕ') into a Kähler-type pair. The orthogonality condition is now equivalent to the four conditions (see Example 3.7):

1. $B' \wedge \omega = 0$,
2. $B' \wedge \omega' = 0$,
3. $\omega \wedge \omega' = 0$,
4. $B \wedge B = \omega \wedge \omega + \omega' \wedge \omega'$.

Note that ω, ω', and B are real closed 2-forms. Moreover, the fourth condition implies that B is also a symplectic 2-form (since $B^2 \neq 0$). In [25], a family of symplectic structures is constructed as follows. Note that Kodaira surfaces are parallelizable and so T_N^* admits at least four linearly independent global sections. One can choose four global sections $\alpha, \beta, \gamma, \delta$ such that $\alpha\wedge\gamma-\beta\wedge\delta$, $\alpha\wedge\delta+\beta\wedge\gamma$, $\alpha\wedge\gamma+\beta\wedge\delta$, and $\alpha\wedge\delta-\beta\wedge\gamma$ are generators of $H^2_{DR}(N, \mathbb{R})$. Then

$$u_1(\alpha\wedge\gamma-\beta\wedge\delta)+v_1(\alpha\wedge\delta+\beta\wedge\gamma)+u_2(\alpha\wedge\gamma+\beta\wedge\delta)+v_2(\alpha\wedge\delta-\beta\wedge\gamma)$$

is symplectic if $u_1^2+v_1^2-u_2^2-v_2^2 \neq 0$.

Therefore, let us write

$$\omega = u_1(\alpha\wedge\gamma-\beta\wedge\delta)+v_1(\alpha\wedge\delta+\beta\wedge\gamma)+u_2(\alpha\wedge\gamma+\beta\wedge\delta)+v_2(\alpha\wedge\delta-\beta\wedge\gamma),$$

$$\omega' = u_1'(\alpha\wedge\gamma-\beta\wedge\delta)+v_1'(\alpha\wedge\delta+\beta\wedge\gamma)+u_2'(\alpha\wedge\gamma+\beta\wedge\delta)+v_2'(\alpha\wedge\delta-\beta\wedge\gamma), \text{ and}$$
$$B' = B_1(\alpha\wedge\gamma-\beta\wedge\delta)+C_1(\alpha\wedge\delta+\beta\wedge\gamma)+B_2(\alpha\wedge\gamma+\beta\wedge\delta)+C_2(\alpha\wedge\delta-\beta\wedge\gamma).$$

Then, the above four Kähler conditions respectively give us:

1. $B_1u_1 + C_1v_1 - B_2u_2 - C_2v_2 = 0$,
2. $B_1u_1' + C_1v_1' - B_2u_2' - C_2v_2' = 0$,
3. $u_1u_1' + v_1v_1' - u_2u_2' - v_2v_2' = 0$,
4. $B_1^2 + C_1^2 - B_2^2 - C_2^2 = u_1^2 + v_1^2 - u_2^2 - v_2^2 + (u_1')^2 + (v_1')^2 - (u_2')^2 - (v_2')^2$.

Therefore, any collection of twelve real numbers $u_1, v_1, u_2, v_2, u_1', v_1', u_2', v_2', B_1, C_1, B_2, C_2$ that satisfy the above four equations as well as the symplectic conditions ($u_1^2+v_1^2-u_2^2-v_2^2 \neq 0$, $(u_1')^2+(v_1')^2-(u_2')^2-(v_2')^2 \neq 0$, and $B_1^2+C_1^2-B_2^2-C_2^2 \neq 0$) will give rise to a generalized

Calabi-Yau structure on a Kodaira surface that is of Kähler-type. To see that the conditions are not inconsistent, observe that

$$\begin{array}{llll} u_1 = 1, & v_1 = 1, & u_2 = 0, & v_2 = 0, \\ u_1' = 1, & v_1' = -1, & u_2' = 2, & v_2' = 2, \\ B_1 = 0, & C_1 = 0, & B_2 = -\sqrt{2}, & C_2 = \sqrt{2}, \end{array}$$

is one solution. Note that this solution gives $\omega^2 > 0$, $(\omega')^2 < 0$, and $(B')^2 < 0$, so the orientations are different. However, this was not a condition in the definition and this is a valid Kähler-type generalized Calabi-Yau structure.

Chapter 4

Deformations of Generalized Complex Structures

In this chapter, we first introduce the notion of generalized complex deformations in Section 4.1 (as presented by Gualtieri in [11]) and examine some general facts regarding generalized complex deformations for symplectic-type structures and B-field transforms. Then, in Sections 4.2 and 4.3, we compute some explicit deformation spaces for even-type generalized complex structures on Kodaira surfaces and a specific family of complex 2-tori. In Section 4.4, we collect the deformation results together as well as our new Global Torelli theorems from Chapter 3 to comment on the moduli space of even-type generalized complex structures on Kodaira surfaces and complex 2-tori. Section 4.5 exhibits some explicit computations that show that certain pairs of generalized complex structures do not give rise to a generalized Kähler structure on Kodaira surfaces. Finally, in Section 4.6 we present an example of an odd-type generalized complex structure on a Kodaira surface.

4.1 Deformation Theory

We begin with a preliminary look at generalized deformation theory in the sense of Kuranishi. See [11] for a complete picture of generalized complex deformations. In general, to deform a generalized complex structure $\mathbb{J}$ one may instead deform its $+i$-eigenbundle L. Say $L(t)$ is a continuous family of generalized almost complex structures with $L(0) = L$. That means that $L(t)$ is maximal isotropic and that $(T \oplus T^*) \otimes \mathbb{C} = L(t) \oplus \overline{L(t)}$ (that is, $L(t) \cap \overline{L(t)} = \{0\}$) for any t. We may encode such small deformations by a smooth homomorphism $\epsilon(t) : L \to \overline{L}$ and consider $L(t) = (1+\epsilon(t))L$, that is, $\nu \mapsto \nu + \epsilon(t)\nu$ for $\nu \in L$. Also, $\overline{L(t)} = \overline{(1+\epsilon(t))L} = (1+\overline{\epsilon}(t))\overline{L}$. The conditions we place on $L(t)$ are that it must be isotropic, have zero intersection with $\overline{L(t)}$, and be closed under the Courant bracket.

First, for the deformed structure to be isotropic, it must satisfy, for all $a, b \in C^\infty(L)$, $< a + \epsilon(t)a, b + \epsilon(t)b >= 0$. But then we get

$$\begin{aligned} 0 &= < a + \epsilon(t)a, b + \epsilon(t)b > \\ &= < a, b > + < a, \epsilon(t)b > + < \epsilon(t)a, b > + < \epsilon(t)a, \epsilon(t)b > \\ &= < a, \epsilon(t)b > + < \epsilon(t)a, b > . \end{aligned}$$

The last step follows becuse both L and $\overline{L}$ are isotropic. Thus, $\epsilon(t)$ must satisfy

$$< a, \epsilon(t)b > + < \epsilon(t)a, b >= 0$$

for all $a, b \in \mathcal{C}^\infty(L)$, or equivalently, $\epsilon(t) \in \mathcal{C}^\infty(\wedge^2 L^*)$ after identifying $\overline{L}$ with L^* via $\langle \cdot, \cdot \rangle$.

Next, $L(t)$ has zero intersection with $\overline{L(t)}$ if and only if the endomorphism on $L \oplus \overline{L}$

$$A(t) := \begin{bmatrix} 1 & \overline{\epsilon(t)} \\ \epsilon(t) & 1 \end{bmatrix}$$

is invertible (see [11]). In this case, $\mathbb{J}(t) = A(t)\mathbb{J}A(t)^{-1}$ is a new generalized almost complex structure.

Before we can discuss integrability of the deformed structures, we need to introduce the concepts *Lie algebroids* and *Lie algebroid differentials*.

Definition 4.1. A **Lie algebroid** over a manifold M is a triple $(E, \rho, [\cdot,\cdot]_E)$ where E is a vector bundle on M, $\rho : E \to T$ is a bundle map called the *anchor map* and $[\cdot,\cdot]_E$ is a Lie algebra structure on sections of E such that

$$[x, fy]_E = \rho(x)f \cdot y + f[x, y]_E,$$

where x, y are sections of E and $f \in C^\infty(M)$. Moreover, Lie algebroids $(E, \rho, [\cdot,\cdot]_E)$ and $(E', \rho', [\cdot,\cdot]_{E'})$ are **isomorphic** if there is a vector bundle isomorphism $\phi : E \to E'$ such that $\rho' \circ \phi = \rho$ and $[\phi(a), \phi(b)]_{E'} = \phi([a, b]_E)$.

Example 4.2. (The Tangent Bundle) The basic example of a Lie algebroid is $(T, 1, [\cdot,\cdot])$ or $(T_{\mathbb{C}}, 1, [\cdot,\cdot])$, where the anchor map is the identity map and $[\cdot,\cdot]$ is the standard Lie bracket.

Example 4.3. (Generalized Complex Structures) If L is a generalized complex structure, then $(L, \pi_T, [\cdot,\cdot]_C)$ is a Lie algebroid. Here π_T is projection onto $T_{\mathbb{C}}$ and $[\cdot,\cdot]_C$ is the Courant bracket.

For more information on general Lie algebroids see [9]. We wish to examine the exterior differential algebra of a Lie algebroid $(E, \rho, [\cdot,\cdot]_E)$. Consider the exterior algebra $\wedge^* E^*$ of E^*.

Definition 4.4. A section of $\wedge^k E^*$ is called a **(homogeneous) form of degree k.** or simply a k-form when the context is clear.

Definition 4.5. Define a differential operator that takes sections of $\wedge^k E^*$ to $\wedge^{k+1} E^*$ as follows: for $\gamma \in C^\infty(\wedge^k E^*)$ and $v_1, v_2, \ldots, v_{k+1} \in C^\infty(E)$:

$$\begin{aligned} d_E\gamma(v_1, \ldots, v_{k+1}) &= \sum_i (-1)^{i+1} \rho(v_i) \cdot \gamma(v_1, \ldots, \hat{v_i}, \ldots, v_{k+1}) \\ &\quad + \sum_{i<j} (-1)^{i+j} \gamma([v_i, v_j]_E, v_1, \ldots, \hat{v_i}, \ldots, \hat{v_j}, \ldots, v_{k+1}). \end{aligned}$$

Here, $\hat{v_i}$ means remove index i. This is the analogue of the invariant definition of the usual exterior derivative d.

One can show that $d_E^2 = 0$. Then, using this differential operator, we can also define analogues of the deRham cohomology groups.

Definition 4.6. For a Lie algebroid $(E, \rho, [\cdot,\cdot]_E)$, the differential cohomology groups are defined as

$$H_E^k(M) = \frac{\{d_E\text{-closed } k\text{-forms}\}}{\{d_E\text{-exact } k\text{-forms}\}}.$$

Example 4.7. In the base case, $(E, \rho, [\cdot,\cdot]_E) = (T_{\mathbb{C}}, 1, [\cdot,\cdot])$, the cohomology groups are the complex deRham groups, that is, $H_E^k(M) = H_{DR}^k(M, \mathbb{C})$.

We will shortly be restricting ourselves to the generalized complex setting, $(L, \pi_T, [\cdot,\cdot]_C)$, but let us first prove a general result regarding isomorphic Lie algebroids, their differentials, and their cohomology groups.

Lemma 4.8. *Let $(E, \rho, [\cdot,\cdot]_E)$ and $(E', \rho', [\cdot,\cdot]_{E'})$ be two isomorphic Lie algebroids over a manifold M and let $\phi : E \to E'$ be the isomorphism. Then the following diagram commutes*

$$\begin{array}{ccc} \wedge^k (E')^* & \xrightarrow{d_{E'}} & \wedge^{k+1}(E')^* \\ \phi^* \downarrow & & \downarrow \phi^* \\ \wedge^k E^* & \xrightarrow{d_E} & \wedge^{k+1} E^*. \end{array}$$

Equivalently, $\phi^* \circ d_{E'} = d_E \circ \phi^*$.

Proof. For a smooth function f and $a \in C^\infty(E)$, we obtain

$$d_E(f)(a) = \rho(a)(f) = \rho'(\phi(a))(f) = d_{E'}(f)(\phi(a)).$$

We now establish a similar relation for any section of $\wedge^k(E^*)$. Let us explicitly compute this when $k = 1$. First note that, since $\phi : E \to E'$ is an isomorphism, so is $\phi^* : (E')^* \to E^*$. Then, for $a, b \in C^\infty(E)$ and $\alpha \in C^\infty(E^*)$, we may find unique $x, y \in C^\infty(E')$ and $\gamma \in C^\infty((E')^*)$ such that $a = \phi^{-1}(x)$, $b = \phi^{-1}(y)$, and $\alpha = \phi^*(\gamma)$. Therefore, we obtain

$$\begin{aligned}
d_E(\alpha)(a,b) &= \rho(a)(\alpha(b)) - \rho(b)(\alpha(a)) - \alpha([a,b]_E) \\
&= \rho(\phi^{-1}(x))(\phi^*(\gamma)(\phi^{-1}(y))) \\
&\quad -\rho(\phi^{-1}(y))(\phi^*(\gamma)(\phi^{-1}(x))) - \phi^*(\gamma)([\phi^{-1}(x), \phi^{-1}(y)]_E) \\
&= \rho'(x)(\gamma(y)) - \rho'(y)(\gamma(x)) - \gamma([x,y]_{E'}) \\
&= d_{E'}(\gamma)(x,y).
\end{aligned}$$

We can extend this argument via induction to $C^\infty(\wedge^k E^*)$ easily. Thus, the two Lie algebroid differentials are related by ϕ^*, namely $\phi^* \circ d_{E'} = d_E \circ \phi^*$. ☐

Let us use the above lemma to prove that two isomorphic Lie algebroids have isomorphic cohomology groups.

Theorem 4.9. *Let $(E, \rho, [\cdot,\cdot]_E)$ and $(E', \rho', [\cdot,\cdot]_{E'})$ be two isomorphic Lie algebroids over a manifold M. Then $H^k_E(M) \cong H^k_{E'}(M)$.*

Proof. Suppose E and E' are isomorphic, let $\phi : E \to E'$ be the isomorphism. Define a map from $H^k_{E'}(M)$ to $H^k_E(M)$ as

$$[\alpha] \mapsto [\phi^*(\alpha)].$$

This is well defined for if $[\alpha_1]_{E'} = [\alpha_2]_{E'}$ then $\alpha_1 - \alpha_2 = d_{E'}\gamma$ for some $(k-1)$-form γ. Then, by the above lemma, we get

$$\phi^*(\alpha_1) - \phi^*(\alpha_2) = \phi^*(d_{E'}\gamma) = d_E(\phi^*(\gamma))$$

and so $[\phi^*(\alpha_1)]_E = [\phi^*(\alpha_2)]_E$. Finally, the map is a bijection because ϕ is a bijection. ☐

We now specialize to the case when our Lie algebroid has the form $(L, \pi_T, [\cdot,\cdot]_C)$ for a generalized complex structure L. Denote the cohomology groups, more precisely, d_L-closed sections of L modulo d_L-exact ones, by $H^k_L(M)$ on the manifold M. Let us first establish some corollaries of Theorem 4.9.

Corollary 4.10. *Let L be a generalized complex structure on M. If L is of type zero everywhere, then the cohomology groups $H^k_L(M)$ are isomorphic to the (complex) deRham cohomology groups $H^k_{DR}(M, \mathbb{C})$.*

Proof. Consider the two Lie algebroids $(T_{\mathbb{C}}, 1, [\cdot,\cdot])$ and $(L, \pi_T, [\cdot,\cdot]_C)$, from Examples 4.2 and 4.3, respectively. Since L has type zero everywhere the projection, $\pi_T : L \to T_{\mathbb{C}}$ is a surjection and therefore an isomorphism of vector bundles. Furthermore, π_T is a Lie algebroid isomorphism since $1 \circ \pi_T = \pi_T$ and, for $a, b \in C^\infty(L)$,

$$[\pi_T(a), \pi_T(b)] = \pi_T([a, b]_C)$$

by the definition of the Courant bracket. Therefore, Theorem 4.9 applies and we obtain $H^k_L(M) \cong H^k_{DR}(M, \mathbb{C})$. □

Corollary 4.11. *Let L be a generalized complex structure and B a real closed 2-form on M. Let $L_B = e^B(L)$ be the B-field transform of L. Then, $H^k_L(M) \cong H^k_{L_B}(M)$.*

Proof. Note that our two Lie algebroids are $(L, \pi_T, [\cdot,\cdot]_C)$ and $(L_B, \pi_T, [\cdot,\cdot]_C)$ in this case. By definition, $e^B : L \to L_B$ is a vector bundle isomorphism. Let us check that it is an isomorphism of Lie algebroids. First,

$$\pi_T \circ e^B(X + \alpha) = \pi_T(X + \alpha + \iota_X B) = X = \pi_T(X + \alpha),$$

which means $\pi_T \circ e^B = \pi_T$. Next, since B is closed, Proposition 2.33 gives

$$[e^B x, e^B y]_C = e^B([x, y]_C)$$

for $x, y \in C^\infty(L)$. Thus, e^B is an isomorphism of Lie algebroids and we may apply Theorem 4.9 to obtain $H^k_L(M) \cong H^k_{L_B}(M)$. □

We can now discuss when a deformed generalized complex structure is integrable. This theorem is established in [11].

Theorem 4.12 (Gualtieri, [11], Section 5.1)**.** *$\mathbb{J}(t)$ is integrable if and only if $\epsilon(t)$ satisfies the Maurer-Cartan equation*

$$d_L(\epsilon(t)) + \frac{1}{2}[\epsilon(t), \epsilon(t)]_S = 0,$$

where $d_L : C^\infty(\wedge^2 L^*) \to C^\infty(\wedge^3 L^*)$ is the Lie algebroid differential of $(L, \pi_T, [\cdot,\cdot]_C)$ and $[\cdot,\cdot]_S$ is the Schouten bracket that extends the Courant bracket.

Let us examine the smooth maps $\epsilon(t)$ more closely. Write $\epsilon(t) = \epsilon_0 + t\epsilon_1 + t^2\epsilon_2 + \cdots$, its power series expansion. Since we would like to obtain the original structure at $t = 0$, we can set $\epsilon_0 = 0$. Further, if $\epsilon(t)$ satisfies the Maurer-Cartain equation, then

$$d_L(\epsilon(t)) + \frac{1}{2}[\epsilon(t), \epsilon(t)]_S = 0.$$

Taking the derivative of both sides with respect to t yields

$$d_L\left(\frac{d\epsilon(t)}{dt}\right) + \left[\frac{d\epsilon(t)}{dt}, \epsilon(t)\right]_S = 0$$

and evaluating at $t = 0$ gives

$$d_L(\epsilon_1) + [\epsilon_1, 0]_S = 0,$$

which implies $d_L(\epsilon_1) = 0$.

Therefore, the first order part of the deformation lies in the kernel of d_L. In [11], it is shown that two first order deformations are equivalent if they differ by an element in the image of d_L and so we will be examining cohomology classes of first order deformations ϵ_1 in $H^2_L(M)$.

Next, we present the deformation theorem for generalized complex structures.

Theorem 4.13 (Gualtieri, [11], Theorem 5.4)**.** *There exists an open neighbourhood $U \subseteq H^2_L(M)$ containing zero, a smooth family $\mathcal{M}' = \{\epsilon_u : u \in U, \epsilon_0 = 0\}$ of generalized complex deformations of $\mathbb{J}$, and an analytic obstruction map $\Phi : U \to H^3_L(M)$ with $\Phi(0) = 0$ and $d\Phi(0) = 0$, such that the deformations in the sub-family $\mathcal{M} = \{\epsilon_z : z \in \Phi^{-1}(0)\}$ are the integrable ones. Furthermore, any sufficiently small deformation ϵ of $\mathbb{J}$ is equivalent to at least one member of $\mathcal{M}$. Finally, if the obstruction map vanishes, then $\mathcal{M}$ is a smooth locally complete family of deformations.*

Remark 4.14. The obstruction map is defined as the projection of $[\epsilon_1, \epsilon_1]_S$ onto a certain subspace of $C^\infty(\wedge^3 L^*)$. What is important for us is that, if the first order deformation $t\epsilon_1$ satisfies $[\epsilon_1, \epsilon_1]_S = 0$, then the obstruction map vanishes. This is of course not surprising since, if the Schouten bracket $[\epsilon_1, \epsilon_1]$ vanishes, then the first order deformation $t\epsilon_1$ is a *full* deformation of our structure, as $t\epsilon_1$ satisfies the Maurer-Cartain equation on its own without the need for higher order terms.

Theorem 4.15. *If L is a generalized complex structure of type zero everywhere, then there exists a small open neighbourhood $U \subseteq H^2_L(M)$ of zero on which the obstruction map Φ vanishes. In other words, U is a smooth locally complete family of deformations.*

Proof. Suppose L has type zero everywhere. Proposition 2.37 tells us that L is the B-field transform of a symplectic structure, say ω. Therefore, a generator for the associated pure spinor line is $e^{B+i\omega}$. Consider infinitesimal deformations ϵ in an open neighbourhood $U \subseteq H^2_L(M)$ containing zero. Since L has type zero everywhere, we know $H^2_L(M) \cong H^2_{DR}(M)$ by Corollary 4.10. Let $\tilde{U}$ be the image of U in $H^2_{DR}(M)$, so that $\tilde{U}$ is a neighbourhood of 0. In addition, for $[\epsilon] \in U$, let $[\tilde{\epsilon}] \in \tilde{U}$ be the image of $[\epsilon]$. Write $\tilde{\epsilon} = \tilde{\epsilon_1} + i\tilde{\epsilon_2}$ where $\tilde{\epsilon_1}$ and $\tilde{\epsilon_2}$ are the real and imaginary parts of $\tilde{\epsilon}$. Choose U small enough so that the following two conditions are satisfied. First, $\omega + \mathrm{Im}(\tilde{\epsilon})$ remains non-degenerate for any $[\tilde{\epsilon}] \in \tilde{U}$. Second, if $[\tilde{\epsilon_2}] = [\tilde{\epsilon_2}']$ then there is a path from $\omega + \tilde{\epsilon_2}$ to $\omega + \tilde{\epsilon_2}'$ in $S_{[\omega+\tilde{\epsilon_2}]}$ (the notation is from Proposition 3.16).

For each $[\tilde{\epsilon}] \in \tilde{U}$, consider $B + i\omega + \tilde{\epsilon} = B + \tilde{\epsilon_1} + i(\omega + \tilde{\epsilon_2})$. Then $B + \tilde{\epsilon_1}$ is a real closed 2-form and, by construction, $\omega + \tilde{\epsilon_2}$ is symplectic. This means that $e^{B+i\omega+\tilde{\epsilon}}$ is a B-field transform of a symplectic structure (here the B-field is $B + \tilde{\epsilon_1}$ and the symplectic structure is $\omega + \tilde{\epsilon_2}$) and is therefore a generalized complex structure on M. Next, if $[\tilde{\epsilon}] = [\tilde{\epsilon}']$, then $e^{B+i\omega+\tilde{\epsilon}}$ and $e^{B+i\omega+\tilde{\epsilon}'}$ are isomorphic generalized complex structures. Indeed, $B + \tilde{\epsilon_1}$ and $B + \tilde{\epsilon_1}'$ differ by an exact form and $\omega + \tilde{\epsilon_2}'$ is symplectomorphic to $\omega + \tilde{\epsilon_2}$ because there is a path from $\omega + \tilde{\epsilon_2}$ to $\omega + \tilde{\epsilon_2}'$ in $S_{[\omega+\tilde{\epsilon_2}]}$, which allows us to apply Proposition 3.16. This proves that we obtain the same generalized complex structure regardless of choice of representative of $[\tilde{\epsilon}]$.

Hence, we have constructed for each $[\tilde{\epsilon}] \in \tilde{U}$ a new generalized complex structure close to L_ω in the moduli space of generalized complex structures. Therefore, $\tilde{U}$ parametrizes infinitesimal deformations of L, and since $\tilde{U} \cong U$ it must be the case that the obstruction map vanishes on U. Therefore, U forms a smooth locally complete family of deformations of L by Theorem 4.13. □

Remark 4.16. Recall that the vanishing of the Schouten bracket $[\epsilon, \epsilon]_S$ implies that the obstruction map Φ vanishes. However, a vanishing Schouten bracket is a much stronger result because it says that the first order part is a full deformation. The above theorem does not guarantee that the Schouten bracket vanishes. On the other hand, in each of the forthcoming symplectic-type computations in Sections 4.2 and 4.3 we will see that the Schouten bracket *does* vanish for any choice of first-order part ϵ. It may be the case that a stronger result holds, and that the Schouten bracket vanishes for any first order deformation in general. This is a question that will be explored more in the future.

We can obtain a similar result regarding B-field transformations of L if we already know that the obstruction map vanishes for infinitesimal deformations of L.

Theorem 4.17. *Let L be a generalized complex structure on M and $U \subseteq H^2_L(M)$ be a small open neighbourhood of zero on which the obstruction map vanishes. Let B*

be a closed real 2-form and L_B the corresponding generalized complex structure. Let $\psi_B : H^2_L(M) \to H^2_{L_B}(M)$ be the isomorphism guaranteed by Corollary 4.11. Then, the obstruction map Φ, vanishes on $\psi_B(U) \subseteq H^2_{L_B}(M)$ and $\psi_B(U)$ is a smooth locally complete family of deformations.

Proof. We use similar arguments to those found in the proof of Theorem 4.15. Since the obstruction map vanishes on U, for $[\epsilon] \in U$, we know that L_ϵ is a generalized complex structure close to L in the moduli space of generalized complex structures on M. Hence, $e^B(L_\epsilon)$ is also a generalized complex structure, and it is close to $L_B = e^B(L)$. If $[\epsilon] = [\epsilon']$, then $L_\epsilon \cong L_{\epsilon'}$, which means $e^B(L_\epsilon) \cong e^B(L_{\epsilon'})$. Hence, for each $[\epsilon] \in U$ we have constructed a generalized complex structure close to L_B. Consequently, U parametrizes infinitesimal deformations of L_B. Therefore, since $U \cong \psi_B(U)$ (because ψ_B is an isomorphism), we obtain that the obstruction map vanishes on $\psi_B(U)$. Finally, Theorem 4.13 tells us that $\psi_B(U)$ is a smooth locally complete family of deformations of L_B. $\square$

We will revisit these structures in the coming sections when we explicitly compute a deformation space and examine generalized holomorphic bundles. We end this section with two results regarding deformations of generalized complex structures of type zero.

Proposition 4.18. *Small infinitesimal deformations of a structure of type zero are of type zero.*

Proof. Let ϵ be an infinitesimal deformation of a type zero generalized complex structure L. Then, the deformed space is

$$L_\epsilon = (1+\epsilon)L = \{x + \epsilon(x) | x \in C^\infty(L)\},$$

which implies

$$\pi_T(x + \epsilon(x)) = \pi_T(x) + \pi_T(\epsilon(x)).$$

Fix $y \in C^\infty(L)$ such that $\pi_T(y) \neq 0$. If $\pi_T(y) + \pi_T(\epsilon(y)) = 0$ then $\pi_T(\epsilon(y)) = -\pi_T(y)$ which would imply that ϵ is not an infinitesimal deformation. This shows that if $\pi_T(y) \neq 0$ then $\pi_T(y) + \pi_T(\epsilon(y)) \neq 0$. Hence, the type of the generalized complex structure cannot increase via infinitesimal deformations. Therefore, the type of L_ϵ is zero. $\square$

This proposition tells us, in particular, that infinitesimal deformations of a symplectic structure $\mathbb{J}_\omega$ are all type zero. Recall that, if ω is a (real) symplectic structure, then the corresponding generalized complex structure has maximal isotropic

$$L_\omega = \{X - i\omega(X) | X \in T_{\mathbb{C}}\},$$

which has type zero everywhere.

Proposition 4.19. *If ω is a (real) symplectic structure then small infinitesimal deformations of L_ω are of the form $L_{\omega_\epsilon} = \{X - i\widetilde{\omega}(X) | X \in T_{\mathbb{C}}\}$ for a complex symplectic structure $\widetilde{\omega}$ (that is, a complex 2-form that is both closed and non-degenerate as a complex-valued 2-form).*

Proof. Fix a small infinitesimal deformation ϵ of L_ω, call the deformed structure L_{ω_ϵ}. Proposition 4.18 tells us that L_{ω_ϵ} has type zero. Further, Proposition 2.37 guarantees the existence of a real closed 2-form B_ϵ and a real symplectic form η_ϵ so that

$$L_{\omega_\epsilon} = \{X + (B_\epsilon - i\eta_\epsilon)X | X \in T_{\mathbb{C}}\} = \{X - i(\eta_\epsilon + iB_\epsilon)X | X \in T_{\mathbb{C}}\}.$$

Moreover, ω is symplectic and therefore non-degenerate which means L_ω has full-rank projection onto $T^*_{\mathbb{C}}$ as well. Since ϵ is an infinitesimal deformation, the same arguments used in the proof of Proposition 4.18 shows that if $x \in C^\infty L_\omega$ and $\pi_{T^*}(x) \neq 0$ then $\pi_{T^*}(x + \epsilon(x)) \neq 0$ (here π_{T^*} is projection onto $T^*_{\mathbb{C}}$). Therefore, L_{ω_ϵ} has full-rank projection onto $T^*_{\mathbb{C}}$. This means that the complex 2-form $\eta_\epsilon + iB_\epsilon$ is non-degenerate. Hence, $\eta_\epsilon + iB_\epsilon$ is a complex symplectic structure. □

Remark 4.20. Before we end this section, it is important to note that, while the above proposition guarantees the existence of such a complex symplectic structure, it is in general difficult to compute $\widetilde{\omega} = \eta_\epsilon + iB_\epsilon$ for a given symplectic structure ω and infinitesimal deformation ϵ. Indeed, ω and ϵ are intricate objects which implies that the generators of L_{ω_ϵ} can be very complicated. Therefore, obtaining a general formula for $\widetilde{\omega}$ can be difficult. However, when we work on Kodaira surfaces in Section 4.2.2, we will provide an explicit description of $\widetilde{\omega}$ for a given ω and ϵ.

4.2 Deformations of Even-Type Structures on Kodaira Surfaces

In this section we compute some explicit deformations of generalized complex structures on a Kodaira surface. We deform a complex structure in Section 4.2.1 and a symplectic structure in Section 4.2.2. We demonstrate that any complex-type structure has a smooth family of deformations (Theorem 4.25). Similarly, Theorem 4.30 shows that the family of deformations of the base symplectic structure forms a smooth locally complete family.

A (primary) Kodaira surface is a compact complex surface of Kodaira dimension 0 and odd first Betti number with trivial canonical bundle. In [22] Kodaira proved that a compact complex surface with trivial canonical bundle is either a K3 surface, a torus, or a Kodaira surface. We will follow the notation presented in [2] and [4] to describe Kodaira surfaces so that we may explicitly compute certain deformation spaces of generalized complex structures.

Let A denote the group of all real affine transformations of $\mathbb{R}^4 = \mathbb{C}^2$ (under the identification $z = x + iy$ and $w = u + iv$) that commute with any transformation of the form

$$g(z, w) = (z + \alpha, w + \overline{\alpha}z + \beta)$$

where $\alpha, \beta \in \mathbb{C}$. We may identify A with $\mathbb{C}^2$ where multiplication is defined as

$$(z, w) * (\alpha, \beta) = (z + \alpha, w + \overline{\alpha}z + \beta).$$

Next, let Γ be a non-abelian group of affine transformations generated by four elements g_1, g_2, g_3, g_4, where

$$g_i g_j g_i^{-1} g_j^{-1} = id$$

for all $(i, j) \neq (3, 4)$ $(i < j)$ and

$$g_3 g_4 g_3^{-1} g_4^{-1} = g_2^m$$

for some $m \in \mathbb{Z}^+$. We may view the g_i as transformations on $\mathbb{C}^2$ and in that case they have the form

$$g_j(z, w) = (z + \alpha_j, w + \overline{\alpha_j}z + \beta_j),$$

where $\alpha_1 = \alpha_2 = 0$ and $\overline{\alpha_3}\alpha_4 - \overline{\alpha_4}\alpha_3 = m\beta_2$. Any Kodaira surface is the compact quotient of A by a discrete subgroup Γ generated by the elements g_i where (α_i, β_i) satisfy the above constraints.

On the other hand, if Γ is any discrete, non-abelian, co-compact subgroup of A, then the complex structure induced on the quotient A/Γ gives a compact complex surface with trivial canonical bundle. It can be shown that this surface is non-Kähler and so is not a torus nor a K3 surface. Thus, it must be a Kodaira surface.

Let us denote a Kodaira surface by $N = \mathbb{C}^2/\Gamma$, and consider the right-invariant tangent vector fields on our Kodaira surface N. We may compute these and determine that T_N is globally generated by invariant vector fields $\{X, Y, U, V\}$ where they are induced by the invariant vector fields on A:

$$X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial u} - y\frac{\partial}{\partial v},$$

$$Y = \frac{\partial}{\partial y} + y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v},$$

$$U = \frac{\partial}{\partial v},$$
$$V = \frac{\partial}{\partial u}.$$

These are chosen to be consistent with the notation presented in [4] and [25]. One readily computes that the only non-zero Lie bracket is

$$[X, Y] = 2U.$$

Further, we define an endomorphism J such that

$$JX = Y \ , \ JY = -X \ , \ JU = V \ , \ JV = -U$$

which is a complex structure on N. This is the base (right-invariant) complex structure corresponding to Γ. If we define

$$T = \frac{1}{2}(X - iY), \text{ and } W = \frac{1}{2}(U - iV)$$

then $\{T, \overline{T}, W, \overline{W}\}$ generate $T_N \otimes \mathbb{C}$ and the only non-zero Lie bracket is

$$[T, \overline{T}] = i(W + \overline{W}).$$

Define the dual basis of forms to be $\{\omega, \overline{\omega}, \rho, \overline{\rho}\}$ which globally generates $T_N^* \otimes \mathbb{C}$.

Next, notice that

$$JT = iT, \ JW = iW, \ J\overline{T} = -i\overline{T}, \text{ and } J\overline{W} = -i\overline{W}$$

which implies that $T_N^{1,0}$ and $T_N^{0,1}$ are globally generated by $\{T, W\}$ and $\{\overline{T}, \overline{W}\}$ respectively. Similarly, $T^*_{N_{1,0}}$ is globally generated by $\{\omega, \rho\}$.

4.2.1 Deformations Starting at a Complex Structure

If we begin with a complex structure J on a manifold M, then the corresponding generalized complex structure $\mathbb{J}_J$ corresponds to the maximal isotropic space

$$L_J = T_{0,1} + T^*_{1,0},$$

which we recall has type n. Let us determine how d_L acts on smooth functions. A typical element of $C^\infty(L)$ is $X + \alpha$ where $X \in T_{0,1}$ and $\alpha \in T^*_{1,0}$. Then, for a smooth function f,

$$d_L(f)(X + \alpha) = \pi_T(X)(f) + \pi_T(\alpha)(f) = X(f) + 0 = \overline{\partial}(f)(X) + 0.$$

We can see that the d_L operator behaves like $\overline{\partial}$ in this case. This extends to $\wedge^k L^*$, in particular, d_L behaves like $\overline{\partial}$ on $\wedge^k(T^*_{0,1})$. This means we may decompose $H^2_L(M)$ as

$$H^2_L(M) = H^2(M, \mathcal{O}) \oplus H^1(M, T_{1,0}) \oplus H^0(M, \wedge^2 T_{1,0}). \tag{4.1}$$

See section 5.3 in [11] for more details. Note that $H^2(M, \mathcal{O}) \cong H^{0,2}_{\overline{\partial}}(M)$ corresponds to complex closed $(0,2)$-forms which are (complex) B-field transforms of the complex structure $\mathbb{J}_J$. Next, note that $H^1(M, T_{1,0})$ encodes the deformations of the base complex structure J and

$$H^{2,0}_{\overline{\partial}}(M) \cong H^0(M, \wedge^2 T_{1,0})$$

corresponds to holomorphic Poisson bivectors which may alter the type of the structure (the Maurer-Cartan equation in this case is satisfied if and only of β is holomorphic and Poisson). We recall some properties of B-fields and holomorphic Poisson bivectors here.

Let B be a closed complex $(0,2)$-form. From Definition 2.31, B acts on L_J by

$$X + \eta \mapsto X + \eta + \iota_X B.$$

Therefore, $L_B = e^B(L_J) = \Gamma_B \oplus T^*_{1,0}$ where $\Gamma_B = \{X + \iota_X(B) : X \in T_{0,1}\}$. Even though we only consider complex $(0,2)$-forms, we actually obtain all B-field transforms of $\mathbb{J}_J$ because $L_J = T_{0,1} + T^*_{1,0}$, which means only the $(0,2)$-part of a 2-form B will have a non-trivial effect on L_J. By definition, B-field transforms do not alter the type of the generalized complex structure.

Next, let β be a holomorphic Poisson bivector. As in example 2.40, we get

$$\mathbb{J}_\beta = \begin{bmatrix} -J & Q \\ 0 & J^* \end{bmatrix}$$

and

$$L_\beta = T_{0,1} \oplus \Gamma_\beta,$$

where $\Gamma_\beta = \{\eta + \beta(\eta) : \eta \in T^*_{1,0}\}$. If β is non-zero at some point of M, then Γ_β will have non-trivial projection onto $T_{\mathbb{C}}$, which means β transforms can alter the type of the generalized complex structure.

Poon examines extended deformations of generalized complex structures on Kodaira surfaces using differential Gerstenhaber algebras in [25]. In particular, a smooth locally complete family of deformations is obtained for generalized deformations of a complex structure. Similarly, Brînzănescu and Turcu ([4]) perform a direct computation of the same deformation space but with some problems (see Remark 4.24). For the convenience of the reader, we perform a corrected version of the computation, as it will be of fundamental importance in later sections.

On a Kodaira surface N and complex structure J, we know that the tangent bundle $T \otimes \mathbb{C}$ is globally generated by invariant vector fields $\{T, \overline{T}, W, \overline{W}\}$ whose only non-zero Lie bracket is

$$[T, \overline{T}] = i(W + \overline{W}).$$

With respect to this complex structure, $T_{1,0}$ and $T_{0,1}$ are globally generated by $\{T, W\}$ and $\{\overline{T}, \overline{W}\}$, respectively. Similarly, $T^*_{1,0}$ is globally generated by $\{\omega, \rho\}$.

The base generalized complex structure to be deformed is the one coming from the complex structure J. Namely, $\mathbb{J}_J : T \oplus T^* \to T \oplus T^*$ is defined to be

$$\mathbb{J}_J := \begin{bmatrix} -J & 0 \\ 0 & J^* \end{bmatrix},$$

which has $+i$-eignenbundle $L := T_{0,1} \oplus T^*_{1,0}$ and is generated by $\overline{T}, \overline{W}, \omega$, and ρ. The $-i$-eigenbundle $\overline{L}$ is generated by $T, W, \overline{\omega}$ and $\overline{\rho}$. Clearly $L \cap \overline{L} = \{0\}$. Referring to Equation 4.1 and the discussion following it, the cohomology space decomposes as:

$$H^2_L(N) \quad = \quad H^{2,0}_{\overline{\partial}}(N) \oplus H^1(N, T_{1,0}) \oplus H^{0,2}_{\overline{\partial}}(N).$$

Note that $H^1(N, T_{1,0})$ corresponds to the usual deformations of complex structures. Therefore, following the notation of [4], we obtain that a typical element of $H^2_L(N)$ is

$$\epsilon = t_{14} T \wedge W + t_{11} T \wedge \overline{\omega} + t_{22} W \wedge \overline{\rho} + t_{32} \overline{\omega} \wedge \overline{\rho}.$$

The coefficient t_{32} corresponds to the complex B-field, t_{14} corresponds to a holomorphic Poisson bivector, and t_{11} and t_{22} parameterize deformations of the base complex structure.

Theorem 4.21. *The above family of deformations $\epsilon \in H^2_L(M)$ is a smooth locally complete family.*

The proof of Theorem 4.21 is an immediate consequence of the following two lemmas.

Lemma 4.22. *$d(\omega) = d(\overline{\omega}) = 0$ and $d(\rho) = d(\overline{\rho}) = -i\omega \wedge \overline{\omega}$.*

Proof. Recall that the only non-zero Lie bracket of pairs of elements of the set $\{T, \overline{T}, W, \overline{W}\}$ is $[T, \overline{T}] = i(W + \overline{W})$. Further, for $\alpha \in \{\omega, \overline{\omega}, \rho, \overline{\rho}\}$ and $X, Y \in \{T, \overline{T}, W, \overline{W}\}$, we know

$$d(\alpha)(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) = -\alpha([X, Y])$$

because $\alpha(X)$ is either 1 or 0. Therefore, if the Lie bracket $[X, Y]$ vanishes, so does $d(\alpha)(X, Y)$, and the only non-zero Lie bracket gives

$$d(\alpha)(T, \overline{T}) = -\alpha([T, \overline{T}]) = -\alpha(i(W + \overline{W})).$$

If $\alpha = \omega$ or $\alpha = \overline{\omega}$, then $d(\alpha)(X, Y) = 0$ for any $X, Y \in \{T, \overline{T}, W, \overline{W}\}$ which means $d(\alpha) = 0$.

On the other hand, if $\alpha = \rho$ or $\alpha = \overline{\rho}$, then $d(\alpha)(T, \overline{T}) = -i$ and $d(\alpha)(X, Y) = 0$ for $\{X, Y\} \neq \{T, \overline{T}\}$, which means $d(\rho) = d(\overline{\rho}) = -i\omega \wedge \overline{\omega}$. □

Lemma 4.23. *For $\epsilon = t_{14} T \wedge W + t_{11} T \wedge \overline{\omega} + t_{22} W \wedge \overline{\rho} + t_{32} \overline{\omega} \wedge \overline{\rho}$, the Schouten bracket $[\epsilon, \epsilon]_S$ vanishes for any $t_{11}, t_{22}, t_{14}, t_{32} \in \mathbb{C}$.*

Proof. We prove that the Schouten bracket of any pair of elements in the set $\mathcal{A} = \{T \wedge W, T \wedge \overline{\omega}, W \wedge \overline{\rho}, \overline{\omega} \wedge \overline{\rho}\}$ is zero. This implies $[\epsilon, \epsilon]_S = 0$ for any such ϵ. Recall for $x \wedge y, w \wedge z \in \mathcal{A}$ that $[x \wedge y, w \wedge z]_S = [x, w]_C \wedge y \wedge z - [x, z]_C \wedge y \wedge w - [y, w]_C \wedge x \wedge z + [y, z]_C \wedge x \wedge w$. Let us first check the Courant brackets of pairs of elements of $\{T, W, \overline{\omega}, \overline{\rho}\}$. Obviously $[T, W]_C = [T, W] = 0$ and $[\overline{\omega}, \overline{\rho}] = 0$ by definition. Also,

$$[T, \overline{\omega}]_C = \mathcal{L}_T \overline{\omega} = \iota_T d\overline{\omega} + d(\overline{\omega}(T)) = 0$$

by the above lemma. Similarly $[W, \overline{\omega}]_C = 0$. Next,

$$[T, \overline{\rho}]_C = \mathcal{L}_T \rho = \iota_T d\overline{\rho} + d(\overline{\rho}(T)) = \iota_T(-i\omega \wedge \overline{\omega}) = -i\overline{\omega}.$$

Finally, the same computation but with T replaced with W proves $[W, \overline{\rho}]_C = 0$.

Therefore, the only non-zero Courant bracket of pairs of elements of $\{T, W, \overline{\omega}, \overline{\rho}\}$ is $[T, \overline{\rho}]_C = -i\overline{\omega}$. Now we have 10 Schouten brackets to compute, but most of the Courant brackets are zero which simplifies the computation. It is easy to see that $[T \wedge W, T \wedge W]_S = [T \wedge W, T \wedge \overline{\omega}]_S = [T \wedge \overline{\omega}, T \wedge \overline{\omega}]_S = [W \wedge \overline{\rho}, W \wedge \overline{\rho}]_S = [W \wedge \overline{\rho}, \overline{\omega} \wedge \overline{\rho}]_S = [\overline{\omega} \wedge \overline{\rho}, \overline{\omega} \wedge \overline{\rho}]_S = 0$. Thus we have only four more brackets to check: $[T \wedge W, W \wedge \overline{\rho}]_S$, $[T \wedge W, \overline{\omega} \wedge \overline{\rho}]_S$, $[T \wedge \overline{\omega}, W \wedge \overline{\rho}]_S$, $[T \wedge \overline{\omega}, \overline{\omega} \wedge \overline{\rho}]_S$.

First,

$$\begin{aligned} [T \wedge W, W \wedge \overline{\rho}]_S &= [T, W]_C \wedge W \wedge \overline{\rho} - [T, \overline{\rho}]_C \wedge W \wedge W \\ &\quad - [W, W]_C \wedge T \wedge \overline{\rho} + [W, \overline{\rho}]_C \wedge T \wedge W \\ &= 0 \wedge W \wedge \overline{\rho} + i\overline{\omega} \wedge W \wedge W - 0 \wedge T \wedge \overline{\rho} + 0 \wedge T \wedge W = 0. \end{aligned}$$

For the second,

$$
\begin{aligned}
[T \wedge W, \overline{\omega} \wedge \overline{\rho}]_S &= [T, \overline{\omega}]_C \wedge W \wedge \overline{\rho} - [T, \overline{\rho}]_C \wedge W \wedge \overline{\omega} \\
&\quad - [W, \overline{\omega}]_C \wedge T \wedge \overline{\rho} + [W, \overline{\rho}]_C \wedge T \wedge \overline{\omega} \\
&= 0 \wedge W \wedge \overline{\rho} + i\overline{\omega} \wedge W \wedge \overline{\omega} - 0 \wedge T \wedge \overline{\rho} + 0 \wedge T \wedge \overline{\omega} = 0.
\end{aligned}
$$

Next, the third,

$$
\begin{aligned}
[T \wedge \overline{\omega}, W \wedge \overline{\rho}]_S &= [T, W]_C \wedge \overline{\omega} \wedge \overline{\rho} - [T, \overline{\rho}]_C \wedge \overline{\omega} \wedge W \\
&\quad - [\overline{\omega}, W]_C \wedge T \wedge \overline{\rho} + [\overline{\omega}, \overline{\rho}]_C \wedge T \wedge W \\
&= 0 \wedge \overline{\omega} \wedge \overline{\rho} + i\overline{\omega} \wedge \overline{\omega} \wedge W - 0 \wedge T \wedge \overline{\rho} + 0 \wedge T \wedge W = 0.
\end{aligned}
$$

Finally, the last Schouten bracket,

$$
\begin{aligned}
[T \wedge \overline{\omega}, \overline{\omega} \wedge \overline{\rho}]_S &= [T, \overline{\omega}]_C \wedge \overline{\omega} \wedge \overline{\rho} - [T, \overline{\rho}]_C \wedge \overline{\omega} \wedge \overline{\omega} \\
&\quad - [\overline{\omega}, \overline{\omega}]_C \wedge T \wedge \overline{\rho} + [\overline{\omega}, \overline{\rho}]_C \wedge T \wedge \overline{\omega} \\
&= 0 \wedge \overline{\omega} \wedge \overline{\rho} + i\overline{\omega} \wedge \overline{\omega} \wedge \overline{\omega} - 0 \wedge T \wedge \overline{\rho} + 0 \wedge T \wedge \overline{\omega} = 0.
\end{aligned}
$$

□

Hence, Theorem 4.13 and, more specifically, Remark 4.14 tell us that $\epsilon \in H^2_L(N)$ are full deformations of the structure and that this collection of ϵ form a smooth locally complete family of deformations (since the obstruction map vanishes). This proves Theorem 4.21.

Remark 4.24. Note that in [4] the elements considered were not in $H^2_L(N)$ but rather arbitrary elements of $\wedge^2 L^*$. This causes problems in the rest of the paper since, for example, they cannot be viewed as the first order part a deformation. Furthermore, the statement of Theorem 4.6 in [4] seems to be incorrect. To be precise, if one considers an $\tilde{\epsilon}$ of the form

$$
t_{32}\overline{T}^* \wedge \overline{W}^* - t_{11}\overline{T}^* \wedge \omega^* - t_{21}\overline{T}^* \wedge \rho^* - t_{12}\overline{W}^* \wedge \omega^* - t_{22}\overline{W}^* \wedge \rho^* + t_{14}\omega^* \wedge \rho^*,
$$

it will fail to satisfy $[\tilde{\epsilon}, \tilde{\epsilon}]_S = 0$ if $t_{12} \neq 0$. Also note that $t_{12}\overline{W}^* \wedge \omega^*$ is not d_L-closed, and so $\tilde{\epsilon}$ cannot be integrable if $t_{12} \neq 0$.

Theorem 4.21 proves that a complex structure admits a smooth locally complete family of deformations. To extend this, we can actually prove that every complex-type structure on N admits a smooth locally complete family of deformations.

Theorem 4.25. *If L is a generalized complex structure of complex-type on a Kodaira surface N, then there is a smooth locally complete family of deformations of L.*

Proof. Let L be a complex-type generalized complex structure on N. Theorem 2.51 tells us that L is of the form $e^B L_J$ for some B-field B and complex structure J on N. If $B = 0$, $L = L_J$ and J has the form presented at the start of this section (Section 4.2) for some group of affine transformations Γ. Then the analysis performed above proves Theorem 4.21 for J. Therefore, if $B = 0$ the proof is done.

On the other hand, if $B \neq 0$ is a real closed 2-form then we may apply Corollary 4.17 to get a smooth locally complete family of deformations of $e^B L_J$ because L_J has a smooth locally complete family of deformations. □

We now understand the structure of these deformations locally. We assumed ϵ was "small" in order for the operator

$$A_\epsilon = \begin{bmatrix} 1 & \overline{\epsilon} \\ \epsilon & 1 \end{bmatrix}$$

to be invertible, which is equivalent to $L_\epsilon \cap \overline{L_\epsilon} = \{0\}$. However, since we have such an explicit description of the deformation space we can obtain "big" deformations by explicitly computing the determinant of the operator A_ϵ.

Proposition 4.26. *For $\epsilon = t_{14} T \wedge W + t_{11} T \wedge \overline{\omega} + t_{22} W \wedge \overline{\rho} + t_{32} \overline{\omega} \wedge \overline{\rho}$, the operator*

$$A_\epsilon = \begin{bmatrix} 1 & \overline{\epsilon} \\ \epsilon & 1 \end{bmatrix}$$

is invertible if and only if the quantity

$$D_C = 1 - |t_{11}|^2 - |t_{22}|^2 + |t_{11}|^2|t_{22}|^2 + t_{14}\overline{t_{32}} + \overline{t_{14}}t_{32} + |t_{14}|^2|t_{32}|^2 - t_{14}t_{32}\overline{t_{11}t_{22}} - t_{11}t_{22}\overline{t_{14}t_{32}}$$

is not zero.

Proof. First, working with the ordered bases $\{\overline{T}, \overline{W}, \omega, \rho\}$ of L and $\{T, W, \overline{\omega}, \overline{\rho}\}$ of $\overline{L}$ we can write ϵ in matrix form

$$\begin{bmatrix} -t_{11} & 0 & 0 & -t_{14} \\ 0 & -t_{22} & t_{14} & 0 \\ 0 & -t_{32} & t_{11} & 0 \\ t_{32} & 0 & 0 & t_{22} \end{bmatrix}.$$

Hence the operator A_ϵ is invertible if and only of $\det(A_\epsilon) \neq 0$. Then a routine (but tedious) computation shows that $\det(A_\epsilon) = D_C \cdot \overline{D_C} = |D_C|^2$. Thus, A_ϵ is invertible if and only if $D_C \neq 0$. □

Here, the notation D_C is for **complex determinant**. Finally, let us compute the generators for L_ϵ. We know that $L_\epsilon = (1+\epsilon)L$, so L_ϵ is generated by

$$(1+\epsilon)\overline{T} = \overline{T} - t_{11}T + t_{32}\overline{\rho},$$

$$(1+\epsilon)\overline{W} = \overline{W} - t_{22}W - t_{32}\overline{\omega},$$

$$(1+\epsilon)\omega = \omega + t_{11}\overline{\omega} + t_{14}W, \text{ and}$$

$$(1+\epsilon)\rho = \rho + t_{22}\overline{\rho} - t_{14}T.$$

From this expression, it is easy to see what the action of a B-field or holomorphic Poisson structure is.

Example 4.27. (B-field Transforms) If $B = t_{32}\overline{\omega}\wedge\overline{\rho}$ is a complex B-field then Proposition 4.26 tells us that any t_{32} is valid because $D_C = 1$. This corresponds to what we know: B-field transforms of generalized complex structures are generalized complex structures. In this case $L_B = \Gamma_B \oplus T^*_{1,0}$ where $\Gamma_B = \{X + \iota_X B | X \in T_{0,1}\}$. Explicitly L_B is generated by

$$\{T + t_{32}\overline{\rho}, W - t_{32}\overline{\omega}, \omega, \rho\}.$$

The type of these structures is 2 globally.

Example 4.28. (Poisson Bivectors) If $\beta = t_{14}T\wedge W$ is a holomorphic Poisson bivector then, as in the B-field case, any t_{14} is valid because $D_C = 1$ again. In this case $L_\beta = T_{0,1}\oplus\Gamma_\beta$ where $\Gamma_\beta = \{\eta + \beta(\eta) : \eta \in T^*_{1,0}\}$. L_β is generated by $\{\overline{T}, \overline{W}, \omega + t_{14}W, \rho - t_{14}T\}$. The type of these structures is 0 globally. This is because β corresponds to the holomorphic symplectic structure $\beta^* = \omega\wedge\rho$ which is, in particular, non-degenerate.

4.2.2 Deformations Starting at a Symplectic Structure

In this section we deform a generalized complex structure coming from a symplectic structure on Kodaira surfaces. We fix a symplectic structure and calculate the deformation space of the structure in the sense of generalized geometry.

Recall that T^* is generated by $\{\omega,\ \rho,\ \overline{\omega},\ \overline{\rho}\}$. A family of real symplectic structures is exhibited in [25] (see Section 3.3, $\omega = \alpha + i\beta$ and $\rho = \gamma + i\delta$): for $a, b \in \mathbb{C}$,

$$\eta = a\ \omega\wedge\rho + \overline{a}\ \overline{\omega}\wedge\overline{\rho} + b\ \omega\wedge\overline{\rho} + \overline{b}\ \overline{\omega}\wedge\rho$$

is non-degenerate if and only if $|a|^2 - |b|^2 \neq 0$. Let us choose the symplectic structure to be $\eta := \omega \wedge \rho + \overline{\omega} \wedge \overline{\rho}$, that is, $a = 1$, $b = 0$. This time, our base generalized complex structure is

$$\mathbb{J}_\eta := \begin{bmatrix} 0 & -\eta^{-1} \\ \eta & 0 \end{bmatrix}.$$

We know that the $+i$-eigenbundle, L, of $\mathbb{J}_\eta$ is $\{X - i\eta(X) : X \in T \otimes \mathbb{C}\}$ and therefore a basis for L is

$$\{e_1, e_2, e_3, e_4\} = \{T - i\rho,\ W + i\omega,\ \overline{T} - i\overline{\rho},\ \overline{W} + i\overline{\omega}\}$$

and similarly a basis for L^* is

$$\{e_1^*, e_2^*, e_3^*, e_4^*\} = \{\omega + iW,\ \rho - iT,\ \overline{\omega} + i\overline{W},\ \overline{\rho} - i\overline{T}\}.$$

Next we wish to obtain a concise description of $H_L^2(N)$.

Lemma 4.29. *$H_L^2(N)$ is generated by*

$$\begin{aligned} e_{12}^* &= (\omega + iW) \wedge (\rho - iT), \\ e_{14}^* &:= (\omega + iW) \wedge (\overline{\rho} - i\overline{T}), \\ e_{23}^* &:= (\rho - iT) \wedge (\overline{\omega} + i\overline{W}), \quad \textit{and} \\ e_{34}^* &:= (\overline{\omega} + i\overline{W}) \wedge (\overline{\rho} - i\overline{T}), \end{aligned}$$

where $e_{ij}^* = e_i^* \wedge e_j^*$.

Proof. We know from Corollary 4.10 that $H_L^2(N) \cong H_{DR}^2(M, \mathbb{C})$ via the dual of the projection map $\pi_T : L \to T_N$. Further,

$$H_{DR}^2(N, \mathbb{C}) = \operatorname{span}_{\mathbb{C}}\{\omega \wedge \rho, \overline{\omega} \wedge \overline{\rho}, \omega \wedge \overline{\rho}, \overline{\omega} \wedge \rho\}.$$

Therefore, using the isomorphism provided by Corollary 4.10, we see that $H_L^2(N)$ is generated by $\{e_{12}^*, e_{14}^*, e_{23}^*, e_{34}^*\}$ as desired. □

Let us consider $\epsilon = Ae_{12}^* + Be_{14}^* + Ce_{23}^* + De_{34}^*$ for some $A, B, C, D \in \mathbb{C}$. As in the complex setting it turns out that all of these are actually deformations themselves, as we now demonstrate.

Theorem 4.30. *The family of deformations of generalized complex structures on a primary Kodaira surface given by $\epsilon = Ae_{12}^* + Be_{14}^* + Ce_{23}^* + De_{34}^*$ with $(A, B, C, D) \in U \subseteq \mathbb{C}^4$ where U is an open neighbourhood of 0, is a smooth locally complete family.*

Proof. We prove that the Schouten bracket $[\epsilon,\epsilon]_S = 0$ for any $A, B, C, D \in \mathbb{C}$. It suffices to prove that the Schouten bracket of any pair of elements among

$$e_{12}^*, e_{14}^*, e_{23}^*, e_{34}^*$$

is zero. We first notice that the only non-zero Courant bracket of pairs of generators of L^*, namely $\{e_1^*, e_2^*, e_3^*, e_4^*\}$, is

$$[e_2^*, e_4^*]_C = -i(e_1^* + e_3^*)$$

which simplifies this computation since it is clear that $[e_{12}^*, e_{23}^*]_S = [e_{14}^*, e_{34}^*]_S = 0$. We do not check all the remaining cases, but present two cases below which will make the result clear.

First, let us check $[e_{12}^*, e_{14}^*]_S$,

$$\begin{aligned}
[e_{12}^*, e_{14}^*]_S &= [e_1^*, e_1^*]_C \wedge e_2^* \wedge e_4^* - [e_1^*, e_4^*]_C \wedge e_2^* \wedge e_1^* \\
&\quad -[e_2^*, e_1^*]_C \wedge e_1^* \wedge e_4^* + [e_2^*, e_4^*]_C \wedge e_1^* \wedge e_1^* = 0
\end{aligned}$$

since $[e_1^*, e_1^*]_C = [e_1^*, e_4^*]_C = [e_2^*, e_1^*]_C = 0$ and $e_1^* \wedge e_1^* = 0$.

Next, we check $[e_{12}^*, e_{34}^*]_S$,

$$\begin{aligned}
[e_{12}^*, e_{34}^*]_S &= [e_1^*, e_3^*]_C \wedge e_2^* \wedge e_4^* - [e_1^*, e_4^*]_C \wedge e_2^* \wedge e_3^* \\
&\quad -[e_2^*, e_3^*]_C \wedge e_1^* \wedge e_4^* + [e_2^*, e_4^*]_C \wedge e_1^* \wedge e_3^* = 0
\end{aligned}$$

since

$$[e_1^*, e_3^*]_C = [e_1^*, e_4^*]_C = [e_2^*, e_3^*]_C = 0$$

and

$$[e_2^*, e_4^*]_C \wedge e_1^* \wedge e_3^* = -i(e_1^* + e_3^*) \wedge e_1^* \wedge e_3^* = 0.$$

The remaining cases are similar to one of the above two cases. Therefore, the obstruction map vanishes for all ϵ, which means they form a smooth locally complete family by Theorem 4.13. This completes the proof. □

The above proof actually shows that every ϵ satisfies the Maurer-Cartan equation. Using this, let us dispense with the small condition and find explicit restrictions on the parameters A, B, C, D so that $L_\epsilon \cap \overline{L_\epsilon} = \{0\}$. This occurs if and only if the matrix

$$A_\epsilon := \begin{bmatrix} 1 & \bar{\epsilon} \\ \epsilon & 1 \end{bmatrix}$$

is invertible.

Proposition 4.31. *The operator A_ϵ is invertible for $\epsilon = Ae^*_{12} + Be^*_{14} + Ce^*_{23} + De^*_{34}$ if and only if*

$$D_S = 1 - |D|^2 - |A|^2 + |A|^2|D|^2 + B\overline{C} + \overline{B}C + |B|^2|C|^2 + AD\overline{BC} + \overline{AD}BC \qquad (*)$$

is not zero.

Proof. It is easy to check that

$$\epsilon = \begin{bmatrix} 0 & -A & 0 & -B \\ A & 0 & -C & 0 \\ 0 & C & 0 & -D \\ B & 0 & D & 0 \end{bmatrix}$$

under the bases $\{e_1, e_2, e_3, e_4\}$ for L and $\{e^*_1, e^*_2, e^*_3, e^*_4\}$ for L^*. Computing the determinant of A_ϵ, we get $D_S \cdot \overline{D_S} = |D_S|^2$. Therefore A_ϵ is invertible if and only if $D_S \neq 0$. □

Remark 4.32. Of course (*) is satisfied in an infinitesimal neighbourhood because the expression will be close to 1. It is interesting to note that even far from zero, as long as (*) is satisfied we obtain an integrable generalized complex structure.

Example 4.33. (Type Analysis) We know by Proposition 4.18 that in an infinitesimal neighbourhood the type of these structures is always zero. Let us explicitly demonstrate this. Recall that $L_\epsilon = (1+\epsilon)L$ and therefore L_ϵ is generated by

$$(1+\epsilon)e_1 = e_1 + Ae^*_2 + Be^*_4,$$

$$(1+\epsilon)e_2 = e_2 - Ae^*_1 + Ce^*_3,$$

$$(1+\epsilon)e_3 = e_3 - Ce^*_2 + De^*_4,$$

and

$$(1+\epsilon)e_4 = e_4 - Be^*_1 - De^*_3.$$

Hence the projection of L_ϵ onto T is generated by

$$\{(1-iA)T - iB\overline{T},\ (1-iA)W + iC\overline{W},\ (1-iD)\overline{T} + iCT,\ (1-iD)\overline{W} - iBW\}. \qquad (**)$$

We wish to prove that the projection of L_ϵ onto T_N has full rank regardless of the (valid) choice of A, B, C, D in a small neighbourhood of zero in $\mathbb{C}^4$. Since only the first and third elements of $(**)$ have T and $\overline{T}$, we work with them. The analysis will be the same using the second and fourth for W and $\overline{W}$.

So we have $(1-iA)T - iB\overline{T}$ and $(1-iD)\overline{T} + iCT$. We must prove that these two vector fields are linearly independent. A quick calculation supposing a linear combination of these two vectors vanishes shows that these are independent as long as

$$BC \neq (1-iA)(1-iD).$$

If we are working with small deformations then the left-hand-side of this expression is close to zero and the right-hand-side is close to 1. Therefore, the expression is satisfied in this case. This is what we wanted to establish.

Moreover, by Proposition 4.19 we know that given an infinitesimal deformation ϵ we should be able to find a complex symplectic structure $\widetilde{\omega}$ such that $L_\epsilon = \{X - i\widetilde{\omega}(X) | X \in T_N \otimes \mathbb{C}\}$. However, as we mentioned in Remark 4.20, this is hard to find in general. Fortunately, we can explicitly determine $\widetilde{\omega}$ for a given ϵ in this case.

Example 4.34. (Finding $\widetilde{\omega}$) Let $\epsilon = Ae^*_{12} + Be^*_{14} + Ce^*_{23} + De^*_{34}$ with A, B, C, D in a small open neighbourhood of zero. Then $BC \neq (1-iA)(1-iD)$, or equivalently

$$\Xi := (1-iA)(1-iD) - BC \neq 0.$$

Also, as in the previous example we get L_ϵ is generated by

$$(1+\epsilon)e_1 = e_1 + Ae^*_2 + Be^*_4,$$

$$(1+\epsilon)e_2 = e_2 - Ae^*_1 + Ce^*_3,$$

$$(1+\epsilon)e_3 = e_3 - Ce^*_2 + De^*_4,$$

and

$$(1+\epsilon)e_4 = e_4 - Be^*_1 - De^*_3.$$

Proposition 4.19 guarantees the existence of a complex symplectic structure $\widetilde{\omega}$ so that $L_\epsilon = L_{\widetilde{\omega}}$. Say

$$\widetilde{\omega} = a_1 \omega \wedge \rho + a_2 \omega \wedge \overline{\rho} + a_3 \overline{\omega} \wedge \rho + a_4 \overline{\omega} \wedge \overline{\rho}$$

for $a_i \in \mathbb{C}$. Then, $L_{\widetilde{\omega}} = \{X - i\widetilde{\omega}(X) | X \in T \otimes \mathbb{C}\}$ is generated by

$$\{T - ia_1\rho - ia_2\overline{\rho},\ W - ia_1\omega + ia_3\overline{\omega},\ \overline{T} - ia_3\rho - ia_4\overline{\rho},\ \overline{W} + ia_2\omega + ia_4\overline{\omega}\}.$$

It is now a matter of solving a system of eight equations which we obtain by determining when of $L_\epsilon = L_{\widetilde{\omega}}$ by examining the basis elements. The equations are:

1. $(1 - iA)a_1 - iBa_3 = 1 + iA$,
2. $(1 - iA)a_2 - iBa_4 = 1 + iB$,
3. $(1 - iA)a_1 + iCa_2 = 1 + iA$,
4. $(1 - iA)a_3 + iCa_4 = -iC$,
5. $iCa_1 + (1 - iD)a_3 = -iC$,
6. $iCa_2 + (1 - iD)a_4 = 1 + iD$,
7. $-iBa_1 + (1 - iD)a_2 = iB$,
8. $-iBa_3 + (1 - iD)a_4 = 1 + iD$.

After employing one's preferred method of solving linear equations we get

$$a_1 = \frac{2(1 - iD)}{\Xi} - 1, \;\; a_2 = \frac{2iB}{\Xi}, \;\; a_3 = \frac{-2iC}{\Xi}, \;\text{ and } a_4 = \frac{2(1 - iA)}{\Xi} - 1.$$

Therefore,

$$\widetilde{\omega} = \left(\frac{2(1 - iD)}{\Xi} - 1\right) \omega \wedge \rho + \left(\frac{2iB}{\Xi}\right) \omega \wedge \overline{\rho} + \left(\frac{-2iC}{\Xi}\right) \overline{\omega} \wedge \rho + \left(\frac{2(1 - iA)}{\Xi} - 1\right) \overline{\omega} \wedge \overline{\rho}.$$

The condition that $\widetilde{\omega}$ is non-degenerate is that $a_1a_4 - a_2a_3 \neq 0$, but

$$a_1a_4 - a_2a_3 = \left(\frac{2(1 - iA)}{\Xi} - 1\right)\left(\frac{2(1 - iD)}{\Xi} - 1\right) - \left(\frac{2iB}{\Xi}\right)\left(\frac{-2iC}{\Xi}\right)$$

which is close to 1 if A, B, C, D are all close to zero. Hence, $\widetilde{\omega}$ is closed and non-degenerate, which means it is a complex symplectic structure. Thus, for ϵ and $\widetilde{\omega}$ as above $L_\epsilon = L_{\widetilde{\omega}}$.

Remark 4.35. Both of the above examples focused on small infinitesimal deformations. Using Proposition 4.31 we can allow A, B, C, D to be large. Then, we are able to obtain generalized complex structures of complex-type as well. For example, choosing $A = D = 0$ and $B = C = -1$ satisfies $D_S \neq 0$ and L_ϵ has a two dimensional projection onto T_N.

4.2.3 An Intersection of Deformation Spaces

In both [11] and [29] a method is examined for interpolating between a complex structure and a symplectic structure via generalized geometry on a Kähler manifold (see, for example, [11] Section 4.6). In our case, these two deformation spaces intersect, which gives us a way of deforming a complex-type structure into a symplectic-type structure. Let us explicitly examine this intersection.

The first step is to understand what parameters $t_{11}, t_{22}, t_{14}, t_{32} \in \mathbb{C}$ in the complex case and $A, B, C, D \in \mathbb{C}$ in the symplectic case are permissable. As we mentioned above, the restriction comes from the fact that the matrix

$$A_\epsilon := \begin{bmatrix} 1 & \bar{\epsilon} \\ \epsilon & 1 \end{bmatrix}$$

must be invertible. Recall the determinant equations were:

1. Complex Case

$$D_C = 1 - |t_{11}|^2 - |t_{22}|^2 + |t_{11}|^2|t_{22}|^2 + t_{14}\overline{t_{32}} + \overline{t_{14}}t_{32} + |t_{14}|^2|t_{32}|^2 - t_{14}t_{32}\overline{t_{11}t_{22}} - t_{11}t_{22}\overline{t_{14}t_{32}},$$

2. Symplectic Case

$$D_S = 1 - |D|^2 - |A|^2 + |A|^2|D|^2 + B\overline{C} + \overline{B}C + |B|^2|C|^2 + AD\overline{BC} + \overline{AD}BC.$$

Proposition 4.36. *A generalized complex structure from the symplectic deformation space given by A, B, C, D coincides with a generalized complex structure from the complex deformation space given by $t_{11}, t_{12}, t_{21}, t_{22}$ if and only if the following seven equations are satisfied:*

(1') $Bt_{11} + (1 + iA)t_{14} = i + A,$

(2') $Bt_{32} + (1 + iA)t_{22} = -iB,$

(3') $(1 - iD)t_{11} + Ct_{14} = -iC,$

(4') $(1 - iD)t_{32} + Ct_{22} = i - D,$

(5') $Ct_{32} + (1 + iA)t_{11} = -iC,$

(6') $(1 - iD)t_{22} + Bt_{14} = -iB,$

(7') $Bt_{11} = Ct_{22}$.

Proof. We will use a C subscript to denote the generalized complex structure coming from a complex structure and an S to denote those coming from a symplectic structure. Then, given the above parameters, we know that in the complex case $L_{\epsilon_C} = (1+\epsilon_C)L_C$ is generated by

$$\overline{T} - t_{11}T + t_{32}\overline{\rho},$$
$$\overline{W} - t_{22}W - t_{32}\overline{\omega},$$
$$\omega + t_{11}\overline{\omega} + t_{14}W, \text{ and}$$
$$\rho + t_{22}\overline{\rho} - t_{14}T.$$

In the symplectic case L_{ϵ_S} is generated by

$$(T - i\rho) - iA(T + i\rho) - iB(\overline{T} + i\overline{\rho}),$$
$$(W + i\omega) - iA(W - i\omega) + iC(\overline{W} - i\overline{\omega}),$$
$$(\overline{T} - i\overline{\rho}) - iD(\overline{T} + i\overline{\rho}) - iC(T + i\rho), \text{ and}$$
$$(\overline{W} + i\overline{\omega}) - iD(\overline{W} - i\overline{\omega}) - iB(W - i\omega).$$

Let us examine when these two spaces coincide for some choice of the parameters A, B, C, D and $t_{11}, t_{22}, t_{32}, t_{14}$. First we stipulate that each generator for L_{ϵ_S} must be in the span of the generators of L_{ϵ_C}. Doing this, we obtain eight equations:

1. $Bt_{11} + (1 + iA)t_{14} = i + A$,
2. $Bt_{32} + (1 + iA)t_{22} = -iB$,
3. $(1 - iD)t_{11} + Ct_{14} = -iC$,
4. $(1 - iD)t_{32} + Ct_{22} = i - D$,
5. $Ct_{22} + (1 + iA)t_{14} = i + A$,
6. $Ct_{32} + (1 + iA)t_{11} = -iC$,
7. $(1 - iD)t_{22} + Bt_{14} = -iB$,
8. $(1 - iD)t_{32} + Bt_{11} = i - D$.

Equations 1 and 5 immediately yield $Bt_{11} = Ct_{22}$. From this we notice that $1 \equiv 5$ and $8 \equiv 4$ so we can reduce our equations to the seven equations $1' - 7'$ as in the statement of the proposition.

$\square$

Proposition 4.36 shows that one can find appropriate choices for A, B, C, D and t_{11}, t_{22}, t_{32}, t_{14} so that the resulting generalized complex structures are the same. It should be noted that these spaces do not intersect *infinitesimally* as is clear from the equations $1' - 7'$, if A, B, C, D are small then the t_{ij} cannot be. Let us examine some explicit examples.

Example 4.37. Let us suppose $B = C = D = 0$ and $A \neq 0$. The symplectic determinant equation becomes $D_S = 1 - |A|^2 \neq 0$, that is $|A| \neq 1$. The above equations tell us that $t_{11} = t_{22} = 0$. Equation 1' tells us that

$$t_{14} = \frac{i + A}{1 + iA}.$$

Note that since $|A| \neq 1$ we get $1 + iA \neq 0$. Further, equation $4'$ gives $t_{32} = i$. Finally we check the complex determinant equation and see that it reduces to

$$D_C = 1 + t_{14}\overline{t_{32}} + \overline{t_{14}}t_{32} + |t_{14}|^2|t_{32}|^2 = |1 + t_{14}\overline{t_{32}}|^2 = \frac{4}{|1 + iA|^2} \neq 0.$$

Therefore we obtain that every symplectic deformation with $B = C = D = 0$ gives rise to a generalized complex structure that can be obtained as a deformation of a complex structure. In other words, all such generalized complex structures lie in the intersection of our two deformation spaces which explicitly demonstrates that the intersection is non-empty.

Example 4.38. Suppose our deformation has the form $A = \overline{D}$ and $B = C = 0$. First, our symplectic determinant is $D_S = (1 - |A|^2)^2 \neq 0$ which again holds if and only if $|A| \neq 1$. Also, $t_{11} = t_{22} = 0$. Then using equations 1' and 4' to solve for t_{14} and t_{32} respectively we get

$$t_{14} = \frac{i + A}{1 + iA}, \text{ and } t_{32} = \frac{i - \overline{A}}{1 - i\overline{A}}$$

which implies $t_{14} = -\overline{t_{32}}$. Lastly the complex determinant equation becomes

$$D_C = 1 - t_{14}^2 - \overline{t_{14}}^2 + |t_{14}|^4 = |1 - t_{14}|^2.$$

But since $|A| \neq 1$ it follows that $t_{14} \neq 1$, which means all of these generalized complex structures lie in the intersection of the two deformation spaces as well.

Remark 4.39. The simple fact that these two deformation spaces intersect has interesting consequences. It implies that whether or not we start at a complex structure or a symplectic structure that we can obtain any other structure in either deformation space via generalized deformations.

4.3 Deformations of Even-Type Structures on Complex Tori

In this section we work with a specific family of complex 2-tori, namely those that are similar to Kodaira surfaces above. By doing this, many results that we obtained for Kodaira surfaces will extend directly to the torus setting. These techniques can be applied to general tori as well.

The complex 2-tori we will be working with are $\mathbb{C}^2/\Gamma$ where Γ is a specific maximal lattice, but let us describe it in a similar manner as we did for Kodaira surfaces. Let A be $\mathbb{C}^2$, with multiplication

$$(z, w) * (\alpha, \beta) = (z + \alpha, w + \beta).$$

Next, let Γ be an abelian group of affine transformations generated by four elements g_1, g_2, g_3, g_4. We may view the g_i as transformations on $\mathbb{C}^2$ and in that case they have the form;

$$g_j(z, w) = (z + \alpha_j, w + \beta_j)$$

where $\alpha_1 = \alpha_2 = 0$ and $\beta_3 = \beta_4 = 0$. Then, our complex torus is $N = \mathbb{C}^2/\Gamma$. The tangent bundle T is globally generated by invariant vector fields

$$< X, Y, U, V >=< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial v}, \frac{\partial}{\partial u} >$$

and all Lie brackets vanish. Then, N admits a complex structure J such that

$$JX = Y,\ JY = -X,\ JU = V,\ JV = -U.$$

If we define

$$T = \frac{1}{2}(X - iY),\ \text{and}\ W = \frac{1}{2}(U - iV)$$

then $\{T, \overline{T}, W, \overline{W}\}$ generate $T \otimes \mathbb{C}$ and all the Lie brackets vanish. Define the dual basis of forms to be $\{\omega, \overline{\omega}, \rho, \overline{\rho}\}$ which globally generate $T^* \otimes \mathbb{C}$.

Then, as in the Kodaira surface case,

$$JT = iT,\ JW = iW,\ J\overline{T} = -i\overline{T},\ \text{and } J\overline{W} = -i\overline{W}$$

which implies that $T_{1,0}$ and $T_{0,1}$ are globally generated by $\{T, W\}$ and $\{\overline{T}, \overline{W}\}$ respectively. Similarly, $T^*_{1,0}$ is globally generated by $\{\omega, \rho\}$.

Deformations Starting at a Complex Structure

We perform a similar analysis of the deformations of a generalized complex structure that comes from a complex structure, as was done by Brînzănescu, Dinuta, and Dinuta in [3]. It should be noted that the same problem outlined in Remark 4.24 is present here, but it does not cause any problems because H^2_L has rank 6 and therefore every generator of $\wedge^2 L^*$ is a generator of H^2_L. The method is the same as for Kodaira surfaces. Define $\mathbb{J}_J : T_N \oplus T^*_N \to T_N \oplus T^*_N$ to be

$$\mathbb{J}_J := \begin{bmatrix} -J & 0 \\ 0 & J^* \end{bmatrix}$$

which has $+i$-eigenbundle $L := T_{0,1} \oplus T^*_{1,0}$, and $-i$-eigenbundle $\overline{L}$.

The cohomology space in question decomposes, as in Equation 4.1 and the discussion after it,

$$H^2_L(N) \quad = \quad H^{2,0}_{\bar\partial}(N) \oplus H^1(N, T_{1,0}) \oplus H^{0,2}_{\bar\partial}(N).$$

and a typical element has the form

$$\epsilon = t_{14} T \wedge W + t_{11} T \wedge \overline{\omega} + t_{22} W \wedge \overline{\rho} + t_{21} T \wedge \overline{\rho} + t_{12} W \wedge \overline{\omega} + t_{32} \overline{\omega} \wedge \overline{\rho}.$$

So the parameters t_{11}, t_{22}, t_{21}, and t_{12} correspond to deformations of the base complex structure, t_{14} is the coefficient of a holomorphic Poisson structure, and t_{32} is the coefficient of a B-field. Once again, as the next theorem illustrates, all such ϵ are full deformations.

Theorem 4.40. *The above family of deformations of generalized complex structures is a smooth locally complete family.*

Proof. Notice, for

$$\epsilon = t_{14} T \wedge W + t_{11} T \wedge \overline{\omega} + t_{22} W \wedge \overline{\rho} + t_{21} T \wedge \overline{\rho} + t_{12} W \wedge \overline{\omega} + t_{32} \overline{\omega} \wedge \overline{\rho},$$

the Schouten bracket $[\epsilon, \epsilon]_S$ is zero for any $t_{ij} \in \mathbb{C}$. This follows immediately from the definition of the Schouten bracket since all the Lie brackets and, consequently, all the Courant brackets vanish in this case.

Therefore, any such ϵ satisfies the Maurer-Cartan equation. Hence, $L_\epsilon = (1+\epsilon)L$ defines a generalized complex structures on N. Applying Theorem 4.13 shows us that this is a smooth locally complete family of deformations. $\square$

We present an analogue of Theorem 4.25 for complex 2-tori now.

Theorem 4.41. *If L is a generalized complex structure of complex-type then there is a smooth locally complete family of deformations of L.*

The proof of the above theorem is similar to the proof of Theorem 4.25 and is omitted.

In addition, we omit the analysis of the invertibiliy of the operator A_ϵ, but one could similarly compute the determinant of the matrix A_ϵ where

$$\epsilon = \begin{bmatrix} -t_{11} & -t_{12} & 0 & -t_{14} \\ -t_{21} & -t_{22} & t_{14} & 0 \\ 0 & -t_{32} & t_{11} & t_{21} \\ t_{32} & 0 & t_{12} & t_{22} \end{bmatrix}.$$

In this case, L_ϵ is generated by

$$(1+\epsilon)\overline{T} = \overline{T} - t_{11}T - t_{12}W + t_{32}\rho,$$

$$(1+\epsilon)\bar{(}W) = \overline{W} - t_{21}T - t_{22}W - t_{32}\omega,$$

$$(1+\epsilon)\omega = \omega + t_{14}W + t_{11}\overline{\omega} + t_{21}\overline{\rho}, \text{ and}$$

$$(1+\epsilon)\rho = \rho + t_{21}\overline{\omega} + t_{22}\overline{\rho} - t_{14}T.$$

Remark 4.42. If the coefficient of the Poisson structure, t_{14}, is zero then the resulting generalized complex structure is of complex-type, and if $t_{14} \neq 0$ it is of symplectic-type. If the B-field is non-trivial ($t_{32} \neq 0$) then the structure still has type 2. The parameters t_{11}, t_{22}, t_{21}, and t_{12} correspond to the usual deformations of complex structures on N.

Both Kähler and hence generalized Kähler structures exist on a torus. In particular, any Kähler structure (J, ω, g) on a complex 2-torus gives rise to a generalized Kähler structure $(\mathbb{J}_J, \mathbb{J}_\omega)$. Therefore, generalized complex structures that come from complex structures give rise to a generalized Kähler structure.

Let us now examine a specific symplectic deformation space.

Deformations Starting at a Symplectic Structure

Let us choose $\eta = \omega \wedge \rho + \overline{\omega} \wedge \overline{\rho}$ as we did for Kodaira surfaces. Then L is generated by $\{e_1, e_2, e_3, e_4\} = \{T - i\rho,\ W + i\omega,\ \overline{T} - i\overline{\rho},\ \overline{W} + i\overline{\omega}\}$ and L^* is generated by $\{e_1^*, e_2^*, e_3^*, e_4^*\} = \{\omega + iW,\ \rho - iT,\ \overline{\omega} + i\overline{W},\ \overline{\rho} - i\overline{T}\}$ as in the Kodaira surface setting.

Lemma 4.43. *The Courant bracket of any two elements of L (or $\overline{L} \cong L^*$) is zero.*

Proof. Recall that all Lie brackets of pairs of elements of T_N are zero. Next, notice that every non-zero element of L (or $\overline{L}$) has non-zero projection onto T_N. Further, we know

$$\pi_T[X + \alpha, Y + \beta]_C = [X, Y]_{Lie}.$$

In particular, the only component in T_N under the Courant bracket is the Lie bracket of the vector field parts (which all vanish). Therefore, if the Courant bracket of two elements of L (or $\overline{L}$) were non-zero, the resulting element would only have a component in T_N^*. But no non-zero element of L (or $\overline{L}$) has zero projection onto T_N which means it would not be an element of L (or $\overline{L}$). But L ($\overline{L}$) is closed under the Courant bracket. This is a contradiction. Hence, the Courant bracket of any two elements of L (or $\overline{L}$) is necessarily zero. □

In order to obtain the deformations $\epsilon \in H_L^2(N) \cong H_{DR}^2(N)$ we recall that since the second Betti number of the torus is 6, all of the generators of $\wedge^2 T^*$ form a basis of $H_{DR}^2(N)$, that is, it is generated by $\{\omega \wedge \overline{\omega},\ \omega \wedge \rho,\ \omega \wedge \overline{\rho},\ \overline{\omega} \wedge \rho,\ \overline{\omega} \wedge \overline{\rho},\ \rho \wedge \overline{\rho}\}$, and therefore, $H_L^2(N)$ is generated by $e_{ij}^* = e_i^* \wedge e_j^*$ for $1 \leq i < j \leq 6$.

Lemma 4.44. *For any $\epsilon \in H_L^2(N)$, $[\epsilon, \epsilon]_S = 0$.*

Proof. This is immediate from Lemma 4.43 since every term will have a Courant bracket. □

Therefore, any sufficiently small element $\epsilon \in H_L^2(N)$ (so that A_ϵ is invertible) is a generalized complex deformation. Further, this is a smooth locally complete family.

Theorem 4.45. *The family of deformations of generalized complex structures on complex 2-tori given by*

$$\epsilon = \sum_{i<j} A_{ij} e_{ij}^*$$

with $A_{ij} \in U \subseteq \mathbb{C}^6$ where U is a small open neighbourhood of 0 is a smooth locally complete family.

Proof. As on Kodaira surfaces, this follows from Theorem 4.13 and Lemma 4.44. □

Small infinitesimal deformations of a generalized complex structure coming from a symplectic structure on a torus all have type zero and also correspond to complex symplectic structures according to Propositions 4.18 and 4.19. Let us proceed to a description of the moduli spaces of even-type generalized complex structures on Kodaira surfaces and complex 2-tori.

4.4 The Moduli Space of Generalized Complex Structures of Even-Type

In this section, we show that moduli spaces of generalized complex structures of even-type on complex two-dimensional nilmanifolds are smooth complex manifolds of dimensions 4 or 6, and present a geometric description of these spaces. Let $\mathcal{M}_{2gen}(N)_{/\cong}$ be the moduli space of generalized complex structures of even-type on a manifold N, up to isomorphism. Huybrechts studies $\mathcal{M}_{2gen}(N)_{/\cong}$ when N is a K3 surface in [21]. We begin this section by summarizing his results and then extend them to Kodaira surfaces and complex 2-tori.

Proposition 4.46 (Huybrechts, [21], Section 3)**.** *Let M be a K3 surface. The moduli space of generalized Calabi-Yau structures on M is a complex 22-dimensional space. For comparison, the space of complex deformations in this case is complex 20-dimensional. There is a hyperspace (complex 21-dimensional space) consisting of all B-field transforms of complex structures, and a real 22-dimensional subspace of real symplectic structures. Every other structure in the space is a B-transform of these symplectic structures.*

Let us first consider the case where N is a complex 2-torus.

Theorem 4.47. *The moduli space of left-invariant generalized complex structures of even-type on a complex 2-torus is a complex six-dimensional manifold. Contained in this manifold is a complex four-dimensional subspace of complex deformations. There is a hyperspace (complex five-dimensional subspace) consisting of all B-field transforms of complex structures, and a real 6-dimensional space of real symplectic structures (coming from $H^2_{DR}(N,\mathbb{R})$). Every other structure in the space is a B-transform of these symplectic structures.*

Proof. Let $L \in \mathcal{M}_{2gen}(N)$. Then L is either a complex-type structure or a symplectic-type structure by Theorem 2.51. If L is a complex-type structure then Theorem 4.41 says that

L admits a smooth locally complete family of deformations. On the other hand, if L is of symplectic-type then Theorem 4.15 gives us such a smooth family. This provides a way to place a smooth manifold structure on $\mathcal{M}_{2gen}(N)$.

If the point corresponds to a symplectic-type structure then $H^2_L(N) \cong H^2_{DR}(N, \mathbb{C}) \cong \mathbb{C}^6$. Moreover, if L is a complex-type point then L is a B-field transform of a complex structure, J. The moduli space of complex structures is parametrized by $H^1(M, T_{1,0}) \cong \mathbb{C}^4$ and B-field transforms lie in $H^0(M, \wedge^2 T^*_{0,1}) \cong \mathbb{C}$. Finally, $H^2_{DR}(N, \mathbb{R}) \cong \mathbb{R}^6$ contains a 6-dimensional family of real symplectic structures. Every other structure in the space is a B-transform of these symplectic structures. □

Next, let N be a Kodaira surface.

Theorem 4.48. *The moduli space of left-invariant even-type generalized complex structures on Kodaira surfaces is a complex four-dimensional manifold. Contained in this manifold is a complex 2-dimensional space of complex deformations and a hyperspace (complex 3-dimensional space) of all B-field transforms of complex structures. There is also a real 4-dimensional space of real symplectic structures coming from $H^2_{DR}(N, \mathbb{R})$.*

Proof. Each point L of the moduli space $\mathcal{M}_{2gen}(N)_{/\cong}$ admits a smooth locally complete family of deformations from Theorems 4.25 and 4.15. This provides a way to place a smooth manifold structure on $\mathcal{M}_{2gen}(N)_{/\cong}$. If the point corresponds to a symplectic-type structure then $H^2_L(N) \cong H^2_{DR}(N, \mathbb{C}) \cong \mathbb{C}^4$. If L is a complex-type point then L is the B-field transform of a complex structure, J. But the moduli space of complex structures is parametrized by $H^1(M, T_{1,0}) \cong \mathbb{C}^2$ and B-field transforms lie in $H^0(M, \wedge^2 T^*_{0,1}) \cong \mathbb{C}$. Finally in Section 3.3 we exhibited a real 4-dimensional family of real symplectic structures coming from $H^2_{DR}(N, \mathbb{R})$. □

Remark 4.49. One particular reason that the above classification theorems are interesting is that it demonstrates that Kodaira surfaces are not so different from K3 surfaces and complex 2-tori. The moduli spaces (of even-type structures) has a very similar shape in all three cases, despite the fact that Kodaira surfaces are non-Kähler. Perhaps this is not surprising as the class of compact complex surfaces with trivial canonical bundle contains only three surfaces: K3 surfaces, complex 2-tori, and Kodaira surfaces.

4.5 Non-Existence of Some Generalized Kähler Structures on Kodaira Surfaces

Recall that a generalized Kähler structure consists of two commuting generalized complex structures $(\mathbb{J}_1, \mathbb{J}_2)$ such that $G := -\mathbb{J}_1\mathbb{J}_2$ induces a positive definite metric. We now illustrate with specific examples that certain pairs of generalized complex structures on Kodaira surfaces fail to induce a positive definite metric.

Complex Deformation Space

First we consider a pair $(\mathbb{J}_1, \mathbb{J}_2)$ of generalized complex structures in the complex deformation space we examined in Section 4.2.1. Recall that $\mathbb{J}_1$ and $\mathbb{J}_2$ are of the form $\mathbb{J}_\epsilon = A_\epsilon \mathbb{J}_J A_\epsilon^{-1}$, where $\mathbb{J}_J$ is a complex structure. Further, $\epsilon : L \to \overline{L}$ has the form

$$\epsilon = \begin{bmatrix} -t_{11} & 0 & 0 & -t_{14} \\ 0 & -t_{22} & t_{14} & 0 \\ 0 & -t_{32} & t_{11} & 0 \\ t_{32} & 0 & 0 & t_{22} \end{bmatrix}$$

and

$$A_\epsilon = \begin{bmatrix} 1 & 0 & 0 & 0 & -\overline{t_{11}} & 0 & 0 & -\overline{t_{14}} \\ 0 & 1 & 0 & 0 & 0 & -\overline{t_{22}} & \overline{t_{14}} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\overline{t_{32}} & \overline{t_{11}} & 0 \\ 0 & 0 & 0 & 1 & \overline{t_{32}} & 0 & 0 & \overline{t_{22}} \\ -t_{11} & 0 & 0 & -t_{14} & 1 & 0 & 0 & 0 \\ 0 & -t_{22} & t_{14} & 0 & 0 & 1 & 0 & 0 \\ 0 & -t_{32} & t_{11} & 0 & 0 & 0 & 1 & 0 \\ t_{32} & 0 & 0 & t_{22} & 0 & 0 & 0 & 1 \end{bmatrix},$$

is an automorphism of $L \oplus \overline{L}$ in the basis $\{\overline{T}, \overline{W}, \omega, \rho, T, W, \overline{\omega}, \overline{\rho}\}$ for $L \oplus \overline{L}$. Since

$$\mathbb{J}_J = \text{diag}\{i, i, i, i, -i, -i, -i, -i\},$$

with respect to this basis, we obtain

$$\mathbb{J}_\epsilon = \begin{bmatrix} * & 0 & 0 & * & * & 0 & 0 & * \\ 0 & * & * & 0 & 0 & * & * & 0 \\ 0 & * & * & 0 & 0 & * & * & 0 \\ * & 0 & 0 & * & * & 0 & 0 & * \\ * & 0 & 0 & * & * & 0 & 0 & * \\ 0 & * & * & 0 & 0 & * & * & 0 \\ 0 & * & * & 0 & 0 & * & * & 0 \\ * & 0 & 0 & * & * & 0 & 0 & * \end{bmatrix},$$

where the entries with a $*$ are not necessarily zero.

Next, the matrix corresponding to the pairing $<,>$ in the above basis is

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Fix any two deformations ϵ_1 and ϵ_2, and consider the corresponding generalized complex structures $\mathbb{J}_{\epsilon_1}$ and $\mathbb{J}_{\epsilon_2}$. Although these generalized complex structures commute for some ϵ_1 and ϵ_2, $G = -\mathbb{J}_{\epsilon_1}\mathbb{J}_{\epsilon_2}$ *never* induces a positive definite metric. Indeed,

$$\mathbb{J}_{\epsilon_i}(\overline{T}) = \begin{bmatrix} * \\ 0 \\ 0 \\ * \\ * \\ 0 \\ 0 \\ * \end{bmatrix} \quad \text{and} \quad M\mathbb{J}_{\epsilon_i}(\overline{T}) = \begin{bmatrix} 0 \\ * \\ * \\ 0 \\ 0 \\ * \\ * \\ 0 \end{bmatrix},$$

implying

$$\begin{aligned}
\widetilde{G}(\overline{T},\overline{T}) &= <G\overline{T},\overline{T}> \\
&= <-\mathbb{J}_{\epsilon_1}\mathbb{J}_{\epsilon_2}\overline{T},\overline{T}> \\
&= <\mathbb{J}_{\epsilon_2}\overline{T},\mathbb{J}_{\epsilon_1}\overline{T}> \\
&= \mathbb{J}_{\epsilon_1}(\overline{T})^t M\mathbb{J}_{\epsilon_2}(\overline{T}) \\
&= 0.
\end{aligned}$$

Hence, no such metric $\widetilde{G}$ is positive definite. We conclude that no pair of such generalized complex structures gives rise to a generalized Kähler structure.

Symplectic Deformation Space

A similar argument to the one we used for the complex deformation space shows that no pair of generalized complex structures from the symplectic deformation space computed in Section 4.2.2 forms a generalized Kähler structure. In particular, we will show that no pair of these generalized complex structures induces a positive definite metric. As in the complex case, let us determine what $\mathbb{J}_\epsilon$ looks like for any deformation $\epsilon \in H^2_L(N)$. In this case, we use the matrix description of ϵ in Section 4.2.2. With respect to the basis $\{T-i\rho,\ W+i\omega,\ \overline{T}-i\overline{\rho},\ \overline{W}+i\overline{\omega},\ \overline{T}+i\overline{\rho},\ \overline{W}-i\overline{\omega},\ T+i\rho,\ W-i\omega\}$ of $L\oplus\overline{L}$ we can directly compute that these generalized complex structures have the form

$$\mathbb{J}_\epsilon = \begin{bmatrix}
* & 0 & * & 0 & * & 0 & * & 0 \\
0 & * & 0 & * & 0 & * & 0 & * \\
* & 0 & * & 0 & * & 0 & * & 0 \\
0 & * & 0 & * & 0 & * & 0 & * \\
* & 0 & * & 0 & * & 0 & * & 0 \\
0 & * & 0 & * & 0 & * & 0 & * \\
* & 0 & * & 0 & * & 0 & * & 0 \\
0 & * & 0 & * & 0 & * & 0 & *
\end{bmatrix}$$

where the entries with a $*$ are not necessarily zero. The matrix corresponding to the pairing $<,>$ in the above basis is

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, fix any two deformations ϵ_1, ϵ_2. A direct computation shows that

$$\mathbb{J}_{\epsilon_i}(T - i\rho) = \begin{bmatrix} * \\ 0 \\ * \\ 0 \\ * \\ 0 \\ * \\ 0 \end{bmatrix} \quad \text{and } M\mathbb{J}_{\epsilon_i}(T - i\rho) = \begin{bmatrix} 0 \\ * \\ 0 \\ * \\ 0 \\ * \\ 0 \\ * \end{bmatrix}$$

which implies, for $G = -\mathbb{J}_{\epsilon_1}\mathbb{J}_{\epsilon_2}$,

$$\begin{aligned} \widetilde{G}(T - i\rho, T - i\rho) &= \ < G(T - i\rho), T - i\rho > \\ &= \ < -\mathbb{J}_{\epsilon_1}\mathbb{J}_{\epsilon_2}(T - i\rho), T - i\rho > \\ &= \ < \mathbb{J}_{\epsilon_2}(T - i\rho), \mathbb{J}_{\epsilon_1}(T - i\rho) > \\ &= \ \mathbb{J}_{\epsilon_1}(T - i\rho)^t M \mathbb{J}_{\epsilon_2}(T - i\rho) \\ &= \ 0. \end{aligned}$$

Therefore, no pair of these generalized complex structures gives rise to a generalized Kähler structure.

Remark 4.50. If one instead considers *generalized pseudo-Kähler structures*, that is, generalized Kähler structures which do not have a positive definite metric, then Kodaira surfaces do admit them. Examining structures of this type on Kodaira surfaces has yet to be completed.

4.6 An Odd-Type Structure on Kodaira Surfaces

All of the above analysis was done for even-type generalized complex structures. These are, in fact, the only structures that exist on K3 surfaces. However, odd-type generalized complex structures do exist on Kodaira surfaces. In this section, we present an example of an odd-type generalized complex structure.

Let us define an odd-type structure by specifying its $+i$-eigenbundle L. Recall that the basis for $T \otimes \mathbb{C}$ is $\{T, W, \overline{T}, \overline{W}\}$ and the dual basis for $T^* \otimes \mathbb{C}$ is $\{\omega, \rho, \overline{\omega}, \overline{\rho}\}$.

Proposition 4.51. *Let L be generated by*

$$T + \rho + \overline{\rho},$$

$$W - \omega + \overline{\rho},$$

$$\overline{W} - \omega - \rho, \text{ and}$$

$$\overline{\omega}.$$

Then, L is a generalized complex structure. Note that the projection of L onto $T_N \otimes \mathbb{C}$ is generated by $\{T, W, \overline{W}\}$ and L is therefore a type 1 structure.

Proof. We need to show that L is isotropic, $L \cap \overline{L} = \{0\}$ and L is closed under the Courant bracket. It is easy to see that L is isotropic by how it was chosen. It is also straightforward to check that $L \cap \overline{L} = \{0\}$. Since $\overline{L}$ is generated by

$$\overline{T} + \overline{\rho} + \rho,$$

$$\overline{W} - \overline{\omega} + \rho,$$

$$W - \overline{\omega} - \overline{\rho}, \text{ and}$$

$$\omega,$$

we show instead that $L \oplus \overline{L} =)(T \oplus T^*) \otimes \mathbb{C}$. Indeed, ω and $\overline{\omega}$ are both elements of $L \oplus \overline{L}$, so are $W, \rho, \overline{W}$, and $\overline{\rho}$. Lastly, we get T and $\overline{T}$ from the first generator of L and $\overline{L}$, respectively. The only condition left to verify is integrability. It is enough to check that the Courant bracket of any two distinct basis elements of L is again an element of L. We will need the identities: $d\omega = d\overline{\omega} = 0$ and $d\rho = d\overline{\rho} = -i\omega \wedge \overline{\omega}$. We first compute

$$\begin{aligned}
[T+\rho+\overline{\rho}, W-\omega+\overline{\rho}]_C &= [T,W] + \mathcal{L}_T(-\omega+\overline{\rho}) - \mathcal{L}_W(\overline{\rho}+\rho) \\
&\quad -\frac{1}{2}d((-\omega+\overline{\rho})(T) - (\overline{\rho}+\rho)(W)) \\
&= \iota_T d(-\omega+\overline{\rho}) - \iota_W d(\overline{\rho}+\rho) \\
&= -\iota_T i\omega \wedge \overline{\omega} + 2\iota_W(i\omega \wedge \overline{\omega}) \\
&= -i\overline{\omega} \in L.
\end{aligned}$$

The proof that $[T+\rho+\overline{\rho}, \overline{W}-\omega-\rho]_C \in L$ is similar. Next,

$$\begin{aligned}
[W-\omega+\overline{\rho}, \overline{W}-\omega-\rho]_C &= [W,\overline{W}] + \mathcal{L}_W(-\omega-\rho) - \mathcal{L}_{\overline{W}}(-\omega+\overline{\rho}) \\
&\quad -\frac{1}{2}d((-\omega-\rho)(W) - (-\omega+\overline{\rho})(\overline{W})) \\
&= \iota_W d(-\omega-\rho) - \iota_{\overline{W}} d(-\omega+\overline{\rho}) \\
&= \iota_W(i\omega \wedge \overline{\omega}) + \iota_{\overline{W}}(i\omega \wedge \overline{\omega}) \\
&= 0 \in L.
\end{aligned}$$

Finally, using the following computation,

$$\begin{aligned}
[T+\rho+\overline{\rho}, \overline{\omega}]_C &= [T,0] + \mathcal{L}_T(\overline{\omega}) - \mathcal{L}_0(\overline{\rho}+\rho) - \frac{1}{2}d((\overline{\omega})(T) - (\overline{\rho}+\rho)(0)) \\
&= \iota_T d(\overline{\omega}) \\
&= \iota_T 0 \\
&= 0 \in L,
\end{aligned}$$

we similarly obtain $[W-\omega+\overline{\rho}, \overline{\omega}]_C = [\overline{W}-\omega-\rho, \overline{\omega}]_C = 0 \in L$.

$\square$

Chapter 5

Generalized Holomorphic Bundles

Generalized holomorphic bundles were introduced by Gualtieri in [15]. These objects are a special class of Lie algebroid connections, where the Lie algebroid is a generalized complex structure. In this chapter, we study some of their general properties and compute explicit examples on Kodaira surfaces and complex 2-tori.

We begin with an examination of general Lie algebroid connections and prove that flat connections are preserved under Lie algebroid isomorphisms (Proposition 5.2). We then study generalized holomorphic bundles for specific generalized complex structures. For a generalized complex structure coming from a complex structure, generalized holomorphic bundles are called *co-Higgs bundles* (see Section 5.1.1). Some of their properties were studied by Hitchin in [19]. Moreover, their moduli spaces have been described by Rayan over $\mathbb{CP}^1$ in [26] and over $\mathbb{CP}^2$ in [27]. In Section 5.1.2, we introduce the notion of *twisted co-Higgs bundle* and prove that such bundles are preserved under B-field transforms (Proposition 5.11). We then explain how generalized holomorphic bundles correspond to Poisson modules when the generalized complex structure comes from a holomorphic Poisson structure (see Section 5.1.3). Finally, in Section 5.1.4, we show that when the generalized complex structure has type zero, generalized holomorphic bundles are in fact flat bundles.

We end the chapter by giving an explicit description of generalized holomorphic bundles on Kodaira surfaces and complex 2-tori, with respect to the families of generalized complex structures we constructed in Sections 4.2 and 4.3. This is done in Sections 5.2 and 5.3.

5.1 Definitions and Properties

Let M be a manifold. We begin with a general result for Lie algebroid connections over M. Recall that we introduced Lie algebroids and Lie algebroid differentials in section 4.1.

Definition 5.1. Let $(E, \rho, [\cdot,\cdot]_E)$ be a Lie algebroid over M. A **Lie algebroid connection** for E is a vector bundle V over M together with an operator $D : C^\infty(V) \to C^\infty(V \otimes E^*)$ such that a Leibniz rule is satisfied, namely

$$D(fs) = d_E(f) \otimes s + fD(s)$$

for section $s \in C^\infty(V)$ and smooth function f on M. If $D^2 = 0$ then the Lie algebroid connection D is called **flat**.

The flatness condition requires an extension of D to elements of $C^\infty(V \otimes E^*)$. To do this, we use the Leibniz rule,

$$D(s \otimes a) = D(s) \wedge a + s \otimes d_E(a) \in C^\infty(V \otimes \wedge^2 E^*).$$

We will introduce a special class of these objects shortly, when E is a maximal isotropic corresponding to a generalized complex structure then a Lie algebroid connection is called a *generalized holomorphic bundle*. Before this, let us prove a result regarding flat Lie algebroid connections.

Proposition 5.2. *Let $(E, \rho, [\cdot,\cdot]_E)$ and $(E', \rho', [\cdot,\cdot]_{E'})$ be two isomorphic Lie algebroids over a manifold M, and let (V, D) be a Lie algebroid connection over E. Denote the isomorphism by $\phi : E \to E'$ which means $\rho' \circ \phi = \rho$ and $[\phi(a), \phi(b)]_{E'} = \phi([a,b]_E)$. Then (V, D) is flat if and only if the resulting bundle (V, D') over E' (via ϕ) is flat.*

Proof. First, we need to recall the result of Lemma 4.8: $\phi^* \circ d_{E'} = d_E \circ \phi^*$. In terms of a local frame $\{s_1, \dots, s_r\}$ of V,

$$D(s_i) = \sum_j \Omega_{ji} s_j$$

(analogous to standard connections), where $\Omega_{ji} \in C^\infty(E^*)$. Then we may write $D = d_E + \Omega$ where Ω is a matrix of smooth sections of E^* (compared with standard connections where Ω is a matrix valued 1-form). With this representation we can also locally interpret the flatness condition as we do in the case of standard connections, namely $D^2 = 0$ if and only if $d_E\Omega + \Omega \wedge \Omega = 0$.

Suppose we have two isomorphic Lie algebroid connections (V, D) and (V, D') for E and E' respectively. We can write $D = d_E + \Omega$ and $D' = d_{E'} + \Omega'$, for $\Omega \in C^\infty(\mathrm{End}(V) \otimes E^*)$ and $\Omega' \in C^\infty(\mathrm{End}(V) \otimes (E')^*)$ where $\Omega = \phi^*\Omega'$. We extend the map ϕ^* to the wedge product in the natural way, that is $\phi^*\Omega' \wedge \phi^*\Omega' = \phi^*(\Omega' \wedge \Omega')$. The curvature matrices are related as well,

$$\begin{aligned} d_E\Omega + \Omega \wedge \Omega &= d_E\phi^*\Omega' + \phi^*\Omega' \wedge \phi^*\Omega' \\ &= \phi^* d_{E'}\Omega' + \phi^*(\Omega' \wedge \Omega') \\ &= \phi^*(d_{E'}\Omega' + \Omega' \wedge \Omega'). \end{aligned}$$

Therefore, $d_E\Omega + \Omega \wedge \Omega = 0$ if and only if $d_{E'}\Omega' + \Omega' \wedge \Omega' = 0$ because ϕ^* is an isomorphism. This is what we wanted to show.

$\square$

Now let us restrict our attention to Lie algebroids of the form $(L, \pi_T, [\cdot,\cdot]_C)$, that is, generalized complex structures. Recall that the anchor map in this case is π_T, projection onto the tangent bundle, and $[\cdot,\cdot]_C$ is the Courant bracket. We will follow [19] in our treatment of these bundles. Again, we identify L^* with $\overline{L}$ via $< \cdot, \cdot >$.

Remark 5.3. It is important to note that we can only identify $\overline{L}$ with L^* when L is real, that is, $L \cap \overline{L} = 0$. Of course this is the case for a generalized complex structure, but for an arbitrary complex Lie algebroid this may not be possible.

Definition 5.4. A generalized holomorphic bundle on a generalized complex manifold (M, L) is a vector bundle V together with a differential operator $\overline{D} : C^\infty(V) \to C^\infty(V \otimes \overline{L})$ such that for a smooth function f and section s

- $\overline{D}(fs) = d_L(f)s + f\overline{D}s$,
- $\overline{D}^2 : C^\infty(V) \to C^\infty(V \otimes \wedge^2\overline{L})$ vanishes.

The first condition is a Leibniz rule in the generalized setting. The second condition is the integrability condition, and requires us to extend the definition of $\overline{D}$ to elements of $C^\infty(V \otimes \overline{L})$. This is exactly the same as the extension we performed on a general Lie algebroid,

$$\overline{D}(s \otimes z) = \overline{D}(s) \wedge z + s \otimes d_L(z) \in C^\infty(V \otimes \wedge^2\overline{L}).$$

In terms of a local frame $\{s_1, \ldots, s_r\}$, we may write

$$\overline{D}(s_i) = \sum_j \Omega_{ji} s_j$$

where $\Omega_{ji} \in C^\infty(\overline{L})$. Then $\overline{D} = d_L + \Omega$ where $\Omega \in C^\infty(\mathrm{End}V \otimes \overline{L})$. The second condition is therefore the flatness condition in this setting. Locally we have $\overline{D}^2$ vanishes if and only if $d_L\Omega + \Omega \wedge \Omega = 0$.

We will examine these bundles in certain special cases in the following sections. We begin with generalized holomorphic bundles corresponding to a generalized complex structure that comes from a complex structure. In this case they are called *co-Higgs bundles*.

5.1.1 Co-Higgs Bundles

Let $(V, \overline{D})$ be a generalized holomorphic bundle corresponding to a generalized complex structure coming from a complex structure J. In this case $\overline{L} = T_{1,0} \oplus T^*_{0,1}$; we may write $\overline{D} = \Phi + \overline{\partial}_A$ where

$$\Phi : C^\infty(V) \to C^\infty(V \otimes T_{1,0})$$

and

$$\overline{\partial}_A : C^\infty(V) \to C^\infty(V \otimes T^*_{0,1}).$$

Then, for a section s, we may locally write:

$$\overline{D}(s) = \Phi(s) + \overline{\partial}_A(s) = \sum_k \phi_k s \frac{\partial}{\partial z_k} + \sum_j \left(\frac{\partial s}{\partial \overline{z_j}} + A_j s \right) d\overline{z_j}. \tag{5.1}$$

Here, ϕ_k and A_j are the components in local holomorphic coordinates. Let us also use the symbol $\overline{\partial}_A$ to denote the induced connection of $\overline{\partial}_A$ on $\mathrm{End}(V) \otimes T_{1,0}$ since we may view Φ as a section of $\mathrm{End}(V) \otimes T_{1,0}$ (by definition, $\overline{\partial}_A^{End}(\Phi) = \overline{\partial}_A \Phi + \Phi \overline{\partial}_A = [\overline{\partial}_A, \Phi]$). We now examine the conditions obtained if we stipulate that $\overline{D}^2 = 0$:

$$0 = \overline{D}^2 = (\Phi + \overline{\partial}_A)^2 = \Phi \wedge \Phi + \overline{\partial}_A \Phi + \overline{\partial}_A^2.$$

Since $\Phi \wedge \Phi$ is the only operator with a component in $\wedge^2 T$, and $\overline{\partial}_A^2$ is the only one with a component in $\wedge^2 T^*$, we get

1. $\overline{\partial}_A^2 = 0 \in C^\infty(\mathrm{End}(V) \otimes \wedge^2 T^*_{0,1})$,

2. $\Phi\overline{\partial}_A = 0 \in C^\infty(\mathrm{End}(V) \otimes T_{1,0} \otimes T^*_{0,1})$,

3. $\Phi \wedge \Phi = 0 \in C^\infty(\mathrm{End}(V) \otimes \wedge^2 T_{1,0})$.

In order to see how we obtained equation 5.1 recall from example 4.1 that for a smooth function f,

$$d_L(f)(X + \alpha) = \pi_T(X)(f) + \pi_T(\alpha)(f) = X(f) + 0 = \overline{\partial}(f)(X) + 0.$$

So the d_L operator behaves like $\overline{\partial}$ on $T_{0,1}$ and the zero operator on $T^*_{1,0}$. This means that the d_L operator only appears in $\overline{\partial}_A$ and not Φ. Locally, our above operators satisfy $\overline{\partial}_A = \overline{\partial} + A$ while $\Phi = 0 + \phi$ for $A \in C^\infty(\mathrm{End}(V) \otimes T^*_{0,1})$ and $\phi \in C^\infty(\mathrm{End}(V) \otimes T_{1,0})$.

Let us examine the three conditions, $\overline{\partial}_A^2 = 0$, $\Phi\overline{\partial}_A = 0$, and $\Phi \wedge \Phi = 0$, in detail. The first condition, $\overline{\partial}_A^2 = 0$, gives V the structure of a holomorphic vector bundle. Next, $\overline{\partial}_A \Phi = 0$ tells us that Φ is a holomorphic section of $\mathrm{End}(V) \otimes T_{1,0}$. Finally, the third condition puts an algebraic restriction on Φ. If we locally write $\Phi = \sum \phi_i \frac{\partial}{\partial z_j}$, then $\Phi^2 = 0$ means

$$\sum_{j<k} [\phi_j, \phi_k] \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_k} = 0.$$

Hence the condition we require is that

$$[\phi_j, \phi_k] = 0$$

for any j, k. When this condition is satisfied we will write $\Phi \wedge \Phi = 0$.

If all three conditions are satisfied, the resulting bundle is called a **co-Higgs bundle**. Hitchin examined some specific examples in [19], and Rayan classified such bundles (with a stability assumption) on $\mathbb{P}^1$ in [26], and on $\mathbb{CP}^2$ in [27]. We list some standard examples here.

1. A co-Higgs line bundle (rank 1 co-Higgs bundle) is a holomorphic line bundle L together with a vector field X. In this case, the condition $X \wedge X = 0$ is trivially satisfied.

2. Over $\mathbb{P}^1$, we may consider the trivial rank-k vector bundle V. Then a co-Higgs field Φ is a $k \times k$ matrix with values in $C^\infty(\mathcal{O}(2))$ since the tangent bundle is $\mathcal{O}(2)$. Thus Φ is a matrix of quadratic polynomials in z. Once again, if $k = 1$ then the integrability condition on Φ is trivially satisfied.

3. If $V = \mathcal{O} \oplus T$ then we may define $\Phi(\lambda, X) = (0, X)$ to get an integrable co-Higgs structure.

4. On a Kodaira surface or a complex 2-torus, since $T_{1,0} \cong \mathcal{O} \oplus \mathcal{O}$, we find that our co-Higgs field lies in the direct sum of two holomorphic sections of $\mathrm{End}(V)$,

$$\Phi \in H^0(\mathrm{End}(V) \otimes T_{1,0}) \cong H^0(\mathrm{End}(V)) \oplus H^0(\mathrm{End}(V))$$

To end our discussion of co-Higgs bundles, we present one method for transforming a co-Higgs bundle using a certain B-transform.

Definition 5.5. Let B be a closed real $(1,1)$-form and $(V, \overline{D})$ (where $\overline{D} = \overline{\partial}_A + \Phi$) be a co-Higgs bundle. The transform of the co-Higgs bundle by B changes the holomorphic structure $\overline{\partial}_A$ and fixes the co-Higgs field Φ. That is, $\overline{D}_B = \overline{\partial}_B + \Phi$ where

$$\overline{\partial}_B = \overline{\partial}_A + \iota_\Phi B.$$

Of course, with $\iota_\Phi B$, we are abusing notation to make the expression more condensed. More explicitly, if $B = \sum B_{i\bar{j}} dz_i \wedge d\bar{z}_j$, we mean

$$\iota_\Phi B = \sum_j \left(\sum_i \phi_i B_{i\bar{j}} \right) d\bar{z}_j.$$

A closed real $(1,1)$-form B is a symmetry of the base generalized complex structure because $L = T_{0,1} \oplus T^*_{1,0} \mapsto L_B = (1+B)(T_{0,1}) \oplus T^*_{1,0}$ but if B is a $(1,1)$-form then $B(T_{0,1}) \subseteq T^*_{1,0}$ and therefore $L_B = L$. These transforms are discussed in detail in [19], where it is proven, in particular, that integrability is preserved under these transforms.

Proposition 5.6 (Hitchin, [19], Section 4.1)**.** *If $(V, \Phi, \overline{\partial}_A)$ is a co-Higgs bundle and B is a closed real $(1,1)$-form then $(V, \Phi, \overline{\partial}_B)$ is again a co-Higgs bundle.*

Remark 5.7. We will examine B-field transforms of the base generalized complex structure in the coming section on twisted co-Higgs bundles and prove an analogue of the above proposition in that setting.

In a certain special case, if B is $\overline{\partial}$-exact, then it was shown in [19] that we can say more about our holomorphic structures:

Proposition 5.8 (Hitchin, [19], Proposition 2)**.** *If B is $\overline{\partial}$-exact, then the two holomorphic structures $\overline{\partial}_A$ and $\overline{\partial}_B$ are gauge equivalent.*

Now let us explore what happens when we deform the base generalized complex structure coming from a complex structure. Recall, we may write any deformation as three components $\epsilon = B + \epsilon_J + \beta$, where B is a *complex* closed $(0, 2)$-form, β is a holomorphic Poisson structure and ϵ_J is a deformation of the complex structure. Of course, if we only deform the base complex structure (that is, $\epsilon = \epsilon_J$) then any generalized holomorphic bundle is still a co-Higgs bundle. Let us examine what happens in the other two cases. In the B-field case, we will obtain *twisted co-Higgs bundles*. In the Poisson structure case we obtain *Poisson modules*.

Remark 5.9. Note that, if $(V, \Phi, \overline{\partial}_A)$ is integrable (that is, a co-Higgs bundle) and ϵ_J is a deformation of J, then it is not necessarily the case that the structure $(V, \Phi_{\epsilon_J}, \overline{\partial}_{A_{\epsilon_J}})$ is integrable. Here we define $\Phi_{\epsilon_J} = \Phi + \iota_\Phi \epsilon_J$ and $\overline{\partial}_{A_{\epsilon_J}} = \overline{\partial}_A + \iota_{\overline{\partial}_A} \epsilon_J$ as we did for the $(1, 1)$-forms B above. The reason for this is that the transformation $(1 + \epsilon_J) : L \to L_{\epsilon_J}$ does not preserve projections onto T. Hence the assumptions of Proposition 5.2 are not satisfied because $\pi_T((1 + \epsilon_J)x) \neq \pi_T(x)$ (here π_T is the anchor map).

5.1.2 Twisted Co-Higgs Bundles

A natural extension of the above B-field discussion is to ask what generalized holomorphic bundles correspond to a given B-field transform if B is *any* closed (possibly complex) 2-form and we transform a generalized complex structure coming from a complex structure $L = T_{0,1} \oplus T^*_{1,0}$ to $L_B = (1 + B)T_{0,1} \oplus T^*_{1,0}$.

First, let us decompose $B = B_{2,0} + B_{1,1} + B_{0,2}$. It is clear that $B_{2,0} + B_{1,1}$ has no effect on L, so only $B_{0,2}$ needs to be considered. Let us fix a closed $(0, 2)$-form B. Then generalized holomorphic bundles for L_B have the form $(V, \Phi_B, \overline{\partial}_A)$ where

$$\overline{\partial}_A : C^\infty(V) \to C^\infty(V \otimes T^*_{0,1}) \text{ and } \Phi_B : C^\infty(V) \to C^\infty(V \otimes (1 + \overline{B})T_{1,0}).$$

In fact, they are all B-field transforms of $(V, \Phi, \overline{\partial}_A)$ where $\Phi_B = \Phi + \iota_\Phi B$. The integrability conditions in this setting are:

1. $\overline{\partial}_A^2 = 0 \in C^\infty(\mathrm{End}(V) \otimes \wedge^2 T^*_{0,1})$,
2. $\overline{\partial}_A \Phi_B = 0 \in C^\infty(\mathrm{End}(V) \otimes (1 + \overline{B})T_{1,0} \otimes T^*_{0,1})$,
3. $\Phi_B \wedge \Phi_B = 0 \in C^\infty(\mathrm{End}(V) \otimes \wedge^2(1 + \overline{B})T_{1,0})$.

These look very similar to the integrability conditions in the co-Higgs setting, which motivates the following definition.

Definition 5.10. Let $L = T_{0,1} \oplus T^*_{1,0}$ be a generalized complex structure corresponding to a complex structure and $L_B = (1+B)L$ for a real closed $(0,2)$-form B. Then a generalized holomorphic bundle for L_B, that is, a bundle of the form $(V, \Phi_B, \overline{\partial}_A)$ is called a **twisted co-Higgs bundle**.

The natural question to ask is if $(V, \Phi, \overline{\partial}_A)$ is integrable, is the transformed structure $(V, \Phi_B, \overline{\partial}_A)$ also integrable? Conversely, if $(V, \Phi_B, \overline{\partial}_A)$ is integrable, is $(V, \Phi, \overline{\partial}_A)$ also integrable? We can give an affirmative answer to both questions.

Proposition 5.11. *If B is a closed $(0,2)$-form, then $(V, \Phi, \overline{\partial}_A)$ is integrable (to a co-Higgs bundle) if and only if $(V, \Phi_B, \overline{\partial}_A)$ is integrable (to a twisted co-Higgs bundle).*

Proof. This is an application of Proposition 5.2, where $\phi = e^B : L \to L_B$. We established in Corollary 4.11 that this is an isomorphism of Lie algebroids.

Hence the result follows from Proposition 5.2.

□

Remark 5.12. The above proposition allows us to reduce the problem of classifying the larger class of B-field transforms of co-Higgs bundles to just classifying co-Higgs bundles themselves, as any twisted co-Higgs bundle can be viewed, in a natural way, as a co-Higgs bundle.

5.1.3 Relationship to Poisson Modules

Poisson modules and their relationship to generalized geometry have already been studied by Gualtieri in [13], [16], and Hitchin in [18]. Here we look at the definition and a specific relation between Poisson modules and generalized holomorphic bundles.

Definition 5.13. A holomorphic **Poisson module** on a holomorphic Poisson manifold (M, β) is a holomorphic bundle V together with a holomorphic structure $\overline{\partial}_A : C^\infty(V) \to C^\infty(V \otimes T^*_{0,1})$ and a map $\Psi : C^\infty(V) \to C^\infty(V \otimes T_{1,0})$ that satisfies, for smooth functions f and g and section s:

1. $\Psi(fs) = f\Psi(s) - \iota_{df}\beta \otimes s$ and
2. $\Psi_{d\{f,g\}}(s) = (\Psi_{df}\Psi_{dg} - \Psi_{dg}\Psi_{df})s$

where $\{f, g\} = \beta(df, dg)$ is the Poisson bracket induced by β, and $\Psi_\alpha(s) = \Psi(s)(\alpha)$.

Remark 5.14. A Poisson module is a holomorphic bundle V together with an action of smooth functions on holomorphic sections $s \mapsto \{f, s\}$ that satisfies

1. $\{f, gs\} = \{f, g\}s + g\{f, s\}$ and
2. $\{\{f, g\}, s\} = \{f, \{g, s\}\} - \{g, \{f, s\}\}$

where we define
$$\{f, s\} := \Psi_{df} s.$$
The conditions in Definition 5.13 are equivalent to the corresponding conditions in the above definition of Poisson module. Let us consider the first condition here. The left-hand side is
$$\{f, gs\} = \Psi_{df}(gs) = \Psi(gs)(df).$$
The right-hand side becomes
$$\{f, g\}s + g\{f, s\} = \beta(df, dg)s + g\Psi_{df} s = g\Psi(s)(df) - (\iota_{dg}\beta)(df)s.$$
Hence the left-hand side and the right-hand side agree if and only if $\Psi(gs) = g\Psi(s) - \iota_{dg}\beta \otimes s$ which is the first condition in the definition of Poisson module above.

For the second condition, the left-hand side is
$$\{\{f, g\}, s\} = \Psi_{d\{f,g\}} s$$
and the right-hand side is
$$\{f, \{g, s\}\} - \{g, \{f, s\}\} = \Psi_{df}(\Psi_{dg} s) - \Psi_{dg}(\Psi_{df} s).$$
Once again, these two expressions are equal if and only if
$$\Psi_{d\{f,g\}}(s) = (\Psi_{df}\Psi_{dg} - \Psi_{dg}\Psi_{df})s.$$

It is useful to see which generalized holomorphic bundles are holomorphic Poisson modules. This question can be answered with the following proposition from [13].

Proposition 5.15 (Gualtieri, [13], Proposition 8). *If*
$$\mathbb{J} = \begin{bmatrix} -J & Q \\ 0 & J^* \end{bmatrix}$$

for J a complex structure and $\beta = P + iQ$ with $P = QJ^*$, a holomorphic Poisson structure then a generalized holomorphic bundle is precisely a holomorphic Poisson module.

Equivalently, a generalized holomorphic bundle corresponding to a generalized complex structure with maximal isotropic $L_\beta = T_{0,1} \oplus \Gamma_\beta$ where $\Gamma_\beta = \{\eta + \beta(\eta) : \eta \in T^*_{1,0}\}$ (for β a holomorphic Poisson structure) is a holomorphic Poisson module.

Proof. In this case $L_\beta = T_{0,1} \oplus \Gamma_\beta$ where $\Gamma_\beta = \{\eta + \beta(\eta) : \eta \in T^*_{1,0}\}$, and as we did in the twisted co-Higgs case above, one can check that $d_L(f)(X + \eta + \iota_\eta \beta) = \overline{\partial}(f)(X) - \beta(df)(\eta)$, thus if we decompose $\overline{D} = \Psi + \overline{\partial}_A$ then $\overline{\partial}_A : C^\infty(V) \to C^\infty(V \otimes T^*_{0,1})$ and $\Psi : C^\infty(V) \to C^\infty(V \otimes \Gamma^*_\beta)$. Further, the Leibniz rule for Ψ in this case is $\Psi(fs) = f\Psi(s) - \iota_{df}\beta \otimes s$ which is the first condition in the definition of Poisson module.

Also, $\overline{D}^2 = 0$ tells us that $\overline{\partial}_A$ is a standard holomorphic structure on our bundle, Ψ is holomorphic with respect to this structure, and a routine (but tedious) computation shows that $\Psi^2 = 0$ implies $\Psi_{d\{f,g\}}(s) = (\Psi_{df}\Psi_{dg} - \Psi_{dg}\Psi_{df})s$ for any section s and smooth functions f and g. This is the second condition in the definition. □

Note. Recall from example 2.40 that any holomorphic Poisson structure $\beta = P + iQ$ has the property that $P = QJ^*$.

Remark 5.16. This gives us a way of viewing certain generalized holomorphic bundles as holomorphic Poisson modules. We shall see in the coming sections that these bundles are flat bundles when we have a holomorphic symplectic structure.

Remark 5.17. We have established above that we may also view a β-transform of a co-Higgs bundle $(V, \overline{\partial}_A, \Phi)$ where $\overline{\partial}_A : C^\infty(E) \to C^\infty(E \otimes T^*_{0,1})$ and $\Phi : C^\infty(E) \to C^\infty(E \otimes T^{1,0})$ as a Poisson module. Explicitly, $(V, \overline{\partial}_A, \Phi)$ gets transformed to $(V, \overline{\partial}_{A_\beta}, \Phi)$ with $\overline{\partial}_{A_\beta} := \overline{\partial}_A + \iota_{\overline{\partial}_A}\beta$. It is natural to ask if integrability is preserved in this setting. In this case, however, the answer is no. If $(V, \overline{\partial}_A, \Phi)$ is a co-Higgs bundle it does not mean that $(V, \overline{\partial}_{A_\beta}, \Phi)$ is integrable and conversely, if $(V, \overline{\partial}_{A_\beta}, \Phi)$ is integrable, it does not mean that $(V, \overline{\partial}_A, \Phi)$ is a co-Higgs bundle. One reason for this is that β changes the action of the Lie-algebroid differential d_L in a radical way. In fact, as we shall see in the next section, while d_L behaves like $\overline{\partial}$ in the co-Higgs setting, it actually behaves like the full exterior derivative d in the holomorphic symplectic setting. The assumptions of Proposition 5.2 are not satisfied in this setting because $x + \iota_x\beta$ changes projections onto T, and therefore $\pi_T((1 + \beta)x) \neq \pi_T(x)$.

On the other hand, in the twisted co-Higgs setting we saw that applying a B-field transform to our generalized holomorphic bundle structure preserved integrability. This is also the case in the Poisson setting because e^B is an isomorphism of vector bundles that satisfies the conditions of Proposition 5.2. Thus $(V, \overline{\partial}_{A_\beta}, \Phi)$ is integrable if and only if $(V, \overline{\partial}_{A_\beta}, \Phi_B)$ is integrable, for any closed $(0, 2)$-form B.

Explicit examples of this will be seen when we analyze the structures on Kodaira surfaces and complex Tori in the next section.

5.1.4 Relationship to Flat Bundles

Flat bundles are well understood, and in some cases we can relate generalized holomorphic bundles to flat bundles. The second condition in the definition, that $\overline{D}^2 = 0$, looks similar to that of flatness. We can actually interpret such bundles as flat bundles when we are in the type zero case as seen in the following proposition.

Proposition 5.18. *If $(V, \overline{D})$ is a generalized holomorphic bundle for a generalized complex structure $\mathbb{J}$ of type zero everywhere, then $(V, \overline{D})$ is isomorphic to a flat bundle (V, ∇).*

Proof. Our proof will follow the same method as the proof of Proposition 5.11. In this case, since L has full-rank projection onto T everywhere, we may let $\phi = \pi_T : L \to T$ and it will be an isomorphism between the the two Lie algebroids $(L, \pi_T, [\cdot, \cdot]_C)$ and $(T, id, [\cdot, \cdot]_{Lie})$. Thus, the resulting bundle after projecting onto T, (V, ∇), is a vector bundle V and a classical connection that can be locally expressed as $\nabla = d + A$. Proposition 5.2 tells us that $dA + A \wedge A = 0$, which means (V, ∇) is flat, as desired.

□

We may immediately employ the above proposition when we are dealing with generalized holomorphic bundles coming from a symplectic structure.

Corollary 5.19. *If $(V, \overline{D})$ is a generalized holomorphic bundle for a generalized complex structure $\mathbb{J}_\omega$ coming from a symplectic structure ω then $(V, \overline{D})$ is isomorphic to a flat bundle (V, ∇).*

Proof. Since $\mathbb{J}_\omega$ has type zero everywhere, we can apply Proposition 5.18. □

Similarly, on a complex surface, if we deform our base complex structure by a holomorphic symplectic structure then the result will also be a flat bundle.

Corollary 5.20. *If β is a holomorphic symplectic structure, and $L_\beta = T_{0,1} \oplus \Gamma_\beta$ where $\Gamma_\beta = \{\eta + \beta(\eta) : \eta \in T^*_{1,0}\}$ then any generalized holomorphic structure for L_β, $(V, \overline{D})$ is isomorphic to a flat bundle (V, ∇).*

Proof. If β is nowhere-vanishing then $\Gamma_\beta \cong T_{1,0}$ and so L_β is everywhere of type zero and we can use proposition 5.18. □

5.2 Generalized Holomorphic Bundles on Kodaira Surfaces

In this section, we classify generalized holomorphic bundles using the deformation space from section 4.2.1. We begin by looking at bundles for generalized complex structures that are B- and β-transforms of a fixed complex structure. Then, we comment on what happens if we allow a deformation of the base complex structure.

5.2.1 Bundles for a Fixed Complex Structure

Recall from Section 4.2 that the parameters t_{11} and t_{22} correspond to the deformations of the complex structure. Therefore, changing these parameters will give rise to a generalized complex structure that comes from a different complex structure. The parameters we focus on here are t_{14} (the Poisson structure's parameter) and t_{32} (the B-field's parameter) and we will set $t_{11} = t_{22} = 0$. Therefore in this case

$$\epsilon = t_{14} T \wedge W + t_{32} \overline{\omega} \wedge \overline{\rho}.$$

Throughout our analysis we will work with L^* rather than identifying it with $\overline{L}$. We know L is generated by $\{T, W, \overline{\omega}, \overline{\rho}\}$ and so using the formula $L_\epsilon = (1+\epsilon)L$ we get that

$$\overline{T} \mapsto \overline{T} + \iota_{\overline{T}}(t_{14} T \wedge W + t_{32} \overline{\omega} \wedge \overline{\rho}) = \overline{T} + t_{32} \overline{\rho}.$$

Similar computations for $\overline{W}$, ω, and ρ show us that L_ϵ is generated by $\{v_1, v_2, v_3, v_4\}$ where

$$v_1 = \overline{T} + t_{32}\overline{\rho},\ v_2 = \overline{W} - t_{32}\overline{\omega},\ v_3 = \omega + t_{14} W,\ v_4 = \rho - t_{14} T.$$

We work with a fixed ϵ for now, and write L without the subscript. Note that $[v_i, v_j]_C = 0$ for all i, j except $[v_1, v_4]_C = i(t_{14} v_2 + v_3)$. Consider the dual basis $\{v_1^*, v_2^*, v_3^*, v_4^*\}$ of L^* where

$$v_1^* = \frac{2}{1 + \overline{t_{14}} t_{32}}(\overline{\omega} + \overline{t_{14}}\overline{W}),\ v_2^* = \frac{2}{1 + \overline{t_{14}} t_{32}}(\overline{\rho} - \overline{t_{14}}\overline{T}),$$

$$v_3^* = \frac{2}{1+t_{14}\overline{t_{32}}}(T+\overline{t_{32}}\rho), \text{ and } v_4^* = \frac{2}{1+t_{14}\overline{t_{32}}}(W-\overline{t_{32}}\omega).$$

Our first goal is to identify explicit restrictions that the integrability condition places on our bundles to make them generalized holomorphic bundles. Let $(V, \overline{D})$ be a rank-r generalized holomorphic bundle. Working locally, we may write $\overline{D} = d_L + \Omega$ where Ω is an $r \times r$ matrix with values in $C^\infty(\overline{L})$, or $\Omega \in C^\infty(\mathrm{End}(V) \otimes \overline{L})$. Say

$$\Omega = Ev_1^* + Fv_2^* + Gv_3^* + Hv_4^*$$

where E, F, G, H are matrices of smooth functions. We must determine conditions on the matrices E, F, G, and H so that the curvature, $d_L\Omega + \Omega \wedge \Omega$, vanishes. To do this, we need to better understand d_L. We will first compute $d_L(f)$ for a smooth function f.

Lemma 5.21. *For a smooth function f and for L^* generated by $\{v_1^*, v_2^*, v_3^*, v_4^*\}$ as above, then*

$$d_L(f) = \overline{T}(f)v_1^* + \overline{W}(f)v_2^* + t_{14}W(f)v_3^* - t_{14}T(f)v_4^*.$$

Proof. Say $d_L(f) = f_1v_1^* + f_2v_2^* + f_3v_3^* + f_4v_4^*$ for some functions f_i. Let us explicitly compute f_1, the rest are similar. We need to substitute elements of L in both sides of the above expression. Input v_1 on both sides. On the left-hand side, $d_L(f)(v_1) = \pi_T(v_1)(f) = \overline{T}(f)$. For the right-hand side of course we get f_1. Thus $f_1 = \overline{T}(f)$. □

Next, we find explicit formulas for $d_L(v_k^*)$.

Lemma 5.22.

$$d_L(v_k^*) = \begin{cases} 0 \textit{ if } k = 1, 4 \\ -it_{14}v_1^* \wedge v_4^* \textit{ if } k = 2 \\ -iv_1^* \wedge v_4^* \textit{ if } k = 3. \end{cases}$$

Proof. Let us find, for example, $d_L(v_2^*)$ to illustrate the process. The the rest will be similar. Say

$$d_L(v_k^*) = \sum_{i<j} V_{ij} v_i^* \wedge v_j^*.$$

Focusing on the left-hand side, we see that

$$d_L(v_k^*)(v_i, v_j) = \pi_T(v_i)(v_k^*(v_j)) - \pi_T(v_j)(v_k^*(v_i)) - v_k^*([v_i, v_j]_C).$$

But $v_k^*(v_i) = \delta_{ki}$ is a constant as is $v_k^*(v_j)$ and therefore

$$\pi_T(v_i)(v_k^*(v_j)) = 0$$

and

$$\pi_T(v_j)(v_k^*(v_i)) = 0.$$

So we have shown $d_L(v_k^*)(v_i, v_j) = -v_k^*([v_i, v_j]_C)$ and we know the only non-zero Courant bracket is $[v_1, v_4]_C = i(t_{14}v_2 + v_3)$, therefore $V_{14} \neq 0$ for $d_L(v_2^*)$ and $d_L(v_3^*)$. Furthermore, $V_{ij} = 0$ if $(i, j) \neq (1, 4)$ for all k.

For $d_L(v_2^*)$ we get $V_{14} = -v_2^*([v_1, v_4]_C) = -it_{14}$ and therefore

$$d_L(v_2^*) = -it_{14}v_1^* \wedge v_4^*.$$

Similarly for $d_L(v_3^*)$ we get $V_{14} = -i$ and so

$$d_L(v_3^*) = -iv_1^* \wedge v_4^*.$$

□

Now we can state the integrability conditions in our setting.

Proposition 5.23. *A matrix $\Omega = Ev_1^* + Fv_2^* + Gv_3^* + Hv_4^*$ with values in $C^\infty(L^*)$ satisfies $d_L\Omega + \Omega \wedge \Omega = 0$ if and only if $C_{ij} = 0$ for $1 \leq i < j \leq 4$ where*

$$C_{12} = \overline{T}(F) - \overline{W}(E) + [E, F],$$

$$C_{13} = \overline{T}(G) - t_{14}W(E) + [E, G],$$

$$C_{14} = \overline{T}(H) + t_{14}T(E) - it_{14}F - iG + [E, H],$$

$$C_{23} = \overline{W}(G) - t_{14}W(F) + [F, G],$$

$$C_{24} = \overline{W}(H) + t_{14}T(F) + [F, H], \text{ and}$$

$$C_{34} = t_{14}W(H) + t_{14}T(G) + [G, H].$$

Proof. First we focus on $d_L\Omega$ as it is simply a matter of using the Leibniz rule on each component as well as lemmas 5.21 and 5.22. We perform one computation here, the rest are similar. We will abuse notation and write E for any entry of E.

$$\begin{aligned} d_L(Ev_1^*) &= d_L(E) \wedge v_1^* + Ed_L(v_1^*) \\ &= \left(\overline{T}(E)v_1^* + \overline{W}(E)v_2^* + t_{14}W(E)v_3^* - t_{14}T(E)v_4^*\right) \wedge v_1^* + E(0) \\ &= \left(-\overline{W}(E)\right) v_1^* \wedge v_2^* - (t_{14}W(E))\, v_1^* \wedge v_3^* + (t_{14}T(E))\, v_1^* \wedge v_4^*. \end{aligned}$$

Repeat the above computation for the remaining three terms in Ω then simply collect the terms together for the six possible wedge products $v_i \wedge v_j$ to obtain

$$d_L\Omega + \Omega \wedge \Omega = \sum_{i<j} C_{ij} v_i \wedge v_j$$

where the C_{ij} denotes the matrix of functions that is the coefficient of that 2-form. After performing the computations one finds that the C_{ij} are as in the statement of this proposition. $\square$

Remark 5.24. Notice that the B-field, which corresponds to t_{32}, does not appear in the above constraints. This is not surprising considering Proposition 5.11 because the B-field will not affect the integrability condition.

This gives a characterization of the possible structures that generalized holomorphic bundles can have in this case. We now analyze what happens in some sub-cases.

5.2.2 Analysis of Special Cases

Base Case: $t_{14} = t_{32} = 0$

First, if $t_{14} = t_{32} = 0$ then we are in the co-Higgs setting outlined in Section 5.1.1. Say $\Omega = Ev_1^* + Fv_2^* + Gv_3^* + Hv_4^*$ and $E, F, G, H \in C^\infty(\mathrm{End}V)$. After substituting $t_{14} = t_{32} = 0$ into the conditions in proposition 5.23, the six conditions become

$$\begin{gathered} \overline{T}(F) - \overline{W}(E) + [E, F] = 0, \\ \overline{T}(G) + [E, G] = 0, \\ \overline{T}(H) - iG + [E, H] = 0, \\ \overline{W}(G) + [F, G] = 0, \\ \overline{W}(H) + [F, H] = 0, \text{ and} \\ [G, H] = 0. \end{gathered}$$

Remark 5.25. One can check that the conditions correspond directly to the three conditions placed on $\overline{\partial}_A = \overline{\partial} + E\overline{\omega} + F\overline{\rho}$ and $\Phi = GT + HW$. Namely

$$\Phi \wedge \Phi = 0 \Leftrightarrow [G, H] = 0,$$

$$\overline{\partial}_A^2 = 0 \Leftrightarrow \overline{T}(F) - \overline{W}(E) + [E, F] = 0, \text{ and}$$

$$\overline{\partial}_A\Phi = 0 \Leftrightarrow \begin{cases} \overline{T}(G) + [E, G] = 0, \\ \overline{T}(H) - iG + [E, H] = 0, \\ \overline{W}(G) + [F, G] = 0, \\ \overline{W}(H) + [F, H] = 0. \end{cases} .$$

Twisted co-Higgs case: $t_{14} = 0$ and $t_{32} \neq 0$

Consider the case where $t_{14} = 0$ and $t_{32} \neq 0$, which means the resulting generalized complex structure has complex-type (type 2).

We know that these correspond to twisted co-Higgs bundles. While we do not have a simple decomposition into tangent and cotangent bundle parts, we may write $\overline{D} = \Phi_B + \overline{\partial}_A$ where $\overline{\partial}_A$ is as above, but $\Phi_B = G(T + \overline{t_{32}}\rho) + H(W - \overline{t_{32}}\omega)$. We know in general that $(\Phi, \overline{\partial}_A)$ is integrable if and only if $(\Phi_B, \overline{\partial}_A)$ is integrable from Proposition 5.11. Therefore, as we would expect, the six conditions placed on E, F, G, H are the same,

$$\begin{gathered} \overline{T}(F) - \overline{W}(E) + [E, F] = 0, \\ \overline{T}(G) + [E, G] = 0, \\ \overline{T}(H) - iG + [E, H] = 0, \\ \overline{W}(G) + [F, G] = 0, \\ \overline{W}(H) + [F, H] = 0, \text{ and} \\ [G, H] = 0. \end{gathered}$$

Remark 5.26. In this case, the conditions in terms of Φ_B and $\overline{\partial}_A$ are

$$\Phi_B \wedge \Phi_B = 0 \Leftrightarrow [G, H] = 0,$$

$$\overline{\partial}_A^2 = 0 \Leftrightarrow \overline{T}(F) - \overline{W}(E) + [E, F] = 0, \text{ and}$$

$$\overline{\partial}_A\Phi_B = 0 \Leftrightarrow \begin{cases} \overline{T}(G) + [E, G] = 0, \\ \overline{T}(H) - iG + [E, H] = 0, \\ \overline{W}(G) + [F, G] = 0, \\ \overline{W}(H) + [F, H] = 0. \end{cases}$$

Poisson Case: $t_{14} \neq 0$ and $t_{32} = 0$

If $t_{14} \neq 0$, and $t_{32} = 0$ it means the resulting generalized complex structure has type zero (symplectic-type). We know these are Poisson modules from proposition 5.15 since t_{14}

is the coefficient of a holomorphic Poisson bivector which gives the deformed generalized complex structure the correct form. This argument proves the following proposition.

Proposition 5.27. *Fix $t_{14} \in \mathbb{C}$, $t_{14} \neq 0$ and $t_{11} = t_{22} = t_{32} = 0$ and consider the resulting generalized complex structure after applying this deformation. Then any generalized holomorphic bundle associated to this generalized complex structure is a Poisson module.*

These are also flat bundles themselves by proposition 5.18 since our Poisson structure is actually a holomorphic symplectic structure and hence the type of the generalized complex structure is everywhere zero in this case. The conditions in this case are:

$$\overline{T}(F) - \overline{W}(E) + [E, F] = 0,$$

$$\begin{aligned}
\overline{T}(G) - t_{14}W(E) + [E, G] &= 0, \\
\overline{T}(H) + t_{14}T(E) - it_{14}F - iG + [E, H] &= 0, \\
\overline{W}(G) - t_{14}W(F) + [F, G] &= 0, \\
\overline{W}(H) + t_{14}T(F) + [F, H] &= 0, \text{ and} \\
t_{14}W(H) + t_{14}T(G) + [G, H] &= 0.
\end{aligned}$$

Compare these conditions to those of the base case. If $(V, \Phi, \overline{\partial}_A)$ satisfies the conditions there, it is not true that $(V, \Phi, \overline{\partial}_{A_\beta})$ will satisfy the above equations, and vice versa. The problem is that there are new terms in these conditions (the ones with a t_{14}). Despite this, we can classify a certain sub-class of these Poisson modules as relating to co-Higgs bundles by stipulating that the terms with a t_{14} vanish.

Proposition 5.28. *If*

$$W(E) = 0,\ T(E) = iF,\ W(F) = 0,\ T(F) = 0,\ \textit{and } W(H) = -T(G)$$

then $(V, \Phi, \overline{\partial}_A)$ is integrable if and only if $(V, \Phi, \overline{\partial}_{A_\beta})$ is integrable.

The General Case: $t_{14} \neq 0$ and $t_{32} \neq 0$

Finally, we examine the most general case where $t_{14} \neq 0$ and $t_{32} \neq 0$. Generalized holomorphic bundles associated to these generalized complex structures can be viewed as flat bundles. This is because the generalized complex structures all have type zero as $t_{14} \neq 0$ implies we actually have a holomorphic symplectic structure. In this case our decomposition has the form $\overline{D} = \Phi_B + \overline{\partial}_{A_\beta}$.

These are all B-field transforms of a Poisson module, and integrability transfers directly. We can see this explicitly if we compare the conditions in the general setting to that of the Poisson setting. Thus, a generalized holomorphic bundle over a generalized complex structure in this general setting is a B-field transform of a Poisson module.

Remark 5.29. Deformations of the Base Complex Structure

Consider the case that t_{11} or t_{22} may be non-zero, but $t_{14} = t_{32} = 0$. These parameters correspond to deforming the base complex structure, and if we only apply such a deformation (with no B-field or Poisson structure) then we are in the co-Higgs setting. However, as we mentioned in the previous section, integrability is not preserved in this setting, much like the Poisson setting above because we are changing the operater d_L. Fortunately, our analysis in the previous section extends directly here, if we allow for non-trivial B- or β-fields, then we obtain flat bundles, Poisson modules, and (twisted) co-Higgs bundles as before. This is because if we change the base complex structure $J \to J_\epsilon$, and then apply one of these B- or β-field transforms to $\mathbb{J}_{J_\epsilon}$ then, regardless of what B or β are, only the $(0,2)$ part of B and the $(2,0)$ part β with respect to J_ϵ will have any effect on $L_{\mathbb{J}_{J_\epsilon}}$. Thus, we have a full picture of generalized holomorphic bundles over generalized complex structures in the complex deformation space.

Remark 5.30. Symplectic Deformations We could perform a similar computation for the deformation space of generalized complex structures coming from symplectic structures in section 4.2.2, but as we mentioned in that section the infinitesimal deformations correspond to generalized complex structures coming from symplectic structures. This means they all have type zero which means we can naturally view them all as flat bundles by proposition 5.18.

5.3 Generalized Holomorphic Bundles on Complex 2-Tori

The results on the torus will be very similar to those on a Kodaira surface. We will not repeat the computations here, but rather collect the results and remark on what changes in each case.

5.3.1 Bundles for a Fixed Complex Structure

We consider the case when $t_{11} = t_{22} = t_{21} = t_{12} = 0$ as these parameters correspond to deforming the base complex structure. Thus for $t_{14}, t_{32} \in \mathbb{C}$, L_ϵ is generated by $\{v_1, v_2, v_3, v_4\}$

where

$$v_1 = \overline{T} + t_{32}\overline{\rho},\ v_2 = \overline{W} - t_{32}\overline{\omega},\ v_3 = \omega + t_{14}W,\ v_4 = \rho - t_{14}T.$$

This is the same as the Kodaira surface setting, but all Courant brackets vanish in this case. Then L_ϵ^* is generated by $< v_1^*, v_2^*, v_3^*, v_4^* >$, which have the same expressions as before. Let $(V, \overline{D})$ be a generalized holomorphic bundle. The computation follows in exactly the same manner. Consider the matrix-valued 1-form corresponding to $\overline{D}$, say locally we write

$$\Omega = Ev_1^* + Fv_2^* + Gv_3^* + Hv_4^*$$

where E, F, G, H are smooth sections of $\mathrm{End}(V)$.

Proposition 5.31. *A matrix-valued 1-form Ω with values in $C^\infty(\overline{L})$ with*

$$\Omega = E(T - \overline{t_{32}}\rho) + F(W + \overline{t_{32}}\omega) + G(\overline{\omega} - \overline{t_{14}}\overline{W}) + H(\overline{\rho} + \overline{t_{14}}\overline{T})$$

satisfies $d_L\Omega + \Omega \wedge \Omega = 0$ if and only if $C_{ij} = 0$ for $1 \leq i < j \leq 4$ where

$$\begin{aligned}
C_{12} &= \overline{T}(F) - \overline{W}(E) + [E, F],\\
C_{13} &= \overline{T}(G) - t_{14}W(E) + [E, G],\\
C_{14} &= \overline{T}(H) + t_{14}T(E) + [E, H],\\
C_{23} &= \overline{W}(G) - t_{14}W(F) + [F, G],\\
C_{24} &= \overline{W}(H) + t_{14}T(F) + [F, H],\ \textit{and}\\
C_{34} &= t_{14}W(H) + t_{14}T(G) + [G, H].
\end{aligned}$$

Remark 5.32. Only C_{14} changes in this case compared to the Kodiara surface case. The same conclusions can be drawn here as was done above.

5.3.2 Analysis of Special Cases

Co-Higgs Bundles

If $t_{14} = t_{32} = 0$ then we have co-Higgs bundles on our torus.

B-Field Transforms and Twisted Co-Higgs Bundles

If $t_{14} = 0$ and $t_{32} \neq 0$ (so we transform then the underlying generalized complex structure by a non-trivial B-field action) then the resulting generalized complex structure is of complex-type (type 2), and any generalized holomorphic bundle will be a twisted co-Higgs bundle.

Poisson Modules and Flat Bundles

If we move the base structure in the Poisson direction, that is, $t_{32} = 0$ and $t_{14} \neq 0$, then the resulting structure will have symplectic-type (type 0) because we will be transforming by a holomorphic symplectic structure. Hence we can either view this as a flat bundle or a Poisson module. So we obtain:

Proposition 5.33. *Fix $t_{14} \in \mathbb{C}$, $t_{14} \neq 0$ and $t_{11} = t_{22} = t_{21} = t_{12} = t_{32} = 0$. Then, after applying this deformation to the base generalized complex structure, any generalized holomorphic bundle associated to this generalized complex structure is a Poisson module.*

General Setting

Once again, in this setting, given a deformation where $t_{14} \neq 0$ and $t_{32} \neq 0$ we get a flat bundle that can be viewed as a B-field transform of a Poisson module.

Remark 5.34. Deforming The Base Complex Structure We may change the base complex structure (by altering the parameters t_{11}, t_{22}, t_{21}, and t_{12}) but the conclusion is the same as in the Kodaira surface case. If we deform with no B-field or Poisson structure then we obtain a co-Higgs bundle. If we have any non-trivial β-field (a holomorphic symplectic structure with coefficient t_{14}) applied to the generalized complex structure then we are in the flat bundle setting because the underlying structure will have type zero. Finally, if we have a non-trivial deformation of the complex structure (t_{11}, t_{22}, t_{21}, and t_{12}) as well as a B-field (t_{32}) but no β-field action then generalized holomorphic bundles are (twisted) co-Higgs with respect to the new complex structure.

Remark 5.35. The symplectic case is the same, as infinitesimal deformations have full-rank projection onto T and can therefore be viewed as flat bundles.

Chapter 6

Future Directions

Here we outline some remaining open problems and future directions this research could take.

6.1 Generalized Complex Structures

1. **(Symplectic Moduli)** Let us begin with a rather lofty goal. While complex structures are well-understood in many settings, symplectic structures are not. There has been considerable amounts of work done in the area, but many open problems still remain. For example, if $[\omega_1] = [\omega_2] \in H^2_{DR}(M, \mathbb{R})$ where M is a surface, does it follow that ω_1 and ω_2 are symplectomorphic? Counterexamples are known in dimension 6 (see [24] for an example), but dimension 4 still proves elusive. Without fully understanding the symplectic moduli space, one cannot expect to fully understand the generalized complex moduli space.

2. **(Odd-Type Deformations)** In our analysis of deformations of generalized complex structures, we focused on even-type structures, that is, complex-type or symplectic-type structures. We also exhibited an odd-type structure on a Kodaira surface. The next step is to compute $H^2_L(N)$ for this structure and use it to understand the deformation space. We could check if all its first-order deformations are integrable as was the case for even-type structures.

3. **(Explicit Description of Complex Symplectic Structures)** In the general setting, we saw in Proposition 4.19 that if one begins at a symplectic structure, and

deforms it in the generalized sense, that infinitesimal deformations give rise to a complex symplectic structure. However, the proof was not constructive. It is difficult, in general, to determine the complex symplectic structure for a given deformation. A more precise understanding of their relationship would be useful.

4. **(A Stronger Torelli Theorem)** While we do not need extra assumptions in the complex-type setting for the Global Torelli Theorem on a Kodaira surface, we do need a connectedness assumption in the symplectic setting. This is because symplectic moduli spaces are not well understood, in general. It would be very enlightening to find a method for relaxing the connectedness assumption (or the hyperKähler assumption for complex 2-tori and K3 surfaces). If we could, then it might be possible to extend the procedure to tori and K3 surfaces, and in particular deduce whether or not every symplectic structure on these surfaces is (hyper)Kähler.

5. **(Odd-Type Generalized Calabi-Yau)** While we have a classification of even-type structures from Theorem 2.51, one could ask if there was a similar way to understand odd-type structures and establish an analogous Global Torelli Theorem.

6. **(Integrable First-Order Deformations)** As we mentioned in Remark 4.16, it would be interesting to determine if the first-order part of a type zero deformation is always integrable on its own without higher-order terms.

7. **(A Topological Description of the Moduli Spaces)** In [2] Borcea was able to provide a topological description of the moduli space of complex structures on a Kodaira surface. He showed that the moduli space was isomorphic to the product of the complex plane and a punctured disk. Obtaining a similar result, even for the subspace of complex-type deformations, would be interesting (see 3.15).

6.2 Generalized Holomorphic Bundles

1. **(Odd-Type Considerations)** Following the theme of the previous section, once we have a better understanding of odd-type generalized complex structures on a Kodaira surface we can further our understanding of generalized holomorphic bundles. In this way we might be able to obtain a class of generalized holomorphic bundles that do not behave like a classical holomorphic bundle, a Poisson module, or a flat bundle.

With the above goals in mind, we end the thesis.

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