Bipartite Distance-Regular Graphs of Diameter Four

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2014

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Using a method by Godsil and Roy, bipartite distance-regular graphs of diameter four can be used to construct $\{0, \alpha\}$-sets, a generalization of the widely applied equiangular sets and mutually unbiased bases. In this thesis, we study the properties of these graphs.

There are three main themes of the thesis. The first is the connection between bipartite distance-regular graphs of diameter four and their halved graphs, which are necessarily strongly regular. We derive formulae relating the parameters of a graph of diameter four to those of its halved graphs, and use these formulae to derive a necessary condition for the point graph of a partial geometry to be a halved graph. Using this necessary condition, we prove that several important families of strongly regular graphs cannot be halved graphs.

The second theme is the algebraic properties of the graphs. We study Krein parameters as the first part of this theme. We show that bipartite-distance regular graphs of diameter four have one "special" Krein parameter, denoted by $q_{3,3}(3)$. We show that the antipodal bipartite distance-regular graphs of diameter four with $q_{3,3}(3) = 0$ are precisely the Hadamard graphs. In general, we show that a bipartite distance-regular graph of diameter four satisfies $q_{3,3}(3) = 0$ if and only if it satisfies the so-called Q-polynomial property. In relation to halved graphs, we derive simple formulae for computing the Krein parameters of a halved graph in terms of those of the bipartite graph. As the second part of the algebraic theme, we study Terwilliger algebras. We describe all the irreducible modules of the complex space under the Terwilliger algebra of a bipartite distance-regular graph of diameter four, and prove that no irreducible module can contain two linearly independent eigenvectors of the graph with the same eigenvalue.

Finally, we study constructions and bounds of $\{0, \alpha\}$-sets as the third theme. We present some distance-regular graphs that provide new constructions of $\{0, \alpha\}$-sets. We prove bounds for the sizes of $\{0, \alpha\}$-sets of flat vectors, and characterize all the distance-regular graphs that yield $\{0, \alpha\}$-sets meeting the bounds at equality. We also study bipartite covers of linear Cayley graphs, and present a geometric condition and a coding theoretic condition for such a cover to produce $\{0, \alpha\}$-sets. Using simple operations on graphs, we show how new $\{0, \alpha\}$-sets can be constructed from old ones.

Acknowledgements

First and foremost, I would like to thank my advisor Chris Godsil, who introduced to me various interesting research problems. During my Ph.D. program, Chris has pointed me toward many important resources and offered me many useful ideas. Without his guidance, this thesis would not have been possible.

I would like to thank my committee members: Kevin Purbhoo, Bruce Richter, Ruxandra Moraru and John Caughman. I appreciate their time for reading this thesis. Their thoughtful comments have also greatly improved the quality of the thesis.

I thank many friends who made my graduate school experience much more wonderful. I also thank all the people who kindly gave hands to me during the apartment fire, especially Cristiane Sato and Marcel Silva, who constantly offered me help in those tough days.

Last and not least, I thank my parents, for their love, and for their support on everything I choose to pursue.

Contents

Chapter 1

Introduction

An *equiangular set* is a set of unit vectors in $\mathbb{C}^m$ such that all the standard inner products between distinct vectors have the same absolute value. A collection of orthonormal bases of $\mathbb{C}^m$ or $\mathbb{R}^m$ are called *mutually unbiased* if all the inner products between vectors from different bases have the same absolute value. Equiangular sets and mutually unbiased bases have applications in various areas, such as quantum information theory (Ivanović [31], Renes et al. [39]) and communication theory (Wu and Parker [49]), for example. A natural generalization of equiangular sets and mutually unbiased bases is sets of unit vectors whose inner products between distinct vectors have absolute value 0 or $\sqrt{\alpha}$ (with $\alpha \neq 0$); we call such a set a $\{0, \alpha\}$-*set*.

While constructions of equiangular sets and mutually unbiased bases have received much attention due to their applications, constructions of $\{0, \alpha\}$-sets in general, received little attention. Bounds for sizes of $\{0, \alpha\}$-sets are proven by Delsarte, Goethals and Seidel [21], and several examples are given. Other than that, little is known about $\{0, \alpha\}$-sets in general. In 2005, Roy presented in his Ph.D. thesis [41] a method, which is a joint work with Godsil, for constructing sets of unit vectors using certain bipartite graphs. Their method can be used to construct $\{0, \alpha\}$-sets. Among the graphs that can be used to construct $\{0, \alpha\}$-sets, bipartite distance-regular graphs of diameter four form a very natural family of candidates. This motivates the study of bipartite distance-regular graphs of diameter four in this thesis.

There are three main themes of the thesis. The first is the connection between bipartite distance-regular graphs of diameter four and strongly regular graphs. It is a consequence of the definition that each bipartite distance-regular graph of diameter four is associated with two halved graphs via the

distance-two relation, each of which is strongly regular. Not all strongly regular graphs can be halved graphs: a result due to van Bon (remarked in [6]) states that conference graphs are never halved graphs, for example. We aim to find out which strongly regular graphs are halved graphs.

New results regarding the first theme are written in Chapter 3, which are briefly summarized as follows. We derive formulae that convert parameters between the bipartite graph and its halved graphs. These formulae allow us to give an elementary proof of van Bon's result. Using these formulae and known necessary conditions on the parameters of distance-regular graphs, we derive a necessary condition for the point graph of a partial geometry (and in particular a graph from an orthogonal array) to be a halved graph. We also prove that several important families of graphs cannot be halved graphs, including graphs from Steiner triple systems and Latin squares (except for one). We will see that halved graphs with a certain parameter being prime have a very restrictive structure.

The second theme is the algebraic properties of bipartite distance-regular graphs of diameter four. Each distance-regular graph is associated with some algebraic parameters called the Krein parameters, which are known to be nonnegative (Scott [43]). Though there are many Krein parameters in general, some of them are trivial in the sense that their non-negativity follows directly from some basic properties of other parameters of the graph. Cameron, Goethals and Seidel [12] showed that strongly regular graphs only have two nontrivial Krein parameters, and that the graphs must possess some "nice" combinatorial structures whenever one of these two parameters vanishes. We aim to find out the nontrivial Krein parameters for a bipartite distance-regular graph of diameter four, and study the properties of the graph when such parameters vanish.

New results regarding Krein parameters are written in Chapter 4. They are briefly summarized as follows. We prove that there is only one nontrivial Krein parameter, which we will denote by $q_{3,3}(3)$, for the bipartite graphs of diameter four. By doing some computations on the parameters, it turns out that $q_{3,3}(3)$ is closely related to a combinatorial parameter of the graph denoted by $p_{4,4}(4)$, which vanishes precisely when the complement of a halved graph contains no cycle of length three. In particular, we prove that for antipodal bipartite distance-regular graphs of diameter four, $q_{3,3}(3) = 0$ if and only if $p_{4,4}(4) = 0$, in which case the graph must be a so-called Hadamard graph. When the graph is not antipodal, we find that the vanishing of the two parameters are equivalent only if the second largest eigenvalue of the graph

has a certain form. Though bipartite distance-regular graphs of diameter four with $p_{4,4}(4) = 0$ appear to be very rare, we present an infinite family of known graphs with $q_{3,3}(3) = 0$. We also present one other infinite family of feasible graph parameter sets satisfying $q_{3,3}(3) = 0$, although only one of which is known to be realized by a graph. Using a theorem by Brouwer, Cohen and Neumaier [6], we show that a graph satisfies $q_{3,3}(3) = 0$ if and only if it satisfies the so-called Q-polynomial property. In the algebraic aspect of the connection between a bipartite graph and its halved graphs, we prove a simple relation between their Krein parameters.

Another algebraic object we investigate is the Terwilliger algebra of a graph, which is the matrix algebra generated by certain matrices associated to the graph. Sometimes the structures of the irreducible invariant subspaces under the Terwilliger algebra of a graph, which we will call irreducible $\mathbb{T}$-modules, encode information for the local structures of the graph. For example, Jurišić, Koolen and Terwilliger [32], and Go and Terwilliger [24] showed that if certain $\mathbb{T}$-modules of a distance-regular graph Γ satisfy a certain condition, then the local graphs of Γ are all strongly regular with the same parameters. In Chapter 6, we describe all the irreducible $\mathbb{T}$-modules of a bipartite distance-regular graph of diameter four, and prove that no irreducible $\mathbb{T}$-module can contain two linearly independent eigenvectors of the graph with the same eigenvalue. While our method used for finding the irreducible modules involves case analysis, results by Curtin [17] help us reduce the number of cases to check.

Finally, going back to the motivation of this thesis, we study constructions and bounds of $\{0, \alpha\}$-sets in Chapter 5 as the third theme of the thesis. We present examples of distance-regular graphs, including an infinite family, that can be used to construct $\{0, \alpha\}$-sets using Godsil and Roy's method. Sets of vectors constructed using their method have all the vector coordinates sharing the same absolute value; we call such a set *flat*. Motivated by this, we improve the bounds for the sizes of $\{0, \alpha\}$-sets in [21] for flat sets. The proof of the improved bounds uses arguments involving Kronecker products very similar to those by Calderbank et al. [10]. By using known necessary conditions on the parameters of a distance-regular graph, we characterize all the distance-regular graphs that produce $\{0, \alpha\}$-sets meeting the improved bounds at equality: there are exactly four such graphs.

Besides distance-regular graphs of diameter four, another family of graphs that can produce $\{0, \alpha\}$-sets comes from linear Cayley graphs, which are closely related to finite projective spaces and projective codes. In order

for such a graph to yield a $\{0, \alpha\}$-set, it has to have exactly five distinct eigenvalues. Using a theorem due to Calderbank and Kantor [11], we give a geometric condition and a coding-theoretic condition that are equivalent to a linear Cayley graph having exactly five distinct eigenvalues. Finally, we present some simple operations on graphs that allow us to construct new $\{0, \alpha\}$-sets from known ones. These new results are also part of Chapter 5.

Chapter 2

Distance-Regular Graphs

In the first section, we review some basic terminology and properties about graph spectra. From the second section on, we summarize some known results that we will need for this thesis about distance-regular graphs. For a comprehensive treatment of distance-regular graphs, we refer the readers to the monograph by Brouwer, Cohen and Neumaier [6]. Readers are also referred to Biggs [3] for a more concise treatment on these graphs. No result in this chapter is new.

2.1 Preliminaries

Let $\Gamma := (V, E)$ be a graph. The *adjacency matrix* of Γ is the zero-one matrix A with rows and columns indexed by V, such that $A_{i,j} = 1$ if and only if i and j are adjacent in Γ. The adjacency matrix of a graph has real entries and is symmetric, so all of its eigenvalues are real, and there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of the adjacency matrix. The *spectrum* of a matrix with complex entries is the multiset of its eigenvalues. We write

$$\{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_t^{(m_t)}\}$$

to mean the multiset that contains m_l copies of λ_l $(1 \leq l \leq t)$ and no other elements. If A is the adjacency matrix of a graph Γ, we simply call the eigenvalues (resp. spectrum) of A the eigenvalues (resp. spectrum) of Γ. We denote the spectrum of Γ by spec(Γ). Regular graphs and bipartite graphs appear frequently in this thesis. Below we state two elementary properties of spectra of such graphs, whose proofs can be found in Godsil and Royle [27].

Lemma 2.1.1. *Let Γ be a k-regular graph. Then k is an eigenvalue of Γ, with multiplicity being the number of connected components in Γ. Furthermore, if Γ is connected and $-k$ is an eigenvalue of Γ, then Γ is bipartite.* $\square$

Lemma 2.1.2. *If Γ is a bipartite graph, then λ is an eigenvalue of Γ if and only if $-\lambda$ is. The multiplicity of λ as an eigenvalue of Γ is equal to that of $-\lambda$.* $\square$

2.2 Basic Properties

In this section, we review some fundamental properties of distance-regular graphs. This section is essentially a short version of Section 20 of Biggs [3]. Whenever there is neither a proof nor citation for a lemma, a proof can be found in Section 20 of [3].

Given a connected graph $\Gamma := (V, E)$, a vertex $v \in V$, and an integer i, let $\Gamma_i(v)$ be the set of vertices of Γ that are at distance i from v. The elements in $\Gamma_1(v)$ are called the *neighbours* of v. Let d be the diameter of Γ. The graph Γ is called *distance-regular* if for any $i \in \{0, 1, \ldots, d\}$, the size of $\Gamma_i(u) \cap \Gamma_1(v)$ depends only on the distance between u and v in Γ. If the distance between u and v is equal to i, we write

$$a_i := |\Gamma_i(u) \cap \Gamma_1(v)|, \quad b_i := |\Gamma_{i+1}(u) \cap \Gamma_1(v)|, \quad c_i := |\Gamma_{i-1}(u) \cap \Gamma_1(v)|.$$

Since b_0 is the valency of any vertex, a distance-regular graph must be regular; we often use K (and sometimes k) to denote the valency of a distance-regular graph. The parameters a_i, b_i and c_i are not independent.

Lemma 2.2.1. *If Γ is a K-regular distance-regular graph of diameter d and a_i, b_i and c_i are as defined above, then*

$$K = a_i + b_i + c_i$$

for all $i \in \{0, 1, \ldots, d\}$.

Proof. Clearly, if u and v are vertices of Γ that are at distance i, then

$$\Gamma_1(v) \subseteq \Gamma_{i-1}(u) \cup \Gamma_i(u) \cup \Gamma_{i+1}(u).$$

Since $K = |\Gamma_1(v)|$, the result follows from the definitions of a_i, b_i and c_i. $\square$

Choose a vertex v of a distance-regular Γ with valency K and diameter d, and let N_i be the size of $\Gamma_i(v)$. In particular, $N_0 = 1$, $N_1 = K$, and $N_i = 0$ for $i < 0$ and $i > d$. The parameters of Γ and the numbers N_i satisfy some nice properties.

Lemma 2.2.2. *Any distance-regular graph with valency K and parameters b_0, b_1, ..., b_{d-1}, c_1, c_2, ..., c_d satisfies*

(i) $N_{i+1} = N_i b_i / c_{i+1}$ *for each $i \in \{0, 1, \ldots, d-1\}$.*

(ii) $1 = c_1 \leq c_2 \leq c_3 \leq \cdots \leq c_d \leq K$.

(iii) $K = b_0 \geq b_1 \geq b_2 \geq \cdots \geq b_{d-1} \geq 1$.

$\hfill\square$

A consequence of (i) above is that the numbers N_i do not depend on the choice of the vertex v. For $i \in \{0, 1, \ldots, d\}$, let A_i be the matrix whose (v, w)-entry equals 1 if v and w are at distance i in Γ and equals 0 otherwise. Note that A_0 is the identity matrix, and A_1 is the adjacency matrix A of Γ. We call the matrices A_i the *distance matrices* of Γ. The following lemma summarizes an algebraic correlation between the distance matrices of a distance-regular graph.

Lemma 2.2.3. *Let Γ be a distance-regular graph with distance matrices $A_0, \ldots, A_d$ and parameters b_0, b_1, ..., b_{d-1}, c_1, c_2, ..., c_d. Then*

$$AA_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1}$$

for $i = 1, 2, \ldots, d-1$. $\hfill\square$

Let $\mathcal{A}(\Gamma)$ be the algebra of polynomials in the adjacency matrix A of Γ over the field $\mathbb{C}$. That is, $\mathcal{A}(\Gamma)$ is the set of all $\mathbb{C}$-linear combinations of the powers of A, equipped with the usual matrix addition and multiplication. We call $\mathcal{A}(\Gamma)$ the *adjacency algebra* of Γ (over $\mathbb{C}$). It follows that the adjacency algebra of a distance-regular graph as a vector space is spanned by its distance matrices.

Lemma 2.2.4. *Let Γ be a distance-regular graph of diameter d. Then the distance matrices of Γ form a basis for its adjacency algebra $\mathcal{A}(\Gamma)$. In particular, the dimension of $\mathcal{A}(\Gamma)$ is $d + 1$.* $\hfill\square$

Using theory of minimal polynomials, one can show the following statement using Lemma 2.2.4.

Lemma 2.2.5. *A distance-regular graph of diameter d has exactly $d + 1$ distinct eigenvalues.* $\qquad\square$

Let ∂ denote the distance function of a graph. Another important property of a distance regular graph Γ is that the size of $\Gamma_i(v) \cap \Gamma_j(w)$ depends only on the distance $\partial(v, w)$ between v and w. These numbers give another algebraic correlation between the distance matrices. These two facts are summarized in the following lemma.

Lemma 2.2.6. *Let Γ be a distance-regular graph of diameter d. Then for any $i, j, r \in \{0, 1, \ldots, d\}$, there exists an integer $p_{i,j}(r)$ such that for any vertices v and w of Γ with $\partial(v, w) = r$, we have*

$$|\Gamma_i(v) \cap \Gamma_j(w)| = p_{i,j}(r).$$

Moreover, if $A_0, \ldots, A_d$ are the distance matrices of Γ, then

$$A_i A_j = \sum_{r=0}^{d} p_{i,j}(r) A_r.$$

$\qquad\square$

The numbers $p_{i,j}(r)$ in Lemma 2.2.6 are called the *intersection numbers* of Γ. Whenever the graph is unclear from the context, we will use the notation $p_{i,j}(r, \Gamma)$ instead.

2.3   Strongly Regular Graphs

A regular graph Γ is called *strongly regular* if it is neither complete nor edgeless, and there exist integers a and c such that in Γ, any two adjacent vertices have a common neighbours and any two nonadjacent vertices have c common neighbours. We often use the 4-tuple (n, k, a, c) to represent the parameters of a strongly regular graph, where n is the number of the vertices in the graph and k is the valency of the graph. Connected strongly regular graphs form an important class of distance-regular graphs as we will see in the following lemma.

Lemma 2.3.1. *Let* Γ *be a connected graph. Then* Γ *is strongly-regular if and only if it is distance-regular and has diameter two.*

Proof. Suppose that Γ is strongly regular with parameters (n, k, a, c). Since Γ is not the complete graph, it must have diameter two for c to be well defined. Note that

$$b_0 = k, \quad c_1 = 1 \quad \text{and} \quad c_2 = c,$$

so we only need to show that b_1 is a constant. Let u and v be adjacent vertices in Γ. Then $|\Gamma_1(u) \cap \Gamma_1(v)| = a$. Since

$$\Gamma_1(v) \subseteq \{u\} \cup \Gamma_1(u) \cup \Gamma_2(u),$$

we have $|\Gamma_2(u) \cap \Gamma_1(v)| = k - 1 - a$, so b_1 is well-defined, and Γ is distance-regular.

Conversely, if Γ is distance-regular with parameters $(k, b_1; c_1, c_2)$, it is easy to check that the strongly regular graph parameters of Γ are well-defined and satisfy $a = k - b_1 - c_1$ and $c = c_2$. $\qquad\square$

The eigenvalues of a strongly regular graph are completely determined by its parameters.

Lemma 2.3.2. *Let* Γ *be a connected strongly regular graph with parameters* (n, k, a, c). *Then* Γ *has three distinct eigenvalues*

$$k, \quad \theta := \frac{a - c + \sqrt{D}}{2} \quad \text{and} \quad \tau := \frac{a - c - \sqrt{D}}{2},$$

where $D = (a - c)^2 + 4(k - c)$. $\qquad\square$

In fact, the converse is also true.

Lemma 2.3.3. *A connected regular graph with exactly three distinct eigenvalues is strongly regular.* $\qquad\square$

For a proof of Lemma 2.3.2 and Lemma 2.3.3, see [27, Sect. 10.2].

A graph of diameter d is called *antipodal* if the binary relation "is equal to or at distance d from" defined on the vertices of the graph is an equivalence relation. Since connected strongly regular graphs have diameter two by Lemma 2.3.1, the relation "is equal to or at distance two from" is just

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the non-adjacency relation. Therefore a connected strongly regular graph is antipodal if and only if non-adjacency is an equivalence relation on its vertices, from which it follows that the antipodal strongly regular graphs are precisely the complete multipartite graphs with equal-size colour classes (excluding the complete graphs and the edgeless graphs). There is also a simple algebraic characterization of the antipodal strongly regular graphs.

Lemma 2.3.4. *A connected strongly regular graph with parameters (n, k, a, c) is antipodal if and only if it has 0 as an eigenvalue.*

Proof. If A is the adjacency matrix of a connected strongly regular graph Γ, then A is nonzero and has trace zero, so its smallest eigenvalue τ is negative. By Lemma 2.3.2, we have $\theta\tau = c - k$, so $\theta \geq 0$, with $\theta = 0$ if and only if $c = k$. Using the fact that Γ has diameter two, it is straightforward to check that $c = k$ if and only if Γ is antipodal; this implies the desired statement. □

Finally, we state a basic fact about the complement of a strongly regular graph.

Lemma 2.3.5. *If Γ is a strongly regular graph with parameters (n, k, a, c), then its complement is a strongly regular graph with parameters $(n, \widetilde{k}, \widetilde{a}, \widetilde{c})$, where*

$$\widetilde{k} = n - k - 1,$$
$$\widetilde{a} = n - 2 - 2k + c,$$
$$\widetilde{c} = n - 2k + a.$$

□

The lemma is proven by counting the number of common non-neighbours of a pair of vertices using elementary arguments. We omit the details here.

2.4   Spectra of Distance-Regular Graphs

A good reference of the well-known results in this section is [3, Ch. 21]. Let Γ be a distance-regular graph of diameter d. By Lemma 2.2.6, the basis $\{A_0, A_1, \ldots, A_d\}$ for the the adjacency algebra $\mathcal{A}(\Gamma)$ satisfies the equation

$$A_i A_j = \sum_{r=0}^{d} p_{i,j}(r) A_r,$$

where the numbers $p_{i,j}(r)$ are the intersection numbers of Γ. Let B_i be a $(d+1) \times (d+1)$ matrix whose (j, r)-entry equals $p_{i,j}(r)$. Then the equation above is equivalent to saying that left-multiplication by A_i, viewed as a linear transformation on $\mathcal{A}(\Gamma)$ with respect to the basis formed by the distance matrices, is faithfully represented by B_i^T. Consequently, A_i and B_i^T have the same minimal polynomial and hence the same set of eigenvalues. Since a square matrix is similar to its transpose, A_i and B_i have the same set of eigenvalues. In particular, the adjacency matrix A of Γ has the same set of eigenvalues as the matrix $B := B_1$. This can be summarized as the following lemma.

Lemma 2.4.1. *The set of eigenvalues of a distance-regular graph with parameters b_0, b_1, ..., b_{d-1}, c_1, c_2, ..., c_d is equal to the set of eigenvalues of the matrix*

$$
B = \begin{bmatrix}
a_0 & c_1 & & & & \\
b_0 & a_1 & c_2 & & & \\
& b_1 & a_2 & \ddots & & \\
& & b_2 & \ddots & \ddots & \\
& & & \ddots & \ddots & c_d \\
& & & & b_{d-1} & a_d
\end{bmatrix},
$$

where $a_i := b_0 - b_i - c_i$. In particular, the eigenvalues of a distance-regular graph are completely determined by its parameters. $\square$

In fact, the multiplicities of the eigenvalues of Γ are also determined by the parameters of Γ. By Lemma 2.2.5, there are exactly $d+1$ distinct eigenvalues of Γ; these are also the eigenvalues of B, each of which has multiplicity 1 with respect to B and hence has a unique eigenvector up to scalar multiplication. These eigenvectors can be used to compute the multiplicities of the eigenvalues.

Lemma 2.4.2 (Biggs)**.** *Let θ_0, θ_1, ..., θ_d be the distinct eigenvalues of a distance-regular graph Γ. For each i, let y_i and z_i be the eigenvector of B^T and B, respectively, with eigenvalue θ_i, normalized so that the first coordinate equals 1. Then the multiplicity of θ_i as an eigenvalue of Γ is equal to*

$$
\frac{N}{y_i^T z_i},
$$

where N is the number of vertices of Γ. $\square$

For a proof of the lemma, see Biggs [3, p.167].

2.5 Adjacency Algebra

Let Γ be a distance-regular graph on N vertices with diameter d. Recall from Lemma 2.2.4 that the distance matrices $A_0, A_1, \ldots, A_d$ form a basis for the adjacency algebra $\mathcal{A}(\Gamma)$. In this section, we describe a second basis for $\mathcal{A}(\Gamma)$. Since the adjacency matrix A is real and symmetric, there exists an orthonormal basis of $\mathbb{R}^N$ whose elements are eigenvectors of A. Fix such a basis B. For each eigenvalue λ of A, let M_λ be the matrix whose columns are those vectors in B with eigenvalue λ, and let

$$E_\lambda := M_\lambda M_\lambda^T.$$

The matrices E_λ are pairwise orthogonal idempotents.

Lemma 2.5.1. *The matrices E_λ defined above satisfy:*

(i) $E_\lambda^2 = E_\lambda$.

(ii) $E_\lambda E_\mu = 0$ *if* $\lambda \neq \mu$.

(iii) $\sum_{\lambda \in \mathrm{ev}(A)} E_\lambda = I$, *where* $\mathrm{ev}(A)$ *is the set of all eigenvalues of* A.

Proof. Since the columns of M_λ are pairwise orthogonal, the matrix $M_\lambda^T M_\lambda$ is the identity matrix. Similarly,

$$M_\lambda^T M_\mu = 0$$

if $\lambda \neq \mu$. Properties (i) and (ii) then follow from the fact that $E_\lambda = M_\lambda M_\lambda^T$. Consider the orthonormal basis of $\mathbb{R}^N$ formed by the columns of the matrices M_λ. For any vector v in the basis, if v has eigenvalue λ, then

$$E_\mu v = M_\mu M_\mu^T v = \begin{cases} v, & \text{if } \mu = \lambda \\ 0, & \text{otherwise} \end{cases}.$$

It follows that $\sum_{\lambda \in \mathrm{ev}(A)} E_\lambda$ must define the identity map, proving property (iii). $\square$

Since Γ has $d+1$ distinct eigenvalues by Lemma 2.2.5, the set

$$\{E_\lambda : \lambda \in \mathrm{ev}(A)\}$$

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has size $d + 1$. It turns out that this set is another basis of $\mathcal{A}(\Gamma)$, which we often call a basis of idempotents. For a proof of this fact, see [25, Sect. 12.2]. Unless specified otherwise, we order the distinct eigenvalues θ_0, θ_1, ..., θ_d of Γ in decreasing order (so that θ_0 is the valency of Γ), and write E_i instead of E_{θ_i}. Since the θ_0-eigenspace is spanned by the all-one vector, we have

$$E_0 = \frac{1}{N} J,$$

where J is the all-one matrix.

Let $\{E_0, E_1, \ldots, E_d\}$ be a basis of idempotents for $\mathcal{A}(\Gamma)$. Then for each i, there exist constants $p_i(r)$ and $q_i(r)$ such that

$$A_i = \sum_{r=0}^{d} p_i(r) E_r \quad \text{and} \quad E_i = \frac{1}{N} \sum_{r=0}^{d} q_i(r) A_r.$$

Note that by Lemma 2.5.1,

$$A_i E_j = \sum_{r=0}^{d} p_i(r) E_r E_j = p_i(j) E_j,$$

so for each i, the $d + 1$ (not necessarily distinct) numbers $p_i(j)$ are the eigenvalues of A_i. In particular, $p_1(j) = \theta_j$ and $p_i(0) = N_i$, where N_i is the valency of the distance-i graph of Γ. With the spectra of the distance matrices of Γ, there is a simple formula to compute the numbers $q_j(i)$.

Lemma 2.5.2. *Let Γ be a distance-regular graph with distance matrices A_0, A_1, ..., A_d, and let E_0, E_1, ..., E_d be idempotents that form a basis of $\mathcal{A}(\Gamma)$. Then for all $i, j \in \{0, 1, \ldots, d\}$, the numbers $p_i(j)$ and $q_j(i)$ defined above satisfy*

$$q_i(j) = \frac{m_i}{N_j} p_j(i),$$

where m_i is multiplicity of the eigenvalue θ_i of A. $\square$

For a proof of this lemma, see [25, Sect. 12.2].

2.6   Krein Parameters

Let Γ be a distance-regular graph on N vertices with distance matrices $A_0, A_1, \ldots, A_d$. In Section 2.2, we saw that the intersection numbers $p_{i,j}(r)$

relate the adjacency matrices in a certain way. There are numbers analogous to the intersection numbers that relate the idempotents E_i. For matrices L and M of the same size that are over the same field, the *Schur product* of L and M, denoted by $L \circ M$, is the matrix of the same size whose (i, j)-entry is equal to $L_{i,j}M_{i,j}$. The Schur multiplication on matrices is commutative, associative, and distributive over addition. Clearly,

$$A_i \circ A_j = \begin{cases} A_i, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}.$$

Since $A_0, A_1, \ldots, A_d$ form a basis of $\mathcal{A}(\Gamma)$, it follows that $\mathcal{A}(\Gamma)$ is closed under Schur multiplication. Hence, if $\{E_0, E_1, \ldots, E_d\}$ is a basis of idempotents for $\mathcal{A}(\Gamma)$, then for any $i, j \in \{0, 1, \ldots, d\}$ there exist real numbers $q_{i,j}(r)$ such that

$$E_i \circ E_j = \frac{1}{N} \sum_{r=0}^{d} q_{i,j}(r) E_r.$$

The numbers $q_{i,j}(r)$ are called the *Krein parameters* of the graph Γ, which can be computed as follows.

Lemma 2.6.1. *The Krein parameters $q_{i,j}(r)$ of a distance-regular graph Γ satisfy*

$$q_{i,j}(r) = \frac{m_i m_j}{N} \sum_{l=0}^{d} \frac{p_l(i)p_l(j)p_l(r)}{N_l^2}, \qquad (2.6.1)$$

where N_l is the valency of the distance-l graph of Γ and m_i is the multiplicity of the eigenvalue θ_i of Γ. □

For a proof of this lemma, see [25, Sect. 12.2]. A remarkable property of the Krein parameters of a distance-regular graph is that they are all non-negative.

Theorem 2.6.2. *All Krein parameters of a distance-regular graph are non-negative real numbers.* □

This remarkable fact is due to Scott [43]. A proof is also given in [25, Sect. 12.4]. The nonnegativity constraint on the Krein parameters, sometimes known as the Krein condition, provides a very useful way to test whether some given integers could be parameters of a distance-regular graph. Suppose we have a finite sequence $(b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d)$ of integers.

We can pretend that it comes from parameters of a distance-regular graph and construct the matrix B as in Lemma 2.4.1. We can use B and Lemma 2.4.2 to compute the eigenvalues of the graph and their multiplicities, and then use the recurrence in Lemma 2.2.3 to compute all the eigenvalues of the distance matrices. We can then use equation (2.6.1) to compute $q_{i,j}(r)$. The numbers $q_{i,j}(r)$ obtained in this way are also called the *Krein parameters* of the sequence. If a Krein parameter of a sequence is negative, then by the Krein condition the sequence cannot come from parameters of a distance-regular graph.

2.7 Q-Polynomial Property

Let Γ be a distance-regular graph with valency K, diameter d, distance matrices A_i, and parameters b_0, ..., b_{d-1}, c_1, ..., c_d. For any eigenvalue λ of Γ, define the *standard sequence* of λ to be the sequence $\sigma := (\sigma_0, \ldots, \sigma_d)$ recursively as follows:

$$\sigma_0 := 1, \quad \sigma_1 := \frac{\lambda}{K}, \quad \sigma_{i+1} := \frac{(\lambda - a_i)\sigma_i - c_i\sigma_{i-1}}{b_i} \quad (1 \le i \le d - 1).$$

If the distinct eigenvalues of Γ can be ordered as $\lambda_0 := K$, λ_1, ..., λ_d such that for each j, the idempotent E_j corresponding to λ_j satisfies

$$E_j = \sum_{i=0}^{d} f_j(\sigma_i) A_i,$$

where σ is the standard sequence of λ_1 and f_j is a polynomial of degree j, then Γ is said to be *Q-polynomial*. Before discussing the motivation of the concept of Q-polynomiality, we first state a characterization for Q-polynomial distance-regular graphs in the bipartite case, due to Brouwer, Cohen and Neumaier [6, Sect. 8.2].

Theorem 2.7.1 (Brouwer, Cohen and Neumaier). *A bipartite distance-regular graph of diameter d is Q-polynomial if and only if the standard sequence σ of some eigenvalue of the graph satisfies the following two conditions:*

(i) $\sigma_i \ne \sigma_j$ whenever $i \ne j$.

(ii) There exist constants s and t such that for each $i \in \{1, 2, \ldots, d-1\}$,

$$\sigma_{i+1} + \sigma_{i-1} = s\sigma_i + t.$$

$\square$

This theorem will be used to investigate bipartite Q-polynomial graphs of diameter four in Section 4.6.

We end this chapter by discussing the motivation of the Q-polynomial property. An *association scheme* with d classes is a set $\{A_0, A_1, \ldots, A_d\}$ of symmetric $(0, 1)$-matrices satisfying all of the following conditions:

(i) $A_0 = I$.

(ii) $\sum_{i=0}^{d} A_i = J$.

(iii) For all i and j, the product $A_i A_j$ is a linear combination of A_0, A_1, ..., A_d.

It follows from the definition that the distance matrices of a distance-regular graph form an association scheme. An association scheme is called *P-polynomial* if there is an ordering of the matrices A_1, ..., A_d such that for each i, the matrix A_i is a polynomial in A_1 of degree i. While Lemma 2.2.3 implies that the distance-matrices of a distance-regular graph always form a P-polynomial association scheme, there are association schemes that are not P-polynomial. In fact, P-polynomial association schemes are exactly distance-matrices of distance-regular graphs (see [25, Sect. 12.3]). This suggests that P-polynomiality is a natural property for association schemes.

On the other hand, it is well-known that a distance-regular graph (or more generally an association scheme) is Q-polynomial if and only if we can order the idempotents E_j such that for each j, the matrix E_j can be written as $g_j(E_1)$ for some Schur polynomial g_j of degree j (a Schur polynomial is a polynomial with ordinary multiplications replaced by Schur multiplications); see Bannai and Ito [1, p. 193] for a proof of this fact. Therefore, Q-polynomiality is analogous to the natural P-polynomiality.

Chapter 3

Bipartite Distance-Regular Graphs

3.1 Introduction

The *distance-two graph* Γ_2 of a bipartite graph Γ is the graph with the same vertex set as Γ, such that two vertices are adjacent in Γ_2 if and only if they are at distance two in Γ. If Γ is connected then it is easy to see that Γ_2 has exactly two connected components (corresponding to the two colour classes), each of which is called a *halved graph* of Γ. When Γ is distance-regular and has diameter four, it follows from Lemma 2.3.1 that each of its halved graphs is strongly regular. Strongly regular graphs are well studied objects. Though there are still interesting open problems (e.g. existence of infinitely many triangle-free strongly regular graphs), a large number of examples of strongly regular graphs are known. Brouwer and van Lint [9] are concerned about those coming from partial geometries, while Calderbank and Kantor [11] described examples coming from projective two-weight codes. In contrast, bipartite distance-regular graphs of diameter four received less attention, though some infinite families are known. It is known that generalized quadrangles of order (q, q) exist for each prime power q ([38]), and the incidence graphs of such generalized quadrangles are non-antipodal bipartite distance-regular graphs of diameter four. Other known infinite families of such graphs include coset graphs of Kasami codes ([6, Sect. 11.2]), dual polar graphs $D_4(q)$ ([6, Sect. 9.4]), and Pasechnik graphs ([8]). An infinite family of antipodal graphs comes from Hadamard matrices ([6, Sect. 1.8]).

We will discuss some of these families in this thesis. It is natural to ask which strongly regular graphs are halved graphs and which ones are not. We will study this problem in this chapter.

We first compute some parametric data for bipartite distance-regular graphs of diameter four and their halved graphs, which we will use frequently in the thesis. We also summarize a list of known necessary conditions for the parameters of a bipartite distance-regular graph, specialized to those of diameter four. As new results, we relate the parameters of a bipartite distance-regular graph of diameter four to those of its halved graph. Using these relations, we derive a necessary condition for the point graph of a partial geometry to be a halved graph, and prove that several important families of strongly regular graphs cannot be halved graphs.

3.2   Graphs of Diameter Four

In this chapter, we will reserve the symbols N and K for the number of vertices and valency of a bipartite distance-regular graph, respectively, and reserve the symbols n and k for those of a halved graph. Recall that N_i is the number of vertices in Γ that are at distance i from a fixed vertex. In this section, we compute some important parametric data that we will use throughout the thesis. The following lemma is well-known.

Lemma 3.2.1. *For any bipartite distance-regular graph Γ with N vertices, valency K, diameter four, and parameters $a_0, \ldots, a_4, b_0, \ldots, b_3, c_1, \ldots, c_4$, the following statements hold:*

(i) $1 + N_2 + N_4 = N_1 + N_3 = N/2$.

(ii) $a_i = 0$ for each $i \in \{0, 1, \ldots, 4\}$.

(iii) $b_i + c_i = K$.

Proof. Let Y and Z be the colour classes of Γ and let v be a vertex in Y. Since any path in Γ has its vertices alternating between Y and Z, we have

$$\{v\} \cup \Gamma_2(v) \cup \Gamma_4(v) = Y \quad \text{and} \quad \Gamma_1(v) \cup \Gamma_3(v) = Z,$$

proving (i) since $|Y| = |Z| = N/2$ by the regularity of Γ. For any i, the set $\Gamma_i(v)$ is contained in a colour class, so no two vertices in $\Gamma_i(v)$ are adjacent, which means $a_i = 0$ and proves (ii). Finally, statement (iii) follows directly from (ii) and Lemma 2.2.1. $\qquad\square$

A consequence of Lemma 3.2.1 (iii) is that the parameters b_i and c_i of a bipartite distance-regular graph of diameter four are completely determined by K, c_2 and c_3, where K is the valency of the graph. When working with bipartite distance-regular graphs of diameter four, we often use the triple (K, c_2, c_3) to represent the parameters. The following lemma gives useful information of the distance partition of such graphs.

Lemma 3.2.2. *If a bipartite distance-regular graph Γ of diameter four on N vertives has parameters (K, c_2, c_3), then*

$$N_2 = \frac{K(K-1)}{c_2}, \quad N_3 = \frac{K(K-1)(K-c_2)}{c_2 c_3},$$

$$N_4 = \frac{(K-1)(K-c_2)(K-c_3)}{c_2 c_3},$$

where N_i is the number of vertices that are at distance i from a fixed vertex in Γ, and

$$N = \frac{2K(K^2 - (c_2+1)K + c_2(c_3+1))}{c_2 c_3}.$$

Proof. By Lemma 2.2.2 (i) and Lemma 3.2.1 (iii), the expressions for N_2, N_3 and N_4 can be directly computed. The desired expression for N then follows from the relation

$$N = 1 + K + N_2 + N_3 + N_4.$$

$\square$

Let Γ be a bipartite distance-regular graph on N vertices, with diameter four and parameters (K, c_2, c_3). Then the matrix B of Γ as in Lemma 2.4.1 is equal to

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ K & 0 & c_2 & 0 & 0 \\ 0 & K-1 & 0 & c_3 & 0 \\ 0 & 0 & K-c_2 & 0 & K \\ 0 & 0 & 0 & K-c_3 & 0 \end{bmatrix}.$$

The eigenvalues of B are

$$\theta_0 := K, \quad \theta_1 := \sqrt{\Delta}, \quad \theta_2 := 0, \quad \theta_3 := -\sqrt{\Delta}, \quad \theta_4 := -K,$$

where $\Delta = (c_2+1)K - c_2(c_3+1)$. We call the number θ_1 above the *nontrivial eigenvalue* of Γ. With the expression of N in Lemma 3.2.2, by computing

the eigenvectors of B^T and B, and applying Lemma 2.4.2, we obtain the multiplicities m_i of the eigenvalues θ_i of Γ, as follows:

$$m_0 = m_4 = 1, \quad m_1 = m_3 = \frac{K^2(K^2 - c_2 c_3 - \Delta)}{c_2 c_3 \Delta},$$

$$m_2 = \frac{2(K^2 - \Delta)(K - c_3)(K - 1)}{c_3 \Delta}.$$

Since $a_i = 0$ by Lemma 3.2.1, the equation in Lemma 2.2.3 becomes

$$A A_i = b_{i-1} A_{i-1} + c_{i+1} A_{i+1},$$

from which we can deduce

$$A_2 = \frac{A^2 - KI}{c_2}, \quad A_3 = \frac{(A^2 - (K + c_2 K - c_2)I)A}{c_2 c_3},$$

$$\text{and} \quad A_4 = \frac{(A^2 - (K + c_2 K - c_2)I)A^2 - c_3(K - c_2)(A^2 - KI)}{c_2 c_3 K}.$$

Consequently, if x is an eigenvector of A with eigenvalue λ, then x is also an eigenvector of A_2, A_3 and A_4, with eigenvalues

$$\frac{\lambda^2 - K}{c_2}, \quad \frac{(\lambda^2 - K - c_2 K + c_2)\lambda}{c_2 c_3},$$

$$\text{and} \quad \frac{(\lambda^2 - K - c_2 K + c_2)\lambda^2 - c_3(K - c_2)(\lambda^2 - K)}{c_2 c_3 K},$$

respectively. Using the spectrum of A, we are then able to obtain the spectra of all the distance matrices, represented by the following matrix of eigenvalues:

$$
\begin{array}{c|ccccc}
 & A_0 & A_1 & A_2 & A_3 & A_4 \\
\hline
m_0 & 1 & K & \frac{K(K-1)}{c_2} & \frac{K(K-1)(K-c_2)}{c_2 c_3} & \frac{(K-1)(K-c_2)(K-c_3)}{c_2 c_3} \\
m_1 & 1 & \sqrt{\Delta} & K - c_3 - 1 & -\sqrt{\Delta} & c_3 - K \\
m_2 & 1 & 0 & -\frac{K}{c_2} & 0 & \frac{K-c_2}{c_2} \\
m_3 & 1 & -\sqrt{\Delta} & K - c_3 - 1 & \sqrt{\Delta} & c_3 - K \\
m_4 & 1 & -K & \frac{K(K-1)}{c_2} & -\frac{K(K-1)(K-c_2)}{c_2 c_3} & \frac{(K-1)(K-c_2)(K-c_3)}{c_2 c_3}
\end{array}
$$

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If an eigenvalue appears more than once in the same column then its multiplicity is the sum of the corresponding m_j.

Let Γ_2 be the *distance-two graph* of Γ. Then the adjacency matrix of Γ_2 is the distance matrix A_2 of Γ. It is easy to see that if Y and Z are the colour classes of Γ then Γ_2 has exactly two connected components, each of which has a colour class of Γ being the vertex set. We called each connected component of Γ_2 a *halved graph* of Γ. Let H be a halved graph of Γ. By induction, if u and v are at distance m in H, then they are at distance $2m$ in Γ. From this it follows that

$$p_{i,j}(r, H) = p_{2i,2j}(2r, \Gamma), \tag{3.2.1}$$

which implies that H is a distance-regular graph of diameter two, which is strongly regular by Lemma 2.3.1. The equation (3.2.1) also implies that the two halved graphs have the same parameters. We will have a more detailed investigation on the strongly regular graphs that could be halved graphs in Section 3.5. To end this section, we express the spectrum of a halved graph by relating it to Γ_2. Since A_2 is the adjacency matrix of Γ_2, after a suitable re-arrangement of the rows and columns, we have

$$A_2 = \begin{bmatrix} A_Z & 0 \\ 0 & A_Y \end{bmatrix},$$

where A_Z and A_Y are the adjacency matrices of the halved graphs H_Z and H_Y of Γ corresponding to the colour classes Z and Y, respectively. If $z, y \in \mathbb{C}^{N/2}$ are vectors such that the vector

$$\begin{bmatrix} z \\ y \end{bmatrix}$$

is an eigenvector of A_2 with eigenvalue λ, then

$$\begin{bmatrix} A_Z z \\ A_Y y \end{bmatrix} = \begin{bmatrix} A_Z & 0 \\ 0 & A_Y \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} = \lambda \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \lambda z \\ \lambda y \end{bmatrix},$$

so λ is also an eigenvalue of each of the halved graphs H_X and H_Y. Since each halved graph is connected and strongly regular, it has exactly three distinct eigenvalues by Lemma 2.3.2. Since Γ_2 has only three distinct eigenvalues, and since each eigenvalue of Γ_2 must be an eigenvalue of each halved graph, the eigenvalues of a halved graph are

$$k = \frac{K(K-1)}{c_2}, \quad \theta = K - c_3 - 1, \quad \text{and} \quad \tau = -\frac{K}{c_2}.$$

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Since both halved graphs are strongly regular with equal parameters, the same eigenvalue has the same multiplicity in both graphs. Therefore, the multiplicities of the eigenvalues k, θ and τ of a halved graph of Γ are m_0, m_1 and $m_2/2$, respectively.

3.3   Parameter Feasibility

Since we frequently deal with bipartite distance-regular graphs of diameter four, in this section, we summarize a list of necessary conditions on the parameters K, c_2 and c_3 of such a graph. Each condition listed below is either known or a straightforward consequence of a known condition.

Lemma 3.3.1. *The parameters K, c_2 and c_3 of a bipartite distance-regular graph Γ of diameter four satisfy all of the following conditions:*

(i) $c_2 \leq c_3 < K$.

(ii) K/c_2 *is an integer.*

(iii) *With the notation in Section 2.5, for all $i, j, r \in \{0, 1, \ldots, 4\}$, the number*

$$p_{i,j}(r) := \frac{1}{NN_r} \sum_{l=0}^{4} p_i(l)p_j(l)p_r(l)m_l$$

computed using K, c_2 and c_3 is a nonnegative integer.

(iv) *The numbers m_τ and m_θ are both integers, where*

$$m_\tau := \frac{(K^2 - \Delta)(K - c_3)(K - 1)}{c_3 \Delta}, \quad m_\theta := \frac{K^2(K - c_2)(K - 1)}{c_2 c_3 \Delta},$$

$$\Delta := (c_2 + 1)K - c_2(c_3 + 1).$$

(v) *If $c_2 > 1$ then $c_3 \geq 3c_2/2$.*

(vi) *The inequality*

$$c_2(K - 2) \leq p_{2,2}(2)\left(1 + \frac{(c_2 - 1)(c_2 - 2)}{K - 2}\right)$$

is satisfied, where $p_{2,2}(2)$ is computed from K, c_2 and c_3 using the formula in (iii). Moreover, if equality holds then the number $K - 2$ divides $(c_2 - 1)(c_2 - 2)$.

(vii) The Krein parameters of (K, c_2, c_3) are nonnegative.

Proof. (i) By Lemma 2.2.2, we have $c_2 \leq c_3$. By Lemma 3.2.1 (iii) and the fact that $b_3 \geq 1$, we have $c_3 < K$.

(ii) Let θ and τ be the nontrivial eigenvalues of a halved graph of Γ, with $\theta > \tau$. By Lemma 2.3.2, we have $\theta + \tau = a + c$, where a and c are strongly regular graph parameters of the halved graph. In Section 3.2, we computed that

$$\theta = K - c_3 - 1 \quad \text{and} \quad \tau = -\frac{K}{c_2}.$$

Since θ is an integer, so is τ.

(iii) The number $p_{i,j}(r)$ computed using the described formula is just a intersection number of Γ described in Lemma 2.2.6. For a proof of this, see Godsil [25, Sect. 12.2], for example. Intersection numbers are always nonnegative integers by definition.

(iv) Both m_τ and m_θ are multiplicities of eigenvalues as computed in Section 3.2.

(v) This is a theorem by Brouwer, Cohen and Neumaier, stated as Theorem 5.4.1 in [6].

(vi) This is a special case of Proposition 5.8.1 in [6], a statement proven by Brouwer, Cohen and Neumaier.

(vii) This is just Theorem 2.6.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The condition (vii) above, sometimes called the Krein condition, at first glance looks tedious to check since there are many Krein parameters in general. However, we shall see in the next chapter that for a bipartite graph of diameter four, it is sufficient to just check the non-negativity of $q_{3,3}(3)$.

3.4   Relations Between Parameters

In Section 3.2, we saw that the halved graphs of a bipartite distance-regular graph Γ of diameter four are strongly regular graphs with the same parameters. In this section, we relate the parameters of Γ to those of its halved graphs.

Lemma 3.4.1. *Let Γ be a bipartite distance-regular graph of diameter four on N vertices, with parameters (K, c_2, c_3). Then the strongly regular graph parameters (n, k, a, c) of a halved graph of Γ satisfy*

$$n = \frac{K(K^2 - (c_2 + 1)K + c_2(c_3 + 1))}{c_2 c_3}, \quad k = \frac{K(K - 1)}{c_2},$$

$$a = \frac{K(c_2 + c_3 - 1) - c_2(c_3 + 1)}{c_2}, \quad c = \frac{c_3 K}{c_2}.$$

Proof. Let H be a halved graph of Γ. Since the $n = N/2$, Lemma 3.2.2 implies that

$$n = \frac{K(K^2 - (c_2 + 1)K + c_2(c_3 + 1))}{c_2 c_3}.$$

Since the valency k of H equals the number N_2 by definition, Lemma 3.2.2 implies

$$k = \frac{K(K - 1)}{c_2}.$$

Note that $c = p_{1,1}(2, H)$. Equation (3.2.1) then implies that

$$c = p_{2,2}(4, \Gamma) = |\Gamma_2(v) \cap \Gamma_2(w)|,$$

where v and w satisfy $w \in \Gamma_4(v)$. Consider the subgraph Γ' of Γ induced by the set $(\Gamma_2(v) \cap \Gamma_2(w)) \cup \Gamma_1(w)$. Note that $\Gamma_1(w) \subseteq \Gamma_3(v)$, so Γ' is bipartite with colour classes $\Gamma_2(v) \cap \Gamma_2(w)$ and $\Gamma_1(w)$. We count the edges of Γ' in two ways. If $u \in \Gamma_2(v) \cap \Gamma_2(w)$, then u and w are at distance 2, so the number of neighbours of u in $\Gamma_1(w)$ is c_2, and the number of edges of Γ' is

$$|\Gamma_2(v) \cap \Gamma_2(w)| \cdot c_2 = c \cdot c_2.$$

Now suppose $u \in \Gamma_1(w)$. Since $\Gamma_1(w) \subseteq \Gamma_3(v)$, we have $u \in \Gamma_3(v)$, so u has c_3 neighbours in $\Gamma_2(v)$. On the other hand, since Γ is bipartite and $u \in \Gamma_1(w)$, any neighbour of u that is not w is at distance two from w. Since $w \notin \Gamma_2(v)$, any neighbour of u in $\Gamma_2(v)$ is in $\Gamma_2(v) \cap \Gamma_2(w)$. This means that u has c_3 neighbours in $\Gamma_2(v) \cap \Gamma_2(w)$, so the degree of u in Γ' is c_3. The number of edges of Γ' is then

$$|\Gamma_1(w)| \cdot c_3 = K \cdot c_3.$$

Double-counting yields $c \cdot c_2 = K \cdot c_3$ and hence

$$c = \frac{c_3 K}{c_2}.$$

To obtain the desired expression for a, we can use a similar yet more complicated double-counting argument. We can also do it in a simpler, algebraic way. In Section 3.2, we computed the two nontrivial eigenvalues of H:

$$\theta = K - c_3 - 1 \quad \text{and} \quad \tau = -\frac{K}{c_2}.$$

By Lemma 2.3.2, we have $a = \theta + \tau + c$. By using the expressions for θ, τ and c obtained above, we can derive

$$a = \frac{K(c_2 + c_3 - 1) - c_2(c_3 + 1)}{c_2}.$$

$\square$

Given a bipartite distance-regular graph (of any diameter), formulae are known for computing its halved graph parameters. In particular, the lemma above can be derived from Proposition 4.2.2 in [6]. However, the proof given above is based on first principles instead of the proposition in [6].

The parameters of a bipartite distance-regular graph of diameter four can also be computed using those of its halved graphs. The formulae in the following lemma are new.

Lemma 3.4.2. *If a strongly regular graph with parameters (n, k, a, c) and nontrivial eigenvalues θ and τ (where $\theta > \tau$) is a halved graph of a bipartite distance-regular graph of diameter four with parameters (K, c_2, c_3), then*

$$K = \frac{k}{|\tau|} + 1, \quad c_2 = \frac{K(K-1)}{k} \quad \text{and} \quad c_3 = \frac{c}{|\tau|}.$$

Proof. In Section 3.2, we saw that $\tau = -K/c_2$. By Lemma 3.4.1, we have $k = K(K-1)/c_2$, so

$$K = \frac{k}{|\tau|} + 1 \quad \text{and} \quad c_2 = \frac{K(K-1)}{k}.$$

Since $c = c_3 K/c_2$ by Lemma 3.4.1, we have

$$c_3 = \frac{c}{|\tau|}.$$

$\square$

3.5 Halved Graphs

In this section, we attempt to answer the following question: which strongly regular graphs are halved graphs and which ones are not? We make use of Lemma 3.4.1 and Lemma 3.4.2 derived in the previous section to investigate several important families of strongly regular graphs. Unless stated otherwise, all results in this section are new.

The first family of graphs we examine is the family of *conference graphs*, which are strongly regular graphs with parameters

$$(n, k, a, c) = \left(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4} \right).$$

Conference graphs are precisely the strongly regular graphs whose nontrivial eigenvalues have the same multiplicity (see [6, Sect. 1.3]), and contain all the so-called Paley graphs as an important sub-family. Readers are referred to [6] and [27] for more details of conference graphs. It is known that a conference graph cannot be a halved graph: it is due to van Bon and is remarked in [6, p.180]. The proof given in [6] involves the Hoffman bound and a non-elementary proposition. Here we give a more elementary proof of this known fact using Lemma 3.4.1.

Theorem 3.5.1. *No conference graph is a halved graph of a bipartite distance-regular graph of diameter four.*

Proof. Suppose that a conference graph with parameters

$$(n, k, a, c) = \left(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4} \right)$$

is a halved graph of a bipartite distance-regular graph of diameter four with parameters (K, c_2, c_3). Since $k = 2c$, Lemma 3.4.1 implies

$$K - 1 = 2c_3$$

and so

$$K - c_3 = \frac{K+1}{2}. \tag{3.5.1}$$

On the other hand, using Lemma 3.4.1 and the fact that $c - a = 1$, straightforward computation shows

$$c_2 = \frac{K}{K - c_3}. \tag{3.5.2}$$

Equations (3.5.1) and (3.5.2) imply

$$c_2 = \frac{2K}{K+1} < 2,$$

so $c_2 = 1$. Substituting $K = 2c_3 + 1$ and $c_2 = 1$ into the parameters obtained in Lemma 3.4.1, we have

$$a = 2c_3^2 - 1 \quad \text{and} \quad c = 2c_3^2 + c_3.$$

Since $c - a = 1$, it follows that $c_3 = 0$, which is impossible. $\square$

Another important class of strongly regular graphs comes from partial geometries. A *partial geometry* with parameters (s, t, α), where s, t and α are positive integers, is an incidence structure with a point set and a line set, together with an incidence relation that relate a point with a line, such that:

(i) any pair of distinct points are incident with at most one common line;

(ii) any line is incident with $s + 1$ points;

(iii) any point is incident with $t + 1$ lines;

(iv) for any line l and any point p not on l, there exist exactly α lines that are incident with p and a point on l.

We use the symbol $PG(s, t, \alpha)$ to mean "partial geometry with parameters (s, t, α)". The *dual* of a partial geometry is the incidence structure obtained by interchanging its the point set and line set without changing the incidence relation. It follows from the definition that the dual of a $PG(s, t, \alpha)$ is a $PG(t, s, \alpha)$. The *point graph* of a partial geometry is the graph with the points of the geometry being the vertices, such that two vertices are adjacent if any only if they are incident with a common line. The *line graph* of a partial geometry is the point graph of its dual. It is well-known that the point graph of a partial geometry is strongly regular.

Lemma 3.5.2. *The point graph of a $PG(s, t, \alpha)$ is strongly regular, with parameters (n, k, a, c) equal*

$$\left(\frac{(s+1)(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1) \right).$$

The nontrivial eigenvalues of this point graph are

$$\theta = s - \alpha \quad \text{and} \quad \tau = -(t+1).$$

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Proof. The strongly regular graph parameters can be obtained by elementary counting arguments. With the parameters, the eigenvalues are obtained by using Lemma 2.3.2. □

A consequence of the lemma above is that the line graph of a partial geometry is also strongly regular. The *incidence graph* of a partial geometry is the bipartite graph whose colour classes are the point set and the line set, with the adjacency relation being the incidence relation. Using the definition of a partial geometry and the fact that s, t and α are all positive, it is routine to check that the incidence graph of a partial geometry has diameter four. If $s = t$, then the incidence graph of a $\mathrm{PG}(s,t,\alpha)$ is distance-regular with parameters $K = s + 1$, $c_2 = 1$ and $c_3 = \alpha$, and its halved graphs are the point graph and the line graph of the geometry. Not every partial geometry has its point graph or line graph being a halved graph.

Theorem 3.5.3. *If the point graph of a partial geometry with parameters (s,t,α) is a halved graph of a bipartite distance-regular graph of diameter four, then $t + 1$ divides $s + 1$. Furthermore, if $s \neq t$, then*

$$\frac{s+1}{t+1} \leq \frac{2}{3}\alpha.$$

Proof. If the point graph of $\mathrm{PG}(s,t,\alpha)$ is a halved graph of a distance-regular graph with valency K, then by Lemma 3.5.2 and Lemma 3.4.2,

$$K = \frac{s(t+1)}{t+1} + 1 = s + 1,$$

and so the parameters of the distance-regular graph satisfy

$$c_2 = \frac{s+1}{t+1} \quad \text{and} \quad c_3 = \alpha.$$

Since c_2 is an integer, $t + 1$ divides $s + 1$. If $s \neq t$ then $c_2 > 1$, so $c_2 \leq 2c_3/3$ by Lemma 3.3.1, which implies the desired inequality. □

There are numerous interesting combinatorial objects that are special cases of partial geometries, including generalized quadrangles, orthogonal arrays and Steiner triple systems, for example. For more details of partial geometries and their graphs, see [9]. We will examine several classes of special cases below.

An *orthogonal array* with parameters (d, m) is a $d \times m^2$ array X with symbols in $[m] := \{1, 2, \ldots, m\}$, such that the columns of any $2 \times m^2$ sub-array of X contain all the ordered pairs in $[m] \times [m]$. Let X be an orthogonal array with parameters (d, m). The *block graph* of X is the graph that has the columns of X being the vertices, such that two columns are adjacent if and only if they agree in some coordinate. For each $(i, j) \in [d] \times [m]$, let $l_{i,j}$ be the set of columns of X whose i-th entry is j. Let us declare the set of columns of X to be the point set, declare the set $\{l_{i,j} : (i, j) \in [d] \times [m]\}$ to be the line set, and declare containment to be the incidence relation. Then it is straightforward to check that this incidence structure is a $\mathrm{PG}(s, t, \alpha)$ with $s = m - 1$ and $t = d - 1 = \alpha$. Note that the block graph of X is the point graph of this partial geometry, and so is strongly regular with (n, k, a, c) equal

$$\big(m^2, (m-1)d, m + d(d-3), d(d-1)\big),$$

by Lemma 3.5.2. A necessary condition for a block graph of an orthogonal array to be a halved graph is obtained immediately from Theorem 3.5.3.

Corollary 3.5.4. *If the block graph of an orthogonal array with parameters (d, m) is a halved graph of a bipartite distance-regular graph of diameter four, then d divides m. Furthermore, if $d \neq m$, then*

$$m \leq \frac{2d(d-1)}{3}.$$

$\square$

An orthogonal array with parameters $(3, m)$ is called a *Latin square* of order m, whose block graph is called a *Latin square graph* of order m. Conventionally, a Latin square is an m by m array with entries in $[m]$ such that no entry is repeated in either a row or a column, but our definition here is equivalent. For a more comprehensive treatment on the theory of Latin squares, see [23]. It turns out that only one Latin square graph is a halved graph.

Corollary 3.5.5. *The complete tripartite graph $K_{3,3,3}$ is the only Latin square graph that is a halved graph of a bipartite distance-regular graph of diameter four.*

Proof. Let H be a Latin square graph of order m that is a halved graph of a bipartite distance-regular graph of diameter four. By Corollary 3.5.4, since

H is the block graph of an orthogonal array with parameters $(3, m)$, it follows that m is a multiple of 3. The second part of the same corollary implies that $m \geq 6$ is not possible, so $m = 3$. Using this and $d = 3$, it follows that the strongly regular graph parameters of H are

$$(n, k, a, c) = (9, 6, 3, 6).$$

Since $k = c = 6$, the graph H must be $K_{3,3,3}$. On the other hand, since $H = K_{3,3,3}$ is the point graph of a $PG(2, 2, 2)$, it is indeed a halved graph of the incidence graph of the geometry, which we know is distance-regular and has diameter four. □

A *Steiner triple system* of order v is a pair $(V, \mathcal{B})$, where V is a non-empty finite set of size $v \geq 4$ and $\mathcal{B}$ is a collection of 3-subsets of V (called *blocks*), such that for any pair of distinct elements of V, there exists a unique block in $\mathcal{B}$ containing the pair. Since any two distinct blocks can share at most one point in common, if x is a fixed element of V then the set

$$\left\{ B \setminus \{x\} : x \in B \right\}$$

forms a partition of $V \setminus \{x\}$. Therefore, the number of blocks that contain a fixed point is $(v-1)/2$. It follows that the incidence structure with point set $\mathcal{B}$, line set V and incidence relation being containment is a $PG(s, t, \alpha)$ with

$$s = \frac{v-1}{2} - 1, \quad t = 2 \quad \text{and} \quad \alpha = 3. \tag{3.5.3}$$

The *block graph* of a Steiner triple system is the point graph of the partial geometry above, which therefore is strongly regular with parameters

$$(n, k, a, c) = \left(\frac{v(v-1)}{6}, \frac{3(v-3)}{2}, \frac{v+3}{2}, 9 \right)$$

by Lemma 3.5.2, where v is the order of the system. These graphs are never halved graphs.

Theorem 3.5.6. *The block graph of a Steiner triple system cannot be a halved graph of a bipartite distance-regular graph of diameter four.*

Proof. Suppose that the block graph H of a Steiner triple system of order v is a halved graph of a bipartite distance-regular graph of diameter four with

parameters K, c_2 and c_3. The negative eigenvalue of H is $\tau := -3$ by Lemma 3.5.2. Using Lemma 3.4.2 and the parameters of H, we can derive that

$$K = \frac{v-3}{2} + 1, \quad c_2 = \frac{K}{3}, \quad \text{and} \quad c_3 = 3.$$

If $c_2 = 1$ then $K = 3$, which is impossible since $c_3 < K$, so $c_2 \geq 2$. Since $c_2 \leq 2c_3/3$ by Lemma 3.3.1, we have $c_2 = 2$, whence $K = 6$. With $K = 6$, $c_2 = 2$ and $c_3 = 3$, the Krein parameter $q_{3,3}(3)$ computed using the formula in Lemma 2.6.1 is negative, so there is no distance-regular graph with such parameters, which is a contradiction. □

The next graphs to examine are the triangular graphs. A *triangular graph* T_m is the graph whose vertices are the 2-subsets of an m-element set, such that two vertices are adjacent if and only if they intersect. It is easy to check that when $m \geq 4$, the graph T_m is a strongly regular graph with parameters

$$(n, k, a, c) = \left(\frac{m(m-1)}{2}, 2(m-2), m-2, 4 \right).$$

Conversely, strongly regular graphs with the parameters above have to be triangular graphs, except when $m = 8$ ([9, Sect. 5A]). These graphs are never halved graphs.

Theorem 3.5.7. *No strongly regular graph with parameters*

$$(n, k, a, c) = \left(\frac{m(m-1)}{2}, 2(m-2), m-2, 4 \right)$$

is a halved graph of a bipartite distance-regular graph of diameter four.

Proof. Suppose there is a halved graph with the given parameters. By Lemma 2.3.2, its negative eigenvalue is $\tau = -2$. Using this and Lemma 3.4.2, the parameters of the bipartite graph can be derived:

$$K = m - 1, \quad c_2 = \frac{m-1}{2} \quad \text{and} \quad c_3 = 2.$$

Since $c_2 \leq c_3 \leq K - 1$ by Lemma 3.3.1 and $c_2 \in \mathbb{Z}$, it follows that $m = 5$, which implies $c_2 = 2$. Now Lemma 3.3.1 implies that $c_3 \geq 3c_2/2 = 3$, which is a contradiction. □

The last class of graphs we are going to investigate comes from generalized quadrangles. A *generalized quadrangle* with parameters (s, t) is a partial geometry with parameters $(s, t, 1)$. By Lemma 3.5.2, the point graph and the line graph of a generalized quadrangle with parameters (s, s) are strongly regular with parameters

$$(n, k, a, c) = \big((s + 1)(s^2 + 1), s(s + 1), s - 1, s + 1\big).$$

Both of these graphs are halved graphs of the corresponding incidence graph, which is distance-regular and has diameter four with parameters $K = s + 1$, $c_2 = 1$ and $c_3 = 1$. In fact, any halved graph that has the parameter c being prime must come from a generalized quadrangle.

Theorem 3.5.8. *If a strongly regular graph H with the parameter c being prime is a halved graph of a bipartite distance-regular graph of diameter four, then H is the point graph of a generalized quadrangle with parameters (s, s), where $s + 1$ is a prime.*

Proof. If H is a halved graph of Γ with parameters (K, c_2, c_3), then by Lemma 3.4.1, $c = c_3 K / c_2$. Since K / c_2 is an integer greater than 1 by Lemma 3.3.1 and c is a prime, we have $c_3 = 1$ and so $c_2 = 1$. It follows that Γ is the incidence graph of a $\mathrm{PG}(K - 1, K - 1, 1)$, that is, a generalized quadrangle with parameters $(K - 1, K - 1)$. By declaring the suitable colour class as the point set, we have H being the point graph. Finally, K is prime since $K = c_3 K / c_2 = c$. $\qquad\square$

There exist strongly regular graphs with the parameter c being prime that do not arise from generalized quadrangles. Examples of such graphs include those with parameters $(16, 6, 2, 2)$ and $(64, 14, 6, 2)$ that arise from projective two-weight codes (see [5]); these graphs cannot be halved graphs by Theorem 3.5.8. On the other hand, generalized quadrangles with parameters (s, s) do exist. When q is a prime power, Payne and Thas [38, Sect. 3.1] described a way of constructing a generalized quadrangle with parameters (q, q) using the projective space over the field with q elements, with further reference [22] cited. When $q \in \{2, 4, 16, 256, 65536\}$, the number $c = q + 1$ is a (Fermat) prime.

Chapter 4

Special Graph Parameters

4.1 Introduction

The Krein condition (Theorem 2.6.2) ensures that all Krein parameters of a distance-regular graph are nonnegative. Though they do not have to be integers in general, the vanishing of certain Krein parameters may have combinatorial implications. For example, a well-known result due to Cameron, Goethals and Seidel [12] says that if either one of the Krein parameters $q_{1,1}(1)$ and $q_{2,2}(2)$ vanishes for a strongly regular graph, then the graph induced by the neighbours of any fixed vertex and the graph induced by the non-neighbours are both strongly regular. The two Krein parameters $q_{1,1}(1)$ and $q_{2,2}(2)$ are special for strongly regular graphs because they are the only two that are necessary to test when checking the Krein condition for parameter feasibility (see Brouwer and Haemers [7, Sect. 11.4], for example). The goal of this chapter is to seek for the "special" Krein parameters of a distance-regular graph of diameter four, and study the properties of the graph when such parameters vanish.

The new results in this chapter are briefly summarized as follows. We first compute the Krein parameters for a bipartite distance-regular graph Γ of diameter four, and show that $q_{3,3}(3)$ is the only one that is necessary to check for the Krein condition, though we will see that $q_{1,1}(1)$ is another special parameter. We will then derive a relation between the Krein parameters of Γ and those of its halved graphs, and show that halved graphs with "special" Krein parameters vanishing all have a very simple structure. We classify all the graphs with $q_{1,1}(1) = 0$ and all the antipodal ones with $q_{3,3}(3) = 0$, both

of which turn out to be the Hadamard graphs. We will see that there is an infinite family of known (non-antipodal) graphs that satisfy $q_{3,3}(3) = 0$, and we will give another infinite family of feasible parameter sets with $q_{3,3}(3) = 0$. In the last section, we will see that having a vanishing $q_{3,3}(3)$ is equivalent to being Q-polynomial for Γ.

All results in this chapter are new, unless stated otherwise.

4.2 Krein Condition

Let K, c_2 and c_3 be positive integers. For K, c_2 and c_3 to be parameters of a bipartite distance-regular graph of diameter four, they have to satsify the following basic conditions:

(a) $c_2 \leq c_3 < K$.

(b) The numbers N_i computed using K, c_2 and c_3 in Lemma 3.2.2 must be positive.

(c) The numbers m_i computed using K, c_2 and c_3 in Section 3.2 must be positive.

(d) The Krein parameters of (K, c_2, c_3) must be nonnegative.

Testing the first three conditions are easy. On the other hand, there are many Krein parameters for a given sequence, so checking the Krein condition seems tedious. The good news is, assuming the first three conditions, it is sufficient to only check the non-negativity of the Krein parameter $q_{3,3}(3)$ in order to verify the Krein condition: this is the main result of this section.

For this section, we assume that the conditions (a), (b) and (c) above are satisfied. Consider the matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
K & \sqrt{\Delta} & 0 & -\sqrt{\Delta} & -K \\
N_2 & K - c_3 - 1 & -\frac{K}{c_2} & K - c_3 - 1 & N_2 \\
N_3 & -\sqrt{\Delta} & 0 & \sqrt{\Delta} & -N_3 \\
N_4 & c_3 - K & \frac{K - c_2}{c_2} & c_3 - K & N_4
\end{bmatrix},
$$

where

$$N_2 = \frac{K(K-1)}{c_2}, \quad N_3 = \frac{K(K-1)(K-c_2)}{c_2 c_3}, \quad N_4 = \frac{(K-1)(K-c_2)(K-c_3)}{c_2 c_3},$$

and

$$\Delta = K + c_2(K - c_3 - 1).$$

Assume that the rows and the columns of the matrix above are indexed by $\{0, 1, 2, 3, 4\}$ in the natural order, and let us use $p_i(j)$ to denote the (i, j)-entry of the matrix. Note that the use of the symbol $p_i(j)$ is consistent with that in Section 2.5, since the entries of the matrix above are just eigenvalues of the distance-matrices if K, c_2 and c_3 are indeed parameters of a graph. Recall from Lemma 2.6.1 that the formula for the Krein parameters is

$$q_{i,j}(r) = \frac{m_i m_j}{N} \sum_{l=0}^{4} \frac{p_l(i) p_l(j) p_l(r)}{N_l^2},$$

where $N = 1 + K + N_2 + N_3 + N_4$. Since the numbers m_i and N are assumed to be positive, for the sake of testing the Krein condition, it is sufficient to check non-negativity for each $\beta(i, j, r)$, where

$$\beta(i, j, r) := \sum_{l=0}^{4} \frac{p_l(i) p_l(j) p_l(r)}{N_l^2}.$$

Note that β is invariant under permutations on i, j and r. Observing this, we use case analysis to show that many numbers $\beta(i, j, r)$ are always non-negative.

Case 1: some index equals 0

First, computation shows that

$$\beta(0, 0, r) = 1 + p_1(r) + p_2(r) + p_3(r) + p_4(r).$$

From the matrix, we see that

$$1 + p_1(r) + p_2(r) + p_3(r) + p_4(r) = \begin{cases} N, & \text{if } r = 0; \\ 0, & \text{otherwise.} \end{cases},$$

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(which is true for distance-regular graphs in general,) so $\beta(0,0,r) \geq 0$.

Now assume that at most one of i, j and r is equal to zero. In this case we compute

$$\beta(0,j,r) = 1 + \frac{p_1(j)p_1(r)}{N_1} + \frac{p_2(j)p_2(r)}{N_2} + \frac{p_3(j)p_3(r)}{N_3} + \frac{p_4(j)p_4(r)}{N_4}.$$

If $j = r$ then $p_l(j)p_l(r) = p_l(r)^2 \geq 0$, so $\beta(0,r,r) \geq 1$, and we may next assume that $j \neq r$.

If $j = 4$ then we compute $\beta(0,4,r)$, and

$$\beta(0,4,r) = 1 - p_1(r) + p_2(r) - p_3(r) + p_4(r)$$
$$= \begin{cases} 0, & \text{if } r \neq 4 \\ N, & \text{otherwise} \end{cases}.$$

If $j = 2$ and $r \in \{1,3\}$ then

$$\beta(0,2,r) = 1 - \frac{K - c_3 - 1}{K - 1} + \frac{c_3(c_3 - K)}{(K-1)(K-c_3)}$$
$$= 1 - \frac{K - c_3 - 1}{K - 1} - \frac{c_3}{K - 1} = 0.$$

Finally,

$$\beta(0,1,3) = 1 - \frac{\Delta}{K} + \frac{(K - c_3 - 1)^2}{N_2} - \frac{\Delta}{N_3} + \frac{(K - c_3)^2}{N_4},$$

where $\Delta = K + c_2(K - c_3 - 1)$. By expanding out, one can see that the right-hand side is equal to zero, so

$$\beta(0,1,3) = 0.$$

Therefore, $\beta(i,j,r) \geq 0$ whenever at least one of i, j and k is equal to 0.

Case 2: no index equals 0

We further divide the possibilities into three sub-cases.

Case 2.1: all indices are 2 or 4

In this case,
$$\beta(4,4,4) = 1 - N_1 + N_2 - N_3 + N_4 = 0,$$
$$\beta(4,4,2) = 1 + (K - c_3 - 1) + (c_3 - K) = 0,$$

and
$$\beta(4,2,2) = 1 + \frac{p_2(2)^2}{N_2} + \frac{p_4(2)^2}{N_4} > 1.$$

Also,
$$\beta(2,2,2) = 1 - \frac{(K/c_3)^3}{N_2^2} + \frac{(K/c_2 - 1)^3}{N_4^2}$$
$$= \frac{K(K - c_2)(K - c_3)(K - c_3 - 1) + K^2(c_3 - 1) - c_3^2(c_2 - 1)}{(K - 1)^2(K - c_3)^2 c_2}.$$

Since $K > c_3 \geq c_2 > 0$, we have $K^2(c_3 - 1) - c_3^2(c_2 - 1) > 0$ and so
$$\beta(2,2,2) > 0.$$

In cases 2.1, we always have $\beta(i,j,r) \geq 0$.

Case 2.2: indices have 2 or 4 mixed with 1 or 3

We break the possibilities in case 2.2 into five groups:

(i) (2,1,1), (2,1,3), (2,3,3).

(ii) (4,1,1), (4,1,3), (4,3,3).

(iii) (2,2,1), (2,2,3).

(iv) (2,4,1), (2,4,3).

(v) (4,4,1), (4,4,3).

For group (i), we have
$$\beta(2,1,1) = \beta(2,1,3) = \beta(2,3,3)$$
$$= 1 - \frac{(K/c_2)(K - c_3 - 1)^2}{N_2^2} + \frac{(k/c_2 - 1)(c_3 - K)^2}{N_4^2}.$$

The negative term above is equal to

$$-\frac{(K - c_3 - 1)^2 c_2}{(K - 1)^2 K},$$

which has absolute value less than 1, so

$$\beta(2, 1, 1) = \beta(2, 3, 3) = \beta(2, 1, 3) > 0.$$

For group (ii), first note that both $\beta(4, 1, 1)$ and $\beta(4, 3, 3)$ are equal to $\beta(0, 1, 3)$, which was shown to be zero in case 1. Also,

$$\beta(4, 1, 3) = 1 + \frac{\Delta}{N_1} + \frac{(K - c_3 - 1)^2}{N_2} + \frac{\Delta}{N_3} + \frac{(c_3 - K)^2}{N_4} > 1.$$

For group (iii), we have

$$\beta(2, 2, 1) = \beta(2, 2, 3) = 1 + \frac{K - c_3 - 1}{(K - 1)^2} - \frac{c_3^2}{(K - 1)^2(K - c_3)}.$$

Since the only negative term here has absolute value at most 1, we find that $\beta(2, 2, 1) \geq 0$. For group (iv), we have

$$\beta(2, 4, 1) = \beta(2, 0, 1) = 0 \quad \text{and} \quad \beta(2, 4, 3) = \beta(2, 0, 3) = 0.$$

Finally, for group (v), we have

$$\beta(4, 4, 1) = \beta(4, 4, 3) = \beta(0, 0, 3) = 0.$$

Hence, in cases 2.2 we always have $\beta(i, j, r) \geq 0$.

Case 2.3: all indices are 1 or 3

In this final remaining case, we have

$$\beta(1, 1, 1) = \beta(1, 3, 3)$$
$$= 1 + \frac{\Delta\sqrt{\Delta}}{K^2} - \frac{\Delta\sqrt{\Delta}}{N_3^2} + \frac{c_2^2(K - c_3 - 1)^3}{K^2(K - 1)^2} - \frac{c_2^2 c_3^2(K - c_3)}{(K - 1)^2(K - c_2)^2}$$

and

$$\beta(3, 3, 3) = \beta(1, 1, 3)$$
$$= 1 - \frac{\Delta\sqrt{\Delta}}{K^2} + \frac{\Delta\sqrt{\Delta}}{N_3^2} + \frac{c_2^2(K - c_3 - 1)^3}{K^2(K - 1)^2} - \frac{c_2^2 c_3^2(K - c_3)}{(K - 1)^2(K - c_2)^2}.$$

All the possible numbers $\beta(i, j, r)$ were computed. Now we can summarize the main result of this section.

Theorem 4.2.1. *Let K, c_2 and c_3 be integers such that $K > c_3 \geq c_2 > 0$. Suppose that the numbers N_i and m_i computed using K, c_2 and c_3 as in Section 3.2 are positive. Then all the Krein parameters of (K, c_2, c_3) are nonnegative if and only if $q_{3,3}(3) \geq 0$*

Proof. By the previous case analysis, all the Krein parameters of (K, c_2, c_3), except for $q_{1,1}(1)$, $q_{1,1}(3)$, $q_{1,3}(3)$ and $q_{3,3}(3)$ (up to permutations of i, j and r), are nonnegative. Since

$$N_3 = \frac{K(K-1)(K-c_2)}{c_2 c_3}, \quad K > c_3 \quad \text{and} \quad \frac{K}{c_2} \geq 2,$$

it follows that $N_3 \geq K$ and so $q_{3,3}(3) \leq q_{1,1}(1)$ by the formulae of these parameters above. Since

$$q_{3,3}(3) = q_{1,1}(3) \quad \text{and} \quad q_{1,1}(1) = q_{1,3}(3),$$

all the Krein parameters are nonnegative if and only if $q_{3,3}(3) \geq 0$. $\quad\square$

The discussion above suggests that the Krein parameters $q_{1,1}(1)$ and $q_{3,3}(3)$ stand out to be "special". Graphs that have one of these two parameters being zero correspond to "extremal" cases, so it is natural to ask what these graphs are. There are simple descriptions for graphs with vanishing $q_{1,1}(1)$.

Theorem 4.2.2. *For a bipartite distance-regular graph of diameter four, the following conditions are equivalent:*

(i) $q_{1,1}(1) = 0$.

(ii) $K = N_3$.

(iii) $K = 2c_2 = c_3 + 1$.

(iv) $N_4 = 1$.

Proof. "(i) $\Rightarrow$ (ii)": If $\beta(1, 1, 1) = 0$, then since $\beta(1, 1, 1) \geq \beta(3, 3, 3) \geq 0$, it follows that $\beta(3, 3, 3) = 0$. This implies that

$$\beta(1, 1, 1) - \beta(3, 3, 3) = 0.$$

By expressing $\beta(1, 1, 1) - \beta(3, 3, 3)$ the formulae above, we have $K = N_3$.

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"(ii) $\Rightarrow$ (iii)": Since

$$N_3 = \frac{K(K-1)(K-c_2)}{c_2 c_3}, \quad K-1 \geq c_3 \quad \text{and} \quad K - c_2 \geq c_2,$$

$K = N_3$ holds only if the two inequalities hold with equalities, which implies $K = 2c_2 = c_3 + 1$.

"(iii) $\Rightarrow$ (i)": Direct computation using the formula of $q_{1,1}(1)$ shows that $q_{1,1}(1) = 0$ if $K = 2c_2 = c_3 + 1$.

"(iii) $\Leftrightarrow$ (iv)": Using the facts that

$$N_4 = \frac{(K-1)(K-c_2)(K-c_3)}{c_2 c_3}, \quad K-1 \geq c_3 \quad \text{and} \quad K - c_2 \geq c_2,$$

the equivalence follows similarly as in "(ii) $\Rightarrow$ (iii)". $\square$

Bipartite distance-regular graphs of diameter four whose parameters satisfy $K = c_3 + 1 = 2c_2$ are called *Hadamard graphs*. These graphs are called Hadamard graphs because they are constructed from Hadamard matrices. For more details about these graphs, the readers are referred to [6]. Theorem 4.2.2 gives alternative descriptions for Hadamard graphs. We will discuss more about graphs with vanishing $q_{3,3}(3)$ later in this chapter.

4.3 Krein Parameters for Halved Graphs

Let Γ be a bipartite distance-regular graph with N vertices, valency K, diameter four, and parameters (K, c_2, c_3). In this section, we order the five eigenvalues $\theta_0, \ldots, \theta_4$ of Γ in decreasing order, and investigate the relation between the Krein parameters of Γ and those of its halved graphs. Since the Krein parameters $q_{i,j}(r)$ satisfy

$$E_i \circ E_j = \frac{1}{N} \sum_{r=0}^{d} q_{i,j}(r) E_r, \tag{4.3.1}$$

where the matrices E_i form a basis of idempotents for the adjacency algebra, we shall first investigate how the idempotents for Γ interact with those for a halved graph. In this section, whenever we express a matrix in block form, all the block matrices have equal size.

Let $n := N/2$, so n is the number of vertices in a halved graph. Since the adjacency matrix A of Γ has the form

$$\begin{bmatrix} 0 & A^{(1,2)} \\ A^{(2,1)} & 0 \end{bmatrix},$$

it is straightforward to check that if x and y are vectors in $\mathbb{R}^n$ such that

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

is an eigenvector of Γ with eigenvalue λ, then

$$\begin{bmatrix} -x \\ y \end{bmatrix}$$

is an eigenvector of Γ with eigenvalue $-\lambda$. Fix an orthonormal basis of the θ_1-eigenspace of Γ (in $\mathbb{R}^N$), say

$$\mathcal{B}_1 := \left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \ldots, \begin{bmatrix} x_m \\ y_m \end{bmatrix} \right\}.$$

Then

$$\mathcal{B}_3 := \left\{ \begin{bmatrix} -x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} -x_2 \\ y_2 \end{bmatrix}, \ldots, \begin{bmatrix} -x_m \\ y_m \end{bmatrix} \right\}$$

is an orthonormal basis of the θ_3-eigenspace of Γ (since $\theta_3 = -\theta_1$). Let us write the distance-two matrix A_2 of Γ in the form

$$\begin{bmatrix} A_2^{(1,1)} & 0 \\ 0 & A_2^{(2,2)} \end{bmatrix},$$

where $A_2^{(1,1)}$ is the adjacency matrix of the halved graph H, and let θ and τ be the nontrivial eigenvalues of H with $\theta > \tau$. In Section 3.2, we saw that the vectors in $\mathcal{B}_1$ are eigenvectors of A_2 with eigenvalue θ, so

$$\theta \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} A_2^{(1,1)} & 0 \\ 0 & A_2^{(2,2)} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} A_2^{(1,1)} x_i \\ A_2^{(2,2)} y_i \end{bmatrix},$$

which implies that x_i is an eigenvector of H with eigenvalue θ. Since different eigenspaces are orthogonal, if we consider $\mathcal{B}_1$ and $\mathcal{B}_3$, then whenever $i \neq j$, we have

$$x_i^T x_j + y_i^T y_j = 0 \quad \text{and} \quad -x_i^T x_j + y_i^T y_j = 0,$$

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which implies that $x_i^T x_j = 0$. Similarly, we have

$$-x_i^T x_i + y_i^T y_i = 0.$$

Since the vectors in $\mathcal{B}_1$ have unit length, it follows that

$$x_i^T x_i + y_i^T y_i = 1,$$

from which we can deduce $x_i^T x_i = 1/2$. Therefore, we just proved that

$$\mathcal{B}_\theta := \left\{ \sqrt{2}x_1, \sqrt{2}x_2, \ldots, \sqrt{2}x_m \right\}$$

is an orthonormal basis of the θ-eigenspace of H.

We first describe a basis of idempotents for $\mathcal{A}(\Gamma)$ using the bases $\mathcal{B}_1$ and $\mathcal{B}_3$. Since the eigenspaces of Γ corresponding to the eigenvalues K and $-K$ are spanned by

$$\begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\mathbb{1} \\ \mathbb{1} \end{bmatrix},$$

respectively, we have

$$E_0 = \frac{1}{N} \begin{bmatrix} J & J \\ J & J \end{bmatrix} \quad \text{and} \quad E_4 = \frac{1}{N} \begin{bmatrix} J & -J \\ -J & J \end{bmatrix},$$

where J is the n by n all-one matrix. By the choice of $\mathcal{B}_1$ and $\mathcal{B}_3$, we can express E_1 and E_3 in the following forms:

$$E_1 = \begin{bmatrix} E_1^{(1,1)} & E_1^{(1,2)} \\ E_1^{(2,1)} & E_1^{(2,2)} \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} E_1^{(1,1)} & -E_1^{(1,2)} \\ -E_1^{(2,1)} & E_1^{(2,2)} \end{bmatrix}.$$

If we set

$$E_2 := I - E_0 - E_1 - E_3 - E_4,$$

then $\{E_0, E_1, E_2, E_3, E_4\}$ is a basis of idempotents for $\mathcal{A}(\Gamma)$. Now we describe a basis of idempotents for $\mathcal{A}(H)$. Set

$$\widehat{E}_0 := \frac{1}{n} J.$$

If we let M_θ to be the matrix whose i-th column is $\sqrt{2}x_i$ (so the columns of M_θ are precisely the vectors in $\mathcal{B}_\theta$), then by setting

$$\widehat{E}_1 := M_\theta M_\theta^T \quad \text{and} \quad \widehat{E}_2 := I - \widehat{E}_0 - \widehat{E}_1,$$

we obtain a basis $\{\widehat{E}_0, \widehat{E}_1, \widehat{E}_2\}$ of idempotents for $\mathcal{A}(H)$. Clearly,

$$\widehat{E}_0 = \frac{2}{N}J = 2E_0^{(1,1)}.$$

By the choice of bases $\mathcal{B}_1$ and $\mathcal{B}_\theta$, we find that

$$\widehat{E}_1 = 2E_1^{(1,1)}.$$

Also,

$$\widehat{E}_2 = I - \widehat{E}_0 - \widehat{E}_1 = I - 2E_0^{(1,1)} - 2E_1^{(1,1)} = E_2^{(1,1)},$$

where the last equality follows from $E_4^{(1,1)} = E_0^{(1,1)}$ and $E_3^{(1,1)} = E_1^{(1,1)}$. Therefore, the relation between the idempotents for $\mathcal{A}(\Gamma)$ and those for $\mathcal{A}(H)$ can be described as follows:

$$\widehat{E}_r = \alpha_r E_r^{(1,1)}, \quad \alpha_0 = \alpha_1 = 2 \quad \text{and} \quad \alpha_2 = 1.$$

Using equation (4.3.1), now we can express the Krein parameters of H in terms of those of Γ.

Theorem 4.3.1. *Let Γ be a bipartite distance-regular graph of diameter four on $2n$ vertices, and let H be a halved graph of Γ. Order their eigenvalues in decreasing order, and with respect to this ordering, let $q_{i,j}(r)$ and $\widetilde{q}_{i,j}(r)$ be the Krein parameters of Γ and H, respectively. If $\alpha_0 = \alpha_1 = 2$ and $\alpha_2 = 1$, then for all $i, j \in \{0, 1, 2\}$, the following equations hold:*

(i) $\widetilde{q}_{i,j}(0) = \alpha_i \alpha_j (q_{i,j}(0) + q_{i,j}(4))/4.$

(ii) $\widetilde{q}_{i,j}(1) = \alpha_i \alpha_j (q_{i,j}(1) + q_{i,j}(3))/4.$

(iii) $\widetilde{q}_{i,j}(2) = \alpha_i \alpha_j (q_{i,j}(2))/2.$

Proof. Earlier in this section, we described bases of idempotents $\{\widehat{E}_0, \widehat{E}_1, \widehat{E}_2\}$

and $\{E_0, E_1, E_2, E_3, E_4\}$ for H and Γ such that $\widehat{E}_r = \alpha_r E_r^{(1,1)}$. It follows that

$$
\begin{aligned}
\widehat{E}_i \circ \widehat{E}_j &= \alpha_i \alpha_j E_i^{(1,1)} \circ E_j^{(1,1)} = \alpha_i \alpha_j \frac{1}{2n} \left(\sum_{r=0}^{4} q_{i,j}(r) E_r^{(1,1)} \right) \\
&= \frac{\alpha_i \alpha_j}{2n} \left((q_{i,j}(0) + q_{i,j}(4)) E_0^{(1,1)} + (q_{i,j}(1) + q_{i,j}(3)) E_1^{(1,1)} + q_{i,j}(2) E_2^{(1,1)} \right) \\
&= \frac{\alpha_i \alpha_j}{2n} \left((q_{i,j}(0) + q_{i,j}(4)) \frac{1}{2} \widehat{E}_0 + (q_{i,j}(1) + q_{i,j}(3)) \frac{1}{2} \widehat{E}_1 + q_{i,j}(2) \widehat{E}_2 \right) \\
&= \frac{1}{n} \left(\frac{\alpha_i \alpha_j}{4} (q_{i,j}(0) + q_{i,j}(4)) \widehat{E}_0 \right. \\
&\quad + \left. \frac{\alpha_i \alpha_j}{4} (q_{i,j}(1) + q_{i,j}(3)) \widehat{E}_1 + \frac{\alpha_i \alpha_j}{2} q_{i,j}(2) \widehat{E}_2 \right).
\end{aligned}
$$

The Krein parameters $\widetilde{q}_{i,j}(r)$ of H are thus as stated. $\qquad\square$

For strongly regular graphs in general, either of the Krein parameters $\widetilde{q}_{1,1}(1)$ and $\widetilde{q}_{2,2}(2)$ can vanish (see [12]). However, only $\widetilde{q}_{1,1}(1)$ can vanish for a halved graph.

Corollary 4.3.2. *If $\widetilde{q}_{i,j}(r)$ denotes the Krein parameters of a halved graph H of a bipartite distance-regular graph Γ of diameter four, then $\widetilde{q}_{2,2}(2) > 0$. Furthermore, $\widetilde{q}_{1,1}(1) = 0$ if and only if Γ is a Hadamard graph, in which case H is a complete graph with a perfect matching removed.*

Proof. Let $q_{i,j}(r)$ denote the Krein parameters of Γ. In Section 4.2 we proved that $q_{2,2}(2) > 0$, so $\widetilde{q}_{2,2}(2) > 0$ by Theorem 4.3.1. In Section 4.2 we also proved that $q_{1,1}(3) = q_{3,3}(3)$ and $q_{3,3}(3) \le q_{1,1}(1)$, so Theorem 4.3.1 implies that $\widetilde{q}_{1,1}(1) = 0$ if and only if $q_{1,1}(1) = 0$. By Theorem 4.2.2, it follows that $\widetilde{q}_{1,1}(1) = 0$ if and only if Γ is a Hadamard graph. Using the fact that $N_4 = 1$ for a Hadamard graph, it is easy to check that H is a complete graph with a perfect matching removed. $\qquad\square$

4.4   Special Krein Parameter: $q_{3,3}(3)$

For a bipartite distance-regular graph Γ of diameter four, we saw that checking the Krein condition is reduced to checking the non-negativity of $q_{3,3}(3)$.

In this section, we attempt to answer the following question: what are the graphs that satisfy $q_{3,3}(3) = 0$? Recall that

$$q_{3,3}(3) = \frac{m_1^2}{N}\beta(3,3,3),$$

where $\beta(3,3,3)$ is equal to

$$1 - \frac{\theta_1^3}{K^2} + \frac{\theta_1^3 c_2^2 c_3^2}{K^2(K-1)^2(K-c_2)^2} + \frac{c_2^2(K-c_3-1)^3}{K^2(K-1)^2} - \frac{c_2^2 c_3^2(K-c_3)}{(K-1)^2(K-c_2)^2}.$$

Since m_1 and N are both positive, $q_{3,3}(3) = 0$ if and only if $\beta(3,3,3) = 0$. Unfortunately, the expression above for $\beta(3,3,3)$ is too complicated to check (or even read), so we seek to find a simpler expression. It turns out that this can be done by re-expressing $\beta(3,3,3)$ in terms of other parameters of the graph. As usual, we use θ_1 to denote the second largest eigenvalue of Γ, and we use θ and τ to denote the nontrivial eigenvalues for its halved graphs, where $\theta > \tau$. Recall that

$$\theta_1^2 = K + c_2(K-c_3-1), \quad \theta = K - c_3 - 1 \quad \text{and} \quad \tau = -\frac{K}{c_2},$$

from which we can deduce

$$c_2 = \frac{\theta_1^2}{\theta-\tau}, \quad K = \frac{-\tau\theta_1^2}{\theta-\tau} \quad \text{and} \quad c_3 = \frac{-\theta_1^2\tau - \theta^2 + \theta\tau - \theta + \tau}{\theta-\tau}. \quad (4.4.1)$$

By making substitutions, the number $\beta(3,3,3)$ can be re-expressed as

$$\frac{(\theta_1\tau - \theta + \tau)(\theta_1\tau + \theta - \tau)^3(\theta_1\tau^2 - \theta_1\theta - \theta^2 + 2\theta_1\tau + \tau^2 - 2\theta + 2\tau)}{\tau^2(\tau+1)^2\theta_1(\theta_1^2\tau + \theta - \tau)^2}.$$

This provides a simpler equivalence condition for $q_{3,3}(3) = 0$.

Lemma 4.4.1. *For a bipartite distance-regular graph of diameter four with second largest eigenvalue θ_1 and nontrivial halved graph eigenvalues θ and τ where $\theta > \tau$, the inequality*

$$\theta_1\tau^2 - \theta_1\theta - \theta^2 + 2\theta_1\tau + \tau^2 - 2\theta + 2\tau \geq 0$$

holds, with equality occurring if and only if $q_{3,3}(3) = 0$.

Proof. Consider the expression of $\beta(3,3,3)$ computed above. Clearly, θ_1, τ^2 and $(\tau+1)^2$ are all positive. On the other hand, since $\theta - \tau = \theta_1^2/c_2$, if K and c_2 are parameters of the graph, then

$$\theta_1\tau \pm (\theta - \tau) = -\theta_1\frac{K}{c_2} \pm \left(\frac{\theta_1^2}{c_2}\right) = \frac{\theta_1}{c_2}(-K \pm \theta_1) < 0,$$

which also implies $(\theta_1^2\tau + \theta - \tau)^2 > 0$. With the expression of $\beta(3,3,3)$, one can see that $\beta(3,3,3)$ has the same sign as

$$\theta_1\tau^2 - \theta_1\theta - \theta^2 + 2\theta_1\tau + \tau^2 - 2\theta + 2\tau,$$

with one being zero if and only if the other also equals zero. $\qquad\square$

Similarly, the number $\beta(1,1,1)$ can be re-expressed as

$$\frac{(\theta_1\tau - \theta + \tau)^3(\theta_1\tau + \theta - \tau)(\theta_1\tau^2 - \theta_1\theta + \theta^2 + 2\theta_1\tau - \tau^2 + 2\theta - 2\tau)}{\tau^2(\tau+1)^2\theta_1(\theta_1^2\tau + \theta - \tau)^2},$$

and the following lemma about $q_{1,1}(1)$ follows.

Lemma 4.4.2. *For a bipartite distance-regular graph of diameter four with second largest eigenvalue θ_1 and nontrivial halved graph eigenvalues θ and τ where $\theta > \tau$, the equation*

$$\theta_1\tau^2 - \theta_1\theta + \theta^2 + 2\theta_1\tau - \tau^2 + 2\theta - 2\tau = 0$$

holds if and only if $q_{1,1}(1) = 0$. $\qquad\square$

In Section 4.2 we saw that the bipartite distance-regular graph of diameter four with $q_{1,1}(1) = 0$ are precisely the Hadamard graphs. While characterizing graphs with $q_{3,3}(3) = 0$ is more complicated in general, it is still very straightforward if we only consider antipodal graphs. We first give a very simple characterization of antipodal graphs in terms of their parameters; this characterization is well-known.

Lemma 4.4.3. *Let (K, c_2, c_3) be parameters of a bipartite distance-regular graph Γ of diameter four. Then $K = c_3 + 1$ if and only if Γ is antipodal.*

Proof. Since the distance-4 relation on the vertices of Γ is the distance-2 relation on the vertices of its halved graphs, it follows that Γ is antipodal if and only if both of its halved graphs are antipodal. Since the halved graphs of Γ are both strongly regular with second largest eigenvalue $K - c_3 - 1$, Lemma 2.3.4 implies that Γ is antipodal if and only if $K - c_3 - 1 = 0$. $\qquad\square$

We now characterize the antipodal graphs with $q_{3,3}(3) = 0$.

Theorem 4.4.4. *For an antipodal bipartite distance-regular graph of diameter four with parameters (K, c_2, c_3), the following conditions are equivalent:*

(i) $K = 2c_2$.

(ii) $q_{3,3}(3) = 0$.

(iii) $p_{4,4}(4) = 0$.

Proof. For an antipodal bipartite distance-regular graph Γ of diameter four, Lemma 4.4.3 implies that $K = c_3 + 1$.

"(i) $\Leftrightarrow$ (ii)": Since the second largest eigenvalue θ of a halved graph equals $K - c_3 - 1$, by Lemma 4.4.1, it follows that $q_{3,3}(3) = 0$ if and only if

$$0 = \theta_1 \tau^2 + 2\theta_1 \tau + \tau^2 + 2\tau = (\theta_1 + 1)(\tau + 2),$$

whence $\tau = -2$, which is equivalent to $K = 2c_2$.

"(i) $\Leftrightarrow$ (iii)": Since being at distance 0 or 4 is a transitive relation on the vertices of Γ, it follows that $p_{4,4}(4) = N_4 - 1$, so $p_{4,4}(4) = 0$ if and only if $N_4 = 1$. Since $K = c_3 + 1$, Lemma 4.2.2 implies that $N_4 = 1$ if and only if $K = 2c_2$. $\square$

Corollary 4.4.5. *An antipodal bipartite distance-regular graph of diameter four satisfies $q_{3,3}(3) = 0$ if and only if it is a Hadamard graph.*

Proof. Since the Hadamard graphs are precisely the bipartite distance-regular graphs of diameter four with $K = 2c_2 = c_3 + 1$, the assertion follows from Lemma 4.4.3 and Theorem 4.4.4. $\square$

The conditions in Theorem 4.4.4 are not equivalent if we remove the antipodal assumption: the folded 8-cube, which has parameters $K = 8$, $c_2 = 2$ and $c_3 = 3$, satisfies $q_{3,3}(3) = 0$ and $p_{4,4}(4) = 18$. There does exist a non-antipodal example in which $q_{3,3}(3) = p_{4,4}(4) = 0$: the graph with parameters $K = 15$, $c_2 = 5$ and $c_3 = 12$, which is described by Brouwer and van Lint [9, p. 108] using the Hoffman-Singleton graph, has vanishing $q_{3,3}(3)$ and $p_{4,4}(4)$. This graph also satisfies $\theta_1 = c_2$, and we will soon see that this condition is related to the vanishing $q_{3,3}(3)$ and $p_{4,4}(4)$. When $q_{3,3}(3) = 0$, the eigenvalue θ_1 can be expressed in terms of θ and τ.

Lemma 4.4.6. *For a non-antipodal bipartite distance-regular graph of diameter four with parameters (K, c_2, c_3), second largest eigenvalue θ_1 and nontrivial halved graph eigenvalues θ and τ where $\theta > \tau$, if $q_{3,3}(3) = 0$ then $\tau^2 - \theta + 2\tau \neq 0$, and*

$$\theta_1 = \frac{(\theta + \tau + 2)(\theta - \tau)}{\tau^2 - \theta + 2\tau}.$$

Proof. Suppose $q_{3,3}(3) = 0$. By Lemma 4.4.1, after re-arranging terms, we have

$$(\tau^2 - \theta + 2\tau)\theta_1 = (\theta + \tau + 2)(\theta - \tau). \qquad (4.4.2)$$

Suppose $\tau^2 - \theta + 2\tau = 0$. Since $\theta - \tau > 0$, we have $\theta + \tau + 2 = 0$. Adding this equation to $\tau^2 - \theta + 2\tau = 0$ yields $\tau^2 + 3\tau + 2 = 0$, whence $\tau \in \{-1, -2\}$. But $\tau = -K/c_2 \neq -1$, so $\tau = -2$. In this case, since $\tau^2 - \theta + 2\tau = 0$, we have $\theta = 0$, so the graph is antipodal by Lemma 4.4.3, which is a contradiction. Therefore $\tau^2 - \theta + 2\tau \neq 0$, and the desired expression of θ_1 follows immediately from equation (4.4.2). $\qquad \square$

To investigate the intersection number $p_{4,4}(4)$, we first make an observation.

Lemma 4.4.7. *Let $\overline{H}$ be the complement of a halved graph H of a bipartite distance-regular graph Γ of diameter four. Suppose that $\widetilde{a}$ is the number of common neighbours of two adjacent vertices in $\overline{H}$. Then $p_{4,4}(4) = \widetilde{a}$.*

Proof. By Lemma 2.3.1, the diameter of H is two, so two vertices are at distance 2 in H if and only if they are adjacent in $\overline{H}$. Since the distance-4 relation on Γ is the distance-2 relation on H, it follows from definitions that $p_{4,4}(4) = \widetilde{a}$. $\qquad \square$

By using the formulae in Lemma 2.3.5 and Lemma 3.4.1, it follows that $\widetilde{a}$ equals

$$\frac{K^3 - (c_2 + 2c_3 + 1)K^2 + (c_3^2 + c_2 c_3 + 2c_3 + c_2)K - 2c_2 c_3}{c_2 c_3}. \qquad (4.4.3)$$

Using this, we can express a condition equivalent to $p_{4,4}(4) = 0$.

Lemma 4.4.8. *Let* (K, c_2, c_3) *be parameters of a bipartite distance-regular graph of diameter four, with second largest eigenvalue* θ_1 *and nontrivial halved graph eigenvalues* θ *and* τ *where* $\theta > \tau$. *Then*

$$\theta_1^2 \theta \tau - \theta^3 \tau + \theta_1^2 \tau^2 + \theta^2 \tau^2 + 2\theta_1^2 \tau + 2\theta^2 - \theta \tau - \tau^2 + 2\theta - 2\tau = 0$$

if and only if $p_{4,4}(4) = 0$.

Proof. By substituting the expressions for K, c_2 and c_3 in (4.4.1) into the number (4.4.3) above, it follows that $p_{4,4}(4)$ equals

$$\frac{\theta_1^2 \theta \tau - \theta^3 \tau + \theta_1^2 \tau^2 + \theta^2 \tau^2 + 2\theta_1^2 \tau + 2\theta^2 - \theta \tau - \tau^2 + 2\theta - 2\tau}{-\theta_1^2 \tau - \theta^2 + \theta \tau - \theta + \tau}. \qquad (4.4.4)$$

□

Now we can derive the relations between θ_1, $q_{3,3}(3)$, and $p_{4,4}(4)$.

Theorem 4.4.9. *For a non-antipodal bipartite distance-regular graph of diameter four with parameters* (K, c_2, c_3) *and second largest eigenvalue* θ_1, *any two of the following conditions imply the third:*

(i) $q_{3,3}(3) = 0$.

(ii) $p_{4,4}(4) = 0$.

(iii) $\theta_1 = c_2$.

Proof. Suppose $\theta_1 = c_2$. Since $c_2 = \theta_1^2/(\theta - \tau)$, we have $\theta_1 = \theta - \tau$. By substituting this into the expression in Lemma 4.4.8, one can see that $p_{4,4}(4) = 0$ if and only if

$$(\tau + 1)(\tau - \theta)(\tau^2 - 2\theta + \tau - 2) = 0,$$

which is equivalent to $\tau^2 - 2\theta + \tau - 2 = 0$ since $\tau \leq -2$. On the other hand, since $\theta_1 = \theta - \tau$, by Lemma 4.4.6 we have $q_{3,3}(3) = 0$ if and only if

$$\frac{(\theta + \tau + 2)(\theta - \tau)}{\tau^2 - \theta + 2\tau} = \theta - \tau. \qquad (4.4.5)$$

By dividing both sides by $\theta - \tau$, one can see that (4.4.5) holds if and only if $\tau^2 - 2\theta + \tau - 2 = 0$. Hence conditions (i) and (ii) are equivalent given (iii). It

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remains to show that conditions (i) and (ii) imply (iii). Suppose $q_{3,3}(3) = 0$ and $p_{4,4}(4) = 0$. By Lemma 4.4.6, we have

$$\theta_1 = \frac{(\theta + \tau + 2)(\theta - \tau)}{\tau^2 - \theta + 2\tau}. \qquad (4.4.6)$$

Substituting this into expression (4.4.4) yields

$$\frac{\theta(\tau + 1)^2(\tau^2 - 2\theta + \tau - 2)}{\tau^3 + \theta^2 - \theta\tau + 2\tau^2 + \theta} = 0.$$

Since the graph is non-antipodal, $\theta \neq 0$, so $\tau^2 - 2\theta + \tau - 2 = 0$. This implies

$$\theta - \tau = \frac{(\theta + \tau + 2)(\theta - \tau)}{\tau^2 - \theta + 2\tau}.$$

Together with (4.4.6) and $c_2 = \theta_1^2/(\theta - \tau)$, we have $c_2 = \theta_1$. $\qquad\square$

Unfortunately, this does not give an equivalence condition for $q_{3,3}(3) = 0$, and even worse, the condition $p_{4,4}(4) = 0$ is very strong. Indeed, if $p_{4,4}(4) = 0$ holds for a bipartite distance-regular graph of diameter four and $\overline{H}$ is the complement of a halved graph, then $\widetilde{a} = 0$ holds for $\overline{H}$ by Lemma 4.4.7, in which case $\overline{H}$ contains no cycle of length three and is called *triangle-free*. Though there exist infinitely many "feasible" parameter sets for triangle-free strongly regular graphs (Biggs [4]), there are only seven known such graphs assuming connectedness and non-antipodality. For concise descriptions for these seven graphs, see [50]. Among these seven graphs, only the the Hoffman-Singleton graph, which has parameters

$$\left(n, \widetilde{k}, \widetilde{a}, \widetilde{c}\right) = (50, 7, 0, 1),$$

is the complement of a halved graph. The corresponding bipartite graph has parameters $K = 15$, $c_2 = 5$ and $c_3 = 12$, and is the only bipartite graph of diameter four with these parameters ([6, Theorem 13.1.1]). Consequently, the graph with parameters $(K, c_2, c_3) = (15, 5, 12)$ is the only known non-antipodal bipartite distance-regular graph of diameter four with $p_{4,4}(4) = 0$. None of the other six triangle-free graphs is the complement of a halved graph: this can be shown by directly computing K, c_2 and c_3 using the formulae in Lemma 3.4.2 and checking the conditions in Lemma 3.3.1. We omit the details of the computations and summarize the parameters of these six graphs, with violated conditions, in Table 4.1.

Table 4.1: Triangle-free strongly regular graphs that are not complements of halved graphs

$(n, \widetilde{k}, \widetilde{a}, \widetilde{c})$	(n, k, a, c)	Violated Condition
(5,2,0,1)	(5,2,0,1)	$c_2 \notin \mathbb{Z}$
(10,3,0,1)	(10,6,3,4)	Lemma 3.3.1 (v)
(16,5,0,2)	(16,10,6,6)	Lemma 3.3.1 (v)
(56,10,0,2)	(56,45,36,36)	$c_2 \notin \mathbb{Z}$
(77,16,0,4)	(77,60,47,45)	$q_{3,3}(3) < 0$
(100,22,0,6)	(100,77,60,56)	$c_2 \notin \mathbb{Z}$

4.5 Parameters and Graphs with $q_{3,3}(3) = 0$

After the unsuccessful attempt of finding a characterization for graphs with $q_{3,3}(3) = 0$, we seek to find the answer of the following question: are there infinitely many bipartite distance-regular graphs of diameter four satisfying $q_{3,3}(3) = 0$? The answer turns out to be affirmative, and in this section we shall see two infinite families of parameter sets that yield a vanishing $q_{3,3}(3)$, one of which contains an infinite sub-family that can be realized by (known) graphs.

A distance-regular graph is said to have *classical parameters* (d, t, α, β) if the graph has diameter d and (ordinary) parameters

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_t \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_t \right),$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_t - \begin{bmatrix} i \\ 1 \end{bmatrix}_t \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_t \right)$$

for $i \in \{0, 1, \ldots, d\}$, where

$$\begin{bmatrix} j \\ 1 \end{bmatrix}_t := 1 + t + t^2 + \cdots + t^{j-1}$$

is the Gaussian binomial coefficient. For more details of graphs with classical parameters, the readers are referred to [6]. With the definition, the potential classical parameters $(4, q, 0, 1)$ then give

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q \quad \text{and} \quad b_i = \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q - \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right).$$

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In particular, $b_i + c_i = b_0$ for all i with $0 \leq i \leq 4$, so any graph realizing the classical parameters $(4, q, 0, 1)$ must be bipartite. For these classical parameters, some important ordinary parameters are computed as follows:

$$K = 1 + q + q^2 + q^3, \quad c_2 = 1 + q, \quad c_3 = 1 + q + q^2,$$

$$\theta_1 = q(q+1), \quad \theta = q^3 - 1, \quad \tau = -(q^2 + 1).$$

Using these parameters and Lemma 4.4.1, it is straightforward to verify that $q_{3,3}(3) = 0$. Moreover, there are graphs realizing these parameters. Indeed, when $q = 1$, we get $K = 4$, $c_2 = 2$ and $c_3 = 3$, and the unique graph realizing these parameters is the 4-cube. For uniqueness, see [6, Sect. 9.2C]. When q is a prime power, the dual polar graph $D_4(q)$ is distance-regular and has the parameters above; this graph is constructed using maximal totally isotropic subspaces of $\mathbb{F}_q^8$ with respect to some quadratic form, and we omit the details here. For a proof of distance-regularity and correctness of the parameters, see [6, Sect. 9.4]. The dual polar graphs $D_4(q)$ then form an infinite family of graphs with $q_{3,3}(3) = 0$.

We look at a different family of parameters. Let m be an integer with $m \geq 2$, and consider the triple (K, c_2, c_3) with

$$K = \frac{2^m (2^{m-2} + 1)(2^m - 1)}{3}, \quad c_2 = \frac{(2^{m-2} + 1)(2^m - 1)}{3},$$

$$c_3 = \frac{(2^{m-1} - 1)(2^{m-1} + 1)(2^m - 1)}{3}.$$

Graphs realizing these parameters must satisfy

$$\theta_1 = \frac{2(2^m - 1)(2^{m-2} + 1)}{3}, \quad \theta = \frac{2^{2m} - 4}{3}, \quad \tau = -2^m,$$

Using which we can verify that $q_{3,3}(3) = 0$. When $m = 2$, the parameters $(K, c_2, c_3) = (8, 2, 3)$ are uniquely realized by the folded 8-cube; see [6, Sect. 9.2D]. Unlike the first case, there is no infinite family of graphs realizing the parameters in the second family within the author's knowledge, though the parameters satisfy all the feasibility conditions in Lemma 3.3.1 except possibly (iii): the numbers N_i and the halved graph parameters all passed the integrality test, but the other intersection numbers are yet to be tested. Whether there exists such an infinite family of graphs is an interesting open problem.

4.6 Q-Polynomial Graphs

Let Γ be a bipartite distance-regular graph of diameter four with parameters (K, c_2, c_3) and second largest eigenvalue θ_1. The goal of this section is to show that Γ is Q-polynomial if and only if $q_{3,3}(3) = 0$. We emphasize that $q_{3,3}(3)$ here is obtained with the usual decreasing ordering on the eigenvalues, which has nothing to do with the Q-polynomial ordering. Recall that the standard sequence of an eigenvalue λ of Γ can be computed as follows:

$$\sigma_0 := 1, \quad \sigma_1 := \frac{\lambda}{K}, \quad \sigma_{i+1} := \frac{(\lambda - a_i)\sigma_i - c_i \sigma_{i-1}}{b_i} \quad (1 \le i \le 3).$$

Since Γ is bipartite, the numbers a_i are all zeroes and $b_i = K - c_i$. Computing using these formulae and numbers, we obtain all the standard sequences of Γ, which are represented as columns by the following matrix:

$$
\begin{array}{c}
 \\
\sigma_0 \\
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4
\end{array}
\begin{array}{ccccc}
K & \theta_1 & 0 & -\theta_1 & -K \\
\left[\begin{array}{c}1 \\ 1 \\ 1 \\ 1 \\ 1\end{array}\right. & \begin{array}{c}1 \\ \frac{\theta_1}{K} \\ \frac{(K-c_3-1)c_2}{K(K-1)} \\ -\frac{\theta_1 c_2 c_3}{K(K-1)(K-c_2)} \\ -\frac{c_2 c_3}{(K-1)(K-c_2)}\end{array} & \begin{array}{c}1 \\ 0 \\ -\frac{1}{K-1} \\ 0 \\ \frac{c_3}{(K-1)(K-c_3)}\end{array} & \begin{array}{c}1 \\ -\frac{\theta_1}{K} \\ \frac{(K-c_3-1)c_2}{K(K-1)} \\ \frac{\theta_1 c_2 c_3}{K(K-1)(K-c_2)} \\ -\frac{c_2 c_3}{(K-1)(K-c_2)}\end{array} & \begin{array}{c}1 \\ -1 \\ 1 \\ -1 \\ 1\end{array}\left.\right]
\end{array}.
$$

We will use this matrix and Theorem 2.7.1 to show the main result. Before doing so, we first simplify condition (ii) in Theorem 2.7.1.

Lemma 4.6.1. *For a sequence $(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ of distinct real numbers, the system*

$$
\begin{cases}
s\sigma_1 + t = \sigma_2 + 1 \\
s\sigma_2 + t = \sigma_3 + \sigma_1 \\
s\sigma_3 + t = \sigma_4 + \sigma_2
\end{cases}
$$

has a solution in (s, t) if and only if

$$(\sigma_1 - \sigma_4)(\sigma_1 - \sigma_2) = (\sigma_2 - \sigma_3)(1 - \sigma_3).$$

Proof. Row reducing the augmented matrix of the system yields

$$\left[\begin{array}{cc|c} \sigma_2 & 1 & \sigma_3 + 1 \\ \sigma_1 - \sigma_2 & 0 & \sigma_2 + 1 - \sigma_3 - \sigma_1 \\ \sigma_3 - \sigma_2 & 0 & \sigma_4 + \sigma_2 - \sigma_3 - \sigma_1 \end{array}\right].$$

From the last two rows of the reduced matrix, it follows that the system has a solution if and only if

$$(\sigma_1 - \sigma_4)(\sigma_1 - \sigma_2) = (\sigma_2 - \sigma_3)(1 - \sigma_3).$$

$\square$

Now we show the main result of this section.

Theorem 4.6.2. *A bipartite distance-regular graph Γ of diameter four is Q-polynomial if and only if $q_{3,3}(3) = 0$.*

Proof. Assume that $q_{3,3}(3) = 0$, and let σ be the standard sequence of the second largest eigenvalue θ_1 of Γ. It is clear that $\sigma_i \neq \sigma_j$ whenever $i \neq j$, except for $(i, j) = (1, 2)$. Since

$$\sigma_1 = \frac{\theta_1}{K} \quad \text{and} \quad \sigma_2 = \frac{(K - c_3 - 1)c_2}{K(K - 1)},$$

by re-expressing σ_1 and σ_2 using (4.4.1), it follows that $\sigma_1 = \sigma_2$ if and only if

$$\theta_1 = \frac{\theta \theta_1^2}{-\tau \theta_1^2 - \theta + \tau},$$

whence

$$-(1 + \theta_1)(\theta_1 \tau + \theta - \tau) = 0.$$

We saw in the proof of Lemma 4.4.1 that $(\theta_1 \tau + \theta - \tau) \neq 0$, so $\sigma_1 \neq \sigma_2$, and therefore σ satisfies condition (i) in Theorem 2.7.1. To show that σ also satisfies condition (ii) in the same theorem, it suffices to show that

$$(\sigma_1 - \sigma_4)(\sigma_1 - \sigma_2) = (\sigma_2 - \sigma_3)(1 - \sigma_3) \qquad (4.6.1)$$

holds, due to Lemma 4.6.1. By rewriting σ in terms of θ_1, θ and τ using formulae (4.4.1), equation (4.6.1) is equivalent to

$$\frac{(\tau - \theta)(\tau \theta_1 - \tau + \theta)^2 (\tau \theta_1 + \tau - \theta)(\theta_1 \tau^2 - \theta_1 \theta - \theta^2 + 2\theta_1 \tau + \tau^2 - 2\theta + 2\tau)}{(\tau + 1)^2 (\tau \theta_1^2 - \tau + \theta)^2 \tau^2 \theta_1^2}$$

being zero, which is indeed the case by Lemma 4.4.1. Therefore σ also satisfies condition (ii) in Theorem 2.7.1, and Γ is Q-polynomial.

Conversely, assume that Γ is Q-polynomial. Then the standard sequence σ of some eigenvalue λ satisfies the two conditions in Theorem 2.7.1. Clearly λ equals θ_1 or $-\theta_1$ (since the others sequences have repeated terms). If $\lambda = \theta_1$, then the argument in the first direction shows that $q_{3,3}(3) = 0$. If $\lambda = -\theta_1$, then by expressing σ in terms of θ_1, θ and τ, equation (4.6.1) is equivalent to

$$\theta_1 \tau^2 - \theta_1 \theta + \theta^2 + 2\theta_1 \tau - \tau^2 + 2\theta - 2\tau = 0,$$

which is equivalent to $q_{1,1}(1) = 0$ by Lemma 4.4.2. Since $q_{1,1}(1) \geq q_{3,3}(3) \geq 0$, it follows that $q_{3,3}(3) = 0$. $\qquad\square$

Chapter 5

Sets of Unit Vectors with Few Inner Products

5.1 Introduction

We equip the vector space $\mathbb{C}^m$ with the usual inner product: the *inner product* of vectors x and y in $\mathbb{C}^m$ is defined to be x^*y, where x^* is the conjugate transpose of x. The *degree set* of a set S of (at least two) unit vectors in $\mathbb{C}^m$ is the set

$$\left\{ |x^*y|^2 : x, y \in S, x \neq y \right\}.$$

For $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$, a set of unit vectors is called a $\{0, \alpha\}$-*set* if its degree set is a subset of $\{0, \alpha\}$.

The concept of $\{0, \alpha\}$-set is motivated by equiangular sets and mutually unbiased bases. A set of unit vectors in $\mathbb{C}^m$ is called *equiangular* if all the inner products between distinct vectors in the set have the same absolute value. A collection of orthonormal bases of $\mathbb{C}^m$ or $\mathbb{R}^m$ are called *mutually unbiased* if all the inner products between vectors from different bases in the collection have the same absolute value. Thus equiangular sets and sets of vectors from a collection of mutually unbiased bases are examples of $\{0, \alpha\}$-sets. Equiangular sets and mutually unbiased bases have applications in various areas. For example, in quantum physics, it is desirable to recover the state of a physical system by means of measurements, and equiangular sets and mutually unbiased bases provide "good" measurements for such purposes. More details about applications of equiangular sets and mutually unbiased bases can be found in Roy's thesis [41]. Recently, Wu and Parker [49] proposed

to use mutually unbiased bases to construct complementary sets, which are sets of sequences satisfying certain correlations; complementary sets have applications in communication theory. Because of the applications, equiangular sets and mutually unbiased bases have received much attention. Many constructions for such sets were found; see [10, 26, 29, 33, 34, 42], for examples. Constructions of general $\{0, \alpha\}$-sets, however, received little attention. Several examples of $\{0, \alpha\}$-sets that are not equiangular sets or mutually unbiased bases are referred in [21]: two examples in $\mathbb{C}^5$ and $\mathbb{C}^9$ come from the regular two-graph described in [28], and another example in $\mathbb{C}^{28}$ comes from the so-called Rudvalis group constructed in [16]. There are four other examples referred in [21] lying in $\mathbb{R}^8$, $\mathbb{R}^{23}$, $\mathbb{C}^4$ and $\mathbb{C}^6$. Constructing $\{0, \alpha\}$-sets in general is one main theme of this chapter.

In 2005, Roy presented in his Ph.D. thesis [41] a method for constructing sets of unit vectors using bipartite graphs; this method is a joint work by Godsil and Roy. In order to construct $\{0, \alpha\}$-sets using Godsil and Roy's construction, we need a connected bipartite graph with exactly five distinct eigenvalues and a certain group of automorphisms. Considering these conditions, there are two natural classes of graphs to investigate, namely the bipartite distance-regular graphs of diameter four and the bipartite covers of linear Cayley graphs: the former ones have the right number of eigenvalues, while the latter ones possess suitable groups of automorphisms.

We first review Godsil and Roy's construction in Section 5.2. We then present new results briefly summarized as follows. Regarding distance-regular graphs, we present examples of such graphs, including an infinite family, that provide new constructions of $\{0, \alpha\}$-sets. We then show that certain upper bounds for the sizes of $\{0, \alpha\}$-sets proven by Delsarte, Goethals and Seidel [21] can be improved for sets of flat vectors, and show that the 8-cycle, the 4-cube, the folded 8-cube, and the coset graph of the extended binary Golay code are the only distance-regular graphs that provide $\{0, \alpha\}$-sets of maximum size with respect to the improved bounds. These results above regarding distance-regular graphs and bounds of $\{0, \alpha\}$-sets are published in [30]. Regarding bipartite covers $\widehat{\Gamma}$ of linear Cayley graphs, we derive a geometric condition and a coding theoretic condition for $\widehat{\Gamma}$ to have five distinct eigenvalues. In the last section, we will see how new constructions of $\{0, \alpha\}$-sets can be derived from old ones.

5.2 A Construction from Graphs

In this section, we review a construction of sets of unit vectors using certain graphs due to Godsil and Roy (presented in Roy's Ph.D. thesis [41, Ch. 4]). No result in this section is new. Let $(G, +)$ be a group. A *character* of G is a group homomorphism from G to the (multiplicative) group of complex numbers with norm 1. For any character χ of G and $S \subseteq G$, let $\chi(S) := \sum_{g \in S} \chi(g)$. Any nontrivial character satisfies a simple property, which we will use later.

Lemma 5.2.1. *If χ is a nontrivial character of G, then $\chi(G) = 0$.*

Proof. Let h be an element of G such that $\chi(h) \neq 1$. Then

$$\chi(G) = \sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(h + g) = \sum_{g \in G} \chi(h)\chi(g) = \chi(h)\chi(G).$$

This is true only if $\chi(G) = 0$. $\qquad\qquad\square$

For characters χ and ϕ of G, let $\chi \circ \phi$ be the function on G defined by $g \mapsto \chi(g)\phi(g)$. It is straightforward to check that $\chi \circ \phi$ is also a character, and that the set $\mathrm{char}(G)$ of characters of G forms an abelian group with the operation $\circ$. In fact, much more can be said if G is finite and abelian.

Lemma 5.2.2. *If G is a finite abelian group, then $\mathrm{char}(G) \cong G$.* $\qquad\square$

See [25, Sect. 12.8] for an outline of a proof. Let $D \subseteq G$. The *Cayley graph* of G with *connection set* D, denoted by $\Gamma(G, D)$, is the directed graph (possibly with loops) that has vertex set G and arc set

$$\{(g, g + h) : g \in G, h \in D\}.$$

Note that a character of G is just a vector in $\mathbb{C}^G$. The following result in [25, Sect. 12.9] describes a relation between the graph $\Gamma(G, D)$ and the characters of G when G is finite and abelian.

Lemma 5.2.3. *Let G be a finite abelian group. Then the characters of G form a set of eigenvectors of $\Gamma(G, D)$ that are linearly independent over $\mathbb{C}$. The eigenvalue of a character χ as an eigenvector is $\chi(D)$.* $\qquad\square$

Since there are as many characters of G as elements of G by Lemma 5.2.2, we immediately have the following lemma.

Lemma 5.2.4. *The set of eigenvalues of $\Gamma(G, D)$ is $\{\chi(D) : \chi \in \mathrm{char}(G)\}$.*

<div align="right">□</div>

For any $\chi \in \mathrm{char}(G)$ and $S \subseteq G$, let $\chi|_S$ be the restriction of χ to S, and let $\overline{\chi}$ be the function on G defined by $g \mapsto \overline{\chi(g)}$; it is easy to check that $\overline{\chi} \in \mathrm{char}(G)$. Using characters, from any Cayley graph on an abelian group, Godsil and Roy constructed a set of unit vectors whose degree set is closely related to the set of eigenvalues of the graph.

Theorem 5.2.5 (Godsil and Roy). *Let G be an abelian group of size n, and let D be a generating subset of G with $|D| = k$. If Γ is the Cayley graph $\Gamma(G, D)$, then*

$$S := \left\{ \frac{1}{\sqrt{k}} \chi|_D : \chi \in \mathrm{char}(G) \right\}$$

is a set of n unit vectors in $\mathbb{C}^k$ with degree set

$$\left\{ \left(\frac{|\lambda|}{k} \right)^2 : \lambda \text{ is an eigenvalue of } \Gamma, \lambda \neq k \right\}.$$

Proof. Since a character maps each group element to a norm-one complex number, the elements of S are indeed unit vectors in $\mathbb{C}^k$ since $|D| = k$. Since D is generating, $|S| = |\mathrm{char}(G)|$. Hence, by Lemma 5.2.2, S has size n. Let χ and ϕ be distinct characters of G. Then

$$\left| (\frac{1}{\sqrt{k}} \chi|_D)^* (\frac{1}{\sqrt{k}} \phi|_D) \right|^2 = \frac{1}{k^2} \left| (\chi|_D)^* (\phi|_D) \right|^2$$

$$= \frac{1}{k^2} \left| \sum_{g \in D} \overline{\chi(g)} \phi(g) \right|^2 = \frac{1}{k^2} \left| (\overline{\chi} \circ \phi)(D) \right|^2.$$

Since $\overline{\chi}$ is the inverse of χ in $\mathrm{char}(G)$, it follows that $\overline{\chi} \circ \phi$ ranges over all the non-identity characters of G as (χ, ϕ) ranges over all the pairs of distinct characters. Since D generates G, it follows that $(\overline{\chi} \circ \phi)(D) = k$ if only if $\overline{\chi} \circ \phi$ is the identity character. By Lemma 5.2.4, the degree set of S is then indeed as stated. □

For a group G and $D \subseteq G$, define the bipartite graph $\widehat{\Gamma}(G, D)$ as follows. The two colour classes of $\widehat{\Gamma}(G, D)$ are $G \times \{0\}$ and $G \times \{1\}$. Vertices $(g, 0)$ and $(h, 1)$ are adjacent in $\widehat{\Gamma}(G, D)$ if and only if $h = g + d$ for some $d \in D$. There is a simple relation between the eigenvalues of $\widehat{\Gamma}(G, D)$ and those of $\Gamma(G, D)$.

Lemma 5.2.6. *If λ is an eigenvalue of $\Gamma(G, D)$, then $|\lambda|$ and $-|\lambda|$ are both eigenvalues of $\widehat{\Gamma}(G, D)$. Furthermore, all eigenvalues of $\widehat{\Gamma}(G, D)$ arise in this manner.*

Proof. Let D^{-1} be the set $\{-d : d \in D\}$, and let A be the adjacency matrix of $\Gamma(G, D)$. Then the adjacency matrix of $\Gamma(G, D^{-1})$ is A^T. If $\chi \in \mathrm{char}(G)$ then Lemma 5.2.3 implies that $A^T \chi = \chi(D^{-1})\chi$. Since

$$\chi(D^{-1}) = \sum_{d \in D} \chi(-d) = \sum_{d \in D} \overline{\chi(d)} = \overline{\chi(D)},$$

$A^T \chi = \overline{\chi(D)}\chi$. Let λ be an eigenvalue of $\Gamma(G, D)$. Then $\lambda = \chi(D)$ for some $\chi \in \mathrm{char}(G)$ by Lemma 5.2.4. By definition, the adjacency matrix of $\widehat{\Gamma}(G, D)$ can be written as

$$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}.$$

It is then straightforward to check that

- if $\lambda \neq 0$ then $(\lambda\chi, |\lambda|\chi)^T$ and $(\lambda\chi, -|\lambda|\chi)^T$ are eigenvectors of $\widehat{\Gamma}(G, D)$ with eigenvalues $|\lambda|$ and $-|\lambda|$, respectively;

- if $\lambda = 0$ then $(\chi, 0)^T$ and $(0, \chi)^T$ are eigenvectors of $\widehat{\Gamma}(G, D)$ with eigenvalue 0.

Hence, $|\lambda|$ and $-|\lambda|$ are both eigenvalues of $\widehat{\Gamma}(G, D)$. Finally, since the characters of G are linearly independent over $\mathbb{C}$ by Lemma 5.2.3, it is straightforward to check that the $2|G|$ vectors described above (with χ ranged over all the characters) are also linearly independent. Therefore all the eigenvalues of $\widehat{\Gamma}(G, D)$ have been described. $\square$

If the group G acts on a set S, we say that G acts on S *regularly* if for any $a, b \in S$, there exists a unique $g \in G$ such that $a^g = b$. Lemma 5.2.6 allows us to give a construction of sets of vectors using bipartite graphs with regular group actions.

Theorem 5.2.7 (Godsil and Roy). *Suppose that a connected bipartite graph $\widehat{\Gamma}$ with colour classes Y and Z has an abelian group G of automorphisms acting regularly on each of Y and Z. Let $y \in Y$ and $z \in Z$, and let*

$$D := \big\{ g \in G : z^g \text{ is adjacent to } y \text{ in } \widehat{\Gamma} \big\}.$$

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If k is the valency of $\widehat{\Gamma}$ and n is the size of a colour class, then

$$S := \left\{ \frac{1}{\sqrt{k}} \chi|_D : \chi \in \mathrm{char}(G) \right\}$$

is a set of n unit vectors in $\mathbb{C}^k$ with degree set

$$\left\{ \frac{\lambda^2}{k^2} : \lambda \text{ is an eigenvalue of } \widehat{\Gamma}, \lambda \neq \pm k \right\}.$$

Proof. We first show that $\widehat{\Gamma}$ is indeed regular. Let u and v be vertices in the same colour class. Then $v = u^h$ for some $h \in G$. Since h is an automorphism of $\widehat{\Gamma}$, it defines an injection from the set of neighbours of u to the set of neighbours of v. Since the choice of u and v are arbitrary within a colour class, it follows that $\widehat{\Gamma}$ must be regular.

Since G acts on each colour class regularly, each vertex in Y has the form y^g for some unique $g \in G$ and each vertex in Z has the form z^h for some unique $h \in G$. Since the neighbours of y^g are z^{g+x} where $x \in G$, by identifying each y^g with g and each z^h with h, it follows that $\widehat{\Gamma} \cong \widehat{\Gamma}(G, D)$. Since $\widehat{\Gamma}$ is connected, D generates G. By Theorem 5.2.5, the degree set of S is

$$\left\{ \left(\frac{|\lambda|}{k} \right)^2 : \lambda \text{ is an eigenvalue of } \Gamma(G, D), \lambda \neq k \right\}.$$

Since $\widehat{\Gamma}$ is connected, k and $-k$ are simple eigenvalues, so the only eigenvalue of $\Gamma(G, D)$ that has absolute value k is k itself by Lemma 5.2.6. Therefore, since all the eigenvalues of $\widehat{\Gamma}$ are real, by Lemma 5.2.6 the degree set of S can be rewritten as

$$\left\{ \frac{\lambda^2}{k^2} : \lambda \text{ is an eigenvalue of } \widehat{\Gamma}, \lambda \neq \pm k \right\}.$$

$\square$

In the construction above, if all the non-identity elements of G have order two, then the range of a character of G is a subset of $\{1, -1\}$; in this case, the set S constructed from G is a subset of $\mathbb{R}^k$.

5.3 $\{0, \alpha\}$-Sets from Distance-Regular Graphs

In this section, we present some distance-regular graphs that can be used to construct $\{0, \alpha\}$-sets using Godsil and Roy's construction in Theorem 5.2.7.

To co-operate with other sections in this chapter, we use k to denote the valency of a distance-regular graph instead of K which was used in previous chapters; we will not talk about halved graphs to avoid confusions.

In order to construct a $\{0, \alpha\}$-set, we need a bipartite graph that has exactly five distinct eigenvalues 0, $\pm\theta_1$ and $\pm k$; distance-regular graphs satisfying this property are precisely those that have diameter four, by Lemma 2.2.5. All graphs described in this section give new constructions of $\{0, \alpha\}$-sets, except for the 8-cycle, which is a special case of the graphs used by Godsil and Roy in [26]. Note that although the constructions are new, the resulting sets may be known. In particular, the $\{0, \alpha\}$-sets that will be constructed from the 8-cycle and the 4-cube are sets of mutually unbiased bases, which were already constructed by others using different methods.

Let X be a set with an element called "zero". The *weight* of an element in X^m is defined to be the number of its nonzero entries. The *Hamming distance* between two elements in X^m is the number of coordinates in which they differ. Let $\mathbb{F}_q$ be the finite field with q elements. Then $\mathbb{F}_q^m$ is a vector space over $\mathbb{F}_q$. Let C be a subspace of $\mathbb{F}_q^m$ such that every nonzero element in C has weight at least two. The *coset graph* of C (with respect to $\mathbb{F}_q^m$) is the graph with vertex set being the set $\mathbb{F}_q^m/C$ of all cosets of C, such that two cosets are adjacent if and only if their difference can be represented by a weight-one element in $\mathbb{F}_q^m$.

8-Cycle

The 8-cycle is usually defined to have vertex set $\mathbb{Z}_8$. To emphasize the group action under $\mathbb{Z}_4$, we give a slightly unconventional, yet equivalent, definition of it. Let $D := \{0, 1\}$ be a subset of $\mathbb{Z}_4$. The 8-cycle is the bipartite graph $\widehat{\Gamma}(\mathbb{Z}_4, D)$, which (by definition) has colour classes $\mathbb{Z}_4 \times \{0\}$ and $\mathbb{Z}_4 \times \{1\}$. It is the unique bipartite distance-regular graph of diameter four with parameters $k = 2$, $c_2 = 1$ and $c_3 = 1$. Its nontrivial eigenvalue is $\theta_1 = \sqrt{2}$. The group $G := (\mathbb{Z}_4, +)$ acts on the vertices of the 8-cycle by addition to the first coordinate (without changing the second), and each colour class is an orbit induced by the actions. Since G is clearly a group of automorphisms of the graph, by Lemma 5.2.7, we can construct a $\{0, 1/2\}$-set of size 4 in $\mathbb{C}^2$, represented by the columns of the matrix

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \end{bmatrix}.$$

In fact, these vectors can be partitioned into two unbiased orthonormal bases of $\mathbb{C}^2$, with a basis containing the first two columns. The construction of vectors from the 8-cycle has essentially been described in [26], in which the 8-cycle is viewed as the incidence graph of the affine plane of order two having the lines with infinite slope removed.

4-Cube

The 4-*cube* is the graph with vertex set $\mathbb{Z}_2^4$, such that two vertices are adjacent if and only if they have Hamming distance 1. It is straightforward to check that the 4-cube is a bipartite distance-regular graph of diameter four having 16 vertices, with parameters $k = 4$, $c_2 = 2$, $c_3 = 3$ and nontrivial eigenvalue $\theta_1 = 2$. Moreover, it is the unique bipartite distance-regular graph of diameter four with such parameters; see [6, Sect. 6.1]. The odd-weight elements form a colour class and the even-weight ones form the other. The set of even-weight elements is an additive abelian group G acting regularly on the colour classes by addition, and is clearly a group of automorphisms of the graph. Since G is a subgroup of $\mathbb{Z}_2^4$, every non-identity element of it has order two. Therefore, using Lemma 5.2.7, we can construct a $\{0, 1/4\}$-set of size 8 in $\mathbb{R}^4$, represented by the columns of the matrix

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

Again, these vectors can be partitioned into two unbiased orthonormal bases of $\mathbb{R}^4$, with a basis containing the first four columns.

Folded 8-Cube

The *folded* 8-*cube* is the graph with vertex set $\mathbb{Z}_2^7$, such that two vertices are adjacent if any only if they have Hamming distance 1 or 7. It is straightforward to check that the folded 8-cube is a bipartite distance-regular graph of diameter four having 128 vertices, with parameters $k = 8$, $c_2 = 2$, $c_3 = 3$ and nontrivial eigenvalue $\theta_1 = 4$. Moreover, it is the unique bipartite distance-regular graph of diameter four with such parameters; see [6, Sect. 9.2D]. The odd-weight elements form a colour class of the even-weight ones form the other. Similar to the 4-cube, the even-weight strings form an abelian group

G of graph automorphisms acting regularly on the colour classes by addition. Using Lemma 5.2.7, we can construct a $\{0, 1/4\}$-set of size 64 in $\mathbb{R}^8$ (since $G \leq \mathbb{Z}_2^7$).

Van Lint-Schrijver Partial Geometry

Let C be the subspace of $\mathbb{F}_3^6$ spanned by the all-one vector. Then any coset of C in $\mathbb{F}_3^6$ has its elements sharing the same coordinate sum (computed in $\mathbb{F}_3$). For $i \in \{0, 1, 2\}$, let V_i be the set of cosets of C whose elements have coordinate sum i. It is easy to check that V_0, V_1 and V_2 all have the same size, which is $3^4 = 81$. Let Γ' be the coset graph of C. Then Γ' is tripartite, with colour classes V_0, V_1 and V_2. Let Γ be the subgraph of Γ' induced by V_0 and V_1. The incidence structure with point set V_0, line set V_1 and incidence graph Γ is called the *van Lint-Schrijver partial geometry*; it was first introduced in [36], and is also discussed in [13] and [6, Sect. 11.5]. The incidence graph Γ is a bipartite distance-regular graph of diameter four having 162 vertices, with parameters $k = 6$, $c_2 = 1$, $c_3 = 2$ and nontrivial eigenvalue $\theta_1 = 3$. See [6, Sect. 11.5]. The set V_0 is a subgroup of the abelian group $\mathbb{F}_3^6/C$; its action on $\mathbb{F}_3^6/C$ by addition induces orbits V_0, V_1 and V_2, so V_0 acts on the colour classes of Γ regularly. Moreover, addition on the vertices of Γ by an element of V_0 is clearly an automorphism of Γ. Therefore, using Lemma 5.2.7, we can construct a $\{0, 1/4\}$-set of size 81 in $\mathbb{C}^6$.

Extended Binary Golay Code

Let I be the 12×12 identity matrix, and let A be the matrix

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

The *extended binary Golay code* is the subspace of $\mathbb{F}_2^{24}$ generated by the rows of the matrix $[I|A]$. In the extended binary Golay code, any element has even weight (and in fact, weight that is divisible by 4), and any nonzero element has weight at least eight. For more details about Golay codes, see [37] and [35]. Let C be the extended binary Golay code, and let Γ be the coset graph of C (with respect to $\mathbb{F}_2^{24}$). Since the elements of C have even weights, the weights of the elements in a coset of C have the same parity; we call a coset of C *even* if its elements have even weights, and *odd* otherwise. It then follows that Γ is bipartite, with the even cosets forming a colour class and the odd cosets forming the other. In fact, Γ is a distance-regular graph of diameter four having 4096 vertices, with parameters $k = 24$, $c_2 = 2$, $c_3 = 3$ and nontrivial eigenvalue $\theta_1 = 8$. Moreover, Γ is the unique bipartite distance-regular graph of diameter four with such parameters; see [6, Sect. 11.3D]. The set G of even cosets of C is a subgroup of the quotient group $\mathbb{F}_2^{24}/C$, and it acts on each colour class of Γ regularly by addition. Note that G acts on Γ as graph automorphisms. Since the non-identity elements of G have order two, by Lemma 5.2.7, we can construct a $\{0, 1/9\}$-set of size 2048 in $\mathbb{R}^{24}$.

Extended Kasami Codes

Let s and t be powers of 2, with $t \leq s$, and let $F := \mathbb{F}_s$. Let $K(s,t)$ be the set of elements x in $\mathbb{F}_2^F$ with even-weight that satisfy

$$\sum_{\alpha \in F} x_\alpha \alpha = \sum_{\alpha \in F} x_\alpha \alpha^{t+1} = 0.$$

The subspace $K(s,t)$ of $\mathbb{F}_2^F$ is called the *extended Kasami code* (with parameters s and t) if one of the following two conditions is satisfied:

  (i) $s = q^{2j+1}$, $t = q^m$, with $q = 2^i$, $m \leq j$, and $gcd(m, 2j + 1) = 1$.

  (ii) $s = q^2$, $t = q$, with $q = 2^i$.

Every nonzero element in an extended Kasami code has weight at least four. As in the extended Golay code, the elements in a coset of $K(s,t)$ have the same parity, so the coset graph $\Gamma(s,t)$ of $K(s,t)$ is bipartite, with the even cosets forming a colour class and the odd ones forming the other. In fact, $\Gamma(s,t)$ is a distance-regular graph of diameter four having $2n$ vertices and k, c_2, c_3 as parameters, where

(i) $(n, k, c_2, c_3) = (q^{4j+2}, q^{2j+1}, q, q^{2j} - 1)$,

(ii) $(n, k, c_2, c_3) = (q^3, q^2, q, q^2 - 1)$,

corresponding to the order above. See [6, Sect. 11.2]. The nontrivial eigenvalue of $\Gamma(s, t)$ is

(i) $\theta_1 = q^{j+1}$.

(ii) $\theta_1 = q$.

As in the extended binary Golay code, the even cosets of $K(s, t)$ form an abelian group G of automorphisms of $\Gamma(s, t)$ that acts on each colour class of $\Gamma(s, t)$ regularly by addition. Since every non-identity element of G has order two, by Lemma 5.2.7, we can construct a $\{0, \alpha\}$-set of size n in $\mathbb{R}^k$, where, corresponding to the order above,

(i) $(\alpha, n, k) = (q^{-2j}, q^{4j+2}, q^{2j+1})$.

(ii) $(\alpha, n, k) = (q^{-2}, q^3, q^2)$.

5.4 Flat Bounds for $\{0, \alpha\}$-Sets

In 1975, Delsarte, Goethals and Seidel [21] proved bounds for sizes of sets of unit vectors with prescribed sets of inner products. One type of bound states the following when applied to $\{0, \alpha\}$-sets.

Theorem 5.4.1 (Delsarte, Goethals and Seidel)**.** *Let S be a $\{0, \alpha\}$-set in $\mathbb{C}^m$ with $0 < \alpha < 1$. Then*

$$|S| \leq \frac{(m+1)m^2}{2}.$$

If $S \subseteq \mathbb{R}^m$ then

$$|S| \leq \frac{(m+2)(m+1)m}{6}.$$

$\square$

Another proof of this theorem using elementary tensor algebra was later given by Calderbank, Cameron, Kantor and Seidel [10] in 1997. We call a vector in $\mathbb{C}^m$ *flat* if all of its entries have the same absolute value. It turns out that by using arguments similar to those by Calderbank, Cameron, Kantor and Seidel, the bounds in Theorem 5.4.1 can be improved if the $\{0, \alpha\}$-set contains only flat vectors.

Theorem 5.4.2. *Let S be a $\{0, \alpha\}$-set of flat vectors in $\mathbb{C}^m$ with $0 < \alpha < 1$. Then*

$$|S| \leq \frac{(m^2 - m + 2)m}{2}.$$

If $S \subseteq \mathbb{R}^m$ then

$$|S| \leq \frac{(m^2 - 3m + 8)m}{6}.$$

Proof. Let $\beta := \sqrt{\alpha}$. Then for distinct x and y in S, we have $|x^*y| \in \{0, \beta\}$. Let M be the matrix whose columns are the vectors in S. Then

$$M^*M = I + \beta C,$$

where C is a Hermitian matrix with zero diagonal that has absolute value 0 or 1 for all off-diagonal entries. For each $x \in S$, let

$$v_x := x \otimes x \otimes \overline{x}$$

be a tensor (or Kronecker) product, where $\overline{x}$ is the vector in $\mathbb{C}^m$ whose i-th entry equals the conjugate of the i-th entry of x. Let $S' := \{v_x : x \in S\}$. For more details about tensor algebra, see [40]. If we let N be the matrix whose columns are the vectors in S', then since

$$v_x^* v_y = (x \otimes x \otimes \overline{x})^*(y \otimes y \otimes \overline{y}) = (x^*y)^2(\overline{x^*y}) = (x^*y)|x^*y|^2,$$

we have

$$N^*N = I + \beta^3 C = (1 - \beta^2)I + \beta^2(I + \beta C).$$

Since $|\beta| < 1$, the matrix $(1 - \beta^2)I$ is positive definite. Since $I + \beta C = M^*M$ is positive semidefinite, so is $\beta^2(I + \beta C)$. Hence N^*N is positive definite and has full rank $|S|$. On the other hand, the rank of N^*N is equal to the rank of N, which is the dimension of $\mathrm{span}(S')$. Let $x \in S$. Since x is a flat unit vector in $\mathbb{C}^m$, each of its entries has absolute value equal $1/\sqrt{m}$. Consider indices of v_x that have forms (i, j, j) or (j, i, j). The entries of v_x corresponding to these indices are

$$x_i x_j \overline{x_j} = \frac{x_i}{m},$$

which depends only on x_i (since m is a constant). If an index of v_x does not have one of the forms above, then it either has form (i, i, j) or is equal

to (i, j, k) for some distinct i, j and k. There are $m(m-1)$ indices of the first type, and there are $m(m-1)(m-2)$ indices of the second type. Since there are m ways to choose i for the indices (i, j, j) and (j, i, j), and since the entry of v_x indexed by (i, j, k) is equal to that by (j, i, k), it follows that the dimension of $\operatorname{span}(S')$ is at most

$$m + m(m-1) + \frac{m(m-1)(m-2)}{2} = \frac{m(m^2 - m + 2)}{2}.$$

This is an upper bound for the rank of N^*N, which is equal to $|S|$, so

$$|S| \leq \frac{m(m^2 - m + 2)}{2},$$

proving the first bound. Now assume further that $S \subseteq \mathbb{R}^m$. Then any $x \in S$ has entries equal to $1/\sqrt{m}$ or $-1/\sqrt{m}$, and

$$v_x = x \otimes x \otimes x.$$

In this case, the entries of v_x corresponding to the indices of the forms (i, j, j), (j, i, j) or (j, j, i) are all equal to x_i/m. Since the entry of v_x indexed by (i, j, k) is invariant under permutations of the components i, j and k, the dimension of $\operatorname{span}(S')$ is at most

$$m + \binom{m}{3} = \frac{m(m^2 - 3m + 8)}{6},$$

proving the second bound

$$|S| \leq \frac{m(m^2 - 3m + 8)}{6}.$$

$\square$

Note that Godsil and Roy's construction in Lemma 5.2.7 yields sets of flat vectors, so the bounds proven above are applicable to the $\{0, \alpha\}$-sets constructed using Godsil and Roy's method. In fact, there are distance-regular graphs that produce $\{0, \alpha\}$-sets of optimal sizes with respect to these bounds. In particular, it is easy to check that the 4-cube, the folded 8-cube and the coset graph of the extended binary Golay code produce sets that meet the flat real bound, while the 8-cycle produces a set that meets the flat complex bound, all at equality. The related parameters for these graphs are summarized in Table 5.1.

Table 5.1: $\{0, \alpha\}$-sets of largest size n from graphs.

Graph	k	c_2	c_3	α	n	Space
4-cube	4	2	3	1/4	8	$\mathbb{R}^4$
Folded 8-cube	8	2	3	1/4	64	$\mathbb{R}^8$
Ext. binary Golay code	24	2	3	1/9	2048	$\mathbb{R}^{24}$
8-cycle	2	1	1	1/2	4	$\mathbb{C}^2$

5.5 Optimal Distance-Regular Graphs

At the end of the previous section, we saw some graphs that produce $\{0, \alpha\}$-sets with the largest sizes. In this section, we shall see that those are the only distance-regular graphs that produce largest $\{0, \alpha\}$-sets. We will make use of the follow result by Brouwer, Cohen and Neumaire [6, Sect. 4.3, p.153-154].

Lemma 5.5.1 (Brouwer, Cohen and Neumaire). *If k, c_2 and c_3 are parameters of a bipartite distance-regular graph of diameter four with $c_2 = 2$ and $c_3 = 3$, then $k \in \{4, 8, 24\}$.* $\qquad\square$

With the lemma, we can prove the main results of this section.

Theorem 5.5.2. *Let S be a $\{0, \alpha\}$-set in $\mathbb{R}^k$ constructed from a distance-regular graph Γ using the construction in Lemma 5.2.7. Then S has the largest size $(k^2 - 3k + 8)k/6$ if and only if Γ is one of the following graphs:*

(i) 4-cube.

(ii) Folded 8-cube.

(iii) Coset graph of the extended binary Golay code.

Proof. The graph Γ has to be bipartite and have diameter four. Let k, c_2 and c_3 be its parameters, and let $2n$ be the number of its vertices. By Lemma 3.4.1,

$$n = \frac{k(k^2 - (c_2 + 1)k + c_2(c_3 + 1))}{c_2 c_3}.$$

Since $|S| = (k^2 - 3k + 8)k/6$, we have

$$\frac{k(k^2 - (c_2 + 1)k + c_2(c_3 + 1))}{c_2 c_3} = \frac{(k^2 - 3k + 8)k}{6},$$

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which, after rearrangement, yields

$$(6 - c_2 c_3)k^2 - (6c_2 + 6 - 3c_2 c_3)k + (6c_2 - 2c_2 c_3) = 0. \qquad (5.5.1)$$

Suppose $6 - c_2 c_3 = 0$. Then (5.5.1) becomes

$$(6c_2 - 12)k + (6c_2 - 12) = 0.$$

Since $c_2 \le c_3$ by Lemma 3.3.1, we have $(c_2, c_3) = (1, 6)$ or $(c_2, c_3) = (2, 3)$. The case where $c_2 = 1$ is impossible, since $k = -1$ has to be true for the above equation to hold. On the other hand, $(c_2, c_3) = (2, 3)$ satisfies (5.5.1) no matter what k is equal to. Now suppose $6 - c_2 c_3 \ne 0$. Applying the quadratic formula to (5.5.1) yields

$$k = \frac{6c_2 + 6 - 3c_2 c_3 \pm (6c_2 - 6 - c_2 c_3)}{12 - 2c_2 c_3}.$$

The case corresponding to the minus sign is discarded, since in that case $k = 1$, which is impossible for a distance-regular graph of diameter four. Hence

$$k = \frac{6c_2 + 6 - 3c_2 c_3 + 6c_2 - 6 - c_2 c_3}{12 - 2c_2 c_3} = \frac{6c_2 - 2c_2 c_3}{6 - c_2 c_3}. \qquad (5.5.2)$$

If $c_2 = 1$ then rewriting (5.5.2) yields

$$c_3 = \frac{6(k-1)}{k-2}.$$

Since $k - 1$ and $k - 2$ are coprime, $k - 2$ must divide 6, whence $k \in \{3, 4, 5, 8\}$. Since $c_3 < k$, it follows $k = 8$, whence $c_3 = 7$. If $c_2 \ge 2$ then let $\rho := k/c_2$. By (5.5.2) we have

$$\rho = \frac{6 - 2c_3}{6 - c_2 c_3},$$

which implies

$$\frac{\rho c_2 - 2}{\rho - 1} = \frac{6}{c_3}.$$

Since $\rho - 1 > 0$,

$$\frac{6}{c_3} = \frac{\rho c_2 - 2}{\rho - 1} \ge \frac{2\rho - 2}{\rho - 1} = 2,$$

whence $c_3 \le 3$. Since $c_2 \ge 2$, condition (v) in Lemma 3.3.1 implies that $c_2 = 2$ and $c_3 = 3$. But this contradicts our assumption that $6 - c_2 c_3 \ne 0$.

The arguments above showed that, if $|S| = (k^2 - 3k + 8)k/6$ then one of the following two conditions holds:

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(i) $c_2 = 2$ and $c_3 = 3$.

(ii) $k = 8$, $c_2 = 1$ and $c_3 = 7$.

However, graphs with $(k, c_2, c_3) = (8, 1, 7)$ do not produce $\{0, \alpha\}$-sets in $\mathbb{R}^8$. Indeed, such a graph has nontrivial eigenvalue $\theta_1 = \sqrt{8}$ and hence $\alpha = 1/8$. On the other hand, any flat unit vectors in $\mathbb{R}^8$ must have entries $\pm 1/\sqrt{8}$, and for such distinct vectors x and y, the only possible values of $|x^* y|^2$ are $9/16$, $1/4$, $1/16$ and 0. So the real flat bound is not applicable to the graphs with $(k, c_2, c_3) = (8, 1, 7)$. Meanwhile, if $c_2 = 2$ and $c_3 = 3$ then $k \in \{4, 8, 24\}$ by Lemma 5.5.1, and the graphs with these parameters are precisely the 4-cube, the folded 8-cube and the coset graph of the extended binary Golay code. $\square$

The case for $\mathbb{C}^k$ is proven similarly.

Theorem 5.5.3. *Let S be a $\{0, \alpha\}$-set in $\mathbb{C}^k$ constructed from a distance-regular graph Γ using the construction in Lemma 5.2.7. Then S has the largest size $(k^2 - k + 2)k/2$ if and only if Γ is the 8-cycle.*

Proof. The graph Γ has to be bipartite and have diameter four. Let k, c_2 and c_3 be its parameters. If $|S| = (k^2 - k + 2)k/2$ then by Lemma 3.4.1,

$$\frac{k(k^2 - (c_2 + 1)k + c_2(c_3 + 1))}{c_2 c_3} = \frac{(k^2 - k + 2)k}{2},$$

which is equivalent to

$$(2 - c_2 c_3)k^2 - (2c_2 + 2 - c_2 c_3)k + 2c_2 = 0. \tag{5.5.3}$$

If $2 - c_2 c_3 = 0$ then $c_2 = 1$, $c_3 = 2$ and $k = 1$, which is impossible, so $2 - c_2 c_3 \neq 0$. Applying the quadratic formula to (5.5.3) yields

$$k = \frac{2c_2 + 2 - c_2 c_3 \pm (2c_2 + c_2 c_3 - 2)}{4 - 2c_2 c_3}.$$

The case with the minus sign is discarded, for otherwise $k = 1$, which is impossible. Therefore

$$k = \frac{2c_2 + 2 - c_2 c_3 + 2c_2 + c_2 c_3 - 2}{4 - 2c_2 c_3} = \frac{2c_2}{2 - c_2 c_3}.$$

The fact that k, c_2 and c_3 are all positive integers then implies that

$$c_2 = c_3 = 1 \quad \text{and} \quad k = 2.$$

The unique bipartite distance-regular graph of diameter four with parameters $k = 2$, $c_2 = 1$ and $c_3 = 1$ is the 8-cycle. $\square$

5.6   Linear Cayley Graphs

Let $\mathbb{F}_q$ be a finite field of q elements. Then $\mathbb{F}_q^l$ is a vector space over $\mathbb{F}_q$ (and hence an abelian group). Consider the graph $\widehat{\Gamma} := \widehat{\Gamma}(\mathbb{F}_q^l, D)$ with some $D \subseteq \mathbb{F}_q^l$. The abelian group $\mathbb{F}_q^l$ acts on each colour class of $\widehat{\Gamma}$ by addition regularly, and these actions define automorphisms of $\widehat{\Gamma}$. By Theorem 5.2.7, if $\widehat{\Gamma}$ is connected and has five distinct eigenvalues, it gives rise to a $\{0, \alpha\}$-set. In this section, we will see conditions for $\widehat{\Gamma}$ to have five distinct eigenvalues when D is *linear*, that is, when D does not contain the zero vector and satisfies $\beta h \in D$ whenever $h \in D$ and $\beta \in \mathbb{F}_q^*$, where $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$.

Since the spectrum of $\widehat{\Gamma}$ is closely related to that of the Cayley graph $\Gamma(\mathbb{F}_q^l, D)$, we start by investigating $\Gamma(\mathbb{F}_q^l, D)$ when D is linear. Since D is closed under taking inverses and does not contain the zero vector, by identifying each pair of arcs of the form (u, v) and (v, u), one can view $\Gamma(\mathbb{F}_q^l, D)$ as an undirected graph. For any chracter χ of the additive group $\mathbb{F}_q$ and any $a \in \mathbb{F}_q^l$, let $\chi_a : \mathbb{F}_q^l \to \mathbb{C}$ be the function defined by $x \mapsto \chi(a^T x)$. There is a convenient description of the characters of $\mathbb{F}_q^l$.

Lemma 5.6.1. *Fix a nontrivial character χ of the additive group $\mathbb{F}_q$. Then for any $a \in \mathbb{F}_q^l$, the function χ_a is a character of $\mathbb{F}_q^l$. Moreover, if $a \neq b$, then $\chi_a \neq \chi_b$. Every character of $\mathbb{F}_q^l$ is equal to χ_a for some a.*

For a proof of the lemma, see [25, Sect. 12.8]. By Lemmas 5.2.4 and 5.6.1, the set of eigenvalues of $\Gamma(\mathbb{F}_q^l, D)$ is

$$\left\{ \chi_a : a \in \mathbb{F}_q^l \right\},$$

where χ is a fixed nontrivial character of $\mathbb{F}_q$. The graph $\Gamma(\mathbb{F}_q^l, D)$ has connections to linear codes and sets of points in the projective space $\mathrm{PG}(l-1, q)$. A *linear code* (or simply a *code*) in $\mathbb{F}_q^m$ is a subspace of $\mathbb{F}_q^m$. The elements of a linear code are called *codewords*. A *generator matrix* of a l-dimensional code C in $\mathbb{F}_q^m$ is an l by m matrix M whose row space is equal to the code; evidently

$$C = \left\{ a^T M : a \in \mathbb{F}_q^l \right\}.$$

Note that $a_1^T M$ and $a_2^T M$ are different codewords of C if $a_1 \neq a_2$, since M has full row rank. The *weight* $w(c)$ of a codeword c is the number of its nonzero entries. Calderbank and Kantor [11] proved the following statement (with a different phrasing), whose partial proof can also be found in [25, Sect. 12.9]. We include a proof here.

Lemma 5.6.2 (Calderbank and Kantor). *Let $D \subseteq \mathbb{F}_q^l$ be a linear set with*

$$D = \{\beta v_i : \beta \in \mathbb{F}_q^*, 1 \le i \le m\},$$

where $v_1, v_2, \ldots, v_m$ are pairwise linearly independent vectors. Let M be the matrix whose i-th column is v_i, and let

$$\Omega := \{\langle v_i \rangle : 1 \le i \le m\}$$

be the set of 1-dimensional spaces associated to D. For $a \ne 0$, let H_a be the set of 1-dimensional subspaces of the hyperplane

$$\{x \in \mathbb{F}_q^l : a^T x = 0\}.$$

If χ is a nontrivial character of $\mathbb{F}_q$, then

$$m - w(a^T M) = |H_a \cap \Omega| = \frac{\chi_a(D) + m}{q}.$$

Proof. Note that $a^T v_i$ is the i-th coordinate of the codeword $a^T M$. Since $a^T v_i = 0$ if and only if $\langle v_i \rangle \in H_a$, we have

$$m - w(a^T M) = |H_a \cap \Omega|,$$

proving the first equality. For the second equality, note that

$$\chi_a(D) = \sum_{i=1}^{m} \sum_{\beta \in \mathbb{F}_q^*} \chi_a(\beta v_i) = \sum_{i=1}^{m} \sum_{\beta \in \mathbb{F}_q^*} \chi(\beta a^T v_i). \qquad (5.6.1)$$

If $\langle v_i \rangle \in H_a$, then $a^T v_i = 0$, and so

$$\sum_{\beta \in \mathbb{F}_q^*} \chi(\beta a^T v_i) = \sum_{\beta \in \mathbb{F}_q^*} 1 = q - 1.$$

If $\langle v_i \rangle \notin H_a$, then,
$$\{\beta a^T v_i : \beta \in \mathbb{F}_q^*\} = \mathbb{F}_q^*,$$
which, together with Lemma 5.2.1, implies that

$$\sum_{\beta \in \mathbb{F}_q^*} \chi(\beta a^T v_i) = \sum_{g \in \mathbb{F}_q^*} \chi(g) = \chi(G) - \chi(0) = -1.$$

It follows from Equation (5.6.1) that the value of $\chi_a(D)$ depends only on $|H_a \cap \Omega|$:

$$\begin{aligned} \chi_a(D) &= (q-1)|H_a \cap \Omega| + (-1)(m - |H_a \cap \Omega|) \\ &= q|H_a \cap \Omega| - m. \end{aligned}$$

The second desired equality then follows. $\qquad\square$

We aim to find graphs $\widehat{\Gamma}(\mathbb{F}_q^l, D)$ that can be used to construct $\{0, \alpha\}$-sets, so we need to consider those that are connected. We first make the following simple observation.

Lemma 5.6.3. *The graph $\Gamma(\mathbb{F}_q^l, D)$ is connected if and only if D spans $\mathbb{F}_q^l$.*

Proof. Let $u, v \in \mathbb{F}_q^l$. By definition of Cayley graphs, there is a path from u to v in $\Gamma(\mathbb{F}_q^l, D)$ if and only if $v - u$ can be written as a sum of elements of D. The lemma then follows. $\qquad\square$

From Lemma 5.6.3, we see that a linear Cayley graph being connected is not a very strong condition. Now we can relate the connectedness of $\widehat{\Gamma}(\mathbb{F}_q^l, D)$ with $\Gamma(\mathbb{F}_q^l, D)$.

Lemma 5.6.4. *The graph $\widehat{\Gamma} := \widehat{\Gamma}(\mathbb{F}_q^l, D)$ is connected if and only if D spans $\mathbb{F}_q^l$ and the graph $\Gamma := \Gamma(\mathbb{F}_q^l, D)$ is not bipartite.*

Proof. Let k be the size of D. By Lemma 2.1.1, the graph $\widehat{\Gamma}$ is connected if and only if k is a simple eigenvalue of $\widehat{\Gamma}$. Since k is also an eigenvalue of Γ, by Lemma 5.2.6, it follows that k is a simple eigenvalue of $\widehat{\Gamma}$ if and only if k is a simple eigenvalue of Γ and $-k$ is not an eigenvalue of Γ. This is true if and only if Γ is connected and not bipartite, by Lemma 2.1.1. The assertion then follows, by Lemma 5.6.3. $\qquad\square$

We use Lemma 5.2.6 to relate the eigenvalues of $\widehat{\Gamma}(\mathbb{F}_q^l, D)$ with those of $\Gamma(\mathbb{F}_q^l, D)$.

Lemma 5.6.5. *Let D be a spanning linear subset of $\mathbb{F}_q^l$ of size k. Then the graph $\widehat{\Gamma}(\mathbb{F}_q^l, D)$ is connected and has exactly five distinct eigenvalues $\pm k$, $\pm \theta_1$ and 0 if and only if the set of eigenvalues of the graph $\Gamma := \Gamma(\mathbb{F}_q^l, D)$ is one of the following:*

  (i) $\{k, 0, -\theta_1\}$.

(ii) $\{k, \theta_1, 0, -\theta_1\}$.

Proof. The graph Γ is connected since D is spanning. If Γ has eigenvalues as described in (i) or (ii) then it is not bipartite by Lemma 2.1.1. The graph $\widehat{\Gamma}$ is then connected and has five distinct eigenvalues as described, by Lemma 5.6.4 and Lemma 5.2.6.

Suppose that $\widehat{\Gamma}$ is connected and has eigenvalues $\pm k$, $\pm \theta_1$ and 0. Since Γ is a k-regular graph, it has only real eigenvalues, one of which being k. Since Γ is connected and not bipartite by Lemma 5.6.3 and Lemma 5.6.4, the number $-k$ is not an eigenvalue of Γ. Since a nonempty graph must have two eigenvalues of different signs (as its adjacency matrix is nonzero and has trace 0), the number $-\theta_1$ must be an eigenvalue of Γ. The set of eigenvalues of Γ is then as described, by Lemma 5.2.6. $\qquad\square$

Linear Cayley graphs with type (ii) eigenvalue set in the lemma above do exist: the complement of the cube has eigenvalues 4, 0 and ± 2 (see van Dam [48]). Graphs with type (i) eigenvalue set have a very simple geometric characterization.

Lemma 5.6.6. *Let $D \subseteq \mathbb{F}_q^l$ be a spanning linear set of size k. Then $\Gamma(\mathbb{F}_q^l, D)$ has exactly three distinct eigenvalues k, 0 and $-\theta_1$ with $\theta_1 \neq k$ if and only if $\overline{D} := \mathbb{F}_q^l \setminus D$ is a nontrivial subspace of $\mathbb{F}_q^l$ with $q \neq 2$ or $\dim(\overline{D}) \neq l - 1$.*

Proof. The graph $\Gamma := \Gamma(\mathbb{F}_q^l, D)$ is connected since D is spanning. Suppose that Γ has exactly three distinct eigenvalues k, 0 and $-\theta_1$ with $\theta_1 \neq k$. By Lemma 2.3.3 and Lemma 2.3.4, Γ is strongly regular and antipodal, so non-adjacency is an equivalence relation on its vertices. Let $u, v \in \overline{D}$. Then $u \nsim 0$ and $v \nsim 0$, where "$\nsim$" denotes the non-adjacency relation in Γ. By transitivity of non-adjacency, $u \nsim v$, so $u - v \in \overline{D}$. On the other hand, $\beta v \in \overline{D}$ for any $\beta \in \mathbb{F}_q$ since D is linear, so $\overline{D}$ is a subspace of $\mathbb{F}_q^l$, which cannot be trivial since Γ is neither complete nor edgeless. If $q = 2$ and $\dim(\overline{D}) = l - 1$, then it is easy to check that Γ is bipartite with colour classes $\overline{D}$ and D, which is impossible since $-\theta_1 \neq -k$. Therefore $q \neq 2$ or $\dim(\overline{D}) \neq l - 1$.

Conversely, suppose that $\overline{D}$ is a t-dimensional subspace of $\mathbb{F}_q^l$ with $t \neq 0$ and $t \neq l$, and either $q \neq 2$ or $t \neq l - 1$. Let Ω be the set of 1-dimensional subspaces of D. Then

$$|\Omega| = (q^l - q^t)/(q - 1).$$

It is easy to check that each hyperplane H of $\mathbb{F}_q^l$ contains b elements of Ω, where

$$b = \begin{cases} (q^{l-1} - q^t)/(q-1), & \text{if } \overline{D} \subseteq H, \\ (q^{l-1} - q^{t-1})/(q-1), & \text{otherwise.} \end{cases}$$

Let χ be a nontrivial character of $\mathbb{F}_q$ and let a be a nonzero vector in $\mathbb{F}_q^l$. By Lemma 5.6.2, straightforward computations show that $\chi_a(D) = 0$ if the hyperplane with normal vector a does not contain $\overline{D}$, and $\chi_a(D) = -q^t$ otherwise. If $a = 0$ then χ_a is the trivial character of $\mathbb{F}_q^l$, so

$$\chi_a(D) = |D| = q^l - q^t.$$

By Lemma 5.2.4 and Lemma 5.6.1, the graph Γ has exactly three distinct eigenvalues k, 0 and $-\theta_1$. Since $q \neq 2$ or $t \neq l-1$, we have $k \neq \theta_1$. $\qquad\square$

Now we can characterize the connected graphs $\widehat{\Gamma}(\mathbb{F}_q^l, D)$ with five eigenvalues.

Theorem 5.6.7. *Let D be the set $\{\beta v_i : \beta \in \mathbb{F}_q^*, 1 \leq i \leq m\}$, where v_1, v_2, ..., v_m are pairwise linearly independent vectors that span $\mathbb{F}_q^l$. Let M be the matrix whose i-th column is v_i. Let k be the size of D and let θ_1 be a real number. Then the following are equivalent:*

(i) The graph $\widehat{\Gamma}(\mathbb{F}_q^l, D)$ is connected and has exactly five distinct eigenvalues $\pm k$, $\pm\theta_1$ and 0.

(ii) Either $\overline{D} := \mathbb{F}_q^l \setminus D$ is a nontrivial subspace of $\mathbb{F}_q^l$ where $q \neq 2$ or $\dim(\overline{D}) \neq l-1$, or for every hyperplane H in $\mathbb{F}_q^l$, if $b := b(H)$ is the number of 1-dimensional spaces among $\langle v_1 \rangle$, $\langle v_2 \rangle$, ..., $\langle v_m \rangle$ that are contained in H, then

$$b \in \left\{ \frac{m}{q}, \frac{m+\theta_1}{q}, \frac{m-\theta_1}{q} \right\},$$

and each of the three possibilities of b occurs for some hyperplane.

(iii) The code C with generator matrix M satisfies one of the following conditions (with $\theta_1 \neq k$):

- *The codewords of C have two weights:*

$$\frac{m(q-1)}{q} \quad or \quad \frac{m(q-1) + \theta_1}{q}.$$

- *The codewords of C have three weights:*

$$\frac{m(q-1)}{q}, \quad \frac{m(q-1) - \theta_1}{q} \quad or \quad \frac{m(q-1) + \theta_1}{q}.$$

Proof. Let χ be a nontrivial character of $\mathbb{F}_q$. By computation using the formulae in Lemma 5.6.2, it follows that condition (iii) is equivalent to

$$\left\{\chi_a(D) : a \in \mathbb{F}_q^l \setminus \{0\}\right\} \in \left\{\{0, -\theta_1\}, \{0, \pm\theta_1\}\right\}. \tag{5.6.2}$$

If a is the zero vector in $\mathbb{F}_q^l$ then $\chi_a(D) = |D| = k$. By Lemma 5.2.4 and Lemma 5.6.1, condition (5.6.2) above is equivalent to saying that the set of eigenvalues of $\Gamma(\mathbb{F}_q^l, D)$ is equal to $\{k, 0, -\theta_1\}$ or $\{k, 0, \pm\theta_1\}$. By Lemma 5.6.5, conditions (i) and (iii) are equivalent. Similarly, the equivalence between conditions (i) and (ii) follows from Lemmas 5.6.2, 5.6.5 and 5.6.6. $\square$

The theorem above translates the eigenvalue condition on a linear Cayley graph into a geometric or coding theoretic condition on the connection set of the graph. It provides alternative perspectives to the problem of finding graphs with five eigenvalues, and provides access to tools in finite projective geometry and coding theory.

5.7 Other Constructions of $\{0, \alpha\}$-Sets

In previous sections, we saw some examples of graphs that can be used to construct $\{0, \alpha\}$-sets. In this section, we discuss variants of known graphs that also yield $\{0, \alpha\}$-sets.

The *bipartite complement* $BC(\Gamma)$ of a bipartite graph Γ is the graph obtained from Γ by changing all the edges (resp. non-edges) between different colour classes to non-edges (resp. edges). If Γ is regular then so is $BC(\Gamma)$, and their spectra are related in a simple way.

Lemma 5.7.1. *If Γ is a k-regular bipartite graph on $2n$ vertices with spectrum*

$$\left\{k^{(m_0)}, \theta_1^{(m_1)}, \ldots, -\theta_1^{(m_1)}, -k^{(m_0)}\right\},$$

then the spectrum of $BC(\Gamma)$ is

$$\left\{(n-k)^{(1)}, k^{(m_0-1)}, \theta_1^{(m_1)}, \ldots, -\theta_1^{(m_1)}, -k^{(m_0-1)}, -(n-k)^{(1)}\right\}.$$

Proof. The adjacency matrix of Γ and the adjacency matrix of $\mathrm{BC}(\Gamma)$ have the forms

$$A := \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \quad \text{and} \quad \overline{A} := \begin{bmatrix} 0 & J_n - B \\ J_n - B^T & 0 \end{bmatrix},$$

respectively, where J_n is the n by n all-one matrix. If $\mathbb{1}$ is the all-one vector of size n, then it is easy to check that the vectors

$$\begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix}$$

are eigenvectors of Γ with eigenvalues k and $-k$, respectively, and are eigenvectors of $\mathrm{BC}(\Gamma)$ with eigenvalues $n - k$ and $k - n$, respectively. Let λ be an eigenvalue of Γ, with eigenvector $[x \ y]$ orthogonal to the two vectors above (where $x, y \in \mathbb{R}^n$). Then the coordinates in each of x and y sum to zero. It follows that

$$\overline{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} J_n y - By \\ J_n x - B^T x \end{bmatrix} = \begin{bmatrix} -By \\ -B^T x \end{bmatrix} = - \begin{bmatrix} By \\ B^T x \end{bmatrix}.$$

On the other hand,

$$\lambda \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} By \\ B^T x \end{bmatrix},$$

so $-\lambda$ is an eigenvalue of $\mathrm{BC}(\Gamma)$. Therefore, by the symmetry of the eigenvalues of a bipartite graph (Lemma 2.1.2), the spectrum of $\mathrm{BC}(\Gamma)$ is the same as that of Γ, except that one copy of k and $-k$ is replaced by $n - k$ and $k - n$ in $\mathrm{BC}(\Gamma)$. $\qquad\square$

For positive integers a and m, let $aK_{m,m}$ be the union of a copies of the complete bipartite graph $K_{m,m}$. The adjacency matrix of $K_{m,m}$ has the form

$$\begin{bmatrix} 0 & J_m \\ J_m & 0 \end{bmatrix},$$

whose spectrum is

$$\left\{ m^{(1)}, 0^{(2m-2)}, -m^{(1)} \right\}.$$

From this it is easy to check that

$$\mathrm{spec}(aK_{m,m}) = \left\{ m^{(a)}, 0^{(2ma-2a)}, -m^{(a)} \right\}.$$

By Lemma 5.7.1, the spectrum of $\mathrm{BC}(aK_{m,m})$ is

$$\left\{(a-1)m^{(1)}, m^{(a-1)}, 0^{(2ma-2a)}, -m^{(a-1)}, -(a-1)m^{(1)}\right\}.$$

Hence, if $a \geq 3$ and $m \geq 2$, then $\mathrm{BC}(aK_{m,m})$ has exactly five distinct eigenvalues and is connected. The colour classes of $\mathrm{BC}(aK_{m,m})$ can be viewed as two copies of $\mathbb{Z}_m \times \mathbb{Z}_a$, with (i,j) and (i',j') in different colour classes being adjacent if and only if $j \neq j'$. It follows that the abelian group $\mathbb{Z}_m \times \mathbb{Z}_a$ acts on each colour class regularly by addition. Moreover, each element of $\mathbb{Z}_m \times \mathbb{Z}_a$ clearly defines an automorphism of $\mathrm{BC}(aK_{m,m})$ when acting on the colour classes. Therefore, using Theorem 5.2.7, a $\{0,\alpha\}$-set in $\mathbb{C}^k$ can be constructed from $\mathrm{BC}(aK_{m,m})$ if $a \geq 3$ and $m \geq 2$, where

$$k = (a-1)m \quad \text{and} \quad \alpha = \frac{m^2}{(a-1)^2 m^2} = \frac{1}{(a-1)^2}.$$

For the remainder of this section, let Γ be a connected k-regular bipartite graph on $2n$ vertices with four or five distinct eigenvalues, whose spectrum necessarily has the form

$$\left\{k^{(1)}, \theta_1^{(m_1)}, 0^{(m_2)}, -\theta_1^{(m_1)}, -k^{(1)}\right\},$$

where $m_1 \neq 0$ and m_2 may be zero (by Lemma 2.1.1 and Lemma 2.1.2). Suppose that G is an abelian group of automorphisms of Γ acting regularly on each colour class. Thus a $\{0,\alpha\}$-set can be constructed from Γ if $m_2 \neq 0$. We will see two simple variants of Γ that also yield $\{0,\alpha\}$-sets.

We first consider the bipartite complement of Γ. Clearly G is also a group of automorphisms of $\mathrm{BC}(\Gamma)$. By Lemma 5.7.1, the spectrum of $\mathrm{BC}(\Gamma)$ is

$$\left\{(n-k)^{(1)}, \theta_1^{(m_1)}, 0^{(m_2)}, -\theta_1^{(m_1)}, -(n-k)^{(1)}\right\}.$$

It follows that $\mathrm{BC}(\Gamma)$ is also connected and has five distinct eigenvalues if $\theta_1 \neq n-k$ and $m_2 \neq 0$, and thus can be used to construct a $\{0,\alpha\}$-set in $\mathbb{C}^{n-k}$, where $\alpha = (\theta_1)^2/(n-k)^2$. Before discussing the second variant of Γ, we first prove a well-known lemma regarding Kronecker products of matrices.

Lemma 5.7.2. *If M_1 and M_2 are real symmetric matrices with spectra*

$$\{\lambda_1, \ldots, \lambda_s\} \quad \text{and} \quad \{\mu_1, \ldots, \mu_t\},$$

respectively, then the spectrum of $M_1 \otimes M_2$ is the multiset

$$\{\lambda_i \mu_j : 1 \leq i \leq s, \ 1 \leq j \leq t\}.$$

Proof. If λ and μ are real numbers and x and y are vectors such that

$$M_1 x = \lambda x \quad \text{and} \quad M_2 y = \mu y,$$

then

$$(M_1 \otimes M_2)(x \otimes y) = M_1 x \otimes M_2 y = \lambda x \otimes \mu y = \lambda\mu(x \otimes y),$$

by properties of Kronecker products. Let $\{x_1, \ldots, x_s\}$ be a basis of $\mathbb{R}^s$ consisting of eigenvectors of M_1, and let $\{y_1, \ldots, y_t\}$ be a basis of $\mathbb{R}^t$ consisting of eigenvectors of M_2. Then

$$\{x_i \otimes y_j : 1 \leq i \leq s, 1 \leq j \leq t\}$$

is a basis of $\mathbb{R}^s \otimes \mathbb{R}^t$ consisting of eigenvectors of $M_1 \otimes M_2$. The spectrum of $M_1 \otimes M_2$ is therefore as described. $\square$

The second variant is $\Gamma^{(m)}$, which is the graph with vertex set $V(\Gamma) \times \mathbb{Z}_m$ such that two vertices are adjacent if and only if their first coordinates are adjacent in Γ. The graph $\Gamma^{(m)}$ is bipartite, with colour classes $Y \times \mathbb{Z}_m$ and $Z \times \mathbb{Z}_m$, where Y and Z are the colour classes of Γ. By definition, if A is the adjacency matrix of Γ, then $A \otimes J_m$ is the adjacency matrix of $\Gamma^{(m)}$. Since the matrix J_m has m as the only nonzero eigenvalue, which is also simple, by Lemma 5.7.2 the spectrum of $\Gamma^{(m)}$ is

$$\left\{ km^{(1)}, \theta_1 m^{(m_1)}, 0^{(2nm - 2m_1 - 2)}, -\theta_1 m^{(m_1)}, -km^{(1)} \right\}.$$

Hence, if $m \geq 2$, then $\Gamma^{(m)}$ has five distinct eigenvalues, even when Γ has only four. Moreover, since G acts on Y and Z regularly, the group $G \times \mathbb{Z}_m$ acts on $Y \times \mathbb{Z}_m$ and $Z \times \mathbb{Z}_m$ regularly (the action on the second coordinates is addition, naturally). Since these actions by the abelian group $G \times \mathbb{Z}_m$ define automorphisms of $\Gamma^{(m)}$, by Theorem 5.2.7 a $\{0, \alpha\}$-set in $\mathbb{C}^{km}$ can be constructed from $\Gamma^{(m)}$, where $\alpha = (\theta_1/k)^2$.

Because a graph Γ with only four distinct eigenvalues also gives rise to a $\{0, \alpha\}$-set using the construction $\Gamma^{(m)}$ above, we briefly summarize such graphs from known results. A *difference set* in a group $(G, +)$ is a subset D of G such that for each $g \in G \setminus \{0\}$, the number of ordered pairs (h_1, h_2) of elements of D for which $h_1 - h_2 = g$ is a constant (that does not depend on g). Beth, Jungnickel and Lenz [2, p. 299-300] showed that if D is a difference set in G and $\mathcal{B}$ is the set

$$\{g + D : g \in G\}$$

of all translates of D, then the incidence structure with point set G, block set $\mathcal{B}$ and incidence relation being containment is a symmetric design, and the group G acts on each colour class of its incidence graph regularly as a group of automorphisms. The same authors [2] also showed the converse: any symmetric design with a group G of automorphisms acting regularly on each colour class of its incidence graph can be represented by a difference set in G in the above manner. On the other hand, Cvetković, Doob and Sachs [20, p. 166-167] showed that connected regular bipartite graphs with exactly four distinct eigenvalues are precisely the incidence graphs of (connected) symmetric designs. Therefore, connected bipartite graphs with exactly four distinct eigenvalues and an abelian group of automorphisms acting regularly on each colour class correspond to difference sets in abelian groups. We will not discuss difference sets and symmetric designs; readers are referred to [2] for details of these objects.

Chapter 6

Terwilliger Algebra

6.1 Introduction

Let Γ be a distance-regular graph with vertex set V and distance matrices A_0, A_1, ..., A_d. Fix a vertex v of Γ, and let $\Gamma_i := \Gamma_i(v)$ be the set of vertices in Γ that are at distance i from v. Let $F_i := F_i(v)$ be the diagonal $(0,1)$-matrix with rows and columns indexed by V, such that the (u,u)-entry of F_i equals 1 if and only if $\partial(u,v) = i$. The *Terwilliger algebra* $\mathbb{T} := \mathbb{T}(v)$ of the graph Γ (with respect to the vertex v) is the matrix algebra generated by A_0, ..., A_d, F_0, ..., F_d over $\mathbb{C}$, so it is an algebra of linear operators acting on $\mathbb{C}^V$ by left multiplication. A $\mathbb{T}$-*module* is a $\mathbb{T}$-invariant subspace of $\mathbb{C}^V$, that is, a subspace W such that $MW \subseteq W$ for each $M \in \mathbb{T}$. Clearly, a subspace is a $\mathbb{T}$-module if and only if it is invariant under A and under each F_i (since each distance matrix is a polynomial in A by Lemma 2.2.3). A nonzero $\mathbb{T}$-module is said to be *irreducible* if it contains no nonzero proper subspace that is a $\mathbb{T}$-module. The Terwilliger algebra of a distance-regular graph is semisimple; consequently, $\mathbb{C}^V$ can be written as an orthogonal direct sum of irreducible $\mathbb{T}$-modules (see [44], for example).

The Terwilliger algebra was first introduced in [44, 45, 46] to study P- and Q-polynomial association schemes. Sometimes, the structures of the irreducible $\mathbb{T}$-modules of a graph encodes information of the combinatorial structures of the graph. For examples, Jurišić, Koolen and Terwilliger [32], and Go and Terwilliger [24] showed that a certain condition on the irreducible $\mathbb{T}$-modules of a distance-regular graph implies a very uniform local structure of the graph, and Collins [15] showed that if a graph satisfies $c_3 = 1$ and all

of its irreducible $\mathbb{T}$-modules are "thin", then the graph must come from a special kind of generalized quadrangle. Decompositions of $\mathbb{C}^V$ into irreducible $\mathbb{T}$-modules have received considerable attention [14, 15, 17, 18, 24, 47].

In hope of learning structural information of bipartite distance-regular graphs of diameter four, we study their irreducible $\mathbb{T}$-modules in an orthogonal decomposition of $\mathbb{C}^V$. As new results, we first describe these modules by considering different endpoints, and we will see that there are eight types of them. The multiplicities of some types of modules are found at the end of the chapter. We describe the eigenvalues of the graph associated to each irreducible module. We will see that the eigenvalues associated to an irreducible $\mathbb{T}$-module depend only on the dimension of the module.

6.2 Irreducible $\mathbb{T}$-Modules

For the remainder of the chapter, we assume that Γ is bipartite and has diameter four, with parameters K, c_2 and c_3. Let V be the vertex set of Γ, fix $v \in V$, and let $\mathbb{T} := \mathbb{T}(v)$ be the Terwilliger algebra of Γ with respect to v. The *endpoint* of a nonzero $\mathbb{T}$-module W is the number

$$\min\{i : F_i W \neq 0,\ 0 \leq i \leq d\}.$$

In this section, we describe the irreducible $\mathbb{T}$-modules of Γ in an orthogonal decomposition of $\mathbb{C}^V$. We proceed by considering modules with different endpoints. If we order the rows and columns according to Γ_0, Γ_1, Γ_2, Γ_3, Γ_4 with respect to v, the adjacency matrix A has the form

$$\begin{bmatrix} 0 & \mathbb{1}_K^T & 0 & 0 & 0 \\ \mathbb{1}_K & 0 & A_{1,2} & 0 & 0 \\ 0 & A_{2,1} & 0 & A_{2,3} & 0 \\ 0 & 0 & A_{3,2} & 0 & A_{3,4} \\ 0 & 0 & 0 & A_{4,3} & 0 \end{bmatrix},$$

where 0 denotes the zero matrix of suitable size (by abuse of notation). Let J^- denote the matrix $A + A_3$. Since only vertices in different colour classes can be at odd distance, the matrix J^- has the form

$$\begin{bmatrix} 0 & J_{0,1} & 0 & J_{0,3} & 0 \\ J_{1,0} & 0 & J_{1,2} & 0 & J_{1,4} \\ 0 & J_{2,1} & 0 & J_{2,3} & 0 \\ J_{3,0} & 0 & J_{3,2} & 0 & J_{3,4} \\ 0 & J_{4,1} & 0 & J_{4,3} & 0 \end{bmatrix},$$

where $J_{i,j}$ is the all-one matrix with rows and columns indexed by Γ_i and Γ_j, respectively. The matrix J^- will be useful in some computations later on.

6.2.1 Modules with Endpoint 0

Let W be a irreducible $\mathbb{T}$-module with endpoint 0. By using the form of A, it is straightforward to show that W must contains the vectors

$$
\begin{bmatrix} \mathbb{1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 \\ \mathbb{1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 \\ 0 \\ \mathbb{1} \\ 0 \\ 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbb{1} \\ 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathbb{1} \end{bmatrix},
$$

where $\mathbb{1}$ denotes the all-one vector of suitable size corresponding to the row labeling of A. By the definition of distance-regular graphs, the vector $A_{i,j}\mathbb{1}$ is equal to $\beta\mathbb{1}$ for some intersection number β. Using this, it is straightforward to show that the span of the five vectors above is a $\mathbb{T}$-module. Since W is irreducible, it must be equal to this span, which is the unique $\mathbb{T}$-module with endpoint 0. We call this module the *standard module*. Now our task is reduced to describing a decomposition of the orthogonal complement of the standard module.

6.2.2 Modules with Endpoint 1

Let W be an irreducible $\mathbb{T}$-module with endpoint 1 that is orthogonal to the standard module. This means the restriction of each vector in W to each Γ_i has its coordinates summed to zero. By a theorem of Curtin [17, Theorem 7.6], $\dim(W) = 3$, and F_1W, F_2W and F_3W are all nonzero (so each of them has dimension 1). Let x be a nonzero vector in F_2W, with x_2 being its restriction to Γ_2. Then

$$
Ax = \begin{bmatrix} 0 \\ A_{1,2}x_2 \\ 0 \\ A_{3,2}x_2 \\ 0 \end{bmatrix}.
$$

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Since W is $\mathbb{T}$-invariant,

$$
F_1 A x = \begin{bmatrix} 0 \\ A_{1,2} x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad F_3 A x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_{3,2} x_2 \\ 0 \end{bmatrix}
$$

are both in W. By observing that $\mathbb{1}_K^T A_{1,2} = c_2 \mathbb{1}_{N_2}^T$ and $\mathbb{1}_{N_2}^T x_2 = 0$, we have

$$
A F_1 A x = \begin{bmatrix} \mathbb{1}_K^T A_{1,2} x_2 \\ 0 \\ A_{2,1} A_{1,2} x_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ A_{2,1} A_{1,2} x_2 \\ 0 \\ 0 \end{bmatrix}.
$$

The vector $A F_1 A x$ is in W, so the span of the vectors $(A F_1 A)^l x$, where l is a nonnegative integer, is a subspace of W. The restriction of this span to Γ_2 is invariant under the symmetric matrix $A_{2,1} A_{1,2}$, so it has a basis consisting of eigenvectors of $A_{2,1} A_{1,2}$; let w be such an eigenvector with eigenvalue λ. From the above, we know that the vectors

$$
y := \begin{bmatrix} 0 \\ A_{1,2} w \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ w \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_{3,2} w \\ 0 \end{bmatrix}
$$

are all in W. Note that $A_{1,2} w \neq 0$, for otherwise the span of the last two vectors would be $\mathbb{T}$-invariant (since $\dim(F_2 W) = 1$), which is impossible since W is irreducible. Similarly, $A_{3,2} w \neq 0$, so the three vectors above form a basis of W. Moreover, $\lambda \neq 0$, for otherwise $A y = 0$, which would mean that y spans a $\mathbb{T}$-module, which is also impossible.

Conversely, let $w \in \mathbb{C}^{\Gamma_2}$ be a vector such that

$$
\mathbb{1}^T w = 0, \quad A_{1,2} w \neq 0 \quad \text{and} \quad A_{2,1} A_{1,2} w = \lambda w,
$$

where $\lambda \neq 0$, and let y be the vector whose restriction to Γ_1 is $A_{1,2} w$ and is

zero elsewhere. Then

$$
\begin{bmatrix} 0 \\ 0 \\ w \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\lambda} A y \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_{3,2}w \\ 0 \end{bmatrix} = \frac{1}{\lambda} A^2 y - y.
$$

Note that $A_{3,2}w \neq 0$, for otherwise $\text{span}\{y, Ay\}$ would be a 2-dimensional $\mathbb{T}$-module, contradicting Theorem 7.6 in [17]. Let U be the span of y, $\lambda^{-1}Ay$ and $\lambda^{-1}A^2y - y$. We show that U is $\mathbb{T}$-invariant. Clearly, U is invariant under each F_i, so we only need to show that U is A-invariant. Note that it suffices to show $A^3y \in U$. By using the formula in Lemma 2.2.3, we deduce that

$$
A^3 = c_2 c_3 A_3 + (K + c_2 K - c_2)A. \tag{6.2.1}
$$

Recall that $J^- := A + A_3$. Using the form of J^-, we have

$$
(A + A_3)y = J^- y = \begin{bmatrix} J_{0,1}A_{1,2}w \\ 0 \\ J_{2,1}A_{1,2}w \\ 0 \\ J_{4,1}A_{1,2}w \end{bmatrix} = \begin{bmatrix} c_2 J_{0,2}w \\ 0 \\ c_2 J_{2,2}w \\ 0 \\ c_2 J_{4,2}w \end{bmatrix} = 0,
$$

so $A_3y = -Ay$. From this and equation (6.2.1), it follows that A^3y is just a scalar multiple of Ay, so $A^3y \in U$. It follows that U is a $\mathbb{T}$-module. Since every irreducible $\mathbb{T}$-module has dimension three, U cannot be reducible. Finally, U is orthogonal to the standard module.

The result in this subsection is summarized in the following lemma.

Lemma 6.2.1. *If $\mathbb{T}$ is the Terwilliger algebra of a bipartite distance-regular graph Γ of diameter four with vertex set V, then a subspace W of $\mathbb{C}^V$ orthogonal to the standard module is an irreducible $\mathbb{T}$-module with endpoint 1 if and only if it is spanned by the vectors*

$$
\begin{bmatrix} 0 \\ A_{1,2}w \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ w \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_{3,2}w \\ 0 \end{bmatrix},
$$

where $w \in \mathbb{C}^{\Gamma_2}$ is a vector satisfying

$$\mathbb{1}^T w = 0, \quad A_{1,2} w \neq 0 \quad and \quad A_{2,1} A_{1,2} w = \lambda w \ (with \ \lambda \neq 0).$$

$\square$

6.2.3 Modules with Endpoint 2

Let W be an irreducible $\mathbb{T}$-module with endpoint 2 that is orthogonal to the standard module. Let x be a nonzero vector in $F_2 W$, and let x_2 be its restriction to Γ_2. Since

$$Ax = \begin{bmatrix} 0 \\ A_{1,2} x_2 \\ 0 \\ A_{3,2} x_2 \\ 0 \end{bmatrix} \in W,$$

it follows that $A_{1,2} x_2 = 0$, so

$$A^2 x = \begin{bmatrix} 0 \\ 0 \\ A_{2,3} A_{3,2} x_2 \\ 0 \\ A_{4,3} A_{3,2} x_2 \end{bmatrix} \quad and \quad F_2 A^2 x = \begin{bmatrix} 0 \\ 0 \\ A_{2,3} A_{3,2} x_2 \\ 0 \\ 0 \end{bmatrix} \in W.$$

The restriction of $\mathrm{span}\{(F_2 A^2)^l x : l \geq 0\}$ to Γ_2 contains an eigenvector of $A_{2,3} A_{3,2}$. Let w be such an eigenvector, with eigenvalue λ. Then W contains the vectors

$$y := \begin{bmatrix} 0 \\ 0 \\ w \\ 0 \\ 0 \end{bmatrix}, \quad Ay = \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_{3,2} w \\ 0 \end{bmatrix} \quad and \quad A^2 y - \lambda y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ A_{4,3} A_{3,2} w \end{bmatrix}.$$

Conversely, suppose w is an eigenvector of $A_{2,3} A_{3,2}$ with eigenvalue λ, such that $A_{1,2} w = 0$ and $\mathbb{1}^T w = 0$. Let y have the form above. There are three types of irreducible $\mathbb{T}$-modules with endpoint 2 defined by w.

1. If $A_{3,2} w = 0$, then $Ay = 0$, so y spans a 1-dimensional $\mathbb{T}$-module.

2. If $A_{3,2}w \neq 0$ and $A_{4,3}A_{3,2}w = 0$, then it is straightforward to check that the subspace U spanned by y and Ay is A-invariant, so it is a 2-dimensional $\mathbb{T}$-module. Note that if $\lambda = 0$, then Ay spans a $\mathbb{T}$-module, so U is not irreducible in this case. On the other hand, it is straightforward to check that U is irreducible as a $\mathbb{T}$-module if $\lambda \neq 0$.

3. Suppose $A_{3,2}w \neq 0$ and $A_{4,3}A_{3,2}w \neq 0$. We show that y, Ay and $A^2 y - \lambda y$ form a basis of a $\mathbb{T}$-module. The three vectors are clearly linearly independent, and the subspace U they span is clearly invariant under the matrices F_i, so it suffices to show that $A^3 y$ is contained in U. Using the form of J^-, we have

$$(A + A_3)y = J^- y = \begin{bmatrix} 0 \\ J_{1,2}w \\ 0 \\ J_{3,2}w \\ 0 \end{bmatrix} = 0,$$

so $A_3 y = -Ay$. By equation (6.2.1) again, $A^3 y \in U$, so U is a $\mathbb{T}$-module. Suppose U is reducible. By Lemma 3.6 in [17], either y spans a $\mathbb{T}$-module or $A^2 y - \lambda y$ does. The former cannot happen since it was case 1, so $A^2 y - \lambda y$ must span a $\mathbb{T}$-module. Since

$$A(A^2 y - \lambda y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_{3,4}A_{4,3}A_{3,2}w \\ 0 \end{bmatrix},$$

we have $A_{3,4}A_{4,3}A_{3,2}w = 0$. In other words, if $A_{3,4}A_{4,3}A_{3,2}w \neq 0$, then U must be irreducible.

All three types are orthogonal to the standard module. Now let us summarize the result in this subsection.

Lemma 6.2.2. *If $\mathbb{T}$ is the Terwilliger algebra of a bipartite distance-regular graph of diameter four with vertex set V, then a subspace W of $\mathbb{C}^V$ orthogonal to the standard module is an irreducible $\mathbb{T}$-module with endpoint 2 if and only if W has one of the following three forms for some eigenvector w of $A_{2,3}A_{3,2}$, with eigenvalue λ, satisfying $\mathbb{1}^T w = 0$ and $A_{1,2}w = 0$:*

(i) the subspace spanned by

$$y := \begin{bmatrix} 0 \\ 0 \\ w \\ 0 \\ 0 \end{bmatrix}, \quad Ay = \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_{3,2}w \\ 0 \end{bmatrix} \quad and \quad A^2 y - \lambda y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ A_{4,3}A_{3,2}w \end{bmatrix},$$

where $A_{3,2}w$, $A_{4,3}A_{3,2}w$ and $A_{3,4}A_{4,3}A_{3,2}w$ are all nonzero.

(ii) the subspace spanned by y and Ay, where $A_{3,2}w \neq 0$, $A_{4,3}A_{3,2}w = 0$ and $\lambda \neq 0$.

(iii) the subspace spanned by y, where $A_{3,2}w = 0$ (and hence $\lambda = 0$).

$\square$

6.2.4   Modules with Endpoint 3

Let W be an irreducible $\mathbb{T}$-module with endpoint 3 that is orthogonal to the standard module. Let x be a nonzero vector in $F_3 W$, and let x_3 be its restriction to Γ_3. Since

$$Ax = \begin{bmatrix} 0 \\ 0 \\ A_{2,3}x_3 \\ 0 \\ A_{4,3}x_3 \end{bmatrix} \in W,$$

it follows that $A_{2,3}x_3 = 0$, so

$$A^2 x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_{3,4}A_{4,3}x_3 \\ 0 \end{bmatrix} \in W.$$

Just as in the previous cases, there exists eigenvector w of $A_{3,4}A_{4,3}$ (with $A_{2,3}w = 0$) such that

$$y := \begin{bmatrix} 0 \\ 0 \\ 0 \\ w \\ 0 \end{bmatrix} \quad \text{and} \quad Ay = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ A_{4,3}w \end{bmatrix}$$

are both in W.

Conversely, let w be an eigenvector of $A_{3,4}A_{4,3}$ with eigenvalue λ, such that $A_{2,3}w = 0$, and let y be defined as above. If $A_{4,3}w = 0$, then y spans a $\mathbb{T}$-module. If $A_{4,3}w \neq 0$, then since $A(Ay) = \lambda y$, the span of y and Ay is an irreducible $\mathbb{T}$-module if and only if $\lambda \neq 0$. Let us summarize the result in this subsection.

Lemma 6.2.3. *If $\mathbb{T}$ is the Terwilliger algebra of a bipartite distance-regular graph of diameter four with vertex set V, then a subspace W of $\mathbb{C}^V$ orthogonal to the standard module is an irreducible $\mathbb{T}$-module with endpoint 3 if and only if W has one of the following two forms for some eigenvector w of $A_{3,4}A_{4,3}$, with eigenvalue λ, satisfying $\mathbb{1}^T w = 0$ and $A_{2,3}w = 0$:*

(i) the subspace spanned by

$$y := \begin{bmatrix} 0 \\ 0 \\ 0 \\ w \\ 0 \end{bmatrix} \quad \text{and} \quad Ay = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ A_{4,3}w \end{bmatrix},$$

where $A_{4,3}w \neq 0$ and $\lambda \neq 0$.

(ii) the subspace spanned by y, where $A_{4,3}w = 0$ (and hence $\lambda = 0$).

$\square$

6.2.5 Modules with Endpoint 4

Let W be an irreducible $\mathbb{T}$-module with endpoint 4 that is orthogonal to the standard module. Let y be a nonzero vector in W whose restriction to Γ_4 is

w. Multiplying y on the left by A shows that $A_{3,4}w = 0$. Conversely, if w is a nonzero vector satisfying $A_{3,4}w = 0$ and y is the vector whose restriction to Γ_4 is w and is zero elsewhere, then $Ay = 0$, so y spans a $\mathbb{T}$-module. We just proved the following lemma.

Lemma 6.2.4. *If $\mathbb{T}$ is the Terwilliger algebra of a bipartite distance-regular graph of diameter four with vertex set V, then a subspace of $\mathbb{C}^V$ orthogonal to the standard module is an irreducible $\mathbb{T}$-module with endpoint 4 if and only if it is spanned by*

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ w \end{bmatrix},$$

where $w \neq 0$, $\mathbb{1}^T w = 0$ and $A_{3,4}w = 0$. □

6.2.6 Summary

The *support* of a subspace W of $\mathbb{C}^V$ is the set

$$\{i : F_i W \neq 0, 0 \leq i \leq 4\}.$$

We can define the support of a vector similarly. In summary, there are eight types of irreducible $\mathbb{T}$-modules for a bipartite distance-regular graph of diameter four.

Theorem 6.2.5. *If $\mathbb{T}$ is the Terwilliger algebra of a bipartite distance-regular graph of diameter four, then the irreducible $\mathbb{T}$-modules orthogonal to the standard module are precisely those described in Lemmas 6.2.1, 6.2.2, 6.2.3 and 6.2.4. The supports of all the irreducible $\mathbb{T}$-modules are given as follows:*

$$\{2\}, \{3\}, \{4\},$$

$$\{2, 3\}, \{3, 4\},$$

$$\{1, 2, 3\}, \{2, 3, 4\},$$

$$\{0, 1, 2, 3, 4\}.$$

□

6.3 Eigenvalues of Modules

Let W be an irreducible $\mathbb{T}$-module of a bipartite distance-regular graph Γ of diameter four. Then W is in particular A-invariant, so it has a basis consisting of eigenvectors of A. Let Λ_W be the multiset of eigenvalues of A corresponding to such a basis of W. In this section we determine Λ_W for each type of W. Recall that the eigenvalues of Γ are $\pm K$, $\pm \theta_1$ and 0, where

$$\theta_1 = \sqrt{K + c_2(K - c_3 - 1)}.$$

Lemma 6.3.1. *If W is the standard module then $\Lambda_W = \{\pm K, \pm \theta_1, 0\}$.*

Proof. Let λ be an eigenvalue of Γ. By computation using the form of A and the standard module, it follows that $\lambda \in \Lambda_W$ if and only if there exist complex numbers $x_0, \ldots, x_4$, not all zeroes, such that

$$\lambda \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} Kx_1 \\ x_0 + (K-1)x_2 \\ c_2x_1 + (K-c_2)x_3 \\ c_3x_2 + (K-c_3)x_4 \\ Kx_3 \end{bmatrix}.$$

This is equivalent to that the matrix

$$\begin{bmatrix} \lambda & -K & 0 & 0 & 0 \\ -1 & \lambda & 1-K & 0 & 0 \\ 0 & -c_2 & \lambda & c_2 - K & 0 \\ 0 & 0 & -c_3 & \lambda & c_3 - K \\ 0 & 0 & 0 & -K & \lambda \end{bmatrix}$$

does not have full rank. But indeed the matrix does not have full rank, since its determinant is

$$(K^2 - \lambda^2)(\theta_1^2 - \lambda^2)\lambda,$$

which is 0 for each λ. Therefore each eigenvalue of Γ is in Λ_W. $\qquad \square$

Since K and $-K$ are simple eigenvalues, if W is not the standard module, then Λ_W can only contain θ_1, 0 or $-\theta_1$. Also, if W is spanned by a single vector y, it has been shown in the previous section that $Ay = 0$, so $\Lambda_W = \{0\}$.

Suppose W has support $\{2, 3\}$. We first show that Λ_W cannot have a repeated element. Suppose otherwise. Then all nonzero vectors in W are

eigenvectors with the same eigenvalue, say μ. Pick a vector y whose support is $\{2\}$. Then $F_2 Ay = 0$. But $\mu y = \mu F_2 y = F_2 Ay$, so $\mu = 0$ and $Ay = 0$. Now y spans a $\mathbb{T}$-invariant subspace, which is impossible since W is irreducible. Hence Λ_W cannot have a repeated element. In Section 6.2.3, we showed that W has a basis $\{y, Ay\}$ such that $A^2 y = \lambda y$ for some $\lambda \neq 0$. Let

$$x := \alpha y + \beta Ay$$

be a vector in W. If $Ax = 0$, then

$$0 = \alpha Ay + \beta A^2 y = \alpha Ay + \beta \lambda y,$$

which implies $\alpha = \beta = 0$, whence $x = 0$. Therefore W contains no eigenvector of A with eigenvalue 0, and so $\Lambda_W = \{\theta_1, -\theta_1\}$. The same argument shows that if W has support $\{3, 4\}$, then $\Lambda_W = \{\theta_1, -\theta_1\}$.

Now suppose W has support $\{1, 2, 3\}$. In Section 6.2.2, we showed that W has a basis consisting of y, $\lambda^{-1} Ay$ and $\lambda^{-1} A^2 y - y$, where $\lambda \neq 0$ and $A_3 y = -Ay$. Clearly another basis of W is given by y, Ay and $A^2 y$. Consider the vector

$$x := \alpha y + \beta Ay + \gamma A^2 y.$$

By applying equation (6.2.1) and $A_3 y = -Ay$, we have

$$A^3 y = -c_2 c_3 Ay + (K + c_2 K - c_2) Ay = \theta_1^2 Ay,$$

from which we can deduce

$$Ax = A(\alpha y + \beta Ay + \gamma A^2 y) = \alpha Ay + \beta A^2 y + \gamma A^3 y$$
$$= \alpha Ay + \beta A^2 y + \gamma \theta_1^2 Ay = (\alpha + \gamma \theta_1^2) Ay + \beta A^2 y.$$

Therefore, $\mu x = Ax$ is equivalent to

$$\mu \alpha y + \mu \beta Ay + \mu \gamma A^2 y = (\alpha + \gamma \theta_1^2) Ay + \beta A^2 y.$$

It turns out that this equation always has a nonzero solution in (α, β, γ) if μ equals 0, θ_1 or $-\theta_1$, so eigenvectors with each of these eigenvalues exist in W, that is, $\Lambda_W = \{0, \pm\theta_1\}$. The same argument shows that $\Lambda_W = \{0, \pm\theta_1\}$ for W with support $\{2, 3, 4\}$.

We summarize the result in this section.

Theorem 6.3.2. *If W is an irreducible $\mathbb{T}$-module of a bipartite distance-regular graph of diameter four with eigenvalues $\pm K$, $\pm \theta_1$ and 0, then*

$$\Lambda_W = \begin{cases} \{0\}, & \text{if } \dim(W) = 1 \\ \{\pm\theta_1\}, & \text{if } \dim(W) = 2 \\ \{0, \pm\theta_1\}, & \text{if } \dim(W) = 3 \\ \{0, \pm\theta_1, \pm K\}, & \text{if } \dim(W) = 5 \end{cases}.$$

$\square$

6.4 Multiplicities of Modules

In Section 6.2, we saw that there are eight types of irreducible $\mathbb{T}$-modules of a bipartite distance-regular graph Γ of diameter four, depending on their supports. Suppose we decompose $\mathbb{C}^V$ into irreducible $\mathbb{T}$-modules. It is known that the number of irreducible $\mathbb{T}$-modules with a fixed support that appear in a decomposition is a constant that does not depend on the decomposition (see [19] and [14]). We call this number the *multiplicity* of the support. For the sake of studying how the combinatorial data of a graph could influence the structure of its $\mathbb{T}$-algebra, we aim to answer the following question: do all the multiplicities of supports for a $\mathbb{T}$-algebra of a bipartite distance-regular graph of diameter four depend only on the parameters of the graph? We do not have an answer to this question, but we show some partial progress in this section.

For each support S described in Theorem 6.2.5, let μ_S be its multiplicity in a decomposition of $\mathbb{C}^V$ into irreducible $\mathbb{T}$-modules. For clarity of notation, we write μ_{12} for $\mu_{\{1,2\}}$, for example. The eight multiplicities are then

$$\mu_{01234}, \quad \mu_{123}, \quad \mu_2, \quad \mu_{23}, \quad \mu_{234}, \quad \mu_3, \quad \mu_{34}, \quad \mu_4.$$

Since the standard module is the unique irreducible $\mathbb{T}$-module with endpoint 0, we have $\mu_{01234} = 1$. Since $F_1\mathbb{C}^V = K$, it follows that $\mu_{123} = K - 1$. In general, $F_i\mathbb{C}^V = N_i$. Using this fact and the two multiplicities we just found, by considering restriction of $\mathbb{C}^V$ to various Γ_i we can deduce the following three equations:

$$\mu_2 + \mu_{23} + \mu_{234} = N_2 - K, \tag{6.4.1}$$

$$\mu_{23} + \mu_{234} + \mu_3 + \mu_{34} = N_3 - K, \tag{6.4.2}$$

$$\mu_{234} + \mu_{34} + \mu_4 = N_4 - 1. \tag{6.4.3}$$

On the other hand, since the multiplicity of 0 as an eigenvalue of Γ is m_2, by Theorem 6.3.2, it follows that

$$\mu_2 + \mu_3 + \mu_4 + \mu_{234} = m_2 - K. \tag{6.4.4}$$

Similarly,

$$\mu_{23} + \mu_{234} + \mu_{34} = m_1 - K, \tag{6.4.5}$$

where m_1 is the multiplicity of θ_1 (and $-\theta_1$) as an eigenvalue of Γ. From Equations (6.4.2) and (6.4.5), we can deduce that

$$\mu_3 = N_3 - m_1.$$

Making substitution, we have a system of five linear equations in five variables. Unfortunately, it does not determine any of the variables uniquely. To summarize the progress in this section, we showed that

$$\mu_{01234} = 1, \quad \mu_{123} = K - 1, \quad \mu_3 = N_3 - m_1,$$

and it is not known whether the remaining five multiplicities are also determined by the graph parameters. The five equations we have in hand come from rather obvious facts. Can we find some less "obvious" equation that could help us solve for more multiplicities? Further investigation on this direction is needed.

Chapter 7

Future Research

We conclude the thesis by discussing open problems related to bipartite distance-regular graphs of diameter four.

In Chapter 4, we saw that the Krein parameter $q_{3,3}(3)$ is special in various aspects. We were able to characterize all the antipodal graphs with vanishing $q_{3,3}(3)$: they are the Hadamard graphs. In general, we saw that $q_{3,3}(3) = 0$ holds for a graph if and only if the graph is Q-polynomial. However, this does not tell anything about the combinatorial properties of a graph with a vanishing $q_{3,3}(3)$. Since strongly regular graphs with vanishing special Krein parameters have some interesting combinatorial structures, it would be natural to ask whether a bipartite distance-regular graph of diameter four has similar structures when $q_{3,3}(3)$ vanishes. This problem requires further attention and effort.

In Section 4.5, two infinite families of parameters satisfying $q_{3,3}(3) = 0$ were presented. One of the families contains an infinite sub-family that is realized by graphs. It is not known whether there are infinitely many graphs lying in the second family. Finding new graphs or ruling out parameters in this family provides a possible future research direction.

In Section 5.3, we presented examples of distance-regular graphs that yield $\{0, \alpha\}$-sets. Among the examples, only the 8-cycle and the 4-cube are antipodal. On the other hand, they are the only examples in the section that give rise to mutually unbiased bases. Godsil and Roy [26] showed that bipartite antipodal distance-regular graphs of diameter four give rise to mutually unbiased bases. However, it is not known whether the converse holds:

Problem. *Suppose that a $\{0, \alpha\}$-set constructed from a bipartite distance-regular graph Γ of diameter four using Theorem 5.2.7 form a collection of*

mutually unbiased bases. Does Γ have to be antipodal?

In Section 5.4, bounds by Delsarte, Goethals and Seidel are improved for flat $\{0, \alpha\}$-sets. These bounds, called the absolute bounds in [21], depend only on the dimension of the space but not the number α. There is a second type of bound in [21], called the special bound, which depends on both the dimension and the number α. The special bounds are described as follows.

Theorem (Delsarte, Goethals and Seidel). *Let S be a $\{0, \alpha\}$-set in $\mathbb{C}^m$ with $0 < \alpha < 1$. Then*

$$|S| \leq \frac{m(m+1)(1-\alpha)}{2 - (m+1)\alpha}.$$

If $S \subseteq \mathbb{R}^m$ then

$$|S| \leq \frac{m(m+2)(1-\alpha)}{3 - (m+2)\alpha}.$$

Although there are infinitely many $\{0, \alpha\}$-sets satisfying the special bounds at equality, none of the maximum examples within the author's knowledge is flat. Can the special bounds also be improved for flat $\{0, \alpha\}$-sets? Further investigation on the special bounds is needed.

In Chapter 6, we described all the irreducible $\mathbb{T}$-modules in an orthogonal decomposition of the Terwilliger algebra for a bipartite graph of diameter four. The following question is a natural continuation.

Problem. *For a bipartite distance-regular graph of diameter four, do the multiplicities of all types of irreducible $\mathbb{T}$-modules only depend on the parameters of the graph?*

In Section 6.4 we saw one direct approach to the problem, and used it to express the multiplicities for some types of modules in terms of the graph parameters. Still, it is not known whether the other multiplicities are also determined by the graph parameters. More investigation on this problem is needed.

Finally, we saw some connections between a bipartite distance-regular graph Γ of diameter four and its halved graphs in Chapters 3 and 4, and it is natural to study the connection between the Terwilliger algebra of Γ and that of its halved graphs. In particular, it would be interesting to study the relation between the irreducible $\mathbb{T}$-modules of Γ and those of its halved graphs.

Appendix A

Graph Terminology

We collect some standard terminology about graphs in this appendix.

A *graph* is an ordered pair (V, E), where V is a nonempty set and E is a set of 2-element subsets of V. The sets V and E are called the *vertex set* and the *edge set* of the graph, respectively, and their elements are called *vertices* and *edges* of the graph, respectively. Sometimes we use a single symbol Γ to denote a graph, in which case we use $V(\Gamma)$ to denote its vertex set. A graph is called *edgeless* if its edge set is empty.

In the rest of the appendix, we fix a graph Γ. If $\{u, v\}$ is an edge, we say that the vertices u and v are *adjacent*, and call u a *neighbour* of v. The number of neighbours of a vertex is called the *valency* of the vertex. If all vertices of Γ have the same valency, we say that Γ is *regular*, and we say that it is *k-regular* if k is the common valency. A *path* is a sequence

$$P := (v_0, v_1, \ldots, v_m)$$

of distinct vertices in which v_i and v_{i+1} are adjacent for all $i \in \{0, \ldots, m-1\}$; in this case v_0 and v_m are said to be *joined* by P, and m is called the *length* of the P. The *distance* between two vertices u and v is defined to be the minimum length among all the paths joining u and v. The graph Γ is called *connected* if every pair of vertices is joined by a path. A *connected component* of Γ is a maximal connected subgraph of Γ. If Γ is connected, then the *diameter* of Γ is the maximum taken over all distances between pairs of vertices of Γ. A *cycle* is a sequence

$$(v_0, v_1, \ldots, v_m, v_0)$$

of vertices such that v_0 and v_m are adjacent and $(v_0, v_1, \ldots, v_m)$ is a path; in this case $m + 1$ is called the *length* of the cycle. A *perfect matching* in Γ is a collection $\mathcal{M}$ of disjoint edges such that every vertex of Γ is contained in some edge in $\mathcal{M}$.

If every pair of distinct vertices are adjacent in Γ, we call Γ a *complete graph*. If $V(\Gamma)$ can be partitioned into at least two parts $C_1, C_2, \ldots, C_t$ (called the *colour classes*) such that vertices in the same part are not adjacent, then we call Γ a *multipartite graph* or a *t-partite graph*, and we call Γ a *complete multipartite graph* if vertices in distinct colour classes are always adjacent. We use the word *bipartite* to mean 2-partite. The *complement* of Γ is the graph $\overline{\Gamma}$ with the same vertex set as Γ such that two distinct vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ.

Finally, a *directed graph* is an ordered pair $(V, \mathcal{A})$, where V is a nonempty set and $\mathcal{A}$ is a subset of $V \times V$, in which case $\mathcal{A}$ is called the *arc set* of the directed graph.

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