On Moments and Related Quantities in Insurance Surplus Analysis

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In risk theory, the time to ruin is one of the central quantities. The Laplace transform, density and moments of the time to ruin have been studied by many authors under different risk model assumptions. The Gerber-Shiu function proposed by Gerber and Shiu (1998) provides an analytic tool in studying these quantities. For example, Dickson and Willmot (2005) inverted the Gerber-Shiu function with respect to the Laplace transform parameter of the time to ruin by Lagrange's implicit function theorem, and hence obtained the density of the time to ruin. The main focus of this thesis is to study the moments involving the time to ruin by using Gerber-Shiu function as the analytic tool. An introduction on the Gerber-Shiu function and different risk models is first given in Chapter 1.

In Chapter 2, the moments of the time to ruin are studied as generalized versions of the Gerber-Shiu function in dependent Sparre Andersen models. It is shown that structural properties of the Gerber-Shiu function hold also for the moments of the time to ruin. In particular, the moments continue to satisfy defective renewal equations. These properties are discussed in detail in Chapter 4 under the model of Willmot and Woo (2012) where Coxian interclaim times and arbitrary time-dependent claim sizes are assumed. In Chapter 3, another very general class of dependent Sparre Andersen models with Coxian claim sizes (e.g. Landriault et al. (2014)) is considered. An analytical form is provided for the moments of the time to ruin under this class, which involves solving linear systems of equations.

In Chapter 5, the number of claims until ruin is taken into consideration under a Sparre Andersen model with exponential claim sizes. The joint density of the time to ruin, the number of claims until ruin and other ruin-related quantities is identified. The joint moments of these quantities can then be obtained from this joint density.

In Chapter 6, the insurance surplus process is studied under a generalized MAP risk

model introduced in Cheung et al. (2011). With Coxian claim sizes, the moments of the time to ruin are in the form of a linear sum of Erlang densities. The associated coefficients of this linear sum are shown to satisfy linear systems of equations.

Finally, a brief conclusion of this thesis and a discussion of future research are given in Chapter 7.

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Dedication

To my parents

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Chapter 1

Introduction and background

In this chapter, the insurance surplus process is first introduced. Details are given on the dependent Sparre Andersen model and the MAP risk model. The Gerber-Shiu function and the moments of ruin-related quantities are then discussed. Mathematical preliminaries that are useful in this thesis are given at the end.

1.1 Dependent Sparre Andersen risk model

The insurance surplus process $\{U_t, t \geq 0\}$ is usually modelled by

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i, \tag{1.1}$$

where u ($u \geq 0$) is the initial surplus and c is the premium rate in one unit of time. $\{N_t, t \geq 0\}$ is a claim number process which is defined through a sequence of independent and identically distributed (iid) interclaim time random variables $\{V_i, i = 1, 2, \ldots\}$, where V_1 is the time until first claim and V_i is the time between $(i-1)$th and ith claim for

$i = 2, 3, \ldots$. $\{Y_i, i = 1, 2, \ldots\}$ is a sequence of claim size random variables which is iid. The pairs $\{(V_i, Y_i), i = 1, 2, \ldots\}$ are iid, but V_i and Y_i may be dependent. (1.1) is known as the dependent Sparre Andersen model (and simply known as the Sparre Andersen model if V_i and Y_i are independent for all $i = 1, 2, \ldots$). See for example Sparre Andersen (1957) and Rolski et al. (1999) for references on this model.

Let the marginal probability density function (pdf) and cumulative distribution function (cdf) of the interclaim time V be $k(t)$ and $K(t)$ respectively, where V is any arbitrary V_i. On the other hand, the marginal pdf and cdf of the claim size Y are denoted by $p(y)$ and $P(y)$ respectively, where Y is any arbitrary Y_i. Also, let $f(t, y)$ be the joint pdf of the pair (V, Y) when $V = t$ and $Y = y$. Finally, let us assume that the positive security loading condition

$$E[cV] > E[Y] \tag{1.2}$$

holds in (1.1).

The classical Poisson risk model is one of the well-known special cases of the dependent Sparre Andersen model. In this model, the joint pdf of the interclaim time and claim size is given by

$$f(t, y) = \lambda e^{-\lambda t} p(y).$$

In other words, the classical Poisson risk model assumes that $\{V_i, i = 1, 2, \ldots\}$ are independent of $\{Y_i, i = 1, 2, \ldots\}$ and $\{V_i, i = 1, 2, \ldots\}$ follow exponential distribution. Readers may refer to e.g. Gerber (1979), Grandell (1991) and Panjer and Willmot (1992) for a complete introduction on the classical Poisson risk model. There are also studies on the dependent Sparre Andersen model with more general interclaim times and claim sizes. Recent examples include Albrecher and Teugels (2006), Boudreault et al. (2006), Cossette et al. (2008), Zhang et al. (2012) and references therein.

1.2 Ruin-related quantities and Gerber-Shiu function

1.2.1 Gerber-Shiu function

Let T be the time to ruin for the process $\{U_t, t \geq 0\}$, which is defined by

$$T = \inf\{t \geq 0 : U_t < 0\} \tag{1.3}$$

and $T = \infty$ if U_t is non-negative for all $t \geq 0$. The Gerber-Shiu function introduced in Gerber and Shiu (1998) is defined as

$$m_\delta(u) = E[e^{-\delta T} w(U_{T-}, |U_T|) I(T < \infty) | U_0 = u], \tag{1.4}$$

where $\delta \geq 0$, the penalty function $w(x, y)$ satisfies mild integrablility conditions and $I(A)$ is an indicator function which takes value 1 if the event A occurs and 0 otherwise. The random variables U_{T-} and $|U_T|$ represent the surplus before ruin and the deficit at ruin respectively. Before the Gerber-Shiu function was introduced, the joint density of U_{T-} and $|U_T|$ had been studied in Dufresne and Gerber (1988) under the classical Poisson risk model. The probability of ruin

$$m_0(u) = E[I(T < \infty) | U_0 = u]$$

is a special case of (1.4) with $\delta = 0$ and $w(x, y) = 1$.

Under the classical Poisson risk model, it was shown in Gerber and Shiu (1998) that (1.4) follows a defective renewal equation. Lin and Willmot (1999) gave the solution to this equation in the form of a compound geometric tail. The Gerber-Shiu function is also considered in more general Sparre Andersen model. For example, Dickson and Hipp (2001), Li and Garrido (2004) and Gerber and Shiu (2005) studied with Erlang interclaim time. Willmot (2007) and Landriault and Willmot (2008) further the studies with arbitrary interclaim time.

There is also literature on the Gerber-Shiu function in dependent Sparre Andersen model. Albrecher and Boxma (2004) assumed a Markovian claim arrival process. Badescu et al. (2009) considered a bivariate phase-type distribution for the interclaim time and the claim size. Albrecher et al. (2011) introduced dependence by mixing distribution.

Next, let us denote the number of claims until ruin by N_T, which is also a widely studied random variable in risk theory. Stanford et al. (2000) developed a recursive method through the number of claims until ruin in order to calculate the probability of ruin. Egidio dos Reis (2002) studied the distribution of the number of claims until ruin under the classical Poisson risk model. In Landriault et al. (2011), the number of claims until ruin is introduced to the Gerber-Shiu function as

$$m_{r,\delta}(u) = E[r^{N_T} e^{-\delta T - sU_{T-}} I(T < \infty) | U_0 = u], \tag{1.5}$$

where $r \in (0, 1]$ and $s \geq 0$. With exponential claim sizes, closed form expression for (1.5) is obtained in the paper.

In Cheung et al. (2010), another generalization of the Gerber-Shiu function that also involves N_T is proposed as

$$m_\delta(u) = E[e^{-\delta T} w(U_{T-}, |U_T|, X_T, R_{N_T-1}) I(T < \infty) | U_0 = u] \tag{1.6}$$

for $\delta \geq 0$. X_t denotes the minimum surplus before time t, i.e. $X_t = \inf_{0 \leq s < t} U_s$. R_n is defined as $R_0 = u$ and $R_n = u + \sum_{i=1}^{n}(cV_i - Y_i)$ for $n = 1, 2, \ldots$. Therefore, X_T is the minimum surplus before ruin. R_{N_T-1} is equal to u if ruin occurs on first claim, and for ruin occurs on claim subsequent to the first, it is the surplus immediately after the second last claim before ruin.

Ruin-related quantities	$N_T = 1$	$N_T > 1$
U_{T-}	U_{T-}	U_{T-}
$\|U_T\|$	$\|U_T\|$	$\|U_T\|$
T	$\frac{U_{T-}-u}{c}$	$\inf\{t \geq 0 : U_t < 0\}$
X_T	u	$\inf\limits_{0\leq s<T} U_s$
R_{N_T-1}	u	$u + \sum_{i=1}^{N_T-1}(cV_i - Y_i)$

Table 1.1: Value of ruin-related quantities when $N_T = 1$ and $N_T > 1$

1.2.2 Moments of ruin-related quantities

In Gerber-Shiu function (1.4), the (joint) moments of the surplus before ruin and the deficit at ruin is easily obtained by considering the penalty function

$$w(x, y) = x^k y^n,$$

where k and n are non-negative integers. Lin and Willmot (2000) showed that the moments of the surplus before ruin and the deficit at ruin can be expressed analytically using compound geometric tails in the classical risk model.

Next, consider the moments of the time to ruin which may be studied in two approaches. The first approach is to determine the (defective) density of the time to ruin and obtain the moments of the time to ruin by integration. To be specific, suppose the (defective) density of the time to ruin given initial surplus u is $g(t|u)$, then the kth moment of the time to ruin can be calculated as

$$E[T^k I(T < \infty)|U_0 = u] = \int_0^\infty t^k g(t|u)dt$$

for $k = 0, 1, 2, \ldots$. The density of the time to ruin has been studied with different model assumptions in the literature. Drekic and Willmot (2003) determined the density of the time to ruin under the classical Poisson risk model with exponential claim sizes. In a classical Poisson risk model with arbitrary claim sizes, Dickson and Willmot (2005) inverted the Gerber-Shiu function to determine the density of the time to ruin by Lagrange's implicit function theorem (e.g. Good (1960) and Goulden and Jackson (1983)). This result was generalized in Landriault and Willmot (2009) where the joint distribution of the time to ruin, the surplus before ruin and the deficit at ruin was given. Recently, Landriault and Shi (2013) assumed combination of n exponentials claim sizes and obtained the density of T by multivariate Lagrange expansion.

The second approach to study the moments of the time to ruin is by noting that they are closely related to the Gerber-Shiu function. To see this, let us consider

$$m_\delta(u) = E[e^{-\delta T} I(T < \infty) | U_0 = u]$$

without loss of generality. (If joint moments of the time to ruin and other ruin quantities are of interest, then consider $m_\delta(u)$ with an appropriate penalty function). Define the discounted kth moment of the time to ruin as

$$m_{k,\delta}(u) = E[T^k e^{-\delta T} I(T < \infty) | U_0 = u], \tag{1.7}$$

where $k = 0, 1, 2, \ldots$, then it is obvious that (1.7) can be obtained by differentiating the Gerber-Shiu function kth times with respect to δ, i.e.

$$m_{k,\delta}(u) = (-1)^k \frac{\partial^k}{\partial \delta^k} m_\delta(u). \tag{1.8}$$

To show that the above differentiation is valid, one can apply the Lebesgue's dominated convergence theorem (e.g. Resnick (2005)). The Lebesgue's dominated convergence theorem can be applied when the integrand of $m_\delta(u) = E[e^{-\delta T} I(T < \infty) | U_0 = u]$ is assumed to satisfy mild integrability conditions. These conditions implicitly follow from the tacit assumption that all moments of the time to ruin considered in this thesis are finite. For more general Gerber-Shiu functions, these integrability conditions impose restrictions on the penalty functions involved. For evaluation of marginal moments of the time to ruin, it may be assumed that the penalty function is 1.

Under the classical Poisson risk model, Lin and Willmot (2000) gave a recursive equation for the moments of the time to ruin. The equation was solved in Willmot (2002) by using the compound geometric distribution and its higher-order equilibrium distributions. These results were recursive in nature and hence involved complicated calculation. Hence, Drekic et al. (2004) and Drekic and Willmot (2005) studied these results from a computational point of view and provided numerical examples by assuming phase-type claim sizes.

There are also studies on the moments of the time to ruin under more general risk models. For example, Dickson and Hipp (2001) considered a Sparre Andersen model with Erlang(2) interclaim times, and Li and Lu (2013) assumed a surplus process with interest.

1.3 MAP risk model

There are also studies on the risk model with a Markovian arrival process (MAP). Interested readers may refer to Neuts (1979) and Latouche and Ramaswami (1999) for introduction on MAP. Many papers, e.g. Ahn and Badescu (2007) and Cheung and Landriault (2010), had analysis of the Gerber-Shiu function in a MAP risk model. In Yu et al. (2010), the moments of the time to ruin were studied in a MAP risk model with phase type claim sizes. A brief description of the MAP risk model, mainly based on Cheung et al. (2011), is given in the following.

For a MAP, it involves a homogeneous continuous-time Markov chain (CTMC). Let this CTMC be $\boldsymbol{Y} = \{Y(t), t \geq 0\}$ defined on a finite state space $S = \{1, \ldots, m\}$. In the context of a risk model, CTMC $\boldsymbol{Y}$ may involve two kinds of transitions which are represented by the transition rate matices $\boldsymbol{D_0}$ and $\boldsymbol{D_1}$ respectively. The (i, j)th entry of

1. $\boldsymbol{D_0}$, where $i \neq j$, is the transition rate the CTMC $\boldsymbol{Y}$ changes from state i to state j with no claim happening;

2. $\boldsymbol{D_1}$ ($i = j$ is also being considered), is the transition rate the CTMC $\boldsymbol{Y}$ changes from state i to state j with a claim happening.

For convenience, either kind of transition will be referrred as a system change in this section and in Chapter 6. The (i, i)th entry of $\boldsymbol{D_0}$ is negative and its absolute value is equal to the rate of a system change given that the CTMC $\boldsymbol{Y}$ is in state i. The sum of the ith row of $\boldsymbol{D_0} + \boldsymbol{D_1}$ should add up to zero. For a MAP risk model, if the CTMC is in state i, the waiting time of a system change follows an exponential distribution with mean equals to the absolute value of the inverse of the (i, i)th entry of $\boldsymbol{D_0}$.

The MAP includes many well-known processes as special cases. When $m = 1$, $\boldsymbol{D_1} = (\lambda)$ and $\boldsymbol{D_0} = (-\lambda)$, the MAP reduces to a homogeneous Poisson process with arrival rate λ. When

$$\boldsymbol{D_1} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix}$$

with $\lambda_i > 0$ for all $i = 1, 2, \ldots, m$ and $\boldsymbol{D_0}$ has non-negative off-diagonal entries, then it is the Markov modulated Poisson process (MMPP). Readers can refer to e.g. He (2014) for more on special cases and applications of the MAP.

1.4 Mathematical notations and preliminaries

In this section, some mathematical notations and preliminaries used in the following chapters are introduced. Also, note the notational convention $\sum_{j=i}^{k} = 0$ for $i > k$ in this thesis.

1.4.1 Laplace transform

For an integrable function $f(\cdot)$ defined on $(0, \infty)$, denote its Laplace transform by

$$\tilde{f}(s) = \int_0^\infty e^{-sx} f(x)dx,$$

where s can be any number with non-negative real part. Unless otherwise specified, this notation of Laplace transform is used throughout the thesis.

For more about the properties of Laplace transform, please refer to Widder (2010).

1.4.2 Dickson-Hipp operator

Given an integrable function $f(\cdot)$ defined on $(0, \infty)$, its Dickson-Hipp operator is denoted by

$$T_r f(u) = \int_u^\infty e^{-r(y-u)} f(y)dy, \quad u \geq 0,$$

where the parameter r can be any number with non-negative real part. One special case is $T_r f(0) = \tilde{f}(r)$. For $r_1 \neq r_2$,

$$T_{r_1} T_{r_2} f(u) = T_{r_2} T_{r_1} f(u) = \frac{T_{r_1} f(u) - T_{r_2} f(u)}{r_2 - r_1}. \tag{1.9}$$

Readers can refer to Dickson and Hipp (2001) for details.

Also, if the operator is applied n times with the same parameter r, where $n = 1, 2, \ldots$, then it is given in Li and Garrido (2004) that

$$T_r^n f(u) = T_r T_r \cdots T_r f(u) = \int_u^\infty \frac{(y-u)^{n-1}}{(n-1)!} e^{-r(y-u)} f(y) dy. \tag{1.10}$$

1.4.3 Initial value theorem

For a continuous function $f(\cdot)$ on $(0, \infty)$, if its derivative $f(\cdot)$ is piecewise continuous on $[0, \infty)$, then

$$\lim_{s \to \infty} s\tilde{f}(s) = \lim_{x \to 0} f(x).$$

Readers can refer to Schiff (1999) for a complete introduction on the initial value theorem.

1.4.4 Coxian distribution

The class of Coxian distributions is now introduced, and it is one of the main classes of distributions considered in later chapters. For a continuous distribution with pdf $f(x)$, it belongs to the class of Coxian-n distributions if its Laplace transform is given by

$$\tilde{f}(s) = \frac{a(s)}{\prod_{i=1}^{m} (\lambda_i + s)^{n_i}}, \tag{1.11}$$

where $\lambda_i, n_i > 0$ for $i = 1, \ldots, m$, $\lambda_i \neq \lambda_j$ for $i \neq j$ and $n = \sum_{i=1}^m n_i$. Moreover, $a(s)$ is a polynomial in s with a degree of at most $n - 1$. It follows from (1.11) that the Coxian-n pdf has the form

$$f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} p_{ij} \frac{\lambda_i (\lambda_i x)^{j-1} e^{-\lambda_i x}}{(j-1)!}.$$

For a detailed discussion on the properties and special cases of Coxian distributions, see e.g. Klugman et al. (2013).

1.4.5 Introduction to defective renewal equation

In this section, defective renewal equations which often arise in risk theory are reviewed. Interested readers may refer to Ross (1996) for a complete introduction on renewal theory.

From e.g. Resnick (1992) and Willmot and Lin (2001), a non-negative function $m(u)$ is said to satisfy defective renewal equation if

$$m(u) = \phi \int_0^u m(u-y)dF(y) + v(u), \quad u \geq 0, \tag{1.12}$$

where $0 < \phi < 1$, $F(y)$ is a distribution function such that $F(0) = 0$ and $v(u)$ is a non-negative continuous function. It was given in Willmot and Lin (2001) that (1.12) has solution

$$m(u) = \frac{1}{1-\phi} \int_0^u v(u-y)dG(y) + v(u), \tag{1.13}$$

where $G(y) = 1 - \bar{G}(y)$ is a compound geometric distribution defined by

$$\bar{G}(y) = \sum_{n=1}^{\infty} (1-\phi)\phi^n \bar{F}^{*n}(y), \quad y \geq 0$$

and $F^{*n}(y) = 1 - \bar{F}^{*n}(y)$ is the n-fold convolution of $F(y)$.

There is also asymptotic solution to (1.12). If there exists an adjustment coefficient $\rho > 0$ such that $\int_0^\infty e^{\rho y} dF(y) = 1/\phi$ where $F(y)$ is a non-arithmetic distribution and $e^{\rho u} v(u)$ is directly Riemann integrable, then (e.g. Willmot and Lin (2001))

$$m(u) \sim \frac{\int_0^\infty e^{\rho y} v(y) dy}{\phi \int_0^\infty y e^{\rho y} dF(y)} e^{-\rho u}, \quad u \to \infty, \tag{1.14}$$

where $f(x) \sim g(x)$, $x \to \infty$, represents $\lim\limits_{x\to\infty} f(x)/g(x) = 1$.

Lower and upper bounds of $m(u)$ in (1.12) are also given in the literature. For example, Willmot et al. (2001) showed that

$$\alpha_1(u)e^{-\rho u} \leq m(u) \leq \alpha_2(u)e^{-\rho u}, \tag{1.15}$$

where $\alpha_1(u) = \inf_{0 \leq z \leq u} \alpha(z)$, $\alpha_2(u) = \sup_{0 \leq z \leq u} \alpha(z)$ and

$$\alpha(z) = \frac{e^{\rho z} v(z)}{\phi \int_z^{\infty} e^{\rho y} dF(y)}.$$

1.5 Outline of the thesis

In Chapter 2, the structural properties of the Gerber-Shiu function are generalized to the moments of the time to ruin. In particular, the moments of the time to ruin are shown to continue satisfy defective renewal equations, which is a useful result for the studies in later chapters.

In Chapter 3, a dependent Sparre Andersen model with Coxian claim sizes is considered. The associated coefficients of the Gerber-Shiu function is first studied as a follow-up of the results in Landriault et al. (2014). Then the moments of the time to ruin are considered and an analytical solution is given for the moments.

In Chapter 4, structural properties of the moments of the time to ruin are studied under a dependent Sparre Andersen model with Coxian interclaim times. The structural quantities needed to determine the moments are specified under this model.

In Chapter 5, the joint density of the time to ruin and other ruin-related quantities is determined under a Sparre Andersen model with exponential claim sizes. Using this joint density, the marginal and joint moments of these ruin-related quantities can be obtained by integration.

In Chapter 6, the moments of the time to ruin are considered under a generalized MAP risk model. By assuming Coxian claim sizes, the moments are shown to be in the form of a linear sum of Erlang densities.

Finally in Chapter 7, a conclusion of this thesis and a discussion of future research is given.

Chapter 2

Structural properties of the moments of the time to ruin

In this chapter, structural properties of the Gerber-Shiu function in dependent Sparre Andersen models are first introduced. These properties are then generalized to the moments of the time to ruin.

2.1 Introduction to structural properties of Gerber-Shiu function

The Gerber-Shiu functions introduced in Section 1.2.1 are shown to satisfy defective renewal equations by many authors. For example, readers can refer to Gerber and Shiu (1998), Cheung et al. (2010) and Landriault et al. (2011) for references. Based on these references, a brief description of the argument is given in the following.

Consider the generalized Gerber-Shiu function (1.6), i.e.

$$m_\delta(u) = E[e^{-\delta T} w(U_{T-}, |U_T|, X_T, R_{N_T-1}) I(T < \infty) | U_0 = u] \tag{2.1}$$

under a dependent Sparre Andersen model introduced in section 1.1. Given initial surplus u and for ruin occured on the first claim, let $h_1(x, y|u)$ be the joint defective density of the surplus before ruin (x) and the deficit at ruin (y). Since ruin is on the first claim, the time of ruin (t) is given by $t = (x-u)/c$ and hence

$$h_1(x, y|u) = \begin{cases} \frac{1}{c} f\left(\frac{x-u}{c}, x+y\right), & x > u \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Also, by definition, $X_T = u$ and $R_{N_T-1} = u$ if ruin is on the first claim. Given initial surplus u and for ruin on claims subsequent to the first, let

$$h_2(t, x, y, v|u) \tag{2.3}$$

where $v < x$, be the joint defective density of the time of ruin (t), the surplus before ruin (x), the deficit at ruin (y) and the surplus immediately after the second last claim before ruin (v).

Then, define the discounted densities

$$h_{1,\delta}(x, y|u) = e^{-\delta\left(\frac{x-u}{c}\right)} h_1(x, y|u), \tag{2.4}$$

$$h_{2,\delta}(x, y, v|u) = \int_0^\infty e^{-\delta t} h_2(t, x, y, v|u) dt \tag{2.5}$$

and

$$h_\delta(x, y|u) = h_{1,\delta}(x, y|u) + \int_0^x h_{2,\delta}(x, y, v|u) dv. \tag{2.6}$$

Cheung et al. (2010) showed that (2.1) satisfy the defective renewal equation

$$m_\delta(u) = \phi_\delta \int_0^u m_\delta(u-y) f_\delta(y) dy + v_\delta(u), \tag{2.7}$$

where $f_\delta(y)$ is the discounted ladder height density defined by

$$f_\delta(y) = \frac{1}{\phi_\delta} \int_0^\infty h_\delta(x, y|0) dx \tag{2.8}$$

with $\phi_\delta = \int_0^\infty \int_0^\infty h_\delta(x, y|0) dx dy$, and

$$\begin{aligned} v_\delta(u) = & \int_u^\infty \int_0^\infty w(x+u, y-u, u, u) h_{1,\delta}(x, y|0) dx dy \\ & + \int_u^\infty \int_0^\infty \int_0^x w(x+u, y-u, u, v+u) h_{2,\delta}(x, y, v|0) dv dx dy. \end{aligned} \tag{2.9}$$

According to (2.7) to (2.9), Cheung et al. (2010) noted that the discounted joint density of $(U_{T-}, |U_T|, R_{N_T - 1})$ characterizes the Gerber-Shiu function with $(U_{T-}, |U_T|, X_T, R_{N_T - 1})$, so one can examine

$$m_\delta(u) = E[e^{-\delta T} w(U_{T-}, |U_T|, R_{N_T-1}) I(T < \infty) | U_0 = u]$$

instead of (2.1) without loss of generality.

2.2 Structural properties of the moments of the time to ruin

In this section, structural properties of the Gerber-Shiu function are generalized to the moments of the time to ruin.

Under the Poisson risk model, Lin and Willmot (2000) showed that the mean time to ruin $E[Tw(U_{T-}, |U_T|) I(T < \infty) | U_0 = u]$ and the higher moments of the time to ruin

$E[T^k I(T < \infty)|U_0 = u]$ for $k = 2, 3, \ldots$ satisfy defective renewal equations. This result will now be generalized in dependent Sparre Andersen models with more general form of the moments of ruin time. Consider the following generalized kth moment of the time to ruin

$$m_{k,\delta}(u) = E[T^k e^{-\delta T} w(U_{T-}, |U_T|, X_T, R_{N_T - 1}) I(T < \infty)|U_0 = u], \tag{2.10}$$

for $k = 0, 1, 2, \ldots$, which includes a four variables penalty function as in (2.1). For representation of the following results, define

$$h_{1,\delta}^{*k}(x, y|u) = \left(\frac{x - u}{c}\right)^k h_{1,\delta}(x, y|u), \tag{2.11}$$

$$h_{2,\delta}^{*k}(x, y, v|u) = \int_0^\infty t^k e^{-\delta t} h_2(t, x, y, v|u) dt \tag{2.12}$$

and

$$h_\delta^{*k}(x, y|u) = h_{1,\delta}^{*k}(x, y|u) + \int_0^x h_{2,\delta}^{*k}(x, y, v|u) dv \tag{2.13}$$

for $k = 0, 1, 2, \ldots$. In fact, (2.11) to (2.13) are functions related to (2.4) to (2.6) respectively by a kth order differentiation with respect to δ.

Theorem 2.2.1. *Consider the dependent Sparre Andersen model as described in Section 1.1 with initial surplus u. The generalized kth moment of the time to ruin, i.e.* $m_{k,\delta}(u)$ *defined in* (2.10)*, satisfies a defective renewal equation. For* $k = 0, 1, 2, \ldots$,

$$m_{k,\delta}(u) = \phi_\delta \int_0^u m_{k,\delta}(u-y) f_\delta(y) dy + v_{k,\delta}(u), \tag{2.14}$$

where $\phi_\delta = \int_0^\infty \int_0^\infty h_\delta(x, y|0) dx dy$, $f_\delta(y) = \frac{1}{\phi_\delta} \int_0^\infty h_\delta(x, y|0) dx$ *and*

$$\begin{aligned} v_{k,\delta}(u) = & \sum_{j=1}^{k} \binom{k}{j} \int_0^u m_{k-j,\delta}(u-y) \int_0^\infty h_\delta^{*j}(x, y|0) dx dy \\ & + \int_u^\infty \int_0^\infty w(x+u, y-u, u, u) h_{1,\delta}^{*k}(x, y|0) dx dy \\ & + \int_u^\infty \int_0^\infty \int_0^x w(x+u, y-u, u, v+u) h_{2,\delta}^{*k}(x, y, v|0) dv dx dy. \end{aligned} \tag{2.15}$$

For $k = 0$, (2.14) *reduces to* (2.7).

Proof. First, rewrite (2.7) as

$$m_\delta(u) = \int_0^u m_\delta(u-y) f_\delta^{**}(y) dy + v_\delta(u), \tag{2.16}$$

where $f_\delta^{**}(y) = \phi_\delta f_\delta(y) = \int_0^\infty h_\delta(x, y|0) dx$.

Differentiate (2.16) k times with respect to δ, which yields

$$\frac{\partial^k m_\delta(u)}{\partial \delta^k} = \sum_{j=0}^{k} \binom{k}{j} \int_0^u \frac{\partial^{k-j} m_\delta(u-y)}{\partial \delta^{k-j}} \frac{\partial^j f_\delta^{**}(y)}{\partial \delta^j} dy + \frac{\partial^k v_\delta(u)}{\partial \delta^k}. \tag{2.17}$$

The first term on the right hand side of (2.17) is obtained by applying the generalized product rule (General Leibniz rule).

Then multiplying $(-1)^k$ on both sides of (2.17) gives

$$m_{k,\delta}(u) = \int_0^u m_{k,\delta}(u-y) f_\delta^{**}(y) dy + \sum_{j=1}^{k} \binom{k}{j} \int_0^u m_{k-j,\delta}(u-y) \left\{ (-1)^j \frac{\partial^j f_\delta^{**}(y)}{\partial \delta^j} \right\} dy$$
$$+ (-1)^k \frac{\partial^k v_\delta(u)}{\partial \delta^k}. \tag{2.18}$$

For $j = 1, \ldots, k$,

$$\begin{aligned} \frac{\partial^j f_\delta^{**}(y)}{\partial \delta^j} &= \int_0^\infty \frac{\partial^j h_\delta(x, y|0)}{\partial \delta^j} dx \\ &= \int_0^\infty \left\{ \left(-\frac{x}{c} \right)^j e^{-\delta x/c} h_1(x, y|0) + \int_0^x \int_0^\infty (-t)^j e^{-\delta t} h_2(t, x, y, v|0) dt dv \right\} dx \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^k v_\delta(u)}{\partial \delta^k} &= \int_u^\infty \int_0^\infty w(x+u, y-u, u, u) \left\{ \left(-\frac{x}{c} \right)^k e^{-\delta x/c} h_1(x, y|0) \right\} dx dy \\ &+ \int_u^\infty \int_0^\infty \int_0^x w(x+u, y-u, u, v+u) \\ &\times \left\{ \int_0^\infty (-t)^k e^{-\delta t} h_2(t, x, y, v|0) dt \right\} dv dx dy, \end{aligned}$$

which yields (2.14) and (2.15) by substituting into (2.18). □

Given a dependent Sparre Andersen model, Theorem 2.2.1 shows that if the functions $h_{2,\delta}^{*k}(x, y, v|0)$ are known for all $k = 0, 1, 2, \ldots$, then $m_{k,\delta}(u)$ can be solved recursively in k. The defective renewal equations (2.14) need to be solved recursively since the function $v_{k,\delta}(u)$ is defined by $m_{j,\delta}(u)$ for $j = 0, 1, 2, \ldots, k-1$ as shown in (2.15).

Finally, if the defective renewal equation (2.14) is completely specified, then its solution is given by

$$m_{k,\delta}(u) = \frac{1}{1-\phi_\delta} \int_0^u v_{k,\delta}(u-y) g_\delta(y) dy + v_{k,\delta}(u), \tag{2.19}$$

where $g_\delta(y) = \frac{d}{dy}G_\delta(y)$ and $G_\delta(y) = 1-\bar{G}_\delta(y)$ is a compound geometric distribution defined by

$$\bar{G}_\delta(y) = \sum_{n=1}^{\infty}(1-\phi_\delta)\phi_\delta^n \bar{F}_\delta^{*n}(y), \quad y \geq 0. \tag{2.20}$$

In (2.20), $F_\delta^{*n}(y) = 1-\bar{F}_\delta^{*n}(y)$ is the n-fold convolution of the distribution function $F_\delta(y) = \int_0^y f_\delta(x)dx$. Readers may refer to Section 1.4.5 for details on solution of defective renewal equation. However, the asymptotic result in (1.14) is of limited applicability in the present situation because the constant $\int_0^\infty e^{\rho y}v(y)dy/\phi\int_0^\infty ye^{\rho y}dF(y)$ is often infinite.

Chapter 3

Dependent Sparre Andersen model with Coxian claim size assumption

In this chapter, a dependent Sparre Andersen model with Coxian claim sizes is considered. The Gerber-Shiu function was shown in Landriault et al. (2014) that it is a linear sum of exponential terms. The associated coefficients of these exponential terms are studied in the first part of this chapter as a follow-up of the results in Landriault et al. (2014).

The moments of the time to ruin are considered in the second part of this chapter. The moments are shown to be in the form of a linear sum. Numerical examples involving the mean and variance of the time to ruin are discussed in detail. These results have been submitted as Lee and Willmot (2014a).

3.1 Background

3.1.1 Model introduction

Recall the dependent Sparre Andersen model introduced in section 1.1, where the interclaim time V and claim size Y are dependent. In this chapter, assume the following joint pdf of (V, Y)

$$f(t, y) = \sum_{i=1}^{m} \sum_{h=1}^{n_i} g_{ih}(t) e_{\beta_i,h}(y), \quad t, y \geq 0, \tag{3.1}$$

with $e_{\beta,h}(y)$ representing the Erlang pdf

$$e_{\beta,h}(y) = \frac{\beta(\beta y)^{h-1} e^{-\beta y}}{(h-1)!}, \quad y > 0.$$

It can be easily seen that the marginal pdf of Y is

$$p(y) = \sum_{i=1}^{m} \sum_{h=1}^{n_i} \left\{ \int_0^\infty g_{ih}(t) dt \right\} e_{\beta_i,h}(y), \tag{3.2}$$

which is a Coxian-n pdf with $n = \sum_{i=1}^{m} n_i$. The class of joint pdfs (3.1) includes a large class of dependency models; interested readers may refer to Landriault et al. (2014) for special cases of (3.1).

The Gerber-Shiu function considered in this chapter is of the form

$$m_\delta(u) = E[e^{-\delta T} w(|U_T|) I(T < \infty) | U_0 = u], \tag{3.3}$$

where the penalty function involves the deficit at ruin only. Interested readers may refer to Landriault and Willmot (2008) for a similar model but with a more general penalty function which includes the surplus before ruin.

3.1.2 Background result

The approach used and the result obtained in Landriault et al. (2014) will now be briefly described as background.

First, Landriault et al. (2014) showed by using probabilistic arguments that (2.6) can be expressed as

$$h_\delta(x, y|u) = \sum_{i=1}^{m} \sum_{h=1}^{n_i} \xi_{\delta,ih}(x|u) e_{\beta_i,h}(y)$$

for some functions $\xi_{\delta,ij}(x|u)$, where $i = 1, \ldots, m; h = 1, \ldots, n_i$, and hence the discounted ladder height density (2.8) becomes

$$f_\delta(y) = \sum_{i=1}^{m} \sum_{h=1}^{n_i} \xi_{\delta,ih} e_{\beta_i,h}(y) \tag{3.4}$$

with $\xi_{\delta,ih} = \phi_\delta^{-1} \int_0^\infty \xi_{\delta,ih}(x|0)dx$ and $\phi_\delta = \int_0^\infty \int_0^\infty h_\delta(x, y|0)dxdy$.

Then as shown in (2.7), it was given in Cheung et al. (2010) that (3.3) satisfies the defective renewal equation

$$m_\delta(u) = \phi_\delta \int_0^u m_\delta(u - y) f_\delta(y)dy + v_\delta(u), \tag{3.5}$$

where

$$v_\delta(u) = \phi_\delta \int_0^\infty w(y) f_\delta(u + y)dy. \tag{3.6}$$

Take Laplace transform of (3.5) yields

$$\tilde{m}_\delta(s) = \frac{\tilde{v}_\delta(s)}{1 - \phi_\delta \tilde{f}_\delta(s)}. \tag{3.7}$$

Using the Laplace transform of (3.4), it follows that (3.7) can be expressed as

$$\tilde{m}_\delta(s) = \sum_{z=1}^{n} \frac{C_{z,\delta}}{s + R_{z,\delta}}. \tag{3.8}$$

Hence, inversion of (3.8) gives

$$m_\delta(u) = \sum_{z=1}^{n} C_{z,\delta} e^{-R_{z,\delta} u}, \quad u \geq 0. \tag{3.9}$$

Assume $\beta_1, \beta_2, \ldots, \beta_m$ and $R_{1,\delta}, R_{2,\delta}, \ldots, R_{n,\delta}$ are all distinct. It was proved in Theorem 1 of Landriault et al. (2014) that $-R_{1,\delta}, -R_{2,\delta}, \ldots, -R_{n,\delta}$ all have negative real parts and are roots of Lundberg's generalized equation (in s)

$$\sum_{i=1}^{m} \sum_{h=1}^{n_i} \left(\frac{\beta_i}{\beta_i + s} \right)^h \tilde{g}_{ih}(\delta - cs) = 1. \tag{3.10}$$

Moreover, $C_{1,\delta}, C_{2,\delta}, \ldots, C_{n,\delta}$ satisfy the system of linear equations

$$\sum_{z=1}^{n} C_{z,\delta} \left(\frac{\beta_i}{\beta_i - R_{z,\delta}} \right)^h = E[w(E_{i,h})] \tag{3.11}$$

for $i = 1, 2, \ldots, m$ and $h = 1, 2, \ldots, n_i$. For notational convenience, $E_{i,h}$ in (3.11) denotes the random variable with Erlang pdf $e_{\beta_i,h}$.

3.2 Explicit form of the associated coefficients

As shown in (3.9), the Gerber-Shiu function is characterized by the roots of Lundberg's generalized equation and the associated coefficients $C_{1,\delta}, C_{2,\delta}, \ldots, C_{n,\delta}$ which satisfy the system of linear equations (3.11). In this section, an approach is employed such that the form for the coefficients $C_{1,\delta}, C_{2,\delta}, \ldots, C_{n,\delta}$ can be determined, and an explicit expression is possible for some special cases of the penalty function.

Theorem 3.2.1. *The coefficient $C_{z,\delta}$ in (3.8), for $z = 1, \ldots, n$, has the form*

$$C_{z,\delta} = \tilde{v}_\delta(-R_{z,\delta}) \frac{\prod_{i=1}^{m} (\beta_i - R_{z,\delta})^{n_i}}{\prod_{j=1, j \neq z}^{n} (R_{j,\delta} - R_{z,\delta})}. \tag{3.12}$$

Proof. Given the ladder height density (3.4), it follows that

$$\left\{\prod_{i=1}^{m}(s+\beta_i)^{n_i}\right\}\tilde{f}_\delta(s)=\left\{\prod_{i=1}^{m}(s+\beta_i)^{n_i}\right\}\sum_{i=1}^{m}\sum_{h=1}^{n_i}\xi_{\delta,ih}\left(\frac{\beta_i}{\beta_i+s}\right)^h$$

is a polynomial in s of degree $n-1$ or less. Thus,

$$\left\{\prod_{i=1}^{m}(s+\beta_i)^{n_i}\right\}\left\{1-\phi_\delta\tilde{f}_\delta(s)\right\}$$

is a polynomial of degree n with the coefficient of s^n equal to 1. In equation (20) of Landriault et al. (2014), it was given that the equation $1-\phi_\delta\tilde{f}_\delta(s)=0$ has roots $-R_{1,\delta},-R_{2,\delta},\ldots,-R_{n,\delta}$ (which can be found out from Lundberg's generalized equation (3.10)). Hence,

$$\left\{\prod_{i=1}^{m}(s+\beta_i)^{n_i}\right\}\left\{1-\phi_\delta\tilde{f}_\delta(s)\right\}=\prod_{j=1}^{n}(s+R_{j,\delta}) \tag{3.13}$$

$$\frac{1-\phi_\delta\tilde{f}_\delta(s)}{s+R_{z,\delta}}=\frac{\prod_{j=1,j\neq z}^{n}(s+R_{j,\delta})}{\prod_{i=1}^{m}(s+\beta_i)^{n_i}} \tag{3.14}$$

for $z=1,2,\ldots,n$.

Now, equate (3.7) and (3.8), i.e.

$$\tilde{m}_\delta(s)=\sum_{h=1}^{n}\frac{C_{h,\delta}}{s+R_{h,\delta}}=\frac{\tilde{v}_\delta(s)}{1-\phi_\delta\tilde{f}_\delta(s)}. \tag{3.15}$$

Then it follows from (3.15) that

$$\begin{aligned}C_{z,\delta}&=\lim_{s\to -R_{z,\delta}}(s+R_{z,\delta})\tilde{m}_\delta(s)\\&=\lim_{s\to -R_{z,\delta}}(s+R_{z,\delta})\frac{\tilde{v}_\delta(s)}{1-\phi_\delta\tilde{f}_\delta(s)}\end{aligned} \tag{3.16}$$

for $z = 1, 2, \ldots, n$. Substitute (3.14) into (3.16) to get

$$C_{z,\delta} = \lim_{s \to -R_{z,\delta}} \tilde{v}_\delta(s) \frac{\prod_{i=1}^{m}(s+\beta_i)^{n_i}}{\prod_{j=1, j\neq z}^{n}(s+R_{j,\delta})},$$

and hence (3.12) follows as a result. □

The expression for the term $\tilde{v}_\delta(-R_{z,\delta})$ in (3.12) is complicated in general, but a simple result is possible with some particular choices of the penalty function $w(y)$ as shown in the following theorem.

Theorem 3.2.2. *Given $w(y) = y^n e^{-zy}$, where $n = 0, 1, 2, \ldots$ and Re $z \geq 0$. Consider the Laplace transform $\tilde{v}_\delta(s) = \int_0^\infty e^{-su} v_\delta(u) du$ and assume $s \neq z$, then it is given by*

$$\tilde{v}_\delta(s) = \frac{n!}{(z-s)^{n+1}} \left\{ \phi_\delta \tilde{f}_\delta(s) - \sum_{j=0}^{n} \frac{(s-z)^j}{j!} \left\{ \frac{\partial^j}{\partial z^j} \phi_\delta \tilde{f}_\delta(z) \right\} \right\} \tag{3.17}$$

with

$$\phi_\delta \tilde{f}_\delta(s) = 1 - \frac{\prod_{j=1}^{n}(s+R_{j,\delta})}{\prod_{i=1}^{m}(s+\beta_i)^{n_i}}. \tag{3.18}$$

Proof. To start with, rewrite (3.6) as

$$v_\delta(u) = \phi_\delta \int_u^\infty w(y-u) f_\delta(y) dy. \tag{3.19}$$

If $w(y) = y^n e^{-zy}$, then (3.19) becomes

$$v_\delta(u) = \phi_\delta \int_u^\infty (y-u)^n e^{-z(y-u)} f_\delta(y) dy. \tag{3.20}$$

Recall the definition of Dickson-Hipp operator in section 1.4.2. According to (1.10), one can express (3.20) as

$$v_\delta(u) = n!\phi_\delta T_z^{n+1} f_\delta(u). \tag{3.21}$$

Take Laplace transform of (3.21) yields

$$\tilde{v}_\delta(s) = n!\phi_\delta T_s T_z^{n+1} f_\delta(0). \tag{3.22}$$

Assume $s \neq z$, induction will be used to show that

$$T_s T_z^{n+1} f_\delta(0) = \frac{1}{(z-s)^{n+1}} \left\{ \tilde{f}_\delta(s) - \sum_{j=0}^{n} (z-s)^j \int_0^\infty \frac{y^j}{j!} e^{-zy} f_\delta(y) dy \right\}. \tag{3.23}$$

First, $T_s T_z f_\delta(0) = \{\tilde{f}_\delta(s) - \tilde{f}_\delta(z)\}/\{z - s\}$ by (1.9), and hence (3.23) is true for $n = 0$. Next, by using (1.9) again,

$$T_s T_z^{n+2} f_\delta(0) = \frac{T_s T_z^{n+1} f_\delta(0) - T_z^{n+2} f_\delta(0)}{z-s}. \tag{3.24}$$

If (3.23) is assumed to be true and by (1.10), then (3.24) becomes

$$\begin{aligned} T_s T_z^{n+2} f_\delta(0) =& \frac{1}{z-s} \left\{ \frac{1}{(z-s)^{n+1}} \left\{ \tilde{f}_\delta(s) - \sum_{j=0}^{n} (z-s)^j \int_0^\infty \frac{y^j}{j!} e^{-zy} f_\delta(y) dy \right\} \right. \\ & \left. - \int_0^\infty \frac{y^{n+1}}{(n+1)!} e^{-zy} f_\delta(y) dy \right\} \\ =& \frac{1}{(z-s)^{n+2}} \left\{ \tilde{f}_\delta(s) - \sum_{j=0}^{n+1} (z-s)^j \int_0^\infty \frac{y^j}{j!} e^{-zy} f_\delta(y) dy \right\}. \end{aligned}$$

Thus, (3.23) is proved by induction. By combining (3.22) and (3.23), one has (3.17) as a result. Also, (3.18) follows directly from (3.13). □

3.3 Moments of the time to ruin

The moments of the time to ruin are the focus of study in this section. Let us define the form of the kth moment of ruin time that will be considered in this chapter as

$$m_{k,\delta}(u) = E[T^k e^{-\delta T} w(|U_T|) I(T < \infty) | U_0 = u], \tag{3.25}$$

for $k = 0, 1, 2, \ldots$, and from which it is obvious that $m_{0,\delta}(u) = m_\delta(u)$. As shown in section 1.2.2, (3.25) is related to (3.3) by a kth-order differentiation, which is formally stated as

$$m_{k,\delta}(u) = (-1)^k \frac{\partial^k}{\partial \delta^k} m_\delta(u). \tag{3.26}$$

The following result is mainly based on this relation.

Theorem 3.3.1. *Consider a dependent Sparre Andersen model introduced in section* (1.1) *with the joint pdf of the interclaim time and the claim size given by* (3.1)*. For* $k = 0, 1, 2, \ldots$*, the* k*th moment of the time to ruin* (3.25) *can be expressed in the form*

$$m_{k,\delta}(u) = \sum_{r=0}^{k} \sum_{z=1}^{n} B_{k,\delta}(r,z) u^r e^{-R_{z,\delta} u}, \quad u \geq 0, \tag{3.27}$$

where $-R_{1,\delta}, -R_{2,\delta}, \ldots, -R_{n,\delta}$ *all have negative real parts and are roots of Lundberg's generalized equation* (3.10)*. Moreover,* $B_{k,\delta}(r,z)$ *for* $r = 0, 1, \ldots, k$ *and* $z = 1, 2, \ldots, n$ *are coefficients with* $B_{0,\delta}(0,z) = C_{z,\delta}$ *in* (3.9)*.*

Proof. For $k = 0$, (3.27) reduces to (3.9) with $B_{0,\delta}(0, z) = C_{z,\delta}$ for $z = 1, 2, \ldots, n$. Now, assume (3.27) is true for k. Then

$$\begin{aligned}
m_{k+1,\delta}(u) =& -\frac{\partial}{\partial\delta} m_{k,\delta}(u) \\
=& -\frac{\partial}{\partial\delta}\left\{\sum_{r=0}^{k}\sum_{z=1}^{n} B_{k,\delta}(r, z) u^r e^{-R_{z,\delta}u}\right\} \\
=& \sum_{r=0}^{k}\sum_{z=1}^{n} B_{k,\delta}(r, z)\frac{\partial R_{z,\delta}}{\partial\delta} u^{r+1} e^{-R_{z,\delta}u} - \sum_{r=0}^{k}\sum_{z=1}^{n} \frac{\partial B_{k,\delta}(r, z)}{\partial\delta} u^r e^{-R_{z,\delta}u} \\
=& -\sum_{z=1}^{n} \frac{\partial B_{k,\delta}(0, z)}{\partial\delta} e^{-R_{z,\delta}u} \\
&+ \sum_{r=1}^{k}\sum_{z=1}^{n}\left\{B_{k,\delta}(r-1, z)\frac{\partial R_{z,\delta}}{\partial\delta} - \frac{\partial B_{k,\delta}(r, z)}{\partial\delta}\right\} u^r e^{-R_{z,\delta}u} \\
&+ \sum_{z=1}^{n} B_{k,\delta}(k, z)\frac{\partial R_{z,\delta}}{\partial\delta} u^{k+1} e^{-R_{z,\delta}u} \\
=& \sum_{r=0}^{k+1}\sum_{z=1}^{n} B_{k+1,\delta}(r, z) u^r e^{-R_{z,\delta}u},
\end{aligned}$$

where $B_{k+1,\delta}(0, z) = -\frac{\partial B_{k,\delta}(0,z)}{\partial\delta}$ for $z = 1, 2, \ldots, n$, $B_{k+1,\delta}(r, z) = B_{k,\delta}(r-1, z)\frac{\partial R_{z,\delta}}{\partial\delta} - \frac{\partial B_{k,\delta}(r,z)}{\partial\delta}$ for $r = 1, 2, \ldots, k$ and $z = 1, 2, \ldots, n$ and $B_{k+1,\delta}(k+1, z) = B_{k,\delta}(k, z)\frac{\partial R_{z,\delta}}{\partial\delta}$ for $z = 1, 2, \ldots, n$. Hence, (3.27) is true by induction. □

The approach used to show (3.11) in Landriault et al. (2014) can be applied here to determine the systems of linear equations satisfied by the coefficients $B_{k,\delta}(r, z)$ in (3.27), and this is given by the following result.

Theorem 3.3.2. *Suppose the conditions of Theorem 3.3.1 hold. In addition, assume that* $\beta_1, \beta_2, \ldots, \beta_m$ *and* $R_{1,\delta}, R_{2,\delta}, \ldots, R_{n,\delta}$ *are all distinct, and also* $\tilde{g}_{in_i}(\delta + c\beta_i)$ *is non-zero for* $i = 1, 2, \ldots, m$. *Then for* $k = 1, 2, \ldots,$ *the coefficients* $B_{k,\delta}(r, z)$ *with* $r = 0, 1, \ldots, k$ *and* $z = 1, \ldots, n$ *satisfy two sets of equations.*

The first set is the following recursive system of linear equations

$$\sum_{x=r}^{k-1} \binom{k}{x} \sum_{y=r}^{x} B_{x,\delta}(y, z) \sum_{i=1}^{m} \sum_{h=1}^{n_i} \sum_{a=r}^{y} Q_{i,h,y,a,r,z,\delta} N_{i,h,z,\delta}(k - x + a - r)$$
$$+ \sum_{y=r+1}^{k} B_{k,\delta}(y, z) \sum_{i=1}^{m} \sum_{h=1}^{n_i} \sum_{a=r}^{y} Q_{i,h,y,a,r,z,\delta} N_{i,h,z,\delta}(a - r) = 0 \tag{3.28}$$

for $r = 0, 1, \ldots, k-1$ *and* $z = 1, \ldots, n$*, where*

$$Q_{i,h,y,a,r,z,\delta} = (-1)^{y-a} c^{a-r} \frac{y!}{r!(a-r)!} \binom{y - a + h - 1}{h - 1} \frac{\beta_i^h}{(\beta_i - R_{z,\delta})^{y-a+h}}, \tag{3.29}$$

and

$$N_{i,h,z,\delta}(k) = \int_0^\infty t^k e^{-(\delta + cR_{z,\delta})t} g_{ih}(t) dt. \tag{3.30}$$

Since (3.28) *is true for* $r = 0, 1, \ldots, k-1$ *and* $z = 1, \ldots, n$*, there are* $k \times n$ *equations in total.*

The second set of equations is

$$\sum_{r=0}^{k} \sum_{z=1}^{n} B_{k,\delta}(r, z) \frac{(-1)^r (j + r - 1)!}{(\beta_i - R_{z,\delta})^{j+r}} = 0 \tag{3.31}$$

for $i = 1, \ldots, m$ *and* $j = 1, \ldots, n_i$. *There are in total* n *equations in* (3.31).

Proof. From equation (26) of Landriault et al. (2014), the Gerber-Shiu function satisfies (by conditioning on the time and the amount of the first claim)

$$m_\delta(u) = \int_0^\infty e^{-\delta t} \int_{u+ct}^\infty w(y - u - ct) f(t, y) dy dt$$
$$+ \int_0^\infty e^{-\delta t} \int_0^{u+ct} m_\delta(u + ct - y) f(t, y) dy dt. \tag{3.32}$$

According to (3.26), one can differentiate (3.32) k times with respect to δ to obtain

$$\begin{aligned} m_{k,\delta}(u) &= \int_0^\infty t^k e^{-\delta t} \int_{u+ct}^\infty w(y-u-ct) f(t,y) dy dt \\ &+ \sum_{x=0}^{k} \binom{k}{x} \int_0^\infty t^{k-x} e^{-\delta t} \int_0^{u+ct} m_{x,\delta}(u+ct-y) f(t,y) dy dt. \end{aligned} \tag{3.33}$$

Putting (3.1) and (3.27) into (3.33) yields

$$\begin{aligned} \sum_{r=0}^{k} \sum_{z=1}^{n} B_{k,\delta}(r,z) u^r e^{-R_{z,\delta} u} &= \int_0^\infty t^k e^{-\delta t} \int_{u+ct}^\infty w(y-u-ct) \left\{ \sum_{i=1}^{m} \sum_{h=1}^{n_i} g_{ih}(t) e_{\beta_i,h}(y) \right\} dy dt \\ &+ \sum_{x=0}^{k} \binom{k}{x} \int_0^\infty t^{k-x} e^{-\delta t} \int_0^{u+ct} \left\{ \sum_{r=0}^{x} \sum_{z=1}^{n} B_{x,\delta}(r,z)(u+ct-y)^r \right. \\ &\left. \times\, e^{-R_{z,\delta}(u+ct-y)} \right\} \left\{ \sum_{i=1}^{m} \sum_{h=1}^{n_i} g_{ih}(t) e_{\beta_i,h}(y) \right\} dy dt \\ &= \sum_{i=1}^{m} \sum_{h=1}^{n_i} \int_0^\infty t^k e^{-\delta t} \left\{ \int_0^\infty w(y) e_{\beta_i,h}(y+u+ct) dy \right\} g_{ih}(t) dt \\ &+ \sum_{x=0}^{k} \binom{k}{x} \sum_{i=1}^{m} \sum_{r=0}^{x} \sum_{z=1}^{n} B_{x,\delta}(r,z) \sum_{h=1}^{n_i} \int_0^\infty t^{k-x} e^{-\delta t} \\ &\times \left\{ \int_0^{u+ct} (u+ct-y)^r e^{-R_{z,\delta}(u+ct-y)} e_{\beta_i,h}(y) dy \right\} g_{ih}(t) dt. \end{aligned} \tag{3.34}$$

However,

$$\begin{aligned} \int_0^\infty w(y) e_{\beta_i,h}(y+u+ct) dy &= \frac{1}{\beta_i} \sum_{q=1}^{h} \left\{ \int_0^\infty w(y) e_{\beta_i,h-q+1}(y) dy \right\} e_{\beta_i,q}(u+ct) \\ &= \frac{1}{\beta_i} \sum_{q=1}^{h} E[w(E_{i,h-q+1})] e_{\beta_i,q}(u+ct), \end{aligned}$$

where $E_{i,h}$ denotes the random variable with Erlang pdf $e_{\beta_i,h}$ as mentioned before. Also, it can be shown that for $R \neq \beta_i$,

$$\int_0^u (u-y)^r e^{-R(u-y)} e_{\beta_i,h}(y)dy = \sum_{a=0}^{r} (-1)^{r-a} \frac{r!}{a!} \binom{r-a+h-1}{h-1} \frac{\beta_i^h}{(\beta_i - R)^{r-a+h}} u^a e^{-Ru}$$
$$+ (-1)^{r+1} \frac{1}{\beta_i} \sum_{q=1}^{h} \frac{(h-q+r)!}{(h-q)!} \frac{\beta_i^{h-q+1}}{(\beta_i - R)^{h-q+r+1}} e_{\beta_i,q}(u).$$

As a result, (3.34) can be rewritten as

$$\begin{aligned}
&\sum_{r=0}^{k}\sum_{z=1}^{n} B_{k,\delta}(r,z) u^r e^{-R_{z,\delta}u} \\
&= \sum_{i=1}^{m}\sum_{h=1}^{n_i} \int_0^\infty t^k e^{-\delta t} \left\{ \frac{1}{\beta_i} \sum_{q=1}^{h} E[w(E_{i,h-q+1})] e_{\beta_i,q}(u+ct) \right\} g_{ih}(t)dt \\
&+ \sum_{x=0}^{k} \binom{k}{x} \sum_{i=1}^{m}\sum_{y=0}^{x}\sum_{z=1}^{n} B_{x,\delta}(y,z) \sum_{h=1}^{n_i} \int_0^\infty t^{k-x} e^{-\delta t} \left\{ \sum_{a=0}^{y} (-1)^{y-a} \frac{y!}{a!} \right. \\
&\times \left. \binom{y-a+h-1}{h-1} \frac{\beta_i^h}{(\beta_i - R_{z,\delta})^{y-a+h}} (u+ct)^a e^{-R_{z,\delta}(u+ct)} \right\} g_{ih}(t)dt \\
&+ \sum_{x=0}^{k} \binom{k}{x} \sum_{i=1}^{m}\sum_{r=0}^{x}\sum_{z=1}^{n} B_{x,\delta}(r,z) \sum_{h=1}^{n_i} \int_0^\infty t^{k-x} e^{-\delta t} \left\{ (-1)^{r+1} \right. \\
&\times \left. \frac{1}{\beta_i} \sum_{q=1}^{h} \frac{(h-q+r)!}{(h-q)!} \frac{\beta_i^{h-q+1}}{(\beta_i - R_{z,\delta})^{h-q+r+1}} e_{\beta_i,q}(u+ct) \right\} g_{ih}(t)dt.
\end{aligned} \tag{3.35}$$

Note that when compared to (3.34), the index of summation for the second and third line of (3.35) is changed from r to y. Then, rearrange (3.35) to get

$$\sum_{r=0}^{k}\sum_{z=1}^{n} B_{k,\delta}(r,z)u^r e^{-R_{z,\delta}u} - \sum_{x=0}^{k}\binom{k}{x}\sum_{i=1}^{m}\sum_{y=0}^{x}\sum_{z=1}^{n} B_{x,\delta}(y,z)\sum_{h=1}^{n_i}\int_0^\infty t^{k-x}e^{-\delta t}$$
$$\times\left\{\sum_{a=0}^{y}(-1)^{y-a}\frac{y!}{a!}\binom{y-a+h-1}{h-1}\frac{\beta_i^h}{(\beta_i - R_{z,\delta})^{y-a+h}}\right.$$
$$\left.\times\left\{\sum_{r=0}^{a}\binom{a}{r}u^r c^{a-r}t^{a-r}\right\}e^{-R_{z,\delta}(u+ct)}\right\}g_{ih}(t)dt$$
$$= \sum_{i=1}^{m}\sum_{h=1}^{n_i}\int_0^\infty t^k e^{-\delta t}\left\{\frac{1}{\beta_i}\sum_{q=1}^{h}E[w(E_{i,h-q+1})]e_{\beta_i,q}(u+ct)\right\}g_{ih}(t)dt$$
$$+\sum_{x=0}^{k}\binom{k}{x}\sum_{i=1}^{m}\sum_{r=0}^{x}\sum_{z=1}^{n} B_{x,\delta}(r,z)\sum_{h=1}^{n_i}\int_0^\infty t^{k-x}e^{-\delta t}$$
$$\times\left\{(-1)^{r+1}\frac{1}{\beta_i}\sum_{q=1}^{h}\frac{(h-q+r)!}{(h-q)!}\frac{\beta_i^{h-q+1}}{(\beta_i - R_{z,\delta})^{h-q+r+1}}e_{\beta_i,q}(u+ct)\right\}g_{ih}(t)dt. \qquad (3.36)$$

Next, since

$$\frac{1}{\beta_i}\int_0^\infty e^{-\delta t}e_{\beta_i,x}(u+ct)g_{ih}(t)dt = \sum_{j=1}^{x} e_{\beta_i,j}(u)\left\{\frac{1}{\beta_i^2}\int_0^\infty e^{-\delta t}e_{\beta_i,x-j+1}(ct)g_{ih}(t)dt\right\}, \qquad (3.37)$$

it can be easily shown by differentiating (3.37) k times with respect to δ that

$$\frac{1}{\beta_i}\int_0^\infty t^k e^{-\delta t}e_{\beta_i,x}(u+ct)g_{ih}(t)dt = \sum_{j=1}^{x} e_{\beta_i,j}(u)M_{i,h,x-j+1,\delta}(k) \qquad (3.38)$$

where

$$M_{i,h,x,\delta}(k) = \frac{1}{\beta_i^2}\int_0^\infty t^k e^{-\delta t}e_{\beta_i,x}(ct)g_{ih}(t)dt.$$

Therefore, using (3.29), (3.30) and (3.38), (3.36) becomes

$$\sum_{r=0}^{k}\sum_{z=1}^{n} B_{k,\delta}(r,z)u^r e^{-R_{z,\delta}u} - \sum_{x=0}^{k}\binom{k}{x}\sum_{i=1}^{m}\sum_{y=0}^{x}\sum_{z=1}^{n} B_{x,\delta}(y,z)\sum_{h=1}^{n_i}\sum_{r=0}^{y} u^r e^{-R_{z,\delta}u}$$
$$\times\left\{\sum_{a=r}^{y} Q_{i,h,y,a,r,z,\delta} N_{i,h,z,\delta}(k-x+a-r)\right\}$$
$$= \sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{q=1}^{h} E[w(E_{i,h-q+1})]\sum_{j=1}^{q} e_{\beta_i,j}(u) M_{i,h,q-j+1,\delta}(k)$$
$$+ \sum_{x=0}^{k}\binom{k}{x}\sum_{i=1}^{m}\sum_{r=0}^{x}\sum_{z=1}^{n} B_{x,\delta}(r,z)\sum_{h=1}^{n_i}(-1)^{r+1}$$
$$\times\left\{\sum_{q=1}^{h}\frac{(h-q+r)!}{(h-q)!}\frac{\beta_i^{h-q+1}}{(\beta_i-R_{z,\delta})^{h-q+r+1}}\sum_{j=1}^{q} e_{\beta_i,j}(u) M_{i,h,q-j+1,\delta}(k-x)\right\}. \qquad (3.39)$$

We further rearrange the summation signs in (3.39) to obtain

$$\sum_{r=0}^{k}\sum_{z=1}^{n} B_{k,\delta}(r,z)u^r e^{-R_{z,\delta}u} - \sum_{x=0}^{k}\binom{k}{x}\sum_{i=1}^{m}\sum_{r=0}^{x} u^r e^{-R_{z,\delta}u}\sum_{y=r}^{x}\sum_{z=1}^{n} B_{x,\delta}(y,z)$$
$$\times\left\{\sum_{h=1}^{n_i}\sum_{a=r}^{y} Q_{i,h,y,a,r,z,\delta} N_{i,h,z,\delta}(k-x+a-r)\right\}$$
$$= \sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{j=1}^{h} e_{\beta_i,j}(u)\sum_{q=j}^{h} E[w(E_{i,h-q+1})] M_{i,h,q-j+1,\delta}(k)$$
$$+ \sum_{x=0}^{k}\binom{k}{x}\sum_{i=1}^{m}\sum_{r=0}^{x}\sum_{z=1}^{n} B_{x,\delta}(r,z)\sum_{h=1}^{n_i}(-1)^{r+1}$$
$$\times\left\{\sum_{j=1}^{h} e_{\beta_i,j}(u)\sum_{q=j}^{h}\frac{(h-q+r)!}{(h-q)!}\frac{\beta_i^{h-q+1}}{(\beta_i-R_{z,\delta})^{h-q+r+1}} M_{i,h,q-j+1,\delta}(k-x)\right\},$$

which finally results in

$$
\begin{aligned}
&\sum_{r=0}^{k}\sum_{z=1}^{n} u^r e^{-R_{z,\delta}u}\Bigg\{ B_{k,\delta}(r,z) \\
&-\sum_{x=r}^{k}\binom{k}{x}\sum_{y=r}^{x} B_{x,\delta}(y,z)\sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{a=r}^{y} Q_{i,h,y,a,r,z,\delta}N_{i,h,z,\delta}(k-x+a-r)\Bigg\} \\
&=\sum_{i=1}^{m}\sum_{j=1}^{n_i} e_{\beta_i,j}(u)\sum_{h=j}^{n_i}\sum_{q=j}^{h}\Bigg\{ E[w(E_{i,h-q+1})]M_{i,h,q-j+1,\delta}(k) \\
&+\sum_{x=0}^{k}\binom{k}{x}\sum_{r=0}^{x}\sum_{z=1}^{n} B_{x,\delta}(r,z)(-1)^{r+1}\frac{(h-q+r)!}{(h-q)!}\frac{\beta_i^{h-q+1}}{(\beta_i-R_{z,\delta})^{h-q+r+1}}M_{i,h,q-j+1,\delta}(k-x)\Bigg\}.
\end{aligned}
\tag{3.40}
$$

Since (3.40) holds for all $u \geq 0$, the coefficients of $u^r e^{-R_{z,\delta}u}$ for $r = 0, 1, \ldots, k$; $z = 1, \ldots, n$ and $e_{\beta_i,j}(u)$ for $i = 1, \ldots, m$; $j = 1, \ldots, n_i$ should be zero. Therefore from the left hand side of (3.40),

$$
\begin{aligned}
0 &= B_{k,\delta}(r,z) \\
&-\sum_{x=r}^{k}\binom{k}{x}\sum_{y=r}^{x} B_{x,\delta}(y,z)\sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{a=r}^{y} Q_{i,h,y,a,r,z,\delta}N_{i,h,z,\delta}(k-x+a-r)
\end{aligned}
\tag{3.41}
$$

for $r = 0, 1, \ldots, k$; $z = 1, \ldots, n$. Then by splitting the summation signs and with definitions (3.29) and (3.30), (3.41) can be written as

$$
\begin{aligned}
0 &= B_{k,\delta}(r,z)\left\{1-\sum_{i=1}^{m}\sum_{h=1}^{n_i}\left(\frac{\beta_i}{\beta_i-R_{z,\delta}}\right)^h \tilde{g}_{ih}(\delta+cR_{z,\delta})\right\} \\
&-\sum_{x=r}^{k-1}\binom{k}{x}\sum_{y=r}^{x} B_{x,\delta}(y,z)\sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{a=r}^{y} Q_{i,h,y,a,r,z,\delta}N_{i,h,z,\delta}(k-x+a-r) \\
&-\sum_{y=r+1}^{k} B_{k,\delta}(y,z)\sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{a=r}^{y} Q_{i,h,y,a,r,z,\delta}N_{i,h,z,\delta}(a-r)
\end{aligned}
\tag{3.42}
$$

for $r = 0, 1, \ldots, k$; $z = 1, \ldots, n$. In (3.42), the first term equals to zero according to (3.10), and by notational convenience $\sum_{i=j}^{k} = 0$ for $j > k$. Hence, (3.42) yields the result (3.28) for $r = 0, 1, \ldots, k-1$; $z = 1, \ldots, n$.

On the other hand, from the right hand side of (3.40),

$$\begin{aligned} 0 = \sum_{h=j}^{n_i} \sum_{q=j}^{h} \Bigg\{ & E[w(E_{i,h-q+1})] M_{i,h,q-j+1,\delta}(k) \\ & + \sum_{x=0}^{k} \binom{k}{x} \sum_{r=0}^{x} \sum_{z=1}^{n} B_{x,\delta}(r, z) \\ & \times (-1)^{r+1} \frac{(h-q+r)!}{(h-q)!} \frac{\beta_i^{h-q+1}}{(\beta_i - R_{z,\delta})^{h-q+r+1}} M_{i,h,q-j+1,\delta}(k-x) \Bigg\} \end{aligned} \tag{3.43}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n_i$. Take out the term $x = 0$ from the summation sign in (3.43), i.e.

$$\begin{aligned} 0 = & \sum_{h=j}^{n_i} \sum_{q=j}^{h} \left\{ E[w(E_{i,h-q+1})] - \sum_{z=1}^{n} B_{0,\delta}(0, z) \left(\frac{\beta_i}{\beta_i - R_{z,\delta}} \right)^{h-q+1} \right\} M_{i,h,q-j+1,\delta}(k) \\ & + \sum_{h=j}^{n_i} \sum_{q=j}^{h} \sum_{x=1}^{k} \binom{k}{x} \Bigg\{ \sum_{r=0}^{x} \sum_{z=1}^{n} B_{x,\delta}(r, z) \\ & \times (-1)^{r+1} \frac{(h-q+r)!}{(h-q)!} \frac{\beta_i^{h-q+1}}{(\beta_i - R_{z,\delta})^{h-q+r+1}} \Bigg\} M_{i,h,q-j+1,\delta}(k-x), \end{aligned}$$

where the first term is equal to zero by (3.11) and so it is left with

$$\begin{aligned} 0 = \sum_{h=j}^{n_i} \sum_{q=j}^{h} \sum_{x=1}^{k} \binom{k}{x} & \Bigg\{ \sum_{r=0}^{x} \sum_{z=1}^{n} B_{x,\delta}(r, z) \\ & \times (-1)^{r+1} \frac{(h-q+r)!}{(h-q)!} \frac{\beta_i^{h-q+1}}{(\beta_i - R_{z,\delta})^{h-q+r+1}} \Bigg\} M_{i,h,q-j+1,\delta}(k-x) \end{aligned} \tag{3.44}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n_i$. If we define

$$\xi_{x,n,i,h}(\delta) = \sum_{r=0}^{x} \sum_{z=1}^{n} B_{x,\delta}(r, z) (-1)^{r+1} \frac{(h+r)!}{h!} \frac{\beta_i^{h+1}}{(\beta_i - R_{z,\delta})^{h+r+1}},$$

then (3.44) can be rewritten as

$$
\begin{aligned}
0 &= \sum_{h=j}^{n_i} \sum_{q=j}^{h} \sum_{x=1}^{k} \binom{k}{x} \xi_{x,n,i,h-q}(\delta) M_{i,h,q-j+1,\delta}(k-x) \\
&= \sum_{x=1}^{k} \binom{k}{x} \sum_{q=j}^{n_i} \sum_{h=q}^{n_i} \xi_{x,n,i,h-q}(\delta) M_{i,h,q-j+1,\delta}(k-x) \\
&= \sum_{x=1}^{k} \binom{k}{x} \sum_{q=j}^{n_i} \sum_{h=0}^{n_i-q} \xi_{x,n,i,h}(\delta) M_{i,h+q,q-j+1,\delta}(k-x) \\
&= \sum_{x=1}^{k} \binom{k}{x} \sum_{h=0}^{n_i-j} \sum_{q=j}^{n_i-h} \xi_{x,n,i,h}(\delta) M_{i,h+q,q-j+1,\delta}(k-x) \\
&= \sum_{x=1}^{k} \binom{k}{x} \sum_{h=0}^{n_i-j} \xi_{x,n,i,h}(\delta) \left\{ \sum_{q=0}^{n_i-h-j} M_{i,h+j+q,q+1,\delta}(k-x) \right\} \\
&= \sum_{x=1}^{k-1} \binom{k}{x} \sum_{h=0}^{n_i-j} \xi_{x,n,i,h}(\delta) \left\{ \sum_{q=0}^{n_i-h-j} M_{i,h+j+q,q+1,\delta}(k-x) \right\} \\
&\quad + \sum_{h=0}^{n_i-j} \xi_{k,n,i,h}(\delta) \left\{ \sum_{q=0}^{n_i-h-j} M_{i,h+j+q,q+1,\delta}(0) \right\}.
\end{aligned} \tag{3.45}
$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n_i$.

Fix any $i \in \{1, 2, \ldots, m\}$, our goal is to prove that

$$\xi_{k,n,i,h}(\delta) = 0, \quad h = 0, 1, 2, \ldots, n_i - 1 \tag{3.46}$$

for $k = 1, 2, 3, \ldots$, which yields (3.31).

Here is the proof. For $k = 1$, (3.45) reduces to

$$0 = \sum_{h=0}^{n_i-j} \xi_{1,n,i,h}(\delta) \left\{ \sum_{q=0}^{n_i-h-j} M_{i,h+j+q,q+1,\delta}(0) \right\} \tag{3.47}$$

for $j = 1, 2, \ldots, n_i$. When $j = n_i$, (3.47) is

$$\xi_{1,n,i,0}(\delta) M_{i,n_i,1,\delta}(0) = 0,$$

and hence $\xi_{1,n,i,0}(\delta) = 0$ since $M_{i,n_i,1,\delta}(0) = \tilde{g}_{in_i}(\delta + c\beta_i)/\beta_i$ is assumed to be non-zero. By considering (3.47) in the reversing order of $j = n_i - 1, n_i - 2, \ldots, 1$, it can be shown that

$$\xi_{1,n,i,h}(\delta) = 0, \quad h = 0, 1, 2, \ldots, n_i - 1. \tag{3.48}$$

It remains to show that (3.46) is true for $k = 2, 3, \ldots$. Assume for $x = 1, 2, \ldots, k-1$,

$$\xi_{x,n,i,h}(\delta) = 0, \quad h = 0, 1, 2, \ldots, n_i - 1, \tag{3.49}$$

then from (3.45),

$$0 = \sum_{h=0}^{n_i-j} \xi_{k,n,i,h}(\delta) \left\{ \sum_{q=0}^{n_i-h-j} M_{i,h+j+q,q+1,\delta}(0) \right\}. \tag{3.50}$$

for $j = 1, 2, \ldots, n_i$. When $j = n_i$, (3.50) gives

$$\xi_{k,n,i,0}(\delta) M_{i,n_i,1,\delta}(0) = 0.$$

Again, since $M_{i,n_i,1,\delta}(0)$ is assumed to be non-zero, we have $\xi_{k,n,i,0}(\delta) = 0$. Next, choose $j = n_i - s$ where $s \in \{1, 2, \ldots, n_i - 1\}$ in (3.50), which yields

$$0 = \sum_{h=0}^{s} \xi_{k,n,i,h}(\delta) \left\{ \sum_{q=0}^{s-h} M_{i,n_i-s+h+q,q+1,\delta}(0) \right\}. \tag{3.51}$$

Assume

$$\xi_{k,n,i,h}(\delta) = 0, \quad h = 0, 1, 2, \ldots, s-1, \tag{3.52}$$

then (3.51) gives

$$\xi_{k,n,i,s}(\delta) M_{i,n_i,1,\delta}(0) = 0 \tag{3.53}$$

and hence $\xi_{k,n,i,s}(\delta) = 0$ by the non-zero assumption of $M_{i,n_i,1,\delta}(0)$. Thus, if steps (3.51) to (3.53) are repeated by choosing s in the order of $s = 1, 2, \ldots, n_i - 1$, it can be shown that

$$\xi_{k,n,i,h}(\delta) = 0, \quad h = 1, 2, \ldots, n_i - 1.$$

Hence, (3.46) is true for $k = 2, 3, \ldots$ under assumption (3.49). Finally, since (3.46) is true for $k = 1$ as shown in (3.48), it can be concluded that (3.46) is true for $k = 1, 2, 3, \ldots$ which results in (3.31). $\square$

Theorem 3.3.2 shows that the associated coefficients of the moments, $B_{k,\delta}(r, z)$, can be solved recursively in k. For example, one can first solve for the associated coefficients of the Gerber-Shiu function, i.e. $C_{z,\delta}$, using (3.11) or (3.12). Then this result can be used to solve for the associated coefficients of the mean, i.e. $B_{1,\delta}(0, z)$ and $B_{1,\delta}(1, z)$. From Theorem 3.3.2, the equations are

$$B_{1,\delta}(1, z) = -\frac{C_{z,\delta}\left\{\sum_{i=1}^{m}\sum_{h=1}^{n_i} Q_{i,h,0,0,0,z,\delta} N_{i,h,z,\delta}(1)\right\}}{\sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{a=0}^{1} Q_{i,h,1,a,0,z,\delta} N_{i,h,z,\delta}(a)} \tag{3.54}$$

for $z = 1, \ldots, n$ and

$$\sum_{z=1}^{n} B_{1,\delta}(0, z)\frac{(j-1)!}{(\beta_i - R_{z,\delta})^{j}} = \sum_{z=1}^{n} B_{1,\delta}(1, z)\frac{j!}{(\beta_i - R_{z,\delta})^{j+1}} \tag{3.55}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. Next, with $C_{z,\delta}$, $B_{1,\delta}(0, z)$ and $B_{1,\delta}(1, z)$, the equations satisfied by the associated coefficients of the second moment are completely specified. From Theorem 3.3.2, the equations to solve for $B_{2,\delta}(0, z)$, $B_{2,\delta}(1, z)$ and $B_{2,\delta}(2, z)$ are

$$B_{2,\delta}(2, z) = -\frac{2B_{1,\delta}(1, z)\left\{\sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{a=r}^{y} Q_{i,h,1,1,1,z,\delta} N_{i,h,z,\delta}(1)\right\}}{\sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{a=1}^{2} Q_{i,h,2,a,1,z,\delta} N_{i,h,z,\delta}(a-1)}$$

for $z = 1, \ldots, n$;

$$
\begin{aligned}
& B_{2,\delta}(1,z) \\
&= -\frac{1}{\sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{a=0}^{1} Q_{i,h,1,a,0,z,\delta}N_{i,h,z,\delta}(a)} \left\{ C_{z,\delta}\sum_{i=1}^{m}\sum_{h=1}^{n_i} Q_{i,h,0,0,0,z,\delta}N_{i,h,z,\delta}(2) \right. \\
&+ 2\sum_{y=0}^{1} B_{1,\delta}(y,z)\sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{a=0}^{y} Q_{i,h,y,a,0,z,\delta}N_{i,h,z,\delta}(1+a) \\
&+ \left. B_{2,\delta}(2,z)\sum_{i=1}^{m}\sum_{h=1}^{n_i}\sum_{a=0}^{2} Q_{i,h,2,a,0,z,\delta}N_{i,h,z,\delta}(a) \right\}
\end{aligned}
$$

for $z = 1, \ldots, n$ and

$$
\sum_{z=1}^{n} B_{2,\delta}(0,z)\frac{(j-1)!}{(\beta_i - R_{z,\delta})^{j}} = \sum_{z=1}^{n} B_{2,\delta}(1,z)\frac{j!}{(\beta_i - R_{z,\delta})^{j+1}} - \sum_{z=1}^{n} B_{2,\delta}(2,z)\frac{(j+1)!}{(\beta_i - R_{z,\delta})^{j+2}}
$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. The above approach can be continued to solve for the associated coefficients of higher moments.

3.4 Numerical Example

In this section, the mean and variance of the time to ruin will be studied under different joint distributional assumption on the interclaim time and the claim size.

First, two cases which have independent interclaim times and claim sizes are considered. The joint pdf of the interclaim time (V) and claim size (Y) are given by

$$
f(t,y) = e^{-t}\left(\frac{2}{3}e^{-\frac{2}{3}y}\right) \quad \text{and} \quad f(t,y) = 4te^{-2t}\left(\frac{2}{3}e^{-\frac{2}{3}y}\right)
$$

respectively in case 1 and case 2. Note that two cases with the same expected interclaim time and expected claim size are chosen ($E[V] = 1$ and $E[Y] = 3/2$). In both cases, let us assume the premium rate of the insurance surplus process $c = 5/2$ (which satisfies the positive loading condition (1.2)).

	$f(t, y)$
Case 1	$e^{-t}\left(\frac{2}{3}e^{-\frac{2}{3}y}\right)$
Case 2	$4te^{-2t}\left(\frac{2}{3}e^{-\frac{2}{3}y}\right)$

Table 3.1: Joint pdf of interclaim times and claim sizes: independent cases

Given case 1 and case 2, two graphs involving the expected value and variance of the time to ruin are plotted as follows.

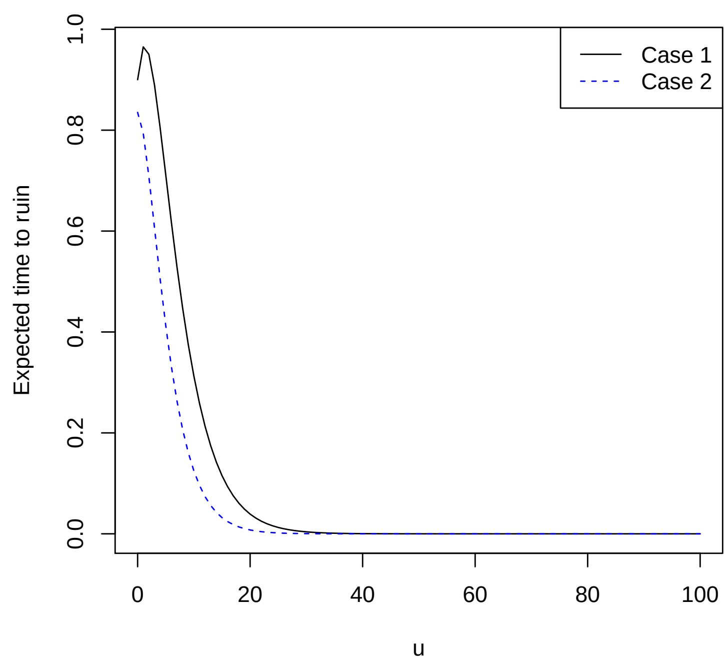

Figure 3.1: Comparison of the expected time to ruin in independent cases

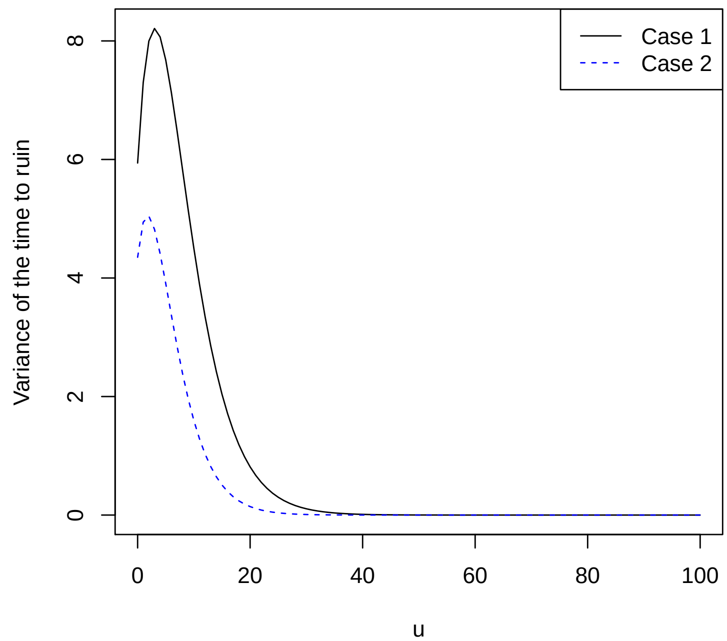

Figure 3.2: Comparison of the variance of time to ruin in independent cases

In Figure 3.1, the y-axis represents the quantity $m_{1,0}(u) = E[TI(T < \infty)|U_0 = u]$ and the x-axis is the initial surplus u. Two observations can be made from the figure. First, the expected time to ruin increases slightly and then decreases fast when initial surplus gets larger. One possible explanation can be obtained from the two factors affecting $E[TI(T < \infty)|U_0 = u]$, namely the time to ruin and the probability of ruin. With larger initial surplus, it should take longer time for the insurance process to become ruin. However, the probability of ruin becomes small if initial surplus is large. Therefore, these two factors are offsetting. According to Figure 3.1, except when initial surplus is small, the probability of ruin should be the dominating factor and therefore the expected time to ruin decreases quickly when initial surplus gets larger. The other observation is that the expected time to ruin in case 2 is shorter than that in case 1 for a given initial surplus u.

In Figure 3.2, the variance of the time to ruin is considered. The quantity $m_{2,0}(u) - \{m_{1,0}(u)\}^2 = E[T^2 I(T < \infty)|U_0 = u] - \{E[TI(T < \infty)|U_0 = u]\}^2$ is plotted against the initial surplus u. Again, in either case 1 or case 2, the variance of the time to ruin increases first and then decreases fast as initial surplus gets larger. Also, the variance of the time to ruin is smaller in case 2 as compared to case 1. These observations are similar to those made from Figure 3.1 which plots the expected time to ruin.

Next, let us study cases where interclaim time and claim size are dependent as another example. Consider two cases with the following joint pdf of the interclaim time (V) and claim size (Y)

$$f(t, y) = \frac{3}{4}e^{-t}\left(\frac{2}{3}e^{-\frac{2}{3}y}\right) + \frac{1}{4}(2e^{-2t})\left(\frac{2}{3}\right)^2 ye^{-\frac{2}{3}y}$$

and

$$f(t, y) = \frac{3}{4}e^{-t}\left(\frac{2}{3}e^{-\frac{2}{3}y}\right) + \frac{1}{4}(2e^{-2t})\left(\frac{1}{3}e^{-\frac{1}{3}y}\right),$$

which are referred to as case 3 and 4 respectively ($E[V] = 7/8$ and $E[Y] = 15/8$ in both cases). Let us assume the premium rate $c = 5/2$ in both case 3 and case 4.

	$f(t, y)$
Case 3	$\frac{3}{4}e^{-t}\left(\frac{2}{3}e^{-\frac{2}{3}y}\right) + \frac{1}{4}(2e^{-2t})\left(\frac{2}{3}\right)^2 ye^{-\frac{2}{3}y}$
Case 4	$\frac{3}{4}e^{-t}\left(\frac{2}{3}e^{-\frac{2}{3}y}\right) + \frac{1}{4}(2e^{-2t})\left(\frac{1}{3}e^{-\frac{1}{3}y}\right)$

Table 3.2: Joint pdf of interclaim times and claim sizes: dependent cases

As in the above independent example, the mean and variance of the time to ruin are plotted against the initial surplus in the following.

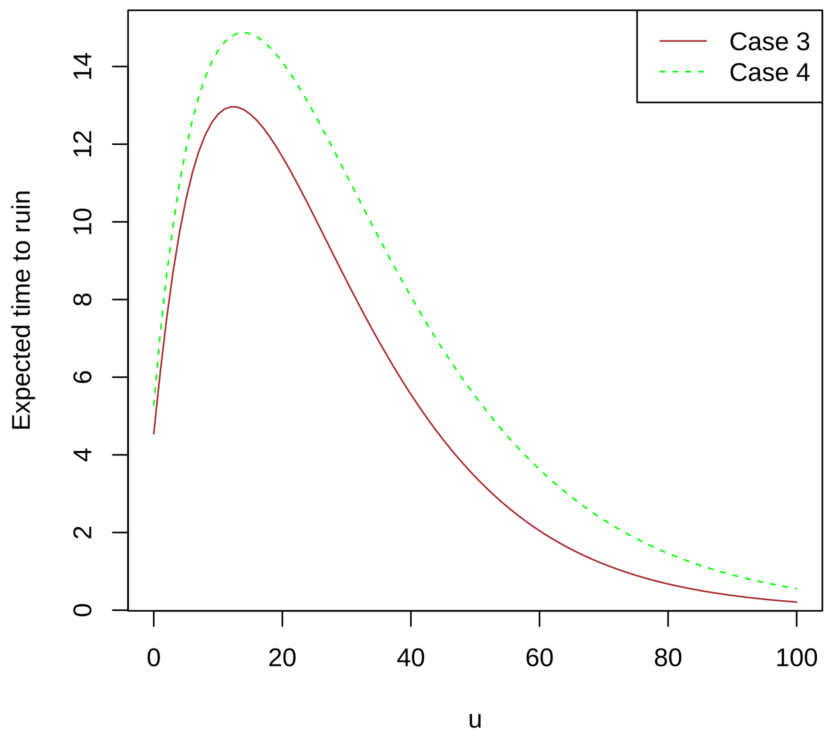

Figure 3.3: Comparison of the expected time to ruin in dependent cases

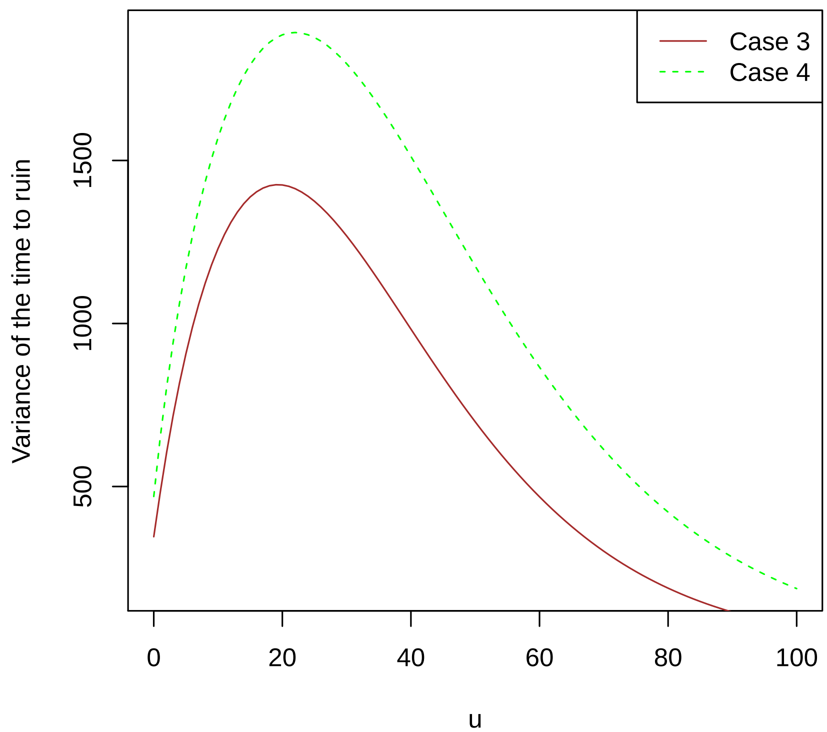

Figure 3.4: Comparison of the variance of time to ruin in dependent cases

The observations from figure 3.3 and figure 3.4 are similar to those in the independent cases. First, in either figure 3.3 or figure 3.4, the curves are concave. Second, the curves for case 3 are below that for case 4, which may be explained by the lower variance of each increment, i.e. $Var(cV - Y)$, in case 3. For detailed explanation of these observations, readers can refer to the analysis in the independent cases.

Chapter 4

Laplace transform of the moments of ruin time and analysis under Coxian interclaim time

In the first part of this chapter, the Laplace transform of the moments of time to ruin is studied in general under dependent Sparre Andersen models. The result generalizes the properties of the Laplace transform of the Gerber-Shiu function shown in Cheung et al. (2010). In the second part, the model of Willmot and Woo (2012) is considered which assumes that the interclaim times are Coxian and the claim sizes are time-dependent. The Laplace transform of the moments of time to ruin and the function $h_{2,\delta}^{*k}(x, y, v|0)$ defined in (2.12) are specified under this model.

The results in this chapter are submitted as Lee and Willmot (2014b).

4.1 Laplace transform of the moments of the time to ruin

Assume an arbitrary dependent Sparre Andersen model introduced in Section 1.1, with the joint pdf of the interclaim time and claim size denoted by $f(t, y)$. In this chapter, consider the Gerber-Shiu function

$$m_\delta(u) = E[e^{-\delta T} w(U_{T-}, |U_T|, R_{N_T - 1}) I(T < \infty) | U_0 = u], \tag{4.1}$$

which includes the surplus before ruin U_{T-}, the deficit at ruin $|U_T|$ and the surplus immediately after the second last claim before ruin R_{N_T-1} in the penalty function. For $k = 0, 1, 2, \ldots$, consider the generalized kth moment of the time to ruin

$$m_{k,\delta}(u) = E[T^k e^{-\delta T} w(U_{T-}, |U_T|, R_{N_T - 1}) I(T < \infty) | U_0 = u]. \tag{4.2}$$

By definition, $m_{0,\delta}(u) = m_\delta(u)$.

Cheung et al. (2010) and Willmot and Woo (2012) showed that the Laplace transform of the Gerber-Shiu function (4.1) satisfies

$$\{1 - \tilde{f}(\delta - cs, s)\} \tilde{m}_\delta(s) = \tilde{\beta}_{0,\delta}(s) - \sigma_{0,0,\delta}(s), \tag{4.3}$$

where

$$\tilde{f}(r, s) = \int_0^\infty \int_0^\infty e^{-rt - sy} f(t, y) dt dy \tag{4.4}$$

is the joint Laplace transform of the interclaim time and the claim size and $\tilde{\beta}_{0,\delta}(s) = \int_0^\infty e^{-su} \beta_{0,\delta}(u) du$ with

$$\beta_{0,\delta}(u) = \int_0^\infty e^{-\delta t} \int_{u+ct}^\infty w(u + ct, y - u - ct, u) f(t, y) dy dt.$$

Also,

$$\sigma_{0,0,\delta}(s) = \int_0^\infty e^{-sx} \varphi_{0,0,\delta}(x, \delta - cs) dx$$

with

$$\varphi_{0,0,\delta}(x, h) = \int_{x/c}^\infty e^{-ht} \int_0^x m_\delta(x-y) f(t, y) dy dt.$$

As mentioned in Cheung et al. (2010),

$$1 - \tilde{f}(\delta - cs, s) = 0 \tag{4.5}$$

is Lundberg's equation (in s).

The above result can be generalized to the Laplace transform of the kth moment of the time to ruin (4.2) for $k = 0, 1, 2, \ldots$, as follows.

Theorem 4.1.1. *Consider an arbitrary dependent Sparre Andersen model as introduced in Section 1.1. The Laplace transform of* (4.2) *satisfies*

$$\{1 - \tilde{f}(\delta - cs, s)\} \tilde{m}_{k,\delta}(s) = \tilde{\beta}_{k,\delta}(s) + \sum_{r=1}^{k} \binom{k}{r} \tilde{f}_r(\delta - cs, s) \tilde{m}_{k-r,\delta}(s) - \sum_{r=0}^{k} \binom{k}{r} \sigma_{k,r,\delta}(s), \tag{4.6}$$

where $\tilde{f}_r(\delta - cs, s) = (-1)^r \frac{\partial^r}{\partial \delta^r} \tilde{f}(\delta - cs, s)$ *and* $\tilde{\beta}_{k,\delta}(s) = \int_0^\infty e^{-su} \beta_{k,\delta}(u) du$ *with*

$$\beta_{k,\delta}(u) = \int_0^\infty t^k e^{-\delta t} \int_{u+ct}^\infty w(u + ct, y - u - ct, u) f(t, y) dy dt. \tag{4.7}$$

Also, for $r = 0, 1, \ldots, k$,

$$\sigma_{k,r,\delta}(s) = \int_0^\infty e^{-sx} \varphi_{k,r,\delta}(x, \delta - cs) dx \tag{4.8}$$

with

$$\varphi_{k,r,\delta}(x, h) = \int_{x/c}^\infty t^r e^{-ht} \int_0^x m_{k-r,\delta}(x - y) f(t, y) dy dt. \tag{4.9}$$

Proof. To prove (4.6), rewrite (4.3) as

$$\tilde{m}_\delta(s) - \tilde{f}(\delta - cs, s)\tilde{m}_\delta(s) = \tilde{\beta}_{0,\delta}(s) - \sigma_{0,0,\delta}(s). \tag{4.10}$$

Then differentiate (4.10) k times with respect to δ, which yields

$$\tilde{m}_{k,\delta}(s) - \sum_{r=0}^{k} \binom{k}{r} \tilde{f}_r(\delta - cs, s)\tilde{m}_{k-r,\delta}(s) = \tilde{\beta}_{k,\delta}(s) - \sum_{r=0}^{k} \binom{k}{r} \sigma_{k,r,\delta}(s),$$

and hence (4.6) follows by rearrangement. □

Note that when (4.6) equals zero, the left hand side also yields Lundberg's equation (4.5). Thus, (4.6) is a generalization of (4.3).

4.2 Coxian interclaim time assumption

In this section, the model of Willmot and Woo (2012) is considered. It is a dependent Sparre Andersen model with the joint pdf of the interclaim time (t) and the claim size (y)

$$f(t, y) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \tau_{ij}(t) b_{ij}(y), \quad t, y \geq 0, \tag{4.11}$$

where $\tau_{ij}(t)$ is Erlang pdf, i.e.

$$\tau_{ij}(t) = \frac{\lambda_i (\lambda_i t)^{j-1} e^{-\lambda_i t}}{(j-1)!}, \quad t \geq 0. \tag{4.12}$$

The marginal pdf of the interclaim time is

$$k(t) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left\{ \int_0^\infty b_{ij}(y) dy \right\} \tau_{ij}(t),$$

which is a Coxian-n pdf with $n = \sum_{i=1}^{m} n_i$. Moreover, given (4.11), Willmot and Woo (2012) noted that (4.5) becomes

$$1 - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left(\frac{\lambda_i}{\lambda_i + \delta - cs} \right)^j \tilde{b}_{ij}(s) = 0. \tag{4.13}$$

In (4.11), if $b_{ij}(y) = b(y)$ for all i, j, then it reduces to a Sparre Andersen model with Coxian interclaim times and time-independent claim sizes. This independent case has been considered in Li and Garrido (2005) with Gerber-Shiu function (1.4), and in Willmot and Woo (2010) with the generalized form (4.1).

4.2.1 Laplace transform of the moments

By assuming that n distinct roots with nonnegative real parts exist for (4.13), Willmot and Woo (2012) specified the Laplace transform of the Gerber-Shiu function. The result will be generalized here to the Laplace transform to the moments of the time to ruin by the approach in Willmot and Woo (2012). In other words, the form of the Laplace transform of the moments will be determined.

Theorem 4.2.1. *Consider a dependent Sparre Andersen model introduced in Section* (1.1) *with joint pdf of the claim size and interclaim time given by* (4.11)*. Furthermore, assume n distinct roots with nonnegative real parts, $\rho_1, \rho_2, \ldots, \rho_n$, exist for* (4.13)*. For $k = 0, 1, 2, \ldots$,* *the Laplace transform of the kth moment of the time to ruin (4.2) is given by*

$$\tilde{m}_{k,\delta}(s) = \frac{1}{1 - \sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\lambda_i}{\lambda_i+\delta-cs}\right)^j \tilde{b}_{ij}(s)}$$
$$\times\left\{\tilde{\beta}_{k,\delta}(s) + \sum_{r=1}^{k}\binom{k}{r}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\lambda_i^j j(j+1)\cdots(j+r-1)}{(\lambda_i+\delta-cs)^{j+r}}\tilde{b}_{ij}(s)\right\}\tilde{m}_{k-r,\delta}(s)\right.$$
$$\left. - \sum_{r=0}^{k}\binom{k}{r}\sigma_{k,r,\delta}(s)\right\}, \tag{4.14}$$

where $\tilde{\beta}_{k,\delta}(s) = \int_0^\infty e^{-su}\beta_{k,\delta}(u)du$ *with*

$$\beta_{k,\delta}(u) = \int_0^\infty t^k e^{-\delta t}\int_{u+ct}^\infty w(u+ct, y-u-ct, u)\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\tau_{ij}(t)b_{ij}(y)\right\}dydt. \tag{4.15}$$

Moreover,

$$\sigma_{k,0,\delta}(s) = \frac{\sum_{h=1}^{n} Q_{k,\delta}(\rho_h)\prod_{j=1,j\neq h}^{n}\left(\frac{s-\rho_j}{\rho_h-\rho_j}\right)}{\prod_{i=1}^{m}(\lambda_i+\delta-cs)^{n_i}}, \tag{4.16}$$

where

$$Q_{k,\delta}(\rho_h) = \left\{\tilde{\beta}_{k,\delta}(\rho_h) + \sum_{r=1}^{k}\binom{k}{r}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\lambda_i^j j(j+1)\cdots(j+r-1)}{(\lambda_i+\delta-c\rho_h)^{j+r}}\tilde{b}_{ij}(\rho_h)\right\}\tilde{m}_{k-r,\delta}(\rho_h)\right.$$
$$\left. - \sum_{r=1}^{k}\binom{k}{r}\sigma_{k,r,\delta}(\rho_h)\right\}\prod_{x=1}^{m}(\lambda_x+\delta-c\rho_h)^{n_x}$$

for $h = 1, 2, \ldots, n$.

Proof. First, with assumption (4.11), it follows that

$$\tilde{f}(\delta - cs, s) = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\lambda_i}{\lambda_i+\delta-cs}\right)^j \tilde{b}_{ij}(s) \tag{4.17}$$

and

$$\tilde{f}_r(\delta - cs, s) = (-1)^r \frac{\partial^r}{\partial \delta^r} \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left(\frac{\lambda_i}{\lambda_i + \delta - cs} \right)^j \tilde{b}_{ij}(s)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\lambda_i^j j(j+1)\cdots(j+r-1)}{(\lambda_i + \delta - cs)^{j+r}} \tilde{b}_{ij}(s). \tag{4.18}$$

Substitute (4.17) and (4.18) into (4.6) gives

$$\left\{ 1 - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left(\frac{\lambda_i}{\lambda_i + \delta - cs} \right)^j \tilde{b}_{ij}(s) \right\} \tilde{m}_{k,\delta}(s)$$

$$= \tilde{\beta}_{k,\delta}(s) + \sum_{r=1}^{k} \binom{k}{r} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\lambda_i^j j(j+1)\cdots(j+r-1)}{(\lambda_i + \delta - cs)^{j+r}} \tilde{b}_{ij}(s) \right\} \tilde{m}_{k-r,\delta}(s)$$

$$- \sum_{r=0}^{k} \binom{k}{r} \sigma_{k,r,\delta}(s), \tag{4.19}$$

which is (4.14) after rearrangement. Moreover, (4.15) follows easily from (4.7) with assumption (4.11).

There is still $\sigma_{k,0,\delta}(s)$ which needs to be determined. To start with, substitute (4.11) into (4.9) which yields

$$\varphi_{k,r,\delta}(x, h) = \int_{x/c}^{\infty} t^r e^{-ht} \int_0^x m_{k-r,\delta}(x - y) \left\{ \sum_{i=1}^{m} \sum_{q=1}^{n_i} \tau_{iq}(t) b_{iq}(y) \right\} dydt$$

$$= \sum_{i=1}^{m} \sum_{q=1}^{n_i} \alpha_{k-r,\delta,iq}(x) \int_{x/c}^{\infty} t^r e^{-ht} \tau_{iq}(t) dt, \tag{4.20}$$

where

$$\alpha_{k,\delta,iq}(x) = \int_0^x m_{k,\delta}(x - y) b_{iq}(y) dy. \tag{4.21}$$

The integral in (4.20) can be simplified as

$$\begin{aligned}
\int_{x/c}^{\infty} t^r e^{-ht}\tau_{iq}(t)dt &= \int_{x/c}^{\infty} t^r e^{-ht}\left\{\frac{\lambda_i(\lambda_i t)^{q-1}e^{-\lambda_i t}}{(q-1)!}\right\}dt \\
&= \frac{(q+r-1)!}{(q-1)!}\frac{\lambda_i^q}{(\lambda_i+h)^{q+r}}\int_{x/c}^{\infty}\frac{(\lambda_i+h)^{q+r}t^{q+r-1}e^{-(\lambda_i+h)t}}{(q+r-1)!}dt \\
&= \frac{(q+r-1)!}{(q-1)!}\frac{\lambda_i^q}{(\lambda_i+h)^{q+r}}\sum_{j=0}^{q+r-1}\frac{\left\{(\lambda_i+h)\left(\frac{x}{c}\right)\right\}^j e^{-(\lambda_i+h)\frac{x}{c}}}{j!} \\
&= \frac{(q+r-1)!}{(q-1)!}\lambda_i^q e^{-(\lambda_i+h)\frac{x}{c}}\sum_{j=1}^{q+r}\frac{x^{q+r-j}(\lambda_i+h)^{-j}}{c^{q+r-j}(q+r-j)!},
\end{aligned}$$

and hence (4.20) can be rewritten as

$$\varphi_{k,r,\delta}(x,h) = \sum_{i=1}^{m}\sum_{q=1}^{n_i}\alpha_{k-r,\delta,iq}(x)\sum_{j=1}^{q+r}\frac{(q+r-1)!}{(q-1)!}\lambda_i^q e^{-(\lambda_i+h)\frac{x}{c}}\frac{x^{q+r-j}(\lambda_i+h)^{-j}}{c^{q+r-j}(q+r-j)!}. \qquad (4.22)$$

Substitute (4.22) into (4.8), which yields

$$\begin{aligned}
\sigma_{k,r,\delta}(s) &= \int_0^{\infty} e^{-sx}\left\{\sum_{i=1}^{m}\sum_{q=1}^{n_i}\alpha_{k-r,\delta,iq}(x)\sum_{j=1}^{q+r}\frac{(q+r-1)!}{(q-1)!}\lambda_i^q e^{-(\lambda_i+\delta-cs)\frac{x}{c}}\right. \\
&\qquad \times \left.\frac{x^{q+r-j}(\lambda_i+\delta-cs)^{-j}}{c^{q+r-j}(q+r-j)!}\right\}dx \\
&= \sum_{i=1}^{m}\sum_{q=1}^{n_i}\sum_{j=1}^{q+r}\frac{(q+r-1)!}{(q-1)!}\frac{\lambda_i^q(\lambda_i+\delta-cs)^{-j}}{c^{q+r-j}(q+r-j)!}\int_0^{\infty}x^{q+r-j}e^{-\frac{\lambda_i+\delta}{c}x}\alpha_{k-r,\delta,iq}(x)dx \\
&= \sum_{i=1}^{m}\sum_{q=1}^{n_i}\sum_{j=1}^{q+r}\frac{(q+r-1)!}{(q-1)!}\frac{\lambda_i^q(\lambda_i+\delta-cs)^{-j}}{(-c)^{q+r-j}(q+r-j)!}\tilde{\alpha}_{k-r,\delta,iq}^{(q+r-j)}\left(\frac{\lambda_i+\delta}{c}\right), \qquad (4.23)
\end{aligned}$$

for $r = 0, 1, \ldots, k$, where

$$\tilde{\alpha}_{k,\delta,iq}^{(r)}(s) = \int_0^{\infty}(-x)^r e^{-sx}\alpha_{k,\delta,iq}(x)dx. \qquad (4.24)$$

Specifically, for $r = 0$ in (4.23),

$$\begin{aligned}\sigma_{k,0,\delta}(s) &= \sum_{i=1}^{m}\sum_{q=1}^{n_i}\sum_{j=1}^{q}\frac{\lambda_i^q(\lambda_i+\delta-cs)^{-j}}{(-c)^{q-j}(q-j)!}\tilde{\alpha}_{k,\delta,iq}^{(q-j)}\left(\frac{\lambda_i+\delta}{c}\right)\\ &= \sum_{i=1}^{m}\sum_{j=1}^{n_i}(\lambda_i+\delta-cs)^{-j}\sum_{q=j}^{n_i}\frac{\lambda_i^q\tilde{\alpha}_{k,\delta,iq}^{(q-j)}\left(\frac{\lambda_i+\delta}{c}\right)}{(-c)^{q-j}(q-j)!}\\ &= \sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\theta_{ij,k,\delta}}{(\lambda_i+\delta-cs)^j} \qquad (4.25)\end{aligned}$$

with

$$\theta_{ij,k,\delta} = \sum_{q=j}^{n_i}\frac{\lambda_i^q\tilde{\alpha}_{k,\delta,iq}^{(q-j)}\left(\frac{\lambda_i+\delta}{c}\right)}{(-c)^{q-j}(q-j)!}.$$

Equivalently, (4.25) can be expressed as

$$\sigma_{k,0,\delta}(s) = \frac{Q_{k,\delta}(s)}{\prod_{x=1}^{m}(\lambda_x+\delta-cs)^{n_x}} \qquad (4.26)$$

with

$$Q_{k,\delta}(s) = \left\{\prod_{x=1}^{m}(\lambda_x+\delta-cs)^{n_x}\right\}\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\theta_{ij,k,\delta}}{(\lambda_i+\delta-cs)^j}.$$

Next, recall the assumption that (4.13) has n distinct roots $\rho_1, \rho_2, \ldots, \rho_n$. For $h = 1, 2, \ldots, n$, put $s = \rho_h$ in (4.19) yields

$$\begin{aligned}\sigma_{k,0,\delta}(\rho_h) = \tilde{\beta}_{k,\delta}(\rho_h) &+ \sum_{r=1}^{k}\binom{k}{r}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\lambda_i^j j(j+1)\cdots(j+r-1)}{(\lambda_i+\delta-c\rho_h)^{j+r}}\tilde{b}_{ij}(\rho_h)\right\}\tilde{m}_{k-r,\delta}(\rho_h)\\ &-\sum_{r=1}^{k}\binom{k}{r}\sigma_{k,r,\delta}(\rho_h). \qquad (4.27)\end{aligned}$$

Again, recall that $\sigma_{k,r,\delta}(\rho_h)$ actually depends on $m_{k-r,\delta}(u)$. Therefore for $r = 1, 2, \ldots, k$, $\sigma_{k,r,\delta}(\rho_h)$ is a function of $m_{x,\delta}(u)$ for at most $x = k-1$. And hence in theory, $\sigma_{k,0,\delta}(\rho_h)$ in

(4.27) can be identified recursively in k, with $\sigma_{0,0,\delta}(\rho_h) = \tilde{\beta}_{0,\delta}(\rho_h)$. Substitute (4.27) into (4.26) gives

$$Q_{k,\delta}(\rho_h) = \left\{ \tilde{\beta}_{k,\delta}(\rho_h) + \sum_{r=1}^{k} \binom{k}{r} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\lambda_i^j j(j+1)\cdots(j+r-1)}{(\lambda_i + \delta - c\rho_h)^{j+r}} \tilde{b}_{ij}(\rho_h) \right\} \tilde{m}_{k-r,\delta}(\rho_h) \right.$$
$$\left. - \sum_{r=1}^{k} \binom{k}{r} \sigma_{k,r,\delta}(\rho_h) \right\} \prod_{x=1}^{m} (\lambda_x + \delta - c\rho_h)^{n_x}. \tag{4.28}$$

Since $Q_{k,\delta}(s)$ is a polynomial with at most degree $n-1$, it can be expressed in Lagrange polynomial form as

$$Q_{k,\delta}(s) = \sum_{h=1}^{n} Q_{k,\delta}(\rho_h) \prod_{j=1, j\neq h}^{n} \left(\frac{s-\rho_j}{\rho_h - \rho_j} \right). \tag{4.29}$$

Substitute (4.29) into (4.26) results in (4.16). □

On the right hand side of (4.14), it involves $\tilde{m}_{k-r,\delta}(s)$ and $\sigma_{k,r,\delta}(s)$ for $r = 1, 2, \ldots, k$. From definition (4.8), $\sigma_{k,r,\delta}(s)$ is a function of $m_{k-r,\delta}(u)$ which can be obtained by inversion of $\tilde{m}_{k-r,\delta}(s)$ in theory. Thus, (4.14) shows that $\tilde{m}_{k,\delta}(s)$ can be determined recursively in k.

4.2.2 Structural quantities related to the moments

Inversion of (4.14) with respect to s gives the kth moment of the time to ruin $m_{k,\delta}(u)$, but it is complicated to invert in general. In this section, an alternative way is provided to solve for $m_{k,\delta}(u)$. The function $h^{*k}_{2,\delta}(x, y, v|0)$ defined in (2.12) will be determined under the Coxian interclaim time assumption, and hence $m_{k,\delta}(u)$ can be solved recursively in k by the defective renewal equations shown in Theorem 2.2.1.

Theorem 4.2.2. *Suppose the conditions of Theorem 4.2.1 holds. For $k = 0, 1, 2, \ldots,$ the* *function* $h^{*k}_{2,\delta}(x,y,v|0)$ *defined in* (2.12) *is given by*

$$\begin{aligned}
&h^{*k}_{2,\delta}(x,y,v|0)\\
&=\sum_{h=1}^{n}\xi_{h,\delta}\Bigg\{e^{-\rho_h v}h^{*k}_{1,\delta}(x,y|v)\\
&+\sum_{r=1}^{k}\binom{k}{r}\Bigg\{\Bigg\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\lambda_i^j j(j+1)\cdots(j+r-1)}{(\lambda_i+\delta-c\rho_h)^{j+r}}\tilde{b}_{ij}(\rho_h)\Bigg\}\\
&\times\Bigg\{e^{-\rho_h v}h^{*k-r}_{1,\delta}(x,y|v)+\int_0^\infty e^{-\rho_h z}h^{*k-r}_{2,\delta}(x,y,v|z)dz\Bigg\}\\
&-\sum_{i=1}^{m}\sum_{q=1}^{n_i}\sum_{j=1}^{q+r}\frac{(q+r-1)!}{(q-1)!}\frac{\lambda_i^q(\lambda_i+\delta-c\rho_h)^{-j}}{(-c)^{q+r-j}(q+r-j)!}\gamma_{k-r,\delta,iq,q+r-j,\frac{\lambda_i+\delta}{c}}(x,y,v)\Bigg\}\Bigg\}\\
&-\sum_{r=1}^{k}\binom{k}{r}\sum_{i=1}^{m}\sum_{q=1}^{n_i}\frac{\lambda_i^q}{(q-1)!(-c)^{q+r}}\gamma_{k-r,\delta,iq,q+r-1,\frac{\lambda_i+\delta}{c}}(x,y,v),
\end{aligned}\tag{4.30}$$

where $h^{*k}_{1,\delta}(x,y|u)$ *is as defined in* (2.11),

$$\xi_{h,\delta}=\frac{\prod_{i=1}^{m}\left(\frac{\lambda_i+\delta}{c}-\rho_h\right)^{n_i}}{\prod_{j=1,j\neq h}^{n}(\rho_j-\rho_h)}\tag{4.31}$$

and

$$\begin{aligned}
\gamma_{k,\delta,iq,r,s}(x,y,v)&=\int_v^\infty(-a)^r e^{-sa}h^{*k}_{1,\delta}(x,y|v)b_{iq}(a-v)da\\
&+\int_0^\infty\int_0^z(-z)^r e^{-sz}h^{*k}_{2,\delta}(x,y,v|z-a)b_{iq}(a)dadz.
\end{aligned}\tag{4.32}$$

Proof. First, with definition (2.11), note that (4.7) can be written as

$$\begin{aligned}
\beta_{k,\delta}(u)&=\int_u^\infty\int_0^\infty w(x,y,u)h^{*k}_{1,\delta}(x,y|u)dydx\\
&=\int_0^\infty\int_u^\infty w(x+u,y-u,u)h^{*k}_{1,\delta}(x,y|0)dydx.
\end{aligned}\tag{4.33}$$

For the moment of the time to ruin (4.2), Theorem 2.2.1 shows that

$$m_{k,\delta}(u) = \phi_\delta \int_0^u m_{k,\delta}(u-y) f_\delta(y) dy + v_{k,\delta}(u) \tag{4.34}$$

for $k = 0, 1, 2, \ldots$, where

$$\begin{aligned} v_{k,\delta}(u) = & \sum_{j=1}^{k} \binom{k}{j} \int_0^u m_{k-j,\delta}(u-y) \int_0^\infty h_\delta^{*j}(x, y|0) dx dy \\ & + \beta_{k,\delta}(u) + \int_u^\infty \int_0^\infty \int_0^x w(x+u, y-u, v+u) h_{2,\delta}^{*k}(x, y, v|0) dv dx dy. \end{aligned}$$

For $u = 0$, (4.34) becomes

$$m_{k,\delta}(0) = \beta_{k,\delta}(0) + \int_0^\infty \int_0^\infty \int_0^x w(x, y, v) h_{2,\delta}^{*k}(x, y, v|0) dv dx dy. \tag{4.35}$$

On the other hand, the initial value theorem can be applied to (4.14). To be specific, multiply both sides of (4.14) by s and let $s \to \infty$, which yields

$$\begin{aligned} & \lim_{s\to\infty} s\tilde{m}_{k,\delta}(s) \\ & = \lim_{s\to\infty} s\tilde{\beta}_{k,\delta}(s) \\ & + \sum_{r=1}^{k} \binom{k}{r} \left\{ \lim_{s\to\infty} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\lambda_i^j j(j+1)\cdots(j+r-1)}{(\lambda_i + \delta - cs)^{j+r}} \tilde{b}_{ij}(s) \right\} \right\} \left\{ \lim_{s\to\infty} s\tilde{m}_{k-r,\delta}(s) \right\} \\ & - \sum_{r=0}^{k} \binom{k}{r} \lim_{s\to\infty} s\sigma_{k,r,\delta}(s). \end{aligned} \tag{4.36}$$

By the initial value theorem, $\lim_{s\to\infty} s\tilde{m}_{k,\delta}(s) = m_{k,\delta}(0)$ and $\lim_{s\to\infty} s\tilde{\beta}_{k,\delta}(s) = \beta_{k,\delta}(0)$. Thus, (4.36) reduces to

$$m_{k,\delta}(0) = \beta_{k,\delta}(0) - \sum_{r=0}^{k} \binom{k}{r} \lim_{s\to\infty} s\sigma_{k,r,\delta}(s). \tag{4.37}$$

By comparing (4.35) and (4.37) yields

$$\int_0^\infty \int_0^\infty \int_0^x w(x, y, v) h_{2,\delta}^{*k}(x, y, v|0) dv dx dy = - \sum_{r=0}^{k} \binom{k}{r} \lim_{s\to\infty} s\sigma_{k,r,\delta}(s). \tag{4.38}$$

To further study (4.38), the terms $\lim_{s\to\infty} s\sigma_{k,r,\delta}(s)$ for $r = 0, 1, 2, \ldots, k$ need to be identified. For $r = 1, 2, \ldots, k$, (4.23) shows that

$$\sigma_{k,r,\delta}(s) = \sum_{i=1}^{m}\sum_{q=1}^{n_i}\sum_{j=1}^{q+r} \frac{(q+r-1)!}{(q-1)!} \frac{\lambda_i^q(\lambda_i+\delta-cs)^{-j}}{(-c)^{q+r-j}(q+r-j)!} \tilde{\alpha}_{k-r,\delta,iq}^{(q+r-j)}\left(\frac{\lambda_i+\delta}{c}\right), \tag{4.39}$$

where

$$\tilde{\alpha}_{k,\delta,iq}^{(r)}(s) = \int_0^\infty (-z)^r e^{-sz} \int_0^z m_{k,\delta}(z-a) b_{iq}(a)\,da\,dz. \tag{4.40}$$

However, by definitions of $h_{1,\delta}^{*k}(x, y|u)$ and $h_{2,\delta}^{*k}(x, y, v|u)$ in (2.11) and (2.12), one has

$$\begin{aligned} m_{k,\delta}(u) = &\int_0^\infty \int_u^\infty w(x,y,u) h_{1,\delta}^{*k}(x,y|u)\,dx\,dy \\ &+ \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) h_{2,\delta}^{*k}(x,y,v|u)\,dv\,dx\,dy. \end{aligned} \tag{4.41}$$

Put (4.41) into (4.40), which gives

$$\begin{aligned} \tilde{\alpha}_{k,\delta,iq}^{(r)}(s) = &\int_0^\infty (-z)^r e^{-sz} \int_0^z \left\{ \int_0^\infty \int_{z-a}^\infty w(x,y,z-a) h_{1,\delta}^{*k}(x,y|z-a)\,dx\,dy \right. \\ &\left. + \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) h_{2,\delta}^{*k}(x,y,v|z-a)\,dv\,dx\,dy \right\} b_{iq}(a)\,da\,dz \\ = &\int_0^\infty \int_a^\infty \int_0^\infty \int_{z-a}^\infty w(x,y,z-a)(-z)^r e^{-sz} h_{1,\delta}^{*k}(x,y|z-a) b_{iq}(a)\,dx\,dy\,dz\,da \\ &+ \int_0^\infty \int_0^z \int_0^\infty \int_0^\infty \int_0^x w(x,y,v)(-z)^r e^{-sz} h_{2,\delta}^{*k}(x,y,v|z-a) b_{iq}(a)\,dv\,dx\,dy\,da\,dz \\ = &\int_0^\infty \int_0^\infty \int_0^\infty \int_v^\infty w(x,y,v)(-v-a)^r e^{-s(v+a)} h_{1,\delta}^{*k}(x,y|v) b_{iq}(a)\,dx\,dy\,dv\,da \\ &+ \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) \int_0^\infty \int_0^z (-z)^r e^{-sz} h_{2,\delta}^{*k}(x,y,v|z-a) b_{iq}(a)\,da\,dz\,dv\,dx\,dy \\ = &\int_0^\infty \int_0^\infty \int_0^x w(x,y,v) \int_0^\infty (-v-a)^r e^{-s(v+a)} h_{1,\delta}^{*k}(x,y|v) b_{iq}(a)\,da\,dv\,dx\,dy \\ &+ \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) \int_0^\infty \int_0^z (-z)^r e^{-sz} h_{2,\delta}^{*k}(x,y,v|z-a) b_{iq}(a)\,da\,dz\,dv\,dx\,dy \\ = &\int_0^\infty \int_0^\infty \int_0^x w(x,y,v) \gamma_{k,\delta,iq,r,s}(x,y,v)\,dv\,dx\,dy \end{aligned} \tag{4.42}$$

with $\gamma_{k,\delta,iq,r,s}(x,y,v)$ given by (4.32). Hence, (4.39) becomes

$$\sigma_{k,r,\delta}(s) = \sum_{i=1}^{m}\sum_{q=1}^{n_i}\sum_{j=1}^{q+r}\frac{(q+r-1)!}{(q-1)!}\frac{\lambda_i^q(\lambda_i+\delta-cs)^{-j}}{(-c)^{q+r-j}(q+r-j)!} \times \int_0^\infty\int_0^\infty\int_0^x w(x,y,v)\gamma_{k-r,\delta,iq,q+r-j,\frac{\lambda_i+\delta}{c}}(x,y,v)dvdxdy, \tag{4.43}$$

and thus for $r = 1,2,\ldots,k$,

$$\begin{aligned}
&\lim_{s\to\infty} s\sigma_{k,r,\delta}(s) \\
&= \lim_{s\to\infty}\left\{ s\sum_{i=1}^{m}\sum_{q=1}^{n_i}\sum_{j=1}^{q+r}\frac{(q+r-1)!}{(q-1)!}\frac{\lambda_i^q(\lambda_i+\delta-cs)^{-j}}{(-c)^{q+r-j}(q+r-j)!}\right. \\
&\left.\times \int_0^\infty\int_0^\infty\int_0^x w(x,y,v)\gamma_{k-r,\delta,iq,q+r-j,\frac{\lambda_i+\delta}{c}}(x,y,v)dvdxdy\right\} \\
&= \lim_{s\to\infty}\sum_{i=1}^{m}\sum_{q=1}^{n_i}\frac{\lambda_i^q\left(\frac{\lambda_i+\delta}{s}-c\right)^{-1}}{(q-1)!(-c)^{q+r-1}}\int_0^\infty\int_0^\infty\int_0^x w(x,y,v)\gamma_{k-r,\delta,iq,q+r-1,\frac{\lambda_i+\delta}{c}}(x,y,v)dvdxdy \\
&+ \lim_{s\to\infty}\left\{\sum_{i=1}^{m}\sum_{q=1}^{n_i}\sum_{j=2}^{q+r}\frac{(q+r-1)!}{(q-1)!}\frac{\lambda_i^q\left(\frac{\lambda_i+\delta}{s}-c\right)^{-1}(\lambda_i+\delta-cs)^{-(j-1)}}{(-c)^{q+r-j}(q+r-j)!}\right. \\
&\left.\times \int_0^\infty\int_0^\infty\int_0^x w(x,y,v)\gamma_{k-r,\delta,iq,q+r-j,\frac{\lambda_i+\delta}{c}}(x,y,v)dvdxdy\right\} \\
&= \sum_{i=1}^{m}\sum_{q=1}^{n_i}\frac{\lambda_i^q}{(q-1)!(-c)^{q+r}}\int_0^\infty\int_0^\infty\int_0^x w(x,y,v)\gamma_{k-r,\delta,iq,q+r-1,\frac{\lambda_i+\delta}{c}}(x,y,v)dvdxdy.
\end{aligned} \tag{4.44}$$

Next, from (4.26) and (4.29),

$$\lim_{s\to\infty} s\sigma_{k,0,\delta}(s) = \lim_{s\to\infty} \frac{s\sum_{h=1}^{n} Q_{k,\delta}(\rho_h) \prod_{j=1,j\neq h}^{n} \left(\frac{s-\rho_j}{\rho_h-\rho_j}\right)}{\prod_{i=1}^{m}(\lambda_i+\delta-cs)^{n_i}}$$

$$= \lim_{s\to\infty} \frac{\sum_{h=1}^{n} Q_{k,\delta}(\rho_h) \prod_{j=1,j\neq h}^{n} \left(\frac{1-\frac{\rho_j}{s}}{\rho_h-\rho_j}\right)}{\prod_{i=1}^{m}\left(\frac{\lambda_i+\delta}{s}-c\right)^{n_i}}$$

$$= \frac{1}{(-c)^n}\sum_{h=1}^{n} Q_{k,\delta}(\rho_h) \prod_{j=1,j\neq h}^{n} \left(\frac{1}{\rho_h-\rho_j}\right). \tag{4.45}$$

Substitute (4.28) into (4.45) gives

$$\lim_{s\to\infty} s\sigma_{k,0,\delta}(s)$$
$$= -\sum_{h=1}^{n} \xi_{h,\delta}\Bigg\{\tilde{\beta}_{k,\delta}(\rho_h)$$
$$+\sum_{r=1}^{k}\binom{k}{r}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\lambda_i^j j(j+1)\cdots(j+r-1)}{(\lambda_i+\delta-c\rho_h)^{j+r}}\tilde{b}_{ij}(\rho_h)\right\}\tilde{m}_{k-r,\delta}(\rho_h)$$
$$-\sum_{r=1}^{k}\binom{k}{r}\sigma_{k,r,\delta}(\rho_h)\Bigg\}, \tag{4.46}$$

where $\xi_{h,\delta}$ is given by (4.31). Let us study (4.46) term by term. First, it follows from (4.33) that

$$\tilde{\beta}_{k,\delta}(\rho_h) = \int_0^\infty e^{-\rho_h v}\int_v^\infty\int_0^\infty w(x,y,v)h_{1,\delta}^{*k}(x,y|v)dydxdv$$
$$= \int_0^\infty\int_0^\infty\int_0^x w(x,y,v)e^{-\rho_h v}h_{1,\delta}^{*k}(x,y|v)dvdxdy. \tag{4.47}$$

Second, by (4.41),

$$\begin{aligned}
\tilde{m}_{k-r,\delta}(\rho_h) &= \int_0^\infty e^{-\rho_h v} \int_0^\infty \int_v^\infty w(x,y,v) h_{1,\delta}^{*k-r}(x,y|v) dx dy dv \\
&+ \int_0^\infty e^{-\rho_h z} \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) h_{2,\delta}^{*k-r}(x,y,v|z) dv dx dy dz \\
&= \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) \\
&\times \left\{ e^{-\rho_h v} h_{1,\delta}^{*k-r}(x,y|v) + \int_0^\infty e^{-\rho_h z} h_{2,\delta}^{*k-r}(x,y,v|z) dz \right\} dv dx dy. \qquad (4.48)
\end{aligned}$$

Thus, by (4.47), (4.48) and (4.43), it follows that (4.46) can be expressed as

$$\begin{aligned}
&\lim_{s\to\infty} s\sigma_{k,0,\delta}(s) \\
&= -\sum_{h=1}^n \xi_{h,\delta} \left\{ \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) \left\{ e^{-\rho_h v} h_{1,\delta}^{*k}(x,y|v) \right.\right. \\
&+ \sum_{r=1}^k \binom{k}{r} \left\{ \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{\lambda_i^j j(j+1)\cdots(j+r-1)}{(\lambda_i+\delta-c\rho_h)^{j+r}} \tilde{b}_{ij}(\rho_h) \right\} \\
&\times \left\{ e^{-\rho_h v} h_{1,\delta}^{*k-r}(x,y|v) + \int_0^\infty e^{-\rho_h z} h_{2,\delta}^{*k-r}(x,y,v|z) dz \right\} \\
&- \sum_{r=1}^k \binom{k}{r} \sum_{i=1}^m \sum_{q=1}^{n_i} \sum_{j=1}^{q+r} \frac{(q+r-1)!}{(q-1)!} \\
&\left.\left. \times \frac{\lambda_i^q (\lambda_i+\delta-c\rho_h)^{-j}}{(-c)^{q+r-j}(q+r-j)!} \gamma_{k-r,\delta,iq,q+r-j,\frac{\lambda_i+\delta}{c}}(x,y,v) \right\} dv dx dy \right\}. \qquad (4.49)
\end{aligned}$$

Finally, substitute (4.44) and (4.49) into (4.38). Let $w(x,y,v) = e^{-s_1 x - s_2 y - s_3 v}$ in (4.38) and (4.30) follows by inversion with respect to s_1, s_2 and s_3. □

Theorem 4.2.2 generalizes the result

$$h_{2,\delta}^{*0}(x,y,v|0) = \sum_{h=1}^n \xi_{h,\delta} e^{-\rho_h v} h_{1,\delta}^{*0}(x,y|v),$$

which is shown in Willmot and Woo (2012). According to Theorem 2.2.1, the defective renewal equation satisfied by $m_{0,\delta}(u)$ is completely specified if $h^{*0}_{2,\delta}(x,y,v|0)$ is given, i.e.

$$m_{0,\delta}(u) = \phi_\delta \int_0^u m_{0,\delta}(u-y) f_\delta(y) dy + v_{0,\delta}(u) \tag{4.50}$$

with

$$\begin{aligned} v_{0,\delta}(u) = & \int_u^\infty \int_0^\infty w(x+u, y-u, u) h^{*0}_{1,\delta}(x,y|0) dx dy \\ & + \int_u^\infty \int_0^\infty \int_0^x w(x+u, y-u, v+u) h^{*0}_{2,\delta}(x,y,v|0) dv dx dy. \end{aligned}$$

Moreover, from (2.19), the solution to (4.50) is

$$m_{0,\delta}(u) = \frac{1}{1-\phi_\delta} \int_0^u v_{0,\delta}(y) g_\delta(u-y) dy + v_{0,\delta}(u). \tag{4.51}$$

On the other hand, it is given in (4.41) that

$$\begin{aligned} m_{0,\delta}(u) = & \int_0^\infty \int_u^\infty w(x,y,u) h^{*0}_{1,\delta}(x,y|u) dx dy \\ & + \int_0^\infty \int_0^\infty \int_0^x w(x,y,v) h^{*0}_{2,\delta}(x,y,v|u) dv dx dy. \end{aligned} \tag{4.52}$$

Equate (4.51) and (4.52) with the penalty function $w(x,y,v) = e^{-s_1 x - s_2 y - s_3 v}$ and it can be shown by inversion with respect to s_1, s_2 and s_3 that $h^{*0}_{2,\delta}(x,y,v|u)$ is a function of $h^{*0}_{2,\delta}(x,y,v|0)$. Readers can refer to Cheung et al. (2010) for detailed steps and results.

In Theorem 4.2.2, it shows that $h^{*k}_{2,\delta}(x,y,v|0)$ is a function of $h^{*r}_{2,\delta}(x,y,v|u)$ for $u \geq 0$ and $r = 0, 1, \ldots, k-1$. Moreover, $h^{*r}_{2,\delta}(x,y,v|u)$ for $u \geq 0$ can be expressed in terms of $h^{*r}_{2,\delta}(x,y,v|0)$ with the approach described in last paragraph. Thus, $h^{*k}_{2,\delta}(x,y,v|0)$ can be solved recursively in k by (4.30). For example, given that $h^{*0}_{2,\delta}(x,y,v|u)$ for $u \geq 0$ is obtained as discussed above, (4.30) can be used to determine $h^{*1}_{2,\delta}(x,y,v|0)$. To continue, by equating (2.19) and (4.41) with $k = 1$ yields $h^{*1}_{2,\delta}(x,y,v|u)$ for $u \geq 0$, and (4.30) may be used again to obtain $h^{*2}_{2,\delta}(x,y,v|0)$, etc.

4.2.3 Numerical example

In this section we present two examples to illustrate the use of the results in Section 4.2.2.

In the first example, we choose a model for which we can compare our results with those obtained by using the results in Chapter 3. The second example illustrating the use of the methodology in situations where other approaches appear not to be readily available. In particular, we assume a claim size distribution which is not of Coxian form.

Example 1 Consider a Sparre Andersen model where the joint pdf of the interclaim time and the claim size is given by

$$f(t, y) = 4te^{-2t}\left(ye^{-y}\right). \tag{4.53}$$

By assuming $\delta = 0$ and $c = 3$, Lundberg's equation (4.13) has non-negative roots 0 and 1. Hence, $\rho_1 = 0$ and $\rho_2 = 1$.

Suppose that the expected time to ruin, i.e. $E[TI(T < \infty)|U_0 = u]$, is of interest. According to Theorem 2.2.1, the functions $h^{*1}_{1,0}(x, y|0)$ and $h^{*1}_{2,0}(x, y, v|0)$ are needed to solve for $E[TI(T < \infty)|U_0 = u]$.

First, one has from (2.11) that

$$h^{*k}_{1,0}(x, y|u) = \frac{4}{3^{k+2}}(x-u)^{k+1}(x+y)e^{-\frac{5}{3}x-y+\frac{2}{3}u}, \quad x > u \tag{4.54}$$

for $k = 0, 1, 2, \ldots$.

Next, $h^{*0}_{2,0}(x,y,v|u)$ for $u \geq 0$ is necessary to specify $h^{*1}_{2,0}(x,y,v|0)$. Note that $h^{*0}_{2,0}(x,y,v|u)$ can be obtained from $h^{*0}_{2,0}(x,y,v|0)$ as follows. To start with, put $k=0$ in (4.30) and one has

$$h^{*0}_{2,0}(x,y,v|0) = \sum_{h=1}^{2} \left\{ \frac{\left(\frac{2}{3}-\rho_h\right)^2}{\prod_{j=1,j\neq h}^{2}(\rho_j-\rho_h)} \right\} e^{-\rho_h v} h^{*0}_{1,0}(x,y|v), \tag{4.55}$$

where $h^{*0}_{1,0}(x,y|v)$ is given in (4.54). Then consider the Gerber-Shiu function

$$m_{0,0}(u) = E[e^{-s_1 U_{T-} - s_2|U_T| - s_3 R_{N_T-1}} I(T<\infty)|U_0=u]. \tag{4.56}$$

According to Theorem 2.2.1, $m_{0,0}(u)$ in (4.56) satisfies

$$m_{0,0}(u) = \phi_0 \int_0^u m_{0,0}(u-y) f_0(y)dy + v_{0,0}(u), \tag{4.57}$$

where $\phi_0 = \int_0^\infty \int_0^\infty \left\{ h^{*0}_{1,0}(x,y|0) + \int_0^x h^{*0}_{2,0}(x,y,v|0)dv \right\} dxdy$,

$$f_0(y) = \frac{1}{\phi_0} \int_0^\infty \left\{ h^{*0}_{1,0}(x,y|0) + \int_0^x h^{*0}_{2,0}(x,y,v|0)dv \right\} dx$$

and

$$\begin{aligned} v_{0,0}(u) &= \int_u^\infty \int_0^\infty e^{-s_1(x+u)-s_2(y-u)-s_3 u} h^{*0}_{1,0}(x,y|0)dxdy \\ &+ \int_u^\infty \int_0^\infty \int_0^x e^{-s_1(x+u)-s_2(y-u)-s_3(v+u)} h^{*0}_{2,0}(x,y,v|0)dvdxdy \\ &= \int_0^\infty \int_u^\infty e^{-s_1 x - s_2 y - s_3 u} h^{*0}_{1,0}(x-u,y+u|0)dxdy \\ &+ \int_0^\infty \int_u^\infty \int_u^x e^{-s_1 x - s_2 y - s_3 v} h^{*0}_{2,0}(x-u,y+u,v-u|0)dvdxdy. \end{aligned}$$

From (2.19), the solution to (4.57) is

$$
\begin{aligned}
m_{0,0}(u) &= \sum_{n=1}^{\infty} \phi_0^n \int_0^u v_{0,0}(z) f_0^{*n}(u-z)dz + v_{0,0}(u) \\
&= \sum_{n=1}^{\infty} \phi_0^n \int_0^u \left\{ \int_0^{\infty} \int_z^{\infty} e^{-s_1x-s_2y-s_3z} h_{1,0}^{*0}(x-z, y+z|0)dxdy \right. \\
&+ \left. \int_0^{\infty} \int_z^{\infty} \int_z^{x} e^{-s_1x-s_2y-s_3v} h_{2,0}^{*0}(x-z, y+z, v-z|0)dvdxdy \right\} \\
&\times f_0^{*n}(u-z)dz \\
&+ \int_0^{\infty} \int_u^{\infty} e^{-s_1x-s_2y-s_3u} h_{1,0}^{*0}(x-u, y+u|0)dxdy \\
&+ \int_0^{\infty} \int_u^{\infty} \int_u^{x} e^{-s_1x-s_2y-s_3v} h_{2,0}^{*0}(x-u, y+u, v-u|0)dvdxdy. \qquad (4.58)
\end{aligned}
$$

On the other hand, from (4.41),

$$
\begin{aligned}
m_{0,0}(u) = &\int_0^{\infty} \int_u^{\infty} e^{-s_1x-s_2y-s_3u} h_{1,0}^{*0}(x, y|u)dxdy \\
&+ \int_0^{\infty} \int_0^{\infty} \int_0^{x} e^{-s_1x-s_2y-s_3v} h_{2,0}^{*0}(x, y, v|u)dvdxdy. \qquad (4.59)
\end{aligned}
$$

Equate (4.59) and (4.58), then it can be shown by inverting with respect to s_1, s_2 and s_3 that

$$
\begin{aligned}
h_{2,0}^{*0}(x, y, v|u) = \sum_{n=1}^{\infty} \phi_0^n \Bigg\{ &h_{1,0}^{*0}(x-v, y+v|0) f_0^{*n}(u-v) \\
&+ \int_0^{v} h_{2,0}^{*0}(x-z, y+z, v-z|0) f_0^{*n}(u-z)dz \Bigg\}
\end{aligned}
$$

for $0 \le v \le \min(x, u)$; $0 \le x < \infty$; $y \ge 0$ and

$$
\begin{aligned}
h_{2,0}^{*0}(x, y, v|u) = &\sum_{n=1}^{\infty} \phi_0^n \int_0^{u} h_{2,0}^{*0}(x-z, y+z, v-z|0) f_0^{*n}(u-z)dz \\
&+ h_{2,0}^{*0}(x-u, y+u, v-u|0)
\end{aligned}
$$

for $u < v \leq x$; $u \leq x < \infty$; $y \geq 0$.

Now with $h^{*0}_{2,0}(x, y, v|u)$, we can determine $h^{*1}_{2,0}(x, y, v|0)$ from (4.30). Finally, with all the quantities calculated in this example and (2.19), the solution for $E[TI(T < \infty)|U_0 = u]$ is given by

$$E[TI(T < \infty)|U_0 = u] = \sum_{n=1}^{\infty} \phi_0^n \int_0^u v_{1,0}(y) f_0^{*n}(u - y)dy + v_{1,0}(u), \tag{4.60}$$

where

$$\begin{aligned} v_{1,0}(u) = &\int_0^u E[I(T < \infty)|U_0 = u - y] \\ &\times \int_0^\infty \left\{ h^{*1}_{1,0}(x, y|0) + \int_0^x h^{*1}_{2,0}(x, y, v|0)dv \right\} dxdy \\ &+ \int_u^\infty \int_0^\infty \left\{ h^{*1}_{1,0}(x, y|0) + \int_0^x h^{*1}_{2,0}(x, y, v|0)dv \right\} dxdy. \end{aligned} \tag{4.61}$$

The solution for $E[TI(T < \infty)|U_0 = u]$ in (4.60) involves an infinite sum, so an approximation is obtained using a finite number of terms, say α. In Figure 4.1, the conditional expected time to ruin $E[T|T < \infty, U_0 = u] = E[TI(T < \infty)|U_0 = u]/E[I(T < \infty)|U_0 = u]$ is approximated with different chosen values of α.

For comparison to the approximation, the exact value of $E[T|T < \infty, U_0 = u]$ is also given in Figure 4.1. Note that the exact value of $E[T|T < \infty, U_0 = u]$ is obtained by using the results in Lee and Willmot (2014a), which are applicable for a different class of models than those considered here.

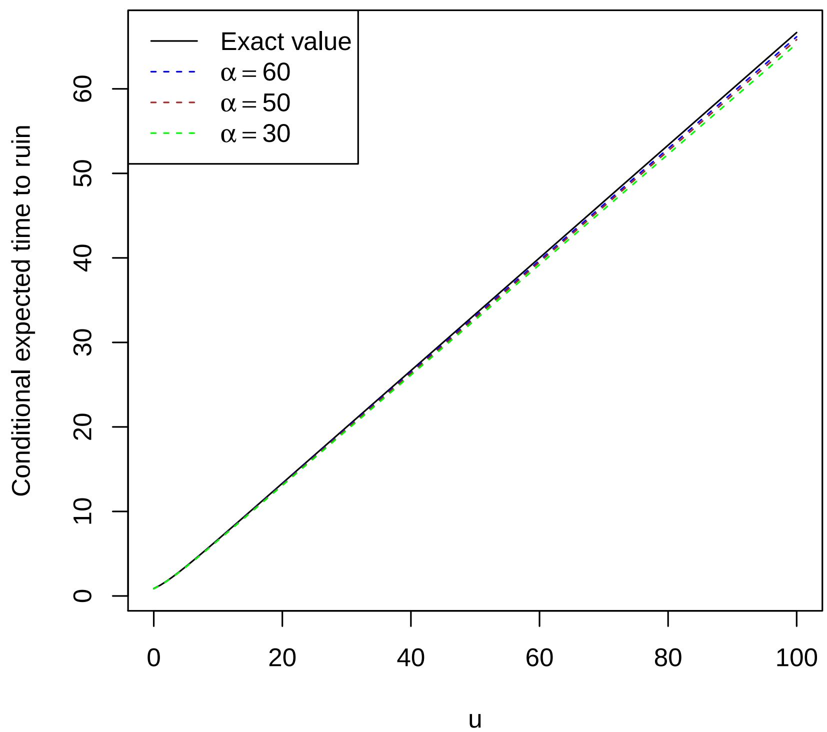

Figure 4.1: Approximate and exact values of $E[T|T < \infty, U_0 = u]$

Example 2 In this example, we consider Erlang(2) interclaim times and Erlang(1/2) claim sizes, i.e. let

$$f(t, y) = 4te^{-2t} \left(\frac{\frac{1}{2} y^{-\frac{1}{2}} e^{-\frac{y}{4}}}{\sqrt{\pi}} \right).$$

Furthermore, assume that $\delta = 0$ and $c = 3$. The value of $E[T|T < \infty, U_0 = u]$ is approximated in Figure 4.2 by using the method in Example 1.

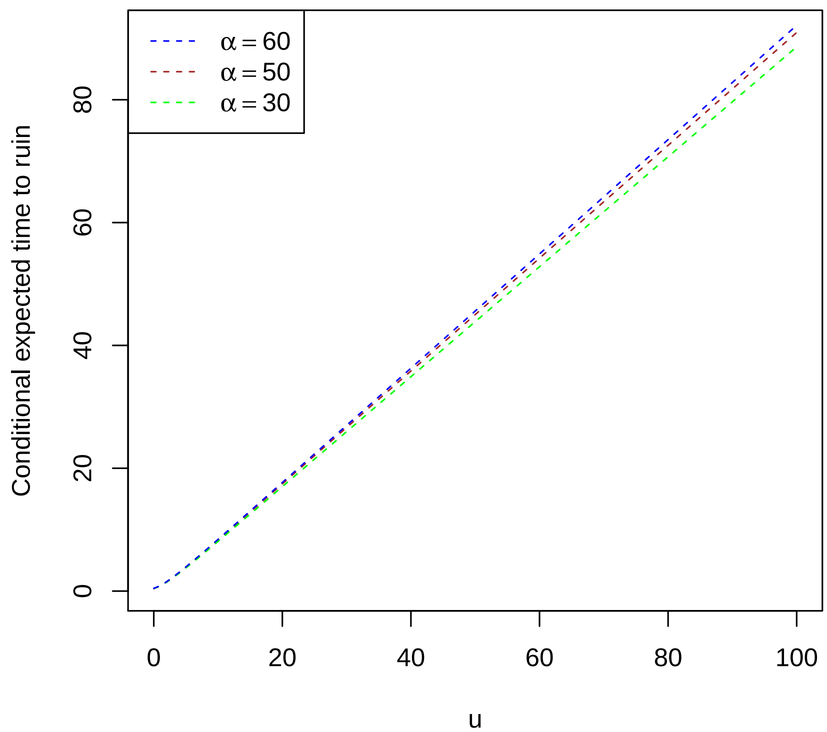

Figure 4.2: Approximation of $E[T|T < \infty, U_0 = u]$

Chapter 5

Joint density of the time to ruin and other ruin quantities in Sparre Andersen models with exponential claims

In Chapter 1, it was mentioned that the joint moments of ruin-related quantities can be obtained through their joint densities by integration. Therefore in this chapter, the joint density of the time to ruin, the number of claims until ruin and other ruin-related quantities is considered under a Sparre Andersen model with the exponential claim sizes.

5.1 Introduction

Consider the dependent Sparre Andersen model described in section 1.1. Recall that the marginal pdfs of interclaim time and claim size are denoted by $k(t)$ and $p(y)$ respectively. Also, the joint pdf of the interclaim time and the claim size is denoted by $f(t, y)$.

Consider also the ruin quantities introduced in Section 1.2.1 which are defined through the time to ruin T and the number of claims until ruin N_T. These include the surplus before ruin U_{T-}, the deficit at ruin $|U_T|$ and the minimum surplus before ruin X_T. Moreover, there is R_{N_T-1} which denotes the surplus immediately after the second last claim before ruin if ruin occurs on claim subsequent to the first, and R_{N_T-1} is equal to u if ruin occurs on first claim. In this chapter, the joint distribution of these quantities is studied through the following generalized Gerber-Shiu function proposed in Shi (2013). For $r \in (0, 1]$ and $\delta \geq 0$, define

$$m_{r,\delta}(u) = E[r^{N_T} e^{-\delta T} w(U_{T-}, |U_T|, X_T, R_{N_T-1}) I(T < \infty) | U_0 = u]. \tag{5.1}$$

5.2 Structural properties of the generalized Gerber-Shiu function

In order to study (5.1), let us first define the following densities involving the ruin quantities in (5.1).

For ruin occuring on the first claim, define

$$g_1(x, y|u) = \begin{cases} \frac{1}{c} f\left(\frac{x-u}{c}, x+y\right), & x > u \\ 0, & \text{otherwise} \end{cases} \tag{5.2}$$

as the joint density of the surplus before ruin (x) and deficit at ruin (y). For ruin occuring on the nth claim, where $n = 2, 3, \ldots$, define

$$g_n(t, x, y, v|u), \quad v < x, \tag{5.3}$$

as the joint defective density of the time to ruin (t), the surplus before ruin (x), the deficit at ruin (y) and the surplus immediately after the second last claim before ruin (v). By definition, $g_1(x, y|u)$ is equivalent to $h_1(x, y|u)$ in (2.2) and $\sum_{n=2}^{\infty} g_n(t, x, y, v|u)$ is equivalent to $h_2(t, x, y, v|u)$ in (2.3).

Based on (5.2) and (5.3), define also the discounted densities

$$g_{1,r,\delta}(x, y|u) = re^{-\delta\left(\frac{x-u}{c}\right)} g_1(x, y|u), \tag{5.4}$$

$$g_{2+,r,\delta}(x, y, v|u) = \sum_{n=2}^{\infty} r^n \int_0^{\infty} e^{-\delta t} g_n(t, x, y, v|u) dt \tag{5.5}$$

and

$$g_{r,\delta}(x, y|u) = g_{1,r,\delta}(x, y|u) + \int_0^x g_{2+,r,\delta}(x, y, v|u) dv.$$

Theorem 5.2.1. *Consider the dependent Sparre Andersen model introduced in Section 1.1. The generalized Gerber-Shiu function* (5.1) *satisfies the defective renewal equation*

$$\begin{aligned} m_{r,\delta}(u) = \phi_{r,\delta} & \int_0^u m_{r,\delta}(u-y) f_{r,\delta}(y) dy \\ & + \int_u^{\infty} \int_0^{\infty} w(u+x, y-u, u, u) g_{1,r,\delta}(x, y|0) dx dy \\ & + \int_u^{\infty} \int_0^{\infty} \int_0^x w(u+x, y-u, u, v+u) g_{2+,r,\delta}(x, y, v|0) dv dx dy, \end{aligned} \tag{5.6}$$

where

$$\phi_{r,\delta} = \int_0^{\infty} \int_0^{\infty} g_{r,\delta}(x, y|0) dx dy \tag{5.7}$$

and

$$f_{r,\delta}(y) = \frac{1}{\phi_{r,\delta}} \int_0^\infty g_{r,\delta}(x, y|0)dx \tag{5.8}$$

is the ladder height density.

Proof. By conditioning on the first drop of the insurance surplus process below initial surplus u,

$$\begin{aligned} m_{r,\delta}(u) = & \int_0^u m_{r,\delta}(u-y) \int_0^\infty g_{r,\delta}(x, y|0)dxdy \\ & + \int_u^\infty \int_0^\infty w(u+x, y-u, u, u) g_{1,r,\delta}(x, y|0)dxdy \\ & + \int_u^\infty \int_0^\infty \int_0^x w(u+x, y-u, u, v+u) g_{2+,r,\delta}(x, y, v|0)dvdxdy, \end{aligned}$$

from which (5.6) follows with definitions (5.7) and (5.8). □

Theorem 5.2.1 generalizes the results in Cheung et al. (2010) and Landriault et al. (2011) where special cases of (5.1) were studied.

For the rest of this chapter, consider the Sparre Andersen model where the interclam times and the claim sizes are assumed to be independent, i.e. $f(t, y) = k(t)p(y)$. Then (5.4) becomes

$$\begin{aligned} g_{1,r,\delta}(x, y|u) &= re^{-\delta(\frac{x-u}{c})} \frac{1}{c} k\left(\frac{x-u}{c}\right) p(x+y) \\ &= \frac{p(x+y)}{\bar{P}(x)} \left\{ re^{-\delta(\frac{x-u}{c})} \frac{1}{c} k\left(\frac{x-u}{c}\right) \bar{P}(x) \right\} \\ &= \frac{p(x+y)}{\bar{P}(x)} \int_0^\infty g_{1,r,\delta}(x, y|u)dy, \end{aligned} \tag{5.9}$$

where $\bar{P}(x) = \int_x^\infty p(y)dy$. Moreover, as argued in Cheung et al. (2010), (5.3) can be written as

$$g_n(t, x, y, v|u) = \frac{p(x+y)}{\bar{P}(x)} \int_0^\infty g_n(t, x, y, v|u)dy,$$

for $n = 2, 3, \ldots$. Thus, it follows that (5.5) has the form

$$\begin{aligned} g_{2+,r,\delta}(x, y, v|u) &= \sum_{n=2}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \left\{ \frac{p(x+y)}{\bar{P}(x)} \int_0^{\infty} g_n(t, x, y, v|u) dy \right\} dt \\ &= \frac{p(x+y)}{\bar{P}(x)} \int_0^{\infty} g_{2+,r,\delta}(x, y, v|u) dy \\ &= \frac{p(x+y)}{\bar{P}(x)} g_{2+,r,\delta}(x, v|u), \end{aligned} \tag{5.10}$$

where

$$g_{2+,r,\delta}(x, v|u) = \int_0^{\infty} g_{2+,r,\delta}(x, y, v|u) dy.$$

Hence,

$$g_{r,\delta}(x, y|u) = \frac{p(x+y)}{\bar{P}(x)} \left\{ \int_0^{\infty} g_{1,r,\delta}(x, y|u) dy + \int_0^{x} g_{2+,r,\delta}(x, v|u) dv \right\}$$

and the ladder height density (5.8) becomes

$$f_{r,\delta}(y) = \frac{1}{\phi_{r,\delta}} \int_0^{\infty} \frac{p(x+y)}{\bar{P}(x)} \left\{ \int_0^{\infty} g_{1,r,\delta}(x, y|0) dy + \int_0^{x} g_{2+,r,\delta}(x, v|0) dv \right\} dx. \tag{5.11}$$

5.3 Joint density of the time to ruin and other ruin quantities under exponential claims

In this section, further assume that the claim sizes follow exponential distribution, i.e.

$$p(y) = \beta e^{-\beta y}, \quad y \geq 0. \tag{5.12}$$

Then (5.11) becomes

$$f_{r,\delta}(y) = \beta e^{-\beta y}, \tag{5.13}$$

which means that the ladder height density is also exponential. Moreover, Landriault et al. (2011) showed that $\phi_{r,\delta}$ defined in (5.7) satisfy

$$\phi_{r,\delta} = r\tilde{k}(\delta + c\beta(1 - \phi_{r,\delta})). \tag{5.14}$$

Now, consider

$$\bar{G}_{r,\delta}(u) = E[r^{N_T} e^{-\delta T} I(T < \infty)|U_0 = u]. \tag{5.15}$$

In Landriault et al. (2011), it was shown that

$$\bar{G}_{r,\delta}(u) = \phi_{r,\delta} \int_0^u \bar{G}_{r,\delta}(u-y)\beta e^{-\beta y} dy + \phi_{r,\delta} e^{-\beta u} \tag{5.16}$$

and $\bar{G}_{r,\delta}(u)$ can be expressed explicitly as

$$\bar{G}_{r,\delta}(u) = \phi_{r,\delta} e^{-\beta(1-\phi_{r,\delta})u}, \tag{5.17}$$

where $\phi_{r,\delta}$ is given by (5.14). By these results for $\bar{G}_{r,\delta}(u)$ and Lagrange's expansion theorem, Landriault et al. (2011) further showed that the joint density of the time to ruin (t) and the number of claims until ruin (n) given initial surplus u is given by

$$h(t, n|u) = \begin{cases} e^{-\beta(u+ct)} k(t), & t \geq 0, n = 1, \\ \frac{nu+ct}{n(n-1)} \left\{ \frac{\beta^{n-1}(u+ct)^{n-2} e^{-\beta(u+ct)}}{(n-2)!} \right\} k^{*n}(t), & t \geq 0, n = 2, 3, \ldots, \end{cases} \tag{5.18}$$

where $k^{*n}(t) = \int_0^t k^{*(n-1)}(t-x)k(x)dx$ with $k^{*1}(t) = k(t)$. (Note that in this chapter, the term "joint density" is used even though the number of claim until ruin is discrete.) The joint density (5.18) was also obtained in Borovkov and Dickson (2008) by a duality argument.

The above results introduced for $\bar{G}_{r,\delta}(u)$ are useful in studying the Gerber-Shiu function

$$m_{r,\delta,1234}(u) = E[r^{N_T} e^{-\delta T} e^{-s_1 U_{T-} - s_2 |U_T| - s_3 X_T - s_4 R_{N_T - 1}} I(T < \infty)|U_0 = u]. \tag{5.19}$$

Theorem 5.3.1. *Consider the Sparre Andersen model described in section 1.1 which has independent interclaim times and claim sizes, i.e.* $f(t, y) = k(t)p(y)$*. Furthermore, assume that the claim sizes have pdf* (5.12)*. With* $\phi_{r,\delta}$ *and* $\bar{G}_{r,\delta}(u)$ *given by* (5.14) *and* (5.17) *respectively, the Gerber-Shiu function* (5.19) *can be explicitly expressed as*

$$m_{r,\delta,1234}(u) = \Theta_{r,\delta}(s_1, s_2, s_3, s_4)\left\{(s_1 + s_3 + s_4)e^{-(s_1+s_3+s_4+\beta)u} + \beta\bar{G}_{r,\delta}(u)\right\}, \tag{5.20}$$

where

$$\Theta_{r,\delta}(s_1, s_2, s_3, s_4) = \frac{\beta}{\beta + s_2}\left\{\frac{\beta\phi_{r,\delta} + s_1 + s_4}{\beta\phi_{r,\delta} + s_1 + s_3 + s_4}\right\} \times \left\{\frac{r\tilde{k}(\delta + c(s_1 + \beta))}{s_1 + s_4 + r\beta\tilde{k}(\delta + c(s_1 + s_4 + \beta))}\right\}.$$

Proof. With (5.12), the discounted densities (5.9) and (5.10) with $u = 0$ are given by

$$g_{1,r,\delta}(x, y|0) = re^{-\delta\left(\frac{x}{c}\right)}\frac{1}{c}k\left(\frac{x}{c}\right)\beta e^{-\beta(x+y)} \tag{5.21}$$

and

$$g_{2+,r,\delta}(x, y, v|0) = \beta e^{-\beta y}g_{2+,r,\delta}(x, v|0). \tag{5.22}$$

Substitute (5.13), (5.21) and (5.22) into (5.6), which yields

$$
\begin{aligned}
m_{r,\delta,1234}(u) &= \phi_{r,\delta} \int_0^u m_{r,\delta,1234}(u-y)\beta e^{-\beta y} dy \\
&+ \int_u^\infty \int_0^\infty e^{-s_1(u+x)-s_2(y-u)-s_3 u - s_4 u} \left\{ r e^{-\delta\left(\frac{x}{c}\right)} \frac{1}{c} k\left(\frac{x}{c}\right) \beta e^{-\beta(x+y)} \right\} dx dy \\
&+ \int_u^\infty \int_0^\infty \int_0^x e^{-s_1(u+x)-s_2(y-u)-s_3 u - s_4(v+u)} \left\{ \beta e^{-\beta y} g_{2+,r,\delta}(x,v|0) \right\} dv dx dy \\
&= \phi_{r,\delta} \int_0^u m_{r,\delta,1234}(u-y)\beta e^{-\beta y} dy \\
&+ \int_0^\infty \int_0^\infty e^{-s_1(u+ct)-s_2 y - s_3 u - s_4 u} \left\{ r e^{-\delta t} k(t) \beta e^{-\beta(ct+y+u)} \right\} dt dy \\
&+ \int_0^\infty \int_0^\infty \int_0^x e^{-s_1(u+x)-s_2 y - s_3 u - s_4(v+u)} \left\{ \beta e^{-\beta(y+u)} g_{2+,r,\delta}(x,v|0) \right\} dv dx dy \\
&= \phi_{r,\delta} \int_0^u m_{r,\delta,1234}(u-y)\beta e^{-\beta y} dy \\
&+ \frac{\beta}{\beta+s_2} r e^{-(s_1+s_3+s_4+\beta)u} \tilde{k}(\delta + c(s_1+\beta)) \\
&+ \frac{\beta}{\beta+s_2} e^{-(s_1+s_3+s_4+\beta)u} \int_0^\infty \int_0^x e^{-s_1 x - s_4 v} g_{2+,r,\delta}(x,v|0) dv dx \\
&= \phi_{r,\delta} \int_0^u m_{r,\delta,1234}(u-y)\beta e^{-\beta y} dy + \frac{\beta}{\beta+s_2} e^{-(s_1+s_3+s_4+\beta)u} \xi_{r,\delta}(s_1,s_4), \quad (5.23)
\end{aligned}
$$

where

$$
\xi_{r,\delta}(s_1,s_4) = r\tilde{k}(\delta + c(s_1+\beta)) + \int_0^\infty \int_0^x e^{-s_1 x - s_4 v} g_{2+,r,\delta}(x,v|0) dv dx.
$$

Take Laplace transform on both sides of (5.23), i.e.

$$
\tilde{m}_{r,\delta,1234}(z) = \phi_{r,\delta} \tilde{m}_{r,\delta,1234}(z) \frac{\beta}{\beta+z} + \frac{\beta}{\beta+s_2} \left\{ \frac{1}{s_1+s_3+s_4+\beta+z} \right\} \xi_{r,\delta}(s_1,s_4). \quad (5.24)
$$

By rearranging and partial fraction, (5.24) gives

$$
\begin{aligned}
\tilde{m}_{r,\delta,1234}(z) &= \frac{\beta}{\beta+s_2} \xi_{r,\delta}(s_1,s_4) \left\{ \frac{1}{s_1+s_3+s_4+\beta+z} \right\} \left\{ \frac{\beta+z}{\beta(1-\phi_{r,\delta})+z} \right\} \\
&= \frac{\beta}{\beta+s_2} \left\{ \frac{\xi_{r,\delta}(s_1,s_4)}{\beta\phi_{r,\delta}+s_1+s_3+s_4} \right\} \left\{ \frac{s_1+s_3+s_4}{s_1+s_3+s_4+\beta+z} + \frac{\beta\phi_{r,\delta}}{\beta(1-\phi_{r,\delta})+z} \right\}. \\
& \qquad (5.25)
\end{aligned}
$$

Then invert (5.25) with respect to z results in

$$m_{r,\delta,1234}(u) = \frac{\beta}{\beta + s_2} \left\{ \frac{\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_3 + s_4} \right\} \left\{ (s_1 + s_3 + s_4) e^{-(s_1+s_3+s_4+\beta)u} + \beta \bar{G}_{r,\delta}(u) \right\}, \tag{5.26}$$

where $\bar{G}_{r,\delta}(u)$ is given by (5.17).

To completely specify (5.26), it remains to identify $\xi_{r,\delta}(s_1, s_4)$. The approach used in Cheung et al. (2010) can be applied here for this purpose. Define

$$m_{r,\delta,14}(u) = E[r^{N_T} e^{-\delta T} e^{-s_1 U_{T-} - s_4 R_{N_T - 1}} I(T < \infty) | U_0 = u].$$

From (5.26) with $s_2 = s_3 = 0$,

$$m_{r,\delta,14}(u) = \frac{\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4} \left\{ (s_1 + s_4) e^{-(s_1+s_4+\beta)u} + \beta \bar{G}_{r,\delta}(u) \right\}. \tag{5.27}$$

On the other hand, by conditioning on the time and amount of the first claim gives

$$\begin{aligned} m_{r,\delta,14}(u) = \int_0^\infty r e^{-\delta t} \Bigg\{ & \int_0^{u+ct} m_{r,\delta,14}(u + ct - y) \beta e^{-\beta y} dy \\ & + \int_{u+ct}^\infty e^{-s_1(u+ct) - s_4 u} \beta e^{-\beta y} dy \Bigg\} k(t) dt. \end{aligned} \tag{5.28}$$

Using (5.27) and (5.16), it follows that (5.28) can be simplified as

$$
\begin{aligned}
m_{r,\delta,14}(u) = & \int_0^\infty re^{-\delta t} \int_0^{u+ct} \frac{\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4} \Big\{ (s_1 + s_4) e^{-(s_1+s_4+\beta)(u+ct-y)} \\
& + \beta\bar{G}_{r,\delta}(u + ct - y) \Big\} \beta e^{-\beta y} dy k(t) dt \\
& + \int_0^\infty re^{-\delta t} \int_{u+ct}^\infty e^{-s_1(u+ct)-s_4 u} \beta e^{-\beta y} dy k(t) dt \\
= & \int_0^\infty re^{-\delta t} \frac{\beta\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4} \Big\{ e^{-\beta(u+ct)} [1 - e^{-(s_1+s_4)(u+ct)}] \\
& + \frac{\bar{G}_{r,\delta}(u + ct)}{\phi_{r,\delta}} - e^{-\beta(u+ct)} \Big\} k(t) dt \\
& + \int_0^\infty re^{-\delta t} \Big\{ e^{-(\beta+s_1)(u+ct)-s_4 u} \Big\} k(t) dt \\
= & \frac{\beta\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4} \int_0^\infty re^{-\delta t} \left\{ \frac{\bar{G}_{r,\delta}(u + ct)}{\phi_{r,\delta}} - e^{-(s_1+s_4+\beta)(u+ct)} \right\} k(t) dt \\
& + e^{-(s_1+s_4+\beta)u} r\tilde{k}(\delta + c(s_1 + \beta)).
\end{aligned}
\tag{5.29}
$$

By (5.14) and (5.17), (5.29) becomes

$$
\begin{aligned}
m_{r,\delta,14}(u) = & \frac{\beta\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4} \Big\{ \int_0^\infty re^{-\delta t} e^{-\beta(1-\phi_{r,\delta})(u+ct)} k(t) dt \\
& - e^{-(s_1+s_4+\beta)u} r\tilde{k}(\delta + c(s_1 + s_4 + \beta)) \Big\} \\
& + e^{-(s_1+s_4+\beta)u} r\tilde{k}(\delta + c(s_1 + \beta)) \\
= & \frac{\beta\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4} \Big\{ \phi_{r,\delta} e^{-\beta(1-\phi_{r,\delta})u} - e^{-(s_1+s_4+\beta)u} r\tilde{k}(\delta + c(s_1 + s_4 + \beta)) \Big\} \\
& + e^{-(s_1+s_4+\beta)u} r\tilde{k}(\delta + c(s_1 + \beta)) \\
= & \frac{\beta\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4} \Big\{ \bar{G}_{r,\delta}(u) - e^{-(s_1+s_4+\beta)u} r\tilde{k}(\delta + c(s_1 + s_4 + \beta)) \Big\} \\
& + e^{-(s_1+s_4+\beta)u} r\tilde{k}(\delta + c(s_1 + \beta)).
\end{aligned}
\tag{5.30}
$$

Then equate (5.27) and (5.30) to get

$$\frac{\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4}(s_1 + s_4)e^{-(s_1+s_4+\beta)u}$$
$$= \left\{ r\tilde{k}(\delta + c(s_1 + \beta)) - \frac{\beta\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4} r\tilde{k}(\delta + c(s_1 + s_4 + \beta)) \right\}$$
$$\times e^{-(s_1+s_4+\beta)u}. \tag{5.31}$$

Since (5.31) is true for all $u \geq 0$, the coefficients of $e^{-(s_1+s_4+\beta)u}$ on both sides are equal, which gives

$$\frac{(s_1 + s_4)\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4} = r\tilde{k}(\delta + c(s_1 + \beta)) - \frac{\beta\xi_{r,\delta}(s_1, s_4)}{\beta\phi_{r,\delta} + s_1 + s_4} r\tilde{k}(\delta + c(s_1 + s_4 + \beta))$$
$$\xi_{r,\delta}(s_1, s_4) = \frac{(\beta\phi_{r,\delta} + s_1 + s_4)r\tilde{k}(\delta + c(s_1 + \beta))}{s_1 + s_4 + r\beta\tilde{k}(\delta + c(s_1 + s_4 + \beta))}. \tag{5.32}$$

Finally, (5.20) follows by substituting (5.32) into (5.26). □

Theorem 5.3.1 provides a generalization to the results in Cheung et al. (2010) and Landriault et al. (2011).

In the following theorem, the joint density of the ruin-related random variables is identified. As mentioned in section 2.1, the joint density of $(N_T, T, U_{T-}, |U_T|, R_{N_T-1})$ instead of $(N_T, T, U_{T-}, |U_T|, X_T, R_{N_T-1})$ can be considered without loss of generality.

Theorem 5.3.2. *Suppose the conditions of Theorem 5.3.1 hold. The joint densities defined in* (5.2) *and* (5.3) *are given by*

$$g_1(x, y|u) = \frac{1}{c}\beta e^{-\beta(x+y)} k\left(\frac{x-u}{c}\right), \quad x > u,$$
$$y \geq 0, \tag{5.33}$$

$$g_2(t,x,y,v|u) = \frac{\beta^2 e^{-\beta(u+ct+y)}}{c} k\left(t - \frac{x-v}{c}\right) k\left(\frac{x-v}{c}\right), \quad t,y \geq 0,$$
$$x \in [0, u+ct],$$
$$v \in [max(x-ct,0), x] \quad (5.34)$$

and

$$\begin{aligned} g_n(t,x,y,v|u) = & \frac{\beta^2 e^{-\beta(x-v+y)}}{c} k\left(\frac{x-v}{c}\right) h\left(t - \frac{x-v}{c}, n-1|u\right) \\ & + \sum_{j=1}^{n-2} \int_0^{\frac{v-max(x-ct,0)}{c}} \frac{(cz-v)^j}{j!} \frac{\beta^{j+2} e^{-\beta(x-v+y+cz)}}{c} k\left(\frac{x-v}{c}\right) \\ & \times k^{*j}(z) h\left(t - z - \frac{x-v}{c}, n-j-1|u\right) dz, \quad t,y \geq 0, \\ & x \in [0, u+ct], \\ & v \in [max(x-ct,0), x], \\ & n = 3,4,\ldots, \end{aligned} \quad (5.35)$$

where $h(t,n|u)$ is given by (5.18).

Proof. Let

$$m_{r,\delta,124}(u) = E[r^{N_T} e^{-\delta T} e^{-s_1 U_{T-} - s_2 |U_T| - s_4 R_{N_T - 1}} I(T < \infty) | U_0 = u].$$

From (5.20),

$$\begin{aligned} m_{r,\delta,124}(u) = & \frac{\beta}{\beta + s_2} \\ & \times \left\{ r\tilde{k}(\delta + c(s_1+\beta)) e^{-(s_1+s_4+\beta)u} \frac{s_1+s_4}{s_1+s_4+r\beta\tilde{k}(\delta + c(s_1+s_4+\beta))} \right. \\ & \left. + \frac{r\beta\tilde{k}(\delta + c(s_1+\beta))}{s_1+s_4+r\beta\tilde{k}(\delta + c(s_1+s_4+\beta))} \bar{G}_{r,\delta}(u) \right\}. \end{aligned} \quad (5.36)$$

In the following, invert (5.36) with respect to r, δ, s_1, s_2 and s_4. To start with, it is obvious that

$$\frac{\beta}{\beta + s_2} = \int_0^\infty e^{-s_2 y} \beta e^{-\beta y} dy. \tag{5.37}$$

Next, using the identity

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1,$$

it can be shown that when $s_1 + s_4 > \beta$,

$$\begin{aligned}
&\frac{s_1 + s_4}{s_1 + s_4 + r\beta\tilde{k}(\delta + c(s_1 + s_4 + \beta))} \\
&= \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{r\beta\tilde{k}(\delta + c(s_1 + s_4 + \beta))}{s_1 + s_4} \right\}^n \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n r^n \beta^n \int_0^\infty e^{-\delta t} \frac{e^{-c(s_1+s_4)t}}{(s_1 + s_4)^n} e^{-c\beta t} k^{*n}(t) dt.
\end{aligned} \tag{5.38}$$

By applying the equation given in Landriault et al. (2011), i.e.

$$\frac{e^{-cst}}{s^n} = \int_{ct}^{\infty} \frac{(x - ct)^{n-1}}{(n-1)!} e^{-sx} dx \tag{5.39}$$

for $n = 1, 2, \ldots$, (5.38) becomes

$$\begin{aligned}
&\frac{s_1 + s_4}{s_1 + s_4 + r\beta\tilde{k}(\delta + c(s_1 + s_4 + \beta))} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n r^n \beta^n \int_0^\infty e^{-\delta t} \left\{ \int_{ct}^{\infty} \frac{(x - ct)^{n-1}}{(n-1)!} e^{-(s_1+s_4)x} dx \right\} e^{-c\beta t} k^{*n}(t) dt \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n r^n \beta^n \int_0^\infty \int_{ct}^{\infty} e^{-\delta t} e^{-(s_1+s_4)x} \frac{(x - ct)^{n-1}}{(n-1)!} e^{-c\beta t} k^{*n}(t) dx dt.
\end{aligned} \tag{5.40}$$

Using (5.40), the term on the second line of (5.36) can be inverted as

$$
\begin{aligned}
& r\tilde{k}(\delta + c(s_1+\beta))e^{-(s_1+s_4+\beta)u}\left\{\frac{s_1+s_4}{s_1+s_4+r\beta\tilde{k}(\delta+c(s_1+s_4+\beta))}\right\} \\
&= \left\{r\int_0^\infty e^{-\delta t}e^{-s_1(u+ct)-s_4u}e^{-\beta(u+ct)}k(t)dt\right\} \\
&\times \left\{1+\sum_{n=1}^\infty(-1)^n r^n\beta^n\int_0^\infty\int_{ct}^\infty e^{-\delta t}e^{-(s_1+s_4)x}\frac{(x-ct)^{n-1}}{(n-1)!}e^{-c\beta t}k^{*n}(t)dxdt\right\} \\
&= r\int_0^\infty e^{-\delta t}e^{-s_1(u+ct)-s_4u}e^{-\beta(u+ct)}k(t)dt \\
&+ \sum_{n=1}^\infty(-1)^n r^{n+1}\beta^n\int_0^\infty e^{-\delta t}\int_0^t\left\{e^{-s_1(u+cv)-s_4u}e^{-\beta(u+cv)}k(v)\right\} \\
&\times \left\{\int_{c(t-v)}^\infty e^{-(s_1+s_4)x}\frac{(x-c(t-v))^{n-1}}{(n-1)!}e^{-c\beta(t-v)}k^{*n}(t-v)dx\right\}dvdt \\
&= r\int_u^\infty e^{-\delta\left(\frac{x-u}{c}\right)}e^{-s_1x-s_4u}\frac{1}{c}e^{-\beta x}k\left(\frac{x-u}{c}\right)dx \\
&- \sum_{n=2}^\infty r^n\int_0^\infty e^{-\delta t}\int_0^t\int_{c(t-v)}^\infty e^{-s_1(x+u+cv)}e^{-s_4(x+u)} \\
&\times \frac{\beta^{n-1}(c(t-v)-x)^{n-2}}{(n-2)!}e^{-\beta(u+ct)}k(v)k^{*(n-1)}(t-v)dxdvdt \\
&= r\int_u^\infty e^{-\delta\left(\frac{x-u}{c}\right)}e^{-s_1x-s_4u}\frac{1}{c}e^{-\beta x}k\left(\frac{x-u}{c}\right)dx \\
&- \sum_{n=2}^\infty r^n\int_0^\infty\int_{u+ct}^\infty\int_{x-ct}^x e^{-\delta t-s_1x-s_4v} \\
&\times \frac{\beta^{n-1}(u+ct-x)^{n-2}}{c(n-2)!}e^{-\beta(u+ct)}k\left(\frac{x-v}{c}\right)k^{*(n-1)}\left(\frac{v+ct-x}{c}\right)dvdxdt. \qquad (5.41)
\end{aligned}
$$

Consider next the term on the last line of (5.36). From (5.38) and (5.39), one has

$$\begin{aligned}
&\frac{1}{s_1+s_4+r\beta\tilde{k}(\delta+c(s_1+s_4+\beta))}\\
&=\frac{1}{s_1+s_4}\left\{1+\sum_{n=1}^{\infty}(-1)^n r^n\beta^n\int_0^\infty e^{-\delta t}\frac{e^{-c(s_1+s_4)t}}{(s_1+s_4)^n}e^{-c\beta t}k^{*n}(t)dt\right\}\\
&=\frac{1}{s_1+s_4}+\sum_{n=1}^{\infty}(-1)^n r^n\beta^n\int_0^\infty e^{-\delta t}\left\{\int_{ct}^\infty\frac{(v-ct)^n}{n!}e^{-(s_1+s_4)v}dv\right\}e^{-c\beta t}k^{*n}(t)dt\\
&=\int_0^\infty e^{-(s_1+s_4)x}dx+\sum_{n=1}^{\infty}r^n\int_0^\infty e^{-\delta t}\int_{ct}^\infty e^{-(s_1+s_4)v}\frac{\beta^n(ct-v)^n}{n!}e^{-c\beta t}k^{*n}(t)dvdt.
\end{aligned}$$

Moreover, by definition of $\bar{G}_{r,\delta}(u)$ in (5.15),

$$\bar{G}_{r,\delta}(u)=\sum_{n=1}^{\infty}r^n\int_0^\infty e^{-\delta t}h(t,n|u)dt,$$

where $h(t,n|u)$ is given by (5.18). Thus, the term on the last line of (5.36) can be written as

$$\begin{aligned}
&\frac{r\beta\tilde{k}(\delta+c(s_1+\beta))}{s_1+s_4+r\beta\tilde{k}(\delta+c(s_1+s_4+\beta))}\bar{G}_{r,\delta}(u)\\
&=\left\{r\beta\int_0^\infty e^{-\delta t}e^{-c(s_1+\beta)t}k(t)dt\right\}\times\left\{\sum_{n=1}^{\infty}r^n\int_0^\infty e^{-\delta t}h(t,n|u)dt\right\}\\
&\times\left\{\int_0^\infty e^{-(s_1+s_4)x}dx+\sum_{n=1}^{\infty}r^n\int_0^\infty e^{-\delta t}\int_{ct}^\infty e^{-(s_1+s_4)v}\frac{\beta^n(ct-v)^n}{n!}e^{-c\beta t}k^{*n}(t)dvdt\right\}\\
&=\left\{\sum_{n=2}^{\infty}r^n\int_0^\infty e^{-\delta t}\int_0^t e^{-c(s_1+\beta)x}\beta k(x)h(t-x,n-1|u)dxdt\right\}\\
&\times\left\{\int_0^\infty e^{-(s_1+s_4)x}dx+\sum_{n=1}^{\infty}r^n\int_0^\infty e^{-\delta t}\int_{ct}^\infty e^{-(s_1+s_4)v}\frac{\beta^n(ct-v)^n}{n!}e^{-c\beta t}k^{*n}(t)dvdt\right\}.
\end{aligned}\tag{5.42}$$

Let us study (5.42) as a sum of two terms. The first term is

$$\left\{\sum_{n=2}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \int_0^t e^{-c(s_1+\beta)x} \beta k(x) h(t-x, n-1|u) dx dt\right\} \left\{\int_0^{\infty} e^{-(s_1+s_4)x} dx\right\}$$
$$= \left\{\sum_{n=2}^{\infty} r^n \int_0^{\infty} e^{-s_1 x} \int_{\frac{x}{c}}^{\infty} e^{-\delta t} \frac{\beta e^{-\beta x}}{c} k\left(\frac{x}{c}\right) h\left(t-\frac{x}{c}, n-1|u\right) dt dx\right\} \left\{\int_0^{\infty} e^{-(s_1+s_4)x} dx\right\}$$
$$= \sum_{n=2}^{\infty} r^n \int_0^{\infty} e^{-s_1 x}$$
$$\times \int_0^x \left\{\int_{\frac{x-v}{c}}^{\infty} e^{-\delta t} \frac{\beta e^{-\beta(x-v)}}{c} k\left(\frac{x-v}{c}\right) h\left(t-\frac{x-v}{c}, n-1|u\right) dt\right\} e^{-s_4 v} dv dx$$
$$= \sum_{n=2}^{\infty} r^n \int_0^{\infty} \int_0^{\infty} \int_{max(x-ct,0)}^{x} e^{-\delta t - s_1 x - s_4 v}$$
$$\times \frac{\beta e^{-\beta(x-v)}}{c} k\left(\frac{x-v}{c}\right) h\left(t-\frac{x-v}{c}, n-1|u\right) dv dx dt$$
$$= r^2 \int_0^{\infty} \int_0^{\infty} \int_{max(x-ct,0)}^{x} e^{-\delta t - s_1 x - s_4 v} \frac{\beta e^{-\beta(u+ct)}}{c} k\left(t-\frac{x-v}{c}\right) k\left(\frac{x-v}{c}\right) dv dx dt$$
$$+ \sum_{n=3}^{\infty} r^n \int_0^{\infty} \int_0^{\infty} \int_{max(x-ct,0)}^{x} e^{-\delta t - s_1 x - s_4 v}$$
$$\times \frac{\beta e^{-\beta(x-v)}}{c} k\left(\frac{x-v}{c}\right) h\left(t-\frac{x-v}{c}, n-1|u\right) dv dx dt. \quad (5.43)$$

The second term is

$$\left\{\sum_{n=2}^{\infty} r^n \int_0^\infty e^{-\delta t}\int_0^t e^{-c(s_1+\beta)x}\beta k(x)h(t-x,n-1|u)dxdt\right\}$$
$$\times\left\{\sum_{n=1}^{\infty} r^n \int_0^\infty e^{-\delta t}\int_{ct}^\infty e^{-(s_1+s_4)v}\frac{\beta^n(ct-v)^n}{n!}e^{-c\beta t}k^{*n}(t)dvdt\right\}$$
$$=\left\{\sum_{n=2}^{\infty} r^n \int_0^\infty e^{-\delta t}\int_0^{ct} e^{-s_1x}\frac{\beta e^{-\beta x}}{c}k\left(\frac{x}{c}\right)h\left(t-\frac{x}{c},n-1|u\right)dxdt\right\}$$
$$\times\left\{\sum_{n=1}^{\infty} r^n \int_0^\infty e^{-\delta t}\int_{ct}^\infty e^{-(s_1+s_4)v}\frac{\beta^n(ct-v)^n}{n!}e^{-c\beta t}k^{*n}(t)dvdt\right\}$$
$$=\sum_{n=3}^{\infty} r^n \sum_{j=1}^{n-2}\left\{\int_0^\infty e^{-\delta t}\int_{ct}^\infty e^{-(s_1+s_4)v}\frac{\beta^j(ct-v)^j}{j!}e^{-c\beta t}k^{*j}(t)dvdt\right\}$$
$$\times\left\{\int_0^\infty e^{-\delta t}\int_0^{ct} e^{-s_1x}\frac{\beta e^{-\beta x}}{c}k\left(\frac{x}{c}\right)h\left(t-\frac{x}{c},n-j-1|u\right)dxdt\right\}$$
$$=\sum_{n=3}^{\infty} r^n \sum_{j=1}^{n-2}\int_0^\infty e^{-\delta t}\int_0^t\left\{\int_{cz}^\infty e^{-(s_1+s_4)v}\frac{\beta^j(cz-v)^j}{j!}e^{-c\beta z}k^{*j}(z)dv\right\}$$
$$\times\left\{\int_0^{c(t-z)} e^{-s_1x}\frac{\beta e^{-\beta x}}{c}k\left(\frac{x}{c}\right)h\left(t-z-\frac{x}{c},n-j-1|u\right)dx\right\}dzdt$$
$$=\sum_{n=3}^{\infty} r^n \int_0^\infty\int_0^\infty\int_{max(x-ct,0)}^x e^{-\delta t}e^{-s_1x}e^{-s_4v}\left\{\sum_{j=1}^{n-2}\int_0^{\frac{v-max(x-ct,0)}{c}}\frac{\beta^{j+1}(cz-v)^j}{j!c}\right.$$
$$\left.\times\, e^{-\beta(x-v+cz)}k\left(\frac{x-v}{c}\right)k^{*j}(z)h\left(t-z-\frac{x-v}{c},n-j-1|u\right)dz\right\}dvdxdt. \qquad (5.44)$$

Hence, with (5.43) and (5.44), (5.42) becomes

$$
\begin{aligned}
&\frac{r\beta\tilde{k}(\delta + c(s_1+\beta))}{s_1+s_4+r\beta\tilde{k}(\delta+c(s_1+s_4+\beta))}\bar{G}_{r,\delta}(u) \\
&= r^2\int_0^\infty\int_0^\infty\int_{max(x-ct,0)}^x e^{-\delta t - s_1 x - s_4 v}\frac{\beta e^{-\beta(u+ct)}}{c}k\left(t-\frac{x-v}{c}\right)k\left(\frac{x-v}{c}\right)dvdxdt \\
&+ \sum_{n=3}^\infty r^n \int_0^\infty\int_0^\infty\int_{max(x-ct,0)}^x e^{-\delta t - s_1 x - s_4 v} \\
&\times\left\{\frac{\beta e^{-\beta(x-v)}}{c}k\left(\frac{x-v}{c}\right)h\left(t-\frac{x-v}{c}, n-1|u\right)+\sum_{j=1}^{n-2}\int_0^{\frac{v-max(x-ct,0)}{c}}\frac{\beta^{j+1}(cz-v)^j}{j!c}\right. \\
&\left.\times\, e^{-\beta(x-v+cz)}k\left(\frac{x-v}{c}\right)k^{*j}(z)h\left(t-z-\frac{x-v}{c}, n-j-1|u\right)dz\right\}dvdxdt. \qquad (5.45)
\end{aligned}
$$

Finally, substitute (5.37), (5.41) and (5.45) into (5.36), which yields

$$
\begin{aligned}
m_{r,\delta,124}(u) = {} & r\int_u^{\infty}\int_0^{\infty} e^{-\delta\left(\frac{x-u}{c}\right)-s_1x-s_2y-s_4u}\frac{1}{c}\beta e^{-\beta(x+y)}k\left(\frac{x-u}{c}\right)dydx \\
& -\sum_{n=2}^{\infty} r^n \int_0^{\infty}\int_0^{\infty}\int_{u+ct}^{\infty}\int_{x-ct}^{x} e^{-\delta t-s_2y-s_1x-s_4v} \\
& \times \frac{\beta^n(u+ct-x)^{n-2}}{c(n-2)!}e^{-\beta(u+ct+y)}k\left(\frac{x-v}{c}\right)k^{*(n-1)}\left(\frac{v+ct-x}{c}\right)dvdxdydt \\
& + r^2\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_{max(x-ct,0)}^{x} e^{-\delta t-s_2y-s_1x-s_4v} \\
& \times \frac{\beta^2 e^{-\beta(u+ct+y)}}{c}k\left(t-\frac{x-v}{c}\right)k\left(\frac{x-v}{c}\right)dvdxdydt \\
& + \sum_{n=3}^{\infty} r^n \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_{max(x-ct,0)}^{x} e^{-\delta t-s_2y-s_1x-s_4v} \\
& \times \left\{\frac{\beta^2 e^{-\beta(x-v+y)}}{c}k\left(\frac{x-v}{c}\right)h\left(t-\frac{x-v}{c}, n-1|u\right)\right. \\
& + \sum_{j=1}^{n-2}\int_0^{\frac{v-max(x-ct,0)}{c}} \frac{\beta^{j+2}(cz-v)^j}{j!c}e^{-\beta(x-v+y+cz)}k\left(\frac{x-v}{c}\right) \\
& \times \left. k^{*j}(z)h\left(t-z-\frac{x-v}{c}, n-j-1|u\right)dz\right\}dvdxdydt. \qquad (5.46)
\end{aligned}
$$

For a given time to ruin t, the surplus before ruin must be less than or equal to $u+ct$. Therefore, (5.46) reduces to

$$\begin{aligned}
m_{r,\delta,124}(u) &= r\int_u^\infty\int_0^\infty e^{-\delta\left(\frac{x-u}{c}\right)-s_1x-s_2y-s_4u}\frac{1}{c}\beta e^{-\beta(x+y)}k\left(\frac{x-u}{c}\right)dydx \\
&+ r^2\int_0^\infty\int_0^\infty\int_0^{u+ct}\int_{max(x-ct,0)}^x e^{-\delta t-s_2y-s_1x-s_4v} \\
&\times \frac{\beta^2e^{-\beta(u+ct+y)}}{c}k\left(t-\frac{x-v}{c}\right)k\left(\frac{x-v}{c}\right)dvdxdydt \\
&+ \sum_{n=3}^\infty r^n\int_0^\infty\int_0^\infty\int_0^{u+ct}\int_{max(x-ct,0)}^x e^{-\delta t-s_2y-s_1x-s_4v} \\
&\times \Bigg\{\frac{\beta^2e^{-\beta(x-v+y)}}{c}k\left(\frac{x-v}{c}\right)h\left(t-\frac{x-v}{c},n-1|u\right) \\
&+ \sum_{j=1}^{n-2}\int_0^{\frac{v-max(x-ct,0)}{c}}\frac{(cz-v)^j}{j!}\frac{\beta^{j+2}e^{-\beta(x-v+y+cz)}}{c}k\left(\frac{x-v}{c}\right) \\
&\times k^{*j}(z)h\left(t-z-\frac{x-v}{c},n-j-1|u\right)dz\Bigg\}dvdxdydt. \qquad (5.47)
\end{aligned}$$

Inversion of (5.47) with respect to r, δ, s_1, s_2 and s_4 yields (5.33), (5.34) and (5.35). □

5.4 Numerical example

In Theorem 5.3.2, the joint density of $(N_T, T, U_{T-}, |U_T|, R_{N_T-1})$ is given. As mentioned at the beginning of this chapter, this joint density can be used to obtain the marginal and joint moments of N_T, T, U_{T-}, $|U_T|$ and R_{N_T-1} by integration. As an example, the following shows how the expected time to ruin may be obtained by using the result in Theorem 5.3.2.

Consider a Sparre Andersen model where the joint pdf of the interclaim time and the claim size is given by

$$f(t,y) = (te^{-t})2e^{-2y}. \tag{5.48}$$

Note that (5.48) is a model with Erlang(2) interclaim times ($k(t) = te^{-t}$) and exponential claim sizes ($p(y) = 2e^{-2y}$). Assume the premium rate $c = 1$ such that the positive loading condition (1.2) is satisfied.

Given (5.48) and $c = 1$, it first follows from (5.33) that

$$\begin{aligned} g_1(x,y|u) &= 2e^{-2(x+y)}(x-u)e^{-(x-u)} \\ &= 2(x-u)e^{-3x-2y+u} \end{aligned} \tag{5.49}$$

for $x > u$ and $y \geq 0$. Next, (5.34) gives

$$\begin{aligned} g_2(t,x,y,v|u) &= 4e^{-2(u+t+y)}(t-x+v)e^{-(t-x+v)}(x-v)e^{-(x-v)} \\ &= 4(t-x+v)(x-v)e^{-3t-2y-2u} \end{aligned} \tag{5.50}$$

for $t, y \geq 0$; $x \in [0, u+ct]$ and $v \in [max(x-ct,0), x]$. Finally, (5.35) gives

$$\begin{aligned} g_n(t,x,y,v|u) &= 4e^{-2(x-v+y)}(x-v)e^{-(x-v)}h\left(t-(x-v), n-1|u\right) \\ &+ \sum_{j=1}^{n-2}\int_0^{v-max(x-t,0)} \frac{(z-v)^j}{j!}2^{j+2}e^{-2(x-v+y+z)}(x-v)e^{-(x-v)} \\ &\times \frac{z^{2j-1}e^{-z}}{(2j-1)!}h\left(t-z-(x-v), n-j-1|u\right)dz \\ &= 4(x-v)e^{-3x-2y+3v}\Bigg\{h\left(t-(x-v), n-1|u\right) \\ &+ \sum_{j=1}^{n-2}\frac{2^j}{j!(2j-1)!}\int_0^{v-max(x-t,0)}(z-v)^j z^{2j-1}e^{-3z} \\ &\times h\left(t-z-(x-v), n-j-1|u\right)dz\Bigg\} \end{aligned} \tag{5.51}$$

for $t, y \geq 0$; $x \in [0, u+ct]$; $v \in [max(x-ct, 0), x]$ and $n = 3, 4, \ldots$, where

$$h(t, n|u) = \begin{cases} e^{-2(u+t)} t e^{-t}, & t \geq 0, n = 1, \\ \frac{nu+t}{n(n-1)} \left\{ \frac{2^{n-1}(u+t)^{n-2} e^{-2(u+t)}}{(n-2)!} \right\} \frac{t^{2n-1} e^{-t}}{(2n-1)!}, & t \geq 0, n = 2, 3, \ldots. \end{cases}$$

Given (5.49), (5.50) and (5.51), the expected time to ruin can be calculated as

$$\begin{aligned} & E[TI(T < \infty)|U_0 = u] \\ & = \int_u^\infty \int_0^\infty \left\{ \frac{x-u}{c} \right\} g_1(x, y|u) dy dx \\ & + \sum_{n=2}^{\infty} \int_0^\infty \int_0^\infty \int_0^{u+ct} \int_{max(x-ct,0)}^{x} \{t\} g_n(t, x, y, v|u) dv dx dy dt. \end{aligned} \tag{5.52}$$

Note that the summation index n on the last line of (5.52) takes values up to ∞. For computational purpose, (5.52) can be approximated as

$$\begin{aligned} & E[TI(T < \infty)|U_0 = u] \\ & \approx \int_u^\infty \int_0^\infty \left\{ \frac{x-u}{c} \right\} g_1(x, y|u) dy dx \\ & + \sum_{n=2}^{r} \int_0^\infty \int_0^\infty \int_0^{u+ct} \int_{max(x-ct,0)}^{x} \{t\} g_n(t, x, y, v|u) dv dx dy dt \end{aligned} \tag{5.53}$$

where r is a finite integer greater than or equal to two. Furthermore, it can be shown numerically that the integral

$$\int_0^\infty \int_0^\infty \int_0^{u+ct} \int_{max(x-ct,0)}^{x} \{t\} g_n(t, x, y, v|u) dv dx dy dt$$

gets sufficiently small when $n > 55$ in this example. This is because $g_n(t, x, y, v|u)$ is a function of $h(t, n|u)$ which tends to zero when n gets large. In Table 5.1, the value of $E[TI(T < \infty)|U_0 = u]$ when $u = 0, 2, 5, 10, 20$ is approximated using (5.53) with different values of r.

For comparison to the approximate values, the exact value of $E[TI(T<\infty)|U_0=u]$ can be obtained using the results in Chapter 3 as follows. From (3.27), one has

$$E[TI(T<\infty)|U_0=u]=B_{1,0}(0,1)e^{-Ru}+B_{1,0}(1,1)ue^{-Ru}, \tag{5.54}$$

where $-R$ is the negative root of the equation (in s)

$$\left(\frac{2}{2+s}\right)\left(\frac{1}{1-s}\right)^2=1. \tag{5.55}$$

By solving (5.55), $R=1.73205$. Moreover, from (3.54) and (3.11),

$$B_{1,0}(1,1)=\frac{-C_{1,0}\left\{\frac{2}{2-R}\right\}\left\{\int_0^\infty te^{-Rt}(te^{-t})dt\right\}}{-\left\{\frac{2}{(2-R)^2}\right\}\left\{\int_0^\infty e^{-Rt}(te^{-t})dt\right\}+\left\{\frac{2}{2-R}\right\}\left\{\int_0^\infty te^{-Rt}(te^{-t})dt\right\}} \tag{5.56}$$

with

$$C_{1,0}\left(\frac{2}{2-R}\right)=1.$$

Finally, from (3.55),

$$\frac{B_{1,0}(0,1)}{(2-R)}=\frac{B_{1,0}(1,1)}{(2-R)^2}. \tag{5.57}$$

By solving (5.56) and (5.57), $B_{1,0}(0,1)=0.122009$ and $B_{1,0}(1,1)=0.032692$. Hence, (5.54) becomes

$$E[TI(T<\infty)|U_0=u]=0.122009e^{-1.73205u}+0.032692ue^{-1.73205u}. \tag{5.58}$$

The values of (5.58) when $u=0,2,5,10,20$ are given in Table 5.1, which can be compared to the approximate values of $E[TI(T<\infty)|U_0=u]$ obtained by (5.53).

u	$r = 25$	$r = 30$	$r = 55$	Exact value
0	0.1219	0.1220	0.1220	0.1220
2	0.5865×10^{-2}	0.5866×10^{-2}	0.5866×10^{-2}	0.5866×10^{-2}
5	4.9482×10^{-5}	4.9483×10^{-5}	4.9483×10^{-5}	4.9484×10^{-5}
10	1.3486×10^{-8}	1.3487×10^{-8}	1.3487×10^{-8}	1.3489×10^{-8}
20	7.0020×10^{-16}	7.0033×10^{-16}	7.0036×10^{-16}	7.0046×10^{-16}

Table 5.1: Approximate and exact values of $E[TI(T < \infty)|U_0 = u]$ by (5.53) and (5.58)

In Table 5.1, it is shown that the approximate values are close to the exact values.

Chapter 6

A generalized MAP risk model with combination of exponentials claim sizes

In Section 1.3, the MAP risk model was reviewed. The waiting times of system changes in the model are exponentially distributed. In this chapter, a generalization of the MAP risk model proposed by Cheung et al. (2011) is introduced, and the moments of the time to ruin are studied under this model.

6.1 Introduction

6.1.1 Generalized MAP risk model

In Cheung et al. (2011), the MAP risk model introduced in Section 1.3 can be generalized as follows. With a homogeneous discrete-time Markov chain (DTMC) $\boldsymbol{Z} = \{Z_i, i = 0, 1, 2, \ldots\}$ defined on finite state space $S = \{1, 2, \ldots, m\}$, the waiting time of a system change is arbitrarily distributed. Note that Z_i denotes the state of the DTMC immediately after the ith system change. The DTMC $\boldsymbol{Z}$ is assumed to have the one-period transition probability matrix $\boldsymbol{P} + \boldsymbol{Q}$. The (i, j)th entry, where $i, j = 1, 2, \ldots, m$, of

1. $\boldsymbol{P}$ is denoted as $p_{i,j}$. It represents the probability that the DTMC $\boldsymbol{Z}$ changes from state i to state j with no claim happening, and therefore $p_{i,i}$ is defined as zero;

2. $\boldsymbol{Q}$ is denoted as $q_{i,j}$. It represents the probability that the DTMC $\boldsymbol{Z}$ changes from state i to state j with a claim happening.

Let the waiting time of the first system change be V_1 and the waiting time between the $(i-1)$th and ith system changes be V_i for $i = 2, 3, \ldots$. For $V_i|Z_{i-1} = j$, where $i = 1, 2, \ldots$, let $k_j(t)$ be its probability density function, $K_j(t)$ be its cumulative distribution function, $\tilde{k}_j(s) = \int_0^\infty e^{-sx} k_j(t)dt$ be its Laplace transform and κ_j be its mean.

Moreover, let the size of the claim that occurs at the ith system change be Y_i for $i = 1, 2, \ldots$. If no claim occurs at the ith sytem change, then $Y_i = 0$. On the other hand, if the ith system change is from state j to state k with a claim occuring, then let $f_{j,k}(y)$, $F_{j,k}(y)$, $\tilde{f}_{j,k}(s) = \int_0^\infty e^{-sy} f_{j,k}(y)dy$ and $\mu_{j,k}$ be the probability density function, the cumulative distribution function, the Laplace transform and the mean of Y_i respectively.

Assume mutual independence of $Y_1, Y_2, \ldots,$ $V_1, V_2, \ldots$ when conditioned on $\{Z_i, i = 0, 1, 2, \ldots\}$. The distribution of the DTMC $\boldsymbol{Z}$ is completely specified as

$$Pr(Y_i \le y, V_i \le v, Z_i = k | Z_{i-1} = j) = K_j(v)(p_{j,k} + q_{j,k} F_{j,k}(y)), \quad y, v \ge 0.$$

Consider the insurance surplus process $\{U_t, t \ge 0\}$ defined by

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i, \tag{6.1}$$

where u is the initial surplus, c is the premium rate, Y_i is the claim size involved in the ith system change and N_t is the number of system changes up to time t. Furthermore, assume that the positive security loading condition

$$c \sum_{i=1}^{m} \pi_i \kappa_i > \sum_{i=1}^{m} \pi_i \sum_{j=1}^{m} q_{i,j} \mu_{i,j} \tag{6.2}$$

holds, where the DTMC $\boldsymbol{Z}$ is assumed to be in state i with a long-run probability of π_i. The long-run probabilities $\{\pi_i, i = 1, 2, \ldots, m\}$ satisfy the system

$$\begin{cases} \pi_j = \sum_{i=1}^{m} \pi_i (p_{ij} + q_{ij}), \quad j = 1, 2, \ldots, m \\ \sum_{i=1}^{m} \pi_i = 1. \end{cases}$$

6.1.2 Gerber-Shiu function

For the rest of this chapter, consider the generalized MAP risk model in Section 6.1.1. Define

$$T = \inf\{t \ge 0 : U_t < 0\} \tag{6.3}$$

and $T = \infty$ if U_t is non-negative for all $t \ge 0$. Cheung et al. (2011) studied the Gerber-Shiu function

$$m_{i,\delta,s}(u) = E[e^{-\delta T} e^{-s U_{T-}} w(|U_T|) I(T < \infty) | U_0 = u, Z_0 = i] \tag{6.4}$$

for $i = 1, 2, \ldots, m$ and $\delta, s \geq 0$, where $w(\cdot)$ is a function that satisfies mild integrable conditions. By assuming that the claim sizes follow combination of exponentials distribution, it was shown in the paper that (6.4) can be completely specified as a linear sum of exponential terms.

Moreover, it was mentioned in concluding remarks of Cheung et al. (2011) that the above result can be extended to more general claim size densities as follows. Assume that the claim sizes are Coxian distributed, i.e. for $j, k = 1, 2, \ldots, m$,

$$f_{j,k}(y) = \sum_{l=1}^{n_{j,k}} \sum_{\gamma=1}^{\nu_{j,k,l}} \pi_{j,k,l,\gamma} \frac{(\beta_{j,k,l})^{\gamma} y^{\gamma-1} e^{-(\beta_{j,k,l})y}}{(\gamma-1)!}, \quad y > 0, \tag{6.5}$$

where $\beta_{j,k,l} > 0$ and $\sum_{l=1}^{n_{j,k}} \sum_{\gamma=1}^{\nu_{j,k,l}} \pi_{j,k,l,\gamma} = 1$. In this case, the Gerber-Shiu function (6.4) has the form

$$m_{i,\delta,s}(u) = \sum_{x=1}^{n} a_{i,\delta,s,x} \{e^{(\rho_{\delta,x})u}\} + \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{h=1}^{\nu_{j,z,\xi}} b_{i,\delta,s,j,z,\xi,h} \left\{u^{h-1} e^{-(s+\beta_{j,z,\xi})u}\right\} \tag{6.6}$$

for $i = 1, 2, \ldots, m$ and $\delta, s \geq 0$, where

$$n = \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \nu_{j,z,\xi}.$$

In (6.6), $\rho_{\delta,x}$ and $s + \beta_{j,z,\xi}$ are assumed to be all distinct. In particular, when $s = 0$, (6.6) reduces to

$$m_{i,\delta,0}(u) = \sum_{x=1}^{n} a_{i,\delta,0,x} \{e^{(\rho_{\delta,x})u}\}, \tag{6.7}$$

or equivalently, $b_{i,\delta,0,j,z,\xi,h} = 0$ for all $i, j, z = 1, 2, \ldots, m$; $\xi = 1, 2, \ldots, n_{j,z}$; $h = 1, 2, \ldots, \nu_{j,z,\xi}$ and $\delta \geq 0$.

6.2 Moments of the time to ruin

The result introduced in last section will now be generalized to the moments of the time to ruin. First define

$$m_{i,\alpha,\delta,s}(u) = E[T^{\alpha} e^{-\delta T} e^{-sU_{T-}} w(|U_T|) I(T < \infty) | U_0 = u, Z_0 = i] \tag{6.8}$$

for $\alpha = 0, 1, 2, \ldots$; $\delta, s \geq 0$ and $i = 1, \ldots, m$, where $m_{i,0,\delta,s}(u) = m_{i,\delta,s}(u)$. The following theorem gives an explicit form of $m_{i,\alpha,\delta,s}(u)$.

Theorem 6.2.1. *Consider the generalized MAP risk model as described in Section 6.1.1 and assume that the claim sizes are distributed as* (6.5). *The generalized* α*th moment of the time to ruin* (6.8) *has the form*

$$m_{i,\alpha,\delta,s}(u) = \sum_{y=0}^{\alpha} \sum_{x=1}^{n} \Theta_{i,\alpha,\delta,s,x,y} \left\{ u^y e^{\rho_{\delta,x} u} \right\} + \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{h=1}^{\nu_{j,z,\xi}} \Lambda_{i,\alpha,\delta,s,j,z,\xi,h} \left\{ u^{h-1} e^{-(s+\beta_{j,z,\xi})u} \right\} \tag{6.9}$$

for $\alpha = 0, 1, 2, \ldots$; $\delta, s \geq 0$ *and* $i = 1, \ldots, m$. *When* $s = 0$, $\Lambda_{i,\alpha,\delta,0,j,z,\xi,h} = 0$ *for* $\alpha = 0, 1, 2, \ldots$; $\delta \geq 0$; $i, j, z = 1, \ldots, m$; $\xi = 1, \ldots, n_{j,z}$ *and* $h = 1, \ldots, \nu_{j,z,\xi}$, *and therefore* (6.9) *simplifies to*

$$m_{i,\alpha,\delta,0}(u) = \sum_{y=0}^{\alpha} \sum_{x=1}^{n} \Theta_{i,\alpha,\delta,0,x,y} \left\{ u^y e^{\rho_{\delta,x} u} \right\}. \tag{6.10}$$

Proof. First, consider the case $s > 0$. For $\alpha = 0$, it follows from (6.6) that (6.9) holds with $\Theta_{i,0,\delta,s,x,0} = a_{i,\delta,s,x}$ and $\Lambda_{i,0,\delta,s,j,z,\xi,h} = b_{i,\delta,s,j,z,\xi,h}$. Now assume (6.9) is true for a chosen α, then

$$
\begin{aligned}
&m_{i,\alpha+1,\delta,s}(u)\\
&= -\frac{\partial}{\partial\delta} m_{i,\alpha,\delta,s}(u)\\
&= -\left\{\sum_{y=0}^{\alpha}\sum_{x=1}^{n}\frac{\partial\Theta_{i,\alpha,\delta,s,x,y}}{\partial\delta}u^y e^{\rho_{\delta,x}u} + \sum_{y=0}^{\alpha}\sum_{x=1}^{n}\Theta_{i,\alpha,\delta,s,x,y}u^{y+1}e^{\rho_{\delta,x}u}\frac{\partial\rho_{\delta,x}}{\partial\delta}\right.\\
&\left. + \sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{\xi=1}^{n_{j,z}}\sum_{h=1}^{\nu_{j,z,\xi}}\frac{\partial\Lambda_{i,\alpha,\delta,s,j,z,\xi,h}}{\partial\delta}u^{h-1}e^{-(s+\beta_{j,z,\xi})u}\right\}\\
&= -\left\{\sum_{x=1}^{n}\frac{\partial\Theta_{i,\alpha,\delta,s,x,0}}{\partial\delta}e^{\rho_{\delta,x}u} + \sum_{y=1}^{\alpha}\sum_{x=1}^{n}\left(\frac{\partial\Theta_{i,\alpha,\delta,s,x,y}}{\partial\delta} + \Theta_{i,\alpha,\delta,s,x,y-1}\frac{\partial\rho_{\delta,x}}{\partial\delta}\right)u^y e^{\rho_{\delta,x}u}\right.\\
&\left. + \sum_{x=1}^{n}\Theta_{i,\alpha,\delta,s,x,\alpha}\frac{\partial\rho_{\delta,x}}{\partial\delta}u^{\alpha+1}e^{\rho_{\delta,x}u} + \sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{\xi=1}^{n_{j,z}}\sum_{h=1}^{\nu_{j,z,\xi}}\frac{\partial\Lambda_{i,\alpha,\delta,s,j,z,\xi,h}}{\partial\delta}u^{h-1}e^{-(s+\beta_{j,z,\xi})u}\right\}\\
&= \sum_{y=0}^{\alpha+1}\sum_{x=1}^{n}\Theta_{i,\alpha+1,\delta,s,x,y}\left\{u^y e^{\rho_{\delta,x}u}\right\} + \sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{\xi=1}^{n_{j,z}}\sum_{h=1}^{\nu_{j,z,\xi}}\Lambda_{i,\alpha+1,\delta,s,j,z,\xi,h}\left\{u^{h-1}e^{-(s+\beta_{j,z,\xi})u}\right\}
\end{aligned}
$$

with $\Theta_{i,\alpha+1,\delta,s,x,0} = -\frac{\partial\Theta_{i,\alpha,\delta,s,x,0}}{\partial\delta}$, $\Theta_{i,\alpha+1,\delta,s,x,y} = -\left(\frac{\partial\Theta_{i,\alpha,\delta,s,x,y}}{\partial\delta} + \Theta_{i,\alpha,\delta,s,x,y-1}\frac{\partial\rho_{\delta,x}}{\partial\delta}\right)$ for $y = 1,\ldots,\alpha$, $\Theta_{i,\alpha+1,\delta,s,x,\alpha+1} = -\Theta_{i,\alpha,\delta,s,x,\alpha}\frac{\partial\rho_{\delta,x}}{\partial\delta}$ and $\Lambda_{i,\alpha+1,\delta,s,j,z,\xi,h} = -\frac{\partial\Lambda_{i,\alpha,\delta,s,j,z,\xi,h}}{\partial\delta}$. Thus by induction in α, it can be concluded that (6.9) holds when $s > 0$.

Next, consider the case $s = 0$. Since (6.7) is true, one can use induction in α as shown above to show that (6.10) holds. However, (6.10) can also be expressed in the form of (6.9) with $\Lambda_{i,\alpha,\delta,0,j,z,\xi,h} = 0$ for $\alpha = 0,1,2,\ldots$; $\delta \geq 0$; $i,j,z = 1,\ldots,m$; $\xi = 1,\ldots,n_{j,z}$ and $h = 1,\ldots,\nu_{j,z,\xi}$.

Finally, from the two cases above, one can conclude that (6.9) is true for all $s \geq 0$. □

In order to completely specify the generalized moments of the time to ruin, the unknown constants in (6.9) need to be determined. The following theorem gives the sets of equations satisfied by these constants.

Theorem 6.2.2. *Suppose the conditions of Theorem 6.2.1 hold. The constants* $\rho_{\delta,1}, \rho_{\delta,2}, \ldots, \rho_{\delta,n}$ *in* (6.9) *and* (6.10) *are the* n *roots with negative real parts of the equation (in* z*)*

$$\det(\boldsymbol{I} - \boldsymbol{\Upsilon}(z)) = 0, \tag{6.11}$$

where $\det(\boldsymbol{I} - \boldsymbol{\Upsilon}(z))$ *is the determinant of* $\boldsymbol{I} - \boldsymbol{\Upsilon}(z)$, $\boldsymbol{I}$ *is a* $m \times m$ *identity matrix and* $\boldsymbol{\Upsilon}(z)$ *is a* $m \times m$ *matrix. The* (i, σ)*th element of* $\boldsymbol{\Upsilon}(z)$ *in* (6.11) *is given by*

$$P_{i,\sigma,0,0,0,0,z} + Q_{i,\sigma,0,0,0,0,z},$$

where

$$P_{i,\sigma,\alpha,a,g,y,z} = p_{i,\sigma} k_{i,\alpha,a,g,y,\delta-cz} \tag{6.12}$$

and

$$Q_{i,\sigma,\alpha,a,g,y,z} = q_{i,\sigma} \sum_{h=y}^{g} \frac{g!(-1)^{g-h}}{h!(g-h)!} k_{i,\alpha,a,h,y,\delta-cz} \sum_{l=1}^{n_{i,\sigma}} \sum_{\gamma=1}^{\nu_{i,\sigma,l}} \frac{(g-h+\gamma-1)!\pi_{i,\sigma,l,\gamma}(\beta_{i,\sigma,l})^{\gamma}}{(\gamma-1)!(z+\beta_{i,\sigma,l})^{g-h+\gamma}} \tag{6.13}$$

with

$$k_{i,r_1,r_2,t_1,t_2,s} = \binom{r_1}{r_2}\binom{t_1}{t_2} c^{t_1-t_2} \int_0^{\infty} t^{r_1-r_2+t_1-t_2} e^{-st} k_i(t) dt, \tag{6.14}$$

and c *is the premium rate of the insurance process* (6.1)*. For notational convenience,* (6.12)*,* (6.13) *and* (6.14) *are used throughout this theorem. In the following, the results are given in two cases, which are* $s > 0$ *and* $s = 0$*.*

<u>*Case 1: Given* $s > 0, \delta \geq 0$ *and* $\alpha = 0, 1, 2, \ldots$ *in* (6.8)</u>

The coefficients $\Theta_{i,\alpha,\delta,s,x,y}$ *satisfy*

$$\Theta_{i,\alpha,\delta,s,x,y} = \sum_{\sigma=1}^{m} \sum_{a=y}^{\alpha} \sum_{g=y}^{a} \left\{ P_{i,\sigma,\alpha,a,g,y,\rho_{\delta,x}} + Q_{i,\sigma,\alpha,a,g,y,\rho_{\delta,x}} \right\} \Theta_{\sigma,a,\delta,s,x,g} \tag{6.15}$$

for $i = 1, 2, \ldots, m$*;* $x = 1, 2, \ldots, n$ *and* $y = 0, 1, 2, \ldots, \alpha$*. Also, the coefficients* $\Lambda_{i,\alpha,\delta,s,j,z,\xi,h}$ *satisfy*

$$\Lambda_{i,\alpha,\delta,s,j,z,\xi,h} = \sum_{\sigma=1}^{m} \sum_{a=0}^{\alpha} \sum_{r=h}^{\nu_{j,z,\xi}} \left\{ \frac{h}{r} \left(P_{i,\sigma,\alpha,a,r,h,-(s+\beta_{j,z,\xi})} + Q_{i,\sigma,\alpha,a,r,h,-(s+\beta_{j,z,\xi})} \right) \right\} \Lambda_{\sigma,a,\delta,s,j,z,\xi,r}$$
$$+ I(i = j) q_{i,z} \sum_{\gamma=h}^{\nu_{i,z,\xi}} \sum_{g=h}^{\gamma} \frac{h\pi_{i,z,\xi,\gamma}(\beta_{i,z,\xi})^{\gamma}}{g!(\gamma - g)!} w_{\gamma-g,(\beta_{i,z,\xi})} k_{i,\alpha,0,g,h,\delta+c(s+\beta_{i,z,\xi})} \quad (6.16)$$

for $i, j, z = 1, 2, \ldots, m$*;* $\xi = 1, 2, \ldots, n_{j,z}$ *and* $h = 1, 2, \ldots, \nu_{j,z,\xi}$*, where*

$$w_{r,\beta} = \int_0^{\infty} y^r e^{-\beta y} w(y) dy. \quad (6.17)$$

Note that for $y = \alpha$*, (6.15) gives*

$$\Theta_{i,\alpha,\delta,s,x,\alpha} = \sum_{\sigma=1}^{m} \left\{ P_{i,\sigma,\alpha,\alpha,\alpha,\alpha,\rho_{\delta,x}} + Q_{i,\sigma,\alpha,\alpha,\alpha,\alpha,\rho_{\delta,x}} \right\} \Theta_{\sigma,\alpha,\delta,s,x,\alpha} \quad (6.18)$$

for $i = 1, 2, \ldots, m$ *and* $x = 1, 2, \ldots, n$*. Given* α*,* δ *and* s*, (6.18) gives a system of* m *linearly dependent equations in* $\{\Theta_{i,\alpha,\delta,s,x,\alpha}\}_{i=1}^{m}$ *for each* $x = 1, 2, \ldots, n$*. Hence for each fixed* $x = 1, 2, \ldots, n$*, one of the* m *equations in* $\{\Theta_{i,\alpha,\delta,s,x,\alpha}\}_{i=1}^{m}$ *should be removed from (6.18). As a result,* n *more equations needed to solve for the coefficients* $\Theta_{i,\alpha,\delta,s,x,y}$ *are given by*

$$\sum_{a=0}^{\alpha} \sum_{\gamma=h}^{\nu_{i,\sigma,l}} \sum_{g=h}^{\gamma} \left\{ \sum_{b=0}^{a} \sum_{x=1}^{n} \Theta_{\sigma,a,\delta,s,x,b} \frac{(-1)^b (b + \gamma - g)!}{(\rho_{\delta,x} + \beta_{i,\sigma,l})^{b+\gamma-g+1}} \right.$$
$$\left. - \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{r=1}^{\nu_{j,z,\xi}} \Lambda_{\sigma,a,\delta,s,j,z,\xi,r} \frac{(-1)^{\gamma-g} (r - 1 + \gamma - g)!}{(s + \beta_{j,z,\xi} - \beta_{i,\sigma,l})^{r+\gamma-g}} \right\}$$
$$\times \frac{h\pi_{i,\sigma,l,\gamma}(\beta_{i,\sigma,l})^{\gamma}}{g!(\gamma - g)!} k_{i,\alpha,a,g,h,\delta+c\beta_{i,\sigma,l}} = 0 \quad (6.19)$$

for $i, \sigma = 1, 2, \ldots, m$*;* $l = 1, 2, \ldots, n_{i,\sigma}$ *and* $h = 1, 2, \ldots, \nu_{i,\sigma,l}$*.*

Case 2: Given $s = 0, \delta \geq 0$ and $\alpha = 0, 1, 2, \ldots$ in (6.8)

The coefficients $\Theta_{i,\alpha,\delta,0,x,y}$ satisfy

$$\Theta_{i,\alpha,\delta,0,x,y} = \sum_{\sigma=1}^{m} \sum_{a=y}^{\alpha} \sum_{g=y}^{a} \left\{ P_{i,\sigma,\alpha,a,g,y,\rho_{\delta,x}} + Q_{i,\sigma,\alpha,a,g,y,\rho_{\delta,x}} \right\} \Theta_{\sigma,a,\delta,0,x,g} \tag{6.20}$$

for $i = 1, 2, \ldots, m$; $x = 1, 2, \ldots, n$ and $y = 0, 1, 2, \ldots, \alpha$. For $y = \alpha$, (6.20) gives

$$\Theta_{i,\alpha,\delta,0,x,\alpha} = \sum_{\sigma=1}^{m} \left\{ P_{i,\sigma,\alpha,\alpha,\alpha,\alpha,\rho_{\delta,x}} + Q_{i,\sigma,\alpha,\alpha,\alpha,\alpha,\rho_{\delta,x}} \right\} \Theta_{\sigma,\alpha,\delta,0,x,\alpha} \tag{6.21}$$

for $i = 1, 2, \ldots, m$ and $x = 1, 2, \ldots, n$. Given α and δ, (6.21) gives m equations in $\{\Theta_{i,\alpha,\delta,0,x,\alpha}\}_{i=1}^{m}$ for each $x = 1, 2, \ldots, n$. Again, one of the m equations in $\{\Theta_{i,\alpha,\delta,0,x,\alpha}\}_{i=1}^{m}$ should be removed from (6.21) for each fixed $x = 1, 2, \ldots, n$. Therefore, n more equations to solve for $\Theta_{i,\alpha,\delta,0,x,y}$ are

$$\begin{aligned} &\sum_{\gamma=h}^{\nu_{i,\sigma,l}} \sum_{g=h}^{\gamma} \left\{ \sum_{a=0}^{\alpha} \sum_{b=0}^{a} \sum_{x=1}^{n} \Theta_{\sigma,a,\delta,0,x,b} \frac{(-1)^b (b+\gamma-g)!}{(\rho_{\delta,x} + \beta_{i,\sigma,l})^{b+\gamma-g+1}} k_{i,\alpha,a,g,h,\delta+c\beta_{i,\sigma,l}} \right. \\ &\left. - w_{\gamma-g,(\beta_{i,\sigma,l})} k_{i,\alpha,0,g,h,\delta+c\beta_{i,\sigma,l}} \right\} \frac{\pi_{i,\sigma,l,\gamma} (\beta_{i,\sigma,l})^{\gamma}}{g!(\gamma-g)!} = 0 \end{aligned} \tag{6.22}$$

for $i, \sigma = 1, 2, \ldots, m$; $l = 1, 2, \ldots, n_{i,\sigma}$ and $h = 1, 2, \ldots, \nu_{i,\sigma,l}$.

Proof. In this proof, the analytic approach used in Cheung et al. (2011) is applied.

By conditioning on the time and amount of the first system change and the state of the insurance surplus process after the system change, Equation (26) of Cheung et al. (2011)

showed that

$$m_{i,0,\delta,s}(u) = \int_0^\infty e^{-\delta t} k_i(t) \left\{ \sum_{\sigma=1}^m p_{i,\sigma} m_{\sigma,0,\delta,s}(u+ct) \right.$$
$$\left. + \sum_{\sigma=1}^m q_{i,\sigma} \int_0^{u+ct} m_{\sigma,0,\delta,s}(u+ct-\varphi) f_{i,\sigma}(\varphi) d\varphi \right\} dt$$
$$+ e^{-su} \int_0^\infty e^{-(\delta+cs)t} k_i(t) \left\{ \sum_{\sigma=1}^m q_{i,\sigma} \int_0^\infty w(\varphi) f_{i,\sigma}(u+ct+\varphi) d\varphi \right\} dt \qquad (6.23)$$

for $i = 1, 2, \ldots, m$ and $\delta, s \geq 0$. To study the moment of the time to ruin, differentiate (6.23) α times with respect to δ, which gives

$$m_{i,\alpha,\delta,s}(u) = \sum_{a=0}^{\alpha} \binom{\alpha}{a} \int_0^\infty t^{\alpha-a} e^{-\delta t} k_i(t) \left\{ \sum_{\sigma=1}^m p_{i,\sigma} m_{\sigma,a,\delta,s}(u+ct) \right.$$
$$\left. + \sum_{\sigma=1}^m q_{i,\sigma} \int_0^{u+ct} m_{\sigma,a,\delta,s}(u+ct-\varphi) f_{i,\sigma}(\varphi) d\varphi \right\} dt$$
$$+ \sum_{\sigma=1}^m q_{i,\sigma} e^{-su} \int_0^\infty t^\alpha e^{-(\delta+cs)t} k_i(t) \int_0^\infty w(\varphi) f_{i,\sigma}(u+ct+\varphi) d\varphi dt \qquad (6.24)$$

for $\alpha = 0, 1, 2, \ldots$. Next, substitute the claim size distribution (6.5) and the general form of the moment (6.9) into (6.24). Thus, (6.24) becomes

$$
\begin{aligned}
&\sum_{y=0}^{\alpha}\sum_{x=1}^{n}\Theta_{i,\alpha,\delta,s,x,y}\{u^{y}e^{\rho_{\delta,x}u}\}+\sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{\xi=1}^{n_{j,z}}\sum_{h=1}^{\nu_{j,z,\xi}}\Lambda_{i,\alpha,\delta,s,j,z,\xi,h}\{u^{h-1}e^{-(s+\beta_{j,z,\xi})u}\}\\
&=\sum_{\sigma=1}^{m}p_{i,\sigma}\sum_{a=0}^{\alpha}\binom{\alpha}{a}\int_{0}^{\infty}t^{\alpha-a}e^{-\delta t}k_i(t)\left\{\sum_{g=0}^{a}\sum_{x=1}^{n}\Theta_{\sigma,a,\delta,s,x,g}(u+ct)^{g}e^{\rho_{\delta,x}(u+ct)}\right.\\
&\left.+\sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{\xi=1}^{n_{j,z}}\sum_{r=1}^{\nu_{j,z,\xi}}\Lambda_{\sigma,a,\delta,s,j,z,\xi,r}(u+ct)^{r-1}e^{-(s+\beta_{j,z,\xi})(u+ct)}\right\}dt\\
&+\sum_{\sigma=1}^{m}q_{i,\sigma}\sum_{a=0}^{\alpha}\binom{\alpha}{a}\int_{0}^{\infty}t^{\alpha-a}e^{-\delta t}k_i(t)\\
&\times\int_{0}^{u+ct}\left\{\sum_{g=0}^{a}\sum_{x=1}^{n}\Theta_{\sigma,a,\delta,s,x,g}(u+ct-\varphi)^{g}e^{\rho_{\delta,x}(u+ct-\varphi)}\right.\\
&\left.+\sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{\xi=1}^{n_{j,z}}\sum_{r=1}^{\nu_{j,z,\xi}}\Lambda_{\sigma,a,\delta,s,j,z,\xi,r}(u+ct-\varphi)^{r-1}e^{-(s+\beta_{j,z,\xi})(u+ct-\varphi)}\right\}\\
&\times\left\{\sum_{l=1}^{n_{i,\sigma}}\sum_{\gamma=1}^{\nu_{i,\sigma,l}}\pi_{i,\sigma,l,\gamma}\frac{(\beta_{i,\sigma,l})^{\gamma}\varphi^{\gamma-1}e^{-\beta_{i,\sigma,l}\varphi}}{(\gamma-1)!}\right\}d\varphi dt\\
&+\sum_{\sigma=1}^{m}q_{i,\sigma}e^{-su}\int_{0}^{\infty}t^{\alpha}e^{-(\delta+cs)t}k_i(t)\int_{0}^{\infty}w(\varphi)\\
&\times\left\{\sum_{l=1}^{n_{i,\sigma}}\sum_{\gamma=1}^{\nu_{i,\sigma,l}}\pi_{i,\sigma,l,\gamma}\frac{(\beta_{i,\sigma,l})^{\gamma}(u+ct+\varphi)^{\gamma-1}e^{-\beta_{i,\sigma,l}(u+ct+\varphi)}}{(\gamma-1)!}\right\}d\varphi dt. \qquad (6.25)
\end{aligned}
$$

Let us simplify the right hand side of (6.25) term by term. The first term is

$$\sum_{\sigma=1}^{m} p_{i,\sigma} \sum_{a=0}^{\alpha} \binom{\alpha}{a} \int_0^\infty t^{\alpha-a} e^{-\delta t} k_i(t) \left\{ \sum_{g=0}^{a} \sum_{x=1}^{n} \Theta_{\sigma,a,\delta,s,x,g} (u+ct)^g e^{\rho_{\delta,x}(u+ct)} \right.$$

$$\left. + \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{r=1}^{\nu_{j,z,\xi}} \Lambda_{\sigma,a,\delta,s,j,z,\xi,r} (u+ct)^{r-1} e^{-(s+\beta_{j,z,\xi})(u+ct)} \right\} dt$$

$$= \sum_{\sigma=1}^{m} p_{i,\sigma} \sum_{a=0}^{\alpha} \binom{\alpha}{a} \sum_{g=0}^{a} \sum_{x=1}^{n} \Theta_{\sigma,a,\delta,s,x,g} \int_0^\infty t^{\alpha-a} e^{-\delta t} k_i(t) \left\{ \sum_{y=0}^{g} \binom{g}{y} u^y (ct)^{g-y} \right\} e^{\rho_{\delta,x}(u+ct)} dt$$

$$+ \sum_{\sigma=1}^{m} p_{i,\sigma} \sum_{a=0}^{\alpha} \binom{\alpha}{a} \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{r=1}^{\nu_{j,z,\xi}} \Lambda_{\sigma,a,\delta,s,j,z,\xi,r} \int_0^\infty t^{\alpha-a} e^{-\delta t} k_i(t)$$

$$\times \left\{ \sum_{h=1}^{r} \binom{r-1}{h-1} u^{h-1} (ct)^{r-h} \right\} e^{-(s+\beta_{j,z,\xi})(u+ct)} dt$$

$$= \sum_{y=0}^{\alpha} \sum_{x=1}^{n} \left\{ \sum_{\sigma=1}^{m} \sum_{a=y}^{\alpha} \sum_{g=y}^{a} P_{i,\sigma,\alpha,a,g,y,\rho_{\delta,x}} \Theta_{\sigma,a,\delta,s,x,g} \right\} u^y e^{\rho_{\delta,x} u}$$

$$+ \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{h=1}^{\nu_{j,z,\xi}} \left\{ \sum_{\sigma=1}^{m} \sum_{a=0}^{\alpha} \sum_{r=h}^{\nu_{j,z,\xi}} \left(\frac{h}{r} P_{i,\sigma,\alpha,a,r,h,-(s+\beta_{j,z,\xi})} \right) \Lambda_{\sigma,a,\delta,s,j,z,\xi,r} \right\} u^{h-1} e^{-(s+\beta_{j,z,\xi})u}. \tag{6.26}$$

Next, the second term on the right hand side of (6.25) is

$$\sum_{\sigma=1}^{m} q_{i,\sigma} \sum_{a=0}^{\alpha} \binom{\alpha}{a} \int_0^\infty t^{\alpha-a} e^{-\delta t} k_i(t)$$
$$\times \int_0^{u+ct} \left\{ \sum_{g=0}^{a} \sum_{x=1}^{n} \Theta_{\sigma,a,\delta,s,x,g} (u+ct-\varphi)^g e^{\rho_{\delta,x}(u+ct-\varphi)} \right.$$
$$\left. + \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{r=1}^{\nu_{j,z,\xi}} \Lambda_{\sigma,a,\delta,s,j,z,\xi,r} (u+ct-\varphi)^{r-1} e^{-(s+\beta_{j,z,\xi})(u+ct-\varphi)} \right\}$$
$$\times \left\{ \sum_{l=1}^{n_{i,\sigma}} \sum_{\gamma=1}^{\nu_{i,\sigma,l}} \pi_{i,\sigma,l,\gamma} \frac{(\beta_{i,\sigma,l})^\gamma \varphi^{\gamma-1} e^{-\beta_{i,\sigma,l}\varphi}}{(\gamma-1)!} \right\} d\varphi dt$$
$$= \sum_{\sigma=1}^{m} q_{i,\sigma} \sum_{a=0}^{\alpha} \binom{\alpha}{a} \int_0^\infty t^{\alpha-a} e^{-\delta t} k_i(t)$$
$$\times \int_0^{\infty} \left\{ \sum_{g=0}^{a} \sum_{x=1}^{n} \Theta_{\sigma,a,\delta,s,x,g} (u+ct-\varphi)^g e^{\rho_{\delta,x}(u+ct-\varphi)} \right.$$
$$\left. + \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{r=1}^{\nu_{j,z,\xi}} \Lambda_{\sigma,a,\delta,s,j,z,\xi,r} (u+ct-\varphi)^{r-1} e^{-(s+\beta_{j,z,\xi})(u+ct-\varphi)} \right\}$$
$$\times \left\{ \sum_{l=1}^{n_{i,\sigma}} \sum_{\gamma=1}^{\nu_{i,\sigma,l}} \pi_{i,\sigma,l,\gamma} \frac{(\beta_{i,\sigma,l})^\gamma \varphi^{\gamma-1} e^{-\beta_{i,\sigma,l}\varphi}}{(\gamma-1)!} \right\} d\varphi dt$$
$$- \sum_{\sigma=1}^{m} q_{i,\sigma} \sum_{a=0}^{\alpha} \binom{\alpha}{a} \int_0^\infty t^{\alpha-a} e^{-\delta t} k_i(t)$$
$$\times \int_0^{\infty} \left\{ \sum_{b=0}^{a} \sum_{x=1}^{n} \Theta_{\sigma,a,\delta,s,x,b} (-\varphi)^b e^{-\rho_{\delta,x}\varphi} \right.$$
$$\left. + \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{r=1}^{\nu_{j,z,\xi}} \Lambda_{\sigma,a,\delta,s,j,z,\xi,r} (-\varphi)^{r-1} e^{(s+\beta_{j,z,\xi})\varphi} \right\}$$
$$\times \left\{ \sum_{l=1}^{n_{i,\sigma}} \sum_{\gamma=1}^{\nu_{i,\sigma,l}} \pi_{i,\sigma,l,\gamma} \frac{(\beta_{i,\sigma,l})^\gamma (u+ct+\varphi)^{\gamma-1} e^{-\beta_{i,\sigma,l}(u+ct+\varphi)}}{(\gamma-1)!} \right\} d\varphi dt. \quad (6.27)$$

Using binomial expansion and

$$\int_0^\infty (-\varphi)^n e^{-s\varphi} d\varphi = \frac{d^n}{ds^n}\int_0^\infty e^{-s\varphi} d\varphi$$
$$= \frac{-n!}{(-s)^{n+1}}, \quad n = 0, 1, 2, \ldots,$$

(6.27) becomes

$$\begin{aligned}
&\sum_{\sigma=1}^{m} q_{i,\sigma} \sum_{a=0}^{\alpha} \binom{\alpha}{a} \int_0^\infty t^{\alpha-a} e^{-\delta t} k_i(t) \\
&\times \int_0^{u+ct} \left\{ \sum_{g=0}^{a} \sum_{x=1}^{n} \Theta_{\sigma,a,\delta,s,x,g} (u+ct-\varphi)^g e^{\rho_{\delta,x}(u+ct-\varphi)} \right. \\
&\left. + \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{r=1}^{\nu_{j,z,\xi}} \Lambda_{\sigma,a,\delta,s,j,z,\xi,r} (u+ct-\varphi)^{r-1} e^{-(s+\beta_{j,z,\xi})(u+ct-\varphi)} \right\} \\
&\times \left\{ \sum_{l=1}^{n_{i,\sigma}} \sum_{\gamma=1}^{\nu_{i,\sigma,l}} \pi_{i,\sigma,l,\gamma} \frac{(\beta_{i,\sigma,l})^\gamma \varphi^{\gamma-1} e^{-\beta_{i,\sigma,l}\varphi}}{(\gamma-1)!} \right\} d\varphi dt \\
&= \sum_{y=0}^{\alpha} \sum_{x=1}^{n} \left\{ \sum_{\sigma=1}^{m} \sum_{a=y}^{\alpha} \sum_{g=y}^{a} Q_{i,\sigma,\alpha,a,g,y,\rho_{\delta,x}} \Theta_{\sigma,a,\delta,s,x,g} \right\} u^y e^{\rho_{\delta,x} u} \\
&+ \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{h=1}^{\nu_{j,z,\xi}} \left\{ \sum_{\sigma=1}^{m} \sum_{a=0}^{\alpha} \sum_{r=h}^{\nu_{j,z,\xi}} \left(\frac{h}{r} Q_{i,\sigma,\alpha,a,r,h,-(s+\beta_{j,z,\xi})} \right) \Lambda_{\sigma,a,\delta,s,j,z,\xi,r} \right\} u^{h-1} e^{-(s+\beta_{j,z,\xi})u} \\
&- \sum_{\sigma=1}^{m} \sum_{l=1}^{n_{i,\sigma}} \sum_{h=1}^{\nu_{i,\sigma,l}} \left\{ q_{i,\sigma} \sum_{a=0}^{\alpha} \sum_{b=0}^{a} \sum_{x=1}^{n} \Theta_{\sigma,a,\delta,s,x,b} \frac{(-1)^b (b+\gamma-g)!}{(\rho_{\delta,x}+\beta_{i,\sigma,l})^{b+\gamma-g+1}} \right. \\
&\left. \times \sum_{\gamma=h}^{\nu_{i,\sigma,l}} \sum_{g=h}^{\gamma} \frac{h\pi_{i,\sigma,l,\gamma}(\beta_{i,\sigma,l})^\gamma}{g!(\gamma-g)!} k_{i,\alpha,a,g,h,\delta+c\beta_{i,\sigma,l}} \right\} u^{h-1} e^{-\beta_{i,\sigma,l} u} \\
&+ \sum_{\sigma=1}^{m} \sum_{l=1}^{n_{i,\sigma}} \sum_{h=1}^{\nu_{i,\sigma,l}} \left\{ q_{i,\sigma} \sum_{a=0}^{\alpha} \sum_{j=1}^{m} \sum_{z=1}^{m} \sum_{\xi=1}^{n_{j,z}} \sum_{r=1}^{\nu_{j,z,\xi}} \Lambda_{\sigma,a,\delta,s,j,z,\xi,r} \frac{(-1)^{\gamma-g}(r-1+\gamma-g)!}{(s+\beta_{j,z,\xi}-\beta_{i,\sigma,l})^{r+\gamma-g}} \right. \\
&\left. \times \sum_{\gamma=h}^{\nu_{i,\sigma,l}} \sum_{g=h}^{\gamma} \frac{h\pi_{i,\sigma,l,\gamma}(\beta_{i,\sigma,l})^\gamma}{g!(\gamma-g)!} k_{i,\alpha,a,g,h,\delta+c\beta_{i,\sigma,l}} \right\} u^{h-1} e^{-\beta_{i,\sigma,l} u}. \qquad (6.28)
\end{aligned}$$

Finally, simplify the last term on the right hand side of (6.25), which is

$$
\begin{aligned}
&\sum_{\sigma=1}^{m} q_{i,\sigma} e^{-su} \int_0^\infty t^\alpha e^{-(\delta+cs)t} k_i(t) \int_0^\infty w(\varphi) \\
&\times \left\{ \sum_{l=1}^{n_{i,\sigma}} \sum_{\gamma=1}^{\nu_{i,\sigma,l}} \pi_{i,\sigma,l,\gamma} \frac{(\beta_{i,\sigma,l})^\gamma (u+ct+\varphi)^{\gamma-1} e^{-\beta_{i,\sigma,l}(u+ct+\varphi)}}{(\gamma-1)!} \right\} d\varphi dt \\
&= \sum_{\sigma=1}^{m} q_{i,\sigma} e^{-su} \int_0^\infty t^\alpha e^{-(\delta+cs)t} k_i(t) \int_0^\infty w(\varphi) \\
&\times \left\{ \sum_{l=1}^{n_{i,\sigma}} \sum_{\gamma=1}^{\nu_{i,\sigma,l}} \frac{\pi_{i,\sigma,l,\gamma} (\beta_{i,\sigma,l})^\gamma}{(\gamma-1)!} \left\{ \sum_{h=1}^{\gamma} \sum_{g=h}^{\gamma} \frac{(\gamma-1)!}{(h-1)!(g-h)!(\gamma-g)!} u^{h-1} (ct)^{g-h} (\varphi)^{\gamma-g} \right\} \right. \\
&\left. \times\, e^{-\beta_{i,\sigma,l}(u+ct+\varphi)} \right\} d\varphi dt \\
&= \sum_{\sigma=1}^{m} \sum_{l=1}^{n_{i,\sigma}} \sum_{h=1}^{\nu_{i,\sigma,l}} \left\{ q_{i,\sigma} \sum_{\gamma=h}^{\nu_{i,\sigma,l}} \sum_{g=h}^{\gamma} \frac{h \pi_{i,\sigma,l,\gamma} (\beta_{i,\sigma,l})^\gamma}{g!(\gamma-g)!} \right. \\
&\left. \times\, w_{\gamma-g,(\beta_{i,\sigma,l})} k_{i,\alpha,0,g,h,\delta+c(s+\beta_{i,\sigma,l})} \right\} u^{h-1} e^{-(s+\beta_{i,\sigma,l})u}. \qquad (6.29)
\end{aligned}
$$

Substitute (6.26), (6.28) and (6.29) into (6.25), which yields

$$\sum_{y=0}^{\alpha}\sum_{x=1}^{n}\Theta_{i,\alpha,\delta,s,x,y}\{u^{y}e^{\rho_{\delta,x}u}\}+\sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{\xi=1}^{n_{j,z}}\sum_{h=1}^{\nu_{j,z,\xi}}\Lambda_{i,\alpha,\delta,s,j,z,\xi,h}\{u^{h-1}e^{-(s+\beta_{j,z,\xi})u}\}$$
$$=\sum_{y=0}^{\alpha}\sum_{x=1}^{n}\left\{\sum_{\sigma=1}^{m}\sum_{a=y}^{\alpha}\sum_{g=y}^{a}P_{i,\sigma,\alpha,a,g,y,\rho_{\delta,x}}\Theta_{\sigma,a,\delta,s,x,g}\right\}u^{y}e^{\rho_{\delta,x}u}$$
$$+\sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{\xi=1}^{n_{j,z}}\sum_{h=1}^{\nu_{j,z,\xi}}\left\{\sum_{\sigma=1}^{m}\sum_{a=0}^{\alpha}\sum_{r=h}^{\nu_{j,z,\xi}}\left(\frac{h}{r}P_{i,\sigma,\alpha,a,r,h,-(s+\beta_{j,z,\xi})}\right)\Lambda_{\sigma,a,\delta,s,j,z,\xi,r}\right\}u^{h-1}e^{-(s+\beta_{j,z,\xi})u}$$
$$+\sum_{y=0}^{\alpha}\sum_{x=1}^{n}\left\{\sum_{\sigma=1}^{m}\sum_{a=y}^{\alpha}\sum_{g=y}^{a}Q_{i,\sigma,\alpha,a,g,y,\rho_{\delta,x}}\Theta_{\sigma,a,\delta,s,x,g}\right\}u^{y}e^{\rho_{\delta,x}u}$$
$$+\sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{\xi=1}^{n_{j,z}}\sum_{h=1}^{\nu_{j,z,\xi}}\left\{\sum_{\sigma=1}^{m}\sum_{a=0}^{\alpha}\sum_{r=h}^{\nu_{j,z,\xi}}\left(\frac{h}{r}Q_{i,\sigma,\alpha,a,r,h,-(s+\beta_{j,z,\xi})}\right)\Lambda_{\sigma,a,\delta,s,j,z,\xi,r}\right\}u^{h-1}e^{-(s+\beta_{j,z,\xi})u}$$
$$-\sum_{\sigma=1}^{m}\sum_{l=1}^{n_{i,\sigma}}\sum_{h=1}^{\nu_{i,\sigma,l}}\left\{q_{i,\sigma}\sum_{a=0}^{\alpha}\sum_{b=0}^{a}\sum_{x=1}^{n}\Theta_{\sigma,a,\delta,s,x,b}\frac{(-1)^{b}(b+\gamma-g)!}{(\rho_{\delta,x}+\beta_{i,\sigma,l})^{b+\gamma-g+1}}\right.$$
$$\left.\times\sum_{\gamma=h}^{\nu_{i,\sigma,l}}\sum_{g=h}^{\gamma}\frac{h\pi_{i,\sigma,l,\gamma}(\beta_{i,\sigma,l})^{\gamma}}{g!(\gamma-g)!}k_{i,\alpha,a,g,h,\delta+c\beta_{i,\sigma,l}}\right\}u^{h-1}e^{-\beta_{i,\sigma,l}u}$$
$$+\sum_{\sigma=1}^{m}\sum_{l=1}^{n_{i,\sigma}}\sum_{h=1}^{\nu_{i,\sigma,l}}\left\{q_{i,\sigma}\sum_{a=0}^{\alpha}\sum_{j=1}^{m}\sum_{z=1}^{m}\sum_{\xi=1}^{n_{j,z}}\sum_{r=1}^{\nu_{j,z,\xi}}\Lambda_{\sigma,a,\delta,s,j,z,\xi,r}\frac{(-1)^{\gamma-g}(r-1+\gamma-g)!}{(s+\beta_{j,z,\xi}-\beta_{i,\sigma,l})^{r+\gamma-g}}\right.$$
$$\left.\times\sum_{\gamma=h}^{\nu_{i,\sigma,l}}\sum_{g=h}^{\gamma}\frac{h\pi_{i,\sigma,l,\gamma}(\beta_{i,\sigma,l})^{\gamma}}{g!(\gamma-g)!}k_{i,\alpha,a,g,h,\delta+c\beta_{i,\sigma,l}}\right\}u^{h-1}e^{-\beta_{i,\sigma,l}u}$$
$$+\sum_{\sigma=1}^{m}\sum_{l=1}^{n_{i,\sigma}}\sum_{h=1}^{\nu_{i,\sigma,l}}\left\{q_{i,\sigma}\sum_{\gamma=h}^{\nu_{i,\sigma,l}}\sum_{g=h}^{\gamma}\frac{h\pi_{i,\sigma,l,\gamma}(\beta_{i,\sigma,l})^{\gamma}}{g!(\gamma-g)!}\right.$$
$$\left.\times w_{\gamma-g,(\beta_{i,\sigma,l})}k_{i,\alpha,0,g,h,\delta+c(s+\beta_{i,\sigma,l})}\right\}u^{h-1}e^{-(s+\beta_{i,\sigma,l})u}\qquad(6.30)$$

for $i=1,2,\ldots,m$; $\alpha=0,1,2,\ldots$ and $\delta,s\geq 0$. Let us study (6.30) in two cases.

Case 1: Given $s > 0, \delta \geq 0$ and $\alpha = 0, 1, 2, \ldots$

Since (6.30) is true for all $u \geq 0$, the coefficients of $u^y e^{\rho_{\delta,x} u}$ should be equal on both sides, which yields (6.15).

In particular, for $y = \alpha$, (6.15) gives (6.18). Given α, δ and s, (6.18) gives a system of m linear equations in $\{\Theta_{i,\alpha,\delta,s,x,\alpha}\}_{i=1}^m$ for each $x = 1, 2, \ldots, n$. For each $x = 1, 2, \ldots, n$, assume that a non-trivial solution exist for $\{\Theta_{i,\alpha,\delta,s,x,\alpha}\}_{i=1}^m$. Then it can be concluded from (6.18) that

$$\det(\boldsymbol{I} - \boldsymbol{\Upsilon}(\rho_{\delta,x})) = 0 \tag{6.31}$$

for $x = 1, 2, \ldots, n$, where $\boldsymbol{\Upsilon}(z)$ is a $m \times m$ matrix with its $(i, \sigma)th$ element equal to $P_{i,\sigma,\alpha,\alpha,\alpha,\alpha,z} + Q_{i,\sigma,\alpha,\alpha,\alpha,\alpha,z}$ or $P_{i,\sigma,0,0,0,0,z} + Q_{i,\sigma,0,0,0,0,z}$ equivalently. There are two points to note from (6.31). First, since (6.31) holds, (6.18) in $\{\Theta_{i,\alpha,\delta,s,x,\alpha}\}_{i=1}^m$ for each fixed $x = 1, 2, \ldots, n$ are m linearly dependent equations. Second, using the argument in Theorem 1 of Cheung et al. (2011), it can be proved that there are n roots with negative real parts to the equation $\det(\boldsymbol{I} - \boldsymbol{\Upsilon}(z)) = 0$. This fact together with (6.31) lead to (6.11).

Furthermore, one can equate the coefficients of $u^{h-1} e^{-(s+\beta_{j,z,\xi})u}$ on both sides of (6.30) which results in (6.16). Finally, (6.19) follows by equating the coefficients of $u^{h-1} e^{-\beta_{i,\sigma,l} u}$ in (6.30).

Case 2: Given $s = 0, \delta \geq 0$ and $\alpha = 0, 1, 2, \ldots$

For $s = 0$, $\Lambda_{i,\alpha,\delta,0,j,z,\xi,h} = 0$ as given in (6.10), and thus (6.30) reduces to

$$\sum_{y=0}^{\alpha}\sum_{x=1}^{n}\Theta_{i,\alpha,\delta,0,x,y}\{u^y e^{\rho_{\delta,x}u}\}$$
$$= \sum_{y=0}^{\alpha}\sum_{x=1}^{n}\left\{\sum_{\sigma=1}^{m}\sum_{a=y}^{\alpha}\sum_{g=y}^{a} P_{i,\sigma,\alpha,a,g,y,\rho_{\delta,x}}\Theta_{\sigma,a,\delta,0,x,g}\right\} u^y e^{\rho_{\delta,x}u}$$
$$+ \sum_{y=0}^{\alpha}\sum_{x=1}^{n}\left\{\sum_{\sigma=1}^{m}\sum_{a=y}^{\alpha}\sum_{g=y}^{a} Q_{i,\sigma,\alpha,a,g,y,\rho_{\delta,x}}\Theta_{\sigma,a,\delta,0,x,g}\right\} u^y e^{\rho_{\delta,x}u}$$
$$- \sum_{\sigma=1}^{m}\sum_{l=1}^{n_{i,\sigma}}\sum_{h=1}^{\nu_{i,\sigma,l}}\left\{q_{i,\sigma}\sum_{a=0}^{\alpha}\sum_{b=0}^{a}\sum_{x=1}^{n}\Theta_{\sigma,a,\delta,0,x,b}\frac{(-1)^b(b+\gamma-g)!}{(\rho_{\delta,x}+\beta_{i,\sigma,l})^{b+\gamma-g+1}}\right.$$
$$\left.\times \sum_{\gamma=h}^{\nu_{i,\sigma,l}}\sum_{g=h}^{\gamma}\frac{h\pi_{i,\sigma,l,\gamma}(\beta_{i,\sigma,l})^{\gamma}}{g!(\gamma-g)!}k_{i,\alpha,a,g,h,\delta+c\beta_{i,\sigma,l}}\right\} u^{h-1}e^{-\beta_{i,\sigma,l}u}$$
$$+ \sum_{\sigma=1}^{m}\sum_{l=1}^{n_{i,\sigma}}\sum_{h=1}^{\nu_{i,\sigma,l}}\left\{q_{i,\sigma}\sum_{\gamma=h}^{\nu_{i,\sigma,l}}\sum_{g=h}^{\gamma}\frac{h\pi_{i,\sigma,l,\gamma}(\beta_{i,\sigma,l})^{\gamma}}{g!(\gamma-g)!}\right.$$
$$\left.\times\, w_{\gamma-g,(\beta_{i,\sigma,l})}k_{i,\alpha,0,g,h,\delta+c\beta_{i,\sigma,l}}\right\}u^{h-1}e^{-\beta_{i,\sigma,l}u}. \tag{6.32}$$

Since (6.32) holds for all $u \geq 0$, the coefficients of $u^y e^{\rho_{\delta,x}u}$ on both sides are equal and this leads to (6.20). For $y = \alpha$, (6.20) gives (6.21). Similarly as in the case $s > 0$, it can be argued that (6.21) represents m linearly dependent equations in $\{\Theta_{i,\alpha,\delta,0,x,\alpha}\}_{i=1}^{m}$ for each fixed $x = 1, 2, \ldots, n$. Moreover, this argument can show that (6.11) also holds in this case when $s = 0$.

Finally, equate the coefficients of $u^{h-1}e^{-\beta_{i,\sigma,l}u}$ on both sides of (6.32) yields (6.22). $\square$

In Theorem 6.2.2, it is shown that the coefficients Θ's and Λ's need to be solved recursively in α, starting with $\alpha = 0$. To see this, consider $m_{i,1,0,0}(u) = E[TI(T < \infty)|U_0 = u, Z_0 = i]$ as an example, i.e. let $\alpha = 1$, $\delta = 0$, $s = 0$ and $w(\cdot) = 1$. From (6.10),

$$m_{i,1,0,0}(u) = \sum_{x=1}^{n} \left\{ \Theta_{i,1,0,0,x,0} \left\{ e^{\rho_{0,x} u} \right\} + \Theta_{i,1,0,0,x,1} \left\{ u e^{\rho_{0,x} u} \right\} \right\}.$$

for $i = 1, 2, \ldots, m$. It is given in (6.20) and (6.21) that the coefficients Θ's satisfy

$$\Theta_{i,1,0,0,x,0} = \sum_{\sigma=1}^{m} \sum_{a=0}^{1} \sum_{g=0}^{a} \left\{ P_{i,\sigma,1,a,g,0,\rho_{0,x}} + Q_{i,\sigma,1,a,g,0,\rho_{0,x}} \right\} \Theta_{\sigma,a,0,0,x,g} \tag{6.33}$$

for $i = 1, 2, \ldots, m$; $x = 1, 2, \ldots, n$ and

$$\Theta_{i,1,0,0,x,1} = \sum_{\sigma=1}^{m} \left\{ P_{i,\sigma,1,1,1,1,\rho_{0,x}} + Q_{i,\sigma,1,1,1,1,\rho_{0,x}} \right\} \Theta_{\sigma,1,0,0,x,1} \tag{6.34}$$

for $i = 1, 2, \ldots, m$; $x = 1, 2, \ldots, n$. In (6.34), the m equations in $\{\Theta_{i,1,0,0,x,1}\}_{i=1}^{m}$ for each fixed $x = 1, 2, \ldots, n$ are linearly dependent. Thus, n more equations to solve for Θ's are given by (6.22), which are

$$\sum_{\gamma=h}^{\nu_{i,\sigma,l}} \sum_{g=h}^{\gamma} \left\{ \sum_{a=0}^{1} \sum_{b=0}^{a} \sum_{x=1}^{n} \Theta_{\sigma,a,0,0,x,b} \frac{(-1)^b (b+\gamma-g)!}{(\rho_{0,x} + \beta_{i,\sigma,l})^{b+\gamma-g+1}} k_{i,1,a,g,h,c\beta_{i,\sigma,l}} \right.$$
$$\left. - w_{\gamma-g,(\beta_{i,\sigma,l})} k_{i,1,0,g,h,c\beta_{i,\sigma,l}} \right\} \frac{\pi_{i,\sigma,l,\gamma} (\beta_{i,\sigma,l})^{\gamma}}{g!(\gamma-g)!} = 0 \tag{6.35}$$

for $i, \sigma = 1, 2, \ldots, m$; $l = 1, 2, \ldots, n_{i,\sigma}$ and $h = 1, 2, \ldots, \nu_{i,\sigma,l}$. Note that (6.33) and (6.35) involve $\Theta_{i,0,0,0,x,0}$ for $i = 1, 2, \ldots, m$ and $x = 1, 2, \ldots, n$. According to (6.10), these are coefficients for $m_{i,0,0,0}(u) = E[I(T < \infty)|U_0 = u, Z_0 = i]$, i.e.

$$m_{i,0,0,0}(u) = \sum_{x=1}^{n} \Theta_{i,0,0,0,x,0} \left\{ e^{\rho_{0,x} u} \right\}$$

for $i = 1, 2, \ldots, m$. Thus, in order to solve for the associated coefficients Θ's of $m_{i,1,0,0}(u)$, the associated coefficients Θ's of $m_{i,0,0,0}(u)$ need to be solved first. This simple example gives an idea on how the coefficients should be solved recursively in α using Theorem 6.2.2.

6.3 Numerical Example

In this section, a numerical example is considered by applying the results in section 6.2. The probability of ruin and the conditional mean time to ruin will be studied under different distributional assumption on the time of a system change.

This example is a two-state generalized MAP risk model given as follows. Let the transition probability matrices be

$$\boldsymbol{P} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{6} & 0 \end{pmatrix} \text{ and } \boldsymbol{Q} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{6} \end{pmatrix}.$$

Also, assume the following claim size densities

$$f_{1,1}(y) = \frac{1}{2}e^{-y} + \frac{1}{2}(\frac{2}{3}e^{-\frac{2}{3}y}),\ f_{1,2}(y) = \frac{3}{4}e^{-\frac{3}{4}y},\ f_{2,1}(y) = 6e^{-6y} \text{ and } f_{2,2}(y) = 9ye^{-3y}.$$

Therefore, the mean claim sizes are $\mu_{1,1} = 5/4$, $\mu_{1,2} = 4/3$, $\mu_{2,1} = 1/6$ and $\mu_{2,2} = 2/3$.

As for the waiting time of a system change, assume three different cases of distribution and each case will be considered together with the model assumption described in last paragraph. The first case is

$$k_1(t) = e^{-t} \text{ and } k_2(t) = 2e^{-2t},$$

where exponential waiting time is assumed in both states. For the second case, the waiting time in state 1 is changed to be gamma distributed while the waiting time distribution in state 2 is kept unchanged, i.e.

$$k_1(t) = 4te^{-2t} \text{ and } k_2(t) = 2e^{-2t}.$$

As for the third case, the waiting time distribution in state 2 is changed to gamma while that in state 1 is kept unchanged as compared to the first case, i.e.

$$k_1(t) = e^{-t} \text{ and } k_2(t) = 108t^2e^{-6t}.$$

Note that in any of the three cases, the mean time in state 1 and state 2 are $\kappa_1 = 1$ and $\kappa_2 = 1/2$ respectively. The variances of the waiting time in state 1 (state 2) are 1 (1/4), 1/2 (1/4) and 1 (1/12) for the first, second and third case respectively.

	$k_1(t)$	$k_2(t)$
Case 1	e^{-t}	$2e^{-2t}$
Case 2	$4te^{-2t}$	$2e^{-2t}$
Case 3	e^{-t}	$108t^2e^{-6t}$

Table 6.1: Waiting time distributions in different cases

Moreover, let the premium rate be $c = 2$ which satisfies the positive loading condition (6.2).

In the following graph, the y-axis gives the probability of ruin, i.e.

$$E[I(T < \infty)|U_0 = u, Z_0 = i] = m_{i,0,0,0}(u)$$

for $i = 1, 2$ and the x-axis is the initial surplus $U_0 = u$.

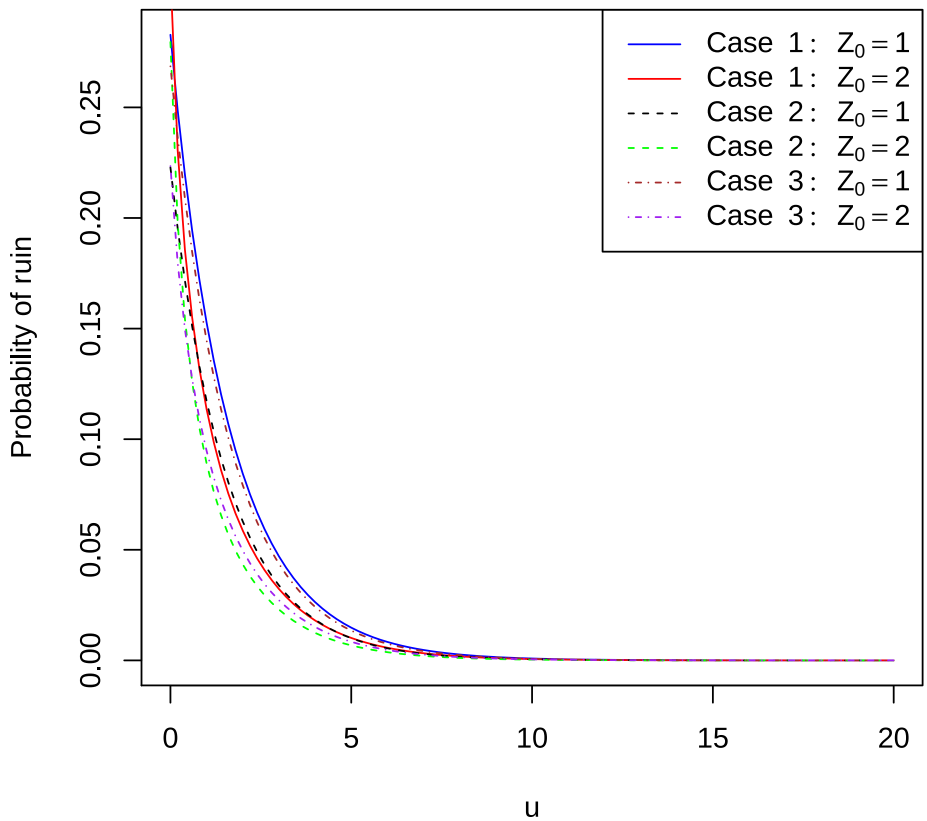

Figure 6.1: Probability of ruin in different cases

In Figure 6.1, the probability of ruin decreases with larger initial surplus. Given $Z_0 = 1$ or $Z_0 = 2$, the probability of ruin is higher in cases with larger variances of the waiting time. This observation is the same as that made in Cheung et al. (2011).

Next, the following graph considers the conditional expected time to ruin. The y-axis represents

$$E[T|T < \infty, U_0 = u, Z_0 = i] = \frac{m_{i,1,0,0}(u)}{m_{i,0,0,0}(u)}$$

for $i = 1, 2$. The x-axis represents the initial surplus $U_0 = u$.

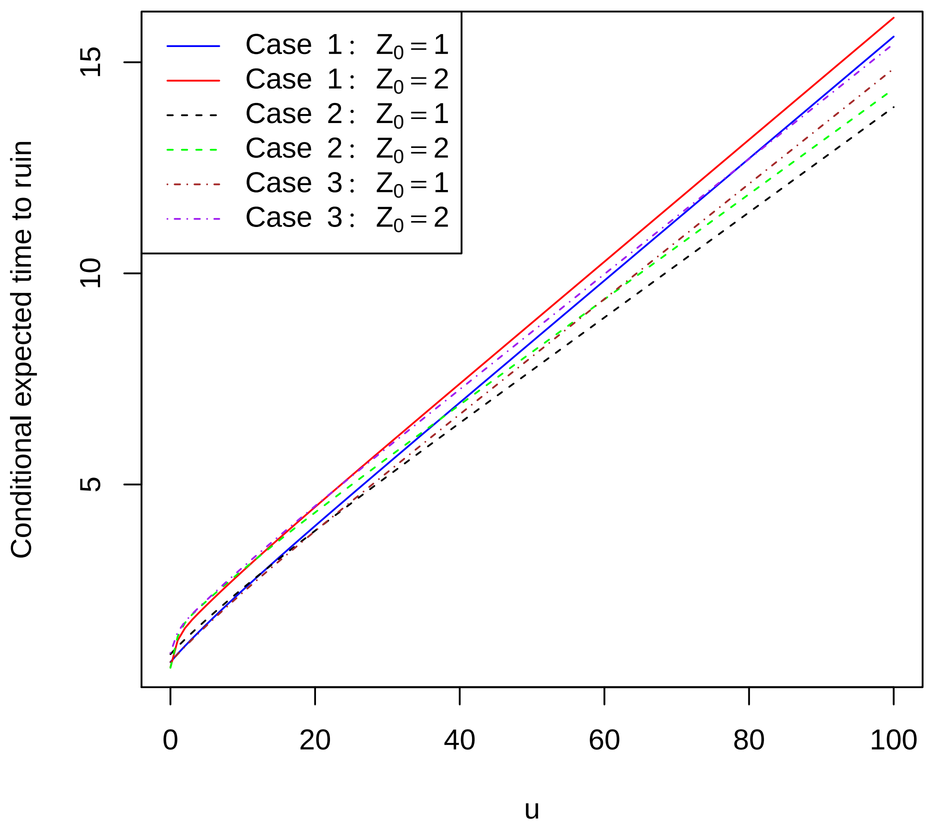

Figure 6.2: Conditional expected time to ruin in different cases

There are several observations made from Figure 6.2. First, the curves are all strictly increasing. In other words, the conditional expected time to ruin increases with the initial surplus. Second, given any of the three cases and the same initial surplus, the expected time to ruin when $Z_0 = 1$ is shorter than that when $Z_0 = 2$. Finally, comparison among cases can be made. For either $Z_0 = 1$ or $Z_0 = 2$, the expected time to ruin is longer in case 1 than that in case 2 except when initial surplus is very small. There is similar observation when comparing case 1 and case 3.

Chapter 7

Conclusion and future research

In this thesis, the generalized moments of the time to ruin are the main focus of study. In dependent Sparre Andersen models, structural properties of the Gerber-Shiu function are shown to continue hold for the moments of the time to ruin in Chapter 2. These properties are useful in further research of the moments of the time to ruin. For example, the bounds for the moments can be studied by using the result introduced in Section 1.4.5. Moreover, it is also of interest to provide good approximation results for the moments since the analytical results usually involve a lot of recursion. There have been approximation results in the classical Possion risk model, readers may refer to e.g. Egidio dos Reis (2000), Dickson and Waters (2002) and Pitts and Politis (2008).

In Chapter 3, dependent Sparre Andersen model with Coxian claim sizes is considered and the form of the moments of the time to ruin is identified as a linear sum of Erlang densities. The coefficients in this sum can be obtained by solving linear systems of equations. Numerical examples are provided for the mean and variance of the time to ruin.

Intuitively, the expected time to ruin and the variance of the time to ruin should be related to the marginal distribution of each increment of the insurance surplus process. The marginal distribution of the interclaim times and the claim sizes, and in particular their dependence structure, will affect the marginal distribution of each increment. Further research is needed as to the manner in which the moments of the time to ruin are related to the dependency between the interclaim times and the claim sizes.

In Chapter 4, structural properties of the moments of the time to ruin are discussed in dependent Sparre Andersen models with Coxian interclaim times. The numerical example shows how the results in Chapter 2 are used recursively in order to determine the expected time to ruin. However, the computation for higher moments of the time to ruin can be intensive due to recursive nature of the method. It appears that the techniques derived in Chapter 4 are applicable primarily to the calculation of lower moments of the time to ruin.

In Chapter 5, the joint density of the time to ruin, the number of claims until ruin and other ruin-related quantities is identified under a Sparre Andersen model with exponential claim sizes. The marginal and joint moments of these quantities can then be obtained by integration. The joint density of these quantities may be considered under more general claim sizes, e.g. Coxian or phase-type claim sizes, in the future. Multivariate Lagrange expansion may be used to obtain the density involving the time to ruin as in Landriault and Shi (2013).

Apart from the dependent Sparre Andersen model, the moments of the time to ruin are also considered in the generalized MAP risk model in Chapter 6. By assuming Coxian claim sizes, the moments are in the form of a linear sum of Erlang densities. The numerical example provided in Chapter 6 has exponential waiting times. More general waiting time distributions are not considered since these models involve intensive computation.

We remark that the penalty functions considered in this thesis are functions of the deficit at ruin (as well as other variables in some cases). Thus, the joint moments of the time to ruin and the deficit at ruin (and in particular their covariance) may be obtained by appropriate choice of the penalty function. This was done in the classical Poisson model by Lin and Willmot (2000). It would be interesting to further study the relationship between the time to ruin and the deficit at ruin in this manner.

By definition, the time to ruin is the first passage time that the insurance surplus process drops below zero. In the literature, the time to absolute ruin has also been considered by many authors. It is the first passage time that the insurance surplus process drops below the level $-c/r$, where $c > 0$ is the premium rate of the insurance surplus process and $r > 0$ is the borrowing rate when the insurance surplus process drops below zero. In fact, both the time to ruin and the time to absolute ruin are special cases of the first passage time that the insurance surplus process drops below an arbitrary level in a risk model with interests and/or dividends.

Topics related to this first passage time have been considered in many papers under the compound Poisson risk model where the interclaim times and claim sizes are assumed to be independent. Interested quantities include the absolute ruin probability, the Laplace transform of the first passage time and generalized Gerber-Shiu functions defined with the first passage time. For example, see Lin et al. (2003), Zhu and Yang (2008), Cai et al. (2009) and Li and Lu (2013).

In future research, the time to absolute ruin and hence the first passage time of the insurance surplus process may be studied in more general risk models with interests and/or dividends. For example, Mitric et al. (2012) considered the Gerber-Shiu function defined with the time to absolute ruin under a Sparre Andersen renewal risk model and Yang and Sendova (2014) studied the time to ruin under a dual risk model. Thus, one may continue the studies of the first passage time in more general risk models, e.g. to consider a renewal claim number process instead of the Poisson claim number process. Moreover, dependency between the interclaim times and the claim sizes may be assumed. Also, the first passage time of the insurance surplus process may be studied in models with jumps. Under these model assumptions, the Laplace transform and the density of this first passage time can be considered.

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