A Process Model of Non-Relativistic Quantum Mechanics

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Physics

Waterloo, Ontario, Canada, 2014

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

A process model of quantum mechanics utilizes a combinatorial game to generate a discrete and finite causal space $\mathcal{I}$ upon which can be defined a self-consistent quantum mechanics. An emergent space-time $\mathcal{M}$ and continuous wave function Φ arise through a uniform interpolation process. Standard non-relativistic quantum mechanics (at least for integer spin particles) emerges under the limit of infinite information (the causal space grows to infinity) and infinitesimal scale (the separation between points goes to zero). This model is quasi-local, discontinuous, and quasi-non-contextual. The bridge between process and wave function is through the process covering map, which reveals that the standard wave function formalism lacks important dynamical information related to the generation of the causal space. Reformulating several classical conundrums such as wave particle duality, Schrödinger's cat, hidden variable results, the model offers potential resolutions to all, while retaining a high degree of locality and contextuality at the local level, yet nonlocality and contextuality at the emergent level. The model remains computationally powerful.

Acknowledgements

I wish to thank Irina Trofimova, partner, fellow psychologist and complex systems traveller, for the many discussions, heated and calm, that we have had over these all too many years. She first suggested than later prodded me into looking at relational models, at generative and developmental models, and at ideas of process, and her ideas of compatibility form an essential component of the theory of process interaction and of process strength, which underlies the emergent probability of the process model. Her inspiration, challenges and criticism helped this to be a better thesis than it might otherwise have been.

I want to thank Robert Mann for his support during the long years that it has taken this project to come to fruition. He was willing to take on a mature student with a most unusual background and a willingness to take a substantial risk in tackling a subject with a very limited audience within the physics community. His encouragement, patience and guidance have been greatly appreciated. Thanks are due to Steve Weinstein for many helpful conversation during the first formative years of this work. I want to thank Rafael Sorkin for his inspiration. I first came across his work on causal sets while I was working on my Ph.D. in mathematics and it stirred my interest in applying these complex systems ideas to physics. I am grateful to have him on my Supervisory Committee. Thanks are due to Achim Kempf, whose work on the application of interpolation theory to the problem of ultraviolet divergencies provided exactly the tools that I needed to bridge the gap between causal tapestries and the Hilbert space. I also want to thank Tony Leggett for the time that he gave being a part of my Committee and for the inspiration that his work on quantum foundations provided.

I wish also to thank my children, Kara, Tegan, Alex, who have put up with me and my research for a very long time. I hope that they will feel their patience rewarded. I also wish to thank my step-daughter Anastasia for lots of distractions.

Finally I would like to thank the anonymous authors whose images I found on Google and incorporated into the various figures illustrating the text. I cannot cite each one individually as there were no personal references listed but they have my gratitude.

Dedication

This thesis is dedicated to Irina Trofimova and to my children.

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Chapter 1

Introduction

1.1 Reality - Why Bother?

The Sept 29 - Oct 5, 2012 issue of New Scientist has the headline '

What *is* Reality?' . Inside one finds a special issue section with small articles titled - Defining Reality, The Bedrock Of It All, Is Matter Real?, Is Everything Made Of Numbers?, If Information...Then Universe, Does Consciousness Create Reality?, How Do We Know?, The Future Of Reality. These articles leave one with a very unsettling picture of our current conception of reality. First of all, it is not even clear what is meant by the term 'reality' . Every definition is left wanting. Like the concept of consciousness, everyone knows what reality is but no one can actually provide a satisfying definition. Every definition either includes something disquieting, or excludes something significant.

The authors suggest two plausible candidates for a definition of reality. The first is that reality is whatever is left after humans are removed. In this regard I particularly like Philip K. Dick's dictum that 'reality is that which, if you stop believing in it, doesn't go away'. Many ideas of reality are decidedly solipsistic, referenced to our perceptions. This notion eliminates solipsism. But it strikes me that it is not the presence of humans that confounds reality - rather, it is linking reality to that which humans can perceive, believe or conceive.

The second definition associates reality with only those entities residing at the bottom of the ontological food chain. Higher level entities, which presumably supervene upon or emerge from these more fundamental entities, are considered as epiphenomena, and therefore unreal. But by that logic, every biological organism is unreal, including each of us.

This notion of reality attributes the status of real *only* to the supposed fundamental material causes. Moreover it accords a status of real *only* to those entities that can be understood as objects. This seems like an unduly simplistic and naïve attitude towards reality, particularly as the fundamental entities do not behave at all like objects. This second definition seems more like a self-serving justification for the reductionist paradigm than a serious definition of reality. The reductionist approach has had its successes, but even the standard model of quantum mechanics, arguably the best candidate for a theory of everything, has been unable to incorporate gravity, which pervades most of the universe, or to model dark matter and dark energy, which together make up 96% of its contents.

Reductionism has been successful in looking to ever smaller spatio-temporal scales but the promised benefits - the ability to explain all phenomena at larger spatio-temporal scales - have been singularly lacking. The account of reality in New Scientist ignores three decades of advances in the field of complex systems theory and our burgeoning knowledge of emergence [128, 200, 352, 355, 429, 430], complex adaptive systems [260], and of upward and downward causation [18]. In complex systems the rules governing lower level entities become progressively less relevant as one moves up spatio-temporal scales. How is it then that mere epiphenomena acquire their own rules and become self-governing, self-regulating, self-defining?

There are some who believe that reality does not exist independent of a conscious observer. One is reminded of Wheeler's famous dictum that

"No elementary phenomenon is a phenomenon until it is a registered (observed) phenomenon". [421, pg. 184]

Some, like Tegmark, believe that reality *is* mathematics [385], an overtly Platonic view and one that, as a mathematician myself, I do not agree with. Mathematics does appear to have a standing that sets it apart from the vicissitudes of the reality that we experience as humans, but that accords mathematics its own unique ontological status. It does not accord our reality the same status. It is indeed surprising that some aspects of reality can be rather accurately described in mathematical terms but it is a very great stretch to assert that *all* of reality can be so described. Moreover, discoveries in complex systems theory, especially as applied to condensed matter, have begun to shed light upon the conditions under which physical systems admit a simple mathematical description, and when they do not [225]. There are many different systems of mathematics, illustrated by the many different versions of set theory shown to exist following the discovery of the independence of the continuum hypothesis [177].

Some believe that reality is information and that science needs to expand to include information as fundamental [409, 68]. Some think that reality is the product of a cellular automaton [429], or a holographic projection onto a boundary in space-time [384]. I am partial to these approaches, but one must be aware that the concept of information is also slippery, like that of consciousness and reality itself. Information is often restricted to the sense of Shannon [320], which relates information and entropy. But Shannon's notion of information is an engineering notion, describing the information carrying capacity of a transmission device. There are other notions of information that are more relevant to other disciplines, especially in computer science [138], biology [406], biosemiotics [129]. Information as form but without meaning is useless. The question of how simple systems generate meaning from form is deep and fascinating and has occupied me for many years [356, 354, 359, 360].

Perhaps, like the problem of consciousness, reality is a primitive construct. Although we all know intuitively what consciousness and reality are, we cannot give them a formal definition, since that requires reference to some other collection of primitives, and there is none. We might be able to say what reality *isn't*, but not what it *is*. As mathematics operates within different foundational assumptions (such as different theories of sets), perhaps we must for the time being operate within different theories of reality, and hope that someday it may be possible to experimentally test these. Indeed, since the work of Einstein-Podolsky-Rosen [424], Bohm [55], Bell [44], Clauser-Horne-Shimony-Holt [321], von Neumann [412], Gleason [150], Kochen-Specker [218], Mermin [255], Ionicioiu-Jennewien-Mann-Terno [191], and many others [157, 9], it has become possible to test some predictions of theories and models that address questions related to fundamental attributes of reality. Indeed experiments to do just that have been carried out by Aspect, Zeilinger and others [24, 421, 157]. It may be that some day the study of the nature of reality will become an empirical science.

Philosophers have studied the problem of the nature of reality for millennia, their ideas forming the fields of metaphysics and ontology. According to Norsen [273], there are four broad types of realism that hold relevance for modern physics. They are

1. Naïve realism - all features of a perceptual experience have their origin in some identical corresponding feature of the perceptual object (this corresponds to non-contextual hidden variable theories) (pg 315)

2. Scientific realism - the doctrine that we can and should accept well established scientific theories as providing a literally-true description of the world (pg 320)

3. Perceptual realism - the idea that sense perception provides a primary and direct access to facts about the world - i.e. that what we are aware of in normal perception is

the world, and not any sort of subjective fantasy, inner theatre, or mental construction (this forms part of the basis for empirical science) (pg 322)

4. Metaphysical realism - the existence of a single, objective, external world “out there” whose existence and identity is independent of anyone’s awareness of it... without necessarily requiring anything specific in regards to its similarity to the world of our perceptual experience or the account of any particular scientific theory (pg 330)

Naïve realism can be rather easily dismissed but without metaphysical realism there is nothing to study and without scientific and perceptual realism there is nothing for science to be founded upon. Norsen argues against the use of the term realism, preferring instead to focus upon the presence or absence of specific attributes that reality may or may not possess.

I think that part of the problem as to why so little headway has been made is the rather poor cross fertilization of ideas between different disciplines. In spite of pleas for interdisciplinary research, academic disciplines remain rather insular, and even within a discipline its various sub-disciplines may exhibit the same tendencies. There is little sharing of ideas between the physical, life and social sciences. The use of mathematics within physics is not as broad as it could be. The lack of attention by mathematicians to problems outside of mathematics and physics is also disheartening. There have been attempts to bring these various communities together through conferences [369, 267] or organizations like The Society for Chaos Theory in Psychology and the Life Sciences, The New England Complex Systems Institute and the Santa Fe Institute. These communities remain small, and progress has been slow. It still remains difficult to publish chaos and complexity oriented research outside of a small number of specialized scientific journals. Even where mathematical sophistication is high, interdisciplinary cooperation and collaboration is the exception rather than the rule.

In order to understand why I have taken the approach to be described in this thesis it might help to know a little of my background. I have always taken an interdisciplinary approach to my research and my career. I have training as a physician and a psychiatrist, and my work as a geriatric psychiatrist involves psychiatry, neurology and internal medicine. I trained in pure mathematics and computer science. My Ph.D. in pure mathematics was supervised by a computer scientist and involved a study of of the confluence of two ideas - that of automata and that of ordered sets [343]. My research into complex systems has involved studies of neural representation, information representation and meaning creation and extraction by complex systems, interacting complex systems, and collective intelligence. I have always found great value in studying what happens when different ideas and disciplines interact and intersect. That basic approach to science informs the approach taken in this thesis. I have utilized ideas from very different domains of research - process theory, archetypal dynamics, combinatorial game theory, causal tapestry theory and interpolation theory, to examine foundational issues in physics in the setting of non-relativistic quantum mechanics. This is a decidedly interdisciplinary approach and it should be understood as such.

I have been much impressed with the writings of the biophysicist Robert Rosen, who inspired my wish to explore the degree to which notions from complex systems theory and emergence could be applied to the understanding of the fundamental levels of reality. He wrote [183]

> "... the basis on which theoretical physics has developed for the past three centuries is, in several crucial respects, too narrow and that, far from being universal, the conceptual foundation of what we presently call theoretical physics is still very special; indeed, far too much to accommodate organic phenomena (and much else besides). That is, I will argue that it is physics, and not biology, which is special; that, far from contemporary physics swallowing biology as the reductionists believe, biology forces physics to transform itself, perhaps ultimately out of all present recognition." (pg. 315)

To move forward requires us to take pride in our successes but remain humble in the face of our failures. More importantly, it requires us to take an open mind towards ideas that may at first seem at odds with our cherished or dogmatically held beliefs. There is the old joke about the drunk and the street lamp. A pedestrian walking along a sidewalk spies a drunk man under a street lamp examining the sidewalk closely, even though there is nothing to be seen there. The man asks the drunk what he is doing, and the drunk replies that he lost his keys. The man asks him if he knows where he lost them and the drunk points further down the sidewalk towards a dark alley. The man then asks why the drunk is looking under the street lamp if the keys are actually over there. The drunk replies that he looks under the street lamp because that is where the light is.

In addition there is a danger, as Mermin pointed out [259], for our mathematical structures to become reified, which risks stifling creativity, constraining the imagination, and turning useful metaphors into dogma. I ask the reader of this thesis to approach its subject matter with an open mind. I approached this thesis with a simple goal in mind - to attempt to put some semblance of reasonableness back into reality as conceived within theoretical physics. I have felt increasingly uncomfortable with the need for multiple dimensions, multiverses, observer driven determinism and such. While I accept that certain common sense ideas about reality might not hold true, and that some nonintuitive ideas might be necessary, I had the feeling that many of the directions that theoretical physics was taking were getting to be too 'far out' . I was reminded of the situation before Copernicus when Ptolemaic epicycles were used to describe the heavens. Effective, geometrically stylish, but cumbersome, they eventually gave way to the elegance, simplicity and parsimony of the Copernican view. For that to happen, it was necessary to understand all of the assumptions that went into the Ptolemaic model. For me to proceed, it was also necessary to first examine the many assumptions that underlie our current formulations of physical theory, and to ask which are known, as much as possible, to be true, which are held to be true based upon opinion and common usage, and which are merely hoped to be true without, or in the face of contrary, evidence. I chose to focus upon NRQM because it has been well tested experimentally and its mathematical form is simpler than that encountered in quantum field theory. I have approached this by attempting to keep the barest minimum of assumptions and as well to minimize as much as possible those aspects, such as non-locality and contextuality, that cause so much consternation when addressing the nature of reality.

I wondered whether it was still possible to have, at the lowest level of reality (what some authors [66] refer to as 'Ultimate Reality'), a set of possibly hidden variables whose evolution in space and time remained more or less local, thus consistent with relativity, and more or less non-contextual, and thus independent of observers. I wondered to what degree it might be possible to consider the non-locality and contextuality of quantum mechanics to be emergent as opposed to being fundamental. In so doing it was necessary to examine the many so-called hidden variable theorems, to carefully examine their underlying assumptions and assess their validity and whether their usual interpretations are necessarily true. It was also necessary to ask which of the underlying assumptions had sufficient empirical support to be accepted as true, and which were assumptions based more on ideology, mathematical convenience or convention, or sociological consensus. I could then ask which assumptions could be safely abandoned and consider what the resulting model of reality would look like. The core assumptions that I considered were

1. That local (classical)(non-contextual) variables *must* obey the rules of Kolmogorov probability theory

2. That systems *must* obey the principle of continuity

3. That presentism -as a model of time *cannot* be valid (no transient now)

4. That local, non-contextual variables *must* give rise to an object centric reality

In this thesis I hope to convince the reader, drawing on the work of Palmer [286], Vorob'ev [413], Khrennikov [207, 208], and Kolmogorov [219], that the first assumption is not so certain, and as a result the conclusions concerning the nonlocal nature of reality may not hold up logically. I also hope to show that contextuality in the sense that measurements determine outcomes may be a feature of the classical world as well as the quantum world. The second assumption is widely held, although empirically it is most easily and clearly demonstrated to be illusory. It has proven very useful when constructing theories and models but at the same time provides a major impediment to progress. It leads to all manner of paradox and mathematical conundrums because with continuity there must come infinity and infinity is a notion as inscrutable as the Cheshire cat in Alice in Wonderland. I am not alone in raising questions about this assumption. A great deal of research in quantum gravity explores the consequences of relaxing this assumption. Gisin [149] has even provided a Bell type inequality whose violation would challenge the validity of the principle of continuity. The third assumption is widely held to be true, in spite of the fact that the experience of a transient now is a deeply ingrained, if not fundamental, feature of our experience of the world. There is no direct experience of either the past or of the future. Closely related to the absence of transience is the belief in time reversal symmetry (unitarity) as a feature of fundamental reality even though there are seven empirically demonstrable arrows of time. The psychological experience of the arrow of time is held to be an illusion, even though Bohr himself made it an essential feature of his theory of measurement. He wrote [58]:

> every atomic phenomenon is closed in the sense that its observation is based on registrations obtained by means of suitable *amplification* devices with *irreversible* functioning (pg 73) (emphasis mine)

The fourth assumption provides an implicit view of the classical world and appears to underlie the presumed constraints that are to be placed upon any possible form of hidden variable theory. I hope to show that this is due in part to a misplaced emphasis upon the characteristics of inanimate matter, and that even the classical level provides abundant examples of systems whose characteristics are anything but object-like.

Based upon these considerations I sought out to construct an explicit quasi-local, quasi-non-contextual model to test out whether a consistent theory could be created. In so doing I have constructed a model which is local, non-contextual and unobservable at the lowest level, which reproduces the predictions of NRQM asymptotically, which assumes that the wave function is ontological, and in which probability, non-locality and contextuality emerge at the level of empirical observation as a result of the role of process as a generative agency.

There have been other attempts to construct hidden variable models of NRQM (and RQM and QFT as well, but those are outside the scope of this thesis) such as the theories of Bohm [55, 56], and of GRW [40, 147, 148, 290], and certain cellular automata models [429, 430]. They have had remarkably little influence in mainstream physics. Indeed the reactions to such models have been quite vitriolic at times. In fact, Streater [341] has suggested that Bohmian theory is a lost cause in physics, irredeemably flawed.

Bell [44] described three 'romantic' and three 'unromantic' models of quantum mechanics, two of the latter being due to Bohm and GRW. Tumulka writes [403]

> The two unromantic pictures, Bohmian mechanics and spontaneous collapse, make it evident that quantum mechanics can be understood in terms of a completely coherent theory with a clear ontology. Regrettably, more than 50 years after Bohm and 20 years after GRW, this is still not very widely known. Given how vague and incoherent orthodox quantum philosophy is, and how radical the claims are that it makes about the intrinsic impossibility to understand physics, one might expect that scientists accept it only if they have to, under the load of incontrovertible evidence. One might thus expect that scientists would immediately give up orthodox quantum philosophy when they learn that theories exist that are understandable and account for all phenomena of quantum mechanics. But historically, the opposite was the case. When David Bohm argued in 1952 [55], perhaps for the first time convincingly, that Bohmian mechanics accounts for quantum mechanics in terms of objective, but non-classical, particle trajectories, the reception was cold. What is it that motivates scientists, rational people who take pride in their ability to understand the most intricate theories, to give up on any serious understanding of quantum mechanics, in favour of the obscure orthodox quantum doctrine? I do not claim to be able to answer this question from the armchair. Indeed, I think that to determine the answer is a research topic for sociologists of science, and a worthwhile one. (pgs 3246-7)

The model that I present in this thesis fits squarely and unapologetically into Bell's category of 'unromantic' . Mathematically it presents an alternative approach to solving Schrödinger's equation, which may be useful in certain cases. More important though is the effort to place at least NRQM on a secure conceptual and mathematical foundation. Once upon a time, physics used to be first and foremost about understanding physical reality. The success of mathematical models, the conceptual difficulties encountered during the early years of quantum mechanics and the later obsession with applications have fostered an attitude of 'shut up and calculate' . The grounding of ideas and understanding has been relegated to the back of the bus, leading some to worry about the viability of current thinking in theoretical physics [28, 404].

Mathematics has benefitted greatly from the effort during the past two centuries to place itself on a firm logical footing [154]. The tension between the Platonic and the constructive world views proved highly productive, though not without cost. Remarkably, in doing so it has been able to turn what at first appeared to be limitations into powerful advantages. It has used the appearance of paradoxes to develop new conceptions, leading to new tools and to new understanding. Mathematics can embrace many different fundamental formulations (particularly in logic, foundations and set theory) and explore the strengths and limitations of each. Mathematics has seen paradoxes as challenges to growth. Physics, on the other hand, seems to treat paradoxes as failings of the human psyche rather than as failings of its conceptual and mathematical structure. All too often the response to a paradox is either to assert that the experienced viewpoint is an illusion, as in the illusion of time passing, the illusion of time's arrows, the illusion of waves and particles, the illusion of order, (indeed sometimes the illusion of reality) or to dismiss the concerns as meaningless.

I hope that the readers of this thesis will consider the conceptual and logical foundations of physics to be of as much importance as the outcomes of calculations and to view this thesis from that perspective. I believe that physics, like mathematics, will advance further and more quickly, if its conceptual house can be put into good order and the paradoxes eliminated. Any theory, whether Bohm, GRW, this thesis, or others, that can do this is worth further attention.

1.2 A Brief Overview of the Model

Conceptually, much of modern physics is founded upon a conception of reality as simply 'out there ' , as simply *being* in its totality - past, present, future are extant together. There is no notion of becoming, no notion of transience. The temporal ordering of events lies merely in how we choose to examine the world. Mathematics, existing in its Platonic universe, provides a lofty, even god-like, vantage point from which to view the world. But mathematics cheats. Since it is not grounded, it allows one to imagine and construct models based upon information that one does not and in some cases for fundamental reasons can not ever possess. Mathematics exists in a ideal world, in which there is perfect information, perfect knowledge, in which there is endless recursiveness upwards and downwards, in which there abounds infinity upon infinity, large and small. But this is patently false when applied to our reality. The non-commutativity and hence, incommensurability, of quantum measurements and the limits on information transfer posed by relativity place fundamental constraints upon what is knowable, as do limitations on the amount of matter in the universe, and thus upon the amount of information that may be carried. Moreover, mathematics *is*, and mathematical structures *are*, quite independent of who is studying them. They are the archetypes for the concept of *object*. Classical physics assumed that the world was populated by material instantiations of these mathematical objects. There was little room for creativity, novelty, development, or agency. Quantum mechanics showed that this conception was incorrect but physics has not really recovered. The world abounds with entities that fail to behave as objects; one merely needs to look to biology, psychology and social science for examples.

My research in complex systems theory [352, 355, 360], particularly the study of cellular automata under the influence of external perturbations or signals [354, 356], has shown again and again how systems manufacture behaviour and form decisions *on the fly*, *in the moment*. These responses are generated in reaction to and in concert with the environment. They do not arise from some explicit set of rules or simple reflexes. Rather, they emerge from the on-going interaction between the automaton and its environment. In the case of adaptive models such as the cocktail party automaton, the cells participating in a decision at one time, and their internal rules, are completely different at a later time, even though the cellular automaton as a whole is able to reliably produce the same behavioral response. I called this *transient induced global response synchronization* (TIGoRS) [356, 359]. This discovery led to what Lumsden et.al. termed Sulis machines [238]. These automata appeared capable of imposing a primitive form of meaning, termed salience, upon their environments and to construct primitive linguistically structured patterns of response accordingly. No explicit internal specification was required, no internal representation. The resulting cognitive-like behaviour was emergent from the automaton-environment interaction. Moreover, the participating constituents are fungible, and in many cases the dynamics that underlies their individual behaviour is irrelevant to understanding their dynamics as part of the collective. They behave like actors in a play, except that the play is implicit in the system-environment interaction rather than being explicitly encoded within the system. These are all features of a process.

Thinking again of a play, an actual play is a mental construct. It exists outside of time and space but can be made manifest in time and space through the intermediary of a director, actors, and a stage crew. A script of a play can be written down, but it forms just words on pages. In itself, it has no time, no space. It is only when it is actualized that it acquires time and space characteristics. But each time this happens, it is a new expression of the play. Different directors, different actors, different stages, different designers, all put different spins on it. All are fungible. Even with the same people involved, each performance is a new performance, and yet each performance is of the play. A play is a process, a coming together of different individuals at different times and in different places, each occurrence transient and unique, yet all linked together by the common thread provided by the script and the playwright.

Living systems, social systems, economic systems, all exhibit rich, complex, adaptive patterns of behaviour, very characteristic of process. Physics on the other hand, has focussed its attention on inanimate matter, which behaves closest to the mathematical object. I began to wonder, following the words of Rosen, whether the problem with physics is that it has taken the notion of object to be fundamental, whereas perhaps it is the notion of process that is fundamental, and objects are merely special kinds of process. The starting point for the thesis thus lay in the possibility that reformulating physics, or at least NRQM, within a process theory framework, might lead to a significant improvement in the structure of the theory. I, of course, am not the first to consider such an idea. Shimony explored this briefly [321] but then abruptly abandoned it, because certain aspects of Whitehead's Process Theory disagreed with certain observations in physics. Others have taken up the challenge [118], but these ideas have never been embraced by the mainstream and progress has been slow.

I have taken literally the challenge provided by Rosen's words, and I asked myself which aspects of complex systems theory might profitably be applied to understanding phenomena at the most fundamental levels. Having studied emergent systems for two decades and having observed evidence of emergence again and again, I felt that the most natural and reasonable starting point was to take emergence to be the most fundamental dynamical principle underlying all of reality. Rather than viewing emergence as special and the reductionist approach as universal, I explicitly turned this on its head. I think of emergence as being the universal means through which simple systems construct more complicated and complex systems, and I think of reductionist situations as being special cases, mostly involving isolated inanimate matter in equilibrium conditions. Therefore I view *all* of the phenomena of experience, indeed all of observable reality, to be a manifestation of emergence (or sometimes supervenience) and I ask what might a lower, ultimate level of reality, capable of giving rise emergently to our observable reality, look like.

Following Whitehead [425], I consider process to be fundamental and the elements of ultimate reality to be the actual occasions generated by process. Observable reality arises in an emergent fashion from actual occasions and from interactions among the processes that generate them. Since observable reality is emergent from actual occasions, these actual occasions are themselves inherently unobservable, even though they do influence the course of observable events and so 'make a difference' .

To motivate this point of view, consider a TV set and its program content. The content of a TV program is separate from the electronic processes that manifest it. The laws of physics describing the behaviours of photons and electrons and semiconductor chips are irrelevant as far as understanding the content of a TV program. They are necessary for that TV program to appear on the screen of the TV, but do not determine the content. One might say that the TV program supervenes on the physical entities that make up the TV, but one could equally well render the TV program on a reel of film and project it onto a wall, and watch it without any TV at all. The TV program knows nothing of the workings of the TV, just as the mind knows nothing of the workings of the brain. In fact the mind is unaware that there even *is* a brain until that brain misbehaves, either through a physiological or structural malfunction.

That actual occasions are held to be inherently and fundamentally unobservable should not be cause for concern. The situation of higher level entities being unaware of lower level entities is commonplace and does not affect their interrelationships one iota. There is no reason to believe or to expect that the level of ultimate reality should be accessible to observation by human observers. It is a conceit of human observers that such a belief hold true. We like to believe in our own, and in Science's, potential omniscience, but reality manages to function perfectly well without our knowledge of it. That everything must be knowable is an assumption, not a necessity, for the Sciences. Nevertheless, the positing of such intrinsically unknowable entities should be done carefully, minimally, with parsimony, and they should make a difference.

Again I refer to Tumulka [403]

> It is often taken as an objection against Bohmian mechanics that it entails the existence of unmeasurable quantities. For example, the velocity of a Bohmian particle cannot be measured if we do not know the wavefunction. *What cannot, not even in principle, be measured*, I hear physicists say, cannot belong to a scientific theory. *Rather, it is to be regarded like angels, ghosts, or the ether.* I would categorize this position as exaggerated positivism, and I find this argument surprising because quantum physicists should know first, of all scientists, that it is wrong.
>
> The quantum formalism itself entails that nature can keep a secret, in the sense that there exist some facts that cannot be revealed by any experiment. To see how, we start from the mathematical fact that different ensembles of wavefunctions (mathematically represented by probability distributions μ over the unit sphere $S(\mathcal{H})$ of Hilbert space) can have the same density matrix $\hat{\rho}$, given by

$$\hat{\rho} = \int_{S(\mathcal{H})} |\Psi >< \Psi| \mu(d\Psi)$$

For example, an ensemble of spin-1/2 particles consisting of 50% spin-up particles and 50% spin-down has the same density matrix

$$\hat{\rho} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

as the ensemble of 50% spin-left and 50% spin-right, or as the ensemble with the spin direction uniformly distributed over all directions. Since the statistics of any quantum experiment depends only on the density matrix, these different ensembles are *empirically indistinguishable*. Nonetheless, they are physically different, since I may have prepared the spin state of every single member of the ensemble, so that I know the state vector of every particle. I can even prove that I know the state vector, and that nature remembers it, by naming, for every member of the ensemble, a direction and predicting with certainty the result of a Stern-Gerlach experiment for the spin component in this direction. Therefore, there is a matter of fact about whether the ensemble is an up-down ensemble or a left-right ensemble; but if I do not tell you, you have no way of determining which it is. It is a variant of this argument to say that one cannot measure the wavefunction of an electron, even though there is, at least in some cases, a matter of fact about what its wavefunction is. (pg. 3247)

The upshot is that it should not be an issue whether or not the actual occasions that comprise the ultimate level of reality are themselves observable or not so long as they exert an influence that makes a difference upon observable events.

In Whitehead's theory, actual occasions come into existence, persist briefly, then fade away, to be replaced by the next actual occasions. Information from the prior actual occasions is passed on, sometimes after transformation, enhancement or editing, to the next actual occasions. There is no interaction between actual occasions that are currently manifesting. The only information transfer is causal, from the causal past through the present to the causal future. In this sense, actual occasions are (causally) local. However, the generation of actual occasions by process itself manifests in a jaunting or saltatory manner and so appears non-local even though no actual information is transferred. Thus there is no violation with relativity.

The basic idea is to postulate an underlying level of ultimate reality consisting of actual occasions and to view all other natural phenomena as emerging from or supervening upon this level. These actual occasions are generated by processes, and in generating the actual occasions these processes also generate space and time and patterns of relationships that give rise to constructs such as mass, energy, position, momentum, angular momentum, spin and so on. These actual occasions are viewed as being fundamentally informational in nature. They are generated by processes but also serve to induce changes in processes which arise as a result of interactions between processes. The processes themselves are viewed as existing outside of space and time. This is analogous to the way in which a play stands apart from any given performance, or a game such as Chess exists apart from any particular mounting of the game. Processes may be active or inactive. Processes can act freely or be in interaction with other processes. Interaction can result in the inactivation of some processes and the activation of others. Interaction can also alter the manner in which processes generate actual occasions, establishing correlations between processes that give rise to non-local effects such as outcome dependence.

There are relationships among the processes that determine which processes may become active or inactive as a result of an interaction. These relationships are formal. Although I do not develop this in any detail in this thesis I imagine that these are probably provided by Lie algebras or Lie groups and constitute a set of symmetry relationships on the collection of processes. Processes are distinguished by virtue of different values of a collection of parameters - such as mass, angular momentum, momentum, spin, energy, lepton number, boson number, particle number and so on. These attributes constrain the types of interactions possible between processes. Processes are thought of as possessors, or perhaps better as generators, of these attributes.

Actual occasions are also assigned parameter values which link them to the processes that generated them. Such information is required to maintain causal consistency and it is just that - information. Actual occasions are thus signifiers rather than possessors of attributes. This ensures that actual occasions are non-contextual. However, such information can only be accessed through an act of measurement, and that involves an interaction between the process generating the actual occasion and the process generating the actual occasions corresponding to the measurement apparatus. The act of measurement is intrinsically contextual.

Actual occasions are discrete. Actual occasions do *not* interact with one another. Only the processes that generate actual occasions interact. Actual occasions merely manifest the outcome of such interactions. Actual occasions do not possess a dynamic of their own. They do not move - they simply arise, persist, and fade away. They themselves are not held subject to the usual laws of physics; rather it is the processes that generate them that are governed by such laws. This is a very important point. If there is some discomfort with such a notion then simply recall the role of virtual particles mediating interactions in QFT. Or think of the generation of a TV image. Objects represented by the image appear to move but the pixels generating them do not. Pixels do not interact - they simply change colour and nothing else. Their rules and those of the depicted images are not the same.

Actual occasions that are currently manifesting form a simultaneous collection of events, that is, they constitute a *transient now*. This model sits firmly within the philosophical tradition known as 'presentism '. Some readers might object that such a notion was proven impossible by such philosophical arguments as that of McTaggart [103] or by the principle of relativity [295]. In Appendix B, I show that this is an entirely plausible construct, compatible with relativity and indeed with everyday experience and practice. The essential relations between actual occasions are via their causal connections. Actual occasions that currently manifest do not interact, do not transfer information among themselves. They are causally isolated and constitute an anti-chain (space-like hypersurface) in some causal manifold. Information between actual occasions is transferred causally and it is only these relations that link actual occasions. As is well known, the causal ordering of a causal manifold is preserved under Lorentz symmetries and is thus relativistically invariant. So too is the causal structure relating actual occasions and the successive generation of actual occasions will be shown to respect relativistic invariance. That it is impossible to assign a universal time valuation to these sets of simultaneously manifesting actual occasions is irrelevant. The only information relevant for an actual occasion is that provided by its past causal cone of prior actual occasions, and it is certainly possible to provide a co-ordinatization of those actual occasions suitable for performing any necessary informational operations. What another simultaneously manifesting actual occasion does with its past causal cone is entirely irrelevant.

In this thesis I have chosen to model actual occasions by means of a "causal tapestry" [381]. I have been developing the notion of a causal tapestry for many years in my work on emergence in complex adaptive systems. They provide a general mathematical framework for studying emergent situations as described within the philosophical setting of archetypal dynamics [380]. Archetypal dynamics concerns itself with emergent situations in which different entities may be understood and described by distinct, mutually irreducible frames of reference termed semantic frames. These semantic frames enable an observer to identify and distinguish various entities, interpret their actions and interactions, and shape interactions of the observer with these entities so as to maintain their coherence. To use a familiar example, a semantic frame provides the frame of reference to ensure that a parent recognizes their child and treats their child in such a way as to foster its growth, development and continued existence, and not to treat it like a stuffed toy.

Physical entities arise in an emergent manner from actual occasions. As a result, no physical entity is capable of directly observing these actual occasions. Actual occasions manifest at a scale vastly below that of any observing entity and they can be experienced only indirectly through their interactions with specialized processes called measurement processes. The reason is fairly obvious. If a physical entity, being emergent upon a collection of actual occasions, were able to resolve individual actual occasions then its physical scale would need to less than the scale of the actual occasions being resolved, and thereby less than the scale of the actual occasions upon which it is emergent. that is simply not possible. Thus the existence of actual occasions may be may be surmised and they must be interpreted within some semantic frame, usually some formal mathematical structure, through which an observer gains understanding. These interpretations are inspired by the actual occasions but not entirely determined by them. There is room for some error and the actual occasions will appear fuzzy or de-localized to a macroscopic observer. More importantly, the physics on the collection of actual occasions is determined entirely by information flow on those occasions and is independent of the interpretations made by the macroscopic observer. The physics on the actual occasions is thus complete and consistent internal to itself. These relationship are expressed within the language of archetypal dynamics.

The fundamental triad of archetypal dynamics distinguishes the realisation of a system (i.e. its manifestation in reality), its interpretation (i.e. the semantic frame that distinguishes, defines, and constrains it), and its representation (i.e. the formal mathematical, linguistic, visual or other medium used to describe the realisation-interpretation relationship). In the present context, the realisation is the 'real' system as it appears *in reality*, the interpretation is the theoretical model used to understand it, the representation of the realisation is by games played out on causal tapestries, while the representation of the interpretation is by a causal manifold $\mathcal{M}$ and its associated Hilbert space $\mathcal{H}(\mathcal{M})$.

A causal tapestry is a formal representation of actual occasions. A causal tapestry is a collection of informons, each of which represents a single actual occasion. An informon has the structure $[n] < \alpha > \{G\}$ where n is a label, α is the interpretation, and G is the informational content of the informon. Beginning from a given causal tapestry one generates a new causal tapestry representing the fading away of the current set of actual occasions and the arising of the next nascent set of actual occasions. This is the action of process. In order to formalize the action of process mathematically within the causal tapestry setting I have chosen to model process as a combinatorial token forcing game. There are several reasons for doing so. First of all, combinatorial games come equipped with a rich set of methods for combining games. In this thesis I use four different forms of sum, $\oplus, \hat{\oplus}, \boxplus, \hat{\boxplus}$, and of product, $\otimes, \hat{\otimes}, \boxtimes, \hat{\boxtimes}$, corresponding to free, exclusive or interactive play, but there are actually many more [162]. This variety of combinations allows for a more nuanced approach to the issue of interactions between processes. Games can be represented by game trees which specify possible moves from any given configuration, and sums and products are specified by their effects on the game trees. The interactive type is actually not a single type but rather is shorthand for a wide variety of types, all defined by the structure of the corresponding game tree. In standard quantum mechanics, interaction is described by the Lagrangian, and there are only two ways in which terms are linked - simple sums or simple products. Wave functions are formed as sums of tensor products of wave functions corresponding to the different systems. The notion of particle number corresponds to the number of informons that may be simultaneously constructed during a given round of play. A process generating a single particle can generate a single informon during any particular round. A process generating N particles generates N informons during a round. Multiple processes generate multiple informons during a single round, each according to their particle number.

The algebraic structure of combinatorial games allows for an identification of particular games with particular numbers and gives rise to the surreal number field [97]. In other cases one can associate games with elements of Lie groups and Lie algebras or other algebraic structures. All interactions lie within a single unifying algebraic structure and there is no need to have two radically different kinds of dynamic taking place. There is no physical 'collapse' taking place. There is a transition among games as a result of different interactions but in all cases there is just a play of games taking place. The informons being generated may vary in terms of the spatial regions into which they embed, and they need not be contiguous, but they most certainly do not collapse.

Restrictions on the game tree that arise due to interactions among different processes, such as occur during the measurement process, give rise to correlations among various observable quantities. This gives rise to outcome dependence in some situations. The non-locality that arises in wave-particle duality and in delayed choice situations is a consequence of the non-local manner in which sites are chosen for the generation of informons (hence actual occasions). This is a reflection that in many games, for example board games, play can take place at any point on the board, not merely adjacent to the point of current play. This does not involve information exchange between informons and so does not violate relativity. It cannot be used for signalling between space-like separated regions. In the causal tapestry model there is never signalling between space-like separated regions. That measurement is a process of interaction between the processes that generate the quantum system and the measurement apparatus gives rise to the apparent contextuality of reality. This is also easily expressed using the language of combinatorial games.

It is important that the reader understand that the use of combinatorial games is purely heuristic and does not imply that reality itself *actually* consists of a collection of combinatorial games. Actual occasions are modeled as being generated by game play but in reality they are thought of as simply coming into being, incorporating the relevant information into their structure, then fading away. The essential features of combinatorial games are their sums, products, the non-locality of game play, and the discreteness of game steps (or at least of game rounds). The role of players is entirely heuristic and no statement about the ontological status of these players is offered or suggested.

The value of the model is that the actual occasions are quasi-local and quasi-non-contextual and the physics is self contained, so that this stands as a complete model in and of itself. It demonstrates that it is possible to have realism at the lowest level. This means that at the lowest level of reality there are actual entities possessing actual characteristics that are local to themselves and observer independent and yet are capable of influencing the generation of observable events. Physics again becomes grounded in something objective even if that something is itself unobservable. Processes act in a discontinuous manner but still utilize only local information. Measurement, being an interaction, is contextual as a result. Observable reality manifests non-local, contextual and subjective aspects because it is emergent from this ultimate reality. For these reasons I have chosen to call this a quasi-local, quasi-non-contextual, realist model of NRQM. The model illustrates that the probabilistic structure of local and non-contextual variables need not be Kolmogorov in structure. In order to achieve this one must give up the principle of continuity and allow for the existence of jump processes (jaunting in the formation of actual occasions) and abandon the idea of time reversibility (more specifically closed time-like curves or temporal loops) and of a block universe and accept a notion of a moving present. In spite of giving up time reversibility in the generation of informons (actual occasions) one can still recover time reversal symmetry as a symmetry in the artificially constructed space of histories of the evolution of the causal tapestries. Indeed my thesis research into order automata showed that is was possible to generate a causal order on a space through an automaton action on the space which was reversible [343]. This is equivalent to a time reversible dynamics generating a causally ordered evolution on a state space, and thus a global time irreversible evolution that is locally time reversible.

The connection to NRQM arises through the process covering map, which associates each process, now viewed as a particular combinatorial token forcing game over spaces of causal tapestries, with a set of elements of Hilbert space. Each play of the game results in the creation of an element of Hilbert space, which is the wave function for that particular evolution of the causal tapestry. The usual NRQM wave function may be thought of as the wave function that would occur under ideal conditions of infinite game play and infinitesimal sampling resolution, none of which are expected to occur in reality. Each play of the game, for a given set of parameter values, gives rise potentially to a distinct wave function. The process covering map gathers these different potential wave functions into a single set. Thus the process covering map is a set valued map. This set can be thought of as a collection of approximate wave functions. As the limit of game play becomes infinite and the sampling resolution becomes infinitesimal, the sets of the process covering map converge to sets having only single points. These points are identical to the usual NRQM wave function.

The model is quite different from the Bohmian and GRW models. Information propagation is causally local - there is no analogue of the Bohmian quantum potential and its troublesome non-locality. This model and that of GRW both involve jump processes. In GRW these are stochastic and determined by the wave function. In my model, these are non-deterministic but not stochastic (taking their meaning from the computer science literature where non-deterministic refers to set-valued functions reflecting free choice and stochastic refers to set valued functions for there is a preassigned probability measure) and they are not determined *by* the wave function but instead *determine* the wave function through the interpolation process. The flashes of GRWf correspond to sequential collapsed versions of the wave function. The jaunts in my model progressively generate a hypersurface in a causal manifold and represent local values of a wave function, the local strength of the generating process, not entire wave functions. The informons in my model represent actual space-time events while in GRW one only has specific wave functions determined - the events, if any, remain unspecified. The wave function in my model is an emergent wave function determined only at the end of game play through interpolation over the entire causal tapestry, whereas in the GRW model the wave function is complete at every individual flash. There are situations (such as with uniform sinc interpolation) in which the process model is similar structurally to certain cellular automaton models [429] but in general the transition rules and the methods of analysis are quite distinct. Cellular automata models focus upon the behaviours of spatio-temporal patterns of activity across the cells of each automaton, whereas in the process model the focus is on the causal manifold and the global function generated on the Hilbert space of the manifold and less so on patterns across the causal tapestry.

I believe that the process model is a significant improvement over Bohm, GRW, and cellular automata models, and presents an even less romantic picture of reality than they do, being local and non-contextual at its lowest levels.

The main goal of the thesis has been to show that it is possible in principle to develop hidden variable models of NRQM which still retain a significant degree of locality and non-contextuality at the lowest levels while still exhibiting non-locality and contextuality at the emergent level of the observer. The process model provides a very general framework for describing dynamical systems. The specific models presented in Chapter 3 and 5 as examples of applications of the process approach are meant to provide an in-principal demonstration suitable for addressing the hidden variables question but are not meant to provide a complete model of every aspect of NRQM. Indeed they apply primarily to the case of single integral spin particles because those can be represented by means of vectors. This is due to the simple fact that the current state of the art in interpolation theory applies to scalar and vector functions whereas its application to spinorial systems is quite under developed. Nevertheless it is reasonable to believe that interpolation techniques can be extended to these as well. Simple multiple particle systems of the type required for the hidden variable theorems are discussed in Chapter 4. More complicated multiple particle systems can, in principle, be described through applications of the algebraic structure of the process space but are not a focus of this thesis.

The models presented are directed specifically to the non-relativistic case. The causal tapestry is a very general construction meant to be compatible with relativistic notions and was chosen with future applications to the relativistic setting in mind. It is a matter for future research, however, to determine whether the process model can be successfully developed to incorporate relativistic and field theoretic features.

The edifice of quantum mechanics was not built in a day and a single thesis cannot possibly encompass all that there is to say about non-relativistic quantum mechanics, let alone the entirety of quantum mechanics including quantum field theory and quantum gravity. Hopefully the reader will find enough of interest in this thesis to spark their curiosity and interest in pushing these ideas further.

If the model (or other models inspired by this approach) helps to put the foundations of physics into better order, then that may allow physics to advance more expeditiously. I think that it is interesting in itself to ask whether our understanding of reality truly rests on solid ground and to test the limits of these ideas by developing alternative models. In this thesis I hope to show that there is indeed some wiggle room in our foundational theories and that fundamental reality may not be as inscrutable as it is normally presented.

Let me end with one comment from Tumulka on why hidden variable approaches may serve better than the standard wave function approach. He writes [403]

> The attitude behind postulating such variables is to be contrasted with the attitude according to which the *wavefunction* describes the state of the matter. The 'description' provided by the wavefunction is, however, in such a vague sense that almost any two physicists disagree about what exactly the reality is like when the wavefunction is such and such. Note how different the sense is

> in which the 'primitive ontology' provides a description of matter. If a theory postulates that matter consists of point particles, and provides the positions of these particles at all times, then it provides a picture that could not be sharper. It may be *wrong*, but there is nothing *vague* about it. But first of all, the primitive ontology makes *explicit* what the reality is, rather than leaving it to everybody's private fantasies. This is a crucial merit of the unromantic pictures. (Sic)(pg. 3250)

The model presented in this thesis is particularly unromantic but I hope that the reader will find it worthy of their attention anyway.

1.3 Organization of the Thesis

For the convenience of the reader, the thesis is constructed as follows:

Part I provides a detailed presentation of the process model together with the main results of the thesis.

Chapter 1 provides a motivational overview of the ideas behind the model, which will hopefully encourage an open mind and a willingness to let go of preconceptions and explore what can be achieved with a shift of viewpoint and of mathematical language.

Chapters 2 and 3 form an integrated whole, and the reader is advised to move back and forth between them. Chapter 2 provides the abstract viewpoint while Chapter 3 describes two specific examples which make the abstract ideas concrete.

Chapter 2 introduces ideas of process and the process theory of measurement. This chapter introduces ideas of process, the algebraic structure of the process space and the important process covering map, which serves as the link between the space of processes and the space of wave functions of NRQM. It introduces the process theory of measurement which is essential for understanding the process approach to the paradoxes. It describes how probabilities are generated, and provides an interpretation of the wave function as a real wave whose values describe the strength of a process in a given region of space-time.

Chapter 3 introduces the process model of NRQM. It provides a detailed description of the combinatorial game used to implement the process. Two different, though related, strategies are described in detail and an argument is provided to show the relationship between the process model and NRQM, in particular to show how NRQM may be viewed as an idealization of the process model when infinite information and infinitesimal sampling may be assumed. It illustrates the details of the causal tapestry structure. It shows how the causal tapestry provides a self contained model of the physics and how the usual structures of physics, the causal manifold of space-time and the Hilbert space of wave functions may be viewed as emergent interpretations rather than actualities. The issue of superpositions is dealt with from the process point of view.

Chapter 4 shows how each of the three types of hidden variable theorems is dealt with in the process framework, and how one is able to have a model with local and non-contextual hidden variables without violating these theorems. An argument is provided showing how several of the paradoxes, particularly wave-particle duality, Schrödinger's cat, and entanglement can be resolved.

Chapter 5 presents a general framework for describing and categorizing various reality game strategies, all capable of realizing NRQM as an asymptotic limit. It also deals with some questions ignored in the simpler presentation in Chapter 2, particularly the problem of initial conditions and how it might be possible to generate a wave function de novo.

Part 2 provides six appendices of philosophical and mathematical results necessary to understand the material in the thesis.

Appendix A reviews the basic features of the hidden variable theorems, their logical structure, current interpretations and implications. It also reviews the measurement problem and some relevant models for attempting its resolution.

Appendix B explores the four main implicit assumptions that underlie most of the arguments described in Appendix A. These are the assumptions of Kolmogorov probability structure for local and non-contextual hidden variables, the principle of continuity, the principle of time reversibility, and the concept of reality as comprised of objects.

Appendix C presents the basic ideas of process theory, archetypal dynamics and emergence. It is not meant to be a learned philosophical discussion of the subjects. Rather, only the essential ideas relevant to their usage in the model are provided. References to more extensive background literature are provided for the inquisitive reader.

Appendix D presents the mathematical structure of causal tapestries and of the reality game. The proof that the causal tapestry structure respects Lorentz symmetries is provided there.

Appendix E presents the basic details of combinatorial game theory and the idea of forcing, which are used to express the actions of process in generating causal tapestries. Again this is far from exhaustive. The literature in combinatorial game theory and logic is vast, dating back more than a century and references to that literature again are provided.

Appendix F presents a basic introduction to the ideas and methods of interpolation theory, particularly sinc interpolation. Brief mention is made of other approaches to interpolation but sinc interpolation is used exclusively in the model because of its elegance, simplicity and intrinsic locality. Issues related to its limitations are discussed as is a brief foray into error theory. Again the modern literature on interpolation is about a century old and key additional references are provided.

The material in Appendix C is derived mostly from the paper Sulis,W.: Archetypal dynamics, emergent situations, and the reality game. Nonlinear Dynamics, Psychology, and Life Sciences. 14(3), 209-238 (2010) while that in Appendix D comes from Sulis,W.: Causal tapestries for psychology and physics. Nonlinear Dynamics, Psychology, and Life Sciences. 16(2), 113-136 (2012). Some material in Chapter 3 appears in Sulis, W.: A Process Model of Quantum Mechanics. J. Mod. Phys. (2014).

The section of references includes not only works cited directly in the thesis but a host of related works which served in various ways to provide background ideas and stimulation and foils in the development of the process model. The basic ideas underlying the process model such as archetypal dynamics and causal tapestries date back to the period from 1999-2007 and appear in a variety of proceedings. Earlier versions of the causal tapestry model were tried during period 2007-2010 while the process model had its gestation from 2010-present. A listing of many of these earlier publications is provided in the references section.

Part I

The Process Approach to Quantum Mechanics

Chapter 2

Process and Measurement

2.1 Actual Occasions and Informons

This chapter introduces the basic concepts of Process Theory needed to understand the underlying conception of the model. More detailed philosophical material is provided in Appendix C, particularly Archetypal Dynamics, a complex systems research program that I began in 1999, which provides the basis for the construction of the causal tapestry representation. The algebra of process is described here along with the pivotal notion of measurement defined from within a process perspective.

The model is based upon several guiding principles based partly upon the Process Theory of Whitehead [425] as described by Shimony [321]. The fundamental primitive construct of Process Theory is the *actual occasion*, which has the following characteristics:

1. Actual occasions are generated by process.
2. Actual occasions underlie both space-time points and the entities manifesting there.
3. Actual occasions are discrete, yet fuzzy or delocalized - they manifest below the level of the entities emergent from them making them ill-defined and unresolvable. They may be finite or infinite in number.
4. Actual occasions are held to possess 'parts' only in so far as the concept serves a heuristic purpose for describing them or their generation - they themselves are holistic entities and must be treated as wholes in any subsequent calculation.

5. Actual occasions do not interact - only the processes that generate them do.
6. Actual occasions are associated with properties that are inherited from the processes that generate them but which are not directly observable. These properties in turn contribute to the determination of a measured property by virtue of a measurement process.
7. Actual occasions express the local strength of the generating process through the local value of their respective wave functions.
8. Actual occasions provide a fuzzy local contribution to the wave function
9. Observable entities, properties, and probabilities are all emergent from actual occasions.

Actual occasions cannot be directly observed, hence their existence and their influence on and relationship to observable entities must be inferred. More about this will be said in Chapter 4. The framework used for understanding this is termed Archetypal Dynamics (see Appendix C). Archetypal dynamics has been used to understand information flow in complex systems, especially in the presence of emergence. Since space-time and physical entities are held to be emergent from actual occasions, this appears to be a natural application. Archetypal dynamics distinguishes between realizations, which represent some system, and interpretations, which agents utilize to interact with the system. In the case of physical systems, the semantic frame is usually provided by physical theory. In the case of quantum mechanics this is taken to be a causal manifold $\mathcal{M}$, representing space-time, and a Hilbert space, $\mathcal{H}(\mathcal{M})$ representing physical entities. These interpretations are extrinsic to the actual occasions, being imposed by some observer, or if one prefers, by some theoretician attempting to analyze the behaviour of the system.

The actual occasions described in process theory are modeled as informons within a causal tapestry. A causal tapestry is a collection of elements called informons, and each informon represents a current actual occasion. More precisely:

Definition: A causal (event) tapestry $\mathcal{I}$ is a causal tapestry with attributes, that is, a 4-tuple (L, K, M, I_p) where K is an index set of cardinality κ, $M = \mathcal{M} \times \mathcal{H}(\mathcal{M}) \times D$, a mathematical structure with $\mathcal{M}$ a causal manifold, $\mathcal{H}(\mathcal{M})$ a Hilbert space over $\mathcal{M}$, D a space of descriptors (properties), I_p a collection of prior causal tapestries.

Informons are elements of L having the form $[n] < \alpha_n > \{G_n\}$ where $n \in K$, $\alpha_n \in M$ and G_n is an acyclic directed graph whose vertex set is a subset of I_p. The interpretation α_n is a tuple $\alpha_n = (\mathbf{m}_n, \phi_n(\mathbf{z}), \mathbf{p}_n)$ where $\mathbf{m}_n \in M$, $\phi_n \in \mathcal{H}(M)$, $\mathbf{p}_n \in D$.

The pair $\mathbf{p}_n, G_n$ consist of *intrinsic* features, meaning that they consist of objective characteristics of each informon that are independent of any observer. The most important of these is Γ_n, the process strength at n. Each informon of $\mathcal{I}$ must be interpreted by some external observer within some formal theoretical frame. The most relevant constructs within NRQM are a spatio-temporal structure and wave functions to represent physical entities. Thus to each informon there will also be associated a pair $\mathcal{M}, \mathcal{H}(\mathcal{M})$ of *extrinsic* features, meaning that they may be observer dependent though not arbitrary. For the systems of interest in this thesis, $\mathcal{M}$ is a causal manifold, and $\mathcal{H}(\mathcal{M})$ is a function (state) space, either Banach or Hilbert. These are understood as being externally imposed upon the informons by some external observer and are not intrinsic features. Moreover, any dynamic imposed upon the informons must not reference these external interpretations.

Each informon n thus embeds into the causal manifold $\mathcal{M}$ at a point $\mathbf{m}_n$. Each $\phi_n(\mathbf{z})$ is a wavelet contribution, a local $\mathcal{H}(\mathcal{M})$ interpretation giving rise to a global $\mathcal{H}(\mathcal{M})$ interpretation on $\mathcal{M}$.

Figure 2.1 depicts the general structure of an informon. Figure 2.2 illustrates a sample content set. Figure 2.3 illustrates the notion of the causal embedding. Figure 2.4 illustrates the local $\mathcal{H}(\mathcal{M})$ interpretation.

In the non-relativistic case discussed in this thesis, the causal manifold $\mathcal{M} = \mathbb{R} \times \mathbb{R}^3$ with causal order given by $(t_1, x) \prec (t_2, y)$ iff $t_1 < t_2$. The local contribution takes the specialized form of a sinc wavelet

$$\phi_n(\mathbf{x}) = \Psi(\mathbf{m}_n) T_{\mathbf{m_n}} sinc_{t_P l_P}(\mathbf{x})$$

where $sinc(x) = \sin x/x$ and for $\mathbf{x} = (t, x, y, z)$,

$$sinc_{t_P,l_P}(\mathbf{x}) = sinc\left(\frac{\pi t}{t_P}\right) sinc\left(\frac{\pi x}{l_P}\right) sinc\left(\frac{\pi y}{l_P}\right) sinc\left(\frac{\pi z}{l_P}\right)$$

and for $\mathbf{m} = (t', x', y', z')$, the 4-d translation operator $T_{\mathbf{m}} sinc_{t_P,l_P}(t, x, y, z) =$

$$sinc\left(\frac{\pi(t-t')}{t_P}\right) sinc\left(\frac{\pi(x-x')}{l_P}\right) sinc\left(\frac{\pi(y-y')}{l_P}\right) sinc\left(\frac{\pi(z-z')}{l_P}\right)$$

The global $\mathcal{H}(\mathcal{M})$ interpretation $\Phi_{\mathcal{I}}(\mathbf{z})$ is generated via

$$\Phi_{\mathcal{I}}(\mathbf{z}) = \sum_{n\in\mathcal{I}} \phi_n(\mathbf{z}) = \sum_{n\in\mathcal{I}} \Psi(\mathbf{m}_n) T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z})$$

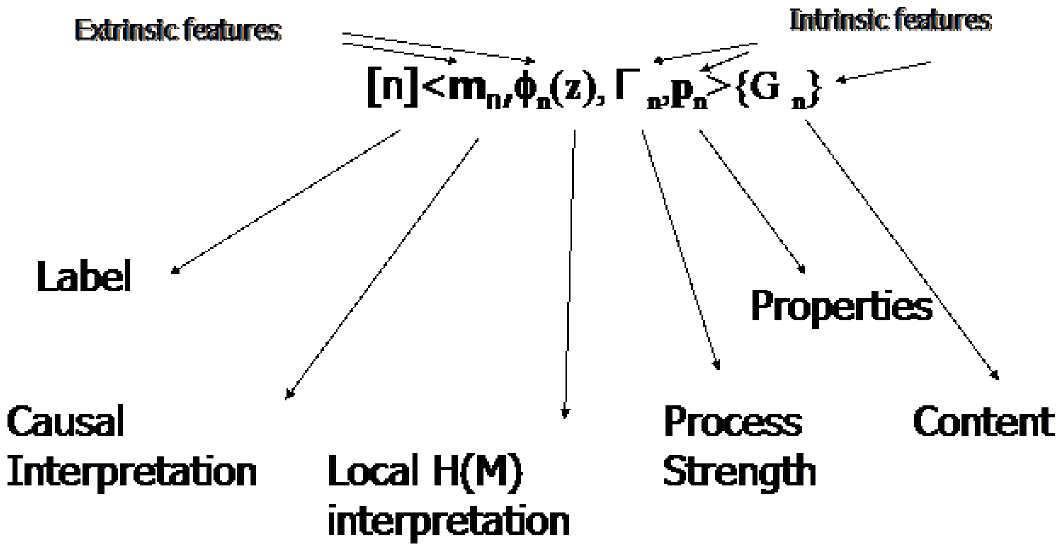

Figure 2.1: The General Structure of an Informon.

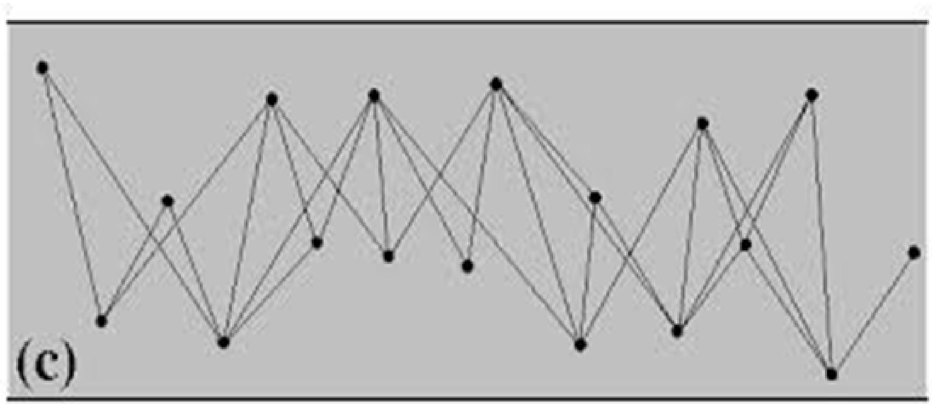

Figure 2.2: General Structure of the Content Set: a causally ordered collection of informons from prior causal tapestries contributing information to the current informon, usually equipped with a (spatial) metric d and a causal function D (i.e. causal distance).

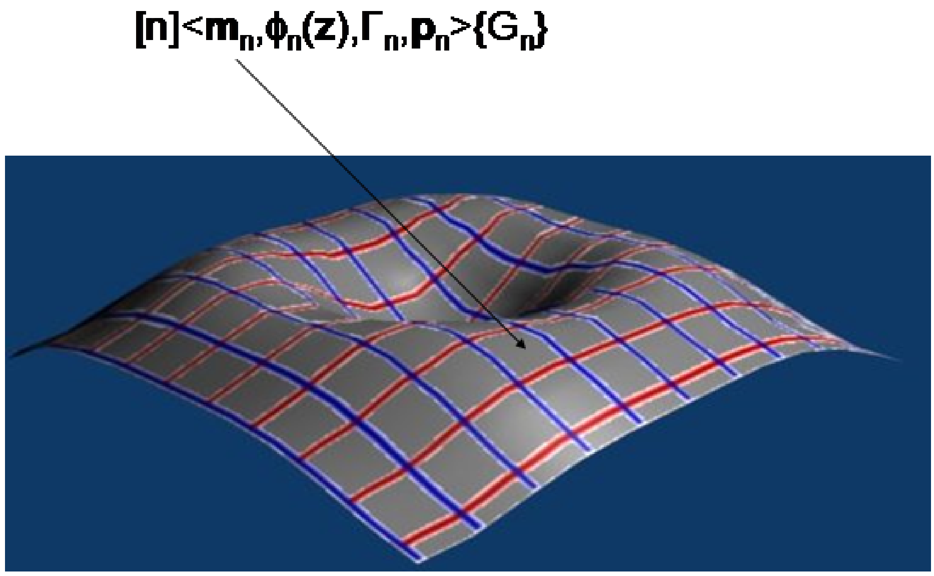

Figure 2.3: The Causal Interpretation: each informon has an interpretation as an element of a causal manifold.

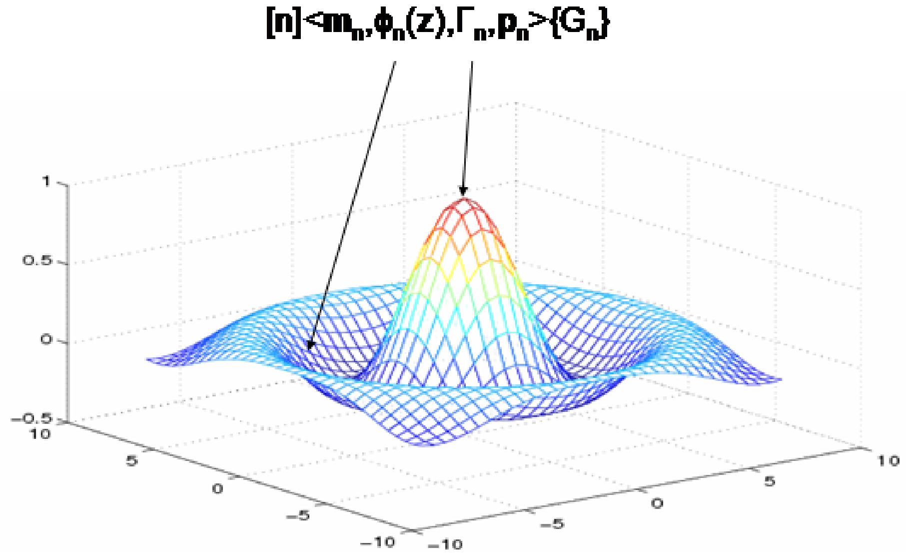

Figure 2.4: The Local H(M) Interpretation: each informon is interpreted as providing a local contribution to a wave function $\Psi(\mathbf{z}) = \Gamma_{\mathbf{m}} T_{\mathbf{m}} g(\mathbf{z})$ where in the non-relativistic model, g is a sinc function sin x/x.

For later reference, I will say that $\Phi_{\mathcal{I}}(\mathbf{z})$ is generated on $\mathcal{I}$ or on the sublattice of $\mathcal{M}$ given by $L_n = \{\mathbf{m}_n | n \in \mathcal{I}\}$ or on the hypersurface $\mathcal{M}_{\mathcal{I}}$ in $\mathcal{M}$ generated by L_n. Note, however, that $\Phi_{\mathcal{I}}(\mathbf{z})$ as a function is defined on all of $\mathcal{M}$. I shall often restrict $\Phi_{\mathcal{I}}(\mathbf{z})$ to the hypersurface $\mathcal{M}_{\mathcal{I}}$ when doing specific calculations. In such a case I shall denote this by $\bar{\Phi}_{\mathcal{I}}(\mathbf{z})$. In keeping with the present centric ontology one can really only speak about $\bar{\Phi}_{\mathcal{I}}(\mathbf{z})$ but mathematically $\Phi_{\mathcal{I}}(\mathbf{z})$ provides a contribution to a global function defined over the entire manifold. Following a succession of processes, this global function takes the form $\sum_i \Phi_{\mathcal{I}_i}(\mathbf{z})$.

This restriction when applied in the present example is obtained by setting $t = t'$ so that the temporal sinc terms vanish. In that case for $\mathbf{m} = (t', x', y', z') = (t', \mathbf{m}')$ and $\mathbf{z} = (t', x, y, z) = (t', \mathbf{x})$

$$T_{\mathbf{m}} sinc_{t_P l_P}(\mathbf{z}) = T_{\mathbf{m}'} sinc_{l_P}(\mathbf{x})$$

where $T_{\mathbf{m}'}$ is the 3d-translation operator defined in the obvious manner.

Each causal tapestry can be thought of as a representation of a transient now, a space-like hypersurface within a causal manifold. Each causal tapestry is generated by a process.

An intuitive sketch of how these various elements fit together to form a tapestry is given in Figure 2.5

It is assumed that some collection I_p of prior causal tapestries exists, but these tapestries are never referenced directly, appearing only within the content sets of the current informons, or as incorporated in the content sets of the nascent informons. The fading of the current causal tapestry $\mathcal{I}$ is recognized by the replacement of the current past causal tapestry collection I_p with the new collection $I_p \cup \{\mathcal{I}\}$. In keeping with the process viewpoint, past information in the form of prior informons appears only as content of informons, never as current informons.

This fading of a prior tapestry and emergence of a nascent tapestry which forms a compound present is depicted in figure 2.6.

2.2 Process

Informons are understood to be generated by processes. Processes possess only algebraic properties: they generate space-time and so cannot be situated in space-time. A process may be active, in which case it acts in a series of rounds to generate informons, or inactive. Each process $\mathbb{P}$ is described by several parameters: one or more tuples of properties

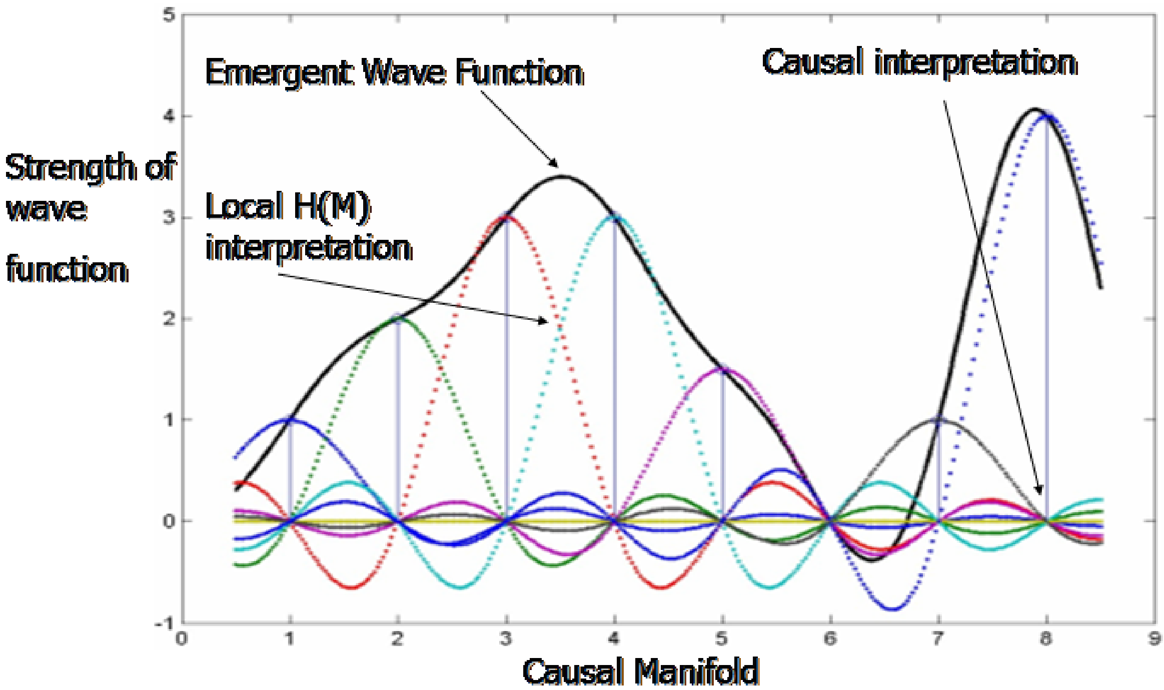

Figure 2.5: Intuitive Picture of a Causal Tapestry.

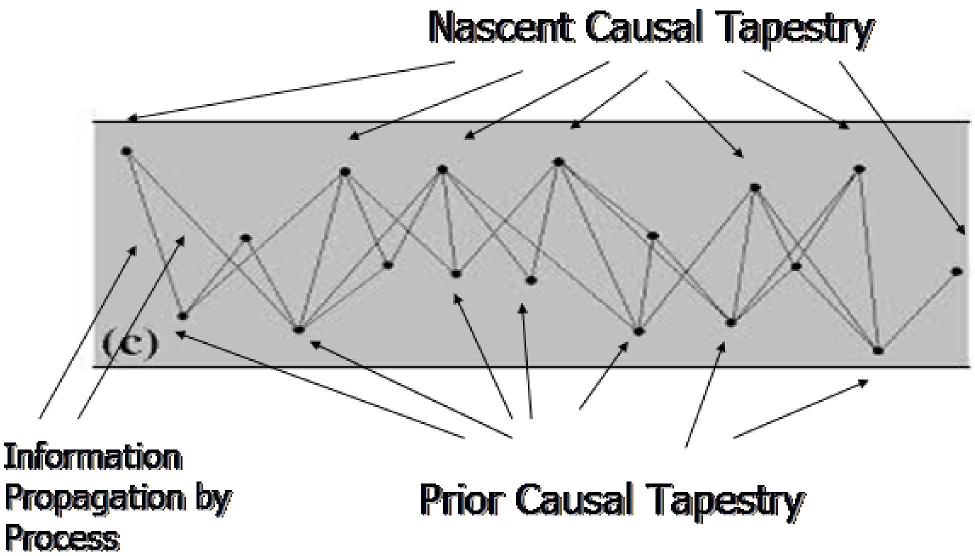

Figure 2.6: The Compound Present.

($\mathbf{p} \in D$), the number of informons generated during a round (R), the number of previous informons whose information is incorporated into a nascent informon (r), the number of rounds needed for a generation (N), the temporal and spatial scales of informons (t_P, l_P). These scales could be universal, applying to all physical entities simultaneously, or they could be individual, applying to each entity separately and linked to the energy ($E \propto 1/t_P$) and momentum ($p \propto 1/l_P$) of the entity. It is left for future experimentation to determine which assumption is more consistent with observation.

For the concept of a property of a process I have in mind either intrinsic characteristics such as charge, spin, lepton number or baryon number, color, or conserved dynamical properties such as mass-energy, linear momentum or angular momentum. I think of the informon as carrying a record of the properties of the process that generated it while the process itself is the bearer of the property since it is the generator of the record. Features such as spacetime location, angular position are associated with the causal interpretation. The distinction will become important later when considering sums of processes.

A process creates a single generation of informons and the history of a system comprises a succession of such generations.

A system unfolds as a succession of such present moments.

$$\cdots \mathcal{I}_n \stackrel{\mathbb{P}_n}{\rightarrow} \mathcal{I}_{n+1} \stackrel{\mathbb{P}_{n+1}}{\rightarrow} \mathcal{I}_{n+2} \cdots$$

The triple of current generation $\mathcal{I}_k$, process $\mathbb{P}_k$, and nascent generation $\mathcal{I}_{k+1}$, forms what philosophers term a compound present.

Each generation $\mathcal{I}_n$ forms a discrete space-like slice of space-time, and being an antichain, embeds into $\mathcal{M}$ as a discrete sampling $\{\mathbf{m}_{n_i}\}$ of a space-like hyper-surface. They also form discrete contributions $\{\phi_{n_i}(\mathbf{z})\}$ to a global function $\Phi(\mathbf{z})$ defined on the causal manifold by $\Phi(\mathbf{z}) \approx \sum_i \phi_{n_i}(\mathbf{z})$ (although $\Phi(\mathbf{z})$ is defined on $\mathcal{M}$ the approximation is best when restricted to the embedding hyper-surface). When the local strengths can be understood as corresponding to the values of some NRQM wave function $\Psi(\mathbf{z})$ at the points $\{\mathbf{m}_{n_i}\}$, it may be the case that the interpolated function $\Phi(\mathbf{z}) \approx \Psi(\mathbf{z})$, especially under certain asymptotic limits. In such a case the dynamics on the causal tapestry may be thought of as a self contained and consistent physical system for which the standard NRQM wave function serves as an idealization under certain limits. Applying the same interpretations to the complete history I_∞ yields a discrete version of $\mathcal{M}$ and a discrete sampling of a global wave function on $\mathcal{M}$. Standard quantum mechanics can be viewed as an idealization or as an effective theory in the limit of infinite information ($r, N \rightarrow \infty$) and infinitesimal spacing ($t_P, l_P \rightarrow 0$). The conditions under which this is possible depend upon the form of the generating function g for the local Hilbert space contribution and the geometry of the embedding into $\mathcal{M}$, and the effectiveness of the interpolation can be determined from various interpolation theories [201, 436]. This will be clearer following the examples given in Chapter 3.

The concept of a primitive process is meant to capture the idea of a single process generating a single physical entity such as a single fundamental particle. A *primitive* process is thus defined as a process which generates a *single* informon during a single round ($R = 1$). Intuitively, the action of a primitive process is to generate, one by one, a succession of informons $n_1, n_2, \ldots$, thus forming a generation $\mathcal{I}_n$. The sequential generation of informons by a process is a key feature of the model. It is essential that informons be generated sequentially to ensure that interactions between processes, especially with measurement processes, be triggered by single informons and thereby exhibit both quantization as well as determinate properties. Measurements are triggered by single informons, resulting in a discrete transfer of information, including that concerning properties such as energy and momentum, and thus imparting a particle-like quality to interactions. At the emergent level, however, the process generates a discrete sampling of a physical wave, thus imparting a wave-like quality to its behaviour. Physical entities from the process perspective possess both particle and wave properties depending upon the scale.

Figure 2.7 depicts the action of a process.

The relationship between process and time is open to multiple interpretations and is worthy of deeper philosophical analysis. Process may be viewed as occurring *outside* of space-time, generating a causal space which can be interpreted as a 4-dimensional causal manifold, one dimension of which the observer associates with time. Process may equally well be considered *as* time, as expressing the arrow of time, and again generating a causal space interpretable as a 4-dimensional causal manifold where the observer associates one dimension with this underlying time. Time could be the usual time, with processes generating the causal space on a near infinitesimal time scale, so that to the observer the generation of the causal space appears instantaneous. Another possibility is that process occurs following a separate, second time, akin to that suggested in the two-time physics of Bars [38] or in the stochastic quantization of Parisi and Wu [104]. Regardless, it is suggested that the compound present formed by the triple $\cdots \mathcal{I}_n \stackrel{\mathbb{P}_n}{\rightarrow} \mathcal{I}_{n+1}$ forms the basis for the personal experience of time and of a moving present. Future research may suggest which, if any, of these possibilities is most reasonable.

Processes are generally considered to act non-deterministically, a term used in computation theory to mean that actions are described by set-valued maps without any intrinsic probability structure. Probabilities arise through two mechanisms: combinatorial prolif-

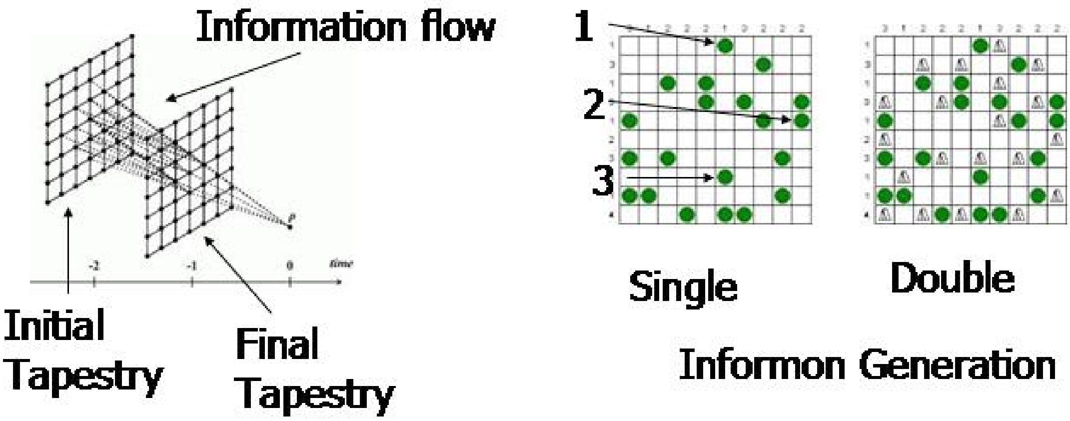

Figure 2.7: Sketch of Process Action: information flows from initial to final tapestry. Informons may be generated singly (single particle case), or multiply (multi-particle case). The numbers denote the order of generation.

eration, similar to the case for iterated function systems, and coupling of processes, in particular through couplings to measurement processes. Of course one can consider deterministic or stochastic processes but these are of less interest. The theory of interaction and measurement will be discussed in Sections 2.8 to 2.10.

In the case that the Lagrangian uses generalized coordinates there may be subtleties in the interpolation procedure depending upon the geometry of these coordinates but that would take this discussion too far afield. It is also important to note that when process is represented by means of combinatorial games as described below, the process covering map will become dependent not only upon the particular interpolation scheme utilized but also upon the particular strategy used to implement the game.

Conservation laws and symmetries applied to the properties of processes provide a set of algebraic constraints upon possible interactions among processes. These are inherited by the wave functions through the process covering map suggesting that processes are primary and wave functions secondary. Quantum mechanics may be best viewed as an effective theory, valid under certain asymptotic limits, but not necessarily the final theory.

In direct analogy to the representation theory of groups and algebras, it proves highly beneficial to have a concrete representation for processes. This facilitates intuition, calculation and theory development. The particular representational framework used in this thesis is that of a combinatorial forcing game with tokens called a reality game. Each informon is generated by a round of game play. The causal tapestry is the collection of informons created by the end of game play and forms a single generation. The end of game play may be determined internally by game parameters or externally by the initiation of an interaction with another game (process). At that point the current game (process) becomes inactive, the next game (process) active, the informons of the current tapestry become information for the next tapestry, and the generation process begins anew under the new process.

2.3 Process and the Wave Function

The standard interpretation of the wave function is based upon Born's rule [1] which associates certain calculations involving the wave function with certain probability calculations. In NRQM, each wave function is normalized to the value 1 and in the position representation the probability that a particle lies within some spatial region R is given by $P(R) = \int_R \Psi^*\Psi dV$. The expectation value of a measurement operator A is given as $E(A) =< \Psi|A|\Psi >$, and so on. This interpretation has proven to be powerful and useful and is now thoroughly ingrained in the minds of most (all) quantum researchers. As a calculational tool, the wave function together with this interpretation has had unparalleled success. Nevertheless philosophers and researchers working in quantum foundations have continued to explore whether or not that is all that there is to say about the wave function [12]. The central problem is whether the wave function should be viewed in purely epistemological terms as placing limits on the knowledge of the quantum system that is available to an external observer or in purely ontological terms as representing some actual physical entity which manifests in reality [27, 93, 298]. Spekkens has written extensively on this subject from a quantum information perspective [173].

Most quantum theorists treat the wave function in the epistemological sense as a calculational tool, reflecting either uncertainty in our knowledge of the quantum system or uncertainty in the behaviour of the quantum system itself [142]. Some researchers, however, have asked whether the wave function could be ontological in nature [271] and recently some have even offered a proof showing that the wave function *must* be ontological [298, 36]. Asserting the reality of the wave function creates problems for the probability interpretation since a probability is not really a thing in itself but rather a measurement of the relative frequency with which some other thing or things will appear under repeated observation, or a degree of confidence or belief in something occurring. Associating a probability with a belief raises even more perplexing problems since how can one wave function manifest the degree of belief of a myriad of disparate observers. A probability needs to be a probability of some thing, and even if the wave function were somehow a probability made real, what would be the thing that the probability referred to and how would that associate to the wave function? Indeed the wave function can be used to calculate the expectation value of *any* measurement operator and so seems to generate the probability distribution of many different things.

Apart from its interpretation there is another issue associated with the wave function, which is whether or not it indeed possesses all of the information required to understand, or at least predict, all of the essential features and behaviours of a quantum system. That is the question of the completeness of the wave function formulation of quantum mechanics. Questioning the completeness of quantum mechanics led Einstein, Podolsky and Rosen to formulate their famous paradox, which in turn led to all of the hidden variable theorems as described in Appendix A.

Quantum mechanics is generally held to be complete in so far as its ability to calculate probabilities and expectation values. But there are subtleties of dynamics that are not easily captured by the standard functional analytic formulation of quantum mechanics. Consider the issue of being 'bound '. Classically it is a fairly straightforward matter to determine whether a particle is bound to another particle because the trajectory of the bound particle will form a closed path with the binding particle lying in the interior of the path. In quantum mechanics this is not so straightforward since particles do not follow trajectories that can be mapped. A free or a bound particle can, in principle, be found anywhere in space. Its mere detection says little about its dynamical state. A detailed determination of the shape of the wave function would help but is unfeasible. Moreover, a free particle could be stationary and have a spherically symmetric wave function just like a bound particle. The main difference is that the probability of the free particle being near the centre is fairly large while for the bound particle is it fairly small. A free particle can propagate but how does one distinguish the random motion of a bound particle and the propagation of a free particle, when both can appear more or less anywhere at any time? In the case of a spherical potential, the potential extends throughout all of space and so in considering when a free particle becomes bound it is not clear when exactly one is to apply the bound equation and not the free particle equation. In principle the free particle could become bound anywhere and at any time. And if it is bound to one particle could it not also become bound to another particle? To every other particle? The equation describing the dynamics of the particle must change between free and bound conditions and so an additional consideration must come into play to determine when this takes place. What is this additional consideration and how does a particle 'know' when to apply it?

The classical interpretation of the wave equation is that it provides a probability distribution for the position of the quantum system. More precisely it provides a probability distribution for a detection by a position measurement apparatus, said detection usually attributed to the presence of a particle in that location at that time. This is not a problem given an ensemble view of the wave function, whereby repeated measurements of an ensemble of identically created particles are conducted and the probability distribution of those measurements calculated. There are problems when one wishes to attribute the probability distribution to a single particle or to attribute a physical reality to the wave function much as one attributes reality to the electromagnetic wave function. In the case of the latter there are demonstrable effects having an electric and magnetic character which can be attributed to the electromagnetic wave so its reality is not really questioned anymore. The Schrödinger wave functions are quite different in character. Although Aharonov and Vaidman [7] have suggested that the wave function of a single particle could be detected using quantum non-demolition measurements, it is only certain statistical measures that can be detected, not the wave function itself.

Consider again the situation of a particle in a spherical potential, this time in the bound state. If the particle is in an eigenstate of the Hamiltonian, say being in energy level n, with angular momentum l and spin angular momentum m, then the wave function takes the form

$$\Psi_{nlm}(r,\theta,\phi) = Ae^{-\frac{i}{\hbar}E_n t} R_{nl}(r) P_l^m(\cos\theta) e^{im\phi}$$

where $R_{nl}(r)$ is the radial wave function (real valued), P_l^m is the associated Legendre polynomial (real valued) and A the normalization constant.

The probability distribution for this particle is given by

$$\Psi^*_{nlm}\Psi_{nlm} = A^2 e^{\frac{i}{\hbar}E_n t} R_{nl}(r) P_l^m(\cos\theta) e^{-im\phi} \times \\ e^{-\frac{i}{\hbar}E_n t} R_{nl}(r) P_l^m(\cos\theta) e^{im\phi} = \\ A^2 R_{nl}^2(r)(P_l^m(\cos\theta))^2$$

Now the Hamiltonian in this case is time independent and so one would expect that the probability distribution would also be time independent and that is indeed the case.

Consider now the case in which the particle is in a superposition of adjacent energy levels. The wave function is this case is given by

$$\frac{1}{\sqrt{2}}\Psi_{nlm} + \frac{1}{\sqrt{2}}\Psi_{(n+1)l'm'} = \\ Ae^{-\frac{i}{\hbar}E_n t} R_{nl}(r) P_l^m(\cos\theta) e^{im\phi} + \\ Be^{-\frac{i}{\hbar}E_{n+1} t} R_{(n+1)l'}(r) P_{l'}^{m'}(\cos\theta) e^{im'\phi}$$

The probability distribution in this case is given as

$$
\begin{aligned}
P(r,\theta,\phi) = (1/2)A^2R_{nl}^2(r)(P_l^m(\cos\theta))^2+ \\
(1/2)B^2R_{(n+1)l'}^2(r)(P_{l'}^{m'}(\cos\theta))^2+ \\
Re\{ABe^{-\frac{i}{\hbar}(E_n-E_{n+1})t}R_{nl}(r)P_l^m(\cos\theta)\times \\
R_{(n+1)l'}(r)P_{l'}^{m'}(\cos\theta)e^{i(m-m')\phi}\} = \\
(1/2)A^2R_{nl}^2(r)(P_l^m(\cos\theta))^2+ \\
(1/2)B^2R_{(n+1)l'}^2(r)(P_{l'}^{m'}(\cos\theta))^2+ \\
Re\{ABe^{-\frac{i}{\hbar}(E_n-E_{n+1})t-\hbar(m-m')}R_{nl}(r)P_l^m(\cos\theta)\times \\
R_{(n+1)l'}(r)P_{l'}^{m'}(\cos\theta)\} = \\
(1/2)A^2R_{nl}^2(r)(P_l^m(\cos\theta))^2+ \\
(1/2)B^2R_{(n+1)l'}^2(r)(P_{l'}^{m'}(\cos\theta))^2+ \\
AB\cos(-\frac{i}{\hbar}(E_n-E_{n+1})t-\hbar(m-m'))R_{nl}(r)P_l^m(\cos\theta)\times \\
R_{(n+1)l'}(r)P_{l'}^{m'}(\cos\theta)
\end{aligned}
$$

In this case even though the Hamiltonian remains time independent the probability distribution function now acquires a temporal fluctuation by virtue of an interaction term between the two eigenstates. Thus one no longer has a stationary probability distribution. However, the time average of this probability distribution is

$$
P'(r,\theta,\phi) = (1/2)A^2R_{nl}^2(r)(P_l^m(\cos\theta))^2 + (1/2)B^2R_{(n+1)l'}^2(r)(P_{l'}^{m'}(\cos\theta))^2
$$

which is the usual probability distribution expected from combining the individual distributions. The loss of stationarity would appear to make any attempt to determine this probability distribution experimentally either difficult or impossible. Even in the case of a non demolition experiment it would be impossible to determine the distribution without synchronizing position sampling to the frequency of the fluctuation and without knowing the phase delay, both of which would require measuring the differences in energy levels and spin angular momenta between the two states which would appear to require a demolition experiment.

If one attempted to measure the probability distribution with a single particle this would have to be done at a series of distinct times, say $t_1, \ldots, t_n$. The functions being sampled at each time would differ, being

$$
\begin{aligned}
&(1/2)A^2R_{nl}^2(r)(P_l^m(\cos\theta))^2 + (1/2)B^2R_{(n+1)l'}^2(r)(P_{l'}^{m'}(\cos\theta))^2 + \\
&\qquad AB\cos(-\frac{i}{\hbar}(E_n - E_{n+1})t_1 - \hbar(m-m'))R_{nl}(r)P_l^m(\cos\theta)\times \\
&\qquad R_{(n+1)l'}(r)P_{l'}^{m'}(\cos\theta), \ldots, \\
&\qquad (1/2)A^2R_{nl}^2(r)(P_l^m(\cos\theta))^2 + \\
&\qquad (1/2)B^2R_{(n+1)l'}^2(r)(P_{l'}^{m'}(\cos\theta))^2 + \\
&\qquad AB\cos(-\frac{i}{\hbar}(E_n - E_{n+1})t_n - \hbar(m-m'))R_{nl}(r)P_l^m(\cos\theta)\times \\
&\qquad R_{(n+1)l'}(r)P_{l'}^{m'}(\cos\theta)
\end{aligned}
$$

If one happened to be sampling at the same frequency as the fluctuation, then one would obtain the mean distribution shifted by a systematic drift term

$$\cos(i\hbar(m-m'))R_{nl}(r)P_l^m(\cos\theta)R_{(n+1)l'}(r)P_{l'}^{m'}(\cos\theta)$$

.

If one knew the phase delay one might offset it, obtaining the average distribution. If one sampled the times uniformly and randomly, then these fluctuations would, on average, cancel each other out, again leaving the average distribution. However, the average distribution is *not* the wave function, since the fluctuating term is not simply a random variation but rather an integral part of the wave function. Indeed the mean wave function is what would be expected from Kolmogorovian probability theory, which we already know to be inconsistent with quantum mechanics.

In this case we see that the only way in which the actual wave function can be detected is if it were possible to carry out a series of quantum non-demolition experiments on an ensemble of particles, not a single particle. One could not simply measure the frequency with which particles appear since such a distribution would actually have to be measured over time, and thus one would not obtain the actual distribution but only a time averaged version. The actual distribution would require an ensemble of particles whose positions could be sampled simultaneously at repeated times, the frequencies being determined for each individual time. The fluctuating distribution thus has meaning only in relation to an ensemble of particles since it is only with an ensemble that it can be measured at all. It is not at all clear how attributing a probability distribution to a single particle in this case would make any sense.

These considerations suggest problems in the interpretation of the wave function, at least in so far as single particles are concerned. For the most part, treating the wave function as an expression of ensemble behaviour is consistent with experiment as well being theoretically consistent. It admits the possibility, at least in principle, of experimental verification. In the case of single particles, however, it appears no longer possible, in general, to verify it experimentally. Suppose for the moment that we consider the possibility that the probability interpretation of the wave function is a consequence of the statistical character of ensemble behaviour and that it simply does not apply to single particles. What then might the wave function represent?

The above discussion suggests that NRQM may indeed be incomplete and that additional features are needed. Suppose that these additional factors arise because the two distinct aspects of ultimate reality - actual occasions and the processes that generate them - were conflated when the original mathematical framework of NRQM was developed. Formally, NRQM takes many of the features of classical mechanics, particularly its Hamiltonian formulation, and attempts to effect a translation to a slightly more general mathematics - from point set analysis to functional analysis. Process per se is not explicitly considered in the functional analytic framework. Perhaps it would be better to look for mathematical systems that are better equipped to represent process and then see whether NRQM could be derived within this setting. Indeed, Palmer has already shown that iterated function systems may reproduce many of the essential features of quantum mechanics, at least spin statistics [286].

Suppose that the wave function actually describes information about the process responsible for the generation of a single particle. Suppose further that the probability interpretation arises in an emergent manner in the context of a statistical ensemble of particles. Note that in most quantum mechanical formulas, particularly in path integral formulations and in quantum field theory, the wave function enters into the Lagrangian, usually coupled either to itself or to the wave function of another particle. Suppose, therefore, that the wave function describes some kind of 'strength' of the generating process. Different processes would then couple through these different process strengths. Positional probability arises when a fundamental particle couples to a position measurement device, and that turn out to be a fairly basic coupling dependent upon a term of the form $\Psi^*\Psi$. In this sense the probability aspect is not an intrinsic feature of the wave function but rather an emergent feature arising out of the interaction between the particle and the measurement device. Given such an interpretation, a single particle could indeed possess a physical wave function, which describes not the particle per se but rather the process that generates the events that we ultimately interpret as a particle. Paradoxes arise because we attribute the wave function incorrectly to the particle rather than to the process.

In the process framework a particle is not a thing in itself but rather is an emergent manifestation of something more primitive. A particle is generated and the links between occurrences of a particle possess an informational aspect. The actual occurrences that are the direct manifestations of these processes occur on a spatio-temporal scale much smaller than that of the particles being generated, so small that they are inherently unobservable by quantum or classical entities. To such an entity, these actual occurrences will appear as ill defined or fuzzy objects. As will be shown in Chapter 3, this fuzziness of actual occasions can be modelled through interpreting their representative informons in the Hilbert space of the causal manifold using frame elements having the functional form (in one dimension)

$$\frac{\sin \pi(\sigma x - kn)}{\pi(\sigma x - kn)}$$

Each informon generates or contributes a sample of the wave function of the physical entity, the coefficient modifying the frame element being the value of the wave function at the causal manifold embedding point of the informon. This coefficient was calculated by determining the action moving forward from the $\mathcal{M}$-embedding point of an informon in the current tapestry to the $\mathcal{M}$-embedding point of the nascent informon in the new tapestry, forming the exponential and then multiplying it by the coefficient of the frame element of the original informon. Repeating the procedure across the current tapestry approximated the path integral calculation of the wave function at the new point. This game theoretic procedure will be shown to provide a highly accurate but discrete approximation to the usual NRQM wave function.

Each informon represents an actual occasion, which in turn is a manifestation of the generating process at a particular space-time point. In the process framework, each informon represents a little piece of ultimate reality and so is a little moment of process made real. Since it is viewed as being real, its value, namely its frame element coefficient, should also express something real, the frame element itself merely reflecting the aforementioned fuzziness. This coefficient is not a probability. It is not a probability of occurrence of something since it itself is made manifest by the generating process. It arises from the various actions involved in actually generating the informon from pre-existing informons. Since it manifests process at that space-time point it should reflect something of that process rather than reflecting some probability of occurrence of something else, especially some global emergent aspect of the particle which should not be possible given the causally local nature of the information that goes into the creation of the coefficient in the first place.

The only entity which is actually local to the space-time point is the process which is generating the actual occasion at the space-time point. Therefore it makes more sense that this coefficient should reflect the process rather than the particle. The simplest process attribute that it could reflect would be the strength or efficacy of the process as it manifests at that space-time point. One could think of this strength as describing the efficacy with which this process might couple to or interact with other processes occurring simultaneously. This is the notion of *compatibility*, introduced by Trofimova in her work on ensembles with variable structures (EVS) [395, 397]. She writes:

> The *compatibility* concept was introduced in simulations of the interactions of diverse agents, acting from the point of view of their own interests, goals and motivations. This concept is based on the fact that connections between elements of any natural system is a form of their cooperation, oriented on joint outcome of their activity. This fact was described on the cellular level within the theory of functional systems by Anochin [23], and is more obvious at the level of individuals, groups, organizations and states interaction.
>
> In order to deal with it on the formal level, we can take all possible motivational factors and characteristics (<<interests>> of the agents) and order up a vector of these interests. We could imagine then the complete vector, which characterizes a certain agent within the space of these interests. Such interests need to be interpreted broadly, as motivation to certain action in economical, physical, psychological, social, aesthetic, intellectual or informational sense (Sic)[395].

Indeed recall my previous comments about how the wave function appears in coupling terms in the Lagrangian for a quantum system. The coefficient is calculated from the Lagrangian and contributes to the Lagrangian in a kind of feedback relationship. Strength serves as a reasonable candidate for an interpretation of the meaning of this coefficient. Strength also serves as a meaningful interpretation of a physically real wave function, since the value of the wave function at a point now corresponds to something directly measurable, the strength of the process, just as an electromagnetic wave reflects the strength of the electric and magnetic fields at a point in space-time. How probabilities enter into this will be discussed below.

Each informon thus contributes a transient of the form

$$\Gamma_{kn/\sigma}\frac{\sin\pi(\sigma x - kn)}{\pi(\sigma x - kn)}$$

where $\Gamma_{kn/\sigma}$ gives the strength of the generating process at the point kn/σ. Summing over all manifesting informons one obtains a global wave function of the form

$$\Psi(x) = \sum_{j=-\infty}^{\infty} \Gamma_{kj/\sigma} \frac{\sin \pi(\sigma x - kj)}{\pi(\sigma x - kj)}$$

providing a distribution of process strength across the space-time slice into which the new causal tapestry embeds. Note that in this representation, $\Psi(kj/\sigma) = \Gamma_{kj/\sigma}$.

2.4 Interactions and the Algebra of Process

The generation of an informon can trigger a coupling between processes or the activation or inactivation of processes, depending upon the compatibility of these informons. Couplings between processes take many forms, providing the space of processes with a rich algebraic and combinatorial structure. Processes may act sequentially (denoted as sums) or concurrently (denoted as products). They may act independently of one another (independent) or their actions may be constrained, so that the action of one forces limitations on the actions of another (interactive). They may act on the same nascent informon (free or bosonic-like) or only on distinct informons (exclusive or fermionic-like). These considerations give rise to 8 general possibilities - a) Sequential sums: $\oplus$ (exclusive, independent), $\hat{\oplus}$ (free, independent), $\boxplus$ (exclusive, interactive), $\hat{\boxplus}$ (free, interactive) b) Concurrent products: $\otimes$ (exclusive, independent), $\hat{\otimes}$ (free, independent), $\boxtimes$ (exclusive, interactive), $\hat{\boxtimes}$ (free, interactive). The interactive case is actually shorthand for a set of possible interactions.

In Appendix E there are descriptions of three basic distinct forms of sum and products in the context of combinatorial games. The existence of an algebraic relationship between the space of processes and that of certain combinatorial games provides the foundation for using games as a representation of process.

Whether processes may form sums or products depends upon their character. The character of a process is determined by its intrinsic properties. The state of a process is determined by the values of its conserved dynamical properties. As a general rule of thumb, processes possessing the same character may conjoin by sum or product but processes having different character may only conjoin by product. Moreover, processes possessing the same character but different states sum through the exclusive sum while processes possessing the same character and state but distinguished solely by their range of interpretations sum by the free sum. That distinction will become important in Chapter 3 in the discussion of superpositions and in Chapter 4 in the discussion of the two slit case.

Processes corresponding to different states of a single entity will possess the same character but different values and combine through the exclusive sum. The reason for this is to ensure that any single informon representing an instance of the entity corresponds to just a single assignment of state properties, thereby assigning a definite reality to the informon. Processes may superimpose but actual occasions corresponding to a single entity *never* superimpose. Bosonic-like processes representing different entities may be conjoined using the free product while fermionic-like processes representing different entities are conjoined using the exclusive product.

In keeping with the interpretation of one informon generated per round representing a process corresponding to a single physical entity, a process which generates multiple physical entities will generate multiple informons per round. A process corresponding to K physical entities will thus generate K informons per round (i.e. $R = K$). Again, keeping with the association of a single physical entity with a primitive process, it is assumed that a process which generates multiple physical entities can be formed by a suitable product of primitive processes. Thus a process corresponding to K physical entities is formed from a product of K primitive processes. The four different kinds of product permit the generation of multiple physical entities having varying degrees of statistical dependence. Complex processes, involving multiple physical entities in superpositions of multiple states are formed from ever more complicated sums and products of simple processes.

A superposition of primitive processes corresponding to a single physical entity (representing a superposition of distinct states of a single physical entity) thus takes the form $\oplus_i \mathbb{P}_i$, meaning that during a given round only *one* of the $\mathbb{P}_i$ is active, generating a single informon. Since this is an exclusive sum this also requires that any given informon n is generated by a *single* subprocess, say $\mathbb{P}_i$. If the properties associated with $\mathbb{P}_i$ are given by the vector of properties $\mathbf{p}_i$, then the properties associated with n will be given as $\Gamma_n^{\mathbb{P}_i}, \mathbf{p}_i$. This interleaving of subprocesses will, through the process covering map described below, nevertheless approximate the standard wave function for the superposition. More generally one has $\oplus_i w_i \mathbb{P}_i$, where $w_i \mathbb{P}_i$ indicates a modification of certain attributes of $\mathbb{P}_i$ by the factor w_i, for example multiplying the strength of the process by w_i, giving a new strength $w_i \Gamma_n^{\mathbb{P}_i}$. In this case the individual weights modify the coefficients of the $\mathbb{H}(\mathcal{M})$-interpretations of the informons and one requires that $\sum_i w_i^* w_i = 1$ to express the fact that the total strength of the process must be distributed over the different subprocesses. Effectively it is a statement of conservation of process strength.

Figure 2.8 illustrates this interleaving of informons generating a superposition.

If $\mathbb{P}$ is a primitive process, thus generating a single entity, then a process generating N such independent entities could be given by either $\overbrace{\mathbb{P}\hat{\otimes}\cdots\hat{\otimes}\mathbb{P}}^{N}$ (bosonic-like) or $\overbrace{\mathbb{P}\otimes\cdots\otimes\mathbb{P}}^{N}$ (fermionic-like).

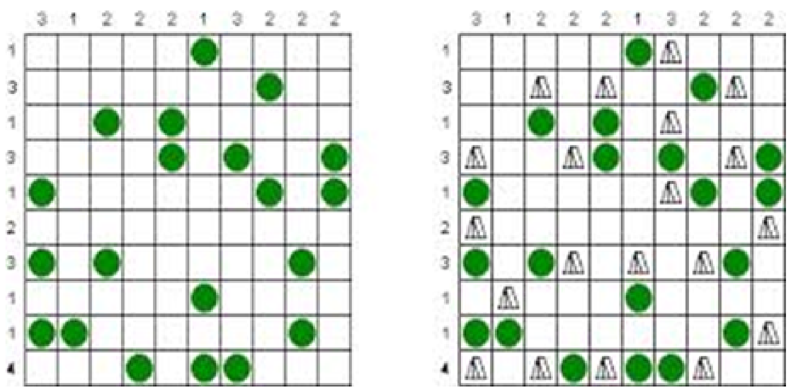

Figure 2.8: Superpositions: informons of a coherent primitive process appear in the left lattice. Informons of a superposition of two coherent primitive processes appear in the right lattice. The informons of the two primitive processes interleave on the lattice while their Hilbert space interpretations superpose.

Entanglement provides an example of an interactive coupling between processes. An entanglement of primitive processes is denoted as $\boxtimes_i \mathbb{P}_i$, (or $\hat{\boxtimes}_i \mathbb{P}_i$) meaning that during a given round, *all* of the $\mathbb{P}_i$ are concurrently generating single informons, but because they are in interactive mode their actions are mutually constrained, and the resulting informons will be correlated. This is more apparent when processes are represented as combinatorial games where the game tree of an interactive product will be a proper subtree of the game tree of the independent products. Note that no information passes among these informons. Each is generated by its own process using only causally local information propagating from current informons linked to that process. There is no “spooky action at a distance”. There is just the realization that the generating sub-processes are not independent of one another but rather are in an interactive mode.

More precisely, suppose that one has two photons, a,b, each in a superposition of polarization states, say $\Psi = \frac{1}{\sqrt{2}}(|0> + |1>)$. Let $\mathbb{P}_0$ denote the process generating $|0>$ and $\mathbb{P}_1$ that for $|1>$. The process generating the superposition for a is given by the exclusive sum $\mathbb{P}_a = \frac{1}{\sqrt{2}}(\mathbb{P}_0^a \oplus \mathbb{P}_1^a)$ and similarly for b. The process for the two photons labelled a, b is thus given by the exclusive product $\mathbb{P}_{ab} = \frac{1}{\sqrt{2}}(\mathbb{P}_a \otimes \mathbb{P}_b) = \frac{1}{\sqrt{2}}(\mathbb{P}_0^a \oplus \mathbb{P}_1^a) \otimes \frac{1}{\sqrt{2}}(\mathbb{P}_0^b \oplus \mathbb{P}_1^b) = \frac{1}{2}[(\mathbb{P}_0^a \otimes \mathbb{P}_0^b) \oplus (\mathbb{P}_0^a \otimes \mathbb{P}_1^b) \oplus (\mathbb{P}_1^a \otimes \mathbb{P}_0^b) \oplus (\mathbb{P}_1^a \otimes \mathbb{P}_1^b)]$. In the case of an entangled pair of photons, the process is now given by an interactive product of these two processes, namely $\mathbb{P}_{ab}^E = \mathbb{P}_a \boxtimes \mathbb{P}_b = \frac{1}{\sqrt{2}}[(\mathbb{P}_0^a \otimes \mathbb{P}_0^b) \oplus (\mathbb{P}_1^a \otimes \mathbb{P}_1^b)]$. This is a simple example of an interactive product but it illustrates the basic idea. This particular product can be described algebraically but in general the product may be described only through the sequence tree. An assumption in NRQM is that all states of physical entities are described by algebraic combinations of sums and tensor products of wave functions, but in the process setting more general products are possible. Whether these more general products correspond to physically realizable processes is a matter for experimental verification or denial.

As an example of this, consider the case, likely physically unrealistic, in which the two photon processes are coupled round for round. In other words, for every informon completed for photon a, an identical informon is completed for photon b. In this case we would write this as $\mathbb{P}_{ab}^{E+} = \mathbb{P}_a \boxtimes \mathbb{P}_b = \frac{1}{\sqrt{2}}[(\mathbb{P}_0^a \boxtimes \mathbb{P}_0^b) \oplus (\mathbb{P}_1^a \boxtimes \mathbb{P}_1^b)]$ and the sequence tree for this can be found by taking the sequence tree for one subprocess, say $\mathbb{P}_0^a$ and appending to each informon the identical corresponding element from the sequence tree of $\mathbb{P}_0^b$, and similarly for the $|1>$ polarizations.

The above describes one possible type of interaction effect between processes when the interaction results in the establishment of a correlation or coupling between the two processes. A second type of interaction occurs when there is a complete or partial change in the process generating the interacting systems. As an example of this form of interaction, consider the binding of one particle to another, as in the capture of an electron by a proton. One may think of the binding process as a three step transition.

Prior to binding, the process is given by $\mathbb{P}_{fe} \otimes \mathbb{P}_{fp}$, the free product of the free electron (fe) and free proton (fp) processes. Let Ψ_{fe} be the free electron wave function. Let Ψ_i^{be} denote the i-th eigenfunction for the electron in the potential well and let $\mathbb{P}_i^{be}$ denote the process generating this eigenfunction. Then we can write $\Psi_{be} = \sum_i w_i \Psi_{bei}$ (assuming that the eigenfunctions form a complete set) for some coefficients w_i. Upon the initial interaction with the proton, we think of the electron process as undergoing a transition from $\mathbb{P}_{fe}$ to $\mathbb{P}_{fe} = \oplus_i w_i \mathbb{P}_i^{be}$. Using the process covering map defined in the next section it is clear that this new process will generate the same wave function as will $\mathbb{P}_{fe}$ but the processes by means of which this wave function is generated will be different. This amounts to a kind of rotation in process space which is reversible and involves only an exchange of information. Once this new process has been established the electron can become bound to the proton and only one process, say $\mathbb{P}_k^{be}$ will be selected, and any excess of energy radiated off by the release of a photon. This last step is irreversible as far as retaining information about the generating process is concerned. More about this point will be discussed below. The steps in the transition can be described as

$$\mathbb{P}_{fe} \otimes \mathbb{P}_{fp} \rightarrow \oplus_i w_i \mathbb{P}_i^{be} \otimes \mathbb{P}_{fp} \rightarrow \oplus_i w_i \mathbb{P}_i^{be} \boxtimes \mathbb{P}_{fp} \rightarrow \mathbb{P}_k^{be} \otimes \mathbb{P}_{fp}$$

The subtleties involved in the binding process become more apparent in the process framework because of the additional forms of linearity and multiplicativity available through the algebraic structure of the process space.

Note that different entanglements lead to different interactive products but for simplicity I refer to them generically via the sign $\boxtimes$ unless they need to be described precisely.

The zero process, $\mathbb{O}$, is the process that does nothing. It generates no informons at all. It is quite possible that certain processes cannot be combined. This might occur because of superselection rules, or because as primitive processes are combined in ever more complex ways configurations may be generated that violate algebraic constraints. For example, it is not reasonable to believe that processes having different characters of strength (scalar, vector, spinorial) could be combined. In such a case a sum of such incompatible processes yields the zero process. For example, a scalar process $\mathbb{P}_s$ and a spinorial process $\mathbb{P}_{sp}$ form the sum $\mathbb{P}_s \oplus \mathbb{P}_{sp} = \mathbb{O}$. Similarly there may be situations in which it is impossible to form the product of two or more distinct processes, in which case the product will be given the value $\mathbb{O}$.

It is conjectured that the emergence of classicality derives not merely from statistical

Superposition of distinct states	$\frac{1}{\sqrt{2}}(\Psi_1(\mathbf{z}) \oplus \Psi_2(\mathbf{z}))$
Two slit (spatial decomposition)	$\frac{1}{\sqrt{2}}(\Psi_L(\mathbf{z})\hat{\oplus}\Psi_R(\mathbf{z}))$
Two free particles	$\frac{1}{\sqrt{2}}(\Psi_1(\mathbf{z}) \otimes \Psi_2(\mathbf{z}))$
Two entangled particles	$\frac{1}{\sqrt{2}}[(\mathbb{P}_1(0) \otimes \mathbb{P}_2(1)) \boxtimes (\mathbb{P}_1(1) \otimes \mathbb{P}_2(0))]$
Schrödinger's cat	$\frac{1}{\sqrt{2}}[(\mathbb{P}(D_n) \otimes \mathbb{P}(C_a)) \boxplus (\mathbb{P}(D_r) \otimes \mathbb{P}(C_d))]$

Table 2.1: Process Descriptions of Quantum Mechanical Situations: illustration of roles of different sum and product operations.

effects or from idealizations in which $\hbar \to 0$ but rather from the appearance of non-superposable processes, particularly when combined in interactive sums and products, as the number of participating components increases. By this is meant the appearance of distinct processes $\mathbb{P}_a, \mathbb{P}_d$ that could, in principle, generate the evolution of a system separately but cannot be superposed so as to generate an evolution in combination, i.e. $\mathbb{P}_a \oplus \mathbb{P}_d = \mathbb{O}$ (or $\mathbb{P}_a \otimes \mathbb{P}_d = \mathbb{O}$). This notion will be discussed further in Chapter 4.

Figure 2.9 depicts different process descriptions of quantum mechanical constructions.

2.5 Process Covering Map

The process viewpoint leads to the insight that the proper setting for quantum dynamics is not the Hilbert space $\mathcal{H}(\mathcal{M})$ but rather the space Π of generating processes. Dynamical information is lost when considering only the wave function. In a process driven world, informons are generated and since interactions such as measurement are held to be triggered by the appearance of informons (which determine the couplings of processes), the sequence of appearance of informons during the unfolding of processes potentially carries dynamically relevant information. Moreover, processes are determined not just by their parameters but also by the strategy by means of which informons are generated. This will be made more apparent when the game representation is discussed below. The process strategy determines the sequence of generation of informons, their causal distances, their properties and their strength. The generation sequence potentially conveys information about the generation strategy.

Beginning with some prior generation $\mathcal{I}$ of informons, a primitive process $\mathbb{P}$, once activated, will generate a sequence of $n_1, n_2, \ldots, n_k, \ldots$ of informons. Due to the non-deterministic nature of the process action, reactivating the process on the same initial generation will, in all likelihood, result in a different sequence of informons, $m_1, m_2, \ldots, m_k, \ldots$. The sequence tree of a process $\mathbb{P}$ and an initial generation $\mathcal{I}$, denoted $\Sigma(\mathbb{P}, \mathcal{I})$ is constructed as follows. Into level 0 put the empty informon $\emptyset$. Into level 1, place all possible informons that can be generated by $\mathbb{P}$ starting from $\mathcal{I}$ in the first round. Form a link between $\emptyset$ and each of these informons. From each n_1 in level 1 form a link to all possible informons that can be generated following $\mathcal{I}$, n_1, and form a link between them and n_1. Level 2 consists of all such informons generated from all informons in level 1. This construction process is repeated for all possible generations. Any actual implementation of the process $\mathbb{P}$ will generate one sequence which constitutes a path through the sequence tree.

The informons $n_1, n_2, \ldots, n_k, \ldots$ along each path of $\Sigma(\mathbb{P}, \mathcal{I})$ constitute a possible next generation of informons. Each informon n_i will provide a local $\mathcal{H}(\mathcal{M})$-contribution $\phi_{n_i}(\mathbf{z})$ to a global $\mathcal{H}(\mathcal{M})$-interpretation $\Phi^1(\mathbf{z}) = \sum_{n_i} \phi_{n_i}(\mathbf{z})$. Thus to the process $\mathbb{P}$ one can associate a set $H_{\mathbb{P}}$ of elements of $\mathcal{H}(\mathcal{M})$ consisting of all global $\mathcal{H}(\mathcal{M})$-interpretations constructed from every possible path in the sequence tree $\Sigma(\mathbb{P}, \mathcal{I})$.

Interpolation theory shows that given certain choices of the interpolation function g, in the limit $N, r \to \infty$, $H(\mathbb{P}) \to \{\Phi^{t_P l_P}(\mathbf{z})\}$, a singleton set. Given two distinct primitive processes $\mathbb{P}_1$ and $\mathbb{P}_2$, it is possible that in the limit $N, r \to \infty$, $H(\mathbb{P}_1) \to H(\mathbb{P}_2) = \{\Phi^{t_P l_P}(\mathbf{z})\}$. In such a case the primitive processes $\mathbb{P}_1, \mathbb{P}_2$ are said to be Ψ-epistemic equivalent. Ψ-epistemic equivalent processes generate distinct "realities" at small scales which are effectively equivalent at large scales, at least statistically. They are indistinguishable from a quantum mechanical point of view since they will asymptotically yield the same wave function. Interpolation theory shows that this asymptotic behaviour holds for a single primitive process so that $\mathbb{P}$ is Ψ-epistemic equivalent to itself. Ψ-epistemic equivalence thus provides a proper equivalence relation on the subset of primitive processes. Ψ-epistemic equivalence coarse grains the space of processes into those that are equivalent quantum mechanically, which narrows the range of strategies to be considered. This imposes a form of strategic "gauge" invariance, so that the strategies considered in models may be chosen for computational or analytical convenience rather than ontological implications. It is always possible that future advances in measurement might permit strategies within Ψ-equivalent classes to be distinguished but that is not the case at present.

The current state of interpolation theory applies mostly to complex vector valued functions while the theory as applied to spinors is much less developed. Thus from a technical standpoint the discussion to follow applies to integral spin particles but it seems reasonable to believe that the argument should hold for half integer spin particles as well. The most general case requires consideration of general function spaces mapping $\mathcal{M}$ to $\mathbb{C}^n$ or $\mathfrak{sp}(n)$ but this would unnecessarily complicate the essential points.

The process covering map (PCM) provides a linkage between the space of processes, Π and the Hilbert space $\mathcal{H}(\mathcal{M})$, and thus to NRQM. In general the PCM will depend upon the strategy used to implement the actions of the processes as well as the initial condition. For simplicity only the initial condition will be referenced. It is constructed first on the set of primitive processes Π_p. For fixed $\mathcal{I}$, and some primitive process $\mathbb{P}$, define the process covering map $\mathfrak{P}_{\mathcal{I}} : \Pi_p \to \mathcal{H}(\mathcal{M})$ by $\mathfrak{P}_{\mathcal{I}}(\mathbb{P}) = H_{\mathbb{P}}$. Note that the process covering map (PCM) is a set-valued map [25, 26], so to be mathematically correct one should write $\mathfrak{P}_{\mathcal{I}} : \Pi_p \to \mathcal{P}(\mathcal{H}(\mathcal{M}))$, where $\mathcal{P}(\mathcal{H}(\mathcal{M}))$ is the power set on $\mathcal{H}(\mathcal{M})$. Nevertheless the PCM will generally (and sloppily) be referred to as a map to $\mathcal{H}(\mathcal{M})$ for convenience.

Under specific conditions, as will be made explicit in Chapter 3, interpolation theory shows that if at every informon n the strength $\Gamma_n \propto \Psi(\mathbf{m}_n)$ for some non-relativistic wave function $\Psi(\mathbf{z})$ which has energy and momenta bounded away from Planck energy and momentum, then $\Psi(\mathbf{z}) = \Psi^{t_P l_P}(\mathbf{z})$, and if $\Psi(\mathbf{z})$ is not band-limited in this way, then this will still be true in the limit $t_P, l_P \to 0$.

The PCM is now extended on Π_p by considering sums. The case of exclusive sums is easily treated. Consider an exclusive sum $\oplus w_i \mathbb{P}_i$ of primitive processes $\mathbb{P}_i$ applied to the causal tapestry $\mathcal{I}$. The elements of the nascent causal space $\mathcal{I}_1$ lie in distinct subsets, $\mathcal{I}_1^i$. Let $j_n = i$ iff $n \in \mathcal{I}_1^i$. Then the global $\mathcal{H}(\mathcal{M})$-interpretation $\Phi(\mathbf{z}) = \oplus_{n \in \mathcal{I}_1} w_{j_n} \phi_n(\mathbf{z}) = \oplus_i w_i \{\oplus_{n \in \mathcal{I}_1^i} \phi_n(\mathbf{z})\} = \oplus_i w_i \Phi^i(\mathbf{z})$ where $\Phi^i(\mathbf{z})$ is the global $\mathcal{H}(\mathcal{M})$-interpretation corresponding to the process $\mathbb{P}_i$. Interpolation theory provides conditions under which, in the asymptotic limit

$$\Phi(\mathbf{z}) = \oplus_i w_i \Phi^i(\mathbf{z}) \to \Psi(\mathbf{z}) = \oplus_i w_i \Psi^i(\mathbf{z})$$

It follows easily that

$$\mathfrak{P}_{\mathcal{I}}(\oplus_i w_i \mathbb{P}_i) = \sum_i w_i \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_i)$$

where for two sets of functions A, B the sum

$$A + B = \{f + g | f \in A, g \in B\}$$

.

This also holds true for free sums, so that

$$\mathfrak{P}_{\mathcal{I}}(\hat{\oplus}_i w_i \mathbb{P}_i) = \sum_i w_i \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_i)$$

.

It is here that the process point of view departs subtly from standard quantum mechanics. It is easy to see that under the limit $N, r \to \infty$, $\mathfrak{P}_{\mathcal{I}}(\oplus_i w_i \mathbb{P}_i) \to \mathfrak{P}_{\mathcal{I}}(\hat{\oplus}_i w_i \mathbb{P}_i)$, so that $\oplus_i w_i \mathbb{P}_i$ and $\hat{\oplus}_i w_i \mathbb{P}_i$ are Ψ-epistemic equivalent, but the processes $\oplus_i w_i \mathbb{P}_i$ and $\hat{\oplus}_i w_i \mathbb{P}_i$ do not possess the same sequence trees. Moreover the local $\mathcal{H}(\mathcal{M})$-contributions which determine the coupling to other processes will differ (since in the free case it is possible for an informon to carry contributions from, two distinct sub-processes, which could result in measurements that differ from the exclusive case).

Non-primitive processes (and thus the case of multiple physical entities) are built from primitive processes by means of sums and products. Although sums generalize easily to Π the case of products is more complicated. The reasons for this, and a detailed treatment of products, are presented in Section 2.11. For the moment consider the exclusive product $\mathbb{P} = \otimes \mathbb{P}_i$ of primitive processes $\mathbb{P}_i$. An informon n^i corresponding to each i is generated during each round. Therefore a local Hilbert space contribution ϕ_{n^i} is generated corresponding to each i. In the exclusive case one may again form a global $\mathcal{H}(\mathcal{M})$-interpretation $\Phi(\mathbf{z}) = \sum_{n^i} \phi_{n^i}(\mathbf{z}) = \sum_i \Phi_i(\mathbf{z})$ and set $\mathfrak{P}_{\mathcal{I}}(\mathbb{P}) = \Phi(\mathbf{z})$. Although such an interpretation describes the observable situation it fails to determine the appropriate couplings when a measurement process is introduced. That is because this interpretation fails to incorporate necessary information about the generation of the informons, in particular that they occur in tuples, not singly. Each round generates a set of informons $\{n^i\}$ and thus a set of local Hilbert space contributions, $\{\phi_{n^i}(\mathbf{z})\}$. This suggests that the proper definition is to associate each complete expression of the process $\mathbb{P}$ with the set of global $\mathcal{H}(\mathcal{M})$-interpretations $\{\Phi_1(\mathbf{z}), \Phi_2(\mathbf{z}), \ldots, \Phi_i(\mathbf{z})\}$. This means that $\mathfrak{P}_{\mathcal{I}}(\mathbb{P}) = \{\{\Phi_1(\mathbf{z}), \Phi_2(\mathbf{z}), \ldots, \Phi_i(\mathbf{z})\}$ over all paths in $\Sigma(\mathbb{P}, \mathcal{I})\}$. Indeed we have $\mathfrak{P}_{\mathcal{I}}(\mathbb{P}) = \mathfrak{P}_{\mathcal{I}}(\otimes \mathbb{P}_i) =$

$$\{\{\Phi_1(\mathbf{z}), \ldots, \Phi_j(\mathbf{z})\} | \Phi_1(\mathbf{z}) \in \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_1), \ldots, \Phi_j(\mathbf{z}) \in \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_j)\}$$

This is a rather inconvenient form mathematically.

One approach, which appears to be little used in physics, is to work with the co-product $\sqcup_i \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_i)$. The co-product is a category theory construction [182] whose main property is that it provides a universal embedding space $\sqcup_i M_i$ for some set of mathematical objects M_i. This is in contrast to the product $\sqcap_i M_i$ which provides a universal covering space for the M_i. Here, the co-product is defined as the set of *formal* sums of functions, one from each of the sets $\mathfrak{P}_{\mathcal{I}}(\mathbb{P}_i)$. By this one means formal terms of the form $f_1 \oplus f_2 \oplus \cdots \oplus f_j$ where $f_i \in \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_i)$. The functions are *not* evaluated at a point. One thus obtains a formal sum of the form $f_1(\mathbf{z}) \oplus f_2(\mathbf{z}) \oplus \cdots \oplus f_j(\mathbf{z})$, which is to be distinguished from the usual algebraic sum of the functions, $f_1 + f_2 + \cdots + f_j$ whose evaluation at a given point would yield a single number $f_1(\mathbf{z}) + f_2(\mathbf{z}) + \cdots + f_j(\mathbf{z})$. Using the co-product one has the relation

$$\mathfrak{P}_{\mathcal{I}}(\otimes_i \mathbb{P}_i) = \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_1) \sqcup \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_2) \sqcup \cdots \sqcup \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_j)$$

.

This appears to be a natural representation as it preserves the existential aspects of multiple processes acting simultaneously (and thus multiple physical entities co-evolving) although it is much less convenient for carrying out calculations.

The standard approach in NRQM, however, has been to keep the contributions from individual processes separate by resorting to the use of the product space. Therefore consider a tuple $(n^1, n^2, \ldots, n^j)$ of informons, a corresponding tuple of causal manifold points $\mathbf{m}_{n^1}, \mathbf{m}_{n^2}, \ldots, \mathbf{m}_{n^j})$ and a corresponding tuple of local Hilbert space contributions $(\phi_{n^1}(\mathbf{z}), \phi_{n^2}(\mathbf{z}), \ldots, \phi_{n^j}(\mathbf{z}))$. This yields $\mathfrak{P}_{\mathcal{I}}(\mathbb{P}) = \mathfrak{P}_{\mathcal{I}}(\otimes_i \mathbb{P}_i) =$

$$\{(\Phi_{n^1}(\mathbf{z}), \Phi_{n^2}(\mathbf{z}), \ldots, \Phi_{n^j}(\mathbf{z}))|\text{ over all paths in } \Sigma(\mathbb{P}, \mathcal{I})\}$$

or

$$\mathfrak{P}_{\mathcal{I}}(\otimes_i \mathbb{P}_i) = \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_1) \times \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_2) \times \cdots \times \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_j)$$

.

where the $\times$ operator means set product. The set will thus be a subset of the j-dimensional tensor product of the Hilbert space $\mathcal{H}(\mathcal{M})$, i.e. $\mathfrak{P}_{\mathcal{I}}(\otimes_i \mathbb{P}_i) \subset \otimes_1^j \mathcal{H}(\mathcal{M})$.

Thus we arrive at a natural relationship between the exclusive product of processes and the associated process covering map and a natural link to the standard formalism of NRQM. It follows the form of the usual product of wave functions. The indistinguishability of physical entities implies the indistinguishability of their generating processes and so products of processes that are to represent physical entities shall need to follow the usual quantum mechanical formulation rules.

The above holds for free products as well. Indeed a moment's reflection will suggest that

$$\mathfrak{P}_{\mathcal{I}}(\hat{\otimes}_i \mathbb{P}_i) = \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_1) \times \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_2) \times \cdots \times \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_j)$$

so that $\otimes_i \mathbb{P}_i$ and $\hat{\otimes}_i \mathbb{P}_i$ are also Ψ-epistemic equivalent. Again the degeneracy in the PCM means that dynamical information is lost when moving over to the wave function, again suggesting that NRQM is incomplete dynamically.

2.6 Process and Operators

The study of the relationship between processes and operators on a Hilbert space is unexplored and promising. Only a few remarks will be offered here.

In the discussion above, the PCM was referenced to a given causal tapestry $\mathcal{I}$ which served as an initial condition upon which the process $\mathbb{P}$ was to act. It is certainly possible that the initial condition could be the empty tapestry but in general it will represent the outcome of the actions of previous processes.

Recall that a causal tapestry $\mathcal{I}$ consists of a collection of informons, each of which has the form $[n] < (\mathbf{m}_n, \phi_n(\mathbf{z}), \mathbf{p}) > \{G\}$. The function $\phi_n(\mathbf{z})$ provides a local contribution from n to a global function $\Phi_{\mathcal{I}}(\mathbf{z})$ defined on the causal manifold $\mathcal{M}$ by $\Phi_{\mathcal{I}}(\mathbf{z}) = \sum_{n \in \mathcal{I}} \phi_n(\mathbf{z})$. This defines a mapping $\mathfrak{I}$ from the space of causal tapestries $\mathbf{I}$ to the Hilbert space $\mathcal{H}(\mathcal{M})$ by

$$\mathfrak{I}(\mathcal{I}) = \Phi_{\mathcal{I}}(\mathbf{z})$$

.

The PCM was defined as a map $\mathfrak{P}_{\mathcal{I}} : \Pi \to \mathcal{P}(\mathcal{H}(\mathcal{M}))$. Here the map is referenced to a given causal tapestry $\mathcal{I}$ but clearly $\mathcal{I}$ is free to roam over the entire space of causal tapestries $\mathbf{I}$. More precisely then we should define $\mathfrak{P}$ as a map $\mathfrak{P} : \Pi \times \mathbf{I} \to \mathcal{P}(\mathcal{H}(\mathcal{M}))$. If we fix some process $\mathbb{P} \in \Pi$ then we can define a tapestry covering map (TCM) $\mathfrak{P}_{\mathbb{P}} : \mathbf{I} \to \mathcal{P}(\mathcal{H}(\mathcal{M}))$ in the obvious manner.

Define a generalized operator $\mathcal{G}$ on $\mathcal{H}(\mathcal{M})$ as a mapping $\mathcal{G} : \mathcal{H}(\mathcal{M}) \to \mathcal{P}(\mathcal{H}(\mathcal{M}))$ such that $\mathcal{G}(f + g) \subset \mathcal{G}(f) + \mathcal{G}(g)$ and let $\mathfrak{G}(\mathcal{H}(\mathcal{M}))$ denote the set of generalized operators on $\mathcal{H}(\mathcal{M})$.

For a fixed process $\mathbb{P}$, define a generalized operator $\mathfrak{G}_{\mathbb{P}}$ on $\mathcal{H}(\mathcal{M})$ as follows:

For every $f \in \mathcal{H}(\mathcal{M})$, set $\mathfrak{G}_{\mathbb{P}}(f) = \cup_{\mathcal{I} \in \mathfrak{I}^{-1}(f)} \mathfrak{P}_{\mathbb{P}}(\mathcal{I})$

One thus obtains the following diagram

$$\begin{array}{ccc} \mathbf{I} & \stackrel{\mathfrak{P}_{\mathbb{P}}}{\rightarrow} & \mathcal{P}(\mathcal{H}(\mathcal{M})) \\ \mathfrak{I} \downarrow & & \downarrow e \\ \mathcal{H}(\mathcal{M}) & \stackrel{\rightarrow}{\mathfrak{G}_{\mathbb{P}}} & \mathcal{P}(\mathcal{H}(\mathcal{M})) \end{array}$$

where e is a map such that $h \subset e(h)$.

The problem is that one cannot guarantee that if two causal tapestries $\mathcal{I}, \mathcal{I}'$ satisfy $\mathfrak{I}(\mathcal{I}) = \mathfrak{I}(\mathcal{I}')$ (that is, they generate the same global Hilbert space interpretation), then $\mathfrak{P}_{\mathbb{P}}(\mathcal{I}) = \mathfrak{P}_{\mathbb{P}}(\mathcal{I}')$ (that is, the process $\mathbb{P}$ generates the same collection of global Hilbert space interpretations). A process $\mathbb{P}$ is said to be Ψ-*faithful* if $\mathfrak{I}(\mathcal{I}) = \mathfrak{I}(\mathcal{I}')$ implies that $\mathfrak{P}_{\mathbb{P}}(\mathcal{I}) = \mathfrak{P}_{\mathbb{P}}(\mathcal{I}')$ for all $\mathcal{I}, \mathcal{I}'$. In the case of a Ψ-faithful process the diagram reduces to the simpler form

$$\begin{array}{ccc} \mathbf{I} & \stackrel{\mathfrak{P}_{\mathbb{P}}}{\to} & \mathcal{P}(\mathcal{H}(\mathcal{M})) \\ \mathfrak{I}\downarrow & & \downarrow id \\ \mathcal{H}(\mathcal{M}) & \stackrel{\to}{\mathfrak{G}_{\mathbb{P}}} & \mathcal{P}(\mathcal{H}(\mathcal{M})) \end{array}$$

where id is the identity.

In either case one can associate each process $\mathbb{P}$ with a generalized operator $\mathfrak{G}_{\mathbb{P}}$ on $\mathcal{H}(\mathcal{M})$.

If we assume that the process $\mathbb{P}$ involves an effective interpolation strategy, then drawing upon the results of Chapter 3 and Appendix F, one can show that in the limit as $N, r \to \infty$ each PCM becomes effectively a map Π to $\mathcal{H}(\mathcal{M})$ (or equally, each TCM becomes a map from $\mathbf{I} \to \mathcal{H}(\mathcal{M})$) since the outcome set is a singleton. Consider now the situation in which the asymptotic limit has been taken. This corresponds to restricting attention to only those processes corresponding to the asymptote, so those processes for which $N, r = \aleph_0$ at least. As an aside, in combinatorial game theory it is quite possible to work with varying infinite types, particularly the ordinals. If we restrict then to the subset Π_∞ of such asymptotic processes then one must also restrict the space of causal tapestries to $\mathbf{I}_\infty$, the subset of causal tapestries that are generated by processes within Π_∞. Hence for $\mathbb{P} \in \Pi_\infty$ and $\mathcal{I} \in \mathbf{I}_\infty$,

$$\mathfrak{P}_{\mathcal{I}}(\mathbb{P}) = \{\Phi^{t_P l_P}(\mathbf{z})\},$$

a singleton set.

It follows that the previous diagrams simplify to

$$\begin{array}{ccc} \mathbf{I}_\infty & \stackrel{\mathfrak{P}_{\mathbb{P}}}{\to} & \mathcal{H}(\mathcal{M}) \\ \mathfrak{I}\downarrow & & \downarrow e \\ \mathcal{H}(\mathcal{M}) & \stackrel{\to}{\mathfrak{G}_{\mathbb{P}}} & \mathcal{P}(\mathcal{H}(\mathcal{M})) \end{array}$$

for general processes and for Ψ-faithful processes to

$$\begin{array}{ccc} \mathbf{I}_\infty & \overset{\mathfrak{P}_\mathbb{P}}{\rightarrow} & \mathcal{H}(\mathcal{M}) \\ \mathfrak{I}\downarrow & & \downarrow id \\ \mathcal{H}(\mathcal{M}) & \overset{\rightarrow}{\mathfrak{G}_\mathbb{P}} & \mathcal{H}(\mathcal{M}) \end{array}$$

In this situation, the generalized operator $\mathfrak{G}_\mathbb{P}$ becomes a standard operator on $\mathcal{H}(\mathcal{M})$. This is the main value for considering Ψ-faithful processes.

2.7 Ontic and Epistemic Equivalence

In order to compare processes (and later their representative games and game strategies) it is useful to have notions of equivalence which can be used to classify processes. Assume that we are working within a single space of causal tapestries. Therefore we may consider the causal manifold $\mathcal{M}$, its Hilbert space $\mathbb{H}(\mathcal{M})$ (or subspaces), the property space D (and the transition space $\mathbb{L}$ - see Appendix D) all fixed in advance. Recall that a causal tapestry $\mathcal{I}$ is just a collection of informons of the form $[n] < (\mathbf{m}_n, \phi_n(\mathbf{z}), \Gamma_n, \mathbf{p}_n) > \{G\}$ where $\mathbf{m}_n \in \mathcal{M}$, $\phi_n(\mathbf{z}) \in \mathcal{H}(\mathcal{M})$, $\mathbf{p}_n \in D$ and G is a suitably structured ordered set of informons. A history of a particle or system is a causally ordered sequence of causal tapestries (more precisely coupled event/transition tapestries) $\mathcal{I}_0 \rightarrow \mathcal{I}_1 \rightarrow \ldots \mathcal{I}_n \rightarrow \ldots \mathcal{I}$ and the informons of G must come from causal tapestries that lie prior to $\mathcal{I}$ in this sequence. Each step $\mathcal{I}_i \rightarrow \mathcal{I}_{i+1}$ is generated by the action of a single process but different steps in this sequence may be generated by different processes, the coupling being described by the duality between the event/transition tapestry and the process tapestry (see Appendix D and Chapter 3). This complexity describes the situation of interaction between systems or of the process of measurement as described in Section 2.10, but this full complexity will not be needed here.

For simplicity the definitions below will be based upon a single causal tapestry space. Isomorphic spaces could also be used but that merely overly complicates the definitions and the extension is fairly obvious. The definitions below are based upon processes but in anticipation of future applications, these same definitions apply to reality games where one simply replaces the process by its representative reality game.

Fix the space $\mathbf{I}$ of causal tapestries and suppose that two processes $\mathbb{P}_1$ and $\mathbb{P}_2$ act on this space. Given an initial causal tapestry $\mathcal{I}$, let the set of causal tapestries generated by the process $\mathbb{P}$ be denoted $H_\mathbb{P}(\mathcal{I})$.

Definition (Deterministic Process): A process $\mathbb{P}$ is said to be *deterministic* if given any initial causal tapestry $\mathcal{I}$, the set $H_\mathbb{P}(\mathcal{I}) = \{\mathcal{I}'\}$ consists of a single causal tapestry. Otherwise the process is said to be non-deterministic.

Definition (Process Equivalence [$\mathcal{PE}$]): Two processes $\mathbb{P}_1$ and $\mathbb{P}_2$ are said to be *process-equivalent* if their respective sequences trees $\Sigma(\mathbb{P}_1,\mathcal{I})$, $\Sigma(\mathbb{P}_2,\mathcal{I})$ satisfy $\Sigma(\mathbb{P}_1,\mathcal{I}) = \Sigma(\mathbb{P}_2,\mathcal{I})$ for all $\mathcal{I}$ (one may also qualify this by restricting to a single $\mathcal{I}$).

Definition (Ontic Equivalence [$\mathcal{OE}$]): Two processes $\mathbb{P}_1$ and $\mathbb{P}_2$ are said to be *ontic-equivalent* if they generate the same collections of causal tapestries, that is, for any initial causal tapestry $\mathcal{I}$, $H_{\mathbb{P}_1}(\mathcal{I}) = H_{\mathbb{P}_2}(\mathcal{I})$.

It follows easily that if two processes are process-equivalent then they are ontic-equivalent. To see this simply follow the path in the sequence tree $\Sigma(\mathbb{P}_1,\mathcal{I})$ from $\mathcal{I}$ to any $\mathcal{I}' \in H_{\mathbb{P}_1}(\mathcal{I})$ and note that the same path appears in $\Sigma(\mathbb{P}_2,\mathcal{I})$ and hence $\mathcal{I}' \subset H_{\mathbb{P}_2}(\mathcal{I})$. Reversing the roles of $\mathbb{P}_1$ and $\mathbb{P}_2$ completes the argument. It is not true, however, that if two processes $\mathbb{P}_1$ and $\mathbb{P}_2$ are ontic-equivalent then they must be process-equivalent. The same causal tapestry may be generated by different sequence trees.

Definition (Strong epistemic equivalence [$\mathcal{SEE}$]): Two processes $\mathbb{P}_1$ and $\mathbb{P}_2$ are said to be *strong epistemic-equivalent* if for every causal tapestry $\mathcal{I}$, the corresponding process covering maps are identical, i.e. $\mathfrak{P}_{\mathcal{I}}(\mathbb{P}_1) = \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_2)$. In other words, the two games generate the same set of wave functions on the subspace of $\mathcal{M}$.

Definition (Weak epistemic equivalence [$\mathcal{WEE}$])(also Ψ-epistemic equivalent): Two processes $\mathbb{P}_1$ and $\mathbb{P}_2$ are said to be *weak epistemic-equivalent* if, for every causal tapestry $\mathcal{I}$, in the asymptotic limit $N, r \to \infty$ (and sometimes $t_P, l_P \to 0$), we have $\mathfrak{P}_{\mathcal{I}}(\mathbb{P}_1) = \mathfrak{P}_{\mathcal{I}}(\mathbb{P}_2)$.

Clearly if two processes $\mathbb{P}_1$ and $\mathbb{P}_2$ are strong epistemic-equivalent then they are weak epistemic equivalent. It is also fairly obvious that if two processes are ontic-equivalent then they are also strong epistemic-equivalent.

Thus we have process equivalence implies ontic equivalence implies strong epistemic equivalence implies weak epistemic equivalence or symbolically

$$\mathcal{PE} \to \mathcal{OE} \to \mathcal{SEE} \to \mathcal{WEE}$$

If two processes are weak epistemic equivalent then they will produce the same wave function in the asymptotic limit, and so they will produce the same emergent NRQM. Clearly given a particular quantum system, its Lagrangian, and a collection of boundary conditions, we will want any process to generate the usual quantum mechanical wave function in the asymptotic limit and so the collection of processes that we wish to consider should all be weak epistemic equivalent to one another. They will differ in many other characteristics, such as the rate of convergence in the asymptotic limit, the levels of accuracy, band limitation, their specific strategies, whether or not they are process or ontic equivalent. It is these properties, together with the presence of absence of physically plausible features and/or observable implications, that enables one to select out those processes (and thus their representations as reality games) which could serve as potential models of a fundamental level of reality.

2.8 The Nature of Measurement

The process of measurement, as understood by Bohr, provides a correspondence between a state of a quantum system and some construct of the classical realm through which one attempts to gain understanding of the quantum system. For Bohr, measurement was intrinsically contextual, referring to the behaviour of a quantum system in conjunction with a measurement apparatus, and the results of such a measurement situation could not be referred to the system alone but only to the quantum system- measurement apparatus pair.

For Bohr, measurement represented a particular type of interaction, that between a quantum system and a measurement apparatus. The process framework also considers measurement to be a particular form of interaction, and its special characteristics derive from certain constraints placed upon the dynamical behavior of the measurement apparatus which permit it to actually serve as a measurement apparatus. A measurement apparatus is a process responsive to the presence of particular properties as expressed by informons. A measurement is the outcome of an interaction between a system process and a measurement apparatus process. The system process generates properties in its informons but as will be seen below, an interaction with the measurement apparatus will transform the system process into a weak epistemic equivalent process capable of generating properties to which the measurement apparatus will respond. If the properties associated with the measurement apparatus are compatible with the original system process, a definite result will occur. If not, an indefinite result (usually meaning multiple inconsistent results) will occur. It is *not* the observer that causes properties to come into existence. It is the processes themselves that do so. The observer merely gets to choose which measurement apparatus to interact with which system process. From a process perspective, reality exists whether or not anyone is looking.

To serve as an effective measurement apparatus, at the very minimum, some component of the measurement apparatus must manifest definite, enduring (or irreversible), distinguishable states which serve as signs which are capable of interpretation by an observer as 'measurements ' , which may be consistently correlated with particular preparations of a quantum system, and which are robust for a particular class of measurement apparatus so that different instances of such an apparatus applied to identically prepared copies of the quantum system will yield identical signs, at least in the case of ideal measurements. There will always be sources of noise so more precisely we should be talking about probability distributions of signs with small variances and minimal overlap, but that would render the discussion unduly complicated. A single measurement of a single quantum system should yield a single sign, although repeated measurements of the same system, or identically prepared copies of the system, might yield a set consisting of multiple signs in the quantum setting, resulting in a multiply peaked probability distribution with small variances. In such a case one needs some assurance that this variation is due to the quantum system or the interaction process and not to the measurement apparatus alone.

Since process generates both the quantum system and the measurement apparatus, it is essential to understand how the measurement apparatus behaves dynamically during the system-apparatus interaction. Each measurement apparatus will be generated by its own distinct process and follow its own particular dynamics. Nevertheless it is possible to formulate some generic constraints on this dynamics due to the considerations described above. In particular, regardless of the quantum system and the details of the interaction process, it is necessary that the measurement apparatus evolve within its own phase space so that at the end of the process it finds itself in a state corresponding to exactly one of its signs. Each such sign state represents dynamically an attracting limit state (point or perhaps a cycle) and in order for this process to occur in a stable and reliable manner each such sign state should possess a basin of attraction that is separable from those of the other sign states. Chaotic and tangled basins are not suitable for a measurement apparatus as tiny fluctuations would become chaotically amplified and the resulting sign state would become inconsistent and random even in the case of a quantum system whose state is a single eigenstate for the measurement process. For example, one could place a dial on one bob of a kicked double pendulum and attempt to measure the momentum of a particle by letting it collide with the kick plate. The subsequent motion of the dial will be chaotic, so that even if a particle with exactly the same momentum hits the plate, the resulting motion of the dial will be effectively random and decidedly non-Gaussian, so no single *measurement* value can be determined. In such a case no consistent, meaningful measurement can be obtained.

Thus a measurement apparatus should possess a dynamical structure which results in it possessing a set of attracting limit states, each possessing distinct non-interpenetrating basins of attraction. These basins need to be stable so that small fluctuations in the state of the measurement apparatus do not shift the trajectory to a different limit state. They also need to be sufficiently broad and separated so that small fluctuations in the quantum system do not result in trajectories to different limit states. These limit states of the measurement apparatus must, in turn, cause a transition in some kind of recording device. The recording device must also possess the same dynamical characteristics as the measurement apparatus and for the same reasons but in addition the limit states of the recording device must be at least transiently irreversible (so that the information encoded in the limit state cannot be erased and so it can serve as a sign for that information) and enduring (so that the information persists for at least a sufficient duration of time so as to enable an external observer to obtain the information). The limit state of the measurement apparatus may be irreversible (one-use, like a photographic plate), or it may be possible under an external influence to return the measurement apparatus to its initial state (i.e. the initial state should be reachable from each of the limit states) so that it can be used to perform new measurements. The limit state should be stable in the absence of such an external influence so that it persists long enough to register a sign on the recording device.

Moreover, in order for these signs to convey meaningful information about the quantum system, it is also necessary for the interaction between the quantum system and the measurement apparatus to generate a stable correspondence between distinct states of the quantum system and distinct signs. In order for this to occur it is necessary that the dynamical behaviour of the quantum system and the measurement apparatus couple so that each state of the quantum system will cause the measurement apparatus to enter into the same, or at least to nearby, basins of attraction so that the measurement apparatus will transition to more or less the same sign. When a measurement apparatus possesses these dynamical characteristics and a stable correspondence can be established between the states of a quantum system and the signs generated by the measurement apparatus following an interaction between them, then one can say that a measurement situation has been established and when a sign is generated, that a measurement has taken place.

There are additional constraints on the quantum system as well. In the classical setting we are most familiar with systems of inanimate matter that remain stable, as far as their physical properties are concerned, over long periods of time, and over many spatial scales. Biological systems are quite unlike these inanimate systems as they develop over time, and indeed vary in their constituents and properties from moment to moment. This is all too familiar to complex systems theorists. The stability of measurements is sustained by the presence of protection. Laughlin and Pines coined the term *protection* to refer to the attractive fixed point of the renormalization group [225, pg 238]. The existence of laws of nature that are insensitive to external destabilizing influences gives rise to protection. Laughlin writes [225]:

> Protection generates exactness and reliability in the physical world... its physical versions have the advantage of being primitive, so that one can unambigu-

> ously identify them as spontaneous self-organizational phenomena involving no intelligence other than the principle of organization itself (pg 144)

In contradistinction to protection, the diminution of a physical property under renormalization is called *irrelevance*. Laughlin describes irrelevant properties as being 'doomed by principles of emergence to be unmeasurably small' . *Relevance* refers to a physical quantity that grows with sample size and whichever phase a system is in. There can arise balanced situations in which additional, rather spurious, factors come into play in determining the transition. This is balanced or unstable protection.

Protection, because it may involve universal principles, that is principles that pertain independent of the constituents that comprise a system, can obscure the underlying elements of the system. Laughlin points to the elastic rigidity of solids which is a consequence of ordering independent of whether or not there are atoms. He suggests that the vacuum also shows signs of universal protection and thus hides its fundamental structure (if any)[225].

This 'dark side of protection' gives rise to what Laughlin [225] describes as the 'Dark Corollaries' to the presence of universality and relevance. These dark corollaries are the *deceitful turkey effect* and *the barrier of relevance*. The deceitful turkey effect refers to situations in which unstable protection makes one think that one has found the underlying microscopic rules when in fact one has not. The barrier of relevance is a spatial scale version of the phenomenon of sensitive dependence on initial conditions in chaos theory. In this case as the spatial scale increases the properties of the material become less and less predictable. This phenomenon is observed in studies of correlated electron systems, in particular the appearance of spectrographic properties that do not reproduce, hypersensitivity to atomic imperfection and ordered phases that come and go depending upon the sample preparation method. Laughlin writes [225]:

> "It is fundamentally impossible to stabilize the experiments against a material's nuances, hence the failure of the experiments to be reproducible " (pgs 152).

A measurement can be made since it is an interaction, and an outcome state of the measurement apparatus will be obtained, but a stable correspondence with the quantum system cannot be achieved because in a real sense the quantum system does not possess the property being measured.

Universality and irrelevance permit the emergence of stable measurements of quantum systems. The dark corollaries, on the other hand, prevent the emergence of such stability, and thus prevent the consistency of measurements, at least of measurements that can be said to reflect intrinsic properties of the system being measured. Thus it is not merely the measurement apparatus that must possess particular dynamical characteristics, it is the quantum system as well. Unless the quantum system possesses dynamical characteristics such as universality and irrelevance, measurements observed at one scale may not persist at larger scales, and the notion of intrinsic property begins to break down. Nevertheless it is still the case, if the measurement apparatus is suitably constructed, that a single sign will be generated from each interaction between the measurement apparatus and the quantum system. Thus a potential property is generated out of this interaction.

From the standpoint of the process perspective, the question as to whether or not the quantum system possesses the property in question is not relevant, since the quantum system is being generated moment to moment by the process. In essence, each property of the quantum system must also be generated moment by moment. The process that generates the actual occasions that manifest the quantum system in reality may be attributed the property in question but not in the sense that the process *possesses* the property. More precisely, the process *generates* the property through its capacity to generate a dynamical interaction between the quantum system and the measurement apparatus in which a stable correspondence is established between the states of the quantum system and certain states of the measurement apparatus that reflect the property.

2.9 The Measurement Process

Let us consider the situation of a single particle interacting with a measurement apparatus. The particle and the apparatus are generated by distinct processes, $\mathbb{P}$ and $\mathbb{M}$ respectively. We assume that the particle is in a superposition of eigenstates $\Psi_i(\mathbf{z})$ (with eigenvalues λ_i) of its Hamiltonian. In NRQM the state of the particle would be represented as $\Psi(\mathbf{z}) = \sum_i w_i \Psi_i(\mathbf{z})$ for some weights w_i. We assign a distinct primitive process $\mathbb{P}_i(\lambda_i)$ to generate each eigenstate and consider the particle process as a weighted combinatorial sum of these primitive processes, $\mathbb{P} = \sum_i w_i \mathbb{P}_i(\lambda_i)$. The weight in this case modifies the strength or value of the local $\mathcal{H}(\mathcal{M})$-interpretation of each informon generated. Likewise treat each basin of attraction of the measurement apparatus as a distinct subsystem and assign to each a distinct process which generates its own dynamic. If the set of possible measurement values is $\{\mu_j\}$, then to each value we assign a process $\mathbb{M}_j(\mu_j)$ and we consider $\mathbb{M} = \sum_j \mathbb{M}_j(\mu_j)$ where the appropriate normalization is subsumed within $\mathbb{M}_j(\mu_j)$.

The fundamental algebraic structure of functional analysis is that of a vector space, and just as a vector in a finite dimensional vector space may be decomposed as a sum of basis elements, so can any element of a Hilbert space. Generally a Hilbert space will possess multiple distinct bases and so given two such bases $\{\phi_i(\mathbf{z})\}$ and $\{\sigma_i(\mathbf{z})\}$ we may write any element $\Psi(\mathbf{z})$ as $\Psi(\mathbf{z}) = \sum_i w_i\phi_i(\mathbf{z}) = \sum_j v_j\sigma_j(\mathbf{z})$. In functional analysis one is free to choose any basis at will and represent any vector within that basis. In the ideal world of mathematics this occurs without any physical implications. Again the functional analytic framework captures an algebraic relationship but without any aspect of ontology. The mathematics demonstrates that which is implicit in the structure of the wave function and what is to be expected should such a change in measurement basis be carried out but it doesn't specify how this change is to be realized in reality. While a change of basis in a finite dimensional vector space can be viewed merely as a change of reference frame, or perspective, a change of basis in quantum mechanics represents a change of measurement serving as the underlying basis for the observation. It does not merely represent a change of observer or of position of an observer. It reflects, rather, a fundamental change in how the observation is being carried out and what kind of measurement apparatus is being utilized to carry out the observation. It i snot merely ideational, it is pragmatic.

In the process perspective, a decomposition of a wave function as a sum of the form $\Psi(\mathbf{z}) = \sum_i w_i\Psi_i(\mathbf{z})$ implies that the informons of the quantum system are being generated by a process $\mathbb{P}$ which can be decomposed into a sum of primitive processes as $\mathbb{P} = \sum_i w_i\mathbb{P}_i$ where each subprocess $w_i\mathbb{P}_i$ generates the wave function contribution $w_i\phi_i(\mathbf{z})$. A change of basis therefore implies not merely a change of perspective but a change in the very processes that are generating the actual occasions. That is, a change of basis carries ontological significance and is not simply epistemological.

The standard quantum mechanical prescription for measurement is to supplement the usual Schrödinger equation with a new equation. Each measurement procedure is associated with an operator A such that the possible values associated to such a measurement are given by the eigenvalues of the equation $A\Psi = \lambda\Psi$. The eigenvectors Ψ_i are those wave functions that satisfy this eigenvalue equation and they are the wave functions that are preserved, up to a scaling factor, under the action of the measurement operator. A general wave function $\Psi(\mathbf{z})$ can be decomposed into a sum of such eigenfunctions, $\Psi(\mathbf{z}) = \sum_i \lambda_i\psi_i(\mathbf{z})$. In process terms, this means decomposing a process $\mathbb{P}$ into an exclusive, independent sum of subprocesses $\mathbb{P}_i$, each of which is preserved under interaction with the measurement process. Rather than being an ad hoc assumption, this becomes an integral part of the process theory of measurement. It is a prescription for determining how a given process will transform due to the boundary conditions imposed by a generic measurement apparatus. It is not a dynamical formula, it is an heuristic formula, however. The dynamic is expressed as the rotation in the process space. The heuristic enables one to determine which subprocesses will form the basis of this rotation, and thus enable the newly rotated process to be calculated. This decomposition does *not* mean that the original process $\mathbb{P}$ was in fact a sum of these subprocesses. Rather it is a transformation of $\mathbb{P}$ into the weak epistemic equivalent process $\oplus w_i\mathbb{P}_i$. They are *different* processes, although they will give rise to the same wave function asymptotically. This is a very important distinction.

Let us fix a context within which to observe a quantum system and let that system experience an evolution described by a wave function $\Psi(\mathbf{z})$. Furthermore, let there be two distinct basis decompositions of $\Psi(\mathbf{z})$, namely, $\Psi(\mathbf{z}) = \sum_i w_i\Psi_i(\mathbf{z}) = \sum_j w'_j\Psi'_j(\mathbf{z}) = \Psi'(\mathbf{z})$. There will be distinct processes $\mathbb{P}_i$ generating $\Psi_i(\mathbf{z})$ and $\mathbb{P}'_j$ generating $\Psi'_j(\mathbf{z})$. Hence $\mathbb{P} = \sum_i w_i\mathbb{P}_i$ generates $\Psi(\mathbf{z})$ and $\mathbb{P}' = \sum_i w'_i\mathbb{P}'_i$ generates $\Psi'(\mathbf{z})$. Since $\Psi(\mathbf{z}) = \Psi'(\mathbf{z})$ it follows that $\mathbb{P}$ and $\mathbb{P}'$ are weak epistemic equivalent.

For simplicity let us ignore details of experimental error. The measurement process is understood as taking place in three stages.

1. In the first stage, prior to the initiation of any interaction between the quantum system and the measurement apparatus, the two games are played freely, so that the combined process can be represented as $\mathbb{P} \otimes \mathbb{M} = (\sum_i w_i\mathbb{P}_i) \otimes \mathbb{M} = (\sum_i w_i\mathbb{P}_i) \otimes (\sum_j \mathbb{M}_j(\mu_j)) = \sum_i \sum_j w_i\mathbb{P}_i \otimes \mathbb{M}_j(\mu_j)$.
2. In the second stage, the quantum system enters into the region of the measurement apparatus. In this stage there is no exchange of physical attributes such as energy, momentum, spin, mass etc. Instead the measurement apparatus establishes a new set of boundary conditions which results in a purely informational effect upon the processes generating the quantum system. This information results in a reversible change in the processes generating the wave function. This occurs, as described above, because it is precisely those system states that are eigenstates of the operator corresponding to the measurement apparatus that are preserved under an interaction with the apparatus. Only these states can survive an interaction with the apparatus for a long enough period of time so that a measurement can take place. Although the processes generating the wave function have changed, the observed wave function remains unchanged. We may now write this new process as $\mathfrak{M}(\mathbb{P}) = \mathbb{P}' = \sum_j w'_j\mathbb{P}'_j(\lambda_j)$ where $\mathbb{P}'_j$ represents the process that generates the eigenfunction $\Psi_j^{\lambda_j}(\mathbf{z})$ corresponding to the eigenvalue λ_j of the measurement apparatus. Here $\mathfrak{M}(\mathbb{P})$ denotes that the process $\mathbb{P}'$ was derived from $\mathbb{P}$ under a change of basis to that of eigenfunctions of the measurement apparatus. Since the wave functions generated by $\mathbb{P}$ and $\mathfrak{M}(\mathbb{P})$ are related by a change of basis, they represent the very same wave function, Ψ and so $\mathbb{P}$ is weak epistemic equivalent to $\mathfrak{M}(\mathbb{P})$. Although this interaction is informational

(perhaps adiabatic), it nevertheless results in a transition of processes, which we write as:

$\mathbb{P} \otimes \mathbb{M} = \sum_i \sum_j w_i \mathbb{P}_i \otimes \mathbb{M}_j(\mu_j) \rightarrow \mathfrak{M}(\mathbb{P}) \otimes \mathbb{M} = \sum_k \sum_j w'_k \mathbb{P}'_k(\lambda_k) \otimes \mathbb{M}_j(\mu_j)$.

3. In the third stage the quantum system enters into the region in which a physical interaction with the measurement apparatus becomes possible. Now each subprocess of $\mathfrak{M}(\mathbb{P})$ can couple only with certain subprocesses of the measurement apparatus (corresponding to nearby measurement values) so that the combined process at the initiation of measurement will be represented more accurately as $\mathfrak{M}(\mathbb{P}) \boxtimes \mathbb{M} = \sum_j \sum_{j' \in H(j)} (\tilde{w}_{jj'} \mathbb{P}'(\lambda_j) \boxtimes \mathbb{M}_{j'}(\mu_{j'}))$ where $H(j)$ is the set of measurement apparatus states to which the system state i couples. The operator $\boxtimes$ represents the interaction between the system process and the measurement apparatus process. In the case of no error this becomes $\mathbb{P} \boxtimes \mathbb{M} = \sum_j \tilde{w}_j \mathbb{P}'(\lambda_j) \boxtimes \mathbb{M}_j(\lambda_j)$.

   The actual initiation of interaction becomes a function of the coupling between system and apparatus. That is, with each play of the game there will manifest a system informon, say $[n] < \alpha > \{G\}$. Let $\alpha = (\mathbf{m}, \phi, \mathbf{p})$. By assumption this informon will be generated by some process of the form $\tilde{w}_{jj'} \mathbb{P}'(\lambda_j)$ for some eigenvalue λ_j, so that its Hilbert space interpretation will be an interpolation contribution to $\tilde{w}_{jj'} \Psi_j^{'\lambda_j}(\mathbf{z})$, the eigenfunction corresponding to λ_j. Note that $\phi(\mathbf{m}) = \tilde{w}_{jj'} \Psi_j^{'\lambda_j}(\mathbf{m})$. There will also manifest a measurement apparatus informon, say $[m] < \beta > \{H\}$ which is being generated by some subprocess, say $\mathbb{M}_{j'}(\mu_{j'})$.

   The likelihood that the system process will couple to the measurement apparatus process will be expected to depend upon the strength of the process at that site and so should be proportional to $\Gamma_n^* \Gamma_n = \phi^*(\mathbf{m})\phi(\mathbf{m}) = \tilde{w}_{jj'}^* (\Psi_j^{'\lambda_j})^*(\mathbf{m}) w_{jj'} \Psi_j^{'\lambda_j}(\mathbf{m})$. The likelihood that it will fail to couple is thus proportional to $p = 1 - \Gamma_n^* \Gamma_n = p = 1 - \phi^*(\mathbf{m})\phi(\mathbf{m})$. If it does not couple then a new informon is created and the likelihood that it will not couple will again be proportional to $1 - x$ for some non-zero x. The probability that it will not couple after two plays is proportional to $p(1 - x)$ and this will rapidly tend to zero with successive game play. Thus almost certainly at some point a coupling will be initiated between the system and the apparatus. item Suppose that this current system informon couples to the current measurement apparatus informon. Then the global process undergoes a transition to an interactive process which is substantially reduced from the original process, namely

$$\tilde{w}_{kk'} \mathbb{P}_k('\lambda_k) \boxtimes \mathbb{M}_{k'}(\lambda_{k'})$$

   or in the case of no error to

$$\tilde{w}_k \mathbb{P}'_k(\lambda_k) \boxtimes \mathbb{M}_k(\lambda_k)$$

The system process has undergone a transition which means that the descriptor no longer refers to the original generative process but only to the current process. As a consequence there will no longer be any play to those informons whose descriptors are for the previous process and its subprocesses. The same is true for the measurement apparatus as its generative process is now restricted to the interactive version of $\mathbb{M}_k(\lambda_k)$.

Note that if the system exits the measurement apparatus and is observed again by an identical measurement apparatus it will again couple to the same component of the second measurement apparatus but if it should be observed by a new apparatus then a new coupling will arise as a result of the transition that will be induced in the generating process by the interaction with the new measurement apparatus.

The strength of the process at an informon thus determines the likelihood of coupling between the quantum system and the measurement apparatus processes. It will also determine the coupling to many different processes. The strength is proportional to the value of the wave function at the corresponding causal manifold point. The interpretation of the wave function or strength as a probability distribution over the causal manifold, however, is contextual, requiring the presence of a suitable measurement device whose measurement values correspond to positions in the causal manifold. The correspondence then is a secondary emergent one, requiring the intermediary measurement apparatus.

At every stage only a single informon is manifest, providing a determinate reality, but as these informons lie below the level of resolution of any physical apparatus only the coherent whole of the resultant tapestry and its state is actually observable. Repeated actions of the process will yield different outcomes. The apparent stochasticity of the quantum system arises from the inherent non-determinism of process, not from an inherent indeterminism in the informons (actual occasions) themselves - the level of ultimate reality. Moreover the collapse that is observed in the wave function is not a collapse at the level of ultimate reality, which only ever manifests single informons, but rather at the level of process and merely reflects that fact that any interaction between processes potentially alters those processes.

Note that in the third stage of the measurement process there is an interaction between the system process and the measurement apparatus process that results in the inactivation of the original system process and the activation of a single subprocess now coupled to a subprocess of the measurement apparatus process. The effect of this on the wave function is to effect a transition from the original wave function, being of the form $\Psi(\mathbf{z}) = \sum_i w_i \Psi_i(\mathbf{z})$ to a lesser wave function, $\Psi_i(\mathbf{z})$ say. It would appear as though the wave function "collapsed", but in reality it is the system process that has undergone the transition. The inactivation of the remaining component subprocesses could be viewed as a form of collapse, but that has a catastrophic quality to it, whereas in reality there has merely been a transition, perfectly natural since an non trivial interaction has taken place between the system process and the measurement apparatus process. This transition is not spontaneous as in GRW, it comes about as a direct result of the interaction, although the precise outcome of the interaction cannot be determined in advance.

2.10 Emergence of Probability and of Properties

In the standard interpretation of quantum mechanics, the wave function is interpreted as a square root of a probability distribution according to Born's rule. In the process framework, the wave function also generates a probability distribution but the relationship between these two is rather more subtle. In standard quantum mechanics, the magnitude of the wave function *is* a probability distribution. That is not true in the process framework. Here the magnitude of the wave function gives the strength of the process in the local region. The wave function is treated as a real wave, much like an electromagnetic wave in which the energy is related to the magnitude of the wave. The strength of the wave function is given by the square of the magnitude of the wave function. When one process couples to another the strength of the coupling depends upon the magnitude of the processes involved at the site at which the coupling initiates. Unlike a normal wave which exists simultaneously throughout space-time, a process manifests at distinct informons, thus generating distinct space-time locations. The effect of interaction is to induce a change in the active process. Since each process acts non-deterministically and in a saltatory manner, a change in the game has widespread effects throughout space-time, even though the change itself is initiated locally. A basic assumption of the process model is that the likelihood of such a change being induced will depend upon the strength of this local coupling - the greater the strength the greater the likelihood that a transition will occur. Observation tells us that this transition must occur stochastically and so it makes sense to assume that the probability of coupling will be proportional to the these strengths.

Born's rule is a specific consequence arising from the connection between a system and a position measurement apparatus. The position probability distribution is also an emergent aspect. The quantum system does not actually possess a location - not the generating process $\mathbb{P}$ nor the $\mathcal{M}$- or $\mathcal{H}(\mathcal{M})$-interpretations, which are multiple since each corresponds to a contributing informon. Moreover, these latter are *interpretations*. Actual occasions are not the quantum system. That status is accorded to the generating process. They are an instance of process. They represent the strength of the generating process at a specific space-time location. They determine the probability of triggering a position measurement event corresponding to that space-time location. This position measurement event is then interpreted as the quantum system physically *occupying* that space-time location when in actuality it is merely the locus of an interaction between the system and the measurement apparatus.

The strategy of process requires that a triggering informon is tested only when it is complete, that is, when it is no longer possible to add tokens to a particular space-time location. Thus a testing of the reality game itself occurs at the completion of each full round whenever an interaction situation is taking place. Since the addition of tokens occurs in a saltatory manner, in a sense the process, through its generated informons, occupies a substantial region of space-time even though on any given round of play tokens are being added to specific sites. When an informon is completed it can then be tested as to whether it triggers a coupling to any other processes being played. This occurs stochastically based on the local magnitude of process given by its $\mathcal{H}(\mathcal{M})$-interpretation value measured at the embedding site. If a coupling does not occur then game play continues and new informons are completed and again tested. Eventually one will trigger a transition, if not during this game then during some subsequent game. The probability interpretation does not actually describe what is happening at the level of the individual informons and is therefore an emergent phenomenon.

More will be said about emergent probability in Chapter 4.

Although informons are assigned properties, these properties connect them to the process that generates them, and this process in turn connects them to other processes, in particular to those that describe the evolution of a measurement apparatus. The properties of an informon are not directly measurable. If one applies a quantum mechanical measurement operator to the $\mathcal{H}(\mathcal{M})$-interpretation of an informon one does not receive a single measurement value even though such a value may form one of its properties. The $\mathcal{H}(\mathcal{M})$-interpretation of an informon is not a wave function per se. It is only the global $\mathcal{H}(\mathcal{M})$-interpretation on the embedding slice which can be interpreted as a wave function. Locally it simply manifests the generating process and the strength of that process at that point.

In order to determine the value of some property associated with an informon it is necessary that a measurement take place. In other words, there must exist some measurement apparatus with which the process generating the informon interacts, and furthermore it is also necessary that said informon trigger off a transition in the measurement process which eventually results in the measurement apparatus producing a sign representing the value of the property in question. Each informon represents a definite actual occasion, a definite fundamental element of reality, to which is assigned a definite space-time embedding point, a definite $\mathcal{H}(\mathcal{M})$-interpretation and definite properties. None of these are directly observable due to the near infinitesimal scale at which these actual occasions manifest. It is only at the emergent collective level that enough informons arise so as to constitute a phenomenon that can be considered to be an observable phenomenon and subsequently couple to a measurement apparatus. Processes serve as generators of actual occasions, and actual occasion serve as triggers for some measurement apparatus, enabling it to transition into a state interpretable as a measurement. In this way, measurement appears as an emergent process, arising out of the interaction between system and measurement apparatus.

Although it is not possible to measure directly the properties of actual occasions they nevertheless do carry the weight of such properties in that they are able to make a difference by triggering an interaction between their generating process and some other process. They also couple only the subprocess of the generating process that actually generates them, and therefore that possesses the same set of properties. The net effect of the coupling between the system process and any other process will be determined by the properties of the process, which are perfectly correlated with the triggering actual occasion. Due to the correlation between the properties of the actual occasion and its generating process, it is reasonable, if slightly inaccurate, to attribute these properties to the actual occasions themselves rather than to their generating processes. Thus we may say that an actual occasion *has* such and such a property, even though it is its generating process which actually expresses such a property. Processes thus manifest themselves indirectly though the actual occasions that they generate, while actual occasions express themselves by inducing transitions among the processes that generate them, thus influencing the subsequent generation of actual occasions. This is a subtle point but it explains the sense in which we understand the assertion that actual occasions as represented by informons possess definite properties, even though these properties are actually emergent outcomes of the interactions between systems and specialized properties called measurement apparatus.

The measurement theory of the process framework thus accords the wave function, in the form of the $\mathcal{H}(\mathcal{M})$-interpretation (and thus effectively the NRQM wave function) an ontological status, as a manifestation of process, in particular its strength. The probabilistic aspects of the wave function are an emergent property which arise in the context of a specific measurement situation. Measurement, in turn, is understood as a particular form of interaction between entities possessing specific dynamical properties. It is not mysterious, and there is no need for an observer to make reality real.

2.11 Process Approach to the Configuration Space

Let us reconsider the process covering map. The discussion of Section 2.5 suggested a rather simple connection between the process covering map and standard NRQM descriptions of fundamental entities. This is far from the truth. That discussion was overly simplified and belies a deep subtlety. In the process framework, space-time and physical entities are generated. Recall that the wave function, which describes physical entities, is considered to represent a real physical wave whose value at a space-time point represents the local strength of the process at that point. This strength determines the likelihood of the process coupling to other active processes or undergoing a transition. As described in Section 2.10 probabilities too are generated. The wave function thus corresponds to real events taking place in space-time and cannot be merely understood as a description of belief, knowledge, possibility or probability. This poses a problem in the case of multiple physical entities since the usual NRQM wave function is defined on an abstract configuration space, and not on real space-time.

Process was described as acting in a series of rounds, during each of which one or more informons are generated. In a sum of primitive processes, single informons are generated sequentially while in a product they are generated simultaneously. Depending upon the precise algebraic construction, the number of informons generated per round may vary. For example, in $\mathbb{P} \oplus (\mathbb{P} \otimes \mathbb{P})$ either $\mathbb{P}$ or $\mathbb{P} \otimes \mathbb{P}$ may be played resulting in either one or two informons being generated per round. To the observer, however, they will see three physical entities, two possibly correlated and one definitely independent. Thus as mentioned in Section 2.5, the co-product formulation would seem to be the natural formulation for the representation of an independent product of primitive processes since it better represents the idea of several active but independent processes acting simultaneously. This is certainly the case when dealing with indistinguishable entities. In the case of distinguishable entities, the product form would appear to be the more natural representation, at least in so far as calculations are concerned. The co-product, however, still provides the more realist representation. Correlations among the informons arise at the level of the generating processes. These correlations manifest indirectly through their influence on the measurement process as discussed in Section 2.9.

The subtlety arises because the measurement process is intimately linked to the appearance of individual informons and not to the wave function per se. The wave function provides information about the likelihood that informons will couple with the measurement apparatus process and thus trigger the series of events leading to an observable measurement value. It is the sequence tree, however, that provides information about the actual generation of informons. Thus if one is concerned with the statistical structure of the measurement values that can be obtained from some process, it makes sense to focus on the sequence tree rather than the wave function. The sequence tree, however, is *not* an ontological entity. It is a formal mathematical construct that does not manifest in reality but provides information about the possible outcomes of the action of a process on a fixed initial causal tapestry. Recall the structure of the sequence tree. The sequence tree summarizes the complete set of possible actions of a process. Beginning with an initial causal tapestry, one complete action of a process is represented by a single path down the sequence tree. If we have n primitive processes $\mathbb{P}_j$ and consider the product representation $\otimes_j \mathbb{P}_j$, then the sequence tree will consist of a tree whose vertices are causal tapestries and each edge of the tree $\mathcal{I}_i \to \mathcal{I}_j$ corresponds to the addition of a set $\{n_1, \ldots, n_n\}$ of at most n new informons, one for each $\mathbb{P}_j$, to $\mathcal{I}_i$. Note that for $\otimes$ these informons must embed to distinct elements of $\mathcal{M}$ but for $\hat{\otimes}$ it is possible for some or all to map to the same element of $\mathcal{M}$. Regardless, $\mathcal{I}_j = \mathcal{I}_i \cup \{n_1, \ldots, n_n\}$. Denote this tree by $\Sigma(\mathcal{I}, \otimes_i \mathbb{P}_i)$ (or $\Sigma(\mathcal{I}, \hat{\otimes}_i \mathbb{P}_i)$ for the free product). Let $\Sigma(\mathcal{I}, \mathbb{P}_i)$ denote the sequence tree for $\mathbb{P}_i$. These sequence trees have a natural structure based upon birthdays. Beginning with $\mathcal{I}$, assigned birthday 0, there will be an initial set of edges corresponding to birthday 1. Each terminal vertex of these edges is assigned birthday 1, and to each of these terminal vertices there will then be appended a new set of edges corresponding to birthday 2. These new terminal vertices are assigned birthday 2. This procedure continues indefinitely. Define the ordered product $\rhd_i \Sigma(\mathcal{I}, \mathbb{P}_i)$ as follows. Form tuples of elements from each sequence tree with the proviso that only elements having the same birthday may form a tuple. Hence a tuple $(\mathcal{I}_1, \ldots, \mathcal{I}_n)$ means that $\mathcal{I}_i \in \Sigma(\mathcal{I}, \mathbb{P}_i)$ and has birthday m (fixed for this tuple). An edge in the product will be a product of edges in each component sequence tree. Clearly then all terminal vertices of these edges will possess the same birthday. Recall that each informon has the form $[n] < \mathbf{m}_n, \phi_n(\mathbf{z}), \Gamma_n, \mathbf{p}_n > \{G_n\}$ where $\mathbf{p}_n$ corresponds to the property of one of the generating processes $\mathbb{P}_i$. In either the free or exclusive cases there is a natural embedding $e : \Sigma(\mathcal{I}, \otimes_i \mathbb{P}_i) \to \rhd_i \Sigma(\mathcal{I}, \mathbb{P}_i)$ obtained by clustering the informons corresponding to each of the subprocesses $\mathbb{P}_i$ and then mapping each such partitioned causal tapestry $\mathcal{I} = \{n_1^1, \ldots, n_{j^1}^1, \ldots, n_1^n, \ldots, n_{j^n}^n\}$ to the product $\{n_1^1, \ldots, n_{j^1}^1\} \times \cdots \times \{n_1^n, \ldots, n_{j^n}^n\}$. In the free case we may define an isomorphism $f : \rhd_i \Sigma(\mathcal{I}, \mathbb{P}_i) \to \hat{\Sigma}(\mathcal{I}, \otimes \mathbb{P}_i)$ by $f((\mathcal{I}_1, \ldots, \mathcal{I}_n)) = \cup_i \mathcal{I}_i$. In the exclusive case we must consider a reduced $\rhd$ product which excludes all elements $(\mathcal{I}_1, \ldots, \mathcal{I}_n)$ such that for some i, j, there are informons $n_i \in \mathcal{I}_i$ and $n_j \in \mathcal{I}_j$ such that $\mathbf{m}_{n_i} = \mathbf{m}_{n_j}$. In the case of interactive products the pruning of vertices can be even more extreme and must be specified for each individual interactive product.

Now each edge of the sequence tree adds a new set of informons to the initial vertex (causal tapestry) and this in turn adds new contributions to the global $\mathcal{H}(\mathcal{M})$ interpretation. Indeed if the new vertices form the set $\{n_1, \ldots, n_n\}$ then the new contributions will be

$$\Gamma_{n_1} T_{\mathbf{m}_{n_1}} sinc_{t_P l_P}(\mathbf{z}), \ldots, \Gamma_{n_n} T_{\mathbf{m}_{n_n}} sinc_{t_P l_P}(\mathbf{z})$$

Each term is incorporated into the global $\mathcal{H}(\mathcal{M})$ for the corresponding process, so

$$\Phi^i(\mathbf{z}) \to \Phi^i(\mathbf{z}) + \sum_j \Gamma_{n_j} T_{\mathbf{m}_{n_j}} sinc_{t_P l_P}(\mathbf{z})$$

These individual global $\mathcal{H}(\mathcal{M})$ interpretations are defined on the causal manifold $\mathcal{M}$ as they represent real wave-like entities on the causal manifold. Note though that in moving from the sequence tree to the wave functions, essential information is lost relating to correlations between informons in their generation. These correlations affect the measurement process and so it is important to see how to access them.

In classical mechanics the way to address this problem is by means of the configuration space. Instead of thinking of n particles moving in 3-space, one thinks of one particle moving in the 3n-space of configuration space. Again, rather than treating the particles as separate and denoting a state of the system as a set of position vectors $\{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$ (which would be the coproduct) one considers the product $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ which represents a single 3n dimensional vector. In NRQM the wave function for such a system is defined on this configuration space. In doing so, it is possible to extract out correlational information but a direct connection to real experience is lost. There are some authors such as Albert [271] who believe that the configuration space is reality but that position, while widely held, is problematic.

As discussed in Section 2.5, it is reasonable from an ontological perspective to define

$$\mathfrak{P}(\otimes_i \mathbb{P}_i) = \mathfrak{P}(\hat{\otimes}_{\mathrm{i}} \mathbb{P}_{\mathrm{i}}) = \mathfrak{P}(\mathbb{P}_1) \times \cdots \times \mathfrak{P}(\mathbb{P}_n)$$

where the $\times$ denotes the set theoretic product and not the function theoretic product.

In NRQM it would be more typical to define this free product wave function as

$$\hat{\Psi}(\mathbf{z}_1, \ldots, \mathbf{z}_n) = \Psi_1(\mathbf{z}_1) \cdots \Psi_n(\mathbf{z}_n)$$

where now the product is indeed the function theoretic product.

The process perspective, however, considers the causal tapestry (or its interpretation as a causal manifold) to be the space-time of reality and physical entities both generate and exist in this reality by virtue of their generating processes. Correlations between physical entities arise due to correlations among their generating processes. NRQM attempts to incorporate all of this additional information into the wave function but the PCM illustrates how much of this information is actually lost in moving from one framework to the other. In the process framework it is quite possible for the sets of informons corresponding to one process to be wholly distinct from those for a different process and thus it is not possible to directly assign a value for a global $\mathcal{H}(\mathcal{M})$ interpretation for the combined processes at every single point.

The problem lies in how the global $\mathcal{H}(\mathcal{M})$ interpretations are generated. In the process framework each global $\mathcal{H}(\mathcal{M})$ interpretation is generated on a subtapestry of a single causal tapestry $\mathcal{I}'$ which embeds as a space-like hypersurface of a causal manifold $\mathcal{M}$ and will have the form

$$\Psi^{\{j\}}(\mathbf{z}_j) = \sum_{n_i \in \mathcal{I}_j} \Gamma_{n_i} T_{\mathbf{m}_{n_i}} sinc_{t_P l_P}(\mathbf{z}_j)$$

The approach from NRQM suggests that one should take the global $\mathcal{H}(\mathcal{M})$ interpretation on the configuration space to be

$$\hat{\Psi}_{qm}(\mathbf{z}_1, \ldots, \mathbf{z}_n) = \sum_{n_i^1 \in \mathcal{I}_1} \Gamma_{n_i^1} T_{\mathbf{m}_{n_i^1}} sinc_{t_P l_P}(\mathbf{z}_1) \cdots \sum_{n_i^n \in \mathcal{I}_n} \Gamma_{n_i^n} T_{\mathbf{m}_{n_i^n}} sinc_{t_P l_P}(\mathbf{z}_n)$$

The product process on the other hand will generate at round i a correlated set of informons $A_i = \{n_i^1, \ldots, n_i^n\}$. If we wish to define a global $\mathcal{H}(\mathcal{M})$ interpretation which retains information about these correlations and their generation then we need to retain information about local process strengths. Since these local strengths determine local probabilities it makes sense to consider a product of these local strengths to correspond to a form of "local strength" for these correlated informon sets. Let $\mathcal{I}' = \{A_1, A_2, \ldots, A_k, \ldots\}$ be the set of generated informons sets, organized by round of generation (or birthday). $\mathcal{I}'$ corresponds to the set of edges along one complete path of the configuration sequence tree.

Thus it makes the most sense to define the global $\mathcal{H}(\mathcal{M})$ interpretation on the configuration space to be

$$\hat{\Psi}_p(\mathbf{z}_1, \ldots, \mathbf{z}_n) = \sum_{\{n_i^1, \ldots, n_i^n\} \subset \mathcal{I}'} \Gamma_{n_i^1} \cdots \Gamma_{n_i^n} T_{\mathbf{m}_{n_i^1}} sinc_{t_P l_P}(\mathbf{z}_1) \cdots T_{\mathbf{m}_{n_i^n}} sinc_{t_P l_P}(\mathbf{z}_n)$$

which is consistent with the formulation of the global $\mathcal{H}(\mathcal{M})$ interpretation for primitive processes.

It should be clear, however, that

$$\hat{\Psi}_{qm}(\mathbf{z}_1,\dots,\mathbf{z}_n) = \sum_{n_i\in\mathcal{I}_1}\Gamma_{n_i}T_{\mathbf{m}_{n_i}}sinc_{t_Pl_P}(\mathbf{z}_1)\cdots\sum_{n_i\in\mathcal{I}_n}\Gamma_{n_i}T_{\mathbf{m}_{n_i}}sinc_{t_Pl_P}(\mathbf{z}_n)\neq$$

$$\sum_{\{n_i^1,\dots,n_i^n\}\subset\mathcal{I}'}\Gamma_{n_i^1}\cdots\Gamma_{n_i^n}T_{\mathbf{m}_{n_i^1}}sinc_{t_Pl_P}(\mathbf{z}_1)\cdots T_{\mathbf{m}_{n_i^n}}sinc_{t_Pl_P}(\mathbf{z}_n) = \hat{\Psi}_p(\mathbf{z}_1,\dots,\mathbf{z}_n)$$

To see this just evaluate both sides for any set of informons that is not generated by the product process.

The NRQM proposal for an interpretation, $\hat{\Psi}_{qm}$, actually contains more information than was generated by the generating process and so cannot be correct within the process framework. The product process, in k rounds, generates k subsets of n informons, for at most a total of kn informons. This is fine for defining $\hat{\Psi}_p$ but not $\hat{\Psi}_{qm}$, which requires k^n tuples of informons when in fact one only has k tuples.

From an ontological perspective it should be clear that neither of these global $\mathcal{H}(\mathcal{M})$ interpretations can be considered to represent a real wave on the causal manifold (or even a space-like hypersurface). The process generates informons upon which a strength can be defined, and which embed into $\mathcal{M}$, while both of the above functions are generated on an abstract configuration space formed from $\mathcal{M}^n$ and involve tuples of points that may be widely separated spatially. The correlations between informons are seen to arise at the process level, not the informon level.

Thus although the first interpretation, $\hat{\Psi}_{qm}$, is closer in spirit to NRQM, it is only the second interpretation, $\hat{\Psi}_p$, which is ontologically and epistemologically consistent with the process framework.

Unfortunately $\hat{\Psi}_p$ is inadequate for determining correlations. The problem with $\hat{\Psi}_p$ is that it is based upon a single complete action of the product process. A single such global interpretation cannot take into account the effects of all possible actions of the process. Different unfoldings of the process may, and likely will, result in the generation of a different causal tapestry having different informons and different local strengths. Indeed it is quite possible for two informons from different unfoldings to embed to the same point in the causal manifold yet possess different local strengths. In order to provide an accurate determination of correlations it is necessary to take into account these subtleties.

The proper way to generate a function expressing these correlations is through the sequence tree. There is another subtlety though. The description of the sequence tree given above is for the space-time version in which we work with the causal tapestries representing the generated space-time. This does not contain information about the correlations since the edges consist of sets of informons and the subsequent causal tapestry is formed via a set union, thereby losing the correlational information. Instead we must form a configuration sequence tree as follows. For n subprocesses, each vertex of the sequence tree will be a causal tapestry consisting of a set of ordered tuples of n informons of the form $(n_i^1, \ldots, n_i^n)$ which is just an tuple formed from the informons generated at round i by the generating process. An edge will consist of an n-tuple of informons $(n_j^1, \ldots, n_j^n)$ so that if this edge takes $\mathcal{I}_i \to \mathcal{I}_{i+1}$ then $\mathcal{I}_{i+1} = \mathcal{I}_i \cup \{(n_j^1, \ldots, n_j^n)\}$. A path down the sequence tree results in the generation of a configurational causal tapestry ${}_1\mathcal{I}$. A different path results in a different causal tapestry ${}_2\mathcal{I}$. The fact that each informon is generated using causally local information which does not involve any other generated informon means that each informon generated by a subprocess $\mathbb{P}_i$ in the sequence tree is generated independently from all of the others generated by $\mathbb{P}_i$. Although the informons within a given edge set in the sequence tree may be correlated by virtue of the structure of the product process, each such edge set is generated independently from every other edge set. This is another subtle but important point. It means that we may artificially extend the causal set ${}_1\mathcal{I}$ by adding informons from ${}_2\mathcal{I}$ so long as we ensure that if we wish to add n_2 to component $\mathcal{I}^i$ of $\mathcal{I}_1$ and there exists in component $\mathcal{I}^i$ of $\mathcal{I}_1$ an n_1 such that $\mathbf{p}_{n_1} = \mathbf{p}_{n_2}$ and $\mathbf{m}_{n_1} = \mathbf{m}_{n_2}$ then $\Gamma_{n_1} = \Gamma_{n_2}$. That is, we cannot have two informons belonging to a single component causal tapestry which manifest the same property and embed to the same causal manifold point but possess different local strengths. An informon of a component of ${}_2\mathcal{I}$ is said to be *admissible* in the corresponding component of ${}_1\mathcal{I}$ if either there is no informon of ${}_1\mathcal{I}$ having the same property and embedding to the same causal manifold point, or the above condition holds. Define the *consistent union* ${}_1\mathcal{I} \triangle {}_2\mathcal{I}$ to be the set ${}_1\mathcal{I} \cup \{n \in {}_2\mathcal{I}$ admissible in ${}_1\mathcal{I}\}$. Admissibility is a symmetric relation so the above definition is consistent. A causal tapestry $\mathcal{I}$ is said to be maximal for a sequence tree if there is no path and no causal tapestry $\mathcal{J}$ generated by this path such that $\mathcal{I} \cup \mathcal{J} \neq \mathcal{I}$.

Denote the configuration sequence tree defined above associated with a product as $\Sigma^C(\mathcal{I}^n, \hat{\otimes}_i \mathbb{P}_i)$. Note that the above definition applies readily to the free product. A similar definition applies to the exclusive product provided that the tuple of informons forming an edge must not contain informons embedding to the same causal manifold point. The configuration sequence tree will again be denoted $\Sigma^C(\mathcal{I}^n, \otimes_i \mathbb{P}_i)$.

Mathematically, notice that a sequence tree is an ordered set, the various causal tapestries forming the vertices of the ordered set. The causal tapestries generated by the complete action of the process correspond to the maximal elements of the order. One may then extend this order by successively taking consistent unions of these maximal vertices.

Given a sequence tree $\Sigma(\mathcal{I}, \mathbb{P})$ let $\mathfrak{I}^M_{\Sigma(\mathcal{I},\mathbb{P})}$ denote the set of all of its maximal causal tapestries. We define the configuration process covering map or PCM^C, denoted $\mathfrak{P}^C_{\mathcal{I}^n}(\otimes_i \mathbb{P}_i)$, to be

$$\mathfrak{P}^C_{\mathcal{I}^n}(\otimes_i \mathbb{P}_i) = \{\Phi^j(\mathbf{z}) = \sum_{(n^1_k,\ldots,n^n_k)\in\mathcal{I}} \Gamma_{n^1_k}\cdots\Gamma_{n^n_k} T_{\mathbf{m}_{n^1_k}} sinc_{t_P l_P}(\mathbf{z}_1)\times\cdots$$

$$\times T_{\mathbf{m}_{n^n_k}} sinc_{t_P l_P}(\mathbf{z}_n) | \mathcal{I} \in \mathfrak{I}^M_{\Sigma^C(\mathcal{I}^n,\otimes_i\mathbb{P}_i)}\}$$

In general, the final causal tapestry from a single unfolding of a multiple system process will consist of a set of n-tuples $\mathcal{I} = \{(n^1_j, \ldots, n^n_j)\}$ and this set will not have a simple algebraic form such as a product of individual tapestries. This is due to the complexification induced by process interactions. If, however, we consider maximal causal tapestries defined on the configuration sequence tree then it may happen that a simpler form will emerge. If we take each component in turn we may create individual causal tapestries $\mathcal{I}^k = \{n^k_1, n^k_2, \ldots\}$. Let $\mathcal{I}' = \mathcal{I}^1 \times \mathcal{I}^2 \times \cdots \times \mathcal{I}^n$. It may well happen (especially in the case of independent products) that there will be one or more maximal causal tapestries $\mathcal{I}_{max}$ which will take the form of a product, so that $\mathcal{I} \subset \mathcal{I}' \subset \mathcal{I}_{max}$. In such a case it is apparent that our two interpretation formulations, $\hat{\Psi}_{qm}$ and $\hat{\Psi}_p$ will coincide. Thus we obtain the usual configuration space wave function of NRQM whenever the generating processes can be decomposed into sums and independent products of primitive processes. In the presence of more complex interactions, however, this need not be the case and $\hat{\Psi}_p$ must be used to generate the correct process based interpretation. This is a situation analogous to the absence of closed form solutions to partial differential equations. The difference in the process formulation is that global $\mathcal{H}(\mathcal{M})$ interpretations will always exist, since the calculations involved in their generation are finite. Whether this interpretation provides a viable wave function becomes a matter of empirical and/or theoretical verification. Finding settings in which such complex interactions occur might provide another means whereby the process model could be tested against NRQM.

Now this may be easily extended to include sums of products by simply summing as follows.

$$\mathfrak{P}^C_{\mathcal{I}^n}((\otimes_i \mathbb{P}_i) \oplus (\otimes_j \mathbb{P}'_j)) = \mathfrak{P}^C_{\mathcal{I}^n}(\otimes_i \mathbb{P}_i) + \mathfrak{P}^C_{\mathcal{I}^n}(\otimes_j \mathbb{P}'_j)$$

where $+$ denotes the set sum as defined in Section 2.5.

This is clearly a more complex association than for the usual NRQM configuration space case. Consider, however, the situation of the path integral strategy defined in Chapter 3.

Without going into details, there is a simple, fixed association between each lattice site on which an informon is defined and the causal manifold point to which that lattice site in associated. Consider the case in which $N, r \to \infty$. First of all let $r \to \infty$. The effect of this is to ensure that the maximal amount of information is incorporated into the generation of each individual informon. This means that the maximum number of tokens are used to form each informon, which in turn means that every time an informon is generated at a given lattice site its local strength will take the same value. This in turn means that there will be only one maximal causal tapestry which means in turn that there will be only one function in the $\mathfrak{P}^C_{\mathcal{I}^n}(\otimes_i \mathbb{P}_i)$ set. This function will take the form

$$\mathfrak{P}^C_{\mathcal{I}^n}(\otimes_i \mathbb{P}_i) = \{\Psi^1_{t_P l_P}(\mathbf{z}_1) \times \cdots \times \Psi^n_{t_P l_P}(\mathbf{z}_n)\}$$

In the event that $N \to \infty$, under suitable conditions this will converge to the NRQM configuration space wave function.

Note that it is important in building these PCM^C sets that function sums are carried out only involving terms having identical properties while terms possessing properties of identical property type will correspond to elements of set sums which in turn correspond to process sums.

In the more general case utilizing a non-uniform interpolation procedure one cannot guarantee convergence to a single map under the condition $r \to \infty$ alone since there will be variations in the embedding points of informons. Assuming that these variations are properly constrained one may be able to show that under the condition that $N \to \infty$ the PCM^C will still converge to a single element set since the different embeddings will still yield functions that interpolate to the same function [252, 436]. The precise determination of these conditions is complicated and beyond the scope of this thesis.

The situation for interactive products is much more complicated. First of all, an interactive product represents a class of possible products between processes. It is a generic product which must be specified in any particular case. This specification might have an algebraic form but most often it will require a specification of the sequence tree of the interaction. An interactive product, whether free or exclusive, will possess a sequence tree that is a proper subset of the sequence tree for the corresponding independent free or independent exclusive product. Being a subtree, in general there may be no simple algebraic representation that generates this sequence tree and which relates the PCM^C of the interactive product to the PCM^{C}s of the individual sub-processes. The PCM^C for the interactive product must be generated from the sequence tree itself. As noted above, in NRQM it is assumed that all possible states of systems can be derived from sums and tensor products of states of sub-systems. That is not true in the process setting. The usual NRQM formulation is more or less certain to arise only when independent products may be used to form the conjoining of the individual subprocess. A global $\mathcal{H}(\mathcal{M})$ interpretation will *always* exist for the process model. Again one sees that the process model is the more general model and provides more dynamical information while potentially (as will be seen in Chapter 3) yielding the same statistical information as NRQM.

For example, the sequence tree for the Schrödinger cat situation is such that every time the cyanide cannister is opened, the cat dies. Until then, the cat could live or die. Once dead, the cat never returns to life. The combined process of cat and cannister thus will generate sequences where one may have closed cannister and live or dead cat informons, but once an open cannister informon manifests there will only be dead cat informons. This does not possess a simple algebraic representation in terms of process sums and products.

The process framework provides an ontological interpretation of the NRQM wave function of a single particle as a continuous idealization of a real, discrete physical wave manifesting on an emergent space-time and which is generated by a primitive process. The PCM extends this interpretation to complicated algebraic combinations of primitive processes. real physical wave which generate a multiplicity of such real waves. These waves represent what is more or less directly observed manifesting in the causal manifold but because they are formed through a summation of local $\mathcal{H}(\mathcal{M})$ contributions, it is not possible to determine the presence of correlations or relations between the individual informons that form these waves. That correlational information originates in the structure of the interactions among the generating processes which in turn can be derived from the sequence tree and the PCM^C of the generating process. The configuration space wave function arises as an asymptotic limit of the PCM^C which demonstrates that this is not an ontological entity but merely an epistemological one.

The process covering map provides a more direct link between the process model and the standard NRQM wave function model. This is not always the case for models based upon different mathematical languages such as category theory [90], topos theory [195, 192, 193, 194], logic [135, 134, 45], Clifford algebras [183] and quantum information [170, 171].

Chapter 3

A Simple Process Model of NRQM

In this chapter I develop the process model of non-relativistic quantum mechanics (NRQM). More precisely I shall present a model appropriate to single integral spin (primarily scalar) particles from which the usual non-relativistic wave function appears under a suitable asymptotic limit. This provides an in-principle demonstration of the approach. Unfortunately given the current state of interpolation theory it is not yet feasible to extend the approach to half integer spin particles though there appears to be no reason a priori why this should not be possible. The goal of the process approach is to develop a realist model of NRQM that is generative, discrete, finite, quasi-local, quasi-non-contextual, and from which NRQM emerges as an effective theory in a suitable asymptotic limit. By doing so, it is possible to examine the weaknesses of the usual interpretations of the hidden variable theorems, to provide NRQM with a realist ontology (which is at least theoretically testable in some details), and to resolve most, if not all, of the paradoxes of NRQM. These aspects will be discussed in Chapter 4.

Detailed definitions of causal tapestries and of the reality game are given in Appendix D, and the reader is encouraged to refer to it for mathematical details. The philosophical background in process theory and archetypal dynamics is provided in Appendix C. The essential background in combinatorial game theory and forcing is provided in Appendix E. An introduction to sinc interpolation is provided in Appendix F. The reader unfamiliar with these subjects should refer to these appendices as needed. Causal tapestries are used for their generality. They were formulated to be compatible with relativistic symmetries but this generality is not required of the process models presented in this thesis, which are restricted to the non-relativistic setting. This restriction simplifies the causal structure and make the models more accessible. The model presented here is based upon Feynman's path integral. It is intentionally simplified so as to highlight the essence of the process approach without overwhelming the reader with the mathematical detail necessary to provide a highly accurate version. The simplicity of the model allows the algebraic aspects which are essential to the resolution of the hidden variable problems and the paradoxes to be most prominent. The more sophisticated mathematics is required to maximize the accuracy of the local wave function valuation and of the interpolation process. That ensures the accuracy of the resulting interpolated wave function but does not materially contribute to the resolution of the fundamental questions.

3.1 Background

There are three standard versions of NRQM. Two are based upon the classical Hamiltonian translated into operator theoretic terms, while the third is based upon the classical Lagrangian and involves the use of path integrals. The Hamiltonian approach provides the usual entry into NRQM. In classical mechanics, the Hamiltonian is understood as the Legendre, or contact, transformation of the Lagrangian. The Hamiltonian describes the total energy of the system and its distribution. It is a function of the generalized coordinates $\mathbf{q}_i$ and momenta $\mathbf{p}_i$. The Hamiltonian is given as $H = \sum_i T_i + V(i_1, \ldots, i_n)$ where the T_i are the kinetic energy terms and $V(i_1, \ldots, i_n)$ is the potential energy, which generally depends on interactions between the constituents of the system. In classical mechanics the Hamiltonian gained prominence because of the beauty and elegance of its symmetry relations, the fact that it generates a set of first order equations, and because it gives rise to Liouville's theorem, which asserts that volumes in the phase space generated by the (q_i, p_i) remain constant under dynamical evolution.

Classical mechanics describes the trajectories of physical systems in the phase space. The time evolution of any function defined on the phase space takes the rather simple form $du/dt = [u, H]$, where $[u, v]$ denotes the Poisson bracket defined as:

$$[u, v] = \sum_{a=1}^{f} (\frac{\partial u}{\partial q_a}\frac{\partial v}{\partial p_a} - \frac{\partial u}{\partial p_a}\frac{\partial v}{\partial q_a})$$

The time evolution of the phase space coordinates takes the rather simple form

$$\frac{dq_a}{dt} = [q_a, H], \frac{dp_a}{dt} = [p_a, H]$$

In classical mechanics the state of a system is represented as a single point in the phase space. The evolution of the system is given as a time dependent function on the phase space.

The usual prescription for creating a quantum mechanical model of the system is to replace the phase space with a Hilbert space $\mathcal{H}(\mathcal{M})$ of L^2 functions defined on a manifold $\mathcal{M}$ (representing either co-ordinates or momenta). The state of a system becomes an element of the Hilbert space, a 'wave function'. The generalized co-ordinates are replaced by Hermitian operators $q_i, i\hbar\frac{\partial}{\partial q_a}$. The evolution of the system is given by a time dependent operator U_{t,t_0} over the Hilbert space.

Despite its important role in NRQM, the Hamiltonian approach is unsuitable for forming a bridge with the process framework. There are two fundamental problems:

1. The Hamiltonian approach is inherently non-local. States $\Psi \in \mathcal{H}(\mathcal{M})$ are defined across $\mathcal{M}$, not just at a single point. Operators are defined on $\mathcal{H}(\mathcal{M})$ and act on Ψ as a whole even if they might extract information at a point such as $x\Psi = \Psi(x)$. The usual representation of Ψ is as a decomposition into a sum of eigenfunctions of some Hermitian operator, usually the Hamiltonian, so that $\Psi = \sum_i w_i \Psi_i$ where w_i is calculated via

$$w_i = \int_X \Psi^* \Psi_i dV,$$

also a non-local operation.

2. The Hamiltonian approach does not permit any form of generativity. The Hilbert space $\mathcal{H}(\mathcal{M})$ is defined on a base manifold $\mathcal{M}$. This manifold, as is typical in classical models, simply *is*. There is no place for the generation of $\mathcal{M}$ as a consequence of the dynamical evolution of the system. Time evolution appears to be generative, being given as a unitary operator U_t, but more properly there is a mapping $U : \mathbb{R} \to L(\mathcal{H}(\mathcal{M}), \mathcal{H}(\mathcal{M}))$, the space of linear operators on $\mathcal{H}(\mathcal{M})$. The time space $\mathbb{R}$ also simply *is*. There is no generation of the temporal manifold as a consequence of the dynamical evolution of the system. This is consistent with the usual block universe conception so prevalent in classical mechanics.

Feynman [65] also pointed out that the Hamiltonian formulation is not the most general formulation possible. In fact, he described a relativistic situation in which the Hamiltonian does not exist (pg. 4-5).

The alternative formulation of quantum mechanics is based upon the space-time approach to quantum mechanics developed by Feynman [65]. The Lagrangian plays a central role in this theory, and an important role in classical mechanics as well on account of the Principle of Least Action. The Lagrangian is defined as the difference between the kinetic and potential energies of the system, that is $L = L(q, \dot{q}, t) = \sum_i T_i - V(i_1, \ldots, i_n)$ where the T_i are the kinetic energy terms and $V(i_1, \ldots, i_n)$ is the potential energy. The Principle of Least Action states that if a particle moves from space-time point $(t, \mathbf{x})$ to space-time point $(t', \mathbf{x}')$ along a trajectory $\phi(t)$ then the action, defined as the time integral of the Lagrangian along the path ϕ, formally $S[\phi] = \int_t^{t'} L(\phi(t), \dot{\phi}(t), t)dt$, takes on an extremal value.

The Lagrangian is useful for the process approach for three reasons:

1. It is a classical function defined on space-time co-ordinates.
2. The action is defined on a space-time path between the initial and terminal points and so depends only upon local information along the path - for small time intervals this will depend primarily upon the initial and terminal points.
3. The action depends only upon prior information - it does not depend upon information from any space-time points contemporaneous with the terminal point. In a generative framework it depends only upon previously generated information.

Feynman, in formulating his path integral approach, imagined a quantum particle as following a stochastic trajectory based on two postulates.

Postulate 1 states

> If an ideal measurement is performed to determine whether a particle has a path lying in a region of space-time, then the probability that the result will be affirmative is the absolute square of a sum of complex contributions, one from each path in the system. [65, pg 79]

Postulate 2 states

> The paths contribute equally in magnitude, but the phase of the contribution is the classical action (in units of $\hbar$); i.e. the time integral of the Lagrangian taken along the path. [65, pg 80]

Feynman and Hibbs, in the case of a single free particle, make the connection between the oscillatory behaviour of its wave function and its energy and momentum as follows:

> ... the concepts of momentum and energy are extended to quantum mechanics by the following rules:
>
> 1. If the amplitude varies in space as e^{ikx}, we say that the particle has momentum $\hbar k$.
> 2. If the amplitude varies in time as $e^{-i\omega t}$, we say that the particle has energy $\hbar\omega$. [65, pg 47]

The total contribution is given by taking the sum of these individual contributions across all possible paths joining the initial and terminal points. The heuristic provided by Feynman and Hibbs for calculating this sum over paths provides an important link between the process formulation and NRQM. Consider the case of a particle moving in one dimension. Subdivide the interval between t_a and t_b into equal intervals of length ϵ so that $N\epsilon = t_a - t_b$ and $t_{i+1} - t_i = \epsilon$. Set $t_0 = t_a$ and $t_N = t_b$ and $x_0 = x_a$ and $x_N = x_b$. The end points (t_a, x_a) and (t_b, x_b) remain fixed throughout the calculation since we are summing over all trajectories linking these two space-time points. Let $a_0 = a, a_N = b, a_i = (t_i, x_i)$. We form straight line paths between adjacent x_i. The action $S[a_{i+1}, a_i] = \int_{(t_i,x_i)}^{(t_{i+1},x_{i+1})} Ldt$ is calculated along the straight line path joining (t_i, x_i) to (t_{i+1}, x_{i+1}). The full action $S[b, a] = \sum_{i=0}^{N-1} S[x_{i+1}, x_i]$. Varying the x_i, we will construct all possible piecewise linear paths connecting the end points. If we allow ϵ to tend to zero, we will progressively approximate all possible continuous paths connecting the end points. Formally we obtain an amplitude for the transition between the endpoints

$$K(t_b, x_b; t_a, x_a) = \lim_{\epsilon \to 0} \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{n/2} \int \cdots \int e^{(i/\hbar)S[b,a]} dx_1 dx_2 \cdots dx_{N-1}$$

This is usually denoted as $K(b, a) = \int_a^b e^{(i/\hbar)S[b,a]} \mathcal{D}x(t)$.

This kernel can be used to calculate the wave function as follows

$$\Psi(t_b, x_b) = \int_{-\infty}^{\infty} K(t_b, x_b; t_a, x_a) \Psi(t_a, x_a) dx_a$$

where $t_a < t_b$. That is, one obtains the current value of the wave function at a particular space-time point (t_b, x_b) by summing contributions from all paths leading from all previous space-time points to the point under consideration.

These transition amplitudes are constructed using information residing in the past causal cone of the terminal point and so remain consistent with a generative framework since they never require information from either the forward light cone of the terminal space-time point or from regions that are space-like separated from it.

As an example of this approach, let us consider a free particle moving in one dimension. The Lagrangian for a free particle is simply given by the kinetic energy, $L = (1/2)m\dot{x}^2$. We calculate the action using a straight line path between adjacent points. The path is given by a succession of line segments of the form $\phi_i(t) = \left(\frac{x_i - x_{i-1}}{t_t - t_{i-1}}\right)(t - t_{i-1}) + x_{i-1} = \left(\frac{x_i - x_{i-1}}{\epsilon}\right)(t - t_{i-1}) + x_{i-1}$ so that $\dot{\phi}_i = (x_i - x_{i-1})/\epsilon$. Then $S = \int_{t_0}^{t_N} L dt = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \frac{m}{2\epsilon^2}(x_i - x_{i-1})^2 dt = \sum_{i=1} \frac{m}{2\epsilon}(x_i - x_{i-1})^2$ and the kernel approximation takes the form

$$K(b,a) = \lim_{\epsilon \to 0} \left(\frac{m}{2\pi i \hbar \epsilon}\right)^{N/2} \int \cdots \int \exp\left\{\frac{im}{2\hbar\epsilon} \sum_{i=1}^{N} (x_i - x_{i-1})^2\right\} dx_1 \cdots dx_{N-1}$$

This is a product of successive Gaussian integrals which can be solved recursively starting from $i = 1$. The first integral above is

$$\int \exp\left\{\frac{im}{2\hbar\epsilon} \sum_{i=1}^{2} (x_i - x_{i-1})^2\right\} dx_1 = \int \exp \frac{im}{2\hbar\epsilon} \left\{(x_2 - x_1)^2 + (x_1 - x_0)^2\right\} dx_1$$

This equals

$$\int \exp \frac{im}{2\hbar\epsilon} \{(2x_1^2 - 2(x_2 + x_0)x_1 + x_2^2 + x_0^2\} dx_1$$

Using the result

$$\int_{-\infty}^{\infty} \exp\{-d(at^2 + bt + c)\} dt = \sqrt{\frac{\pi}{ad}} \exp d \left(\frac{b^2 - 4ac}{4a}\right)$$

and substituting into the above we obtain

$$\sqrt{\frac{\pi}{-2(im/2\hbar\epsilon)}} \exp \frac{im}{2\hbar\epsilon} \left(\frac{4(x_2+x_0)^2 - 4(-2)(-x_2^2 - x_0^2)}{-8} \right) =$$

$$i\sqrt{\frac{\pi\hbar\epsilon}{m}} \exp \frac{im}{2\hbar\epsilon} \left(\frac{4x_2^2 + 8x_2x_0 + 4x_0)^2 - 8 - x_2^2 - 8x_0^2)}{-8} \right) =$$

$$i\sqrt{\frac{\pi\hbar\epsilon}{m}} \exp \frac{im}{2\hbar\epsilon} \left(\frac{x_2^2 - 2x_2x_0 + x_0^2}{2} \right) = i\sqrt{\frac{\pi\hbar\epsilon}{m}} \exp \frac{im}{2\hbar 2\epsilon}(x_2 - x_0)^2$$

One continues recursively in this fashion until one obtains

$$K(b,a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)} \right)^{1/2} \exp \left\{ \frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} \right\}$$

Remarkably in this simple example the factor of ϵ disappears from the calculation and the limiting operation is avoided.

The connection to the usual Hamiltonian formulation is through the transition amplitude. Given an initial state $|q_i>$ and final state $|q_f>$, the amplitude for the transition between these two states is calculated in the Hamiltonian formalism as $< q_f|e^{iHT/\hbar}|q_i>$ where T is the time interval between the two states. Zee [438] shows that

$$< q_f|e^{iHT/\hbar}|q_i >= K(q_f, q_i) = \int_{q_i}^{q_f} e^{(i/\hbar)S[q_f,q_i]}\mathcal{D}q(t)$$

The calculation of the path integral becomes very complicated and difficult as soon as a potential V is included in the Lagrangian. If V contains only terms that are quadratic in the position variables it is merely complicated while other forms of V require sophisticated perturbation techniques. These are beyond the scope of this thesis which merely attempts to provide a demonstration in principle of this approach and to motivate future research.

A great advantage to starting with non-relativistic quantum mechanics rests in the simpler causal structure that arises when velocities are small compared to c. In such circumstances, the metric relating space-time points can be taken to be the usual Euclidean metric of classical mechanics,

$$\rho(t_a, \mathbf{x}_a, t_b, \mathbf{x}_b) = [(t_a - t_b)^2 + \sum_i (\mathbf{x}_{a,i} - \mathbf{x}_{b,i})^2]$$

The manifold of space-time events is thus flat, and the causal structure is also simplified, so that $(t_a, \mathbf{x}_a) \prec (t_b, \mathbf{x}_b)$ iff $t_a < t_b$. Note that 3-vectors are denoted in boldface, while 4-vectors, informon interpretations and Hilbert space interpretations are in greek. The context provides the distinction.

The casual manifold $\mathcal{M}$ is therefore just $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ (where we have singled out the time parameter) equipped with the Euclidean metric and causal structure. This also holds if $\mathbb{R}$ is given the usual ordering, $\mathbb{R}^3$ is treated as an antichain, and $\times$ is interpreted as the order product. In this context each causal tapestry will embed into a space-like hypersurface (antichain) of the form $\{t\} \times \mathbb{R}^3$ for some t, fixed for each causal tapestry.

The Hilbert space $\mathcal{H}(\mathcal{M})$ will be taken to be the usual space of entire analytic L^2 functions on $\mathcal{M}$. This is the usual setting for NRQM. We assume that these functions are scalar since that suffices in this case. The state of current research in the interpolation theory of vectorial or spinorial functions is much less well developed than is the case for scalar functions.

3.2 The Reality Game

Chapter 2 introduced the basic notions of process theory together with the idea of the process space and the process covering map. This chapter presents a simplified process model of NRQM to illustrate these ideas and to provide an in-principle demonstration of the emergence of NRQM under certain asymptotic limits, namely infinite information propagation and infinitesimal sampling.

Recall from Chapter 2 that the fundamental elements of reality, the actual occasions, are conceived as being generated by processes. Space-time and all physical entities are understood as being emergent upon these actual occasions.

Actual occasions are represented by the informons of a causal tapestry while the generating process is represented as a combinatorial game, the reality game.

A causal tapestry (L, K, M, I_p) consists of a collection of informons L, an index set K, an interpretation space M and I_p a collection of *prior* causal tapestries (initial conditions). The space M is further divided into an intrinsic component D, which describes attributes of the informons per se (intrinsic interpretation or property), and an extrinsic component M' which refers to the interpretation by some external agent or observer (extrinsic interpretation). Here one may think of this observer as a quantum physicist studying NRQM, who wishes to interpret these informons within a theoretical language that is familiar. In such a case we may take the extrinsic component to have the form $M' = \mathcal{M} \times \mathcal{H}(\mathcal{M})$, where $\mathcal{M}$ is a causal manifold and $\mathcal{H}(\mathcal{M})$ is a Hilbert space on $\mathcal{M}$. The component D consists of a set of properties (k-tuples) which are inherited from the process that generated the informon. Hence if the informon n has intrinsic interpretation $\mathbf{p}_n$, then its generating process $\mathbb{P}$ also has intrinsic property $\mathbf{p}_n$. For simplicity we denote each informon in the form $[n] < \alpha > \{G\}$ where $n \in K$ (label), $\alpha = (\mathbf{m}_n, \Phi_n(\mathbf{z}), \mathbf{p}_n)$ (interpretation) and G is an acyclic directed graph over informons of I_p (content). $\mathbf{m}_n$ is referred to as the causal or $\mathcal{M}$-interpretation while $\Phi_n(\mathbf{z})$ is referred to as the local Hilbert space interpretation or $\mathcal{H}(\mathcal{M})$-interpretation. The reader should remember that only $\mathbf{p}_n$ is intrinsic to the informon, the pair $(\mathbf{m}_n, \Phi_n(\mathbf{z}))$ is extrinsically imposed and may well vary from observer to observer.

Since this is an in-principle demonstration and not meant to provide the ideal or definitive version of a process model of NRQM, I shall focus upon a simple, primitive process $\mathbb{P}$ generating a single eigenstate of a Hamiltonian H. In this case I shall ignore the details of the property vector $\mathbf{p}_n$ and allow it to simply represent Γ_n, the local strength or coupling effectiveness of the generating process at the informon n, and λ, the eigenvalue of the eigenstate. The full property vector will play a role in the case of superpositions of multiple subprocesses, discussed later in section 3.9.

As with many mathematical structures it is beneficial to work with a concrete representation which possesses the formal mathematical structure but also features that make it amenable to analysis. A physical analogy would be the representations of Lie Algebras and Lie Groups. A particularly useful heuristic representation of process is as a two player, co-operative, combinatorial game [97], based on the forcing games used in mathematical logic to generate models [185]. A detailed discussion of combinatorial games is provided in Appendix E to which the reader is referred for more information. Combinatorial games are quite distinct from the more familiar economic games. Combinatorial games derive their power and significance from their combinatorial structure. The most common combinatorial games involve the application of various *tokens* to different *sites*, combinations of which form configurations. Chess, Checkers, Go are familiar examples of combinatorial games. A move alters the tokens associated with a position. Most commonly one considers games involving two players who alternate in making moves and each of which possess a distinct set of tokens and moves. Moves are made non-deterministically, meaning that from any position more than one move is possible and the move is freely chosen without any preassigned probability. It is required that games have a definite end either in the form of some set of configurations or by reaching a predetermined limit of individual moves. Usually the last player to move is said to *win* the game. Games are often described by their game tree which connects a given game configuration to the set of configurations that can be obtained from it by applying a move of either player I or player II. A complete play of the game becomes represented as a path through the game tree beginning with some initial configuration and ending with a final configuration. The game tree is the combinatorial game equivalent of the process sequence tree used previously to construct the process covering map. Combinatorial games come equipped with a remarkable variety of algebraic operations, including many different forms of sum and product [96]. Combinatorial games are intuitive and capture all of the essential algebraic structure of the process space. It is customary in the study of combinatorial games to assign each player a strategy which consists of a set of rules determining their moves. Combinatorial games are ideal for describing the generation of causal tapestries in which as much as possible is to be emergent, not specified in advance.

The combinatorial game approach provides a purely heuristic representation of a process which facilitates calculations, and is not meant to be ontological. When applied to implementing a process, the players are not meant to represent real entities but merely to facilitate the description of the generation of informons and they play no role once an informon has been created. As an aide to intuition one may associate Player I with the system under consideration and Player II with some human observer. Alternatively one may think of Player I as propagating past information forward while Player II builds new informons.

The use of tokens is also purely heuristic. They enable the game to keep track of relevant information required for the generation of informons during the course of game play. Once created, information concerning individual tokens is lost and only the interpretation and content of each informon can be used for the generation of subsequent informons in the next causal tapestry.

The game used to generate causal tapestries is termed a *reality game*. It is a combinatorial token forcing game. Two players cooperatively construct virtual tokens which are placed on virtual sites which eventually become complete informons. Game play proceeds in a series of short rounds, in which a token is constructed. These short rounds are organized into sequences which constitute full rounds, from which a single complete informon arises. In a general reality game a short round unfolds as follows:

1. Player I first selects an informon from the current tapestry $\mathcal{I}$ not previously chosen, or any informon if a new round is beginning

2. Player II then selects either the informon in the new causal tapestry $\mathcal{I}'$ currently under construction, or if a new short round has begun, selects a label n not previously

used and creates a new informon $[n] < (,,,\lambda) > \{\}$, where λ is the eigenvalue of the eigenstate.

3. Player I then provides or updates the content set G_n.

4. Player II then selects a point $\mathbf{m}_n$ in the causal manifold and assigns n a partial interpretation $< (\mathbf{m}_n,,,\lambda) >$.

5. Player I then creates and places a token on the site $\mathbf{m}_n$.

6. Player II then updates the Hilbert space component $\phi_n(\mathbf{z})$ and the local strength Γ_n, and then the interpretation $< (\mathbf{m}_n, \phi_n(\mathbf{z}), \Gamma_n, \lambda) >$.

7. If no further tokens can be added (either no other contributing sites exist or an external limit has been reached), the round ends and a new round begins, otherwise the above process repeats

8. If a round ends the entire process of short rounds and rounds repeats. Game play ends when either no informon can be constructed or an external limit is reached

Each reality game is specified in advance by several parameters: N (length of game play) (which must be an ordinal for technical reasons), R (number of informons generated during a single round), r (number of prior informons contributing information to a nascent informon), ρ (approximation measure), δ (approximation accuracy), t_P (temporal sampling wavelength), l_P (spatial sampling wavelength), ω (band limit frequency), L (Lagrangian), Σ (specification of strategy), p (set of properties). Denote a reality game by $\mathfrak{R}(N, r, \rho, \delta, t_P, l_P, \omega, L, \Sigma, p)$. The values of these parameters must be empirically determined depending upon their physical interpretation.

Starting with a proper causal tapestry, the goal of the reality game is to generate a new causal tapestry which accurately (meaning within accuracy δ) approximates the evolution of a quantum system from one space-like hypersurface to another (one transient now to the next). In mathematical logic the goal is usually equated with a win by one player, traditionally Player II. Thus Player II is said to win the game if, at the end of play, a new causal tapestry has been created that satisfies all of the causal tapestry criteria, ensures that the new set of informons embeds into a space-like hypersurface of $\mathcal{M}$, and that a suitable approximation to a wave function is defined on this hypersurface (technically it is defined on all of $\mathcal{M}$). By suitable I mean that there is some predefined criterion of accuracy, δ, and some wave function satisfying the Schrödinger equation such that the measure of difference ρ between this wave function Ψ and the interpolated function generated by the game Φ satisfies $\rho(\Psi, \Phi) < \delta$. ρ could be the L^2 metric or any other chosen measure of accuracy. (ρ, δ) constitute an additional pair of game parameters[1].

The description of the reality game provided above is quite general. Within the context of mathematical logic and forcing it is common to specify a *strategy* for the two players and to consider the kinds of structures that can be generated by different strategies. The usual distinction in both mathematical logic and computer science is between deterministic, non-deterministic and stochastic strategies. In the standard theory of combinatorial games, stochastic strategies and features are forbidden and any stochastic features arise from the combinatorics. Each player will have their own strategy.

A deterministic strategy for a player is a rule $\mathfrak{D}$ which assigns to any game position $\mathfrak{p}$ a definite position $\mathfrak{p}' = \mathfrak{D}(\mathfrak{p})$. A non-deterministic strategy is a rule $\mathfrak{N}$ which assigns to any game position $\mathfrak{p}$ a set of possible positions $\{\mathfrak{p}_i\} = \mathfrak{N}(\mathfrak{p})$. A stochastic strategy is a rule $\mathfrak{S}$ which assigns to each position a probability distribution (or a measure) f on the set $\mathfrak{P}$ of all positions, i.e. $\mathfrak{S}(\mathfrak{p}) = f$ and the choice of a next position is selected with probability given by f. In the reality game, only deterministic and non-deterministic strategies are allowed, again in keeping with the emergentist paradigm. However, as was discussed in Chapter 2, limited stochastic features come into play when considering interactions between informons and their generating games (i.e. between actual occasions and their generating processes) particularly in the context of interactions between processes which are precipitated or influenced by the actual occasions being generated.

3.3 Path Integral Strategy

For ease of presentation, consider a single non-relativistic particle with mass m, in a single eigenstate of the Hamiltonian H, and interacting with a potential $V(\mathbf{z})$, which summarizes the effect of the environment. The particle is modeled as a process while the environment process is ignored (being incorporated into V). This simplification allows the value of V associated with an informon n to simply be a property of the informon. Since the setting is non-relativistic one can set $\mathcal{M} = \mathbb{R}^4$ and take the causal order to be given by $(t_1, x_1, y_1, z_1) \prec (t_2, x_2, y_2, z_2)$ iff $t_1 < t_2$.

For simplicity, the strength of the particle process at a given informon n is assumed to be given by a complex scalar Γ_n, so that the coupling effectiveness or compatibility is proportional to $t_P l_P^3 \Gamma_n^* \Gamma_n$ for the global $\mathcal{H}(\mathcal{M})$ interpretation (and $l_P^3 \Gamma_n^* \Gamma_n$ if restricted to

[1] As an aside, another possible definition is to generate a Hilbert space interpretation Φ such that $i\hbar(\partial\Phi/\partial t) + ((\hbar^2/2m)\nabla^2\Phi) - V\Phi = \mathcal{O}(\delta)$ but I won't use this here.

the hypersurface). The methods employed here readily generalize to the case where the strength is given by a complex valued finite dimensional vector and so in principle this method should extend to the case of non-relativistic bosons and hopefully to photons as well. The situation becomes very complicated whenever there are interactions between the degrees of freedom in the strength vector. The state of the art of interpolation theory does not yet permit a robust application to the spinorial setting but there has been some initial work [22, 21, 206]. In addition there has been some early work applying interpolation theory in the setting of graphs, particularly flows on graphs [201].

Player I propagates information forward to the nascent generation while Player II uses this to construct the new informons. Let $\mathcal{I}_n$ be the current generation and $\mathcal{I}_{n+1}$ the nascent generation. Let $\mathcal{I}_n^p = \cup_{i<n}\mathcal{I}_i$. The causal tapestry will be constructed as a sublattice of a uniform lattice. This is a game in which the game board is effectively constructed with each play of the game. Informons may thus be labelled by their site value, which will be of the form $l_n = (mt_P, il_P, jl_P, kl_P)$ where m is fixed, referring to the generation number, and $-\infty \leq i, j, k \leq \infty$. The metric distance between $n \in \mathcal{I}_m, n' \in \mathcal{I}_{m'}$ is given by the Euclidean metric applied to the spatial lattice site values. Hence if $l_n = (mt_P, il_P, jl_P, kl_P)$ and $l_{n'} = (m't_P, i'l_P, j'l_P, k'l_P)$ then $d(n, n')^2 = d(l_n, l_{n'})^2 = ((i-i')^2 + (j-j')^2 + (k-k')^2)l_P^2$. The causal relation is given by $d_c(n', n) = t_{n'} - t_n = (m' - m)t_P$. Hence $n \prec n'$ if $d_c(n', n) > 0$, or $m' > m$. Note that one could easily set $\mathcal{M} = \mathbb{M}^4$, the 4-dimensional Minkowski space with causal order given via the Minkowski metric to obtain the relativistic case.

The process strength Γ_n of an informon n is thought of as propagating forward to subsequent informons as a discrete (possibly dissipative) wave. The contribution to the next generation informon n' will depend upon the metric distance between them. If the Lagrangian for the particle is $\mathcal{L} = \frac{p^2}{2m} + V$, then let $\mathcal{L}(n, n') = \frac{md(n,n')^2}{2t_P^2} + V(n)$. then each contribution will take the form $e^{(i/\hbar)\{\frac{m^2 d(n,n')^2}{2mt_P^2} + V(n)\}t_P}$ or alternatively, $e^{(i/\hbar)\{\frac{m^2 d(l_n, l_{n'})^2}{2mt_P^2} + V(n)\}t_P}$. Note that this is derived from information residing solely within the causal tapestry and does not depend upon the causal manifold or $\mathcal{H}(\mathcal{M})$-interpretations. This is in keeping with the view that the physics takes place on the causal tapestry and the interpretations reflect the point of view of the human observer or of some theory and are not real in themselves but simply constitute idealized descriptions.

The $\mathcal{H}(\mathcal{M})$-interpretation is constructed by means of sinc interpolation, chosen because of its simplicity, its effectiveness for a large class of L^2 functions on $\mathcal{H}(M)$, and for the richness of the available literature dealing with the range of functions that can be interpolated, convergence properties, error properties and so on. Sinc interpolation requires the use of a lattice embedding into $\mathcal{M}$, which admittedly is unrealistic, but Maymon and Oppenheim [252] have shown that even if the actual embedding point is off lattice, for small errors this will still provide a highly accurate approximation of the actual function, and thus in the situations to be considered in the free path integral strategy, this is a 'good enough' assumption. A more realistic model would require the use of non-uniform embeddings and more sophisticated interpolation techniques, such as Fechtinger-Gröchenik theory [436], but the details would unnecessarily complicate the presentation. The important point is that the physics is independent of the interpolation scheme which merely serves to translate it into a standard quantum mechanical form, and thus it is somewhat arbitrary and dependent upon the particular circumstances. Sinc interpolation is good enough for the in-principle demonstration presented here.

The class of functions that can be interpolated by sinc interpolation depends upon the values of t_P and l_P and will lie within the class $B^n_\sigma \subset L^2(\mathcal{H}(\mathcal{M})$ (these are the so-called band-limited functions, meaning that their Fourier transforms have support within the bounded region $[-\sigma, \sigma]$). This class is smaller than $L^2(\mathcal{H}(\mathcal{M}))$ but in general this is not a problem because in reality all quantum systems possess bounded energy and momentum which determine the value of σ through $E = \hbar\sigma = 2\hbar\pi/t_P$ and $p = \hbar\sigma = 2\hbar\pi/l_P$. Fixing σ to be finite is equivalent to asserting the existence of an ultraviolet cutoff, a reasonable assumption when considering the behaviour of any physically real particles. Thus one advantage of sinc interpolation is that a natural ultraviolet cutoff exists by virtue of the nature of the interpolation process and does not need to be specified as an ad hoc assumption. Since quantum mechanics deals mostly with L^2 functions we need consider only the classes B^2_σ and I will denote $L^2(\mathcal{H}(\mathcal{M}))$ restricted to B^2_σ by $\mathcal{H}_\sigma(\mathcal{M})$.

A basic strategy which provides an in-principle demonstration of the power of the method is that of the Bounded Radiative Uniform Sinc Path Integral Strategy ($\mathfrak{PI}$). The path integral strategy is specified by the parameters R, r, N, t_P, l_P and by

1. Δ (distance bound): arbitrary, determines maximum causal distance of information transmission.

2. ρ (approximation measure): arbitrary, set by the observer according to mathematical or experimental considerations.

3. δ (approximation accuracy): arbitrary but bounded by experimental measurements

4. ω (band limit frequency): bounded by upper limits of energy and momentum of the quantum system

5. $\mathcal{L}$ (Lagrangian): determined by the particulars of the quantum system

6. p (set of properties): here energy, momentum

The informons of the causal tapestry $\mathcal{I}_n$ will be embedded into a sub-lattice of the space-like hyper-surface (or time slice) $\{nt_P\} \times \mathbb{R}^3$ in $\mathcal{M}$. The embedding lattice in $\mathcal{M}$ will thus take the general form (nt_P, il_P, jl_P, kl_P) for integers n, i, j, k. The embedding point in $\mathcal{M}$ of an informon n will be denoted $\mathbf{m}_n$. In the simplified version presented here, $l_n = \mathbf{m}_n$ but this is not true generally. In fact it is only necessary that the embedding into $\mathcal{M}$ preserve the causal order and that the metric or causal distance lies within certain tolerance limits from that on the causal tapestry. In the relativistic case $\mathcal{I}_n$ will embed into the causal manifold $\mathcal{M}$ as a sub-lattice of some space-like hyper-surface and the embedding may or may not preserve causal distances (the error being bound by some parameter ϵ. An informon will be referred to either by its index n or by its lattice site l_n.

The path integral strategy for a single short round proceeds as follows:

1. Player I moves first. Player I non-deterministically chooses any informon $[l_n] < (\mathbf{m}_n, \phi_n(\mathbf{z}), \Gamma_n, \lambda, V(n)) > \{G_n\}$ from the current tapestry $\mathcal{I}_m$ which has not previously been played in this round, where $\mathbf{m}_n$ is the $\mathcal{M}$ embedding, $\phi_n(\mathbf{z})$ is the local Hilbert space contribution, l_n is a lattice site, Γ_n is the strength of the generating process at n and $V(n)$ is the value of the potential at n .

2. If there is an informon $[n'] < \alpha_{n'} > \{G_{n'}\}$ currently in play in the new tapestry $\mathcal{I}_{m+1}$ then Player II tests whether $d(l_n, l_{n'}) < \Delta$ in the new tapestry $\mathcal{I}'$. If the bound is exceeded, play reverts back to step I, otherwise it proceeds. If there is no current informon then Player II chooses a label n' not previously used and selects a lattice site $((m+1)t_P, i'l_P, j'l_P, k'l_P)$ not previously used such that $d(l_n, l_{n'}) < \Delta$ and creates a new informon $[n'] <, , , \lambda, > \{\}$.

3. Player I next updates the content set. If the new inform already possesses a content set $G_{n'}$, then Player I replaces $G_{n'}$ with $G_{n'} \cup \hat{G}_n \cup \{[n] < \alpha_n > \{G_n\}\}$ ($\hat{G}_n$ is an order theoretic up-set of G_n) and checks to ensure that all necessary order conditions are satisfied. If the new informon is nascent, then Player I simply sets $G_{n'} = \hat{G}_n \cup \{[n] < \alpha_n > \{G_n\}\}$. The content set determines what prior information is permitted in constructing tokens. It only includes informons from the past causal cone of the informon. In the case of NRQM, it turns out that only informons from the current tapestry are needed since the relevant information is already incorporated into their $\mathcal{H}(M)$-interpretations. Thus it suffices if $G_{n'}$ or $\emptyset$ is replaced with $G_{n'} \cup \{[n] < \alpha_n > \{G_n\}\}$ or $\{[n] < \alpha_n > \{G_n\}\}$ respectively. Note that in this case the causal consistency criteria are trivially satisfied.

4. Player II next determines the causal manifold embedding. If the nascent informon n' already possesses a causal manifold embedding, then Player II does noth-

ing. Otherwise Player II sets $\mathbf{m}_{n'} = l_{n'}$ and the nascent informon becomes $[n'] < \mathbf{m}_{n'},,,\lambda, V(n') > \{G_{n'}\}$ where $V(n') = V(\mathbf{m}_{n'})$.

5. Player I next constructs a token representing the information passing from n to n' and to be used to form the local $\mathcal{H}(\mathcal{M})$ interpretation at n'. Denote this token as $\mathcal{T}_{n'n}$. Let $\tilde{S}[n', n] = \mathcal{L}(n', n)t_P$. Let T_n denote the set of tokens on n. Let Γ_n denote the sum of the tokens on n, that is $\Gamma_n = \sum\{\mathcal{T}_{nm} | \mathcal{T}_{nm} \in T_n\}$. The relationship between these two is $\Gamma_n = (1/A^3)\Phi_n(\mathbf{m}_n)$, where A is the path integral normalization factor described by Feynman and Hibbs [132] which is appropriate to the current Lagrangian and initial and boundary conditions. The reason for this will become apparent later. Define the propagator $P_{n'n} = (l_P^3/A^3)e^{i\tilde{S}[l_{n'}, l_n]/\hbar}$. Then Player I places a token $\mathcal{T}_{n'n} = P_{n'n}\Gamma_n$ on the site $l_{n'}$. If there already is a set $T_{n'}$ of tokens on informon n' then replace it by $T_{n'} \cup \{T_{n'n}\}$.

6. Finally Player II must determine the local $\mathcal{H}(\mathcal{M})$ interpretation. If $\mathbf{z} = (t, x, y, z)$ and $\mathbf{m}_{n'} = ((nt_P, ml_P, rl_P, sl_P))$ define $T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z}) =$

$$sinc\left(\frac{\pi(t - nt_P)}{t_P}\right) sinc\left(\frac{\pi(x - ml_P)}{l_P}\right)$$
$$\times sinc\left(\frac{\pi(y - rl_P)}{l_P}\right) sinc\left(\frac{\pi(z - sl_P)}{l_P}\right)$$

   Player II constructs the local $\mathcal{H}(\mathcal{M})$ interpretation by coupling the tokens on the site to a suitable interpolation function, which in the current strategy utilizes a sinc function given as $A^3 T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z})$. If the new informon has just been formed, then the $\mathcal{H}(\mathcal{M})$-interpretation is given as $\phi_{n'}(\mathbf{z}) = \mathcal{T}_{n'n} A^3 T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z})$. If the informon already possesses a local $\mathcal{H}(\mathcal{M})$ interpretation, $\phi_{n'}(\mathbf{z})$, then replace it by the newly updated local $\mathcal{H}(\mathcal{M})$ interpretation $\phi_{n'}(\mathbf{z}) + \mathcal{T}_{n'n} A^3 T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z})$.

   In other words, add the new token to the collection, sum the token values and couple the sum to the interpolation wavelet.

7. If no further tokens can be added (either no other contributing sites exist or an external limit has been reached), the round ends and a new round begins.

Play continues until the allotted number of allowed game steps has been reached. At the end of play a new causal tapestry $\mathcal{I}_{m+1}$ has been created and the old causal tapestry $\mathcal{I}_m$ is eliminated, formally becoming a part of $\mathcal{I}^p_{m+1}$, the collection of prior tapestries. Any relevant information from $\mathcal{I}_m$ now resides within the content sets of the informons of $\mathcal{I}_{m+1}$.

Let n' denote an informon of $\mathcal{I}_{m+1}$. Let $L_{n'}$ denote the set of all informons from $\mathcal{I}_n$ that contribute tokens to the formation of n'. Equally, $L_{n'}$ is the set of all informons from $\mathcal{I}_n$ that form vertices in $G_{n'}$. The local $\mathcal{H}(\mathcal{M})$ interpretation of n' may now be written as $\phi_{n'}(\mathbf{z}) = \sum_{n \in L_{n'}} \mathcal{T}_{n'n} A^3 T_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z})$

The global $\mathcal{H}(\mathcal{M})$ interpretation on $\mathcal{M}$ is formed by summing the local contributions over all of $\mathcal{I}_{m+1}$, that is $\Phi^{m+1}(\mathbf{z}) = \sum_{n' \in \mathcal{I}_{m+1}} \phi_{n'}(\mathbf{z})$. One may restrict this to the $t = (m+1)t_P$ hyper-surface, obtaining, as will be shown below, a highly accurate approximation to the standard quantum mechanical wave function on the hyper-surface. Note that fixing $t = m + 1$ causes the time based sinc term to take the value 1 and one indeed obtains a function on the hyper-surface. This approximation will be less accurate when extended to the entirety of $\mathcal{M}$. To achieve greater accuracy requires either summing over the content sets of $\mathcal{I}_{m+1}$, i.e. $\Phi^{m+1,c}_{n'}(\mathbf{z}) = \sum_{n \in G_{n'}, n' \in I_{n'}} \phi_n(\mathbf{z})$ or over all of $\mathcal{I}_{m+1} \cup \mathcal{I}_p$, $\Phi^{m+1,p}_{n'}(\mathbf{z}) = \sum_{n \in \mathcal{I}_{m+1} \cup I_p} \phi_n(\mathbf{z})$.

A superposition of eigenstates may be generated by an exclusive sum of primitive processes, each corresponding to a single eigenstate. The global wave function may be obtained by interleaving the informons generated by the sub-processes, either on the same lattice, in which case a slightly different interpolation function must be used (or the missing values need be determined using the scheme of [247]) or the informons of distinct processes may be placed on distinct lattices.

The astute reader of Appendix D will note that the reality game is actually played out on a pair of causal tapestries, an event tapestry and a transition tapestry. So far the only causal tapestry that we have looked at has been the event tapestry. The transition tapestry is not needed to formulate and prove the previous arguments and its discussion would only add unnecessary complications to the presentation. Moreover, the content set is not used given the such a simple dynamic. It becomes needed when interactions occur or in time dependent or relativistic settings which are outside of the scope of this thesis.

Figure 3.1 illustrates some basic ideas involved in the path integral strategy.

3.4 Formal Proof of Emergent NRQM

In the previous section the assertion was made that NRQM can be viewed as an effective theory arising in the asymptotic limit as $N, r \to \infty$ and $t_P, l_P \to 0$. To prove this consider the following.

Assume that the particle is generated by a primitive process in an eigenstate of its Hamiltonian. Let $\mathcal{I}_0$ denote the initial generation for the particle process $\mathbb{P}$ and assume

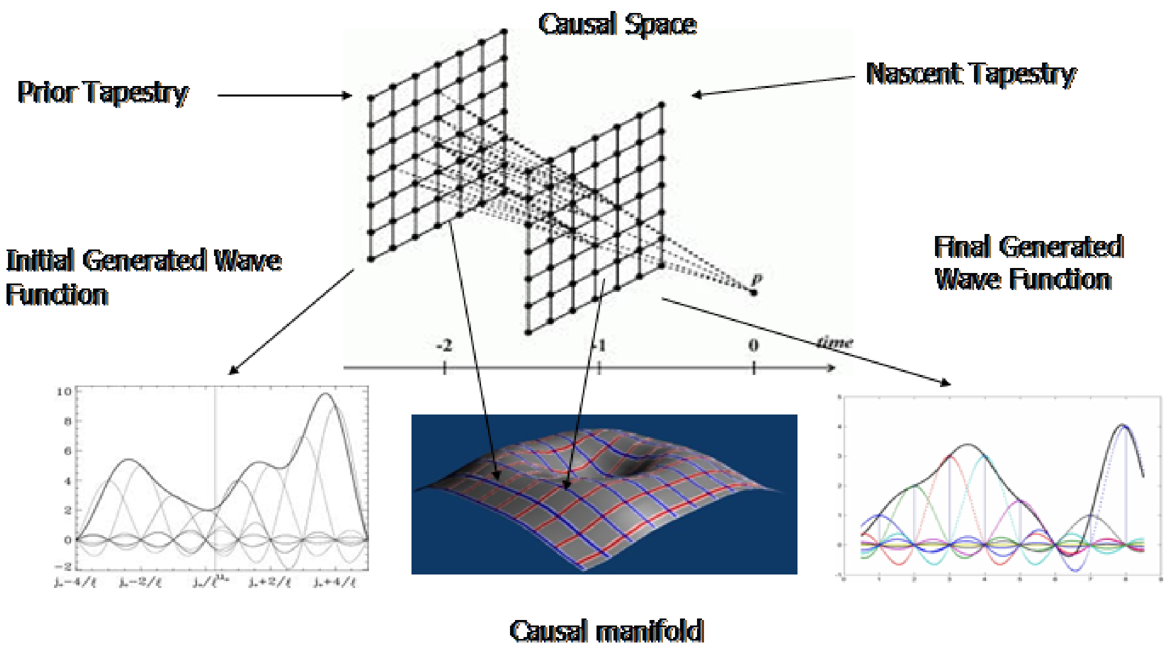

Figure 3.1: Depiction of Strategy Action.

that on this generation the process strengths Γ_n correspond to the values of the initial wave function sampled at the embedding points, i.e $\Gamma_n = \Psi_0(\mathbf{m}_n)$ (or $\Phi^0(\mathbf{m}_n) = \Psi_0(\mathbf{m}_n)$).

Parzen's theorem [287, 436] states that if $f(t_1, \ldots, t_N)$ is a function band limited to the N-dimensional rectangle $B = \prod_{i=1}^{N}(-\sigma_i, \sigma_i)$, $\sigma_i > 0$, $i = 1, \ldots, N$ so that its Fourier transform $F(\omega_1, \ldots, \omega_N)$ is such that

$$\int_{-\sigma_1}^{\sigma_1} \cdots \int_{-\sigma_N}^{\sigma_N} |F(\omega_1, \ldots, \omega_N)|^2 d\omega_1 \cdots d\omega_N < \infty,$$

$F(\omega_1, \ldots, \omega_N) = 0$ for $|\omega_k| > \sigma_k$, $k = 1, \ldots, N$, and $\pi k_i/\sigma_i = \hat{k}_i$, then $f(t_1, \ldots, t_N) =$

$$\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} f\left(\hat{k}_1, ., \hat{k}_N\right) s(\sigma_1(t_1 - \hat{k}_1\pi))..s(\sigma_N(t_N - \hat{k}_N\pi))$$

Therefore by Parzen's theorem, on $\mathcal{I}_0$, $\Phi^0(\mathbf{z}) =$

$$\sum_{n \in \mathcal{I}_0} \Gamma_n A^3 T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z}) =$$

$$\sum_{n \in \mathcal{I}_0} \Psi_0(\mathbf{m}_n) A^3 T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z}) \approx \Psi_0(\mathbf{z})$$

.

Assume that the process has generated all generations up to and including $m+1$. Let $g_{n'}(\mathbf{z}) = A^3 T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z})$. Recall that $\Phi^{m+1}(\mathbf{s}) = \sum_{n \in \mathcal{I}_{m+1}} \phi_n(\mathbf{z})$.

Hence one can write

$$\Phi^{m+1}(\mathbf{z}) = \sum_{n' \in \mathcal{I}_{m+1}} \sum_{n \in L_{n'}} \mathcal{T}_{n'n} g_{_{n^{m+1}}}(\mathbf{z})$$

If we assume the convention that $\mathcal{T}_{n'n} = 0$ if n does not propagate information to n' then we can rewrite the above as

$$\Phi^{m+1}(\mathbf{z}) = \sum_{n^{m+1+} \in \mathcal{I}_{m+1}} \sum_{n^m \in \mathcal{I}_m} \mathcal{T}_{n^{m+1} n^m} g_{n^{m+1}}(\mathbf{z})$$

$$= \sum_{n^{m+1}\in\mathcal{I}_{m+1}} \sum_{n^m\in\mathcal{I}_m} \mathcal{P}_{n^{m+1}n^m}\Gamma_{n^m} g_{n^{m+1}}(\mathbf{z})$$

$$= \sum_{n^{m+1}\in\mathcal{I}_{m+1}} \sum_{n^m\in\mathcal{I}_m} \mathcal{P}_{n^{m+1}n^m} \sum_{n^{m-1}\in\mathcal{I}_{m-1}} \mathcal{T}_{n^m n^{m-1}} g_{n^{m+1}}(\mathbf{z}) =$$

$$\sum_{n^{m+1}\in\mathcal{I}_{m+1}} \sum_{n^m\in\mathcal{I}_m} \mathcal{P}_{n^{m+1}n^m} \sum_{n^{m-1}\in\mathcal{I}_{m-1}} \mathcal{P}_{n^m n^{m-1}}\Gamma_{n^{m-1}} g_{n^{m+1}}(\mathbf{z})$$

Continuing one arrives at

$$\begin{aligned}
\Phi_{m+1}(\mathbf{z}) = &\sum_{n^{m+1}\in\mathcal{I}_{m+1}} \cdots \sum_{n^0\in\mathcal{I}_0} \mathcal{P}_{n^{m+1}n^m}\cdots\mathcal{P}_{n^1n^0}\Gamma_{n0} g_{n^{m+1}}(\mathbf{z}) = \\
&\sum_{n^{m+1}\in\mathcal{I}_{m+1}} \cdots \sum_{n^0\in\mathcal{I}_0} \mathcal{P}_{n^{m+1}n^m}\cdots\mathcal{P}_{n^1n^0}\Psi_0(\mathbf{m}_{n^0}) g_{n^{m+1}}(\mathbf{z}) = \\
&\sum_{n^{m+1}\in\mathcal{I}_{m+1}} \sum_{n^m\in\mathcal{I}_m} \cdots \sum_{n^0\in\mathcal{I}_0} \frac{l_P^3}{A^3} e^{\frac{i}{\hbar}\tilde{S}[\alpha_{n^{m+1}},\alpha_{n^m}]} \frac{l_P^3}{A^3} e^{\frac{i}{\hbar}\tilde{S}[\alpha_{n^m},\alpha_{n^{m-1}}]} \times\cdots\times \\
&\frac{l_P^3}{A^3} e^{\frac{i}{\hbar}\tilde{S}[\alpha_{n^1},\alpha_{n^0}]} \Psi_0(\mathbf{m}_{n^0}) g_{n^{m+1}}(\mathbf{z}) = \\
&\sum_{n^{m+1}\in\mathcal{I}_{m+1}} \sum_{n^m\in\mathcal{I}_m} \cdots \sum_{n^0\in\mathcal{I}_0} e^{\frac{i}{\hbar}\tilde{S}[n^{m+1},n^m]+\tilde{S}[n^m,n^{m-1}]+\cdots+\tilde{S}[n^1,n^0]} \times \\
&\overbrace{\frac{l_P^3}{A^3}\frac{l_P^3}{A^3}\cdots\frac{l_P^3}{A^3}}^{m+1} \Psi_0(\mathbf{m}_{n^0}) g_{n^{m+1}}(\mathbf{z}) = \\
&\sum_{n^{m+1}\in\mathcal{I}_{m+1}} \sum_{n^m\in\mathcal{I}_m} \cdots \sum_{n^0\in\mathcal{I}_0} e^{\frac{i}{\hbar}\mathcal{L}[n^{m+1},n^m]t_P+\mathcal{L}[n^m,n^{m-1}]t_P+\cdots+\mathcal{L}[n^1,n^0]t_P} \times \\
&\overbrace{\frac{l_P^3}{A^3}\frac{l_P^3}{A^3}\cdots\frac{l_P^3}{A^3}}^{m+1} \Psi_0(\mathbf{m}_{n^0}) g_{n^{m+1}}(\mathbf{z}) \approx
\end{aligned}$$

$$\sum_{n^{m+1}\in\mathcal{I}_{m+1}}\sum_{n^m\in\mathcal{I}_m}\cdots\sum_{n^0\in\mathcal{I}_0} e^{\frac{i}{\hbar}S[\alpha_{n^{m+1}},\alpha_{n^0}]}\overbrace{\frac{l_P^3}{A^3}\frac{l_P^3}{A^3}\cdots\frac{l_P^3}{A^3}}^{m+1}\Psi_0(\mathbf{m}_{n^0})g_{n^{m+1}}(\mathbf{z}) \approx$$

$$\sum_{n^{m+1}\in\mathcal{I}_{m+1}}\sum_{n^0\in\mathcal{I}_0}\int_{I_{i-1}}\cdots\int_{I_1} e^{\frac{i}{\hbar}S[\alpha_{n^{m+1}},\alpha_{n^0}]}\overbrace{\frac{d\alpha_{n^{m+1}}}{A^3}\cdots\frac{d\alpha_{n^1}}{A^3}}^{m}\frac{l_P^3}{A^3}\Psi_0(\mathbf{m}_{n^0})g_{n^{m+1}}(\mathbf{z})$$

where the I_i refers to the continuous extension of the sub-lattice upon which $\mathcal{I}_i$ is defined, i.e. $I_i = \{it_P\} \times \mathbb{R}^3$, the action integral has been taken over the piecewise linear path $\alpha_{n^0}, \alpha_{n^1}, \ldots, \alpha_{n^{m+1}}$ on the continuous lattice extension $[0,(m+1)t_P] \times \mathbb{R}^3$ and where the final step is obtained approximating each discrete sum by an integral.

Now as $N, r \to \infty$, the number of informons from $\mathcal{I}_k$ contributing to any informon of $\mathcal{I}_{k+1}$ grows to infinity for each k and a moment's reflection will suggest therefore that in the limit every possible path between the informons of $\mathcal{I}_0$ and the informons of $\mathcal{I}_{m+1}$ will be included in the calculation. The entirety of each causal tapestry will be connected to all of the other causal tapestries. As $t_P \to 0$, the temporal spacing between lattice slices decreases, so not only does the total number of lattice slices increase, but the number of lattice slices between α_0 and α_i increases while their distance decreases. Under such circumstances, according to Feynman and Hibbs [132] the product of the integrals above converges to the path integral between the points α_0 and α_i. Substituting this into the above one obtains $\Phi^{m+1}(\mathbf{z}) \approx \sum_{n^{m+1}\in\mathcal{I}_{m+1}}\sum_{n^0\in\mathcal{I}_0} K(\alpha_{n^{m+1}},\alpha_{n^0})\Psi_0(\mathbf{m}_{n^0})\frac{l_P^3}{A^3}g_{n^{m+1}}(\mathbf{z}) =$

$$\sum_{n^{m+1}\in\mathcal{I}_{m+1}}\sum_{n^0\in\mathcal{I}_0} K(\alpha_{n^{m+1}},\alpha_{n^0})\Psi_0(\mathbf{m}_{n^0})l_P^3 T_{\mathbf{m}_{n^{m+1}}} sinc_{t_P,l_P}(\mathbf{z}) \approx$$

$$\sum_{n^{m+1}\in\mathcal{I}_{m+1}}\int_{I_0} K(\alpha_{n^{m+1}},\alpha_{n^0})\Psi_0(\alpha_0)d\alpha_0 T_{\alpha_i} sinc_{t_P l_P}(\mathbf{z}) =$$

$$\sum_{n^{m+1}\in\mathcal{I}_{m+1}}\Psi_{m+1}(\alpha_{n^{m+1}})T_{\alpha_i} sinc_{t_P l_P}(\mathbf{z})$$

If one restricts the calculation to the hypersurface for $\mathcal{I}_{m+1}$ then one obtains

$$\overline{\Phi^{m+1}(\mathbf{z})} = \overline{\sum_{n^{m+1}\in\mathcal{I}_{m+1}}\Psi_{m+1}(\alpha_{n^{m+1}})T_{\alpha_i} sinc_{t_P l_P}(\mathbf{z})} = \Psi_{m+1}(\mathbf{z})$$

where the final equality follows from Parzen's theorem. If $\Psi(\mathbf{z})$ refers to the wave function defined on the entire manifold $\mathcal{M}$ then one has $\Phi^{m+1}(\mathbf{z}) \approx \Psi(\mathbf{z})$, although the approximation may be quite poor. A better approximation is obtained by $\sum_0^{m+1} \Phi^i(\mathbf{z}) \approx \Psi(\mathbf{z})$.

3.5 Errors

The next question to address is the goodness of fit between the wave function as determined by this model and the wave function as calculated using the usual path integral or Schrödinger equation methods. Goodness of fit is a more accurate term because the truly important question is not whether it accurately matches the NRQM wave function but rather how well it satisfies the Schrödinger equation and provides the essential statistical relations. Comparison to the NRQM wave function is made by using the discrepancy measure ρ or by substituting into the appropriate Schrödinger equation and examining for goodness of fit. This in turn determines whether or not the game is a win for Player II. If it is, then we can say that the reality game $\mathfrak{R}(N, r, \rho, \delta, t_P, l_P, \omega, L, \Sigma, p)$ generates the wave function to accuracy δ.

The discrepancy between the global $\mathcal{H}(\mathcal{M})$-interpretation given above and the standard NRQM wave function depends upon the accuracy of the approximation to the integral $\int_{\mathcal{M}_t} K(\mathbf{x}_{j'}, \mathbf{x}_j)\phi_j(\mathbf{x}_j)d\mathbf{x}_j$, the deviations from uniformity of the causal embedding points, the number r of current informons contributing information to any nascent informon as well as the values of t_P, l_P. This is a difficult problem to assess in general but results are available in special cases. For example, in one dimension, if the wave function Ψ satisfies $|\Psi| \leq M|t|^{-\gamma}$ for $0 < \gamma \leq 1$, $|\int_{\mathcal{M}_t} K(x_{j'}, x_j)\phi_j(x_j)dx_j - \Psi(x_{j'})| \leq \epsilon$, the discrepancy between each embedding point and its ideal lattice embedding point is less that δ, and the truncation number $r = 2[W^{1+1/\gamma} + 1] + 1$, then according to a theorem of Butzer [75, 436], the error E satisfies

$$||E||_\infty \leq -K(\Psi, \gamma, \epsilon/l_P, \delta/l_P) l_P \ln l_P$$

where

$$K = (1 + \tfrac{1}{\gamma}) \left\{ \sqrt{5}e \left[(\tfrac{14}{\pi} + \delta/l_P + \tfrac{7}{3\sqrt{5\pi}})||\Psi^{(1)}||_\infty + \epsilon/l_P \right] + 6e(M + ||\Psi||_\infty) \right\}$$

Hence, $||E||_\infty \approx 10^{-33}K$ if l_P is the Planck length. In three dimensions one finds that $||E||_\infty \approx 10^{-99}K^3$.

In the ideal case in which the kernel sum equals the kernel integral, and the causal embedding is uniform the accuracy of the global $\mathcal{H}(\mathcal{M})$ interpretation will depend upon the total number of informons contributing to the interpretation so that one can use the Thomas-Yao theorem [436].

Theorem (Thomas-Yao): Let $f(x, y, z)$ be band limited to $[-\sigma_x, \sigma_x] \times [-\sigma_y, \sigma_y] \times [-\sigma_z, \sigma_z]$, with no frequencies higher than $(r_x\sigma_x/2\pi), (r_y\sigma_y/2\pi), (r_z\sigma_z/2\pi)$ and $|f(x, y, z)| \leq M$. Then for

$$T(x, y, z) = f(x, y, z) - \sum_{k_x=N_1^x}^{N_2^x} \sum_{k_y=N_1^y}^{N_2^y} \sum_{k_z=N_1^z}^{N_2^z} f(x_{k_x}, y_{k_y}, z_{k_z}) \times$$

$$sinc\sigma_x(x - k_x) sinc\sigma_y(y - k_y) sinc\sigma_z(z - k_z)$$

then

$$T(x, y, z) \leq \frac{M|\sin \sigma_x x||\sin \sigma_y y||\sin \sigma_z z|}{(2\pi)^3 \cos(\pi r_x/2) \cos(\pi r_y/2) \cos(\pi r_z/2)} \times$$

$$\left[\frac{1}{N_1^x} + \frac{1}{N_2^x}\right] \left[\frac{1}{N_1^y} + \frac{1}{N_2^y}\right] \left[\frac{1}{N_1^z} + \frac{1}{N_2^z}\right]$$

Consider the global $\mathcal{H}(\mathcal{M})$ interpretation generated on $\mathcal{I}$ and restricted to its embedding hypersurface $\mathcal{M}_{\mathcal{I}}$ and let $\Psi(x, y, z)$ denote the NRQM wave function on $\mathcal{M}_{\mathcal{I}}$. The generating frequencies of the reality game are $\frac{2\pi}{l_P}, \frac{2\pi}{l_P}, \frac{2\pi}{l_P}$ respectively, N_x, N_y, N_z are the bounds of the (symmetric) intervals about $x = 0, y = 0, z = 0$. Translating the theorem into our reality game parameters, one obtains a bound on the truncation error,

$$|T(x, y, z)| \leq$$

$$\frac{\max_{\mathcal{M}'} |\Psi(x, y, z)|}{(2\pi)^3 \cos(\pi\sigma_x l_P/2) \cos(\pi\sigma_y l_P/2) \cos(\pi\sigma_z l_P/2)} [\frac{2}{N_x}][\frac{2}{N_y}][\frac{2}{N_z}]$$

The frequency bounds are related to the momenta of the particle by $\sigma_x = p_x/\hbar, \sigma_y = p_y/, \hbar\sigma_z = p_z/\hbar$. One can only roughly approximate the N bounds because they will depend upon how the embedding sites are selected. Let the accent $\hat{}$ applied to a parameter indicate the numerical value of the parameter with its units stripped away. Thus $\hat{l}_P$ represents the numerical value of l_P with the units stripped away. Let us assume some origin site and that along each lattice axis there are $2/\hat{l}_P$ informons, meaning that the spread of the physical entity generated by the process is a metre from the origin in each lattice direction. Substituting into the above gives

$$|T(x,y,z)| \leq \frac{\max_{\mathcal{M}'}|\Psi(x,y,z)|}{(2\pi)^3\cos(\pi p_x l_P/2\hbar)\cos(\pi p_y l_P/2\hbar)\cos(\pi p_z l_P/2\hbar)}(\frac{l_P^3}{2^3})$$

If we let p_P equal the Planck momentum ($\hbar/l_P$) then the above becomes

$$\frac{\max_{\mathcal{M}'}|\Psi(x,y,z)|}{(2\pi)^3\cos(\pi p_x/2p_P)\cos(\pi p_y/2p_P)\cos(\pi p_z/2p_P)}(\frac{l_P^3}{2^3})$$

Most particles will possess energy and momentum well below the Planck energy and momentum and so the cosine values above will be pretty close to 1. Therefore we have

$$|T(x,y,z)| \leq \frac{\max_{\mathcal{M}'}|\Psi(x,y,z)|}{(2\pi)^3}(\frac{l_P^3}{2^3}) \approx 10^{-95}\max_{\mathcal{M}'}|\Psi(x,y,z)|(\text{units } m^{-3})$$

Thus for an observation at the smallest spatial and temporal intervals that can currently be measured, the discrepancy between the Schrödinger wave function and the interpretation function is so small as to render them effectively identical.

In standard NRQM the wave function of a fundamental particle is held to extend over all of space-time. A metre is thus a rather small distance compared to that, but it is rather enormous from the standpoint of a generative model such as this process model. For a slightly more realistic example, note that if there are n informons proceeding from the origin along a lattice direction then the distance from the origin will be nl_P. If information were to propagate from the furthest informon to a nascent informon embedding at the origin, then that information would propagate with speed nl_P/t_P. If we set this speed to be the maximum possible according to relativity, then $nl_P/2t_P = c$. If t_P is chosen to be the Planck time, then $n = 1$ which results in a very poor accuracy. If on the other hand we choose $\hat{t}_P = \hat{l}_P$ then $n = \hat{c}$. In that case in the Thomas-Yao formula we have

$$|T(x,y,z)| \leq \frac{\max_{\mathcal{M}'}|\Psi(x,y,z)|}{(2\pi)^3}(\frac{8}{c^3}) \approx 3\times 10^{-25}\max_{\mathcal{M}'}|\Psi(x,y,z)|(\text{units } m^{-3})$$

which is still rather accurate.

A more accurate estimation of error requires estimating the accuracy of the generation of the discrete approximation to the path integral or to the kernel integral. This is beyond the scope of this thesis but in general will depend upon the number of informons contributing information to each nascent informon. In the setting of NRQM discussed here this can be the entire prior causal tapestry since there is no limit to the speed of information transfer. If more realistic relativistic considerations are taken into account then the problem becomes more difficult. For example in the second error situation discussed above, one has only $2\hat{c}$ informons which can send information to the informon generated at the origin and lesser numbers will contribute to the outlying informons. The informons at the extremities can only receive information from $\hat{c}$ informons. This will certainly lead to errors in the approximations to the calculation of the local strengths which are based on these integrals.

3.6 Discussion

Before I proceed, a few comments are in order. First of all it is important to note that causal tapestries are discrete structures. The number of informons generated by a single process may be finite although vast (which will be the usual interpretation) or countably infinite. Particularly in the finite case, the global $\mathcal{H}(\mathcal{M})$ interpretation function, whether on each hypersurface or on the entire causal manifold, will be a well defined, bounded, entire function with bounded Fourier transforms, meaning that it corresponds to the wave function of a particle having bounded energy and momentum. Divergences and infinities do not occur. The reality game implementing the dynamics is played using only information residing within the intrinsic interpretation of the prior informons, together possibly with information residing within their content sets. In the simple model presented here the content information has not been used but it becomes relevant when dealing with time dependent dynamics, with process interactions and with relativistic dynamics. The causal manifold and its Hilbert space are to be understood as *interpretations*, idealizations or archetypal representations of the informons that comprise the causal tapestry and external to the them (though associated to them, hence the notion). It is the causal tapestry which is meant to represent the ultimate reality which is why it is important that the process dynamics be ontologically and epistemologically closed on the collection of causal tapestries and not involve the extrinsic interpretations. The extrinsic interpretations are needed, however, to connect this model to the standard mathematical structures of NRQM.

As will be shown in the next section, NRQM can be viewed as an effective, emergent theory arising in the continuum limit as $N, r \to \infty$ and $t_P, l_P \to 0$. From the process perspective, NRQM is an idealization and not a veridical depiction of the fundamental level of reality. Quantum systems are emergent, appearing in the global $\mathcal{H}(\mathcal{M})$ interpretation function. Space, and time are also to be viewed as emergent, arising from the actions and interactions of these processes. This is not readily apparent from the path integral strategy but it should be possible to enable the reality game to endow the collection of causal tapestries with a metric structure and more explicitly generate the causal space-time structure. That is beyond the scope of this thesis and would take us into the realm of quantum gravity studies. The process model has a natural field aspect to it and so it should also be possible to extend this into QFT as well.

Second, note that the only information required to construct these functions comes from the current causal tapestry and the nascent causal tapestry. Indeed all of the necessary information resides effectively within the 'transient now' that consists of the triad of current tapestry, nascent tapestry and process. Thus the approach remains consistent with the ideas of process theory and of a moving present as captured in the reality game. The notion of a transient now developed in this model is more subtle than the idea of a simple space-like hypersurface such as $\{t\} \times \mathbb{R}^3$. The idea of the transient now is best ascribed to the game itself rather than to an individual causal tapestry. There is a succession of games being played out and the outcome of such game play is a causal tapestry but if one were to observe this manifest in reality then as soon as one causal tapestry is completed one would observe the previous causal tapestry disappearing and the next being created so that there is actually always a little overlap between the old and the new. Thus our transient now as observed in the flow of actual occasions is actually an extended present, but the succession of games is discrete and so the now is actually better marked by the succession of games rather than by the succession of causal tapestries. This can be depicted as

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\mathfrak{PI}} & \mathcal{I}' \\ & \uparrow & \\ & \mathbb{P} & \end{array}$$

where $\mathcal{I}$ and $\mathcal{I}'$ are the current and nascent tapestries respectively, $\mathfrak{PI}$ is the game and $\mathbb{P}$ is the process. The triad of $\mathcal{I}, \mathbb{P}, \mathcal{I}'$ forms what philosophers call a compound present (see Appendix B). The game play $\mathcal{I} \xrightarrow{\mathfrak{PI}} \mathcal{I}'$ is ongoing but in a succession of processes we have

$$\begin{array}{ccccccc} \mathcal{I}_0 & \xrightarrow{\mathfrak{PI}} & \mathcal{I}_1 & \xrightarrow{\mathfrak{PI}} & \mathcal{I}_2 & \xrightarrow{\mathfrak{PI}} & \mathcal{I}_3 \\ & \uparrow & & \uparrow & & \uparrow & \\ & \mathbb{P}_1 & & \mathbb{P}_2 & & \mathbb{P}_3 & \end{array}$$

These mark the transient now. The reader is referred to Appendix B for a discussion of the invariance of causal tapestries under Lorentz transformations. So long as the generating game respects causal relations, the system of causal tapestries will respect Lorentz invariance and so there is no violation of relativity in the present centered structure of this model. In stating this I am in no way suggesting that the specific process models to be described in this chapter and in Chapter 5 are Lorentz invariant. These models represent processes possessing very specific dynamical structure meant to reproduce the main features of non-relativistic quantum mechanics. Here I am specifically referring to the fact that the mathematical structures upon which the process dynamics plays out are based upon a conception of a compound present, a notion that is generally rejected by modern physics. The discussion on Unitarity in Appendix B provides a brief overview of arguments dealing with the nature of time, and serves to point out that the matter has not been settled for good. The model just presented is thus offered as a viable present centric model of NRQM. It is hoped that future research may show that a relativistically invariant particle dynamics can be formulated within the process framework so that both the causal tapestries as well as their generating processes may be relativistically invariant and thus suitable for QFT. That research is left to the future and is well beyond the scope of this thesis.

There are different ways to understand how game play proceeds. I prefer to think of it as occurring outside of our perceptual time, in the more abstract world of process. This could be thought of as a second time as occurs in the theories of Parisi and Wu [104] and Bars [38]. It might also be viewed as taking place within normal time but on a scale vastly smaller than even the Planck time. However one wishes to imagine it, the important point is that it occurs on a scale which is much smaller than that at which the entities of observable physical realty manifest themselves, and so is inherently unobservable, just as the actual occasions that the informons of the causal tapestry represent are inherently unobservable.

Third, the information used to construct each step of the game is causally local information. It is information that can be transferred along a causal path between the sender and the receiver. The $\mathcal{H}(\mathcal{M})$ interpretation is built up from the actual value of the interpretation calculated at the origination point coupled with a phase shift determined solely on the basis of local information acquired along the path linking the originating point to the terminating point. This information is integrated into the final function, but each component piece of information is local. The only non-locality arises from the non-deterministic nature of game play. The players are allowed to choose the initial and final informons non-deterministically, the only constraints being that the pairing has not been previously selected (which would create redundant information), and that the pairing is causal. This latter constraint is not an issue here due to the simple nature of the causal structure but becomes significant in the relativistic case.

Game play, to borrow a term from the science fiction writer Alfred Bester [51], *jauntes* from site to site. The current causal tapestry extends like a percolation process into the next causal tapestry, but only one site is ever extended during any given round of game play. This latter condition models the idea that our reality game is describing the evolution of a single particle and that a single particle can only ever manifest a single informon during any single round. This accounts for the apparent locality of a particle when interacting in spite of it also appearing as a diffuse wave (the $\mathcal{H}(\mathcal{M})$ interpretation. More about this will be said in Chapter 4. If one likes, one may picture the particle as manifesting informons that appear to jaunte from site to site in the hypersurface of the causal manifold, leaving a fuzzy trace behind them which sums to give the wave-like effect even though at any instant only a particle-like effect manifests. The particle-like effect asserts itself during interactions between processes while we observe only the wave-like effect because the scale of the individual informons is below any possible quantum or classical entity. This jaunting behaviour violates the principle of continuity but our observable reality retains a continuous character on account of the interpolation process that renders the global $\mathcal{H}(\mathcal{M})$ interpretation. Moreover, the jaunting movement of game play across the lattices does not involve the transfer of any information. It merely determines where the next game play is to take place. No signal is transferred and so relativistic constraints are never violated. This mixture of locality at the level of informons and non-locality at the process level is termed *quasi-locality.*

Properties have been discussed so far in this chapter because our model describes only single particle generation by a simple process. Multiple particle generation and superposition processes will be described later. Anticipating those results, one should note that individual informons are assigned definite individual properties that are inherited from (or generated by) the process that generates the informon. These specific properties are non-contextual. They are fixed properties of the informon independent of any observer. As discussed in Chapter 2, they influence the outcomes of interactions between processes so that they ‘make a difference’, and thus possess ‘reality’ . Nevertheless, they are inherently unobservable and do not violate Type II hidden variable theorems. They manifest themselves only in the context of a measurement situation and in such a case properties are seen to be generated by the act of measurement with measured values depending upon the structure of the generating process. If the generating process does not endow the actual occasion with additional properties then they can be said to possess them only as potentialities, not as actualities. Thus the determination of the specific collection of properties that any given informon possesses is contextual as it arises from the actions of processes and these are contextual. Still, the properties that an actual occasion does possess, once they possess them, are in themselves non-contextual. This mixture of non-contextuality at the level of informons and contextuality at the process level is termed *quasi-non-contextuality.*

Finally it is important to note that all of the essential information defining the $\mathcal{H}(\mathcal{M})$ interpretation resides in the value of the interpretation on each individual lattice site. The sinc function does not provide any real information of its own. It merely serves as a bridge to link across distinct lattice sites. This interpolation function could actually take many forms and the main aspect that will be affected is the accuracy of the interpolation. Recall that any interpolation has the form $\sum_i f(i)g_i(\mathbf{z})$ where $f(i)$ is the value of the function to be interpolated, sampled at the point i, and $g_i(\mathbf{z})$ is the wavelet used to form the interpolation at all other points. The sampling value contains all of the information concerning the function, whereas the choice of wavelet is somewhat more arbitrary. The reason why there is a separation between the laying down of tokens by Player I and the coupling to the interpolating wavelets by Player II is on account of the difference in informational character between the sampling values that form an interpolation and the wavelets that carry out the interpolation. This distinction disappears when any actual value for the interpretation is being determined at a particular site off the lattice but it is essential to keep this distinction in mind when carrying out constructive calculations on these interpretations. There is an important distinction between working in the process or game space and working in the Hilbert space, which is reflected in the process covering map.

In the causal tapestry framework it is therefore the token values on the informons, the actual occasions, that contain the essential information and the wavelets generate the interpretation which provides the link to other forms of representation of this information, here in particular the Hilbert space of wave functions. By means of these tokens, the causal tapestry contains sufficient information to recreate everything else, including the symmetries that can be observed. It is not necessary that there be continuity in the causal tapestry, nor is it necessary that there be time reversal symmetry. These become emergent properties. Indeed, fundamentally, causal tapestries are both discontinuous and time asymmetric, proving that these are ideological or pragmatic assumptions and not ontological necessities.

The value of the process model is that it grounds physics in a realist ontology without the need for observers or multiverses or perhaps even multiple dimensions. Those features which appear to be at odds with a grounded reality such as non-locality and contextuality are seen to be features of the processes that generate the ground, and not the ground itself. Thus one can have a reality which is 'out there' without that reality being 'OUT THERE! ' . The weirdness arises because we think that the ground is the generator, since in most models of physics nothing develops at all. Viewing reality from a process perspective, and realizing that observers are not necessary components because processes already intrinsically manifest a kind of a-locality (through the jaunting nature of the generation process) and contextuality (since everything that is observable is emergent and arises as a result of interactions between processes) in the way in which measurements are performed, the actual occasions themselves can therefore take on local, non-contextual characteristics. The process viewpoint paints a picture of reality in which there are definite observer independent somethings at the fundamental level, but as we can only interact with these somethings through the mediation of process, our *experience* of them adopts the characteristics of process. The use of the process perspective, as will be discussed in Chapter 4, resolves many, and I suspect, all of the paradoxes of NRQM. The price is to abandon the principle of continuity and time reversal symmetry at the fundamental level but this is not a great loss since we recover them at the observable level.

Additional strategies are considered in Chapter 5. Although the $\mathfrak{PI}$ strategy suffices to show that a quasi-local, quasi-non-contextual realist model of NRQM is indeed possible, and therefore that the standard interpretations of the hidden variable theorems are incorrect (or at least overstated), it does not provide an effective model of a sub-quantum reality. There are problems with convergence and with the need to start the evolution from an initial causal tapestry in a rather special form. More sophisticated models are described in Chapter 5 as material for future research. That even the simple $\mathfrak{PI}$ strategy works provides motivation to drive future efforts.

3.7 The Kernel Strategy

A more accurate and more transparent game for NRQM can be obtained if we take the $\mathfrak{PI}$ strategy as described in Section 3.3 and modify the rule for constructing tokens, keeping all other details unchanged. Feynman points out [65] that the definition of the propagator as the path integral of the exponential of the action works well in the non-relativistic case but may fail under relativistic conditions. Under such conditions one must resort to using the propagator itself. Although the propagator may often be found by means of a path integral it actually contains only local information. Feynman and Hibbs show [132] that at least in the case of a time independent Hamiltonian in which the Schrödinger equation possesses a collection of orthonormal eigenfunctions $\Lambda_i(x)$, each corresponding to the energy E_i, the (one dimensional) kernel may be written as

$$K(t_b, x_b; t_a, x_a) = \begin{cases} \sum_{n=1}^{\infty} \Lambda_n(x_b)\Lambda_n^*(x_a)e^{-(i/\hbar)E_n(t_b-t_a)} & \text{for } t_b > t_a \\ 0 & \text{for } t_b < t_a \end{cases}$$

In this modified game, Player I adds tokens of the form $l_P^3 K(\beta, \alpha)\Gamma_\alpha$ to the site β. Player I needs only have local information about the informon in the current tapestry and the informon in the nascent tapestry that are being causally connected during this short round of game play. Player II then constructs the $\mathcal{H}(\mathcal{M})$ interpretation more or less as before using the new tokens, so that if the new informon is nascent, then the $\updownarrow\wr\rfloor\dashv\updownarrow\mathcal{H}(\mathcal{M})$ interpretation is given as $\phi_{n'}(t_n + t_P, \mathbf{z}) = l_P^3 K(\alpha_{n'}, \alpha_n)\Gamma_{\alpha_n} T_{\alpha_{n'}} sinc_{t_P,l_P}(t_n + t_P, \mathbf{z})$ and if the informon already possesses a $\mathcal{H}(\mathcal{M})$ interpretation, $\phi_{n'}(t_n + t_P, \mathbf{z})$, then replace it by the new $\mathcal{H}(\mathcal{M})$ interpretation,

$$\phi_{n'}(t_n + t_P, \mathbf{z}) + l_P^3 K(\alpha_{n'}, \alpha_n)\Gamma_{\alpha_n} T_{\alpha_{n'}} sinc_{t_P,l_P}(t_n + t_P, \mathbf{z}).$$

Keeping the notation of the previous chapter and repeating some of the argument in Section 4.4 one can show that in the case where N, r are sufficiently large, t_P, l_P sufficiently small, and with repeated use of Parzen's Theorem, the global $\mathcal{H}(\mathcal{M})$ interpretation generated on the nascent causal tapestry $\mathcal{I}_{m+1}$ is given by

$$\Phi_{m+1}(\beta) \approx \sum_{\alpha_{n'} \in \tilde{L}_{m+1}} \sum_{\alpha_n \in \tilde{L}_m} l_P^3 K(\alpha_{n'}, \alpha_n)\Gamma_{\alpha_n} T_{\alpha_{n'}} sinc_{t_P,l_P}(\beta) =$$

$$\sum_{\alpha_{n'} \in \tilde{L}_{m+1}} \sum_{\alpha_n \in \tilde{L}_m} l_P^3 K(\alpha_{n'}, \alpha_n)\Gamma_{\alpha_n} T_{\alpha_n} sinc_{t_P,l_P}(\alpha_n) T_{\alpha_{n'}} sinc_{t_P,l_P}(\beta) =$$

$$\sum_{\alpha_{n'} \in \tilde{L}_{m+1}} \sum_{\alpha_n \in \tilde{L}_m} l_P^3 K(\alpha_{n'}, \alpha_n)\Phi_m(\alpha_n) T_{\alpha_{n'}} sinc_{t_P,l_P}(\beta) \approx$$

$$\sum_{\alpha_{n'} \in \tilde{L}_{m+1}} \sum_{\alpha_n \in \tilde{L}_m} l_P^3 K(\alpha_{n'}, \alpha_n)\Psi_m(\alpha_n) T_{\alpha_{n'}} sinc_{t_P,l_P}(\beta) \approx$$

$$\sum_{\alpha_{n'} \in \tilde{L}_{m+1}} \int_{\mathcal{M}_{t_m}} K(\alpha_{n'}, \alpha_n)\Psi_m(\alpha_n) d\alpha_n T_{\alpha_{n'}} sinc_{t_P,l_P}(\beta)$$

According to Feynman and Hibbs [132]

$$\Psi_{m+1}(\beta) = \int_{M_{t_m}} K(\beta, \alpha)\Psi_m(\alpha) d\alpha$$

where $M_{t_{\{m\}}}$ is the $\{t_m\} \times \mathbb{R}^3$ hypersurface, so substituting we obtain $\overline{\Phi_{m+1}(\beta)} \approx$

$$\overline{\sum_{\alpha_{n'} \in \tilde{L}_{m+1}} \int_{M_{t_m}} K(\alpha_{n'}, \alpha_n) \Psi_m(\alpha_n) d\alpha_n T_{\alpha_{n'}} sinc_{t_P, l_P}(\beta)} =$$

$$\overline{\sum_{\alpha_{n'} \in \tilde{L}_{m+1}} \Psi_{m+1}(\alpha_{n'}) T_{\alpha_{n'}} sinc_{t_P, l_P}(\beta)} \approx \Psi_{m+1}(\beta).$$

Hence in the limit as $N, r \to \infty$ and $t_P, l_P \to 0$ we have

$$\overline{\Phi_{m+1}(\beta)} \to \Psi_{m+1}(\beta)$$

Again NRQM emerges in the asymptotic limit.

This kernel strategy $\mathfrak{K}$ is certainly simpler to set up and to analyze than the $\mathfrak{PI}$ strategy, but it is less directly intuitive. The proof that NRQM emerges in the asymptotic limit is certainly easier. Again it relies only upon knowledge that exists in the moment between the current and the nascent tapestries, thus it remains consonant with the viewpoint that causal tapestries represent a form of transient now, in which the current tapestry fades away to be replaced by the nascent tapestry, until it in turn fades away and the cycle repeats.

The astute reader may notice that there is a problem in the case that the potential $V(\mathbf{z}) = 0$. The subtlety here is that we have not paid attention to the nature of the relationship between the environment process $\mathbb{E}$, whose influence is summarized in the potential $V(\mathbf{z})$. There are at least three possibilities for the relationship between $\mathbb{P}$ and $\mathbb{E}$. First of all, they may be independent, in which case the combined process is described using the exclusive product, $\mathbb{P} \otimes \mathbb{E}$. The global $\mathcal{H}(\mathcal{M})$ interpretation for $\mathbb{P}$ will be generated using the free particle kernel for the Schrödinger equation, as defined in Section 3.1. The case treated above in this section is the bound case, in which the kernel is given by the Green's function for the Schrödinger equation. Finally the scattering case is a different interactive product $\mathbb{P} \boxtimes' \mathbb{E} = \mathbb{P} \oplus \mathbb{P} \boxtimes \mathbb{E}$ which involves a superposition of a free process and an interactive process. The kernels necessary for each of these games may vary because they involve differing types of interactions. No more will be said of this here since the main focus of the thesis is on the application of the process approach to dealing with the paradoxes and hidden variable problems but the construction of strategies for dealing with complicated interactive products is a potentially rich and fascinating undertaking.

3.8 Strategies and Operators

The relationship between processes and Hilbert space operators was discussed in Section 2.6. There it was shown that general processes give rise to generalized linear operators on the Hilbert space $\mathcal{H}(\mathcal{M})$ and that in the case of Ψ-faithful processes, these operators correspond to standard linear operators.

In this chapter, processes have been represented by combinatorial games and two specific strategies, the path integral and kernel strategies, have been explored. In both cases, these strategies used uniform sinc interpolation, so that all n-th generation causal tapestries would embed into the same lattice decomposition L_n of the same hypersurface $\mathcal{M}_n$ of the causal manifold $\mathcal{M}$. Many advantages to using the uniform sinc interpolation approach were discussed (along with several disadvantages from a physical standpoint). Another important advantage to uniform sinc interpolation is that it gives rise to Ψ-faithful processes.

Consider a process $\mathbb{P}$ being represented by a combinatorial game using a uniform sinc interpolation strategy. Let $\mathcal{I}, \mathcal{I}'$ denote two distinct causal tapestries arising at the $n-1$-th generation and let $\mathbb{P}$ act on them. The global $\mathcal{H}(\mathcal{M})$ interpretations, $\Phi(\mathbf{z})$ and $\Phi'(\mathbf{z})$ respectively will take the form

$$\Phi(\mathbf{z}) = \sum_{n\in\mathcal{I}} \Gamma_n T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z})$$

$$\Phi'(\mathbf{z}) = \sum_{n'\in\mathcal{I}'} \Gamma'_{n'} T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

If $\mathfrak{I}(\mathcal{I}) = \mathfrak{I}(\mathcal{I}')$ then

$$\sum_{n\in\mathcal{I}} \Gamma_n T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z}) = \sum_{n'\in\mathcal{I}'} \Gamma'_{n'} T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

On the lattice L_n if follows that for $\mathbf{m} \in L_n$

$$T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{m}) = \begin{cases} 1 \text{ if } \mathbf{m} = \mathbf{m}_n \\ 0 \text{ if } \mathbf{m} \neq \mathbf{m}_n \end{cases}$$

Hence substituting $\mathbf{m}_n$ into both sides of the equation yields

$$\sum_{n\in\mathcal{I}} \Gamma_n T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{m}_n) = \sum_{n'\in\mathcal{I}'} \Gamma'_{n'} T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{m}_n)$$

so that if $\mathbf{m}_n = \mathbf{m}_{n'}$ for some n' then $\Gamma_n = \Gamma'_{n'}$, otherwise $\Gamma_n = 0$. Substituting $\mathbf{m}_{n'}$ into both sides yields if $\mathbf{m}_{n'} = \mathbf{m}_n$ for some n then $\Gamma'_{n'} = \Gamma_n$, otherwise $\Gamma_{n'} = 0$. Hence it follows that there is an isomorphism between $\mathcal{I}$ and $\mathcal{I}'$ given by $n \to n'$ iff $\mathbf{m}_n = \mathbf{m}_{n'}$ and in such a case, $\Gamma_n = \Gamma'_{n'}$. This implies that $\mathcal{I}$ and $\mathcal{I}'$ are identical causal tapestries and so $\mathbb{P}$ acting on them will generate identical outcomes. Thus $\mathbb{P}$ is Ψ-faithful.

The path integral and kernel strategies described in the preceding sections both utilized uniform sinc interpolation. Let us suppose for the moment that a suitable non-uniform interpolation strategy has been found and applied together with the kernel strategy for constructing the propagator. A similar argument can be expected to hold in the path integral case but the kernel argument is simpler in construction.

Assume therefore that a suitable non-uniform interpolation strategy has been found which works effectively with the kernel (or path integral) strategy so as to ensure that in the asymptotic limit as $N, r \to \infty$, the process covering map yields a single global $\mathcal{H}(\mathcal{M})$ interpretation beginning with any suitable causal tapestry.

Assume then that there are two distinct causal tapestries $\mathcal{I}, \mathcal{I}'$ formed in the n-th generation such that their respective global $\mathcal{H}(\mathcal{M})$ interpretations,

$$\Phi(\mathbf{z}) = \sum_{n \in \mathcal{I}} \Gamma_n T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z})$$

$$\Phi'(\mathbf{z}) = \sum_{n' \in \mathcal{I}'} \Gamma'_{n'} T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

respectively satisfy

$$\sum_{n \in \mathcal{I}} \Gamma_n T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z}) = \sum_{n' \in \mathcal{I}'} \Gamma'_{n'} T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

The causal manifold embeddings $\{\mathbf{m}_n\}$ and $\{\mathbf{m}_n\}$ are uncorrelated beyond lying within the same hypersurface $\mathcal{M}_n$ in $\mathcal{M}$ and satisfying the requirements of the interpolation method, which usually places some restriction on the density of the embedding points in $\mathcal{M}$. In the kernel strategy, the strengths Γ_n and $\Gamma'_{n'}$ are derived from the sum of tokens which are in turn derived from causal tapestries $\mathcal{J}, \mathcal{J}'$ formed at the $n-1$-st generation and it is assumed that their global $\mathcal{H}(\mathcal{M})$ interpretations form good approximations to some NRQM wave function. Expanding these strengths one obtains

$$\Phi(\mathbf{z}) = \sum_{n \in \mathcal{I}} \sum_{m \in \mathcal{J}} l_P^3 K(\mathbf{m}_n, \mathbf{m}_m) \Gamma_m T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z})$$

$$\Phi'(\mathbf{z}) = \sum_{n'\in\mathcal{I}'}\sum_{m'\in\mathcal{J}'} l_P^3 K(\mathbf{m}_{n'},\mathbf{m}_{m'})\Gamma'_{m'} T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

By assumption, $\{\Gamma_m\}$ and $\{\Gamma'_{m'}\}$ form approximate samplings of the same NRQM wave function $\Psi(\mathbf{z})$. A moment's reflection shows that

$$\sum_{m\in\mathcal{J}} l_P^3 K(\mathbf{m}_n,\mathbf{m}_m)\Gamma_m \approx \sum_{m\in\mathcal{J}} l_P^3 K(\mathbf{m}_n,\mathbf{m}_m)\Psi(\mathbf{m}_m)$$

and

$$\sum_{m'\in\mathcal{J}'} l_P^3 K(\mathbf{m}_{n'},\mathbf{m}_{m'})\Gamma'_{m'} \approx \sum_{m'\in\mathcal{J}'} l_P^3 K(\mathbf{m}_{n'},\mathbf{m}_{m'})\Psi(\mathbf{m}_{m'})$$

so that each sum constitutes a discrete approximation of the same integral over the hyypersurface $\mathcal{M}_{n-1}$, namely

$$\int_{\mathcal{M}_{n-1}} K(\mathbf{y},\mathbf{z})\Psi(\mathbf{z})dV$$

Therefore in the limit $t_P, l_P \to 0$ these will converge to the same values, which means that the process $\mathbb{P}$ will generate different interpolations of the very same NRQM wave function $\Psi(\mathbf{z})$, which in turn implies that $\mathbb{P}$ is Ψ-faithful.

The above is a rather causal argument which needs to be made more rigorous. It does suggest that the kernel and path integral strategies are worthy of further study because they are most likely to lead to Ψ-faithful processes whether using uniform or non-uniform interpolation strategies. This is important because it implies that in the asymptotic limit $N, r \to \infty$, and possibly also $t_P, l_P \to 0$ as well, the process may be idealized or represented as a standard Hilbert space operator.

Thus the usual NRQM formalism arises in a natural manner. This is consistent with the idea promoted throughout this thesis that NRQM is an emergent theory, an idealization of the discrete system when the asymptotic limits can be taken to be reasonably justified.

3.9 The Momentum Representation

In standard quantum mechanics the momentum representation is obtained from the position representation by means of the Fourier transform. If $\Psi(\mathbf{x})$ is the wave function in the position representation, then

$$\hat{\Psi}(\mathbf{k}) = \frac{1}{(\sqrt{2\pi\hbar})^3} \int_V \Psi(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}/\hbar} dV$$

gives the wave function in the momentum representation.

The wave function $\hat{\Psi}(\mathbf{k})$ in the momentum representation may be defined by a kernel $\mathcal{K}$ just as for the position representation. Feynman and Hibbs [132] show that this kernel may be derived from the kernel in the position representation as follows:

$$\mathcal{K}(t_b, \mathbf{p}_b; t_a, \mathbf{p}_a) = \iint e^{-(i/\hbar)\mathbf{p}_b\cdot\mathbf{x}_b} K(t_b, \mathbf{x}_b; t_a, \mathbf{x}_a) e^{+(i/\hbar)\mathbf{p}_a\cdot\mathbf{x}_a} d^3\mathbf{x}_b d^3\mathbf{x}_a$$

To see this,

$$\hat{\Psi}_{t_1}(t_1, \mathbf{p}_1) = \int \mathcal{K}(t_1, \mathbf{p}_1; t, \mathbf{p}) \hat{\Psi}_t(t, \mathbf{p}) d\mathbf{p}$$

Rewrite this as

$$\int e^{-(i/\hbar)\mathbf{p}_1\cdot\mathbf{x}_1} \Psi_{t_1}(t_1, \mathbf{x}_1) d\mathbf{x}_1 = \int e^{-(i/\hbar)\mathbf{p}_1\cdot\mathbf{x}_1} \int K(t_1, \mathbf{x}_1; t, \mathbf{x}) \Psi_t(t, \mathbf{x}) d\mathbf{x}_1 d\mathbf{x} =$$

$$\int \mathcal{K}(t_1, \mathbf{p}_1; t, \mathbf{p}) \int e^{-(i/\hbar)\mathbf{p}\cdot\mathbf{x}} \Psi_t(t, \mathbf{x}) d\mathbf{x} d\mathbf{p}$$

Using the inverse Fourier transform to substitute for Ψ_t yields

$$\int\int\int e^{-(i/\hbar)\mathbf{p}_1\cdot\mathbf{x}_1} K(t_1, \mathbf{x}_1; t, \mathbf{x}) e^{(i/\hbar)\mathbf{p}\cdot\mathbf{x}} \hat{\Psi}_t(t, \mathbf{x}) d\mathbf{x}_1 d\mathbf{x} d\mathbf{p} = \int \mathcal{K}(t_1, \mathbf{p}_1; t, \mathbf{p}) \hat{\Psi}(t, \mathbf{x}) d\mathbf{p}$$

Equating the kernels gives

$$\int\int e^{-(i/\hbar)\mathbf{p}_1\cdot\mathbf{x}_1} K(t_1, \mathbf{x}_1; t, \mathbf{x}) e^{(i/\hbar)\mathbf{p}\cdot\mathbf{x}} d\mathbf{x}_1 d\mathbf{x} = \mathcal{K}(t_1, \mathbf{p}_1; t, \mathbf{p})$$

As Feynman and Hibbs point out, one must actually take into account the transformation from t to E, from time to energy. Including this one obtains

$$\mathcal{K}(t_b,\mathbf{p}_b;t_a,\mathbf{p}_a) = \iint \int_{-\infty}^{\infty}\int_{t_a}^{\infty} e^{-(i/\hbar)\mathbf{p}_b\cdot\mathbf{x}_b} e^{+(i/\hbar)E_b t_b} K(t_b,\mathbf{x}_b;t_a,\mathbf{x}_a) \times e^{+(i/\hbar)\mathbf{p}_a\cdot\mathbf{x}_a} e^{-(i/\hbar)E_a t_a} dt_b dt_a d^3\mathbf{x}_b d^3\mathbf{x}_a$$

In the causal tapestry setting the global $\mathcal{H}(\mathcal{M})$ interpretation is defined on the entire causal manifold $\mathcal{M}$, but as the accuracy relative to the NRQM wave function is poorer, it is usual to focus just upon the restriction of the global $\mathcal{H}(\mathcal{M})$ interpretation to the embedding hypersurface corresponding to the generating causal tapestry. The causal tapestry $\mathcal{I}_{t_1}$ embeds to the hypersurface $\{t_1\}\times\mathbb{R}^3$ with embedding lattice L_1. For simplicity let me consider only the case of one spatial dimension, so that $\mathcal{M}$ has $(1+1)$ dimensions. On this surface the global $\mathcal{H}(\mathcal{M})$ interpretation takes the form

$$\Phi_{t_1}(t_1,z) = \sum_{(t_1,y)\in L_1} \Phi_{t_1}(t_1,y) T_{(t_1,y)} sinc_{t_P,l_P}(t_1,z)$$

If we take the Fourier transform of this function we obtain (see Stenger [340])

$$\hat{\Phi}_{t_1}(t_1,k_1) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \sum_{(t_1,y)\in L_1} \Phi_{t_1}(t_1,y) T_{(t_1,y)} sinc_{t_P,l_P}(t_1,z) e^{-ik_1 z/\hbar} dz =$$

$$\frac{1}{\sqrt{2\pi\hbar}} \sum_{(t_1,y)\in L_1} \Phi_{t_1}(t_1,y) e^{-ik_1 y/\hbar} =$$

$$\frac{1}{\sqrt{2\pi\hbar}} \sum_{(t_1,y)\in L_1} \int K(t_1,y;t,x)\Phi_t(t,x) dx e^{-ik_1 y/\hbar}$$

Now Φ is the inverse Fourier transform of $\hat{\Phi}$ so

$$\Phi_t(t,x) = \frac{1}{\sqrt{2\pi\hbar}}\int \hat{\Phi}_t(t,k) e^{+iky/\hbar} dk$$

Substituting we obtain

$$\hat{\Phi}_{t_1}(t_1,k_1) = \frac{1}{\sqrt{2\pi\hbar}} \sum_{(t_1,y)\in L_1} \int K(t_1,y;t,x) \frac{1}{\sqrt{2\pi\hbar}} \int \hat{\Phi}_t(t,k) e^{+iky/\hbar} dk dx e^{-ik_1 y/\hbar} =$$

$$\frac{1}{\sqrt{2\pi\hbar}}\sum_{(t_1,y)\in L_1}\int\int e^{-ik_1y/\hbar}K(t_1,y;t,x)\frac{1}{\sqrt{2\pi\hbar}}e^{+iky/\hbar}\hat{\Phi}_t(t,k)dxdk \approx$$

$$\frac{1}{\sqrt{2\pi\hbar}}\int\int\int e^{-ik_1y/\hbar}K(t_1,y;t,x)\frac{1}{\sqrt{2\pi\hbar}}e^{+iky/\hbar}\hat{\Phi}_t(t,k)dydxdk =$$

$$\int \mathcal{K}(t_1,k_1;t,k)\hat{\Phi}_t(t,k)dk$$

So the momentum representation based upon the $\mathcal{H}(\mathcal{M})$-interpretation remains consistent with the usual NRQM momentum representation. The basis functions are quite different, however. Note that the Fourier transform formula for the interpolation actually only holds for $k < 2\pi/l_P$, reflecting the fact that our particles possess bounded momentum (and energy as well).

3.10 Multiple Subprocesses-Superposition

The previous sections considered the case in which the state of the quantum system can be described by a single eigenfunction of some self adjoint operator such as the Hamiltonian. The process for such a state, implemented using the free path integral strategy, generates a global $\mathcal{H}(\mathcal{M})$-interpretation pwhich provides a highly accurate approximation to the Schrödinger wave function and so can serve as a candidate model for a lower level of reality than that at the quantum level. It is well known, however, that fundamental particles and even more complicated quantum systems, can exist in a superposition of more primitive quantum states. An understanding of this situation requires some of the subtleties of combining combinatorial games.

Consider first the case of a quantum system in a superposition state, the wave function being given as $\Psi = \sum_{i=0}^{k} w_i\Psi_i$ where each Ψ_i is an eigenfunction of some operator A corresponding to the eigenvalue λ_i. The causal tapestry describing this situation must now be extended to include properties, so that the interpretation of an informon now takes the form $\alpha = (\mathbf{m}_n, \phi_n(\mathbf{z}), \Gamma_n, \mathbf{p}_n, \lambda_i)$ for some eigenvalue i. I am only considering non degenerate eigenvalues here. If the eigenvalue is degenerate and there several wave functions corresponding to the same eigenvalue then they will need to be distinguished by some label j and that label added as an additional property.

Just as the entire wave function is a superposition of individual eigenfunctions, so the process generating the entire wave function is considered as a sum of subprocesses, each generating an individual eigenstate. Each eigenstate Ψ_i corresponds to a single sub-process thought of as being generated by its own reality game $\mathfrak{R}(N, r, \rho, \delta, t_P, l_P, \omega, L, \Sigma, p, \lambda_i)$, where the eigenvalue λ_i has now been included as a property of the game. The reality game $\mathfrak{R}(N, (\rho, \delta), t_P, l_P, \Sigma, L)$ for the entire wave function may then be considered as the sum of the reality games generating the individual eigenfunctions. As discussed in Chapter 2, only the exclusive sum is used to interrelate processes in order to ensure that informons correspond to ontologically unique states.

A proper analysis of superpositions requires the use of non-uniform interpolation techniques. but the essence of the argument, interleaving, can be exhibited in the simpler case where each sub-process generates its own sub-lattice. So consider the case in which the causal tapestry is subdivided into a collection of regular sublattices, each with lattice spacing t_P, l_P as before. I think of these lattices as being separated from one another at even smaller scales, at least $t_P/k, l_P/k$. Recall that the exclusive sum requires that the play of each game take place on sites distinct from that of the other games. In this case, play of game $\mathfrak{R}(N, r, \rho, \delta, t_P, l_P, \omega, L, \Sigma, p, \lambda_i)$ takes place on lattice L^i. In this case we may write the processes as

$$\mathbb{P} = \bigoplus_{i=0}^{k} w_i \mathbb{P}_i$$

and the associated games as

$$\mathfrak{R}(N, r, \rho, \delta, t_P, l_P, \omega, L, \Sigma, p) = \sum_{i=0}^{k} w_i \mathfrak{R}(N, r, \rho, \delta, t_P, l_P, \omega, L, \Sigma, p, \lambda_i)$$

The weights are singled out because they induce a distinct effect upon the interpretations being generated by the individual games and this point requires emphasis in the construction.

The exclusive sum ensures that each informon corresponds to an actual occasion arising from the action of a single subprocess, a single eigenstate, and therefore corresponds to a single set of properties. This in turn implies that the particle itself is never in a superposition of ontologically distinct states; it is only the process that generates the particle that can be in a superposition. The property component of the interpretation keeps track of which subprocess is contributing to the generation of each informon and maintains consistency over time as the play of different games jauntes around the lattice. Further it ensures that only one process acts on any single site, so that only one game is ever played on any site.

Play proceeds more or less as in the previous case. On any given play one of the subprocesses is chosen to be played, say $\mathbb{P}_i$. When Player 1 makes their first move they may only extend from an informon whose interpretation is of the form $\alpha = (\mathbf{m}_n, \phi_n(\mathbf{z}), \Gamma_n, \mathbf{p}_n, \lambda_i)$. In the event that this is the first time that the process $\mathbb{P}_i$ is active and no such current informon exists then play moves to Player II. If Player II selects a token naive site $\mathbf{m}'_n$ then the new informon n' is given the partial form $[n'] < (\mathbf{m}'_n, , \Gamma_{n'}, \mathbf{p}_n, \lambda_i) > \{\}$. If the site is already associated with an informon then that informon must already be assigned a property interpretation $\mathbf{p}_i, \lambda_i$.

There are two ways in which the property may be reflected in the evolution of the particle. First of all, it may be the case that the Lagrangian is parameterized by the property. In that case, the Lagrangian to be used for the play of the reality game $\mathfrak{R}(N, (\rho, \delta), t_P, l_P, \Sigma, L, \lambda_i)$ will not be L but rather L_i, reflecting the choice of the parameter to correspond to the eigenvalue λ_i. In the second case, the Lagrangian itself may be universal for the entire collection of reality games but the value of the eigenvalue is incorporated into the form of the initial wave function, that is, it forms part of the initial conditions for the particle. Of course there is a third possibility, namely, that both conditions apply.

Regardless of how the interpretation is then constructed, the global game is structured so that although the particular choice of which reality game is to be played during any given round is non-deterministic and jauntes from site to site, only one reality game will be played out on any given site and any particular sublattice. Moreover, if the reality game $\mathfrak{R}(N, (\rho, \delta), t_P, l_P, \Sigma, L, \lambda_i)$ is modified by the weight w_i in the game expression, then each token being laid down during game play will have its value multiplied by w_i. In other words, if Player I would normally lay down a token of the form $(l_P^3/A^3)e^{iS[\beta,\alpha]/\hbar}\phi_n(\mathbf{m}_n)$ then in the modified game Player I lays down a token of the form $w_i(l_P^3/A^3)e^{iS[\beta,\alpha]/\hbar}\phi_n(\mathbf{m}_n)$. Player II will then form a local $\mathcal{H}(\mathcal{M})$-interpretation in the usual manner, either $w_i(l_P^3/A^3)e^{iS[\alpha_{n'},\alpha_n]/\hbar}\phi_n(\mathbf{m}_n)A^3T_{\mathbf{m}_{n'}}sinc_{t_P,l_P}(\mathbf{z})$ if the informon is nascent or

$$\phi_{n'}(\mathbf{z}) + w_i(l_P^3/A^3)e^{iS[\alpha_{n'},\alpha_n]/\hbar}\phi_n(\mathbf{m}_n)A^3T_{\mathbf{m}_{n'}}sinc_{t_P,l_P}(\mathbf{z}).$$

if the informon already has a partial interpretation $\phi_{n'}(\mathbf{z})$.

Then the global $\mathcal{H}(\mathcal{M})$ interpretation with generators on the lattice slice $\tilde{L}_2$ takes the form $\Phi_{t_2}(\mathbf{z}) =$

$$\sum_{n' \in \mathcal{I}'} \phi_{n'}(\mathbf{z}) = \sum_{n' \in \mathcal{I}'} \sum_{n \in I_{n'}} w_i (l_P^3/A^3) e^{iS[\alpha_{n'},\alpha_n]/\hbar} \phi_n(\mathbf{m}_n) A^3 T_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z})$$

But $\mathcal{I}'$ embeds only into lattice $\tilde{L}_2^i$ where i refers to the property λ_i of n'. Since we do not assume any condition yet on the length of game play N, then transforming into lattice terms we obtain

$$\Phi_{t_2}(\mathbf{z}) = \sum_{\alpha_{n'} \in L_2} \sum_{\alpha_n \in L_{n'}} w_i (l_P^3/A^3) e^{iS[\alpha_{n'},\alpha_n]/\hbar} \phi_n(\mathbf{m}_n) A^3 T_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z})$$

$$\sum_{\alpha_{n'} \in \sum_{i=0}^{k} L_2^i} \sum_{\alpha_n \in L_{n'}} w_i (l_P^3/A^3) e^{iS[\alpha_{n'},\alpha_n]/\hbar} \phi_n(\mathbf{m}_n) A^3 T_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z}) =$$

$$\sum_{i=0}^{k} \sum_{\alpha_{n'} \in L_2^i} \sum_{\alpha_n \in L_{n'}^i} w_i (l_P^3/A^3) e^{iS[\alpha_{n'},\alpha_n]/\hbar} \phi_n(\mathbf{m}_n) A^3 T_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z}) =$$

$$\sum_{i=0}^{k} \sum_{\alpha_{n'} \in L_2^i} \sum_{\alpha_n \in L_{n'}^i} w_i (l_P^3/A^3) e^{iS[\alpha_{n'},\alpha_n]/\hbar} \Phi_{t_1}^i(\mathbf{m}_n) A^3 T_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z}) = \sum_{i=0}^{k} w_i \Phi_{t_2}^i(\mathbf{z})$$

Under the limits $N \to \omega_2$ and $t_P, l_P \to 0$ it follows that $\Psi_{t_2}(\mathbf{z}) = \sum_{i=0}^{k} \Psi_{t_2}^i(\mathbf{z})$. Thus the reality game constructed above yields the appropriate superposition in the emergent limit. Note though that at every round of game play only a single informon is constructed and each such informon is associated with only one property λ_i so that the informons are themselves never in a superposition state. It is only the process which generates them that exists in a superposition. In this way the reality game model retains the usual sense of realism. As play proceeds the particle will manifest all of the eigenstates individually and distributed throughout the causal tapestry, and so effectively throughout the causal manifold. At the level of the observer one will observe only the interpolated wave function, which will correspond to a superposition of these eigenstates. There is no paradox here. The problem arises as a result of the inherent inability to resolve reality at the level of the individual actual occasion. One can have a realist dynamic operating at the lowest level even though at the level at which we can observe reality one sees only these seemingly paradoxical superpositions. The superpositions reflect the nature of the generating processes, not the nature of the reality being generated.

3.11 The Process Space and Process Tapestry

In the previous sections the behaviour of a single quantum system generated by a single process was described by means of a causal tapestry and its associated reality game. More precisely, the system was described by means of an event tapestry which described the actual occasions generated by the process. For simplicity the transition tapestry was ignored since the case of NRQM could be developed solely within the setting of an event tapestry. The case of multiple non-interacting systems may also be described by an event tapestry, so long as the processes that generate the individual systems do not change within the generation of a single tapestry or between generated tapestries. In the case, however, that the systems are capable of interacting with one another, it becomes necessary to consider not just the succession of causal tapestries but the succession of processes that generate these tapestries. Strictly this is true of systems whose generating processes exist in a superposition of primitive processes since different sub-processes generate different elements at different stages. This important but subtle aspect was omitted for the sake of clarity from the description of the superposition case in the previous chapter. That oversight will be rectified here and requires the introduction of the process space and of the transition tapestry which keeps track of the changes in reality games that occurs as a result of interactions..

Recall from Appendix D the descriptions of the event and transition tapestries.

Definition: An event tapestry Ω is a causal tapestry with attributes, that is, a 4-tuple $(L, K, \mathfrak{M}, I_p)$ where K is an index set of cardinality κ, $\mathfrak{M} = \mathcal{M} \times F(\mathcal{M}) \times D \times P(\mathcal{M})$ a mathematical structure with $\mathcal{M}$ a causal space, $F(\mathcal{M})$ a function (state) space, either Banach or Hilbert, D a space of descriptors (properties), $P(\mathcal{M}')$ either a Lie algebra or tangent space on a manifold $\mathcal{M}'$, I_p a union of event tapestries. The event tapestry serves as a generalization of the notion of a position space.

Likewise we may define a transition tapestry as follows:

Definition: A transition tapestry Π is a causal tapestry with attributes, that is, a 4-tuple $(L', K', \mathfrak{M}', I'_p)$ where K' is an index set of cardinality κ, $\mathfrak{M}' = \mathcal{M}' \times F(\mathcal{M}') \times D' \times P'(\mathcal{M})$ a mathematical structure with $\mathcal{M}'$ a manifold (sometimes a causal space), $F(\mathcal{M}')$ a function (state) space, either Banach or Hilbert, D' a space of descriptors, $P'(\mathcal{M})$ either a Lie algebra or tangent space on a causal manifold $\mathcal{M}$, I'_p a union of transition tapestries. The transition tapestry serves as a generalization of the notion of a tangent space (and perhaps as a generalized momentum space as well).

Together these tapestries form an interlinked dyad which satisfies the Tableau Consistency Criterion.

The transition tapestry relates the transitions from current to nascent informons in terms of transformations or operations on the causal manifold $\mathcal{M}$ and the Hilbert space $\mathcal{H}(\mathcal{M})$. To some extent this is redundant information but it provides a correspondence between strictly game theoretic actions and more traditional mathematical or physical operations.

The process tapestry provides a concordance between processes and their games, and the particular moves being implemented, which could be associated with many different games. The process tapestry is modelled on the transition tapestry, since technically, the process is responsible for the transition from the current to the next causal tapestry, and therefore for the generation of a nascent informon out of one or more current informons. The action of a process therefore lies within the space between informons and so must be associated with the edges relating informons within the collection of content sets. Processes are represented within the process framework as combinatorial games. Although games may act at a space-time point during one round of game play, this point will change during subsequent rounds and, as I have shown, do so in a jaunting manner. Moreover a process that generates multiple informons during a given round of play (as in an entangled or multi-particle process) will therefore act at multiple space-time points simultaneously. Both of these considerations imply that although a process may act at a space-time point it itself does not actually *exist* at a definite space-time point. Indeed since these reality games generate the events that we interpret as *being* space-time points, it makes sense to consider these games as existing in a universe *outside* of space-time.

There is a causal structure that arises when a process is activated or inactivated, meaning that the corresponding game is played, or dropped. It is reasonable to represent these relationships between processes and games using causal tapestries but we do not expect the interpretation structures to be similar. Indeed we have no analogue of the causal manifold or Hilbert space. We do however have the space of processes Π and of reality games, $\mathbb{G}(\mathcal{H})$. There will be a projection from Π to $\mathbb{G}(\mathcal{H})$ but we do not expect them to be equivalent since plausibly many different processes might be describable by similar games. There is also the space of properties P attributed to these processes and the space of game trees $\mathbb{T}(\mathcal{H})$ that represent how each particular instantiation of a game unfolds. Since the generation of each informon (or at least the generation of a token that will eventually become an informon) occurs during a round of game play that means that the transition from one informon to another can be represented by one or more subtrees of the full game tree corresponding to the moves involved in that round of play.

Definition (Process Space): A *process space* Π is an abstract space of processes closed under the operations of $w, \oplus, \boxplus, \otimes, \hat{\otimes}, \boxtimes, \hat{\oplus}, \hat{\boxplus}, \hat{\boxtimes}$. An element of $\mathbb{P}$ is thus a string of processes joined by any of the above operators, or multiplied by the value w. A process may be active or inactive. A history is a sequence of elements of Π conjoined by the above operations)

$$\mathbb{P}_0 \to \mathbb{P}_1 \to \mathbb{P}_2 \cdots$$

where for a given process $\mathbb{P}$ being an element of $\mathbb{P}_i$ means that $\mathbb{P}$ was active during step i, and conjoined to the other elements of $\mathbb{P}_i$. Otherwise $\mathbb{P}$ is understood to be inactive at step i.

Definition (Game Space): The game space $\mathbb{G}(\mathcal{H})$ is the closure of a set $\mathcal{H}$ of reality games under the game operations $w, \oplus, \boxplus, \otimes, \hat{\otimes}, \boxtimes, \hat{\oplus}, \hat{\boxplus}, \hat{\boxtimes}$. An element of $\mathbb{G}(\mathcal{H})$ is thus a string of games joined by any of the above operators, or multiplied by the value w. The set $\mathcal{H}$ is called the base of $\mathbb{G}(\mathcal{H})$.

Definition: A process tapestry Σ is a causal tapestry with attributes, that is, a 4-tuple $(L, K', \mathfrak{H}, \Sigma_p)$ where K' is an index set of cardinality κ, $\mathfrak{H}' = \mathbb{G}(\mathcal{H}) \times \mathbb{E}(\mathcal{H}) \times \mathfrak{P}(H) \times D$ a mathematical structure with $\mathbb{G}(\mathcal{H})$ a space of weighted games with basis $\mathcal{H}$, $\mathbb{E}(\mathcal{H})$ a set of generators for $\mathbb{G}(\mathcal{H})$, $\mathfrak{P}(\mathcal{H})$ a collection of (partial) game trees of combinatorial games, D a space of descriptors, I'_p a union of process tapestries. The process tapestry serves as a bookkeeping device to keep track of which particular games generate which informons and which global reality games generate which causal tapestries, and to keep track of changes in the reality games being played as a consequence of interactions occurring during prior play. The set of generators $\mathbb{E}(\mathcal{H})$ form the primitive games out of which all other games are built. These are akin to eigenfunctions of the Hilbert space.

From the perspective of the process space Π, the succession of processes generates a succession of causal tapestries. We may depict this as

$$\mathcal{I}_0 \xrightarrow{\mathbb{P}_1} \mathcal{I}_1 \xrightarrow{\mathbb{P}_2} \mathcal{I}_3 \xrightarrow{\mathbb{P}_3} \cdots \xrightarrow{\mathbb{P}_n} \mathcal{I}_n$$

Through the association of each process with a reality game there will also be a corresponding sequence in the game space $\mathbb{G}(\mathbb{H})$ where H_i is the game associated with the process $\mathbb{P}_i$.

$$\mathcal{I}_0 \xrightarrow{H_1} \mathcal{I}_1 \xrightarrow{H_2} \mathcal{I}_3 \xrightarrow{H_3} \cdots \xrightarrow{H_n} \mathcal{I}_n$$

Thus we have an association between causal tapestry edges and games (or processes) and between games (or process edges) and causal tapestries. In the above we have

$e(\mathcal{I}_{i-1}, \mathcal{I}_i) \equiv H_i$ and $e(H_{i-1}, H_i) \equiv \mathcal{I}_{i-1}$. This sets up a form of duality between these spaces.

This is a macro-level viewpoint. However, at the micro-level of the informons the picture is more complicated because it is sub-processes that actually relate individual informons. An informon of the process tapestry takes the form $[n] < \alpha > \{G\}$ where as before n is an identifier label, G is the content set, being an ordered set of prior process informons, and the interpretation has the form $\alpha = (H, F, \mathfrak{h}, p,)$ where H is a reality game, F a primitive sub-game of H, $\mathfrak{h}$ is a subtree of the game tree $\mathfrak{H}$ of H and p a set of properties.

Let $[n] < \alpha > \{G\}$ lie in the current event tapestry and $[n'] < \alpha' > \{G'\}$ lie in the event tapestry being generated. Assume that this generation takes place under the reality game $\hat{H}$. Let $\alpha = (\mathbf{x}, \phi, p)$ and $\alpha' = (\mathbf{x}', \phi', p')$. In the absence of interaction, $p = p'$, but in the presence of interaction there will exist some kind of superselection rule relating these two properties. In the first case there will exist an informon in the process tapestry of the form $[m] < \beta > \{X\}$ where $\beta = (H, \hat{H}, \mathfrak{h}, p)$ and $\hat{H}$ is a component of H. In the case of interaction the situation is more complicated. In this case the interpretation of the informon of the current event tapestry must have the extended form $\alpha = (\mathbf{x}, \phi, p, \rho)$ where ρ is a set valued map on $F(H)$ and there must exist a transition tapestry element which allows for the transition $p \to p'$.

There will exist a process informon of the form $[m] < \beta > \{X\}$ where $\beta = (H, \hat{H}, \mathfrak{h}, p')$ such that $\hat{H}$ generated $[n'] < \alpha' > \{G'\}$ from $[n] < \alpha > \{G\}$ but there must also have existed an informon of the form $[m'] < \beta' > \{X'\}$ where $\beta' = (H, H', \mathfrak{h}', p)$ where H' generated $[n] < \alpha > \{G\}$ from some prior informon. Moreover $H' \in \rho(H)$.

The process tapestry couples to the event (or transition) tapestry as follows. Let $[n] < (\mathbf{x}, \phi, p, \rho) > \{G\}$ be an informon of the current event tapestry and let $[n'] < (\mathbf{x}', \phi', p', \rho') > \{G'\}$ denote an informon of the nascent event tapestry which is causally subsequent to the informon n. Then there will exist an informon in the process tapestry, $[m] < (H, \hat{H}, \mathfrak{h}, p') > \{X\}$ such that $\mathfrak{h}$ is the subtree of the reality game H that created the informon n' and $H \in \rho$.

Chapter 4

Process Approach to the Paradoxes

The focus of this chapter is on process model explanations and resolutions of many of the paradoxes and conundrums plaguing non-relativistic quantum mechanics. In particular I wish to focus upon the issues of non-Kolmogorov probability, wave-particle duality, the two slit experiment, the Schrödinger cat problem, contextuality, non-locality, and Type I, II, and III hidden variable theorems. The process model provides a realist model of NRQM which is local and non-contextual at the fundamental level, yet non-local and contextual at observable levels, i.e. quasi-local and quasi-non-contextual. In this Chapter I hope to demonstrate that the process and game approach results in a significant conceptual simplification of ultimate reality.

4.1 Non-Kolmogorov Probability Structure

In Appendix A it is shown that one of the mains result of both Type I and Type II hidden variable theorems is to demonstrate that the probability structure of NRQM is non-Kolmogorov in character. For Type I theorems the argument took the form: if we assume locality (which implies classicality) then it follows that the probability structure *must* be Kolmogorov in structure. This leads to an inequality on correlations that NRQM violates, and so NRQM possesses a non-Kolmogorov probability structure. From this it was concluded that reality at the ultimate level must be non-local (at least in the sense of parameter independence). However, in Appendix B it is shown that the argument from locality to Kolmogorov probability structure was false. Classical events can be described by three distinct probability structures: Kolmogorov, trigonometric contextual and hyperbolic contextual [207]. Kolmogorov probability structures hold only when the individual subprocesses are probabilistically compatible so that the individual spaces can be combined into a single consistent Kolmogorov probability space which enables correlation calculations to be carried out. This cannot be done, in general, in a classical context, a fact that unfortunately has been overlooked form more than 50 years and has led to much confusion and error. Thus the Type I hidden variable theorems only demonstrate that the probability structure of NRQM is non-Kolmorogov. Type II hidden variable theorems come in two forms. The theorems of von Neumann and Gleason [150, 412] show directly that the probability structure of Hilbert space must be non-Kolmorogov due to the absence of dispersion free measures, and therefore cannot be represented by a Kolmogorov probability distribution function. The second set of Type II theorems purport to show that quantum systems cannot be assigned definite values for properties prior to their measurement, so that their properties are contextual. That question will be addressed in a later section.

Thus if the process framework is to be acceptable, it must be capable of generating a non-Kolmogorov probability structure. In Chapter 2, the value of the local and global Hilbert space interpretations at a specific space-time point were interpreted as expressing the “strength” of the generating process at that point. This strength is not a simple real value. Rather, it describes the coupling between the generating process and other processes that might be simultaneously active, and so it influences whether or not an interaction between these processes takes place and the form that it takes. Thus this strength carries additional information so that formally it may be expressed as a real or complex scalar (possibly even a quaternion), a vector, a spinor, or perhaps a tensor. There is a magnitude associated to this strength. In the case of NRQM, the magnitude of the Hilbert space interpretation is related to the probability of coupling between the generating process and the process generating a measurement device. The informons of the causal tapestry are inherently unobservable and generated by the reality game associated with the generating process. The reality game, being a combinatorial game, is not, in itself, stochastic, and so there are no probabilities per se associated with the generation of individual informons. In classical probability theory (at least in von Mises theory), probabilities are about the frequencies of observed events, so probabilities should be linked in some manner to the measurement situation. To determine a probability one needs to place the quantum system into an interaction with a measurement apparatus and in such a situation the process strength determines the likelihood of a coupling to the measurement apparatus which in turn determines a measurement. This is a subtle but important distinction. The Hilbert space interpretation does not directly determine a probability except in the context of a measurement situation (or at least an interaction).

In the discussion that follows all probabilities are calculated relative to the embedding hypersurface of the current causal tapestry.

Recall from Chapter 2 that a sum of processes of the form

$$\mathbb{P} = \bigoplus_i w_i \mathbb{P}_i$$

is interpreted as a sum of their respective games

$$\bigoplus_i w_i \mathfrak{R}_i(N, r, \rho, \delta, t_P, l_P, \omega, L, \Sigma, p)$$

where multiplication by the weight w_i modifies the *tokens* of the reality game $\mathfrak{R}_i$, and $\sum_i w_i^* w_i = 1$. Effectively this means that the (relative) strength of each subprocess $\mathbb{P}_i$ has been modified by $w_i^* w_i$. Suppose that one wishes to determine the probability of observing a particular value of some property, P. One must first introduce a measurement apparatus capable of carrying out the observation of the property P and place it in interaction with the quantum system under consideration. Let $\{\lambda_j\}$ represent the set of possible values of the property P. From the measurement apparatus frame, each subprocess $\mathbb{P}_i$ can be represented as a sum of subprocesses, each corresponding to a distinct value of the parameter. Thus $\mathfrak{M}(\mathbb{P}_i) = \oplus w_{ij} \mathbb{P}_j(\lambda_j)$. Let $\mathbb{M}$ denote the measurement apparatus process and $\mathbb{M}_j(\lambda_j)$ the i-th sub-process corresponding to the observable state λ_j. Assume that the measurement subprocesses occur concurrently. The simultaneous case follows fairly easily.

Next one must play out the sequence of processes or their respective games as the measurement interaction unfolds. Recall from Chapter 2 that this is presumed to occur in three stages (here assuming that the process $\mathbb{P}$ is primitive):

1. Prior to interaction the process structure is $\mathbb{P} \otimes \mathbb{M} = (\oplus_i w_i \mathbb{P}_i) \otimes \mathbb{M} = (\oplus_i w_i \mathbb{P}_i) \otimes (\oplus_j \mathbb{M}_j(\lambda_j)) = \oplus_j \oplus_i w_i \mathbb{P}_i \otimes \mathbb{M}_j(\lambda_j)$

2. The processes initially interact resulting in a rotation to the measurement frame. This is written as

   $$\mathbb{P} \otimes \mathbb{M} = \oplus_j \oplus_i w_i \mathbb{P}_i \otimes \mathbb{M}_j(\lambda_j) \to \mathfrak{M}(\mathbb{P}) \otimes \mathbb{M} = \oplus_j \oplus_k \oplus_i w_i w_{ik} \mathbb{P}_k(\lambda_k) \otimes \mathbb{M}_j(\lambda_j) =$$

   $$\oplus_j \oplus_k (\oplus_i w_i w_{ik}) \mathbb{P}_k(\lambda_k) \otimes \mathbb{M}_j(\lambda_j) \to \mathfrak{M}(\mathbb{P}) \boxtimes \mathbb{M} = \oplus_k \oplus_{j \in H(k)} (\oplus_i w_i w_{ik}) \mathbb{P}_k(\lambda_k) \boxtimes \mathbb{M}_j(\lambda_j)$$

   where the λ_k are the eigenvalues and the $\mathbb{P}_k$ are the generating sub-processes corresponding to the eigenfunctions. $H(k)$ is the collection of informons for which there is a non-zero compatibility between every $j \in H(k)$ and k. The final transition is an interactive product because not all eigenprocesses of the measurement apparatus couple to every eigenprocess of the quantum system. In the ideal case of error free measurement, this becomes $\sum_k (\oplus_i w_i w_{ik}) \mathbb{P}_k(\lambda_k) \boxtimes \mathbb{M}_k(\lambda_k)$

3. Finally interaction between the quantum system and the measurement apparatus takes place. Following the completion of a full round of the process, a single informon n is created through the action of one of the subprocesses $(\oplus_i w_i w_{ik}) \mathbb{P}_k(\lambda_k)$. If this does not couple to the measurement process then a new round is initiated. The probability that this particular informon n couples to the measurement apparatus is proportional to the strength Γ_n of the generating process associated with the informon. In this case that strength is determined as the magnitude of the sum of the strengths (which correspond to tokens in the associated reality game) associated with the process $\mathbb{P}_k(\lambda_k)$, i.e the local strength will be given as $\Gamma'_n = (\sum_i w_i w_{ik}) \Gamma_n$ so that the probability will be proportional to $l_P^3 |(\sum_i w_i w_{ik})|^2 |\Gamma_n|^2$. If such a coupling occurs then a transition to the reduced subprocess $(\oplus_i w_i w_{ik}) \mathbb{P}_k(\lambda_k) \otimes \mathbb{M}_k(\lambda_k)$ takes place and a measurement is generated (at least in the case of demolition measurements).

Clearly $|\sum_i w_i w_k|^2 \neq \sum_i |w_i w_k|^2$ which would be the result if the process model obeyed Kolmogorov probability theory. Thus one sees that the probability structure is non-Kolmogorov.

If one applies the process covering map to the measurement sequence described above and considers the asymptotic limit as $N, r \to \infty$ then it should be fairly clear that the transitions:

$\mathbb{P} \otimes \mathbb{M} = \oplus_j \oplus_i w_i \mathbb{P}_i \otimes \mathbb{M}_j(\lambda_j) \to \oplus_j \oplus_k \oplus_i w_i w_{ik} \mathbb{P}_k(\lambda_k) \otimes \mathbb{M}_j(\lambda_j) = \oplus_j \oplus_k (\oplus_i w_i w_{ik}) \mathbb{P}_k(\lambda_k) \otimes \mathbb{M}_j(\lambda_j) \to \oplus_k \oplus_{j \in H(k)} (\oplus_i w_i w_{ik}) \mathbb{P}_k(\lambda_k) \otimes \mathbb{M}_j(\lambda_j)$ will map to:

$$\begin{aligned}
\Psi^{t_P,l_P} \otimes \Psi_M^{t_P,l_P} = \sum_j \sum_i w_i \Psi_i^{t_P,l_P} \otimes \Psi_M^{t_P,l_P}(\lambda_j) \to \\
\sum_j \sum_k \sum_i w_i w_{ik} \Psi_k^{t_P,l_P}(\lambda_k) \otimes \Psi_M^{t_P,l_P}(\lambda_j) = \\
\sum_j \sum_k (\sum_i w_i w_{ik}) \Psi_k^{t_P,l_P}(\lambda_k) \otimes \Psi_M^{t_P,l_P}(\lambda_j) \to \\
\sum_k \sum_{j \in H(k)} (\sum_i w_i w_{ik}) \Psi_k^{t_P,l_P}(\lambda_k) \otimes \Psi_M^{t_P,l_P}(\lambda_j)
\end{aligned}$$

and it follows that the emergent probabilities as generated by the process model correspond to the probabilities as determined by the usual NRQM wave functions in the event that the above limit yields proper NRQM wave functions (with or without $t_P, l_P \to 0$).

For simplicity consider the case of error free measurement. The complete case obscures the main ideas and follows from careful bookkeeping.

There are two main types of probability under consideration. In the first case, the measurement apparatus responds to the property value λ_i associated with an informon n and not to its causal manifold embedding point $\mathbf{m}_n$.

Suppose that under the process $\mathbb{P} = \oplus_i w_i \mathbb{P}_i$ the causal tapestry $\mathcal{I}$ is generated and that the informons associated with the subprocess $\mathbb{P}_i$ lie in the subcollection $\mathcal{I}_i$. The probability that the measurement apparatus will register a value of λ_i is thus proportional to the coupling probabilities of the informons in subcollection $\mathcal{I}_i$. Each informon n has token $w_i \Gamma_n$ and thus strength $l_P^3 w_i^* w_i \Gamma_n^* \Gamma_n$ so that the total probability will be given as

$$\sum_{n \in \mathcal{I}_i} l_P^3 w_i^* w_i \Gamma_n^* \Gamma_n$$

The global $\mathcal{H}(\mathcal{M})$-interpretation corresponding to this subcollection is given by

$$\Phi_i(\mathbf{z}) = \sum_{n \in \mathcal{I}_i} w_i \Gamma_n T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z})$$

It follows from interpolation theory that the $T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z})$ in the path integral and kernel strategies form an orthonormal set so that

$$\int_{-\infty}^{\infty} \Phi_i^*(\mathbf{z}) \Phi_i(\mathbf{z}) d\mathbf{z} = \sum_{n \in \mathcal{I}_i} l_P^3 w_i^* w_i \Gamma_n^* \Gamma_n$$

Thus in the asymptotic limit in which $\Phi_i(\mathbf{z}) = \Psi(\mathbf{z})$, a proper NRQM wave function. one sees that the probability determined by the process model agrees with that determined by NRQM.

Away from the asymptotic limit there will be some discrepancy because the causal tapestry $\mathcal{I}_i$ will not satisfactorily sample the entirety of the relevant hyper-surface and so there will be regions of the causal manifold that effectively fail to contribute to the measurement process because there is no corresponding informon to couple to the measurement process and thereby contribute their process strength to the measurement probability. Therefore we may interpret

$$P = \sum_{n \in \mathcal{I}_i} l_P^3 \Gamma^*_n \Gamma_n$$

to be proportional to the total probability that an informon will couple to the measurement apparatus and a measurement value obtained and therefore $1 - P$ is simply proportional to the probability that a measurement will not occur. In the case that the causal tapestry embeds evenly throughout the causal manifold $\mathcal{M}$ this probability will go to 0 because effectively there will always be an informon somewhere so as to interact with the measurement apparatus process.

In the second case the measurement apparatus responds to the causal manifold embedding points of the informons. Consider the probability that a measurement apparatus will record an event generated by the process $\mathbb{P}$ occurring within some region R of the hypersurface M into which $\mathcal{I}$ embeds. Let $\mathcal{I}_R$ be the subcollection of informons whose causal manifold embeddings lie within the region R, i.e. $n \in \mathcal{I}_R$ iff $\mathbf{m}_n \in R$.

From the process point of view, this probability is given by

$$\sum_{n \in \mathcal{I}_R} l_P^3 \Gamma^*_n \Gamma_n$$

If the NRQM wave function in the position representation is given by $\Psi(\mathbf{x})$, then in the position representation one may calculate the probability of the particle being observed in some region R by integrating the associated probability density function $\Psi^*\Psi$ over the region R, i.e.

$$P(R) = \int_R \Psi^*(\mathbf{x})\Psi(\mathbf{x}) d\mathbf{x}$$

The comparable calculation via the global $\mathcal{H}(\mathcal{M})$-interpretation is given by

$$\int_R \Phi^*(\mathbf{z})\Phi(\mathbf{z}) dV =$$

$$\sum_{n \in \mathcal{I}} \sum_{n' \in \mathbb{I}} \Gamma^*_n \Gamma_{n'} \int_R T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z}) T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z}) dV$$

These integrals have no closed form solutions in general but because the wavelet function $T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z})$ takes a zero value at every $\mathbf{m}_{n'} \neq \mathbf{m}_n$ and decays rapidly from its maximum, one can see that

$$\int_R \Phi^*(\mathbf{z})\Phi(\mathbf{z})dV \approx \sum_{n \in \mathcal{I}_R} l_P^3 \Gamma_n^* \Gamma_n$$

The probabilities of coupling between the system process and the measurement apparatus depend upon the local strengths associated with individual informons and can be determined by simple summations just as in Kolmogorov probability theory. The non-Kolmogorov aspects of the process model arise from the manner in which the strengths are calculated and from the fact that the strengths are complex valued and so the calculation of the square of the amplitude upon which the coupling probabilities are directly proportional potentially involves cross terms between distinct contributions, which is distinct from the Kolmogorov case.

Since the embedding lattice is discrete, while the regional decomposition of $\mathcal{M}'$ is continuous, it is not surprising that some discrepancy may arise between the calculation of this probability on the lattice and its calculation using the interpolated wave function and the corresponding NRQM wave function in those situations in which the reality game generates errors. In the asymptotic limit, as the Hilbert space interpretation approaches the usual NRQM wave function, the probability as calculated by the Hilbert space interpretation will agree with that calculated using the usual NRQM wave function. This asymptotic behaviour must be understood as an idealization and not a dynamical feature of the model. In the process model, t_P, l_P are fixed for a given process (even if they may vary from process to process) and so the physics takes place solely on the causal tapestries. Again, one sees that the dynamics of the process model is self contained and consistent. The discrepancy that therefore arises between the probabilities and correlations calculated using the process model and using the standard NRQM formalism could be used in principle to test whether the process model provides a more accurate model of reality than standard NRQM. Unfortunately the degree of accuracy associated with some process models, as shown in Chapter 3, clearly exceeds current experimental methods but experimental measurement of these probabilities would at least place some bounds on possible reality game strategies.

As an aside note that if time is taken into account so that there is some uncertainty as to the timing of the measurements then the global $\mathcal{H}(\mathcal{M})$ interpretation on the entire manifold must be considered. Hence contributions from multiple prior causal tapestries must be included in the sum and an additional factor of t_P must multiply the local strengths in their calculations.

4.2 Wave-Particle Duality

The earliest recognized non-intuitive aspect of quantum mechanics was that of wave-particle duality. Classically, waves and particles have completely different mathematical descriptions, the former as functions over a manifold, the latter as points of the manifold. Formally then, particles have a description using maps from $\mathbb{R} \to \mathbb{R}^3$ while waves are described using maps from $\mathbb{R} \to (\mathbb{R}^3)^{\mathbb{R}}$. They exhibit completely different behaviours. Waves exhibit constructive and destructive interference while particles do not. Waves can exchange energy and exert influences simultaneously at multiple spatial locations while particles can only do so local to a single point.

In quantum mechanics there are many situations in which a quantum system behaves much like a particle, and other situations in which it behaves like a wave [157, 325]. The two slit experiment provides a striking demonstration of how small changes in the experimental setup can result in significant changes in the resulting behaviour. This is generally explained by information or knowledge based arguments. If a system is presented with different distinguishable alternatives then the system behaves as if it were a particle. If the alternatives are indistinguishable then it behaves as like a wave, or at least as if all possibilities are expressed simultaneously. However, even in the situation in which the system appears to behave as if it were a wave, it will only interact with a measuring device as if it were a localized particle.

The photoelectric effect provides a striking example of the problems associated with this wave-particle dichotomy. In the photoelectric effect, light is projected onto a reactive metal surface and electrons are released upon exposure. The problem is that electrons are only ever released individually, locally, and with definite discrete energies. This should not happen if light is a wave that distributes over the entirety of the metallic surface.

The duality apparent in both the behaviour and the description of quantum systems created deep conceptual problems for the founders of quantum mechanics. Recall Bohr's comments on this question [58]

> ... how flawed the simple wave-particle description is. Once light [or a material particle] is in an interferometer, we simply cannot think of it as either a wave or a particle. Nor will melding the two descriptions onto some strange hybrid work. All these attempts are inadequate. What is called for is not a composite picture of light, stitched together out of bits taken from various classical theories. Rather we are being asked for a new concept, a new point of view that will be fundamentally different from those developed from the world of classical physics.

Bohr and his followers proposed a decidedly anti-realist conception of reality. Whether a quantum system exhibits wave or particle behaviour became dependent upon the choice of an observer or of an experimental setup. This behaviour was no longer an intrinsic aspect of the quantum system but dependent upon the context that the system found itself in. Wheeler's delayed choice experiment poses an even greater conceptual challenge since the choice of whether a quantum system exhibited particle or wave like behaviour could be left until *after* the system entered the experimental apparatus, and thus presumably after the system itself should have made its choice.

Underlying all of these conundrums is the assumption that a quantum system must be *either* particle-like or wave-like and *not* anything else. Even Bohr believed that. But this is a problem with the manner in which such behaviour is represented mathematically. Particles are always represented as mathematical *points*, even though in reality nothing physical is actually a point. Every physical entity possesses at least a tiny amount of spatial and temporal extension, everything is always just a little bit fuzzy. Waves too are always represented by functions that extend across all of space-time, that is, they are continuous in the mathematical sense, manifesting at *every* point of space and time. Both viewpoints represent idealizations and extremes - particles having no spatio-temporal extension while waves have complete spatio-temporal extension.

Contrary to Bohr's unduly pessimistic view, there is in fact a middle ground, as first pointed out by Kempf [203, 204, 205], and which derives from the theory of interpolation. Representing an entire function f using an interpolation expansion of the form

$$f(x) = \sum_{x_i} f(x_i) T_{x_i} sinc_{\omega}(x)$$

neatly incorporates both discrete features, arising from the discrete nature of the sampling set $\{x_i\}$, and continuous features, arising from the coupling to the sinc wavelets and the global summation.

The process model resolves the wave-particle duality problem by incorporating one additional aspect - namely it imposes a dynamic on the creation of these interpolation samples such that they are generated sequentially and not simultaneously. As a consequence, at each step in the generation process there is a single localized expression of the quantum system - a discrete, particle-like entity, but due to the extremely small scale at which these entities, these actual occasions, manifest they are unobservable to the emergent entities that form our observable reality (although they are observable to the processes that generate them) and as a result it is only the global $\mathcal{H}(\mathcal{M})$-interpretation that is observable, and that interpretation is an interpolation of a continuous, wave-like entity. The process model eliminates the false dichotomy between particle and wave. In the process tapestry model, each informon possesses both particle and wave aspects. The particle aspects are represented by the embedding into the causal manifold, which interprets the informon as being associated with a specific space-time location. The wave aspects are represented by the local $\mathcal{H}(\mathcal{M})$-interpretation, which interprets each informon as being a fuzzy wave like entity whose intensity is highly concentrated around the embedding point. The local $\mathcal{H}(\mathcal{M})$-interpretation serves as a frame element contributing to a global interpolation of a wave function over the space-time surface into which the informons embed. Both the embedding and the $\mathcal{H}(\mathcal{M})$-interpretation are emergent aspects of the quantum system.

Recall that the dynamic of the process model is represented by means of a combinatorial game, the reality game. The play of the reality game consists of a series of short rounds which form a full round, at the end of which a new informon is created. One can picture this as follows:

$$\begin{array}{rcccccccc} & & R_1 & R_2 & R_3 & \cdots & R_N & \cdots & \\ & & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\ \mathcal{I} = & \{ & n_1 & n_2 & n_3 & \cdots & n_N & \cdots & \} \\ & & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\ & & \mathbf{m}_1 & \mathbf{m}_2 & \mathbf{m}_3 & \cdots & \mathbf{m}_N & \cdots & \\ & & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\ \Phi = & \sum & \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_N & \cdots & \end{array}$$

where the R_i are the individual rounds of game play, $\mathcal{I}$ the causal tapestry being generated, n_i the individual informons being generated, $\mathbf{m}_i$ the causal manifold embedding points of each informon, ϕ_i the local Hilbert space interpretation of each informon, and Φ the global $\mathcal{H}(\mathcal{M})$-interpretation (wave function). Interactions between processes are triggered by single informons and result in exchanges of information (and correspondingly of properties) specific to the triggering informon. Such exchanges are therefore discrete (hence particle-like) even though the entire causal tapestry is still associated with a global Hilbert space interpretation (hence wave-like) which governs the spatial distribution of interactions and their frequencies. Triggering events result in the activation and inactivation of processes and therefore the ending and beginning of individual games (unless of course these games are allowed to run their full course, in which case they will automatically repeat since the generating process will remain active). In this manner, there appears to be a different dynamic in the case of free quantum systems from that of interacting systems. The case of the free system can be depicted thus:

$$
\begin{array}{lccccccccccccc}
 & & & \Pi & & & & & & & \Pi & & & \\
 & R_1 & R_2 & \cdots & R_N & \cdots & \rightarrow & & R'_1 & R'_2 & \cdots & R'_N & \cdots & \\
 & \downarrow & \downarrow & \cdots & \downarrow & \cdots & & & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\
\mathcal{I} = \{ & n_1 & n_2 & \cdots & n_N & \cdots \} & \rightarrow & \mathcal{I}' = \{ & n'_1 & n'_2 & \cdots & n'_N & \cdots & \} \\
 & \downarrow & \downarrow & \cdots & \downarrow & \cdots & & & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\
 & \mathbf{m}_1 & \mathbf{m}_2 & \cdots & \mathbf{m}_N & \cdots & \rightarrow & & \mathbf{m}'_1 & \mathbf{m}'_2 & \cdots & \mathbf{m}'_N & \cdots & \\
 & \downarrow & \downarrow & \cdots & \downarrow & \cdots & & & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\
\Phi = \sum & \phi_1 & \phi_2 & \cdots & \phi_N & \cdots & \rightarrow & \Phi' = \sum & \phi'_1 & \phi'_2 & \cdots & \phi'_N & \cdots &
\end{array}
$$

where a single process remains active from one game to the next and the entire game is played out. In the case of interacting systems, game play terminates prematurely as a result of the interaction. The original process is inactivated and a new process activated resulting in the play of a different game. This may be depicted as:

$$
\begin{array}{lcccccccccccc}
 & & & \Pi_1 & & & & & & \Pi_2 & & & \\
 & R_1 & R_2 & \cdots & R_N & \rightarrow & & R'_1 & R'_2 & \cdots & R'_N & \cdots & \\
 & \downarrow & \downarrow & \cdots & \downarrow & & & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\
\mathcal{I} = \{ & n_1 & n_2 & \cdots & n_N \} & \rightarrow & \mathcal{I}' = \{ & m'_1 & m'_2 & \cdots & m'_N & \cdots & \} \\
 & \downarrow & \downarrow & \cdots & \downarrow & & & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\
 & \mathbf{m}_1 & \mathbf{m}_2 & \cdots & \mathbf{m}_N & \rightarrow & & \mathbf{m}'_1 & \mathbf{m}'_2 & \cdots & \mathbf{m}'_N & \cdots & \\
 & \downarrow & \downarrow & \cdots & \downarrow & & & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\
\Phi = \sum & \phi_1 & \phi_2 & \cdots & \phi_N & \rightarrow & \Phi' = \sum & \phi'_1 & \phi'_2 & \cdots & \phi'_N & \cdots &
\end{array}
$$

where the $R'_i, m'_i, \mathbf{m}'_i, \phi'_i$ serve to indicate that a different game is being played out and the interaction occurs during round R_N with the generation of informon n_N.

Each informon $[n] < (\mathbf{x}_n, \phi_n, \mathbf{p}_n) > \{G\}$ connects to the process that generates it via the property $\mathbf{p}_n$ and this allows the process to generate informons in a non-local, jaunting manner. It is this jaunting game play that violates the principle of continuity and provides the quantum system with the freedom to explore alternatives even when they appear to be space-like separated. In spite of this, information is transmitted locally and causally and informons are themselves are accorded process specific attributes and so are non-contextual. The origins of non-locality and contextuality will be discussed in a later section.

The process model resolves the wave-particle duality by showing that quantum systems possess *both* particle-like and wave-like aspects simultaneously - there is no either-or.

Observation of these two aspects depends upon the particular experimental arrangement and which aspects are emphasized. In the end, the particle or wave behaviour that is empirically observed is an emergent phenomenon arising out of the interaction between the quantum system and the measurement apparatus. This point will be illustrated in the next section where the two-slit experiment is discussed.

4.3 A Simple Two Slit Experiment

To illustrate the ideas above let us consider a simple model of a free non relativistic particle passing through a two slit detector and then impinging on a plate detector. This will illustrate the basic ideas of the game approach. There are two games here, one corresponding to the particle and the other to the detector. For simplicity, assume that the space is two dimensional, the source of the particle is at the point on the manifold denoted $(0,0)$, the two slits lie in the $x = a$ plane while the detector lies in the $x = b$ plane. The slits span the two regions from $y = c$ to $y = d$ and from $y = -c$ to $y = -d$. Again for simplicity we shall assume that the embedding to the manifold is to a regular lattice whose spatial spacing is on the order of Planck length (l_P) and temporal spacing on the order of Planck time (t_P).

Denote the game of the particle as $\mathbb{P}$ and that of the detector as $\mathbb{D}$. The Lagrangian for the particle is $L = mv^2/2$. Let us implement a $\mathfrak{PI}$ strategy. The game to be played initially is $\mathbb{P} \oplus \mathbb{D}$. Since the detector does not undergo any significant change until it interacts with the particle during the detection process, its play is simple until detection occurs. Thus we can ignore the details of the detector and focus just on the particle.

The initial causal tapestry will be that corresponding to the release of the particle from the source. Assume for simplicity that this initial wave function takes the form of a delta function. Then we may take it to have a value of 1 at the source. Let $L_1 \subset \{(t, \mathbf{x}) | \mathbf{x} \in \mathbb{R}^3\}$ denote the set of lattice sites in $\mathcal{M}$ to which $\mathcal{I}$ embeds and $L_2 \subset \{(t + t_P, \mathbf{x}) | \mathbf{x} \in \mathbb{R}^3\}$ is likewise the embedding lattice for $\mathcal{I}'$ and for each $n' \in \mathcal{I}'$ let $L_{n'}$ denote those elements of L_1 that are embedding points for the set $I_{n'}$.

Hence one can write $\Phi^{t+t_P}(\mathbf{z}) =$

$$\sum_{\alpha_{n'} \in L_2} \sum_{\alpha_n \in L_{n'}} (l_P^3/A^3) e^{iS[l_{n'}, l_n]/\hbar} \phi_n(l_n) A^3 T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z})$$

Set $t_1 = t$ and $t_2 = t + t_P$.

So let us first assume that N is large enough so that the second lattice is completely covered by informons and all elements of both lattices are interrelated, corresponding actions calculated and interpretations created. At a minimum this will require $4\omega_0^2$ steps where ω_0 is the first countable ordinal, so assume that $N > 4\omega_0^2$. In this case both lattices will be fully covered so that $L_1 = \tilde{L}_1$, $L_2 = \tilde{L}_2$ and $L_{n'} = \tilde{L}_1$ for all $n' \in \mathcal{I}'$ so that we may reference each lattice point by the informon that embeds to it. Therefore we may rewrite the equation as

$$\Phi^{t_2}(\mathbf{z}) = \sum_{\mathbf{m}_{n'} \in \tilde{L}_2} \sum_{\mathbf{m}_n \in \tilde{L}_1} \frac{l_P^3}{A^3} e^{iS[l_{n'}, l_n)]/\hbar} \phi_n(\mathbf{m}_n) A^3 T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z})$$

But by the definition of the Hilbert space interpretation, $\Phi^{t_1}(\mathbf{z}) = \sum_{n \in \mathcal{I}} \phi_n(\mathbf{z})$ and for a point $\mathbf{y}$ on the lattice L_1 to which informon $n \in \mathcal{I}$ embeds (i.e. $\mathbf{y} = \mathbf{m}_n$) it follows that $\Phi^{t_1}(\mathbf{y}) = \phi_n(\mathbf{y})$, so I can rewrite the above as

$$\Phi^{t_2}(\mathbf{z}) = \sum_{\mathbf{m}_{n'} \in \tilde{L}_2} \sum_{\mathbf{m}_n \in \tilde{L}_1} \frac{l_P^3}{A^3} e^{iS[l_{n'}, l_n]/\hbar} \Phi^{t_1}(\mathbf{m}_n) A^3 T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z})$$

Let me now apply this to the simple case of a free particle moving in one dimension using the $\mathfrak{P}\mathfrak{I}$ strategy.

The action along a single straight line path between (t_1, x_1) and (t_2, x_2) is given as $S[(t_2, x_2), (t_1, x_1)] = \frac{m}{2\epsilon}(x_2 - x_1)^2$. Substituting into the formula for Φ^{t_2} one obtains

$$\Phi^{t_2}(\mathbf{z}) = \sum_{l_{n'} \in L_2} \sum_{l_n \in L_1} (l_P/A) e^{iS[l_{n'}, l_n]/\hbar} \Phi^{t_1}(l_n) A T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z}) =$$

$$\sum_{l_{n'} \in L_2} \sum_{l_n \in L_1} (l_P/A) e^{\frac{im}{2\hbar\epsilon}(x_{n'} - x_n)^2} \Phi^{t_1}(l_n) A T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z}) =$$

$$\sum_{l_{n'} \in L_2} \sum_{l_n \in L_1} (2\pi i\hbar t_P l_P/m) e^{\frac{im}{2\hbar\epsilon}(x_{n'} - x_n)^2} \Phi^{t_1}(l_n) A T_{\mathbf{m}_{n'}} sinc_{t_P, l_P}(\mathbf{z})$$

At time $t = 0$ the Hilbert space interpolation is simply

$$\Phi^0(\mathbf{z}) = sinc_{t_P l_P}(\mathbf{z})$$

so that $\phi^0 = 1$ and $\phi^n(n) = 0$ for all n on the lattice. If one now generates the next tapestry, the Hilbert space interpretation will be

$$\Phi^{t_2}(\mathbf{z}) = \sum_{l_{n'} \in L_1} \sum_{l_n \in L_0} (2\pi i \hbar t_P l_P/m) e^{\frac{im}{2\hbar\epsilon}(x_{n'}-x_n)^2} \Phi^{t_1}(l_n) A T_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z}) =$$

$$\sum_{l_{n'} \in L_1} (2\pi i \hbar t_P l_P/m) e^{\frac{im}{2\hbar\epsilon}(x_{n'}^2)} A T_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z})$$

which is simply a Gaussian wave packet which will begin to expand outwards in space as each new causal tapestry is created.

This will continue until the causal tapestry reaches the space-like hypersurface corresponding to the plate containing the slits, that is the $x = a$ plane. At this point there is an interaction between the particle and the detector. This interaction does not affect the detector game in any way but it does affect the particle game. This is due to the fact that the physical barrier of the detector prevents any possible of information transfer across the barrier. As a result the free particle game must become inactivated since it is no longer possible to form causal links between points where the path of information transfer must cross the barrier. Information may only be transferred at the two slits. Let the time that the particle reaches the detector plate be t. If, for simplicity, we take the slits to be infinitesimal and located at position c, then the Hilbert space interpretation along the detector plate must be of the form

$$\Phi^t(\mathbf{z}) = \frac{1}{\sqrt{2}}(\Phi^t(-c) T_{-c} sinc_{t_P l_P}(\mathbf{z}) + \Phi^t(c) T_c sinc_{t_P l_P}(\mathbf{z}))$$

The individual functions

$$\Phi^{t-}(\mathbf{z}) = \Phi^t(-c) T_{-c} sinc_{t_P l_P}(\mathbf{z})$$

and

$$\Phi^{t+}(\mathbf{z}) = \Phi^t(c) T_c sinc_{t_P l_P}(\mathbf{z})$$

are effectively generators for eigenfunctions for the detector since only these can actually transit the slits and propagate through the detector. These two functions thus correspond to the rotation into an eigenbasis for the detector. The introduction of the $1/\sqrt{2}$ is necessary to preserve the total strength of the process, since effectively a single process is being subdivided into two sub-processes. It would be incorrect to simply take the new Hilbert space interpretation on the detector plate to be

$$\Phi^t(\mathbf{z}) = \Phi^t(-c)T_{-c}sinc_{t_P l_P}(\mathbf{z}) + \Phi^t(c)T_c sinc_{t_P l_P}(\mathbf{z})$$

with that on the immediately preceding causal tapestry having the form

$$\Phi^{t-t_P}(\mathbf{z}) = \sum_{l_n \in L_{t-t_P}} \Phi^{t-t_P}(l_n)T_{\mathbf{m}_n} sinc_{t_P,l_P}(\mathbf{z})$$

This would ignore the fact that an interaction with the detector has taken place. In the free case one would calculate

$$\Phi^t(\mathbf{z}) = \sum_{l_{n'} \in L_t} \sum_{l_n \in L_{t-t_P}} (2\pi i\hbar t_P l_P/m)e^{\frac{im}{2\hbar\epsilon}(x_{n'}-x_n)^2}\Phi^{t-t_P}(l_n)AT_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z})$$

which would imply that there would be residual Hilbert space interpretation on *all* lattice points lying along the detector plate, whereas there is actually non-zero interpretation *only* at the sites corresponding to the slits.

In the process framework, the reason for the discrepancy is that the game corresponding to the free particle process comes to an end because the free particle process is inactivated and a new process begins corresponding to the initiation of activity at the two slits. The particle is not simply a particle but is also an emergent wave and this has consequences since the transition from the original generating process to the two new subprocesses must preserve the strength of the original process, which must in turn be be balanced between the two subprocesses. Hence there must be the $1/\sqrt{2}$ factor. Moreover, these two subprocesses possess the same character, being aspects of a single particle, and the same sate values, since the only difference between them lies in the generation of the causal and $\mathcal{H}(\mathcal{M})$-interpretations. Recalling the discussion in Chapter 2 on interactions, this means that these subprocesses conjoin using the free independent sum. The importance of this is that infomons may receive informational contributions from prior informons generated by either subprocess or by both. This permits the necessary superposition of information to take place.

At the detector plate the global game is no longer the independent product $\mathbb{P} \otimes \mathbb{D}$ but rather the interactive product $\mathbb{P} \boxtimes \mathbb{D}$.

Thus we may write

$$\mathbb{P} \boxtimes \mathbb{D} = \frac{1}{\sqrt{2}}((\mathbb{P}_+ \otimes \mathbb{D})\hat{\oplus}(\mathbb{P}_- \otimes \mathbb{D}))$$

.

This new game may be rewritten as $\frac{1}{\sqrt{2}}(\mathbb{P}_+\hat{\oplus}\mathbb{P}_-) \otimes \mathbb{D}$. For simplicity, assume that the subprocesses lie on different sublattices L^l and L^r.

Each subprocess $\mathbb{P}_+$ and $\mathbb{P}_-$ is just a free particle process arising from different initial conditions and both are freely coupled to the detector since no further interaction with the detector takes place until the particle reaches the detection surface. Thus the Hilbert space interpretation on the subsequent $t + t_P$ - hypersurface will take the form

$$\Phi^{t+t_P}(\mathbf{z}) = \sum_{l_{n'} \in L_{t+t_P}} \sum_{l_n \in L_t} \frac{2\pi i\hbar t_P l_P}{m} e^{\frac{im}{2\hbar\epsilon}(x_{n'}-x_n)^2} \Phi^t(l_n) AT_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z}) =$$

$$\sum_{l_{n'} \in L^l_{t+t_P}} \frac{2\pi i\hbar t_P l_P}{m} e^{\frac{im}{2\hbar\epsilon}(x_{n'}-x_n)^2} \frac{1}{\sqrt{2}} \Phi^t(c) AT_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z}) +$$

$$\sum_{l_{n'} \in L^r_{t+t_P}} \frac{2\pi i\hbar t_P l_P}{m} e^{\frac{im}{2\hbar\epsilon}(x_{n'}-x_n)^2} \frac{1}{\sqrt{2}} \Phi^t(-c) AT_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z}) =$$

$$\sum_{l_{n'} \in L^l_{t+t_P}} \frac{2\pi i\hbar t_P l_P}{m} e^{\frac{im}{2\hbar\epsilon}(x_{n'}-x_n)^2} \frac{1}{\sqrt{2}} \Phi^{t+}(c) AT_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z}) +$$

$$\sum_{l_{n'} \in L^r_{t+t_P}} \frac{2\pi i\hbar t_P l_P}{m} e^{\frac{im}{2\hbar\epsilon}(x_{n'}-x_n)^2} \frac{1}{\sqrt{2}} \Phi^{t-}(-c) AT_{\mathbf{m}_{n'}} sinc_{t_P,l_P}(\mathbf{z}) =$$

$$\frac{1}{\sqrt{2}}(\Phi^+_{t+t_P}(\mathbf{z}) + \Phi^-_{t+t_P}(\mathbf{z}))$$

and so the subsequent evolution of the new process will be identical with the sum of the evolutions of the individual subprocesses. In particular it is clear that the local strengths will be given as a sum of the local strengths of the individual subprocess and that they will then interfere as expected in the usual NRQM formulation of the problem. It should also be clear that each subprocess is equivalent to the evolution of a free particle originating at each slit, or to the evolution of two wave fronts originating at each slit which subsequently interfere. Referring to the measurement theory of Chapter 2, the probability of coupling to the detector plate will depend upon the strength of the process at each location along the surface of the detector plate, which the process reaches at time t'. This in turn will be proportional to the square of the magnitude of the Hilbert space interpretation at that site. This in turn will be given as

$$P(\text{ detection at x }) = |\Phi^{t'*}(x)\Phi^{t'}(x)| = \frac{1}{2}|\Phi^{t'+}(x) + \Phi^{t'-}(x)|^2$$

in accord with the usual NRQM prediction. More precisely of course this just gives the probability density at the point x and the actual probability must be determined over some small region surrounding x. At time t' when the particle interacts with the detector plate the game will transition again to an interactive product, this time of the form

$$\frac{1}{\sqrt{2}}(\mathbb{P}_+\hat{\oplus}\mathbb{P}_-) \boxtimes \mathbb{D} =$$

$$\frac{1}{\sqrt{2}}(\mathbb{P}_+\hat{\oplus}\mathbb{P}_-) \boxtimes \sum \mathbb{D}_i$$

where the $\mathbb{D}_i$ refer to the individual locations along the surface of the detector plate capable of registering a particle detection. At time t' some informon at location j will interact with the detector plate at a nearby region i and the game will change to

$$\frac{1}{\sqrt{2}}(\mathbb{P}_+\hat{\oplus}\mathbb{P}_-)_j \otimes \mathbb{D}_i$$

where $\frac{1}{\sqrt{2}}(\mathbb{P}_+\hat{\oplus}\mathbb{P}_-)_j$ denotes whatever process the particle may continue to evolve under after the interaction of the detector plate with the particle informon at location j. It may well be that the particle is absorbed and its process ceases entirely, or it might be scattered off. In either event it becomes irrelevant as far as the subsequent detector measurement is concerned. The important point though is that one obtains the usual NRQM result even though at any step of game play only a single informon is ever being generated and the information used to generate each informon is causally local to that informon and non-contextual. Nevertheless one still has the usual interference pattern appearing on the detector plate and at every detection only a single site is ever registered since only a single informon ever couples to the detector.

4.4 Schrödinger's Cat and Classicality

The Schrödinger cat paradox is another archetypal quantum mechanical conundrum. Imagine a sealed room. Within it is a cat, initially known to be alive, and a canister of cyanide gas set to be released upon detection of alpha decay of a radioactive isotope. Obviously when the room is unsealed at some future date the decay might have happened, in which case the cat is dead, or not, in which case the cat is still alive. The usual argument is to form the quantum mechanical superposition of these two possibilities and imagine a wave function for the contents of the room as $\Psi = (1/\sqrt{2})(|\text{no decay} > |\text{alive} > +|\text{decay} > |\text{dead} >)$. It is then argued that this is an ontologically valid state and therefore prior to the observer opening the room the cat actually exists in a superposition of dead and alive.

Of course it is not actually clear that processes such as being alive or being dead constitute valid quantum mechanical states. In both states the constituents of the cat are the same, but it is not at all clear that the Hamiltonian function is the same in both states, in which case they do not satisfy the same wave equation. Even if it were the case that these states could be described by a single Hamiltonian, there are clearly additional factors at play since there are many properties of these two cat states that are decidedly distinct from one another, especially as regards behaviour. The Hamiltonian under-determines the problem. It is also true that α-decay is not a state but a process as well, and the α-decay process occurs stochastically over a period of time. So even the declaration of the NRQM wave function as $(1/\sqrt{2})(|\text{no decay} > |\text{alive} > +|\text{decay} > |\text{dead} >)$ belies the actual complexity of the situation.

However there is another issue which arises because in standard NRQM there is only one form of sum and one form of product. If we convert this to process terms, the standard NRQM formulation asserts that the description of the room comprises a process $\mathbb{P}(D)$ for the detector and a process $\mathbb{P}(C)$ for the cat. These two may be further subdivided into the subprocesses $\mathbb{P}(D_n)$ for the detector in the non-release state, $\mathbb{P}(D_r)$ for the detector in the release state, $\mathbb{P}(C_a)$ for the alive cat, and $\mathbb{P}(C_d)$ for the dead cat. It is then presumed that these can be independently summed and that an independent product links them. That is, it is assumed that one may write

$$\Psi = \frac{1}{\sqrt{2}}[(\mathbb{P}(D_n) \otimes \mathbb{P}(C_a)) \oplus (\mathbb{P}(D_r) \otimes \mathbb{P}(C_d))]$$

The use of the independent sum implies that on any given step it is possible to play either game, up until the game ends (which here means that the observer unseals the room). In such a case the cat would appear to oscillate randomly between a state of being alive and a state of being dead, or as some would have it, in a weird combination of both.

The problem, however, is that it is impossible for the cat to ever effect a transition from the dead state to the alive state. It can remain indefinitely (more or less if the observer doesn't wait too long) in either the alive or dead state, or transition from alive to dead, but never the converse. As a result it is simply impossible to play the independent sum because there are selection rules operating which prevent certain transitions in the game tree from taking place. Thus the proper sum is the interacting sum. Moreover it is simply not possible to form a state for the cat such as $\frac{1}{\sqrt{2}}[\mathbb{P}(C_a) \oplus \mathbb{P}(C_d)]$ on account of these transition rules. Thus the only proper description for the combined state is as an interactive sum $\frac{1}{\sqrt{2}}[\mathbb{P}(C_a) \boxplus \mathbb{P}(C_d)]$. The game tree allows repeated play of the alive game but once the dead game gets played the only allowable moves are of the dead game.

The proper description of the process in the room is therefore

$$\Psi = \mathbb{P}(D) \boxtimes \mathbb{P}(C) = \frac{1}{\sqrt{2}}[(\mathbb{P}(D_n) \otimes \mathbb{P}(C_a)) \boxplus (\mathbb{P}(D_r) \otimes \mathbb{P}(C_d))]$$

Note that the interactive product is used in the initial expansion because there is no interaction between the canister prior to decay and the cat, but there is an interaction between the canister after decay and the cat, namely the cat dies, so the canister and cat games cannot be played freely.

Note that the choice of the play of the game is entirely driven by the game for the canister since as soon as it is played the cat dies. The play of the cannister game allows for a spontaneous transition from the no release game to the decay game, this transition being determined stochastically by the decay rate. The probability of finding a live cat thus depends upon the spontaneous decay rate and will be equal to the probability that no decay occurs during the period of observation. This will equal the likelihood that over the play of the entire game one only plays the no release game for the cannister.

This problem of classicality more generally is a problem worthy of a thesis in its own right and so only a few tentative conjectures are offered here to foster further study.

In NRQM it is simply assumed that the linearity of the Schrödinger equation implies that any two solutions Ψ_1 and Ψ_2 may be summed to give an ontologically realizable state. The discussion in Chapter 2 of the PCM and in Chapter 5 showed that it is quite possible to have processes that are weak epistemic equivalent, meaning that they yield the same wave function asymptotically, yet are not equivalent otherwise. The NRQM formalism as it stands does not distinguish between them and so does not necessarily take into account the presence of super-selection rules that might prevent certain operations being performed on them.

These super-selection rules effectively mean that some combinations might yield the zero process, for example

$$\mathbb{P}_1 \oplus \mathbb{P}_2 = \mathbb{O}$$

In the Schrödinger cat example, $\mathbb{P}(D_r) \otimes \mathbb{P}(C_a) = \mathbb{O}$.

How do these super-selection rules arise in the first place? They could simply be postulated, although that seems to be a bit of a cheat. Let us look more closely at what sums imply.

Within the process framework, the presence of a sum conjoining two processes $\mathbb{P}_1, \mathbb{P}_2$ implies that they act sequentially. Any sequence of processes is possible, but only one process ever acts during a single round. Staying with the exclusive sum, it is also the case that the two processes never act on the same informon. Generally the exclusive sum is used to represent the situation in which one has a single process type, governed by a single strategy type but possibly where there may be different values available to the properties that may be generated. The individual processes in such a sum are meant to represent different instances of the same process type but with possibly different property values being generated. For example a sum of eigenstates is meant to represent a sum of states for the *same* physical system. The conundrum for classicality is that if we apply the same constraint and assume that in the sum we are representing states of the *same* classical system, then we are faced with asserting that the system exists simultaneously in two distinct classical states, something that simply is never observed.

The way out in the discussion of the Schrödinger cat problem was to insist that in the classical setting one must use the interactive sum, rather than the independent sum. But why exactly is this necessary? One approach is to assert that this is a scale phenomenon, not manifesting at quantum mechanical scales but manifesting at classical scales, when $\hbar \rightarrow 0$. However, the arguments in Chapter 3 showed that this limit is necessary to obtain NRQM as an idealization of the process model. The same argument cannot be used to obtain classicality at the same time. So another mechanism must be in play.

One thing that distinguishes classical from quantum systems is their size. Another feature is their complexity. A classical system consists of a large number (more often vast number) of components which engage in complex interactions with one another.

Each process taking part in a classical superposition is in fact a complex algebraic tangle of primitive processes, some of which will represent different states of single physical systems. If there are M distinct systems comprising the classical system then we may consider a complex process $\mathbb{P}$ to be an element of Π formed from the set $\{\{\mathbb{P}_i^1\}, \{\mathbb{P}_j^2\}, \ldots, \{\mathbb{P}_k^M\}\}$. If we have a second classical process $\mathbb{Q}$ based upon the same component subsystems (another issue for the Schrödinger cat example since a live cat is continually renewing its subsystems, something that a dead cat does not) then it will be an algebraic combination based on the set $\{\{\mathbb{Q}_{i'}^1\}, \{\mathbb{Q}_{j'}^2\}, \ldots, \{\mathbb{Q}_{k'}^M\}\}$

In the realization of the conjoined process $\mathbb{P} \oplus \mathbb{Q}$ there is no issue until following a round in which an informon of $\mathbb{P}$ is generated, there follows a round in which an informon of $\mathbb{Q}$ is generated. The converse situation can be described analogously. The informon of $\mathbb{P}$ will consist of a collection of informons $\{n^1, n^2, \ldots, n^M\}$ corresponding to each of the component subprocesses.

When $\mathbb{Q}$ now acts, the information residing in these new informons, being informons of the same component subsystems as governed by $\mathbb{Q}$, may trigger changes in the subprocesses that comprise $\mathbb{Q}$, thus inducing a transition to a new classical process $\mathbb{Q}'$, as in the Schrödinger cat example where a transition from $\mathbb{P}(C_a) \to \mathbb{P}(C_d)$ can take place.

It is equally possible that the information, although it does not result in a wholesale change of classical process, may preclude possible moves on the part of $\mathbb{Q}$, so that only a subtree of the subsequent sequence tree may actually be implemented. If such a restriction occurs, then it immediately follows that we are no longer operating within the independent sum $\mathbb{P} \oplus \mathbb{Q}$ but instead have transitioned to the interactive sum $\mathbb{P} \boxplus \mathbb{Q}$. At some point in the evolution of these processes there may arise a condition under which it is possible that either $\mathbb{P}$ or $\mathbb{Q}$ be unable to act, and thus rendered inactive, leaving only a single classical process. This may also be the case from the beginning, which would force $\mathbb{P} \oplus \mathbb{Q} = \mathbb{P} \oplus \mathbb{Q} = \mathbb{O}$

The above is clearly sketchy, but it does suggest that the absence of macroscopic superpositions is a expression of the complexity of the conjoining of the individual subprocessees that form a macroscopic or classical object. Indeed, it might be reasonable to *define* a (macroscopic or classical) object to be a complex process for which sums of distinct states are not permitted. In other words, an object would be a collection of complex processes $\{\mathbb{C}_i\}$, each generating a distinct state of the object, such that $\mathbb{C}_i \oplus \mathbb{C}_j = \mathbb{O}$ for all i, j.

4.5 Quasi-Locality and Continuity

A significant amount of attention in both the physical and the philosophical literature is devoted to the question of whether reality at its ultimate level is local or non-local. The usual interpretation of the hidden variable theorems, in particular the Type I hidden variable theorems, is that they the demonstrate that reality at the ultimate level must be non-local. The exact nature of this presumed non-locality has not been fully determined as of yet. Moreover, as argued in Appendix A, this conclusion is far from certain. Type I hidden variable theorems are predicated on an assumption that any model of local reality, and this includes classical models, must of necessity be Kolmogorov is structure. The Type I hidden variable argument constructs an explicit inequality involving correlations among three entities that must be satisfied if the probability structure is Kolmogorov, and is violated by quantum mechanics, which is known to be non-Kolmogorov. The argument of Appendix B showed that this assumption is in fact false. There are situations, abundant situations in fact, in the classical realm which admit non-Kolmogorov probability structure. The results of Vorob'ev [413] provide a complete characterization of the conditions that are necessary for a combined system to admit a Kolmogorov probability structure. Khrennikov [208, 211, 215] has provided a wealth of examples in which non-Kolmogorov probability structure appears in a classical setting and has developed a contextual probability theory, the Växjö model, which in fact provides three distinct models - Kolmogorov, complex contextual and hyperbolic contextual, similar to the three different geometries, Euclidean, Riemannian and Hyperbolic [208, 213]. The Type I theorems fail because they rest upon a false premise.

Let us consider, though, that the conclusion is still true even if the Type I theorems fail to prove it. First of all, we must address the question of what exactly is meant by the term locality. Shimony [321], extending the work of Jarrett, pointed out that the notion of locality in every Type I hidden variable theorem reduces to two types of independence among correlations. Consider a situation involving the production of a pair of particles that are allowed to move to significant space-like separated points where they interact separately with measurement apparati. Following Shimony's terminology let $p(m/k, a, b)$ refer to the probability of obtaining a measurement m given the complete state k with the two measurement apparati set for a and b respectively. Let $p(m/k, a, b, n)$ refer to the probability of obtaining a measurement of m given the complete state k with the two measurement apparati set for a and b respectively and the second particle yielding measurement n.

Parameter independence means that $p(m/k, a, b)$ is independent of b and may be written as $p(m/k, a)$. *Outcome* independence means that $p(m/k, a, b, n)$ is independent of n and may be written as $p(m/k, a, b)$. The assertion that a hidden variable model cannot be local means that it cannot be both parameter and outcome independent. NRQM clearly violates outcome independence on account of the phenomenon of entanglement but Shimony argued that this "passion at a distance" does not violate special relativity because it does not allow for superluminal signalling. NRQM does respect parameter independence because to do otherwise would permit superluminal signalling.

Consider the case of a pair of entangled particles in the entangled state $\Psi = \frac{1}{\sqrt{2}}(|0 > |0 > + |1 > |1 >)$. The meaning of these states is unimportant for the argument. Outcome independence is easily violated here because measurement of one particle implies the measurement for the other particle.

Consider how this is formulated within the process framework. First of all, begin with free processes generating two entangled particles, say $\mathbb{P}_A$ and $\mathbb{P}_B$. If these particles were independent of one another then the conjoined process would be described as $\mathbb{P}_A \otimes \mathbb{P}_B$. In an entangled state, however, they are not independent and so one must consider the interactive sum $\mathbb{P}_A \boxtimes \mathbb{P}_B$ which in this case is equal to $\frac{1}{\sqrt{2}}[(\mathbb{P}_A(0) \otimes \mathbb{P}_B(0)) \oplus (\mathbb{P}_A(1) \otimes \mathbb{P}_B(1))]$. Note that the independent sum and product can be used here because there is no interaction between these conjoined subprocesses.

Game play proceeds as usual. During any given round, one may be playing either the game generating $(\mathbb{P}_A(0) \otimes \mathbb{P}_B(0))$ or the game generating $(\mathbb{P}_A(1) \otimes \mathbb{P}_B(1))$. The informons created during each round will therefore either correspond to the processes giving rise to measurements $0, 0$ for A,B respectively, or to those giving rise to measurements $1, 1$. Consider particle A and apply a measurement apparatus for a to particle A and for b to particle B. Suppose that on this round the process being played is $(\mathbb{P}_A(0) \otimes \mathbb{P}_B(0))$. Each informon will interact with the respective measurement apparatus and yield a positive or negative result which is independent of the other. Measurements of outcomes for A are given in the following table (rows for parameter of A, columns for parameters of B)

parameter	0	1
0	+	+
1	-	-

A similar table holds for B. It should be obvious then that the probabilities obtained are independent of the parameter value chosen for the other particle.

If we attempt to do this to test outcome independence we find that any informon for A corresponding to 0 is matched by an informon for B corresponding to 0 and similarly for 1. Thus it is obvious that there is no outcome independence as far as the emergent probability is concerned.

It is important to note that the processes for A and B are coupled using the independent product and so the informons that are being generated by these processes are being generated independent of one another. The only factor that they have in common is the ability to generate the same measurement outcome when coupled to measurement apparati having the same parameter settings. The generation of each informon is still based upon causally local information which does not involve the other informon. The content sets of every informon will be based solely upon the past causal cone for that specific site and will contain informons associated only with its own process, that is, A informons associate with A processes while B informons associate with B processes. Although there may be outcome dependence at the level of the Hilbert space interpretation, the informons associated to each particle remain causally independent of one another and are generated using local information. The apparent non-local connections arise at the process level as a result of the nature of the interaction between the processes.

The entangled process may be viewed in two ways. First of all, it may be thought of as an interactive product of two separate processes, as described above. Viewed in this way, this entanglement appears to give rise to spooky at-a-distance effects, even though as has been shown, there is actually no information passing between the two processes or between the informons that are associated with the two processes. The coupling is not at the level of the informons, i.e. the level of ultimate reality, but rather at the level of the processes, and we have seen in Chapter 2 that the space of process lacks any kind of spatio-temporal structure, so there is no reference against which notions of locality or non-locality can be measured. There is a second manner of viewing the entanglement situation which makes a great deal more sense, and that is to consider the entangled system as being due to a single process that on any given step of game play acts upon two different informons simultaneously. In other words, following completion of a full round, such a process (and its game) will create two distinct informons. This is somewhat akin to a Grandmaster of Chess playing two games at once. It is also akin to the situation in quantum field theory in which a quantum field may exist in a multiple particle state and so creates several particles simultaneously which then subsequently evolve more or less independently of one another. Allowing for a single process to make multiple simultaneous moves is not a great stretch to consider. Correlations can then occur between the generated informons even while no information passes between the informons in their generation.

In the process framework there are two different kinds of locality (or non-locality) taking place. The non-locality observed in the phenomenon of outcome dependence is an emergent level phenomenon, appearing in correlations between events at the observable level. This non-locality is induced by the nature of the generating process, to which notions of locality and non-locality do not apply. This non-locality arises because the nature of game play violates the principle of continuity and because the action of process in generating informons is free to jaunt to any point of the causal manifold, without there being any exchange of information in the process. Parameter independence holds because there is no information passing between individual processes - only indirectly between informon and process. This again ensures that information is only ever transferred locally and no superluminal signalling can ever take place.

At the lowest level, the ultimate level of reality, locality still pertains. This occurs because informons residing within a single causal tapestry do not exchange information with one another. They do not know of one another's existence unless and until, at some future date, information from one is causally propagated to the other. The informons being generated to form the new causal tapestry utilize information that is entirely local to each individual. The content set refers to prior informons, but this refers only to the original source of the information. It is presumed that there is a forward causal chain linking every informon within the content set to the informon and it is understood that such information has been passed forward along that causal chain to be incorporated into the informon as it comes into existence. Thus the information may have originated in the past but it reaches the informon in the present as it is formed. Therefore it is actually local to the informon at hand. Moreover, the informons that constitute any actual occasion are space-like separated, and this is true of the informons as they are formed to create the new causal tapestry. No information passes between informons within the same causal tapestry. Again, whatever information they use in their formation is local. Thus the informons form a set of local hidden variables. Thus the process based hidden variable model provides a local generative dynamic at the fundamental (informon) level while generating apparent non-local correlations at the quantum or classical levels as a result of interactions between processes, which are neither local nor non-local since they have no spacetime associations at all.

The term *quasi-local* is used to describe the process model on account of this phenomenon of lowest level locality and emergent level non-locality reflecting the inherent non-locality (more precisely the absence of any spatiality) of process. This quasi-locality arises as a result of the violation of the principle of continuity inherent in the jaunting nature of the selection of informons to be created during given rounds. The only information involved in the selection of an informon to be generated lies in the knowledge that no informon has currently been associated with the selected site in the causal manifold. The actual information that is utilized in the creation of the informon is derived from the informons that ultimately end up represented within the content set of the informon. These informons reside in the past causal cone of the informon being generated and represent information available to the generating process at the stage at which the informon is generated. To my mind, the loss of continuity is a less significant problem than are the problems associated with reconciling special relativity and the non-locality as supposedly demanded by the hidden variable theorems. Maudlin [249] has eloquently argued that this conflict cannot be easily swept away. Continuity, even by simple observation, can frequently be shown to be an idealization, a mathematical convenience, enabling one to take advantage of the powerful tools available to continuous models. Special relativity is much more difficult to avoid. The empirical evidence in support of special relativity is convincing and seemingly irrefutable. Given the choice, I chose to abandon the principle of continuity, but the structure of the models shows that many aspects of continuity are regained through the use of the interpolation procedures which (potentially) recreate the wave function, and through the causal embedding, which, at least in the case that t_P, l_P are the Planck time and length respectively, creates a fairly good representation of the continuous causal manifold.

This causal structure is not specifically required of the models described in Chapters 3 and 5, which as was argued, were developed solely to reproduce the results of NRQM. In the non-relativistic case the causal structure is particularly simple. Nevertheless the causal structure is in place as a structural attribute of the causal tapestry, and it is hoped that in the future it may be possible to extend the process model to include aspects of the relativistic case as well.

Closely linked to the notion of continuity is the notion of infinity. Continuity requires the ability to infinitely subdivide some entity. This ability is inherent in the definition of the real numbers, which form the prototypical example of a continuous structure. It is a deep and controversial question whether or not infinity exists as a reality or whether it is merely a mathematical construct. Indeed infinity, while a useful concept in mathematics, creates problems when it shows up in inconvenient places in physics, especially in calculations. A recent article in new Scientist, Aug 17-23, 2013 (pg 32-35) suggests that it might be time to dispense with infinity and return to a study of finite models. The article points to the controversy with which such an idea is met. Infinity and continuity are powerful, seductive and at times arcane notions. Very romantic in Bell's terms. However we do not observe infinity in actuality. There are limits to the accuracy with which we can measure and resolve things. It may be that infinity and continuity are like the fabled Siren, singing a song we cannot resist but in the end leading us to our doom.

The process model is finite, discrete, self contained, consistent. The physics, at least the physics that can be done with the current models, can be carried out entirely on the informons without reference to the causal and Hilbert space interpretations. One simply runs the model and skips the steps involving the construction of the interpretations. The process strength is the important quantity and it can be calculated entirely on the causal tapestry. Our usual NRQM arises as a idealization in the infinite limits but perhaps the computational problems with quantum mechanics arise because it is an idealization rather than a fully realistic model. If so then there may be value in studying discrete and finite models such as the process model presented in this thesis.

4.6 Quasi-Non-Contextuality and Completeness

The issue of contextuality was addressed in Appendix A in the discussion of Type II hidden variable theorems which explicitly demonstrate that NRQM is a contextual theory. Mermin defines non-contextuality as follows [255, 257]:

> This tacit assumption that a hidden-variables theory has to assign to an observable A the same value whether A is measured as part of the mutually commuting set $A, B, C, \ldots$ or a second mutually commuting set $A, L, M, \ldots$ even when some of the $L, M, \ldots$ fail to commute with some of the $B, C, \ldots$, is called "noncontextuality" by the philosophers.

Given this definition, the Mermin and GHZ results demonstrate that NRQM is a contextual theory. The von Neumann and Gleason results primarily demonstrate that the probability structure of NRQM is non-Kolmogorov. That issue was dealt with at the beginning of this chapter. Here I shall deal specifically with the notion of contextuality as defined above.

The Bell and Kochen-Specker results essentially demonstrate that which Vorob'ev showed, namely, that one can construct probabilistically incompatible sets of events whenever the dimension of the space being constructed is greater than 2. As discussed in Appendix B, this result is not restricted to quantum mechanics but rather is about probability theory generally, even in the classical setting. If that point had been known earlier perhaps there would have been a lot less fuss made over the contextuality of NRQM. The fact is that the vast majority of situations, whether classical or quantum mechanical are contextual from a probabilistic point of view and so it is truly quite unsurprising that ultimate reality should also prove to be contextual.

The extreme interpretation of this result is that reality possesses no definite properties until some observer observes and causes properties to come into being. That decidedly solipsistic idea gained purchase in quantum mechanics but perhaps it would have been more quickly abandoned had it been realized that contextuality is classical as well, even in circumstances in which definite physical properties can be ascribed to classical objects. Emergent situations such as arise in psychology, sociology, economics all possess contextual aspects, which is the focus of archetypal dynamics [380]. The process model is less extreme. In the process model, informons are accorded a limited non-contextuality with properties attributed to them by virtue of the process generating them. The presence of these properties serves to link informons to generating processes and to determine which informons can propagate information by means of which processes to which nascent informons. These informon properties are determinate in the sense of Bunge [71] in that they serve to determine the outcomes of certain measurement procedures. They do not, however, determine the outcomes of all measurement procedures as is assumed in the classical case. For this reason, the process model is termed *quasi-contextual*. More about this will be said in the section below on Type II hidden variable theorems.

The issue of the completeness of quantum mechanics dates back to Einstein, Podolsky and Rosen [119]. Much has been made about the completeness of quantum mechanics but these arguments have always to my mind seemed a little rigged, in the sense that the standard against which any theory is to be judged is the statistical structure of quantum mechanics. To be complete from this point of view means to possess a statistical structure which is more akin to a classical theory, meaning more akin to a Kolmogorov probability theory, which we already know cannot be the case. It seems to me, however, that significant dynamical information is lacking in the quantum mechanical framework, and this point was emphasized in the discussion of the process covering map and the ideas of process, ontological, strong epistemic and weak epistemic equivalence, Two processes may be weak epistemic equivalent, meaning that they will yield the same wave function in the asymptotic limit but they need not be process, ontological or strong epistemic equivalent. The fine structure is absent in the quantum mechanical formulation. The previous discussions of entanglement and of classicality suggested that the origin of these lies in the algebraic structure of the interactions among contributing processes and that some of this information is missing from the quantum mechanical depiction. For these reasons it is suggested that while quantum mechanics may provide a complete theory as far as the statistics of certain interactions, particularly measurement interactions are concerned, it remains nevertheless incomplete since it is lacking important dynamical information which the process model may provide.

4.7 Uncertainty Relations

Heisenberg's famed uncertainty relations provide another striking contrast between classical and quantum mechanical depictions of reality. They describe correlations that exist between certain pairs of measurement operators. Given an operator A on a Hilbert space $\mathcal{H}(\mathcal{M})$ of wave functions, define the expectation value of A for some state $\Psi \in \mathcal{H}(\mathcal{M})$ to be

$$< A > = < \Psi|A|\Psi >$$

Define the variance (or dispersion) σ_A^2 as

$$\sigma_A^2 = < \Psi|(A - < A >)^2|\Psi >$$

Abers [1] gives a general statement of the uncertainty principle which is that if A, B are any two operators on the Hilbert space of wave functions, then

$$\sigma_A^2 \sigma_B^2 \geq -\frac{1}{4} < \Psi|[A, B]|\psi >^2$$

In the case that A, B do not commute, this forces a lower bound upon the product of their variances. The two most famous uncertainty relations link position and momentum and energy and time and are usually stated as

$$\sigma_p \sigma_q \geq \frac{1}{2}\hbar$$

$$\sigma_E \sigma_t \geq \frac{1}{2}\hbar$$

The uncertainty relations are often interpreted to imply that at a fundamental level, particles are delocalized in position and momentum or in energy and time. The latter is exploited in QFT to justify the use of virtual particles in calculations, or to describe the quantum vacuum as a seething cauldron of fluctuating virtual particles.

A similar problem arises in signal theory. If a signal is represented as an entire function (meaning a complex valued function which is analytic over the entire domain, usually $\mathbb{R}^n$) then such a signal cannot be both band-limited and time limited (equally, spatially frequency limited and spatially limited). There is a paradox here since in practice real signals are in fact both band and time limited. Slepian [327] attempted to resolve this paradox by recourse to the Heisenberg uncertainty principles. There is a disquieting issue in quantum mechanics as well, namely that in the cases of position-momentum, angular position-angular momentum, and time-energy uncertainty, there is a correlation between a dynamically free property (position, angular position, time) and a dynamically conserved quantity (momentum, angular momentum, energy). These conservation laws form the cornerstones of physics and are not merely empirically validated but form essential guiding principles in the structure of theories and reflect deep symmetries of nature. The dynamical characters of these two classes of properties are distinct, yet similar consequences for the distributions of these properties in physical systems are often inferred from these uncertainty relations.

The process viewpoint considers the uncertainty relations, not as statements about the nature of the actual occasions, but rather as statements concerning the correlations among measurement processes. Recall from Chapter 4, that an interaction between a system process $\mathbb{P}$ and a measurement process $\mathbb{M}$ potentially involves a transition from the original system process $\mathbb{P}$ to a weak epistemic equivalent, rotated process $\mathcal{M}(\mathbb{P}) = \sum_i w_i \mathbb{P}_i$, where each $\mathbb{P}_i$ is a process that can independently act in conjunction with the measurement apparatus until a definitive interaction with the measurement apparatus leading to a definitive measurement value takes place. Interacting the new system process with a second measurement apparatus $\mathbb{M}'$ will result in a second rotation $\mathcal{M}'(\mathcal{M}(\mathbb{P})) = \sum_i w_i \mathcal{M}'(\mathbb{P}_i) = \sum_i \sum_j w_i w_{ij} \mathbb{P}'_j$.

The notion of commutation in process terms is subtly different from that in NRQM. This is because the decomposition of a process into subprocesses is considered to be a change in the generating process to one which is at least weak epistemic equivalent to the original, but not necessarily process equivalent to it. Thus in the process framework, two measurement processes $\mathbb{M}, \mathbb{N}$ are said to commute if, for any process P with which they interact, $\mathcal{M}(\mathcal{N}(\mathbb{P})) = \mathcal{N}(\mathbb{P})$ and $\mathcal{N}(\mathcal{M}(\mathbb{P})) = \mathcal{M}(\mathbb{P})$. Since $\mathcal{N}(\mathbb{P})$ is weak epistemic equivalent to $\mathbb{P}$ it follows that $\mathcal{M}(\mathcal{N}(\mathbb{P}))$ is also weak epistemic equivalent to $\mathbb{P}$ and so is $\mathcal{N}(\mathcal{M}(\mathbb{P}))$. This means that $\mathcal{M}(\mathcal{N}(\mathbb{P}))$ is weak epistemic equivalent to $\mathcal{N}(\mathcal{M}(\mathbb{P}))$. This implies that a rotation of $\mathbb{P}$ into the $\mathbb{M}$ frame implies that the rotated process is already invariant in the $\mathbb{N}$ frame and vice-versa. This in turn implies that $\mathbb{M}$ properties are preserved by $\mathbb{N}$ and vice-versa.

Processes that fail to commute in this way will not preserve property information. Since it is impossible to determine the values σ_N^2, σ_M^2 without interacting the process $\mathbb{P}$ with the measurement processes $\mathbb{M}, \mathbb{N}$, this implies that sequential interactions of these two processes will force the property values to spread out, even if they were definite for one or the other of the measurement processes in the beginning. Thus the uncertainty relations, in the process framework, tell us about the interactions between measurement processes, not about the ontological nature of the informons. Informons acquire property values from the processes that generate them and non-commuting measurement processes fail to preserve these property values when they act. In a sense, non-commuting measurement processes are *rivals*, since they erase knowledge of the property values generated by the other process.

The paradox of band and time limitation is resolved in the process framework because it arises only in the idealization afforded by the asymptotic limit. This provides another example where the use of continuous methods introduces complications that are not present in the model itself. Far from the asymptote, the causal tapestries involved in the process model are large but finite. Each informon is assigned a set of definite, although not complete, property values. Each informon n is also assigned a local $\mathcal{H}(\mathcal{M})$-contribution $\phi_n(\mathbf{z})$ of the form $\Gamma_n T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z})$. Although the informons are localized to the finite set of points $\{\mathbf{m}_n\}$ in the causal manifold $\mathcal{M}$, their local $\mathcal{H}(\mathcal{M})$-contributions are defined over the entirety of the causal manifold (although they decay rapidly and so effectively behave like localized functions). Consider the global $\mathcal{H}(\mathcal{M})$-interpretation, $\Phi(\mathbf{z}) = \sum_n \phi_n(\mathbf{z})$ and take its Fourier transform. One obtains

$$\hat{\Phi}(\mathbf{t}) = \begin{cases} \sum_n \Gamma_n e^{i\mathbf{m}_n \cdot \mathbf{p}_n} & \text{for } |\mathbf{p}_n(0)| \leq \pi/t_P, |\mathbf{p}_n(i)| \leq \pi/l_P, i = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

Thus the Fourier transform of the global $\mathcal{H}(\mathcal{M})$-interpretation has compact support while the interpretation itself extends across all of $\mathcal{M}$. Nevertheless this interpretation may still correspond to a physical system with a definite momentum and energy. That is because the spread in the Fourier space reflects the fact that the interpretation is being generated at frequencies that are proportional to t_P, l_P, and not to the specific energy and momentum of the physical system that it represents. This is another subtlety of the process approach.

In Chapter 2 in the general description of actual occasions I suggested that they were inaccessible to the emergent entities that they manifest. The arguments suggested for this statement were rather informal. Two mathematical reasons are given here. First, in the process models presented in Chapter 3 it was suggested that we take t_P, l_P to be the Planck time and length respectively. If we do so, then we obtain a single interpolation model capable of representing a wide range of, at the very least, bosonic particles. The assumption, clearly, is that emergent physical systems have energy and momentum bound well away from those at the Planck scales. If we attempt to increase the energy of such a system to bring its resolving ability down to Planck scales then the frequency band corresponding to this energy will bring it perilously close to that of $2\pi/t_P$ and from interpolation theory we know that the degree of agreement between the interpolation and the corresponding ideal wave function will begin to break down. Effectively this means that the process model is no longer capable of generating the system. In other words, if one were to attempt to bring a system to such extreme energy levels it would effectively decohere.

A second argument is provided by the Uncertainty relations which show that in order to resolve an informon at the Planck scales there would be an unacceptably large variation in energy and momentum. This is similar to the arguments used in quantum gravity research to suggest that space-time at the Planck scales may be discrete.

4.8 Relationship to Hidden Variable Theorems

4.8.1 Type I Theorems

Recall that the Type I hidden variable theorems are characterized by inequalities involving correlations among measurements, inequalities that are purportedly satisfied by certain classes of hidden variable models but violated by NRQM. In Appendix A it was shown that these models all presume that the probabilistic structure associated with the hidden variables *must* be Kolmogorov in nature and that NRQM is described by a non-Kolmogorov probability theory. The arguments involved in these theorems are all meant to show that any form of local hidden variable model, which is assumed to possess a Kolmogorov probability structure, *must* satisfy a certain inequality involving correlations between measurements between three or distinct quantum systems. NRQM can be shown to give predictions for these correlations which violate the inequalities, which demonstrates that no local hidden variable model can reproduce all of the predictions of NRQM. In Appendix B it was shown that the assumption that a local hidden variable model *must* follow a Kolmogorov probability structure is false as a general statement, and the results of Vorob'ev show that when three or more distinct systems are involved, there are additional criteria on the structure of the relations between the event spaces of these systems that must be satisfied if the probability structure is to be Kolmogorov. Otherwise the probability structure will be non-Kolmogorov and a probability model such as the Växjö contextual probability model must be considered. The probability structure of NRQM fits into the Växjö structure.

In Chapter 3, a specific process model was presented in which only causally local information was used to generated a set of hidden variables (informons) from which the usual NRQM wave function (or a close approximation thereof) emerged globally on a particular hypersurface of a causal manifold. The probability structure of these process generated hidden variables was discussed in Chapter 2 and in Section 3.1 and shown to be non-Kolmogorov in structure. Thus this model defeats the basic assumptions of the Type I hidden variable theorems.

In order to achieve this, a shift in mathematical language from one based simply in either traditional functions on topological spaces, or operators on function spaces, to a language based on the notion of process and of combinatorial games was made. In this language, quantum systems become emergent entities arising out of a lower level dynamics which is inherently unobservable and generated by processes in a manner similar to the play of a combinatorial game. As shown in the preceding two sections, in order to achieve this one has to abandon, at least at this lowest level, the principle of continuity and adopt a presentist attitude towards time. These are not great losses since continuity is recovered as an emergent property at the observable level and it is possible to formally create a growing block model of time if one uses the information contained within the content sets of individual informons. The process model generates in an emergent manner, most of the phenomenology that is observed empirically and reproduces the statistical predictions of NRQM.

In Chapter 3 it was shown, at least for the $\mathfrak{PI}$ strategy, that the standard wave function of NRQM emerges in the asymptotic limit as the temporal and spatial spacing of informons becomes infinitely small. Therefore in the asymptotic limit the process model will yield all of the usual predictions of NRQM, and so this demonstrates that it is indeed possible to have a local hidden variable which reproduces all of the usual results of NRQM. However, the idea underlying the process model is that at the level of ultimate reality, the space of events is modeled as a succession of causal tapestries, discrete in both space and time, with a characteristic spacing given by the Planck time and Planck length (at least for $\mathfrak{PI}$ strategy). The Hilbert space interpretation of the current causal tapestry will be identical to the usual wave function provided that the embedding lattice of the informons of the causal tapestry covers all of the hypersurface of the causal manifold and the wave function is a band-limited function, meaning that it corresponds to a system of bounded energy and momentum. There will be a discrepancy between the Hilbert space interpretation and the usual NRQM wave function if these conditions are not met, as discussed in Chapter 3 and in Appendix F. Since t_P and l_P are held to be fixed in principle, it is necessary to examine whether it might be the case that the Type I hidden variable inequalities are still not violated on account of the discrepancy arising from the finitary assumptions.

Unfortunately the theory of interpolation has not been fully developed in the case of spinorial functions so that calculations cannot be carried out precisely, but it is reasonable to expect that such functions may be treated simply as vector functions of dimension 2. If that is true the we may at least motivate a calculation of Bell's inequality in the process case even if a precise answer cannot be given. The original Bell argument was presented in Appendix A. Consider a pair of spin half particles prepared in the singlet state and moving off in opposite directions. Measure selected spin components σ_1, σ_2 and take some unit vectors $\mathbf{a}, \mathbf{b}$ respectively and consider the expectation value $<\sigma_1 \cdot \mathbf{a}, \sigma_2 \cdot \mathbf{b}>= -\mathbf{a} \cdot \mathbf{b}$. Let Ψ represent the wave function of this singlet state. Then the expectation value is calculated as

$$E(\mathbf{a},\mathbf{b}) =< \Psi|\sigma_1 \cdot \mathbf{a}\sigma_2 \cdot \mathbf{b}|\Psi >$$

Consider the Hilbert space interpretation on a causal slice $\mathcal{M}_t$ within the causal manifold $\mathcal{M}$. Let the embedding lattice be L so that $L \subset \mathcal{M}_t \subset \mathcal{M}$ and $\Phi(\mathbf{z}) = \sum_{n \in L} \Phi_n(\mathbf{z}) = \sum_{n \in L} \phi_n T_{\mathbf{m}_n} sinc_{t_P l_P}(\mathbf{z})$. The sinc functions on the lattice form an orthonormal basis. Therefore the norm of this function is $\Phi^*\Phi = t_P l_P^3 \sum_n \phi^*(n)\phi(n)$ [436].

The important quantity is the difference between the NRQM wave function Ψ on the slice and the Hilbert space interpretation Φ on the slice, $T = \Psi - \Phi$.

Then

$$E(\mathbf{a},\mathbf{b}) = E_\Phi(\mathbf{a},\mathbf{b}) + < T|\sigma_1 \cdot \mathbf{a}\sigma_2 \cdot \mathbf{b}|\Psi > + < \Psi|\sigma_1 \cdot \mathbf{a}\sigma_2 \cdot \mathbf{b}|T > + < T|\sigma_1 \cdot \mathbf{a}\sigma_2 \cdot \mathbf{b}|T >$$

Then

$$|E(\mathbf{a},\mathbf{b})| \leq |E_\Phi(\mathbf{a},\mathbf{b})| + | < T|\sigma_1 \cdot \mathbf{a}\sigma_2 \cdot \mathbf{b}|\Psi > |+$$

$$| < \Psi|\sigma_1 \cdot \mathbf{a}\sigma_2 \cdot \mathbf{b}|T > | + | < T|\sigma_1 \cdot \mathbf{a}\sigma_2 \cdot \mathbf{b}|T > | \leq$$

$$|E_\Phi(\mathbf{a},\mathbf{b})| + |T||\Psi| + |\Psi||T| + |T|^2 = |E_\Phi(\mathbf{a},\mathbf{b})| + 2|T| + |T|^2$$

In the asymptotic limit, $|T| \to 0$ and so the process model expectation value becomes arbitrarily close to the NRQM expectation value and so the Bell inequalities will eventually be violated. In Chapter 3 a particular case was presented in which the discrepancy T was of the order of $10^{-97} \max|\Psi|$. So in that case the Type I inequalities will clearly be violated by the process model. There may be cases, however, in which the discrepancy becomes appreciable and this could serve to develop a test to place bounds on the values of t_P, l_P, rendering the process model, and the various different strategies described in Chapter 5, testable, at least in principle.

4.8.2 Type II Theorems

In the process model, informons are assigned definite properties. These properties link them to the process or subprocess that generates them. The Type II hidden variable theorems purport to demonstrate that this is impossible. They do so by constructing a collection of mutually commuting observables contrived in such a manner that if one were to replace the operators defining these observables by definite numerical values then a contradiction is obtained. At first glance it appears that the process model blatantly contradicts these theorems. But there are subtleties that need to be examined before leaping to such a conclusion.

As they stand, the Type II theorems are obviously valid from a mathematical point of view. However, in order to apply them within the process framework one must look more closely. Each informon of the causal tapestry does indeed possess a collection of properties, which could include properties such as mass, energy, momentum, spin, angular momentum and so on. These properties are used within the reality game to determine which informons can be extended in the new causal tapestry by which processes. In the case of a superposition of eigenstates (the case of a single eigenstate being trivial), it is the case that during each full round of play a single informon related to a single process is created. However, during the next full round of play, any process comprising the superposition may be played, and therefore an informon possessing the properties related to any eigenfunction could be created. Even if one could detect the properties associated to a particular informon, these properties are likely to change from round to round. In the Type II theorems, there is no mention as to when the values of the properties that are substituted into the operator equations are to be determined. It seems as though they are supposed to exist for the duration of existence of the particles involved. But these properties change from moment to moment, so which is to be used?

Suppose that the operator equations described in section A.2 of Appendix A are considered to represent that situation at just one moment. Since the time frame is not specified this would seem to be a legitimate possibility. According to the usual interpretation of NRQM, since the operators involved all commute with one another it is possible to find a single basis of eigenfunctions within which all of the operators have an eigenvalue. This eigenvalue is to be measured by applying the operator to the wave function and examining the magnitude of the resulting dilatation. This must be carried out operator by operator. Each operator is formed as a triple product of operators, one for each of three particles. According to the measurement theory, in order to actually obtain a measured value it is necessary for the process generating each particle to interact with the appropriate measurement apparatus. Each triple corresponds to a product of three measurements, one for each particle. Presumably these are to be carried out simultaneously. This doesn't actually matter since the particles are assumed to be generated by processes in the free product game. After each of these operators has been applied, the particles will rotate into the eigenbasis of the measurement operator and then a measurement will take place. Assume that the measurement leaves each particle intact. One applies the next operator and again the processes must rotate into the new eigenbasis corresponding to the new operator. This will not be the same basis as yielded the original measurement. In other words, these are not the same processes. It is impossible to apply these operators simultaneously to the particles in order to measure the values simultaneously, and yet that is what is necessary in order for these Type II theorems to apply.

If that argument fails to convince then an even simpler argument holds. According to NRQM if one is to measure a quantum system one applies the appropriate operator to the wave function of the system and then reads off the eigenvalues. A system in a single eigenstate will yield a single eigenvalue and so can be said to possess the relevant property whose value will be the eigenvalue. In the case that the system is in a superposition of eigenstates, no definite eigenvalue can be associated with the system. Instead, repeated measurements will yield each of the eigenvalues with frequencies determined by the weights of the corresponding eigenfunctions in the expansion of the system wave function. If one examines the Hilbert space interpretation of each informon one sees that each is a dilatation of a sinc function. Sinc functions are not eigenfunctions of any of the usual NRQM measurement operators. If one attempts to apply a measurement operator to the Hilbert space interpretation of an informon one does not obtain a single eigenvalue but rather a collection of eigenvalues, so it is impossible to assign any given eigenvalue and accord it the status of the measured value of the property since this value will change upon repeated measurements. It is simply impossible to assign a definite value using the usual NRQM prescription.

Now each informon is assigned a definite set of properties. Is this a contradiction? Not at all. As noted in Section 3.6 a full range of properties is not assigned to each informon. Only properties associated with the generating informon may be assigned. Measurement processes, just as measurement operators, need not commute and so processes do not possess complete sets of properties. The properties associated with informons are internal to the informon and cannot be ascertained by any external observer. They are solely for the use of the informon within the play of the game. They have meaning, however, since they determine which process or processes may propagate the information of the informon into future informons. Moreover, they also determine the value of the property should it be measured since it will determine, with an emergent probability based on the weighting, the value of the property should it be measured by an appropriate measurement process. That is clear from the description of the measurement process within the process framework, since each informon will trigger a transition into the measurement subprocess linked to the informon's property. Thus each informon carries the information necessary to determine a particular value of a property if the appropriate measurement is carried out, and the informon can thus be said to possess this property but the property itself cannot be directly observed and evaluated. Thus it is simply never possible to carry out the kinds of associations of numerical values to measurement operators required of these Type II theorems. and these reality games skirt around these theorems.

4.8.3 Type III Theorems

In a series of papers, Terno and Ioniciou and et.al. [191] introduced a quantum control version of Wheeler's classic delayed choice experiment. Their goal was to create a scenario in which, contrary to Bohr, complementary aspects of a quantum situation could be measured simultaneously, if surreptitiously. This allowed them to test whether or not properties such as 'wave' or 'particle' and 'deterministic' could be applied simultaneously in a model of the system. They showed that indeed they could not.

They couple together two distinct ideas. First of all, that any set of hidden variables describing the system should be deterministic in nature. Bell himself did not require that the hidden variables be deterministic in nature [44]. That requirement would seem to go back to Einstein who remained dissatisfied with the probabilistic nature of quantum mechanics [119]. Even at the classical level there are so many events in which randomness and non-determinism play major roles that the loss of determinism does not appear to be very significant. Indeed, it opens the door to the existence of free will and that seems psychologically comforting [98]. The process model incorporates non-determinism as an essential feature in the play of individual games, and allows for a stochastic element in the transition from the play of one game to the next.

Second, they define field and particle as the ability (resp. inability) to create interference, and then describe realism as the assertion that a photon must possess either, but not both, of these properties throughout its lifetime. One must consider this as a rather strong form of realism. A weaker form of realism would be to require merely that the photon possess some characteristic which enables it to determine particle-like or wave-like behaviour. In the reality game model, the photon is always discrete, i.e. particle-like at its most fundamental level of description, and the wave-like aspects of its behaviour are an emergent property of whatever measurement situation it finds itself in. It is not a case of either-or. It is a case that all measurements are ultimately contextual and emergent. Each quantum particle serves as a generator of properties but possesses properties only within specified measurement contexts.

Coupled together in this way, these two requirements allow for the proof of a constraint which cannot be satisfied by such models.

Strictly speaking, the process formulation does not contradict this Type III theorem because its hidden variables are not deterministic (although as noted in Chapter 5 some of them could be) and they do not satisfy the strong realism assumption (but they do satisfy the weaker form). Thus this Type III theorem does not actually apply to the process model, but it is instructive to see how this scenario is formulated within this framework.

The situation described consists of two photon sources, one generating single photons (photon) and one a pair of entangled photons (ancillas). The single photon passes through a Hadamard gate, then a phase shifter gate, then another Hadamard gate which is active or inactive according to the state of one of the entangled ancillary photons, and then on to a detector. The entangled ancillary photons follow different paths, one going to the quantum control of the second Hadamard gate and the other to a detector.

Pictorially we have

$$\begin{array}{ccccccccc} \mathrm{S} & \stackrel{A}{\rightarrow} & \boxed{H} & \rightarrow & \stackrel{\phi}{\bullet} & \rightarrow & \boxed{H^b} & \rightarrow & \mathrm{D}_a \\ & & & & & & B\uparrow & & \\ & & & & & & E & \stackrel{C}{\rightarrow} & D_c \end{array}$$

The simplest implementation of the causal tapestry approach is in many ways the more stringent. Consider the situation in which only a single photon is allowed to transit the apparatus at any time. In this case we need to consider only a single photon process coupled to the apparatus and can ignore details of the photon source. We shall also ignore details of the detector. In Chapter 5, the theory of measurement is described and it suffices that the detector will register photons as distributed according to the Born rule applied to the photon wave function arriving at the detector.

There are a number of processes to consider. There are the single photon source S, the entangled photon source E, the first and second Hadamard gates H, H^b, the phase shift gate G^ϕ, the photon detector D_a, the ancilla C rotation R^α, the ancilla detector D_c and finally the three processes for the photons themselves, A, B, C. All of these co-exist throughout each of the experiment. Each run begins with the emission of a single photon. The timing of the entangled photons is unimportant except that it may or may not trigger a shift in the Hadamard gate control.

Since the role of the sources is merely to trigger off photonic processes and they have no subsequent role to play we shall ignore them and focus only upon the remaining processes.

At the start of the game the product of the various processes can be written as

$$A \otimes H \otimes G^{\phi} \otimes ((B \boxtimes C) \boxtimes H^{b}) \otimes D_a \otimes R^{\alpha} \otimes D_c$$

The game played between the two entangled photons is an interactive product because the play of these two games is coupled through the entanglement interaction. These two entangled photonic processes are coupled to the Hadamard gate control through an interactive product because the specific states being manifested by the entangled photon game determine states of the Hadamard control, hence their interactive product.

When the single photon interacts with the first Hadamard gate it undergoes a rotation. Such a transition is understood within the causal tapestry setting to imply a change in generating processes, from a single process into a sum of two processes - this case the single process with a reduced efficacy and an orthogonal process, both within the eigenspace of the detector. This interaction is purely informational. Following the interaction these new subprocesses couple to the remaining processes freely. The transformation is written as

$$(A \boxtimes H) \otimes G^{\phi} \otimes ((B \boxtimes C) \boxtimes H^{b}) \otimes D_a \otimes R^{\alpha} \otimes D_c = \frac{1}{\sqrt{2}}(A_0 \oplus A_1) \otimes G^{\phi} \otimes ((B \boxtimes C) \boxtimes H^{b}) \otimes D_a \otimes R^{\alpha} \otimes D_c$$

The next interaction takes place between this pair of photon processes and the phase shift gate. Again it involves a primarily informational change, this time applying a phase change to the second process. This transformation is described as

$$\frac{1}{\sqrt{2}}(A_0 \oplus A_1) \boxtimes G^{\phi} \otimes ((B \boxtimes C) \boxtimes H^{b}) \otimes D_a \otimes R^{\alpha} \otimes D_c \rightarrow \frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes ((B \boxtimes C) \boxtimes H^{b}) \otimes D_a \otimes R^{\alpha} \otimes D_c$$

Describing the next interaction is rather more complicated. First, let us expand the entanglement game. We may write

$$B \boxtimes C = \frac{1}{\sqrt{2}}[(B_0 \otimes C_0) \oplus (B_1 \otimes C_1)]$$

Now the C photon of the entangled pair undergoes a σ_y rotation of α. This induces the transition

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes ((B \boxtimes C) \boxtimes H^b) \otimes D_a \otimes R^\alpha \otimes D_c \\
&\quad \to \frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes ((B \boxtimes (C \boxtimes R^\alpha)) \boxtimes H^b) \otimes D_a \otimes D_c \to \\
&\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes [\{\frac{1}{\sqrt{2}}(B_0 \otimes (C_0 \boxtimes R^\alpha)) \oplus \frac{1}{\sqrt{2}}(B_1 \otimes (C_1 \boxtimes R^\alpha))\} \boxtimes H^b] \otimes D_a \otimes D_c = \\
&[\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes [\{\frac{1}{\sqrt{2}}(B_0 \otimes (C_0 \boxtimes R^\alpha))\} \boxtimes H^b] \otimes D_a \otimes D_c] \oplus \\
&[\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes [\{\frac{1}{\sqrt{2}}(B_1 \otimes (C_1 \boxtimes R^\alpha))\} \boxtimes H^b] \otimes D_a \otimes D_c] = \\
&[\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes [\{\frac{1}{\sqrt{2}}(B_0 \otimes (c\alpha C_0 \oplus -s\alpha C_1))\} \boxtimes H^b] \otimes D_a \otimes D_c] \oplus \\
&[\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes [\{\frac{1}{\sqrt{2}}(B_1 \otimes (s\alpha C_0 \oplus c\alpha C_1))\} \boxtimes H^b] \otimes D_a \otimes D_c] = \\
&[\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes [\{\frac{1}{\sqrt{2}}(C_0 \otimes (c\alpha B_0 \oplus s\alpha B_1))\} \boxtimes H^b] \otimes D_a \otimes D_c] \oplus \\
&\quad [\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes [\{\frac{1}{\sqrt{2}}(C_1 \otimes (-s\alpha B_0 \oplus c\alpha B_1))\} \boxtimes H^b] \otimes D_a \otimes D_c]
\end{aligned}$$

where $s\alpha = \sin\alpha$ and $c\alpha = \cos\alpha$. Next note that the coupling between the entangled photon pair and the second Hadamard gate is actually only between the B photon and the gate, and that detector D_a only couples with A, B while detector D_c only couples with C. Therefore we can rearrange the terms in the above equation to obtain the process in the form

$$\begin{aligned}
&[\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes [\frac{1}{\sqrt{2}}(c\alpha B_0 \oplus s\alpha B_1) \boxtimes H^b] \otimes D_a \otimes (C_0 \otimes D_c)] \oplus \\
&\quad [\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi} A_1) \otimes [\frac{1}{\sqrt{2}}(-s\alpha B_0 \oplus c\alpha B_1) \boxtimes H^b] \otimes D_a \otimes (C_1 \otimes D_c)]
\end{aligned}$$

Now we assume that the interaction between the B photon and the Hadamard gate H^b occurs prior to the arrival of the photon A. The game for H^b is actually a sum two games, H^b_o (inactive) and H^b_c (active). The interactive product will couple B_0 with H^b_o and B_1 with H^b_c. Substituting into the above yields, following the interaction between B and H_b, that the process now takes the form

$$
\begin{aligned}
&[\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi}A_1) \otimes \frac{1}{\sqrt{2}}[(c\alpha B_0 \otimes H^b_o) \oplus (s\alpha B_1 \otimes H^b_c)] \otimes D_a \otimes (C_0 \otimes D_c)] \oplus \\
&\quad [\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi}A_1) \otimes \frac{1}{\sqrt{2}}[(-s\alpha B_0 \otimes H^b_o) \oplus (c\alpha B_1 \otimes H^b_c] \otimes D_a \otimes (C_1 \otimes D_c)] \rightarrow \\
&\quad [\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi}A_1) \boxtimes \frac{1}{\sqrt{2}}[(c\alpha B_0 \otimes H^b_o) \oplus (s\alpha B_1 \otimes H^b_c)] \otimes D_a \otimes (C_0 \otimes D_c)] \oplus \\
&\qquad [\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi}A_1) \boxtimes \frac{1}{\sqrt{2}}[(-s\alpha B_0 \otimes H^b_o) \oplus (c\alpha B_1 \otimes H^b_c] \otimes D_a \otimes (C_1 \otimes D_c)]
\end{aligned}
$$

the transition occurring when the A photon interacts with the gate. The effect of H^b_o on the processes of A is to leave them unchanged. The effect of H^b_c is to effect a rotation of the A processes. We have

$$
\begin{aligned}
\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi}A_1) \boxtimes H^b_c &= \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(A_0 \oplus A_1) \oplus e^{i\phi}\frac{1}{\sqrt{2}}(A_0 \oplus -A_1)) = \\
&\quad \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi}A_0) \oplus \frac{1}{\sqrt{2}}(A_1 \oplus -e^{i\phi}A_1)) = \\
&\qquad \frac{1}{2}((1+e^{i\phi})A_0 \oplus (1-e^{i\phi})A_1)) = e^{i\phi/2}(\cos\frac{\phi}{2}A_0 - i\sin\frac{\phi}{2}A_1)
\end{aligned}
$$

For clarity, let us use the notation of Terno et.al. [388] and set

$$\frac{1}{\sqrt{2}}(A_0 \oplus e^{i\phi}A_1) = P_A$$

and

$$e^{i\phi/2}(\cos\frac{\phi}{2}A_0 - i\sin\frac{\phi}{2}A_1) = W_A$$

the ‘particle ’ and ‘wave ’ states of A respectively.

Then following the interaction with the Hadamard gate H^b the process becomes

$$[\frac{1}{\sqrt{2}}[\cos\alpha P_A B_0 \oplus \sin\alpha B_1 W_A] \otimes D_a \otimes (C_0 \otimes D_c)] \oplus$$
$$[-\frac{1}{\sqrt{2}}[\sin\alpha B_0 P_A \oplus -\cos\alpha B_1 W_P] \otimes D_a \otimes (C_1 \otimes D_c)]$$

In order to calculate relevant probabilities and correlations one must utilize the theory of measurement and determine the strengths of the couplings between the photon processes and the detector processes. One then uses the covering map to project these processes onto their sets of Hilbert space interpretations, which allows these coupling values to be calculated.

$$[\frac{1}{\sqrt{2}}[\cos\alpha P_A B_0 \oplus \sin\alpha B_1 W_A] \otimes D_a \otimes (C_0 \otimes D_c)] \oplus$$
$$[-\frac{1}{\sqrt{2}}[\sin\alpha B_0 P_A \oplus -\cos\alpha B_1 W_P] \otimes D_a \otimes (C_1 \otimes D_c)] \rightarrow$$
$$[\frac{1}{\sqrt{2}}[\cos\alpha \{\Phi_{P_A}\}\{\Phi_{B_0}\} \oplus \sin\alpha \{\Phi_{B_1}\}\{\Phi_{W_A}\}] \otimes \{\Phi_{D_a}\} \otimes (\{\Phi_{C_0}\} \otimes \{\Phi_{D_c}\})] \oplus$$
$$[-\frac{1}{\sqrt{2}}[\sin\alpha \{\Phi_{B_0}\}\{\Phi_{P_A}\} \oplus -\cos\alpha \{\Phi_{B_1}\}\{\Phi_{W_P}\}] \otimes \{\Phi_{D_a}\} \otimes (\{\Phi_{C_1}\} \otimes \{\Phi_{D_c}\})]$$

Since technically the B photon is lost after interacting with the Hadamard gate H^b, its presence in the process description is superfluous once the interaction has taken place and so it can be eliminated from the above formula to obtain

$$[\frac{1}{\sqrt{2}}[\cos\alpha P_A \oplus \sin\alpha W_A] \otimes D_a \otimes (C_0 \otimes D_c)] \oplus$$
$$[-\frac{1}{\sqrt{2}}[\sin\alpha P_A \oplus -\cos\alpha W_P] \otimes D_a \otimes (C_1 \otimes D_c)] \rightarrow$$
$$[\frac{1}{\sqrt{2}}[\cos\alpha \{\Phi_{P_A}\} + \sin\alpha \{\Phi_{W_A}\}]\{\Phi_{D_a}\}(\{\Phi_{C_0}\}\{\Phi_{D_c}\})] +$$
$$-\frac{1}{\sqrt{2}}[\sin\alpha \{\Phi_{P_A}\} + -\cos\alpha \{\Phi_{W_P}\}]\{\Phi_{D_a}\}(\{\Phi_{C_1}\}\{\Phi_{D_c}\})]$$

In the asymptotic limit one expects that these covering sets will converge to singleton functions identical to the NRQM wave functions in which case the above converges to

$$\frac{1}{\sqrt{2}}(\cos\alpha|p>_A+\sin\alpha|w>_A)|0>_C-\frac{1}{\sqrt{2}}(\sin\alpha|p>_A+-\cos\alpha|w>_A)|1>_C$$

and the usual NRQM predictions should follow. A precise determination must await further development of the model.

Let us return to the process as it appears prior to detection. The processes for A and C are already eigenprocesses for their respective detectors and so the result of a detection is merely to assert that a play from the corresponding process took place. Thus we may eliminate the detector processes from the overall process and consider only the play of the corresponding reality game. So let us consider the reduced process

$$\frac{1}{\sqrt{2}}[(\cos\alpha P_A B_0 \ \oplus \ \sin\alpha B_1 W_A) \ \otimes \ C_0] \ \oplus \ (\sin\alpha B_0 P_A \ \oplus \ -\cos\alpha B_1 W_P]) \ \otimes \ C_1)]$$

If we now think in terms of the corresponding reality game and its sub-games, we can understand the evolution of the system as follows. The $\oplus$ linking the terms $\frac{1}{\sqrt{2}}(\cos\alpha P_A B_0 \oplus \sin\alpha B_1 W_A) \otimes C_0]$ and $\frac{1}{\sqrt{2}}(\sin\alpha B_0 P_A \oplus -\cos\alpha B_1 W_P]) \otimes C_1)$ ensures that these two sub-games are played sequentially and independent of one another. That is, on any given round of play, the informon created could correspond to one generated by either sub-game. That is, a C_0 informon could be created or a C_1 informon could also be created and the $\frac{1}{\sqrt{2}}$ term will modify the strength of the Hilbert space interpretations of the respective informons so that on average each will couple to the D_c detector with probability $\frac{1}{2}$. The $\otimes$ ensures that the associated subgame must also be played, that is, if C_0 is played then $(\cos\alpha P_A B_0 \oplus \sin\alpha B_1 W_A)$ is played and if C_1 is played then $-(\sin\alpha B_0 P_A \oplus -\cos\alpha B_1 W_P)$ is played. Note that the sub-sub-processes within each of these sub-processes are related by $\oplus$ which means that they are played sequentially and independent of one another. That means on following any given round of play an informon from either game could be created. The weight associated with each game will modify the strength of the coupling to its detector so as to be detected with probability equal to the square of the weight. If we restrict consideration to the case in which the detector D_c registers only a C_0 informon, then the coupled A game is given by $(\cos\alpha P_A B_0 \oplus \sin\alpha B_1 W_A)$ and it follows that the probability of the detector D_a detecting a particle-like state will be $\cos^2\alpha$ and $\sin^2\alpha$ for the wave-like state, just as predicted by the usual NRQM argument.

The process model avoids running afoul of the Type III theorem as well, in this case because it is non-deterministic and incorporates a weak form of realism which only requires that at each moment of time a particular measurement is determined, not that it is specified for all time. The wave-particle duality is not a rigid wave-particle dichotomy. Instead, processes and games within the process framework possess a dual structure reflected in the dual embeddings to the causal manifold and its Hilbert space. Informons possess both particle and wave aspects, and which dominates is a reflection of the particular game being played and the nature of the measurement apparatus to which it must be coupled in order for a measurement to be carried out. Informons possess properties that suffice to enable them to generate or to determine specific measurements when coupled to a suitable measurement apparatus. Informons are generated non-deterministically, their generating processes couple to each other stochastically depending upon their respective strengths, and they are generated in a jaunting, non-local manner all the while arising from purely causally local information.

Chapter 5

A Miscellany of Reality Games

5.1 Construction of Strategies

The description of a reality game as given in Chapter 3 and Appendix D outlines the general structure of a generic reality game together with the the goal of the game - the construction of a new causal tapestry from a pre-existing tapestry which consistently extends the causal structure and the $\mathcal{H}(\mathcal{M})$-interpretation so as to provide an interpolation of the NRQM wave function of the quantum system on the causal manifold $\mathcal{M}$ (and with greatest accuracy on the embedding hypersurface of the causal tapestry). As described, a reality game could be played in *almost* any manner imaginable. This is akin to describing the trajectory of a particle as a time parameterized curve in a 3-manifold. Something more needs to be specified in order to make the formalism useful. Lacking from the general description is a specification of the strategy to be used by each player as they play the game. This strategy then determines the form of the game tree which defines the specific game being played. This is akin to specifying the partial differential equation to be satisfied by the trajectory. Choosing the initial conditions defines a particular trajectory as does choosing an initial causal tapestry for the play of the reality game.

The path integral strategy $\mathfrak{PI}$ presented in Chapter 3 is rather simplistic but it served its purpose to demonstrate the limitations on the hidden variable theorems as providing constraints on a fundamental reality. More physically realistic strategies are needed. A detailed discussion of these is beyond the scope of this thesis but it is worth examining some general considerations to be taken into account when seeking new strategies. Several strategies will be presented here to provide an understanding of the use of the notion of a strategy to generate, to distinguish, and to classify reality games.

Recall that a reality game is a token game. Players cooperatively construct tokens that are placed on virtual sites which eventually become complete informons. Game play proceeds in a series of short rounds, in which a token is constructed. These short rounds are organized into sequences which constitute full rounds, from which a single complete informon arises. Recall that in a general reality game a short round unfolds as follows:

1. Player I first selects an informon m from the current tapestry $\mathcal{I}$ not previously chosen, or any informon if a new round is beginning
2. Player II then selects either the informon in the new causal tapestry $\mathcal{I}'$ currently under construction, or if a new short round has begun, selects a label n not previously used and creates a new informon $[n] < (, , \mathbf{p}_n, \lambda_n) > \{\}$.
3. Player I then provides or updates the content set G_n.
4. Player II then selects a point $\mathbf{m}_n$ in the causal manifold and assigns it a partial interpretation $< (\mathbf{m}_n, , , \mathbf{p}_n, \lambda_n) >$.
5. Player I then creates and places a token $\mathcal{T}_{nm}$ on the site $\mathbf{m}_n$.
6. Player II then updates the local $\mathcal{H}(\mathcal{M})$ interpretation $< (\mathbf{m}_n, \Phi_n(\mathbf{z}), \Gamma_n, \mathbf{p}_n, \lambda_n) >$.
7. If no further tokens can be added (either no other contributing sites exist or an external limit has been reached), the round ends and a new round begins, otherwise the above process repeats
8. If a round ends the entire process of short rounds and rounds repeats. Game play ends when either no informon can be constructed or an external limit is reached

Each reality game is specified by several parameters: N (length of game play), R (number of informons created per round), r (number of current informons contributing information to any nascent informon), ρ (approximation measure), δ (approximation accuracy), t_P (temporal sampling wavelength), l_P (spatial sampling wavelength), ω (band limit frequency), L (Lagrangian), Σ (specification of strategy), p (set of properties). We denote a reality game by $\mathfrak{R}(N, r, \rho, \delta, t_P, l_P, \omega, L, \Sigma, p)$.

Player I determines which information from the current tapestry is to be carried forward while Player II constructs the novel elements of the new informons - their labels and interpretations. These steps are summarized in Table 5.1.

Table 5.1: Information Actions of Players.

Step	Player I	Player II
Task	Propagate Past Information	Create New Information
1	Propagate Current Informon	
2		Create New Informon
3	Propagate Content	
4		Create causal M extension
5	Generate Token	
6		Create $\mathbb{H}(M)$ extension

A play of these steps constitutes a short round which results in the generation of a new token, and these short rounds are repeated up to the limit specified by r which determines the duration of a round (hence the amount of information incorporated into an informon). Specifying how each of these individual steps is to be carried out determines the strategy for the game and specifies a particular reality game. The strategy Σ will be given as a 6-tuple which specifies how each step above is to be carried out.

For example, the Path Integral Game $\mathfrak{PI}$ can be described as (non-deterministic, non-deterministic, complete, non-deterministic, action integral, sinc translate). This means that Player I chooses the current informon non-deterministically. Player II chooses the new informon (or creates a new label) non-deterministically, Player I creates the new content by adding the current informon to its own content set and passing that forward). If need be, Player II chooses a causal manifold interpretation point non-deterministically. Player I generates a token by constructing the exponential of the action between the causal embedding points of the current and new informons. Finally Player II forms the Hilbert space interpretation by coupling the tokens to a sinc function translated to the new causal manifold embedding point.

The table can be used to construct and describe a host of different game strategies.

In constructing strategies for reality games it is useful to keep in mind the particular ontology one is attempting to model. There are three basic interpretations that can be applied to the process model. The *conservative* interpretation presumes that there is a pre-existing background space-time in which actual occasions manifest. These actual occasions constitute the fundamental elements of reality. Particles and fields arise in an emergent manner from these actual occasions. The wave function represents an actual physical attribute, namely the *strength* of the generating process and the properties of physical entities arise from interactions between processes. The effect of a property is to induce or to determine a change in the generating process according to a set of rules or dynamical laws or more likely dynamical symmetries or constraints. The usual functions of physics such as the potential and the Lagrangian are also to be considered as emergent functions because they summarize the effects of interactions with a vastly large number of systems constituting the physical environment of the system under consideration, The conservative approach is rather close in spirit to the standard approaches in physics but it is realist.

The *process* approach, which is the approach taken in this thesis, again asserts that actual occasions form the fundamental elements of reality but it does not presume the existence of any background space-time. Instead it presumes that space-time itself is an emergent manifestation of actual occasions as are all of the other observables of physics. It is a realist viewpoint since it does assert the essential reality of actual occasions, but the remainder closely follows ideas of complex systems theory and archetypal dynamics as described in Appendix C. It also is close in spirit to many approaches to quantum gravity and to the study of quantum fields in curved space-time. Functions such as the Lagrangian are also considered as emergent because technically they too should be manifestations of actual occasions (since these are the elements underlying our observable reality) and therefore they should themselves be interpolations over the embedding lattices of the causal tapestries. This subtle point was omitted from previous discussions again for simplicity of presentation but it is an important point.

The third approach, the *relational* approach is more radical. It draws inspiration from a more radical version of archetypal dynamics, but also from ideas of Trofimova [395, 397] related to the origins of functionality and diversity in complex adaptive systems. It also shares some aspects with relational approaches to physics [306]. In the relational approach it is process alone which carries any reality and everything else arises from interactions and relations between processes. Actual occasions are themselves considered to be emergent entities and space-time, particles, fields, are not thought of as emergent entities but rather as consistencies or coherences or symmetries that arise out of relations between processes. These latter entities are *interpretations* of actual occasions but do not represent entities in their own right.

The process formulation permits interpretation by all three viewpoints, although it was developed with the process viewpoint particularly in mind. The conservative viewpoint is certainly a more readily accessible viewpoint but it does not take advantage of the power of the process game approach. It does, however, make the development of strategies for reality games somewhat easier since actual occasions may be viewed in a more literal manner as a kind of primitive fundamental *particle*. The process viewpoint takes advantage of the

Table 5.2: Ontology of Strategies.

Interpretation	Actual Occasions	Space-time	Particles-Fields
Conservative	Realist	Background	Emergent
Process	Realist	Emergent	Emergent
Relational	?Anti-Realist	Interpreted	Interpreted

generative capacity of the process model allowing for everything that is observable to be emergent. As was suggested in Chapter 4, this point of view offers a possible resolution to many of the paradoxes of NRQM and quite possibly resolves many problems in quantum field theory because of the finitary nature of actual occasions and of information flow among them. Although not a focus of this thesis, one hopes that the discrete and finite nature of actual occasions might eliminate the many infinities that appear in QFT calculations.

The relational viewpoint takes full advantage of the process model, particularly as regards the meaning of the interpretations. In the process viewpoint these interpretations are emergent but to a large degree independent of any specific observing entity. They arise due to the macroscopic character of any observable physical entity but this emergent form is to a large extent independent of the observer. The relational viewpoint exploits the understanding of interpretation as described by archetypal dynamics. It reflects a system of meaning attributed to any entity or collection of processes which interacts with the processes that are being modeled. The relational viewpoint arises out of studies in psychology, economics, and the social sciences, where one must deal with complex adaptive systems. It is less directly relevant to the study of a fundamental level of reality underlying NRQM based upon our current understanding, but I don't believe that one should be too quick to dismiss the possibility that information and meaning, albeit in a primitive form such as salience [380], might play a role at very lowest levels of reality as well as at the highest. Nevertheless these ideas will not be addressed specifically here, other than to provide an understanding of some of the background of ideas out of which the process approach was developed. These thoughts are summarized in Table 5.2.

Table 5.1 above describes the different tasks that must be completed by each player in the course of a single short round in which a single new token is applied to a nascent informon. A series of short rounds then constructs a complete informon. It is possible to structure the reality game so as to build each informon piecemeal, but that approach, while mathematically tenable, is not very intuitive from a physical standpoint. It is far more reasonable to expect that each informon comes into existence as a complete whole and this is also consistent with the process view of the actual occasions that these informons are meant to represent. Therefore in defining different strategies the short round format will remain invariant and the focus will instead be on creating variations in the manner in which each player carries out the various tasks. As noted above, there are 6 tasks that need to be performed and the two players divide these between themselves. Player I concentrates on determining what information should be passed forward into the newly constructed informon, while Player II concentrates upon forming that new informon. There are many different possibilities for how each of these tasks is to be carried out.

1. Player I and Player II must both choose informons to interrelate. Player I may chose an initial informon in a deterministic or non-deterministic manner and there may or may not be some kind of general collection of guidelines to be used in doing so. The choices depend upon whether the informons are being embedded into the causal manifold in a regular structure such as a regular lattice, or into a space-like hypersurface corresponding to a single time, or to a space-like hypersurface which exhibits temporal extension. The more that the embedding departs from uniformity the more realistic the model becomes. Player II has the easier task because at this stage it either picks a new informon label and builds an informon around it, or it simply remains at the current informon until it has been completely constructed. Essentially there is no strategy needed for Player II and only a strategy for Player I needs to be stated.

2. Player I must select the content to be carried forward into the new informon. There are four basic choices. First, it can preserve all information from the past. In this case, if the current informon is $[n] < \alpha > \{G\}$ and the new partially constructed informon is $[n'] < \alpha' > \{G'\}$, then the new content becomes $G' \cup G \cup \{[n] < \alpha > \{G\}\}$. Second, it can forget all but the current information and the new content becomes $G' \cup \{[n] < \alpha > \{G\}\}$. Third, it can pass forward only a subset of the current information so that the new content becomes $G' \cup \hat{G} \cup \{[n] < \alpha > \{G\}\}$ where $\hat{G}$ is an up-set of G, or fourth, it can forget all information and the content remains G'. That can occur in the context of an interaction or when the particle or system is in a superposition state.

3. After the initial choice of a new informon to be constructed, Player II must choose the causal manifold embedding point. This can again be made in a deterministic, non-deterministic or stochastic manner. The choice will determine whether the embedding is uniform or non-uniform and thus the nature of the space-like hypersurface

into which the global causal tapestry embeds. This in turn impacts on the nature of the interpolation which is to be subsequently carried out and on the goodness of fit of the resulting wave function. From a modeling point of view, a uniform embedding strategy makes it more amenable to mathematical analysis. It allows for the use of sinc interpolation, for which there is a rich literature. It allows for easier estimations of accuracy. It allows for easier calculations of token values. However, a non-uniform strategy is more likely to be accurate as a model of a fundamental reality. Although a non-uniform strategy can be used with sinc interpolation [252], one really needs to consider the theory of reproducing kernel Hilbert spaces [84], Feichtinger-Gröchenig theory [436], and the theory of interpolation on graphs [201]. It is a much greater challenge to find *local* non-uniform interpolation schemes. There are several non-uniform interpolation procedures in existence but the frame functions in such instances all contain non-local information. These are satisfactory mathematically but not if one wishes to support he search for quasi-local models of the fundamental level of reality.

4. Player I must construct the tokens which are then placed on the embedding point of the causal manifold (metaphorically, not literally of course) and from which the $\mathcal{H}(\mathcal{M})$-interpretation is built. These tokens provide information linking a current informon to a new informon. In Chapter 3, the $\mathfrak{PI}$ strategy was presented. There each token represents a contribution to the path integral calculation, being a weighted exponential of the action linking the current and new embedding points. Other methods of constructing these tokens are possible such as having each token represent the kernel or propagator between the current and new embedding points, direct sampling of the wave function (not ideal since it is presumed to be generated, not given, but useful for formal comparisons) or using local approximants based upon the wave function. It is also possible to develop an interpolation theory using the value of the wave function at a point together with its derivative there (or an approximation to both). The selection of tokens determines the degree to which the weight on the interpolation element approximates the ideal value of the wave function at the embedding point, and thus the accuracy of the interpolation, particularly in the non-asymptotic region.

5. Finally Player II must construct the $\mathcal{H}(\mathcal{M})$-interpretation. It may do so in part during each short round or it may wait until all of the short rounds have been completed and use the final token set. However it does it, it will determine a weight based upon the set of tokens present on the embedding site and combine it with a local interpolation function. The choice of function depends upon the totality of strategies preceding its incorporation. Certain strategies lend themselves to sinc interpolants.

Table 5.3: General Strategy Constraints.

Informon	Content	Embedding	Token	Interpolant
Det	Complete	Uniform	Action	Sinc
Non-Det	Minimal	Nonuniform	Kernel	Sp Function
Guidelines	Selective	Mixed	Approximate	F-G Method

Other strategies are best implemented using other special functions or sometimes infinite products or series. The presence of certain boundary conditions or symmetries may also make particular frame functions more useful. In every case, these interpolants are meant to express the fuzziness of these informons to an observer that exists at higher ontological levels and which is fundamentally incapable of resolving each informon exactly. This fuzziness can be expected to take on different forms depending upon the physical symmetries present in the situation and the boundary conditions and Player II must take these into account. As well, the choice of particular frame elements will also determine the degree of accuracy of the interpolation in both the asymptotic and far from asymptotic regions.

The specification of strategies for each of these steps provides a specification of the strategy for the global game, and hence the game itself. Let us consider some possible choices of strategies. Later some specific strategies will be examined in more detail and in particular how they relate to one another through the different equivalence relations defined above.

Table 5.3 provides a collection of considerations which may be used to construct a general strategy. Simply select one entry from each column to choose a category of strategy. For example, the free path integral game of Chapter 3 would be described as (Nondeterministic, Complete, Uniform, Action, Sinc). Note that there are $3^5 = 243$ different classes. More classes are possible but these cover the most promising model types.

Not all possible strategy classes are expected to yield plausible models. The most promising would appear to lie within the collections

$$\text{Non-Det} \times \left\{\frac{\text{Minimal}}{\text{Selective}}\right\} \times \text{Nonuniform} \times \left\{\frac{\text{Action}}{\text{Kernel}}\right\} \times \left\{\frac{\text{Sinc}}{\text{F-G Method}}\right\}$$

5.2 Radiative Models

In this section I wish to focus briefly upon a collection of strategies that appear to be worthy of future research. I have termed them radiative models, although a more accurate term might be bounded information models, but the radiative adjective provides a better intuitive association. These are all models in which information spreads outwards from a current informon but in a localized or limited fashion. In some models this information spreads just to a neighbourhood surrounding the current informon. In others it spreads more widely but to only a limited number of sites. In all cases, the radiative strategies determine the method by which informons are selected for information transfer and place different constraints on these pairings. In particular there are bounds placed upon the number of nascent informons to which information from a current informon may spread. Thus unlike the free strategies $\mathfrak{P}\mathfrak{I}$ and $\mathfrak{K}$, these are all bounded strategies and the value of the r parameter becomes significant. The method for selecting tokens is left unspecified in general but for illustrative purposes I shall generate tokens using the kernel. I wish to consider three specific models in order to illustrate the equivalence ideas described in the previous section.

The three models are:

1. Deterministic radiative strategy
2. Non-deterministic radiative strategy
3. Bounded information strategy

As before, for ease of analysis I am going to consider all of these in the case of uniform embedding, kernel based tokens and sinc interpolants.

5.2.1 Deterministic Radiative Strategy

The deterministic radiative strategy $\mathfrak{D}\mathfrak{R}$ has the same essential components as the $\mathfrak{K}$ strategy but with an additional integer parameter R' which represents the half diameter of a cube centered on a lattice site. The parameter $r = (2R' + 1)^3$.

Let (t_n, x, y, z) denote a generic lattice site in which the current causal tapestry embeds. Construct an ordered sequence of adjacent lattices sites

$$\{(t_n + t_P, x + il_P, y + jl_P, z + kl_P)| - R' \le i, j, k \le R'\}$$

where the ordering is fixed for all x, y, z.

A short round of the game is played as follows:

1. Player I first selects an informon from the current tapestry $\mathcal{I}$ not previously chosen.
2. Player II then selects the informon which embeds to the site next in the site sequence for the current informon, or if none exists, selects a label n not previously used and creates a new informon $[n] < (, , , \mathbf{p}_n, \lambda_n) > \{\}$.
3. Player I then provides or updates the content set G_n.
4. Player II then assigns the new informon the site value according to the sequence unless it has been previously assigned.
5. Player I then creates and places a token on the site l_n using whatever kernel method.
6. Player II then updates the $\mathcal{H}(\mathcal{M})$ interpretation $< (\mathbf{m}_n, \Phi_n, \Gamma_n, \mathbf{p}_n, \lambda) >$ using a sinc interpolant.
7. The short round is then repeated using the next site in the sequence and continues until the end of the sequence has been reached. The round then ends.
8. If a round ends then the entire process repeats with Player I picking a new unused current informon and the rounds continue until either no current informons remain or the limit r in the number of rounds has been reached.

Suppose that the process begins at a single informon. The information associated with that informon will spread in the next game play to the $(2R' + 1)^3$ adjacent lattice sites. In the next play of the game it will extend to the $[(2R' + 1)^3]^2$ nearby lattice sites and so on. Information thus radiates outward from each current informon with each game play.

The tokens on a site in the new lattice will come from the $(2R' + 1)^3$ adjacent sites referenced back to the current tapestry. If the new informon $[n'] < \mathbf{m}_{n'} > \{G'\}$ lies at site $\mathbf{m}_{n'} = (t_n + t_P, x, y, z)$ then set $S_{ijk} = (t_n, x + il_P, y + jl_P, z + kl_P)$ so that its tokens will consist of the set

$$\{l_P^3 K(\mathbf{m}_{n'}, S_{ijk})\phi_n(S_{ijk})| - R' \le i, j, k \le R'\}$$

so that the local $\mathcal{H}(\mathcal{M})$-interpretation will have the form

$$\Phi_{n'}(\mathbf{z}) = \left\{ \sum_{-R' \leq i,j,k \leq R'} l_P^3 K(\mathbf{m}_{n'}, S_{ijk}) \phi_n(S_{ijk}) \right\} T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

and the global interpretation on the embedding slice L' will have the form

$$\Phi_{n'}(\mathbf{z}) = \sum_{\alpha' \in L'} \left\{ \sum_{-R \leq i,j,k \leq R} l_P^3 K(\mathbf{m}_{n'}, S_{ijk}) \phi_{n'}(S_{ijk}) \right\} T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

It should be obvious from the argument in Section 7.3 that in the asymptotic limit this will converge to the usual NRQM wave function on the embedding slice. Hence $\mathfrak{DR}$, $\mathfrak{PI}$ and $\mathfrak{K}$ are weak-epistemic-equivalent. It should be obvious that they will not be process, ontic, or strong epistemic equivalent, nor isomorphic, since in the $\mathfrak{DR}$ strategy the number of informons in the new causal tapestry grows as $((2R'+1)^3)^n$ whereas in the $\mathfrak{FA}$ and $\mathfrak{K}$ strategies it remains constant, bounded by r. Moreover, in the latter strategies, the new informons may be distributed randomly throughout the lattice, whereas in the $\mathfrak{DR}$ strategy they cluster around embedding sites of the current tapestry.

5.2.2 Non Deterministic Radiative Strategy

The non deterministic radiative strategy game $\mathfrak{NR}$ has the same essential components as the game $\mathfrak{DR}$ but in this case instead of a sequence, the adjacent lattice sites are formed into a set which is accessed non deterministically.

Let (t, x, y, z) denote a generic lattice site in which the current causal tapestry embeds. Construct a set of adjacent lattices sites

$$\{(t + t_P, x + il_P, y + jl_P, z + kl_P) | -R' \leq i, j, k \leq R'\}$$

A short round of the game is played as follows:

1. Player I first selects an informon from the current tapestry $\mathcal{I}$ not previously chosen.
2. Player II then selects an informon which embeds to a site in the site set for the current informon which has not been used, or if none exists, selects a label n not previously used and creates a new informon $[n] < (,,, \mathbf{p}_n, \lambda_n) > \{\}$.

3. Player I then provides or updates the content set G_n.
4. Player II then assigns the new informon the site value according to the sequence unless it has been previously assigned.
5. Player I then creates and places a token on the site α_n using whatever method.
6. Player II then updates the $\mathcal{H}(\mathcal{M})$ interpretation $<(\mathbf{m}_n, \Phi_n, \Gamma_n, \mathbf{p}_n, \lambda_n)>$ using a sinc interpolant.
7. The short round is then repeated using the next site in the sequence and continues until the end of the sequence has been reached. The round then ends.
8. If a round ends then the entire process repeats with Player I picking a new unused current informon and the rounds continue until either no current informons remain or the limit r in the number of rounds has been reached.

As before, if the process begins at a single informon then the information associated with that informon will spread in the next game play to the $(2R'+1)^3$ adjacent lattice sites. In the next play of the game it will extend to the $[(2R'+1)^3]^2$ nearby lattice sites and so on. Information thus radiates outward from each current informon with each game play.

The tokens on a site in the new lattice will again come from the $(2R'+1)^3$ adjacent sites referenced back to the current tapestry. If the new informon $[n'] < \alpha' > \{G'\}$ lies at site $\mathbf{m}_{n'} = (t_n + t_P, x, y, z)$ then set $S_{ijk} = (t_n, x + il_P, y + jl_P, z + kl_P)$ so that its tokens will consist of the set

$$\{l_P^3 K(\mathbf{m}_{n'}, S_{ijk})\phi_n(S_{ijk}) | -R' \leq i,j,k \leq R'\}$$

so that the local $\mathcal{H}(\mathcal{M})$-interpretation will have the form

$$\Phi_{n'}(\mathbf{z}) = \left\{ \sum_{-R' \leq i,j,k \leq R'} l_P^3 K(\mathbf{m}_{n'}, S_{ijk})\phi_n(S_{ijk}) \right\} T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

and the global interpretation on the embedding slice L' will have the form

$$\Phi_{n'}(\mathbf{z}) = \sum_{\alpha' \in L'} \left\{ \sum_{-R' \leq i,j,k \leq R'} l_P^3 K(\mathbf{m}_{n'}, S_{ijk})\phi_{n'}(S_{ijk}) \right\} T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

It is obvious that $\mathfrak{DR}$ and $\mathfrak{NR}$ generate the same set of informons from a given current causal tapestry, although the order in which they are generated will be different. Thus $\mathfrak{DR}$ and $\mathfrak{NR}$ are ontic, strong epistemic and weak epistemic equivalent but they are not process equivalent.

$\mathfrak{DR}$ and $\mathfrak{NR}$ will share the same asymptotic and accuracy properties as regards the Hilbert space interpretation.

5.2.3 Bounded Information Strategy

The bounded information strategy $\mathfrak{BI}$ again bears many similarities to $\mathfrak{DR}$ and $\mathfrak{NR}$. From any given current informon, connections can form to at most $(2R'+1)^3$ new informons, but these informons may be distributed anywhere in the new embedding lattice. The bounded information strategy fully exploits the jaunting nature of non deterministic selection during game play.

At the beginning of game play, each current informon is given exactly $(2R'+1)^3$ blank tokens. Recall that each round generates a single new informon. So each short round contributes tokens to a single informon. Suppose that each short round unfolds as follows:

1. Player I nondeterministically selects an informon from the current tapestry $\mathcal{I}$ not previously chosen and still possessing blank tokens.
2. If a new informon is in play, Player II does nothing. Otherwise Player II either nondeterministically selects an informon from the new tapestry which has not been previously linked to the current informon and which has fewer than $(2R'+1)^3$ tokens or selects a label n not previously used and creates a new informon $[n] < (,,,\mathbf{p}_n, \lambda_n) > \{\}$. In either case this new informon is put into play.
3. Player I then provides or updates the content set G_n.
4. Player II then nondeterministically assigns a site value to any newly created informon and leaves any existing value unchanged.
5. Player I then creates and places a token on the site α_n using the kernel method.
6. Player II then updates the $\mathcal{H}(\mathcal{M})$ interpretation $< (\mathbf{m}_n, \Phi_n, \Gamma_n, \mathbf{p}_n, \lambda_n) >$ using a sinc interpolant.

7. The short round is then repeated until either a) the total number of tokens on the site of the new informon currently in play reaches the maximum of $(2R'+1)^3$, b) there are no informons in the current tapestry that have not been matched to the informon under construction or c) there are no more current informons that still possess tokens. At that point the current round ends and a new round is begun.

8. The entire process then repeats with Player I picking a current informon still possessing tokens and the rounds continue until either no current informons possess tokens or the limit r in the number of rounds has been reached.

As before information will spread out in a random appearing non-deterministic manner. Each current informon begins with $(2R'+1)^3$ blank tokens and short rounds continue until the current round ends, which occurs when the new informon under construction has acquired at most $(2R'+1)^3$ tokens.

It is apparent that in such a model, a system starting as a single informon will generate $(2R'+1)^3$ new informons. In the case that there are multiple informons in the current tapestry, each informon in the current tapestry will give rise to at most $(2R'+1)^3$ new informons in the new tapestry. However it is also possible that the will simply contribute to a much smaller set but any new informon will only receive tokens from at most $(2R'+1)^3$ current informons. The current and new tapestries can be formed into a graph such that every current informon has order $(2R'+1)^3$ and every new informon has order at most $(2R'+1)^3$.

The local $\mathcal{H}(\mathcal{M})$-interpretation no longer has a simple form because the number of tokens may vary from informon to informon. The best one can say is

$$\phi_{n'}(\mathbf{z}) = \sum \{\text{tokens at site } \alpha'\} l_P^3 T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

and the global interpretation on the embedding slice L' will have the form

$$\Phi_{n'}(\mathbf{z}) = \sum_{\alpha' \in L'} \sum \{\text{tokens at site } \alpha'\} l_P^3 T_{\mathbf{m}_{n'}} sinc_{t_P l_P}(\mathbf{z})$$

One should note that the games trees for $\mathfrak{DR}$ and $\mathfrak{NR}$ are subtrees of the game tree for $\mathfrak{BI}$. They are all weak epistemic equivalent but neither $\mathfrak{DR}$ nor $\mathfrak{NR}$ is process, ontic or strong epistemic equivalent to $\mathfrak{BI}$.

5.3 Initial Conditions and Time Dependence

The astute reader may have noticed in Section 3.4 that the proof of the emergence of NRQM from the process model in the asymptotic limit required that the initial causal tapestry I_0 possess a global $\mathcal{H}(\mathcal{M})$-interpretation, $\Phi_0(\mathbf{z})$ such that

$$\Phi_0(\mathbf{z}) = \sum_{a_0 \in I_0} \phi_{a_0} T_{\mathbf{m}_{a_0}} sinc_{t_P l_P}(\mathbf{z}) = \sum_{a_0 \in I_0} \Psi_0(a_0) T_{\mathbf{m}_{a_0}} sinc_{t_P l_P}(\mathbf{z}) = \Psi_0(\mathbf{z})$$

where $\Psi_0(\mathbf{z})$ is a standard NRQM wave function satisfying the Schrödinger equation,

$$i\hbar \frac{\partial \Psi_0}{\partial t} = H\Psi_0$$

The necessity for the quantum system to already exist in a proper quantum state when the game is initialized is a defect of the $\mathfrak{PI}$ strategy but it is a problem for NRQM in general. This problem is better addressed within the framework of quantum field theory but that is beyond the scope of this thesis which focusses solely upon NRQM. Consider the case in NRQM of a time independent Hamiltonian, H. If the quantum system begins in a superposition of energy eigenstates $\sum_i w_i \Psi_i$ then the solution is given as

$$\Psi = \sum_i w_i e^{-i\lambda_i t/\hbar} \Psi_i(\mathbf{x})$$

Note that on account of time reversal symmetry, the wave function exists for *all* of time, to the infinite past and to the infinite future. It *never* comes into being and it *never* fades from existence. Of course the Hamiltonian may be time dependent, but even in that case it is a general principle of quantum mechanics that the evolution of the wave function will be unitary, *unless* it undergoes a collapse under the influence of a measurement. Again the wave function should exist for all times.

This is ideal from the standpoint of signal theory since it is not possible for an entire function that is band-limited in spatial and temporal frequencies to have bounded spatial and temporal domains (that is, to be zero outside of bounded regions). This is a bit of a problem for photons at least, which appear to come into being and then disappear, especially if they are to have bounded energy and momentum. It is also a problem for a generative theory such as in the process framework. One way around this is to abandon the principle of continuity, to abandon the notion that wave functions must be entire.

Thus NRQM does not easily describe the situation of particle creation and annihilation. If it tries, then the particle must arise and be expressed throughout the whole of space, from its creation to its annihilation. If the particle can be localized to some bounded region of space, then it must exist throughout all of time. Neither possibility holds true for many fundamental particles.

One advantage for a generative theory like those within the process framework is that, causal tapestries being discrete, the $\mathcal{H}(\mathcal{M})$-interpretations do not need to exist for all time. They can come into existence, they can fade from existence. Indeed it is only necessary that they agree with NRQM in those regions where the process model posits the existence of informons. Outside of those regions the quantum system can be viewed as non-existent, and the apparent manifestation of the wave function is an artifact of the fuzziness of the $\mathcal{H}(\mathcal{M})$-interpretations. Indeed, if the informons of the causal tapestry are restricted to a bounded region of either space or time, the $\mathcal{H}(\mathcal{M})$-interpretation will become vanishingly small as one moves further and further from the region. Thus in the process models, the wave function is *effectively* time and space bounded even if not strictly so.

Note that causal tapestries, being discrete structures, do not suffer the problems associated with most continuous models. Any association of interpolation weights with the points on the embedding lattice will, through the interpolation process, correspond to a complex function in the Hilbert space over the causal manifold. They will not necessarily correspond to a function within a specific class of functions such as $L^p(\mathcal{M})$ or $B^n(\mathcal{M})$. Since NRQM is viewed as an effective theory emerging under certain asymptotic limits, it is not necessary that every Hilbert space interpolation for every game strategy be itself a satisfactory NRQM wave function. We do require that such functions should provide acceptable approximations to the NRQM wave function, the definition of acceptable depending upon the particular circumstances. In the end one would hope that empirical evidence would suggest what degree of approximation is necessary in order for the game strategy to be considered acceptable.

In the case of a time dependent Hamiltonian the usual approach is through perturbation theory. The development of a full perturbation is well beyond the scope of this present work. But recall from the discussion of Section 2.3 that it is not possible to experimentally determine the wave function of a time dependent Hamiltonian system based upon the observation of a single particle, unless the time evolution is extremely slow or the Hamiltonian is periodic. This is because at any given time a single particle will only ever manifest at a single spatial location and from this it is impossible to deduce a probability distribution or indeed any spatially distributed function. Moreover, unless the Hamiltonian is periodic, in which case sampling could occur at the frequency of the Hamiltonian and a probability distribution built up from repeated observations, the Hamiltonian will never repeat, and so one can never obtain repeat samplings of the same probability distribution. It is only in the case of an ensemble that a measurement of the wave function of a time dependent Hamiltonian appears possible. The perturbation techniques become possible because in a certain sense the wave function is thought of as static object in spacetime to which various manipulations become possible. In the case of a causal tapestry, however, the wave function is presumed real, and is generated non-deterministically across each space-like hypersurface of the causal manifold. The wave function of a single particle remains a real wave, expressing the strength of the generating process at each space-time location and as such a method is needed which effectively allows the wave function of a time dependent Hamiltonian to be generated in 'real ' time.

The problem remains as to how such an initial wave function arises in the first place. The discussion presented in the beginning of this section shows that the wave function must exist throughout any space-time slice and this cannot be generated in NRQM, it simply *is.* In the process model, however, we want to be able to actually generate this from an initial seed. The next section provides a tentative model for such a bootstrapping process.

5.4 Radiative Bootstrap Strategy

The idea of the radiative bootstrap strategy is to provide a strategy capable of generating, at least under some conditions, a proper NRQM wave function in the asymptotic limit beginning with a local seed function that need not itself be a NRQM wave function. This section is included in the interest of thoroughness and to promote future studies but is not necessary for the main argument. Hence only a sketch of the approach is presented. The idea is that a process begins by initiating a single informon and then the relevant information spreads out during subsequent plays of the reality game. As a result of game play, an $\mathbb{H}(M)$-interpretation is built up on subsequent causal tapestries and converges to an approximation of a proper NRQM wave function. One would hope that the rate of convergence to such an approximation is sufficiently fast and that in the asymptotic limits a proper NRQM wave function will emerge.

The Feynman-Hibbs approach requires that the NRQM wave function satisfy the integral equation

$$\Psi(t', \mathbf{x}') = \int_{\mathcal{M}} K((t', \mathbf{x}'), (t, \mathbf{x}))\Psi(t, \mathbf{x})d(t, \mathbf{x})$$

where $K((t', \mathbf{x}'), (t, \mathbf{x}))$ is the usual path integral kernel derived from the Lagrangian.

An alternative approach can be derived by first converting the Schrödinger equation itself to an integral form. Byron and Fuller [76] derive this form, which is given as

$$\Psi(\mathbf{r}) = \xi(\mathbf{r}) + \int G_0(\mathbf{r}, \mathbf{r}', E)V(\mathbf{r}')\Psi(\mathbf{r}')d\mathbf{r}' = \xi(\mathbf{r}) + \int K(\mathbf{r}, \mathbf{r}', E))\Psi(\mathbf{r}')d\mathbf{r}'$$

where

$$G_0(\mathbf{r}, \mathbf{r}', E) = -\frac{m}{2\pi\hbar^2}\frac{e^{iq|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

is the Green's function and $\xi(\mathbf{r})$ is the solution to the homogeneous equation

$$(\nabla^2 + q^2)\xi(\mathbf{r}) = 0$$

Note that $\xi(\mathbf{r}) = 0$ in the bound case and $\xi(\mathbf{r}) = \frac{A}{(2\pi)^{3/2}}e^{i\mathbf{q}\cdot\mathbf{r}}$ in the scattering case.

Let the operator K be defined by

$$K\Psi(\mathbf{r}) = \int K(\mathbf{r}, \mathbf{r}', E)\Psi(\mathbf{r}')d\mathbf{r}' = \int -\frac{m}{2\pi\hbar^2}\frac{e^{iq|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}V(\mathbf{r}')d\mathbf{r}'$$

Under certain conditions on the potential V it may be possible to solve the above equation through the Born series

$$\Psi(\mathbf{r}) = \xi(\mathbf{r}) + K\xi(\mathbf{r}) + K^2\xi(\mathbf{r}) + K^3\xi(\mathbf{r}) + \cdots$$

Consider again the free kernel strategy of the previous chapter.

Player I laid down tokens on the new embedding site in the causal manifold which were derived from information involving the current and new informons that were in play during the current short round. This current informon was added to the content set of the new informon at an earlier step. Player II then constructed the $\mathcal{H}(\mathcal{M})$-interpretation of the informon based upon the tokens. Note that the content set was not used in the creation of the interpretation of the informon. Moreover, since only the informons of the current tapestry are required to construct the interpretation, the content set could consist of all prior causally related informons, or just those form the current tapestry. The global $\mathcal{H}(\mathcal{M})$-interpretation, the interpolated wave function, is constructed from the $\mathcal{H}(\mathcal{M})$-interpretations of the individual informons.

There is another way of constructing the global interpolation and that is to use the information residing in the content set, which actually is the information of the informon. The full power of the content set was not used in the simple case shown in Section 3.3. There the content of the content set was ignored but note that if one were to construct a content interpretation by summing the $\mathcal{H}(\mathcal{M})$-interpretations of all informons in the content, then one would obtain the global $\mathcal{H}(\mathcal{M})$-interpretation of the now prior causal tapestry. In the time independent situation this past information is not needed for the global interpretation. But what of the case where we wish to bootstrap from an initial condition or where the Hamiltonian is time dependent? There it is likely that past information will be important in the generation of the current global interpretation.

Let us associate each term of the Born series with a causal tapestry, so that each application of the integral operator corresponds to a complete play of a game. The time dependence then translates into there potentially being a different game played in the creation of each successive tapestry.

The Born series is constructed on the spatial component $\mathbb{R}^3$ of the causal manifold $\mathbb{R} \times \mathbb{R}^3$. The interpretations will be defined solely on the spatial component and so our sinc functions will be 3d.Consider the Born series up to the nth term.

$$\Psi(\mathbf{r}) = \xi(\mathbf{r}) + K\xi(\mathbf{r}) + K^2\xi(\mathbf{r}) + K^3\xi(\mathbf{r}) + \cdots + K^{n-1}\xi(\mathbf{r})$$

If one samples this series on the sublattice L' of $\{(n-1)t_P\} \times \mathbb{R}^3$ then using Parzen's theorem one can write $\Psi(\mathbf{r}) =$

$$\left[\sum_{n \in L'} \xi(\mathbf{x}_n) + K\xi(\mathbf{x}_n) + K^2\xi(\mathbf{x}_n) + K^3\xi(\mathbf{x}_n) + \cdots + K^{n-1}\xi(\mathbf{x}_n)\right] T_{\mathbf{x}_n} sinc_{l_P}(\mathbf{r})$$

Let us apply the free kernel strategy. Let the initial causal tapestry $\mathcal{I}_0$ possess informons whose interpretations take the form $\alpha_n = ((0, \mathbf{x}_n), \xi(n) T_{\mathbf{x}_n} sinc_{l_P}(\mathbf{r}))$ and embed into the lattice L_0. After game play there will be a new causal tapestry $\mathcal{I}_1$ whose informons will have interpretations of the form $((t_P, \mathbf{x}_n), \Phi^1_n(\mathbf{r}))$ where $\Phi^1_n(\mathbf{r}) =$

$$\left[\sum_{m \in L_0} K(n,m)\right] l_P^3 \Phi^0_m T_{\mathbf{x}_n} sinc_{l_P}(\mathbf{r}) = \left[\sum_{m \in L_0} K(n,m)\right] l_P^3 \xi(\mathbf{x}_n) T_{\mathbf{x}_n} sinc_{l_P}(\mathbf{r})$$

$$\approx K\xi(\mathbf{x}_n) T_{\mathbf{x}_n} sinc_{l_P}(\mathbf{r})$$

Continuing in this way we see that after n plays, $\Phi^n_{\mathbf{x}_m}(\mathbf{r}) \approx K^n \xi(\mathbf{x}_m) T_{\mathbf{x}} sinc_{l_P}(\mathbf{r})$.

Let the content of the informon be $G_{\mathbf{x}}$. We may form the Hilbert space interpretation of the content as $\Phi_{G_{\mathbf{x}}} = \sum_{k=0}^{n-1} \sum_{y \in C_{\mathbf{x}}} \Phi^k_{\mathbf{y}}$ where for simplicity each informon is referred to by its causal manifold embedding point. If we evaluate this interpretation at the point $\mathbf{x}$ then notice that because of the *sinc* terms, only the contributions at the point $\mathbf{x}$ will survive. Hence $\Phi_{G_{\mathbf{x}}}(\mathbf{x}) = \sum_{k=0}^{n-1} \Phi^k_{\mathbf{x}}(\mathbf{x}) \approx \sum_{k=0}^{n-1} K^k \xi(\mathbf{x})$. Therefore we take the $\mathcal{H}(\mathcal{M})$-interpolation of the wave function to be

$$\Phi(\mathbf{z}) = \sum_{\alpha \in L'} [\Phi^n_\alpha(\alpha) + \Phi_{G_\alpha}(\alpha)] T_{\mathbf{x}_\alpha} sinc_{l_P}(\mathbf{z})$$

As $t_P, l_P \to 0$ the sums will more closely approximate the integrals while as the number of games played tends to infinity the wave function will trend to the sum of the Born series. If the series converges rapidly then the difference between the interpreted wave function and the NRQM wave function may nearly vanish after only a few plays of the game. If t_P is close to the Planck time, then there will be approximately 10^{44} plays of the game every second, so that in all likelihood the interpreted wave function will effectively match the NRQM wave function. Of course in general the problem will not be simple. The important point though is that, being discrete, there will always be an evolution of the causal tapestries regardless of whether or not there is convergence in the asymptotic regime. Where NRQM breaks down it is possible that the causal tapestry may remain viable and the question becomes under what conditions one has the appropriate convergence and under what conditions NRQM might break down. There is also a need to develop a proper collection of perturbation methods to deal with a broad range of potentials, initial and boundary conditions and time dependencies. Those questions are beyond the scope of this present work.

5.5 Absolute Versus Relative Generation

In the models presented above there has been little discussion about the sampling parameter values t_P and l_P beyond the fact that NRQM emerges in the asymptotic limit as $t_P, l_P \to 0$ and that in the $\mathfrak{PI}$ strategy (and in most strategies generally) they are taken to be the Planck time and Planck length respectively. There are two major approaches to these values. In the absolute approach, one takes these as universal values holding for all physical processes at the fundamental level. This approach is similar to that used in cellular automaton models of the fundamental level [429], where there is a fixed lattice decomposition of space-time and events unfold upon this lattice. Of course, such an approach fails when the lattice is no longer uniform, and indeed sinc interpolation fails generally in the case of non-uniform sampling (although it still does remarkably well [252]) and must be replaced by a more sophisticated approach. The absolute approach is advantageous because it places all games on an equal footing. The frequencies associated with each individual process lie within the dynamics of the process and the choice of absolute values for t_P and l_P that lie close to the Planck values accords nicely with the idea that below the Planck scales reality as we know it breaks down. These values then acquire a more significant status as representing fundamental scales for space and time. That is natural and appealing.

There is an alternative approach which links the sampling frequencies to the dynamical properties of the process. In this case, one links t_P to the energy of the process and each momentum component p_x becomes linked to a spatial scale, l_P^x. So long as these sampling frequencies exceed twice the Nyqvist frequency, one will obtain an adequate NRQM wave function in the asymptotic limit. These can be set via the usual Feynman prescription of $\omega = E/\hbar$ and $\lambda_x = p_x/\hbar$, so the sampling frequencies could be 2ω and $2\lambda_x$. Other functional relationships could be imagined, however.

There is some appeal to using the relative approach since it more closely links dynamical aspects of process to the generation of causal tapestries. There is some danger, however, that this approach might run afoul of the uncertainty relations since it posits the possibility of definite energy and momentum for quantum systems, whereas in the absolute approach, position, time, energy and momentum are all allowed to exist across some range (though in the case of energy this range may be discrete). It might be possible to test these two possibilities since in the relative approach, especially if the Nyqvist frequencies are used, the accuracy is expected to improve as the frequencies increase, i.e. as energy and momentum increase, since $E = 2\pi\hbar/t_P$ and $p = 2\pi\hbar/l_P$. This is an area for future research.

Chapter 6

Conclusions

Richard Feynman once said that nobody understands quantum mechanics. I believe that this statement holds to the present day. Most interpretations of quantum mechanics either argue that questions regarding the ontological status of quantum mechanical notions are unanswerable, or they paint a picture of reality which is far removed from personal experience. Following upon a series of seminal results constituting the "hidden variable" theorems, quantum mechanics paints a picture of reality which is nonlocal, contextual and non-Kolmogorov. This has created a schism between the classical and quantum depictions of reality that to date has found no reconciliation. The attempts to reconcile these opposing perspectives have resulted in ever more mathematically complicated and convoluted theories which run counter to intuition and personal experience and have little if any experimental support.

The main goal of this thesis was to explore whether the conception of reality as proposed by quantum mechanics was truly necessary and demanded by experiment and theory. In the Appendices I have tried to show that many of the assumptions underlying our formal conception of the classical realm are false in general and are based on a primitive notion of object which in turn was based on observations of inanimate matter. The study of biological, psychological and social systems that began in the 19th century, leading in the 20th century to the development of complex systems theory, has rather strikingly shown that the concept of object is too narrow and confined to be useful generally. Instead the description and understanding of these systems requires the use of ideas such as development, generation, emergence, and process. Following the lead of Robert Rosen, I wondered to what extent these complex systems ideas could be brought down and applied to the fundamental levels of reality and if so, what they might say about the nature of ultimate reality.

Starting with Whitehead's theory of process, which was an early 20th century attempt to philosophically ground these other disciplines, I constructed a formal model of process and described the basic mathematical structures involved in their specification and analysis. I proposed that process be viewed as constructing a finite (but large), discrete collection (causal tapestry) of predominantly informational elements (informons), from which emerges both space-time and the physical entities manifesting within it. I constructed a self contained dynamic on this collection of informons and showed how these elements could be interpreted as elements of a causal manifold (space-time) and as contributing interpolation elements to a wave function defined on this manifold (physical entity). This wave function was interpreted as a real wave propagating from one causal tapestry to another (and hence on the causal manifold) akin to a discrete wave or diffusion process.

The algebraic structure of the process space was described. Processes were shown to be either active or inactive and possessing a rich set of interactions. Processes generate space-time but are not themselves situated in space-time. They are alocal. It was shown that the connection to the standard wave function of NRQM was through the process covering map, which is a set-valued map from the process space to some Hilbert space. Two distinct process covering maps were described. The first map was defined on the Hilbert space over the causal manifold and provided a space-time representation of the entities emergent on the space of informons. These wave functions are interpreted as real physical waves, the value (strength) of the wave function at a given informon being used to determine the likelihood of this process coupling with a second active process when that informon manifests. The second map was defined on the Hilbert space over the configuration space and provided a tool by which correlational and statistical information could be calculated. In the process model this second map, and the associated configuration space, do not have any ontological status and are purely epistemological in character.

Processes were then represented as combinatorial games, taking advantage of the algebraic isomorphism available between these two spaces. Several game strategies were discussed, the most important being the path integral strategy and the kernel strategy. Concrete models of primitive processes were presented. These consisted of a game played out in a series of rounds during each of which a single informon was created. This informon was generated using only causally local information derived from prior causal tapestries (never from informons within the causal tapestry being generated and so space-like separated) and using a classical representation of information propagation. The choice of the location of each informon to be generated was determined in a non-deterministic, jaunting manner, which appeared temporally and spatially discontinuous or non-contiguous. This jaunting action never involved space-like information transfer and so does not represent a form of nonlocality as postulated by the hidden variable theorems. There is no conflict with special relativity. Thus the model provided a set of classical looking hidden variables which evolved using only local information, albeit by means of an alocal process. This gives rise to the term quasi-local to describe the reality generated by process. Note that this is possible because of the discontinuous action of process. This is actually not a great departure from classical ideas except as regards evolution in time, which is usually held to be continuous. The only quantum aspect of these process models is that the action is discontinuous.

Repeated actions of process generate a succession of causal tapestries which embed as space-like hypersurfaces in the causal manifold and so constitute a form of compound present. This too does not depart from classical notions although it does depart from the usual view that reality is organized into a block universe. That this presentist view is also consistent with special relativity was discussed in the Appendices.

Each informon was assigned a set of definite properties having definite values, and these were inherited from the process generating them. It was shown that this set could not be complete, however, due to the non-commutative nature of process concatenation. Contextuality arises whenever one attempts to measure something as it involves an interaction between a measurement apparatus process and the system under consideration. That these two processes interact, and that such an interaction will generically alter the process generating the system and therefore some of its properties, should not be so surprising. After all, this is a fact of life in biology, psychology and the social sciences. It is important to note that this limited non-contextuality is a result of informons being generated by processes rather than simply existing.

In addition it was shown that the probability structure applies not to individual informons but rather to interactions between processes as outlined in the process theory of measurement. There it was shown that process strength served to determine the likelihood of an interaction taking place between or among a collection of processes. These interactions give rise to measured values, and correlations among these measured values could be determined from an emergent non-Kolmogorov probability structure. The process model thus provided an explicit example of a classical type hidden variable model which possesses a non-Kolmogorov probability structure.

I showed that using either the path integral or kernel strategies, the NRQM wave function appears in the asymptotic limit of infinite spatial distribution and infinite information. From a process perspective, NRQM is to be understood as an idealization or effective theory, valid whenever the infinite limit approximation can be assumed.

I suggested that the distinction between the classical and quantum realms was not a matter of scale (which determines the accuracy of the interpolation), but rather was a reflection of the complexity of process interactions, in particular classicality appeared whenever the process interactions did not permit the formation of superpositions.

The process model suggests that reality at the lowest level, just like reality at macroscopic levels, can remain local and non-contextual while still behaving according to the rules of quantum mechanics. The process framework suggest that by adopting a generative view we can have, at the lowest level, a generated reality which is local and non-contextual. Nevertheless, because it is a generated reality there will arise correlations between these elements due to the interdependence, interrelatedness and interactivity of the generating processes. What is observed -classical, quantum - depends upon the form of the generating process. The process framework thus offers a unifying common language which holds across spatiotemporal scales. The appearance of nonlocality, contextuality and non-Kolmogorovness lies not in the basic elements but rather in the generators, the processes.

I showed that the path integral and kernel strategies led to models that behaved in respect of the hidden variable theorems exactly as NRQM does. Therefore I showed that one can have a form of local and (partially) non-contextual hidden variable (albeit with an alocal generator) theory capable of reproducing the results of NRQM. I also suggested how the process approach could resolve many, if not all, of the paradoxes associated with quantum theory. In particular I showed how the process approach resolves the wave-particle duality problem and how this might be used to resolve other paradoxes such as the two slit, Schrödinger's cat, and delayed choice cases.

I proposed a theory of measurement based upon a notion of interaction between processes, in particular specialized processes termed measurement processes. I showed how this measurement process could reproduce the usual results of NRQM utilizing the emergent non-Kolmogorov probability structure and that measurements could occur without the intervention of observers (other than to choose the measurement apparatus).

A variety of process models were introduced. These models are theoretically testable, as different parameter values will give rise to different degrees of accuracy in interpolating the NRQM wave function and in determining the eigenvalues and probabilities of different measurements. The parameter values chosen to illustrate the models in the thesis give rise to degrees of accuracy well below experimental feasibility, but different models and different parameter values can result in significantly lower degrees of accuracy which might eventually be testable and thus capable of placing bounds on models classes and parameters.

In brief, I suggest that the process model of NRQM provides a unitary framework capable of reproducing, with significant accuracy, the wave functions of NRQM. I believe that it holds promise as a framework within which to develop theories of quantum gravity, though it clearly needs to prove capable of being extended to relativistic and quantum field situations. In its present form it provides a realist interpretation of the wave function and considers the configuration space and the wave function defined upon it to be merely computational tools, while the true source of correlations between systems resides in the interaction structure of processes and thus in the process space. The process model thus completes NRQM.

The process model provides a realist interpretation of NRQM, and that reality unfolds using only local information without the need for spooky action at a distance or superluminal space-like information propagation. It is non-contextual, with limitations imposed by the generating processes. It is observer independent, thus restoring a degree of objectivity back into reality. Nevertheless there is still a role for the observer in that it is the observer who makes the choice of which measurement apparatus to couple with the system under observation. The model propagates information akin to a discrete classical wave or diffusion process. The only quantum aspects lie in its discreteness. It gives rise to an emergent non-Kolomogorov probability structure. To do all of this it is necessary that this reality be generated, emergent, and that processes act in an alocal manner but without nonlocal information transfer. In spite of this it relates to the hidden variable theorems exactly as does NRQM.

The process model shows that the significant issues for a realist model of reality are not nonlocality and contextuality but rather the emergence of space-time and physical entities, and the interdependence, interrelatedness and interactivity of the generating processes.

Part II

Philosophical and Mathematical Background

Appendix A

Hidden Variable Theorems

There was no issue with the notion of reality in classical mechanics. Reality was that which could be directly observed. The observables of physics were generally properties that were either directly accessible to the senses or which could be measured using some apparatus to yield definite and repeatable results. Quantum mechanics changed that, presenting phenomena that could not be readily experienced but instead had to be inferred. Experience was indirect, second hand - theory became the bridge between our experience and the phenomena.

Philosophers lost to classical physicists when it came to metaphysics. With the advent of quantum mechanics, physicists appeared to abandon metaphysics altogether, questioning not just whether metaphysics was possible but whether the concept of reality itself held any meaning. Consider Wheeler's famous dictum:

> No elementary phenomenon is a phenomenon until it is a registered (observed) phenomenon [421, pg. 184].

This point of view has led some to question whether there is a reality which exists apart from us, leading to a form of solipsism.

For a long time physics ignored questions of metaphysics, perhaps put off by Bohr's theory of complementarity, or by simply giving in to the all time frequent exhortation to "shut up and calculate" . Still, some physicists such as Einstein and Bohm persisted in asking questions of metaphysics and with Bell's discovery of his celebrated inequality, the era of experimental metaphysics was ushered in. Since then there have been an ever increasing number of studies directed to questions about the nature of ultimate reality.

Many of these so-called hidden variable theorems are interpreted as being rejections of one form of reality or another, or sometimes of reality entirely. For example, one often reads that the Bell theorems show that either one must accept that quantum mechanics is complete or reject local realism. In this context Norsen points out that when one speaks of realism one cannot speak of naïve realism since that has clearly been shown to be false (nevertheless the model to be presented below exhibits a subtlety in the understanding of what it means for a theory to be contextual). It cannot be about either scientific or perceptual realism, since in the first case one would lose the basis upon which to accept the validity of the argument, and in the second case to accept the validity of the empirical evidence presented in support of the argument. Norsen argues that only metaphysical realism is left. He shows that Bell's theorems support the argument that locality $\rightarrow$ metaphysical realism, so that it is impossible to save locality by rejecting realism since the existence of locality implies metaphysical realism. And in rejecting metaphysical realism Norsen [273] writes:

> and with it any meaningful claims about Locality, the causal structure of the world, and literally everything else that every concept and theory in the entire history of physics has purported to be about (pg 332)

The issue then, for the hidden variable theorems, should not be about whether or not there is realism - there must be realism otherwise science has nothing to talk about - but rather what kind of realism - local, non-local, contextual, non-contextual, objective, process-like etc. Moreover, is it necessary that the kind of realism manifesting at one ontological level be true across all ontological levels and all entities? Might different entities display different characteristics?

To some extent discussing the nature of any fundamental level of reality is challenging from a quantum mechanical viewpoint since quantum mechanics itself is agnostic about any such level - it merely provides a remarkably effective means of determining the statistical properties of the outcomes of various measurements. It enables one to determine probability distributions, expectation values, correlations and the like, but does not address the question of what 'actually happens'. In writing about quantum mechanics, Einstein said:

> What does not satisfy me in that theory, from the standpoint of principle, is its attitude toward that which appears to be the programmatic aim of all physics: the complete description of any (individual) real situation (as it supposedly exists irrespective of any act of observation or substantiation)[180, pg. 2]

It is prima facie evident from abundant experimental research that there are nonzero correlations between measurements of space-like separated quantum systems (as in the case of entangled photons). Schrödinger wrote as early as 1935 that:

> Attention has recently been called to the obvious but very disconcerting fact that even though we restrict the disentangling measurements to *one* system, the representative obtained for the *other* system is by no means independent of the particular choice of observations which we select for that purpose and which by the way are *entirely* arbitrary (Sic) [180]

Measurements depend upon the context of measurement (as evidenced by spin measurements along different axes). Bohr wrote:

> ... a closer examination reveals that the procedure of measurement has an essential influence on the conditions on which the very definition of the physical quantities in question rests [180, pg 27]

There is no dispute over the fact that there are non-local correlations and contextual measurements in quantum mechanics. The question that EPR posed was whether quantum mechanics said all that was possible to be said about the nature of reality (which is remarkably little), meaning that quantum mechanics is complete as a theory, or whether there exits a level of reality below that of quantum mechanics, or a theory of quantum reality distinct from quantum mechanics, which possesses a grounded ontology and which still provides the statistical predictions of quantum mechanics. The hidden variable theorems were devised to address this question. In each case they assume certain features that any deeper model of quantum reality could possess, and then show that any theory based upon such features is incapable of reproducing all of the results of quantum mechanics. These models generally assume that by reality one means a reality possessing features assumed of classical reality, a reality based upon the concept of the object. It is not clear though that any theory describing this more fundamental level of reality must itself obey either the laws of quantum mechanics or of classical mechanics. It is equally possible that quantum and classical mechanics could be emergent from some more fundamental theory.

Mathematics learned quite some time ago that it is the problem that is important and any tool that helps to bring insight or resolution to the problem is useful. This has certainly been true in number theory, which has availed itself of arithmetic, algebra, logic, differential calculus, integral calculus, geometry. Logic has long since moved past the Aristotelean type of linguistic argument and embraced techniques from ordered sets, combinatorial games and computer science. There is no reason, a priori, to presume that either quantum mechanics or classical mechanics should hold at all levels of physical reality. There are ideological reasons for doing so but as these lowest levels have yet to be explored experimentally, and as studies into quantum gravity, dark matter, dark energy, the mesoscopic boundary, complex adaptive systems, condensed matter systems and so on have progressed, the question remains open, at least to my mind.

A student of biology, psychology or of complex adaptive systems would not be overly surprised at the notion of contextuality in measurement, nor necessarily of non-local correlations. The study of inanimate and of animate entities provides the observer with very different intuitions about the structure of reality. Approaching the fundamental level with intuitions derived from inanimate entities has resulted in an effective theory, but one full of paradoxes and lacking a firm ontological foundation. The present thesis explores what might be obtained by approaching the fundamental level with intuitions gleaned from the study of complex and emergent systems, and as will be shown, the result is an effective theory with a firm ontological foundation.

The notion of a reality derived from object is expressed in the Einstein-Podolsky-Rosen condition for an element of reality [180, pg. 47]:

> If without in any way disturbing the system, we can predict with certainty (i.e. with probability equal to unity the value of a physical quantity, then there exists an element of physical reality corresponding to this quantity.

For Einstein, an element of reality must be a possessor of a property. In the viewpoint proposed in this thesis, an element of reality need only be a generator or a determinant of a property. The EPR statement includes aspects of both locality, in that no disturbance of any kind may occur, and of non-contextuality, in that these elements possess definite predetermined properties.

In the present context, the principle of locality refers to the notion that local events must have local causes and that influences between space-like separated entities must be transmitted by some intervening agency whose speed must be limited by that of light in order to be consistent with the principle of relativity. That signals are limited in speed, the upper bound being by the speed of light, has been confirmed repeatedly by experiment. A paradox ensues, however, because experiments on space-like separated entangled photons have demonstrated correlations that should not occur if nature is local, and have also shown that any possible signal passing between the entangled pair must have a speed of at least $50,000c$ [149]. Maudlin [251, pg 240] has examined the relationship between quantum mechanical non-locality and relativity and arrived at four unequivocal (his adjective) constraints:

> Violation of Bell's inequality does not require superluminal matter or energy transport
> Violation of Bell's inequality does not entail the possibility of superluminal signalling
> Violation of Bell's inequality does require superluminal causal connections
> Violation of Bell's inequality can be accomplished only if there is superluminal information transmission

I shall show in Chapter 6 that the last two violations are in fact not necessary.

Non-contextuality refers to the idea that if a classical element of reality is measured for a particular property and a specific value v is obtained, then *prior* to the measurement being performed the element actually possessed that property and its value was v. In particular it means that a property and its value exist independent of any measurement performed on the system to determine both. Any actual measurement experiment invariable perturbs the quantum system resulting in noise. But as the classical system already possesses the property having value v, such noise merely results in observations following a probability distribution and the law of large numbers assures one that the mean of the distribution, given enough measurements, will provide the true value v. Quantum mechanics, however provides many situations, such as involving spin measurements, in which the order of measurement has a profound effect upon the measured values obtained, and this effect cannot be accounted for merely as noise introduced by the measurement apparatus. This very real contextuality of the measurement process forms the basis for Bohr's notion of complementarity. This contextuality arises from the identification of quantum states with elements of a Hilbert space and measurements with non-commuting Hermitian operators on the Hilbert space. Some authors have argued that the root of the problem is an excessive over-reliance upon the association of measurement with Hermitian operators. For example, Daumer et.al. write:

> .. the basic problem with quantum theory...more fundamental than the measurement problem and all the rest, is a naïve realism about operators,,, by (this) we refer to various, not entirely sharply defined, ways of taking too seriously the notion of operator-as-observable, and in particular to the all too casual talk about 'measuring operators' which tends to occur as soon as a physicist enters quantum mode [180, pg 38]

I am not going to debate the merits of the operator approach here. I will accept that the outcomes of various measurements are given as the eigenvalues of certain Hermitian operators, and that their eigenfunctions play important roles in the measurement process. I will, however, try to avoid the use of the term observable in relation to these operators and speak only of them as representing an aspect of the measurement process. This will be particularly true in Chapter 5.

The main thrust of this thesis is to show, by providing an explicit model of NRQM, that there are subtle flaws in the hidden variable arguments due to the implicit assumption of several additional propositions for which the grounds supporting adherence are more ideological than empirical. These in turn are derived from an implicit notion of reality as being comprised of objects, and a particular conception of what a reality founded on objects implies. By shifting to a process viewpoint, in which the reality of our experience is a generated reality, one can have an ultimate level of reality consisting of local, non-contextual hidden variables, while the reality that is observable displays features of non-locality and contextuality.

These issues will be discussed in detail in the next chapter. Here I wish to describe the basic hidden variable theorems and their structure. For convenience I have divided the hidden variable theorems into three basic forms.

A.1 Type I Theorems

Type I hidden variable theorems are those of Bell [44], Clauser-Horne-Shimony-Holt [321], and Leggett-Garg [227]. They result in the derivation of an inequality involving correlations among measurements that must hold if certain conditions on the nature of the underlying 'reality' hold true and are violated by quantum mechanics.

The classic Bell argument [44] goes as follows. Consider a pair of spin-1/2 particles prepared in the singlet state and allowed to move apart freely in opposite directions. Assume that spin measurements σ_1, σ_2 are made on the particles respectively. If, for any unit vector $\mathbf{a}$, a measurement of $\sigma_1 \cdot \mathbf{a}$ yields a value of $+1$ then measurement of $\sigma_2 \cdot \mathbf{a}$ must necessarily yield a value of -1.

Bell assumes that a complete specification of the state of these particles is effected by means of parameters λ. Crucially Bell [44] points out:

> It is a matter of indifference in the following whether λ denotes a single variable or a set, or even a set of functions, and whether the variables are discrete or continuous.

It does *not* matter whether these hidden variables are deterministic, non-deterministic or even stochastic. Indeed in formulating the argument Bell presumed the existence of a probability distribution for the parameter values of the λ. It has long been a matter of debate whether the Type I hidden variable theorems refer specifically to theories involving deterministic hidden variables or more general kinds of variables. The requirement of determinism has long been attributed to Einstein but Hemmick and Shakur [180] point out that Einstein actually gave up this position to focus upon more fundamental concerns. Bell himself argued frequently that his theory was *not* about determinism. He wrote:

> It is remarkably difficult to get this point across, that determinism is not a *presupposition* of the analysis. There is a widespread and erroneous conviction that for Einstein determinism was always *the* sacred principle. (Sic) [273]

but his words have tended to fall upon deaf ears and one still reads that the Bell theorems are about deterministic hidden variable models. The discovery of deterministic chaos has rendered the distinction between determinism and stochasticity somewhat moot [44].

In what follows we assume that σ_1, σ_2 are fixed and allow the unit vectors $\mathbf{a}, \mathbf{b}$ to vary. Let the result of measuring $\sigma_1 \cdot \mathbf{a}$ be denoted $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$ for the result of measuring $\sigma_2 \cdot \mathbf{b}$. Assume that $A(\mathbf{a}, \lambda) = \pm 1, B(\mathbf{b}, \lambda) = \pm 1$.

Assume that $A(\mathbf{a}, \lambda)$ is independent of the choice of $\mathbf{b}$ and that $B(\mathbf{b}, \lambda)$ is independent of the choice of $\mathbf{a}$. Let $\rho(\lambda)$ denote the probability distribution of λ so that the expectation value of the product of the two components $\sigma_1 \cdot \mathbf{a}$ and $\sigma_2 \cdot \mathbf{b}$ is given as

$$E(\mathbf{a}, \mathbf{b}) = \int d\lambda \rho(\lambda) A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda)$$

This is the crucial step actually and will be discussed in detail in the next chapter. It assumes that the probability structure of the additional parameter space of the λ *must* follow the laws of Kolmogorov probability theory.

NRQM predicts that in the singlet state this value is

$$< \sigma_1 \cdot \mathbf{a}, \sigma_2 \cdot \mathbf{b} >= -\mathbf{a} \cdot \mathbf{b}$$

Assuming that $B(\mathbf{a}, \lambda) = -A(\mathbf{a}, \lambda)$, write

$$E(\mathbf{a}, \mathbf{b}) = -\int d\lambda \rho(\lambda) A(\mathbf{a}, \lambda) A(\mathbf{b}, \lambda)$$

If $\mathbf{c}$ is another unit vector write

$$E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{c}) = -\int d\lambda\rho(\lambda)[A(\mathbf{a},\lambda)A(\mathbf{b},\lambda) - (A(\mathbf{a},\lambda)A(\mathbf{c},\lambda)] =$$

$$\int d\lambda\rho(\lambda)[A(\mathbf{a},\lambda)A(\mathbf{b},\lambda)[(A(\mathbf{b},\lambda)A(\mathbf{c},\lambda) - 1]$$

hence

$$|E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{c})| \leq \int d\lambda\rho(\lambda)[1 - (A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)]$$

and thus

$$1 + E(\mathbf{b},\mathbf{c}) \geq |E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{c})|$$

Shimony [321] extended this argument to include a more detailed analysis of the locality condition following upon the work of Jarrett. Consider a source capable of emitting a pair of entangled particles $1, 2$ which move in opposite directions towards detectors having controllable parameters a, b respectively. The possible outcomes are given by the sets of values $\{s_m\}$ and $\{t_n\}$ respectively, assumed for convenience to lie within the set $[-1, 1]$. One assumes that the pair of particles is described by some complete state k which belongs to a space K of complete states. The only assumption on K is that a probability measure can be defined upon it.

Let

1. $p^1(m/k, a, b)$ be the probability of the outcome s_m for particle 1 given the complete state k and parameters a, b
2. $p^2(n/k, a, b)$ be the probability of the outcome t_n for particle 2 given the complete state k and parameters a, b
3. $p(m, n/k, a, b)$ be the joint probability of the outcome s_m for particle 1 and t_n for particle 2 given the complete state k and parameters a, b
4. $p^1(m/k, a, b, n)$ be the joint probability of the outcome s_m for particle 1 given the complete state k , parameters a, b and the outcome t_n for particle 2

5. $p^2(n/k, a, b, m)$ be the joint probability of the outcome t_n for particle 2 given the complete state k , parameters a, b and the outcome s_m for particle 1

Shimony [321] then states (pg 92):

> The general principles of probability theory, with no further assumptions, impose the following product rule:

$$p(m, n/k, a, b) = p^1(m/k, a, b)p^2(n/k, a, b, m) = p^2(n/k, a, b)p^1(m/k, a, b, n)$$

Note again the implicit assumption of Kolmogorov probability theory. Shimony then defines:

Definition (Parameter Independence):

1. $p^1(m/k, a, b)$ is independent of b and may be written as $p^1(m/k, a)$
2. $p^2(n/k, a, b)$ is independent of a and may be written as $p^2(n/k, b)$

Definition (Outcome Independence):

1. $p^1(m/k, a, b, n) = p^1(m/k, a, b)$
2. $p^2(n/k, a, b, m) = p^2(n/k, a, b)$

The conjunction of parameter and outcome independence leads to the following factorization:

$$p(m, n/k, a, b) = p^1(m/k, a)p^2(n/k, b)$$

Shimony defines expectation values:

1. $E^1(k, a) = \sum_m p^1(m/k, a)s_m$, the outcome expectation value for particle 1 given complete state k and parameter a
2. $E^2(k, b) = \sum_n p^2(n/k, b)t_n$, the outcome expectation value for particle 2 given complete state k and parameter b

3. $E(k,a,b) = \sum_{m,n} p(m,n/k,a,b) s_m t_n$, the outcome expectation value for the product of outcomes, given the complete state k, for particle 1 given parameter a and particle 2 given parameter b

One obtains:

$$E(k,a,b) = E^1(k,a)E^2(k,b)$$

Note that again this accords with Kolmogorov probability theory.

Using the following lemma, namely, if x', y', x'', y'' all belong to the interval $[-1,1]$ then S belongs to the interval $[-2,2]$ where:

$$S = x'y' + x'y'' + x''y' - x''y'',$$

Shimony obtained a generalization of the Clauser-Horne-Shimony-Holt (CHSH) inequality:

$$-2 \leq E(k,a',b') + E(k,a'b'') + E(k,a'',b') - E(k,a'',b'')$$

or following integration over the probability distribution w on K:

$$-2 \leq E_w(a',b') + E_w(,a'b'') + E_w(,a'',b') - E_w(a'',b'')$$

If both parameter independence and outcome independence hold, then this result is true but in violation of the predictions of quantum mechanics and with experimental observations. Shimony points out that it is not possible to violate parameter independence as that would permit non-relativistic signalling to occur, which would contradict the special theory of relativity. Quantum mechanics violates outcome independence in the experimentally verified phenomenon of entanglement. Shimony argues that outcome independence cannot be used to send signals to space-like separated observers and so relativity is not violated, at least in spirit. He called this ‘passion at a distance’.

This does not quite let quantum mechanics off the hook. Suppose that the particles are in the entangled state

$$\Psi = \frac{1}{\sqrt{2}}(|+>|->+|->|+>)$$

Assume further that they are space-like separated at the time that the actual measurements are performed. Experiments have shown [149] that if there were a signal propagating from one particle to the other when a measurement is performed, that signal would have to propagate at a speed exceeding $50,000c$. From a probabilistic point of view, the first particle could appear in either state $|+>$ or $|->$ with equal probability as could the second particle. If these two are space-like separated and cannot communicate with one another, then how is it that the first particle being found in state $|+>$ ensures that the second particle must be in state $|->$? How is it that the correlation created by the entanglement is maintained in spite of their space-like separation? Type I theorems aim to demonstrate that reality must be non-local, but what exactly does non-locality mean in this context? Note carefully that the theorems are formulated in terms of the correlations among the outcomes of *measurements* of systems. These theorems do not describe the physical nature of reality but rather place constraints on the structure of the probability theory used to determine those values. In particular, they assume that any fundamental reality based upon a set of complete states *must* obey a Kolmogorov probability structure. Moreover they assume the statistical independence of any hidden variables describing these entangled particles even though by their very definition they are not statistically independent entities.

The arguments all have the form: Assume that hidden variables imply Kolmogorov probability structure. Then an inequality may be constructed that is violated by quantum mechanics.

If this assumption is incorrect, then the conclusion does not logically follow. The Type I theorems use an argument of the form

$$\begin{array}{c} \alpha \rightarrow \beta \\ \neg\beta \\ ------------ \\ \neg\alpha \end{array}$$

If α is false then logically the argument becomes

$$\begin{array}{c} \alpha \rightarrow \beta \\ \neg\alpha \\ ------------ \\ \beta \vee \neg\beta \end{array}$$

and no conclusion is possible.

Both Bell and CHSH inequalities apply to the case of systems that are space-like separated. Leggett and Garg [227] considered a different case in which the systems are time-like separated.

Leggett and Garg considered two conditions:

1. Macroscopic Realism - a macroscopic system with two or more macroscopically distinct states will at any given time *be* in one or the other of these states

2. Macroscopic Noninvasive Measurability - it is possible in principle to determine the state of the system with arbitrarily small perturbations on its subsequent dynamics

To demonstrate that these two conditions are incompatible with NRQM they considered a system governed by some potential having two wells sufficiently distant from one another but where the system oscillates back and forth between these two minima. Denote the regions around the minima as L, R respectively. They assumed that the system spends so little time in the region between these minima that the probability of observing it there can be taken to be zero. A measurement Q of the system thus finds it in either the L region ($Q = -1$) or the R region ($Q = 1$).

We can construct random variables Q_i that reflect the measurement value of Q at times $t_0, t_1, t_2, \ldots$. Based upon the first assumption they argued that it is possible to construct joint probability distributions

$$\rho(Q_1, Q_2), \rho(Q_1, Q_2, Q_3), \ldots$$

and correlation functions $K_{ij} =< Q_i, Q_j >$.

They assumed that $\sum_{Q_2=\pm 1} \rho(Q_1, Q_2, Q_3) = \rho(Q_1, Q_3)$, which is again the total probability according to Kolmogorov probability theory. They showed that different inequalities may be obtained, for example,

$$1 + K_{12} + K_{23} + K_{13} \leq 0$$

or

$$|K_{12} + K_{23}| + |K_{14} + K_{24}| \leq 2$$

and that these can be violated by quantum mechanics under certain conditions.

The Macroscopic realism condition is similar to that of non-contextual hidden variables. The second assumption seems unduly strong. In dealing with biological, psychological or social systems it is pretty much the norm that interactions with the system alter the state of the system. That systems react otherwise seems to arise from the fact that physics has focussed primarily upon inanimate matter when constructing its theories. Just as there is no reason to believe that such an assumption holds for all observations of all macroscopic objects, it is not clear that it need hold true to microscopic entities either. Taken together these two assumptions make for a very strong form of non-contextuality. The model to be presented below upholds the first assumption, avoids the second assumption at the level of ultimate reality but violates it at the level of process.

A.2 Type II Theorems

Typical of Type II hidden variable theorems are those based on the work of von Neumann [412], Gleason [150], Kochen-Specker [218], Ghirhardi-Horne-Zeilinger and Mermin [255]. Taken together they demonstrate that any collection of hidden variables must be contextual, meaning that measurement values can only be assigned by taking the entire quantum system and measurement apparatus into consideration. Conversely they imply that it is impossible to assign definite measurement values to the states of a quantum system prior to a measurement being carried out. It was Bohr who first conceived of the idea that a quantum system cannot be considered to possess a property in advance of a measurement of that property being carried out. This is expressed in the complementarity principle, which describes observables as arising in complementary pairs, such as position and momentum. It is impossible to simultaneously measure complementary observables as a result of the fact that complementary observables correspond to non-commuting operators in the Hilbert space of the system.

Von Neumann showed that there is an isomorphism between the algebra of observables and the algebra of self-adjoint operators on a Hilbert space $\mathcal{H}$. Instead of observables, whose structure is rather complicated, one focusses upon "eventualities" , which form a subclass $\mathcal{L}_H$ of bivalent observables with the value 1 representing true and 0 representing false. This subclass is isomorphic to the lattice of projectors (or equivalently to the closed linear subspaces) of the Hilbert space $\mathcal{H}$. These projectors have only the eigenvalues $0, 1$. For any eventuality e, the associated projector is denoted by E_e. Mackey [240] introduced a notion of a measure on $\mathcal{L}_H$, being a mapping $p : \mathcal{L}_H \to [0, 1]$ such that $p(1) = 1$ (1 being the identity operator) and if one has a set $\{a_i\}$ such that $a_i \leq a'_j$ whenever i, j are distinct (here $'$ means orthocomplementation) then $p(\vee a_i) = \sum p(a_i)$.

As an example let $\Psi \in \mathcal{H}$ and $||\Psi|| = 1$. Define p_Ψ for all $e \in \mathcal{L}_H$ by $p_\Psi(e) =< \Psi|E_e|\Psi >$. Complex combinations of measures of the form p_Ψ also form a measure in Mackey's sense.

A dispersion free measure p is a measure which assigns either 0 or 1 to every $e \in \mathcal{L}_H$.

Type II hidden variable theorems fall into two forms. The first form is due to von Neumann [412] and Gleason [150], who demonstrated that the probabilities of quantum mechanics did not conform to the classical Kolmogorov probability distributions. Von Neumann argued that dispersion free states (or hidden variables) were impossible because any real linear combination of any two Hermitian operators is an observable, and the same linear combination of expectation values is the expectation value of the combination. That is $< \Psi|\alpha A + \beta B|\Psi >=< \Psi|\alpha A|\Psi > + < \Psi|\beta B|\Psi >$. If one has dispersion free states, for example single eigenstates of an operator A, then the expectation values correspond to eigenvalues, but eigenvalues do not necessarily sum so this cannot be true. Bell criticized von Neumann's argument as being too general since he included the case of non-commutative operators for which the necessary measurements could not be properly carried out.

Gleason's work refines that of von Neumann by focussing on commutative observables. He proved the non-existence of dispersion free measures whenever the dimension of the Hilbert space is greater than 2. Gleason proved that if the dimension of $\mathcal{H} > 2$ and p is a measure then there is a finite or denumerable sequence of normalized vectors Ψ_i and of non-negative real numbers w_i summing to 1 such that for all $e \in \mathcal{L}_H$, $p(e) = \sum w_i p_{\Psi_i}(e)$. As a corollary, there is no p which assigns only a 1 or 0 to each $e \in \mathcal{L}_H$, in other words, there are no dispersion free measures.

More precisely, what Gleason [150] showed is given by the following theorem:

Theorem (Gleason): Let μ be a measure on the closed subspaces of a separable (real or complex) Hilbert space $\mathcal{H}$ of dimension at least three. Then there exists a positive semi-definite self-adjoint operator T of the trace class such that for all closed subspaces A of $\mathcal{H}$

$$\mu(A) = \text{trace}(TP_A)$$

where P_A is the orthogonal projection of $\mathcal{H}$ onto A.

The following is quite pointed so let me quote Shimony directly. He writes that the program of noncontextual hidden variables envisaged a space Λ of complete states of S, each member λ of which determines a dispersion-free measure on $\mathcal{L}_H$. It envisaged, furthermore, a σ-algebra $\Sigma(\Lambda)$ of subsets of Λ and *classical* probability functions ρ such that each triple $< \Lambda, \Sigma(\Lambda), \rho >$ constitutes a *classical* probability space. And finally it envisaged that for each normalized $\Psi \in \mathcal{H}$ there is a *classical* probability measure function ρ_Ψ on $\Sigma(\Lambda)$ such that for all $e \in \mathcal{L}_H$

$$p_\Psi(e) = \int_\Lambda p_\lambda(e) dp_\Psi$$

(emphasis mine)

According to Shimony [321] "Gleason's theorem and its corollary doomed the program of non-contextual hidden variables theories" (pg 107).

Again notice that the argument shows the non-existence of a classical, i.e. Kolmogorov probability structure on the space of hidden variables but it does not rule out the possibility of non-Kolmogorov probability structures (indeed quantum mechanics provides one such structure) on these hidden variables. It is simply assumed that any form of hidden variable must of necessity follow a Kolmogorov probability structure. In Chapter 3 additional arguments are provided against the conflation of Kolmogorov probability structure and classicality.

The second form of Type II theorems is due to Kochen-Specker [218], Greenberger-Horne and Zeilinger and Mermin [255], who developed a different set of results expressing the same point about the non-existence of contextual hidden variables. They present algebraic arguments to demonstrate that there are measurement situations in quantum mechanics which are at least theoretically feasible to implement and in which the attempt to preassign definite measurement values to all of the observables fails, leading to a contradiction. Mermin's presentation [255] is by far the shortest, simplest, and most restrictive, so let me repeat it here.

Mermin assumes the existence of three spin 1/2 particles and works in the 8-dimensional spin space. The spin operators are σ_a^n, where $n = 1, 2, 3$ refers to the particles and $a = x, y, z$ refers to the axis of spin.

Consider the ten spin operators

$$\sigma_x^1,\ \sigma_y^1,\ \sigma_x^2,\ \sigma_y^2,\ \sigma_x^3,\ \sigma_y^3,\ \sigma_x^1\sigma_y^2\sigma_y^3,\ \sigma_y^1\sigma_x^2\sigma_y^3,\ \sigma_y^1\sigma_y^2\sigma_x^3,\ \sigma_x^1\sigma_x^2\sigma_x^3$$

Assume that a value v can be assigned to each of these operators. In theory they commute so that in principle it should be possible to measure them. In any event the commutation relations together with constraints on the possible values leads to the following general constraints:

1. $v(\sigma_x^1\sigma_y^2\sigma_y^3)v(\sigma_x^1)v(\sigma_y^2)v(\sigma_y^3) = 1$
2. $v(\sigma_y^1\sigma_x^2\sigma_y^3)v(\sigma_y^1)v(\sigma_x^2)v(\sigma_y^3) = 1$
3. $v(\sigma_y^1\sigma_y^2\sigma_x^3)v(\sigma_y^1)v(\sigma_y^2)v(\sigma_x^3) = 1$
4. $v(\sigma_x^1\sigma_x^2\sigma_x^3)v(\sigma_x^1)v(\sigma_x^2)v(\sigma_x^3) = 1$
5. $v(\sigma_x^1\sigma_y^2\sigma_y^3)v(\sigma_y^1\sigma_x^2\sigma_y^3)v(\sigma_y^1\sigma_y^2\sigma_x^3)v(\sigma_x^1\sigma_x^2\sigma_x^3) = -1$

Since the eigenvalues of all ten operators are ± 1 it is impossible to assign values to these terms which are compatible with the constraints. Since the ten operators commute it is possible to find a state Ψ which is a simultaneous eigenvector for the operators $\sigma_x^1\sigma_y^2\sigma_y^3$, $\sigma_y^1\sigma_x^2\sigma_y^3$, $\sigma_y^1\sigma_y^2\sigma_x^3$ all having eigenvalue 1, while $\sigma_x^1\sigma_x^2\sigma_x^3$ takes the eigenvalue -1.

Mermin transforms this into a question about locality. Note that the above situation implies that if the three particles are space-like separated then it is possible to measure the value of m_x (or m_y) for any particle by measuring either m_x or m_y for the other two particles and doing the calculations on the eigenvalue products given by the above terms. This would suggest, if nonlocal effects are to be avoided, that each must have a pre-assigned value. But then these values must satisfy the following constraints

1. $m_x^1 m_y^2 m_y^3 = 1$
2. $m_y^1 m_x^2 m_y^3 = 1$
3. $m_y^1 m_y^2 m_x^3 = 1$
4. $m_x^1 m_x^2 m_x^3 = -1$

which again is impossible.

This again appears to eliminate any form of locality and non-contextuality.

A.3 Type III Theorems

At present there is only one Type III theorem published [191]. It will be discussed in more detail in Chapter 7. It produces a constraint on any possible hidden variable theory in which the variables are deterministic and satisfy a strong form of realism, namely that a particle is always either particle or wave, although the choice can vary from particle to particle in an apparatus. This Type III hidden variable theorem is based on Wheeler's concept of a delayed choice experiment. The delayed choice paradigm involves the use of a Mach-Zehnder interferometer. A beam of photons impinges on a beam splitter, sending it along two paths at right angles to one another. These two beams are then reflected by 90 and 270 degrees respectively, allowing them to converge at a point and then pass on to two detectors. Assuming that each detector will measure a single photon at any given time, they will be anticorrelated, meaning that when one detector fires the other detector never fires. This supports the idea of a photon as a particle traveling along only one of the two paths to a detector. Suppose now that a second beam splitter is introduced at the convergence point prior to the interaction with the detectors. In that case one now observes interference, in that the number of counts at each detector varies as a function of the phase difference between the two paths induced by the second beam splitter. The count diagrams for the two detectors are 180 degrees out of phase with one another. [157].

The interesting question is what happens if the second beam splitter is introduced after the particle has passed the first beam splitter and entered the interferometer. According to the usual conception of a particle and of a wave, a physical entity could be either a particle or a wave depending upon conditions, but once it has determined which it is to be, it should remain so until some additional interaction changes it again. In particular in the interferometer experiment, if the photon starts out only along one path, then it cannot also show up along the other path at a later time, In other words, once it has chosen to behave like a particle it should remain like a particle throughout the experiment. The presence of the second beam splitter should not affect this in any way - in particular there should be no observable effect of the phase difference induced by the second splitter since there is no photon propagation along the other path with which the propagation along the first path could interfere. Quantum mechanics, however, predicts that once the second beam splitter is introduced, regardless of the time so long as that time is prior to the arrival of the photon at the detector, then an interference pattern will be observed. Somehow, in spite of the absence of the beam splitter at the start of the experiment when the photon should behave like a particle, the photon can recover its wave-like properties and again interfere. This is a particularly puzzling result, especially if one wishes to preserve some classical notions like particle or wave at the quantum level.

Bohr, in developing his notion of complementarity, said [157, pg. 82]:

> ... how flawed the simple wave-particle description is. Once light [or a material particle] is in an interferometer, we simply cannot think of it as either a wave or a particle. Nor will melding the two descriptions onto some strange hybrid work. All these attempts are inadequate. What is called for is not a composite picture of light, stitched together out of bits taken from various classical theories. Rather we are being asked for a new concept, a new point of view that will be fundamentally different from those developed from the world of classical physics.

The present work takes up this challenge of Bohr's and is an attempt to address these many questions by adopting a wholly new approach based upon the notion of process as being the primary determinant of physical reality.

A.4 The Measurement Problem

In classical physics a physical system, at each point along its world line, is assumed to exist in a *state*, and this state determines all past and future states (at least in the absence of interactions with other systems). In classical physics it is assumed that this state can be described as a vector over a space of parameters directly linked to physical constructs such as position, linear momentum, angular momentum, mass, energy, etc. and that a knowledge of the values of these parameters for a given state suffices to predict these values for past and future states. Such a description of a state is said to be *complete*, since no additional information is required to enable these predictions to be carried out. In classical physics these parameters are assumed to be intrinsic ontological characteristics of the system itself, meaning that if the state of the system should posit that the particle is at position $\mathbf{x}$ with momentum $\mathbf{p}$, and if one applied a measuring device capable of measuring position and momentum, then it would register an entity at position $\mathbf{x}$ having momentum $\mathbf{p}$. The particle is described as *being* at position $\mathbf{x}$ and *having* momentum $\mathbf{p}$, or any other physical parameter. These parameter values are held to be attributes of the particle, independent of the particular measurement apparatus employed. Moreover it is often held that the particle possesses macroscopic non-invasive measurability, as defined by Leggett-Garg, so that an interaction with the measurement apparatus can take place which does not alter the intrinsic state of the particle.

As mentioned previously, this latter property applies in special cases involving particular measurements, since biological, social, economic and psychological systems manifestly fail generally to possess this characteristic. It is only necessary that the measurement apparatus be able to detect and reflect the state of the system immediately prior to the measurement, and that the measurement value itself should be independent of the interaction with the measurement apparatus regardless of what happens to the particle after the measurement has taken place. It is also assumed, and this is critical for much of the formulation and analysis of classical physics, that it is possible, at least in principle, to measure all of the relevant parameters *simultaneously* and *exactly*, so that this complete state vector can actually be determined. This notion of simultaneous measurability is necessary for the mathematical and probabilistic formulation of classical mechanics as it currently stands.

The development of quantum mechanics provided serious challenges to this classical approach. The discovery of the uncertainty relations appeared to place fundamental limits on the degree of accuracy with which measurements could be carried out. Moreover, it appeared to be impossible to determine a complete set of parameter values attributable to a given state. Certain parameters appeared to be in what Bohr termed 'complementary pairs ' and it was impossible to measure these complementary pairs both exactly and simultaneously. In addition it appeared to be impossible, at least until the notion of weak measurements was developed [8], to measure a quantum system without disturbing it, in particular, without causing it to undergo a catastrophic transition. Weak, or non-demolition measurements, are thought to provide a form of microscopic non-invasive measurability. A sequence of measurements involving complementary pairs of parameters, such as A, B, A would result in a sequence of measured values, a, b, a'. but $a' \neq a$ in general. This would not be expected of a classical particle since A and B are quire different parameters, such as position and momentum, or rotation about the x and y axes. Observing one should not affect the observation of the other and repeating the observation of the first should leave its value unchanged. Nevertheless it is fundamentally impossible to ensure otherwise.

Bohr suggested that measurements could not in principle be separated from the conditions under which they were obtained. He wrote [58, pgs. 73,90]

> every atomic phenomenon is closed in the sense that its observation is based on registrations obtained by means of suitable *amplification* devices with *irreversible* functioning such as, for example, permanent marks on the photographic plate, caused by the penetrations of electrons into the emulsion... the quantum mechanical formalism permits well-defined applications referring only to such

> closed phenomena and must be considered a rational generalization of *classical* physics. (emphasis mine)

The important points in Bohr's statement are that the measurement process involves a) a form of amplification which is b) irreversible and c) that can be classically interpreted. More will be said about this in Chapter 5 when measurement is discussed from the process perspective.

It was no longer possible to meaningfully assign a classical state vector to quantum particles. Instead, the quantum state generalized from a finite to an infinite dimensional vector - based upon the observation of wave-like properties for electrons leading to the Schrödinger equation, or the use of Heisenberg matrix methods to enable calculations of possible parameter values and their probabilities of detection. Born interpreted the Schrödinger wave function as providing a probability distribution from which the probabilities of obtaining different values for these parameters could be determined. A quantum state was considered as providing a complete specification of the probability structure of measurements obtained from observations of a quantum particle.

Aside from the problem of the observer, there is another perplexing problem plaguing the quantum theory of measurement. This the problem of *wave function collapse*. In part this is a consequence of the manner is which von Neumann formulated his mathematical theory of quantum mechanics [412]. von Neumann wanted to place quantum mechanics onto a solid mathematical foundation, and he chose to use functional analysis as the language within which to situate its constructs. Functional analysis is a natural generalization from finite dimensional to infinite dimensional vector spaces. von Neumann's formulation, however, divides quantum mechanics into two disconnected components.

On the one hand, NRQM posits that the temporal evolution of the state of a quantum system is describable by a wave function Ψ which is governed by the Schrodinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

where H is an operator version of the classical Hamiltonian function for the system. H is a linear operator, which is significant because it means that if Ψ_1 and Ψ_2 are both solutions of the Schrodinger equation then so is $w_1\Psi_1 + w_2\Psi_2$ for arbitrary complex weights w_1, w_2. When the Hamiltonian is time independent, such as is expected for systems evolving freely, there exists a unitary operator $U = e^{-iHt}$ such that the wave function at time t, Ψ_t, given the initial wave function at time 0, Ψ_0, can be determined by

$$\Psi_t = U(t)\Psi_0 = e^{-iHt}\Psi_0$$

Measurement is described entirely differently. To each measurement situation there corresponds an Hermitian operator A, and the possible results of a measurement are given by the eigenvalues of this operator, i.e. those values which satisfy the equation

$$A\Psi = \lambda\Psi$$

Suppose that Ψ_1 is a solution corresponding to eigenvalue λ_1 and Ψ_2 is a second solution corresponding to eigenvalue λ_2. Then the sum is a solution, but what value is ascribable to a measurement of such a system? Applying A to the sum yields

$$A\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_2) = \frac{1}{\sqrt{2}}[A\Psi_1 + A\Psi_2] = \frac{1}{\sqrt{2}}(\lambda_1\Psi_1 + \lambda_2\Psi_2) \neq \lambda\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_2)$$

The sum is *not* an eigenvector of A so it would appear that no measurement can be ascribed to the superposition. In fact, if such a superposition is measured it does yield definite measured values, λ_1 and λ_2 but these appear randomly, each with probability $\frac{1}{2}$.

If an actual measurement of the parameter associated to the operator A is carried out, and repeated, the quantum system generally yields the same value, so that given a superposition $\frac{1}{\sqrt{2}}(\Psi_1+\Psi_2)$ a measurement of A will consistently yield either λ_1 or λ_2, which would be the case if the system were originally in either the state Ψ_1 or the state Ψ_2. This suggests that *prior* to the measurement the system is in state $\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_2)$ but *after* the measurement the system is in either state Ψ_1 or state Ψ_2. It is said that the wave function has 'collapsed' .

There is nothing, however, in the Schrödinger form of quantum mechanics that allows for such a 'collapse' of the wave function. The time evolution operator is unitary, meaning that it is linear and invertible, so that

$$U(t)\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_2) = \frac{1}{\sqrt{2}}(U(t)\Psi_1 + U(t)\Psi_2)$$

and neither $U(t)\Psi_1$ nor $U(t)\Psi_2$ can ever take the value 0.

As noted above, the measurement operator A also does not collapse the wave function. This is the measurement problem.

von Neumann, and London and Bauer who followed him, considered, according to Shimony [321], that Type I discontinuous transitions due to the performance of a measurement were an uneliminable aspect of quantum theory. Following Shimony [321], their argument goes as follows:

Suppose that one has a quantum system x described by a wave function $u(x)$. Let $u_i(x)$ denote an eigenstate with respect to the measurement apparatus which measures F in x. The measurement apparatus y, being a physical system, should have a state and evolution describable by quantum mechanics (this is an assumption rather than a necessity, particularly if quantum mechanics is viewed as an effective theory rather than as an absolute theory). Let this state be $\phi(y)$. If the initial state of the combined system is

$$\Psi(t_0) = u_i(x)\phi_0(y)$$

then in time t_1 the combined system evolves to

$$\Psi(t_0 + t_1) = u_i(x)v_i(y)$$

where $v_i(y)$ is an eigenstate of some observable G of y such that $g_i \neq g_j$, if $i \neq j$. This implies that a determination of the value of G for the apparatus y indirectly but unequivocally determines the value of F in x. The time transition is given according to the Schrödinger for the combined system by a linear unitary operator.

If the system x begins in a superposition state, the combined initial state will take the form

$$\Psi(t_0) = \Psi_1 u_1(x)\phi_0(y) + \Psi_2 u_2(x)\phi_0(y) + \ldots$$

so that under time evolution, the operator being unitary, the combined system evolves to the state

$$\Psi(t_0 + t_1) = \Psi_1 u_1(x)v_1(y) + \Psi_2 u_2(x)v_2(y) + \ldots$$

This implies that the initial indefiniteness in the measurement of F becomes an indefiniteness in the measurement of G. This can be solved by a direct measurement of G, but that would involve a collapse of the form

$$\Psi(t_0 + t_1) \rightarrow u_n v_n$$

for some n. But this is just the sort of collapse that one wishes to avoid. So suppose one introduces an additional device y' sensitive to the states of $x + y$, then following the same argument one obtains a new combined system and a new evolved state of the form

$$\Xi(t_0 + t_1) = \Psi_1 u_1(x) v_1(y) v_1'(y') + \Psi_2 u_2(x) v_2(y) v_2'(y') + \dots$$

This just leads to an infinite regress from which there appears to be no recourse except to a Type I discontinuous transition.

Shimony generalized this result and showed [321, pg. 34] that even with approximate measurements, the quantum state of a system will not undergo a transition to an eigenstate under the usual Schrodinger dynamics.

von Neumann and London and Bauer consider this Type I transition to be effected somehow by the action of consciousness, which somehow registers the results of a measurement and in so doing collapses the wave function. I am not going to enter into a discussion of the possible role of mind in quantum mechanics [338]. I will only comment that as a practicing psychiatrist with over 30 years of experience, there is no place for a notion of mind which is somehow causally separated from the body and brain that embody it. Mind can and should be viewed as emergent from the physical processes of body and brain, but mind requires the existence of a physical body to manifest itself. That physical body is subject to the laws of physics and so, in a sense, must be mind as well. Thus there is no recourse to some mystical property of mind and consciousness to resolve the measurement problem. Being an emergent phenomenon, the laws of quantum mechanics may be irrelevant to the functioning of mind. But mind cannot act upon matter without the intervention of a physical body, and that body does not act upon the quantum system directly. It only acts upon the measurement apparatus and even then, mind generally merely registers the outcome of the measurement apparatus *after* the interaction with the quantum system has taken place. How this post hoc experience is supposed to affect the wave function of the quantum system, either *before* or *during* the measurement process remains a mystery. In chapter 6 I will suggest an approach to the measurement problem from the point of view of a different mathematical foundation for NRQM.

There have been several other attempts to resolve the measurement problem. The most commonly cited would seem to be those of Bohm and followers [55, 57, 244], decoherence and consistent histories [158], the multiverse [313] and Ghirardi-Rimini-Weber (GRW)[147].

The approaches closest in spirit to the ideas in this thesis are those of Bohm and GRW, especially the flash variant [290].

Bohmian theory provides a non-local realist model of NRQM. Realist models of quantum mechanics have been around since De Broglie first proposed the idea of a 'pilot wave', later developed by Bohm into a fully realized ontological model [55]. The basic idea is to rewrite the wave function as a product, $\Psi = R\exp(iS\hbar)$ where R and S are subject to the differential equations

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = 0$$

which can be understood as a Hamilton-Jacobi equation for a particle with momentum $\mathbf{p} = \nabla S$ moving normal to a wave front S = constant under an additional quantum potential $Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$ while

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left(R^2 \frac{\nabla S}{m} \right) = 0$$

describes the conservation of probability for an ensemble of such particles having a probability density $P = R^2$.

Bohm interprets these equations as representing the motion of a definite particle coupled to a quantum field given by Ψ (which satisfies the Schrodinger equation). The equation of motion of the particle is given by

$$m\frac{d\mathbf{v}}{dt} = -\nabla(V) - \nabla(Q)$$

The quantum potential does not depend upon the amplitude of the wave in the sense that multiplying the wave function by a constant does not change the value of the quantum potential. It depends only upon the *form* of the wave. A similar situation pertains in signal theory where the information content of a signal depends upon its form but not upon its intensity. Bohm and Hiley make the important point that the quantum potential acts upon the particle not as a form of energy but rather as a form of *information*. Such information is not to be understood in the Shannon sense as a reduction of uncertainty but rather in the semantic sense as providing knowledge about the physical environment within which the particle is moving. This usage of information is uncommon in physics and engineering but virtually universal in most other fields of human endeavor.

Bohm's theory is generally described as a hidden variable model although as Bohm himself points out there are actually no hidden variables in the model. Rather, the most important aspect of Bohm's model is that it provides a realist ontological interpretation of quantum mechanics in its assertion that actual particles exist whose motion is guided by the addition of the quantum potential which imparts a stochastic character to their motion as a result of nonlinear effects.

This quantum potential is decidedly nonlocal in its effect which appears to occur instantaneously. This apparent violation of relativistic constraints has limited acceptance of Bohm's approach but to be fair nothing is being stated as to the ontological character of this quantum potential. While relativity limits the speed at which any signal may propagate there is no suggestion that the information provided by the quantum potential propagates as a signal.

Following Tumulka [402, 403], the GRW model posits that the wave function follows a stochastic jump process in the Hilbert space. If there are N 'particles' , then the wave function is given as $\Psi(q_1, \ldots, q_N)$ for $q_i \in \mathbb{R}^3$. For any point $x \in \mathbb{R}^3$ define a *collapse rate operator*

$$\Lambda_i(x) = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{(\hat{Q}_i - x)^2}{2\sigma^2}}$$

where $\hat{Q}_i$ is the position operator of particle i. Let Ψ_{t_0} denote the initial wave function at some initial time t_0. Then Ψ evolves as follows:

1. Initially it evolves unitarily according to the Schrödinger equation until some random time $T_1 = t_0 + \Delta T_1$, so that

$$\Psi_{T_1} = U_{\Delta T_1} \Psi_{t_0}$$

   where $U_t = e^{-iHt/\hbar}$ is the usual time evolution operator for a Hamiltonian H and the random time ΔT_1 is exponentially distributed with rate parameter $N\lambda$.

2. At time ΔT_1 it undergoes an instantaneous collapse with random centre X_1 and random label I_1 so that

$$\Psi_{T_1} \to \Psi_{T_1^+} = \frac{\Lambda_{I_1}(X_1)^{1/2} \Psi_{T_1}}{||\Lambda_{I_1}(X_1)^{1/2} \Psi_{T_1}||}$$

   where I_1 is chosen randomly and uniformly from the set $\{1, \ldots, N\}$ and the collapse centre is chosen randomly according to the distribution

$$\mathbb{P}(X_1 \in dx_1 | \Psi_{T_1}, I_1 = i_1) =< \Psi_{T_1} | \Lambda_{i_1}(x_1) | \Psi_{T_1} > dx_1 = ||\Lambda_{i_1}(x_1)^{1/2} \Psi_{T_1}||^2 dx_1$$

3. The algorithm is then repeated, evolving $\Psi_{T_1^+}$ unitarily to a random time $T_2 = T_1 + \Delta T_1$

The constants σ, λ are considered to be constants of nature.

In the flash model, GRWf, the ontology is built upon primitive events termed *flashes*, the space-time location of these flashes being just the space-time locations of each collapse of the wave function. The history becomes a sequence of discrete space-time points

$$(X_1, T_1), \ldots, (X_k, T_k), \ldots, \quad T_1 < T_2 < \cdots$$

denoting the locations of each collapse.

In the GRW model, the wave function appears necessary in order to generate the probability distribution required to drive the spontaneous collapses. That does appear to be a drawback. There has been some work, at least in the quantum field theoretic and relativistic settings to suggest that the wave function itself might not be required [114]. There is still a kind of dual process taking place though, one for the wave function and one for the flashes. Moreover the wave function is not generated in these models, it is somehow given, and the flashes are in a sense driven by it.

Appendix B

Philosophical Assumptions

This chapter addresses some fundamental assumptions underlying many models in physics and explores whether they are known to be factually true based upon empirical evidence or whether they have simply become a part of the philosophical basis and ideology of physics. If the latter then it is argued that there is value is studying what is possible when these assumptions are abandoned even if merely to generate counterfactual predictions that could be disconfirmed by experiment (such as occurs with Bell's inequality). At the very least this permit further limits to be placed upon the types of models that are permissible or useful in theoretical physics.

B.1 Kolmogorovian Probability

The concept of probability is fundamental to many branches of science, and statistical analysis is crucial in determining the outcomes of experiments and in testing various models and formal theories. Probability theory plays a fundamental role in the formulation of quantum mechanics, especially in the non-relativistic case which is the focus of this work. Probability theory is fundamental to the interpretation of quantum mechanics. It is also the source of much of the conceptual confusion and many of the paradoxes that confound quantum mechanics. Hidden variable theorems based upon Bell type inequalities involving relations between various correlation functions depend intimately upon the structure of the probability theory within which these functions are defined and constructed. Indeed that is the very point used by Palmer [286] in his demonstration of a local deterministic hidden variable model of quantum mechanics using iterated function systems. In that model he showed that it was impossible to construct the required three state correlation functions on account of chaotic effects. He produced a local deterministic model of quantum spin correlations that defeated the Bell inequalities. This result has been largely ignored by the physics community.

Modern probability theory is based upon the axioms of Kolmogorov. This theory has achieved the status of dogma, often being held to be *the* theory of probability. Although there may be different interpretations of what the theory describes, the mathematics is considered to be a veridical description of the classical world. To assert otherwise is to bring derision and knee-jerk rejections upon whomever dared to do so. The situation in probability today is similar to that in geometry in the 19th century. For two thousand years, Euclidean geometry had held sway as the one true model of geometry. This conception eventually gave way to the realization that there were actually many different types of geometry, just as there turned out to be many different forms of logic and of set theory. It turns out that there are many different forms of probability theory as well. Probabilities associated to classical events are generally held to be modeled exclusively by Kolmogorov type probability theory. That assumption forms a necessary part of the logical foundation of the various Bell inequalities. That this assumption turns out not to be true forms the central thesis of this section.

Probability theory began in the 17th century in the correspondence between Pascal and Fermat on games of chance. There the goal was to aid the gambler in making decisions that would enable them to maximize their profit from playing these games. Probability was linked directly to the idea of frequency. The probability of an outcome was the limiting value of the fraction of times that the event occurred during the play of the game as the number of plays was extended to infinity. Any individual play of a game resulted in fractions that varied from this number but with repeated play the average of these fractions would tend to the limiting value. This is a consequence of Gauss's celebrated law of large numbers which shows that in repeated measurements of this value the distribution of individual measurements of the value follows a normal distribution. The beautiful mathematical properties of the normal distribution led to its widespread misapplication for more than a century even though natural phenomena manifest events that follow a diversity of probability distributions. This has had serious consequences in fields such as psychology, medicine and economics [418]

Probability theory was placed on firm mathematical ground in the early 20th century by Kolmogorov, who formulated a set of formal axioms based on set theory and analysis, and subsequently elaborated by Carathéodory with his work on measure theory. Kolmogorov's theory of probability has over the past century become widely accepted as *the* theory of probability for the classical realm. It forms the basis for modern statistics. Kolmogorov theory has achieved iconic status and is rarely if ever questioned today.

In order to understand the problem it is first necessary to recall the essential elements of Kolmogorov theory.

First of all, the modern setting for probability theory is that of a measure space. Let Ω be a set and define an algebra of subsets F to be a collection of subsets of Ω such that $\emptyset$ and Ω are in F and the union, intersection and difference of any two sets in F also lies in in F. F is a σ-algebra if the countable union of sets in F also lies in F.

A measure on F is a map $\mu : F \to \mathbb{R}^+$ such that $\mu(A \cup B) = \mu(A) + \mu(B)$ for $A \cap B = \emptyset$. μ is said to be σ-additive if for every pair-wise disjoint sequence of sets $\{A_n\}_{n=1}^{\infty}$, if $A = \cup_{n=1}^{\infty} A_n$ lies in F then $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$.

Given sets Ω_1, Ω_2 and algebras of subsets F_1, F_2, a map $\zeta : \Omega_1 \to \Omega_2$ is termed measurable if for any set $A \in F_2$, the set $\zeta^{-1}(A) \in F_1$.

A Kolmogorov probability space is a triple (Ω, F, P) where Ω is a set, F a σ-additive algebra on Ω and P a σ-additive measure on F taking values in the interval $[0, 1]$.

Random variables on the probability space $\mathcal{P} = (\Omega, F, P)$ are measurable functions from $(\Omega, F) \to (\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. The collection of random variables is denoted $RV(\mathcal{P})$. The probability distribution of a random variable ζ is defined by $P_\zeta(B) = P(\zeta^{-1}(B))$ for $B \in \mathcal{B}$. This defines a σ-additive measure on the Borel space $\mathcal{B}$. Each random variable generates a probability space $(\mathbb{R}, \mathcal{B}, P_\zeta)$.

Crucial to Kolmogorov's theory are three axioms.

1. Let Ω be a set and $E \subset \mathcal{P}(\Omega)$. A probability measure P is a map from E to $[0, 1]$ such that $P(\Omega) = 1$ and $P(A_1 \cup \ldots \cup A_n \cup \ldots) = P(A_1) + \ldots + P(A_n) + \ldots$ for disjoint sets A_i
2. The conditional probability of A given B is defined as $P(A|B) = P(A \cap B)/P(B)$
3. Events A, B are said to be independent if $P(A \cap B) = P(A)P(B)$.

Axiom 2 can be considered to be analogous to the parallel postulate of geometry. Axiom 3 plays a critical role in all hidden variable theorems.

The definition of conditional probability uses Bayes' formula. Note that it is not a theorem, rather it is an assumption, of Kolmogorov probability theory. Using it one obtains as a theorem the following formula of total probability.

Theorem (Law of Total Probability): For any pair-wise disjoint partition $\{A_i\}$ of E, and any $B \in E$,

$$P(B) = P(A_1)P(B|A_1) + \ldots + P(A_n)P(B|A_n) + \ldots.$$

The axiomatic approach, while valuable mathematically, suffers from the problem that information about the intrinsic structure of any real experiment is lost. This turns out to be of vital importance when one attempts to study the statistical relationships among results obtained from multiple, distinct experiments. Indeed in order to Kolmogorov theory to multiple experiments it is first necessary to create a single event space within which all possible outcomes can be described. Formally this means that given several experimental situations, each having their own probability space (Ω_i, F_i, P_i) one wishes to construct a single Kolmogorov probability space (Ω, F, P) in which these smaller spaces arise as special cases. For example one might define $(\Omega, F, P) = (\prod_i \Omega_i, \prod_i F_i, \prod_i P_i)$ where $\prod_i P_i(\prod_i A_i) = \prod_i P_i(A_i)$.

The problem is that, in general, this programme fails. It fails in the case of quantum mechanics for two reasons - interference, and non-commutativity of observables. The frequency interpretation of probability still holds in quantum mechanics. But as Feynman and Hibbs assert:

> The concept of probability is not altered in quantum mechanics. When we say that the probability of a certain outcome of an experiment is p, we mean the conventional thing, i.e., that if the experiment is repeated many times, one expects that the fraction of those which give the outcome in question is roughly p. What is changed, and changed radically, is the method of calculating probabilities [132, pg 2].

The wave function of non-relativistic quantum mechanics is most often viewed as giving rise to a probability distribution of the form $P = \Psi^*\Psi$. This simple interpretation, attributed to Born, actually belies a deep subtlety. Consider the case in which one has a system upon which one may perform two different measurements a, b resulting in the dichotomous outcomes $a = \{a_1, a_2\}$ and $b = \{b_1, b_2\}$. Kolmogorov theory shows that the sum of probabilities takes the form

$$P(b = \beta) = \sum_{a_i} P(a = a_i)P(b = \beta | a = a_i)$$

However, if one attempts the same calculation in a quantum mechanical setting using the Born rule then one obtains the formula

$$P(b=\beta) = \sum_{a_i} P(a=a_i)P(b=\beta|a=a_i) +$$

$$2\cos\theta\sqrt{P(a=a_1)P(b=\beta|a=a_1)P(a=a_2)P(b=\beta|a=a_2)}$$

instead. From this result alone it is clear that quantum probability theory is of a non-Kolmogorov type. The theorems of von Neumann and of Gleason show that the measures on a Hilbert space, and thus the probabilities in quantum mechanics, must be of a non-Kolmogorov type since they arise from the trace of an operator product. In particular, for any closed subspace A of the Hilbert space and every positive semi-definite self-adjoint trace class operator T, one can define a measure on A by $\mu(A) = \text{trace}(TP_A)$, where P_A is the orthogonal projection on A. All measures take this form and thus cannot arise from a simple probability distribution function f in the form $\mu(A) = \int_A f(x)dx$.

Indeed as noted by Khrennikov [208], inequalities of Bell type were developed much earlier in probability theory dating back to the time of Boole and arise in situations in which one attempts to determine correlations when it is impossible to define those correlations using a single Kolmogorov probability space. Khrennikov [208] comments that even Kolmogorov in his original writings on probability understood this point and thus was more sophisticated than later writers. Indeed although Kolmogorov created a purely mathematical model of probability based upon the algebra of sets, he understood that for it to be useful it needed to be connected with the empirical nature of actual experimental measurements. For that, he drew inspiration from the work of von Mises [411]. Von Neumann in his famous treatise on quantum mechanics [412] also incorporates the ideas of von Mises into his statistical interpretation of quantum mechanics (pg 298), as clearly does Feynman as he himself expressed in the quotation above. Von Mises was one of the foremost philosophers of probability at the turn of the Twentieth century. Nearly a century later his influenced has waned as more subjective philosophies of probability gained prominence. In the context of scientific experimentation, and not human inference or belief, the frequentist formulation of von Mises seems closest to empirical practice. Although quite popular at present, subjectivist interpretations [142] of quantum probabilities, especially those determined by the Schrödinger equation, seem problematic. Why should individual beliefs about reality be determined for all observers by a single deterministic partial differential equation? How can it be that individual beliefs and knowledge vary from individual to individual in so many domains of inquiry, but not in the case of quantum mechanics? Von Mises rejected all subjectivist accounts. First and foremost, von Mises viewed probability theory as an empirical science, and not merely a category of logic independent of reality. He wrote [411]:

> Like all the other natural sciences, the theory of probability starts from observations, orders them, classifies them, derives from them certain basic concepts and laws and, finally, by means of the usual and universally applicable logic, draws conclusions which can be tested by comparison with experimental results. In other words, in our view the theory of probability is a normal science, distinguished by a special subject and not by a special method of reasoning. (pg 31)

von Mises describes the purpose of probability theory and in doing so he makes an important and critical point which has been overlooked by subsequent generations of probabilists and statisticians (and physicists as well). He wrote [411]:

> ... we can describe the purpose of the theory of probability as follows: Certain collectives exist which are in some way linked with each other, e.g., the throwing of one or the other of two dice separately and the throwing of the same two dice together form three collectives of this kind. The first two collectives determine the third one, i.e., the one where both dice are thrown together. This is true so long as the two dice, in falling together, *do not influence each other in any way.* If there is no such interaction, experience has shown that the two dice thrown together give again a collective such that its probabilities can be derived, in a simple way, from the probabilities in the first two collectives. This derivation *and nothing else* is here the task of probability calculus. (pg 31)(emphasis mine)

Thus von Mises emphasized the very important point that the usual rules for applying probability theory to a situation require first of all that the entities involved in the various individual collectives be independent of one another. That is, they must not influence or interact with one another in any manner. In the real world, and especially in the quantum mechanical world, this is not always the case. Entanglement is a case in point. In an entangled wave function of the form

$$\frac{1}{\sqrt{2}}(|0>|0|-|1>1>)$$

the wave functions of the individual particles in the individual terms $\frac{1}{\sqrt{2}}|0>|0|>$ and $-\frac{1}{\sqrt{2}}|1>1>$ might be independent of each other, but this is not true for the global wave function.

Kolmogorov also made this point clear in his famous treatise, though unfortunately he failed to examine its ramifications in detail. He wrote [219]

> The concept of mutual *independence* of two or more experiments holds, in a certain sense, a central position in the theory of probability....
>
> We thus see, in the concept of independence, at least the germ of the peculiar type of problem in probability theory. In this book, however, we shall not stress that fact, for here we are interested mainly in the logical foundation for the specialized investigations of the theory of probability.
>
> In consequence, one of the most important problems in the philosophy of the natural sciences is - in addition to the well known one regarding the essence of the concept of probability itself - to make precise the premises which would make it possible to regard any given real events as independent. This question, however, is beyond the scope of this book. (pgs. 8-9)(emphasis his)

Thus Kolmogorov understood that the attribution of the notion of independence to two collectives was not a simple matter of assumption but rather was determined by the details of the relationship between the collectives - what comprises them, what interrelates them, what generates them, what dynamics drives them.Independence appears under specific *conditions*, and these conditions are a matter of empirical observation and analysis. Moreover, at the time, Kolmogorov was speaking of the classical realm, not quantum mechanics.

In discussing the notion of independence, von Mises writes [411]

> However, to obtain a *definition* of independence, we must return to the method that we have already used in defining the concepts 'collective' and 'probability'. It consists in choosing that property of the phenomenon which promises to be the most useful one for the development of the theory, and postulating this property as the fundamental characteristic of the concept which we are going to define. Accordingly, subject to a slight addition to be made later on, we now give the following definition: A collective II will be said to be independent of another collective I if the process of sampling from collective II by means of collective I, using any of its attributes, does not change the probabilities of the attributes of collective II, or, in other words, if the distribution within any of the sampled collectives remains the same as in the original collective II. (Sic)(pg 51)

Note that this is *not* true of entangled particles so it is not really much of a surprise that the hidden variable model used to describe the situation of entanglement cannot use independent variables. The wave function for the pair of entangled particles noted above belies the fact that such entanglement requires specific *conditions* for their creation - it cannot arise between any two particles chosen simply at random. It describes a particular dynamical state between the particles - a point that appears to be frequently missed when such a state is described in the literature. the quantum mechanical formulation describes the probability relationship between the particles but not the dynamical relationship or the conditions that gave rise to the state. Correcting that state of affairs is the subject of part II of this thesis.

From this definition (together with an additional technical requirement on the independence of limiting probabilities from the choice of place selections), von Mises derives the product rule for independent collectives, namely that if A, B are independent, then

$$P(A \cap B) = P(A)P(B)$$

Von Mises writes [411]:

> From two independent collectives of this kind, a new collective can be formed by the process of 'combination', i.e. by considering simultaneously both the elements and the attributes of the two initial collectives.
>
> The result of this operation is that the distribution in the new collective is obtained by multiplying the probabilities of the single attributes in the two initial collectives. (pg. 53)

In consideration of the question of locality and independence, von Mises comments [411]:

> If, finally, two dice are thrown by two different persons from separate boxes, perhaps even in two distant places, the assumption of independence becomes an *intuitive certainty*, which is an outcome of a still more general human experience. In *each concrete case, however, the correctness of the assumption of independence can be confirmed only by a test*, namely, by carrying out a sufficiently long sequence of observations of the dice under consideration, or of another system considered to be equivalent to the one in which we are interested. The results of this test are compared with the predictions deduced from

> the assumption of the multiplication rule, and the collectives are considered as independent if agreement between theory and experiment is found. (emphasis mine)(pg 54-55)

Note that although von Mises asserts that non-locality should imply independence he was an empiricist and even though one might be strongly guided by intuition, ultimately it is only through empirical observation that independence can be assumed. Indeed in the formulation of Bell's argument, it is assumed that if particles are space-like separated then they must be probabilistically independent, and therefore their probabilities must obey the product rule. The argument leads to a contradiction with quantum mechanics, which shows that they weren't probabilistically independent in fact.

The notion of probabilistic independence is actually much more subtle, a fact generally ignored by most discussions of the subject in probability and statistical texts. The notion of independence is closely related to the concept of the object, discussed later in this chapter. Objects are held to simply *be*. They exist complete in themselves, and interact with the world around them either directly or through an intermediary. They are not generated, they simply are. In the process view being advocated here, physical entities are not objects but rather represent emergent aspects of processes. They are generated de novo at each instant and the apparent continuity in form of an entity is a reflection of the persistence of certain information through time by the generating process. The object-like properties of a physical entity are thus determined by the generating process and therefore may or may not be present and may or may not be characteristic of the entity. Furthermore, the independence or non-independence of the entities is not determined by the entities themselves but rather by the processes that generate the entities. Without knowledge of the generating processes, independence cannot be assumed even for space-like separated entities and becomes a matter of empirical observation. This point will be examined in more detail in Part II.

Kolmogorov defines independence as follows [219]:

Definition (Mutual Independence): N experiments $\mathfrak{U}_1, \mathfrak{U}_2, \ldots, \mathfrak{U}_N$ are called *mutually independent*, if for any $q_1, q_2, \ldots, q_N$ the following equation holds true:

$$P(A^1_{q_1} A^2_{q_2} \cdots A^N_{q_N}) = P(A^1_{q_1}) P(A^2_{q_2}) \cdots P(A^N_{q_N})$$

where $A^i_{q_i}$ refers to the q_i-th event in collective $\mathfrak{U}_i$.

Note that the product rule must hold for *all* combinations of events across *all* of the collectives. It is simply inadequate to determine this for pairwise combinations and then to presume that it holds true generally. Kolmogorov presents a simple example from Bernstein to show this [219]:

> This can be shown by the following simple example (S. N. Bernstein): Let the set E be composed of four elements $\xi_1, \xi_2, \xi_3, \xi_4$; the corresponding elementary probabilities p_1, p_2, p_3, p_4 are each assumed to be $1/4$ and
>
> $$A = \{\xi_1, \xi_2\},\ B = \{\xi_1, \xi_3\},\ C = \{\xi_1, \xi_4\}$$
>
> It is easy to compute that
>
> $$P(A) = P(B) = P(C) = 1/4$$
>
> $$P(AB) = P(BC) = P(AC) = 1/4 = (1/2)^2$$
>
> $$P(ABC) = 1/4 \neq (1/2)^3$$
>
> (pg 11)

Independence is frequently simply assumed in most applications of probability theory to real events. In fact it must be proven to be true.

Problems related to independence are closely related to the problem of combining collectives. Von Mises [411] describes four different ways of creating new collectives from old ones. These are selection, mixing, partitioning and combining. Mixing assumes that no interaction occurs between the events being mixed, otherwise the simple rules may fail.

Von Mises points out [411]:

> The clarification and the ensuing correct formulation of the mixing operation can only be achieved by having recourse to the concept of the collective. The difference between the correct formulation of the addition rule and the incorrect one follows from the principle that only such probabilities can be added as are attached to different attributes in one and the same collective. The operation consists in mixing only attributes of this kind.... *It is not permissible to mix together attributes belonging to two different collectives.* (pgs. 40-41)(emphasis mine)

The operations of selection, mixing and partitioning all share the common feature that they create a new collective from exactly *one* original collective. In combining, two or more collectives are formed into a single collective. The situation is no longer simple.

Khrennikov writes "For him (Kolmogorov) it was totally clear that it is very naive to expect that all experimental contexts can be described by a single (perhaps huge) probability space"[208, pg. 26]. It may not to possible to measure all of these observables simultaneously.

Nevertheless even Kolmogorov was aware that there was a contextual aspect to his theory and that every experiment should be described by its own Kolmogorov probability space. He also used Bayes formula to define conditioning which, in Khrennikov's opinion [208], resulted in a very special notion of contextuality and conditioning. Indeed Kolmogorov was very clear about what he meant when constructing a probability space. He clearly accepted the considerations of von Mises when discussing the relationship between his theory and experimental data. He wrote [411]:

> We apply the theory of probability to the actual world of experiments in the following manner:
>
> 1. There is assumed a complex of conditions, $\mathfrak{C}$, which allows of any number of repetitions.
> 2. We study a definite set of events which could take place as a result of the establishment of the conditions $\mathfrak{C}$....We include in set E all the variants which we regard *a priori* as possible.
> 3. If the variant of the events which has actually occurred upon realization of conditions $\mathfrak{C}$ belongs to the set A (defined in any way), then we say that the event A has taken place....
> 4. Under certain conditions, which we shall not discuss here, we may assume that to an event A which may or may not occur under conditions $\mathfrak{C}$, is assigned a real number $P(A)$...

It is clear from Kolmogorov's description that the probability will depend upon the particular conditions $\mathfrak{C}$, thus on the context within which the experiments were conducted. It is also important to note that even for Kolmogorov, probability theory was a contextual theory. In a strict sense the question of whether or not a hidden variable theory is contextual or non-contextual is irrelevant since as Kolmogorov realized, all probability theory is contextual even in a classical setting. Von Mises emphasized this point as well in his discussion of mutually independent, combinable but interdependent and noncombinable collectives. The latter is of critical importance. Von Mises writes [411]:

> It is, finally, not without interest to give an example of two collectives which are neither independent nor dependent in the above sense, collectives which we consider altogether uncombinable.... The initial single sequences of observations have the property of randomness; they have, however, a certain mutual relation which precludes their being combined into a new collective. We call collectives of this kind *noncombinable*.
>
> This example illustrates again the insufficiency of the well known elementary form of the multiplication rule, which does not take into account the possible relations between the two collectives, A reliable statement of the multiplication rule can only be based on a rational concept of probability whose foundation is the analysis of the collective. (Sic)(pgs. 56-57)

Von Mises emphasized that it was not enough to simply have two or more collectives and then naïvely combine them into a single large collective. In order to do this it is first necessary that they be compatible (and hence combinable). The important issue is then whether or not a set of observables exhibits probabilistic compatibility or incompatibility, that is, whether it is possible to construct a single probability space serving for the entire family. This point was lost on subsequent generations of probability theorists. It is also important to note that the issue of probabilistic compatibility is a generic issue, that applies to the combination of any conditions, classical or quantum. There is nothing unique or special about quantum mechanics is this regard. The reasons for probabilistic incompatibility may differ between classical and quantum conditions, but the end result is the same. One cannot naïvely apply the rules of Kolmogorov probability theory in such cases.

A complete analysis of the problem of probabilistic compatibility was provided by Vorob'ev in 1962 [413], though according to Khrennikov, this work was largely ignored by the probability community until 2001. Vorob'ev's paper is quite detailed. He states the problem in simple terms. Suppose that one is given three two-dimensional random variables $(X_1, Y_1), (X_2, Z_2), (Y_3, Z_3)$ and the distribution functions of X_1 and X_2, Y_1 and Y_3, Z_2 and Z_3 coincide respectively. Is there a three dimensional random variable (X, Y, Z) whose two dimensional projections have distributions matching those of (X_1, Y_1),(X_2, Z_2),(Y_3, Z_3)? The answer, in general, is no, even though the general teaching in probability theory, and certainly in most every discipline in which probability theory is applied, states otherwise.

As in virtually every branch of mathematics, there are qualifications and prerequisite conditions. Vorob'ev's result is so important that I will describe it in some detail.

Vorob'ev set his theorem within the framework of combinatorial topology, a forerunner of modern algebraic topology.

To state Vorob'ev's result requires some background definitions.

Definition: A complex (abstract or simplicial complex) is a pair (K, Σ) where K is a set and Σ is a set of subsets of K such that

1. If $s \in \Sigma$ and $s \supset s'$, $s' \neq \emptyset$, then $s' \in \Sigma$
2. Every set consisting of a single element of K is in Σ and the empty set is not in Σ.

Usually one thinks of a Euclidean *n-simplex* $\mathfrak{S}$ as defined by a finite set $\{a_i\}$ of n points in some N-dimensional Euclidean space called vertices and consisting of the set of all points $\{\sum_i \lambda_i a_i | \sum_i \lambda_i = 1, \lambda_i \geq 0\}$. A q-simplex, $q \leq n$ defined by some q element subset of $\{a_i\}$ is called a q-face of $\mathfrak{R}$.

A Euclidean (simplicial) *complex* $\mathfrak{K}$ is a set of simplexes such that

1. Every face of a simplex in $\mathfrak{K}$ belongs to $\mathfrak{K}$
2. The intersection of two simplexes in $\mathfrak{K}$ is either empty or a face of each of them
3. Each point in a simplex in $\mathfrak{K}$ has a neighbourhood that intersects only a finite number of simplexes in $\mathfrak{K}$

Definition: Given a complex $\mathfrak{K}$, an *r-skeleton* is a sub-complex of $\mathfrak{K}$ consisting of all n-simplexes in $\mathfrak{K}$ ($n \leq r$.

Definition: A *complex* $\mathfrak{A}$ is a finite, unrestricted complex in the combinatorial-topological sense of the term. As an example take the set I_n consisting of all subsets of an n-element set. The set of all vertices of $\mathfrak{A}$ is denoted $|\mathfrak{A}|$.

Definition: A *skeleton* T of a complex $\mathfrak{A}$ is called *maximal* if it is not a proper face of some other skeleton of $\mathfrak{A}$. The number of maximal skeletons of the complex $\mathfrak{A}$, called its *length* is denoted $\kappa(\mathfrak{A})$.

Definition: Let T_1 and T_2 be two distinct maximal skeletons of $\mathfrak{A}$. T_1 yields a *maximal intersection* with T_2 if the intersection $T_1 \cap T_2$ is not a proper face of the intersection $T_1 \cap T_3$ for some skeleton $T_3 \in \mathfrak{A}$.

Definition: A *maximal skeleton* $T \in \mathfrak{A}$ is called *extreme* in $\mathfrak{A}$ if all maximal intersections of skeletons of $\mathfrak{A}$ are equal. Denote this unique maximal intersection by A. By definition, an empty complex has no extreme skeletons.

Lemma: If T is an extreme skeleton of the complex $\mathfrak{A}$, then there is a vertex $a \in T$ which does not belong to any maximal skeleton of $\mathfrak{A}$ different from T. The vertex a is called a *proper vertex* and the set of all proper vertices of T is denoted $\sigma_{\mathfrak{A}} T$.

Definition: Let T be an extreme skeleton in the complex $\mathfrak{A}$. Consider the sub-complex $\mathfrak{N}_T$ of $\mathfrak{A}$ consisting of all skeletons of $\mathfrak{A}$ which do not intersect $\sigma_{\mathfrak{A}} T$. $\mathfrak{N}_T$ is called the *normal* sub-complex of $\mathfrak{A}$ corresponding to the extreme skeleton T. A sub-complex is called, generically, a normal sub-complex if it is a normal sub-complex of $\mathfrak{A}$ corresponding to some extreme skeleton.

Definition: A *normal series* is a sequence of sub-complexes

$$\mathfrak{A} = \mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_r$$

where for each i, $\mathfrak{A}_{i+1}$ is a normal sub-complex of $\mathfrak{A}_i$ and the complex $\mathfrak{A}_r$ does not possess any extreme skeletons.

Definition: The complex $\mathfrak{A}$ is called *regular* if it possesses a normal series in which the last term is the empty set.

Definition: r is called the *length* of the normal series and one can show that $r < \kappa(\mathfrak{A})$ if the normal series is non regular and $r = \kappa(\mathfrak{A})$ if the series is regular.

The class $\mathfrak{R}$ of regular complexes can be characterized as follow:

1. $\mathfrak{R}$ contains all complexes of the form I_n, $n \geq 1$.
2. If $\mathfrak{A} \in \mathfrak{R}$, $T \in \mathfrak{A}$, $R \cap |\mathfrak{A}| = \emptyset$, then $\mathfrak{A} \cup T_{T \cup R} \in \mathfrak{R}$.
3. $\mathfrak{R}$ is the smallest class of complexes satisfying conditions 1 and 2.

Definition: A set S together with a system Σ of σ-algebras of subsets is called a generalized measurable space, $< S, \Sigma >$. If there exists a family of measures μ_Σ such that to each σ-algebra $\mathfrak{S} \in \Sigma$ there exists a measure $\mu_{\mathfrak{S}} \in \mu_\Sigma$ then the triple $< S, \Sigma, \mu_\Sigma >$ is called a generalized space with measures.

Definition: Let $< S, \mathfrak{S}_1 >$ and $< S, \mathfrak{S}_2 >$ be two measurable spaces with measures μ_1, μ_2 respectively. These measures are called *consistent* if $\mu_1(A) = \mu_2(A)$ for any $A \in$

$\mathfrak{S}_1 \cap \mathfrak{S}_2$. Given a generalized space with measures $< S, \Sigma, \mu_\Sigma >$, the family of measures μ_Σ is called a *consistent* family of measures if all of the measures in μ_Σ are pairwise consistent.

Clearly when Σ consists of a single σ-algebra $\mathfrak{S}$ the generalized measurable space turns out to be the usual measurable space.

Definition: Let $[\Sigma]$ denote the smallest σ-algebra which contains $\cup_{\mathfrak{S}\in\Sigma}\mathfrak{S}$. The measure μ on the measurable space $< S, [\Sigma] >$ is called an *extension* of the family of measures μ_Σ on the generalized measurable space $< S, \Sigma >$ if it is consistent with every measure of Σ. If such a measure exists, then the family of measures is called *extendable.*

Definition: Let M be a set and to each $m \in M$ associate a measurable space $< S_m, \mathfrak{S}^0_m >$. Let $K \subset M$ and let $S_K = \prod_{m\in K} S_m$. Let $\mathfrak{B}_K$ consist of the set of all measurable rectangular K-cylinders in S_M. This is the family of all subsets of the form

$$\prod_{m\in K} A_m \times S_{M-K},\ A_m \in \mathfrak{S}^0_M$$

Let $\mathfrak{S}_K$ denote the smallest σ-algebra of subsets of S_M which contains $\mathfrak{B}_K$. Therefore to each $K \subset M$ there exists a measurable space $< S_M, \mathfrak{S}_K >$. Given any arbitrary family of subsets $\mathfrak{A}$ of M one may form a generalized measurable space $< S_M, \Sigma_{\mathfrak{A}} >$ where $\Sigma_{\mathfrak{A}}$ consists of all σ-algebras $\mathfrak{S}_K$ for $K \in \mathfrak{A}$.

The main result of Vorob'ev is the following:

Theorem: Let M and S_M be finite, $\mathfrak{A}$ some complex of subsets of M and $< S_M, \Sigma_{\mathfrak{A}}, \mu_{\mathfrak{A}} >$ a generalized measure space with consistent measures. Then the consistent family of measures $\mu_{\mathfrak{A}}$ is extendable if and only if the complex $\mathfrak{A}$ is regular.

The proof is long but the structure of the extended measure can be described. For each $K \subset M$ and $x_K \in S_K$ set $x^*_K = x_K \times S_{M-K}$. The sets x^*_K are minimal subsets of the σ-algebra $\mathfrak{S}_K$. For $x \in S_M$, let x_K denote the projection of x on S_K. Given a regular complex $\mathfrak{A}$, let T be one of its extreme skeletons and let $\mathfrak{A}_T$ denote the normal sub-complex of $\mathfrak{A}$ corresponding to T. Let $V = |\mathfrak{A}_T| \cap T$ and for any $x \in S_{|\mathfrak{A}|}$ for which $\mu_V(x^*_{|\mathfrak{A}_T|}) \neq 0$ set

$$\mu_{|\mathfrak{A}|}(x) = \frac{\mu_{|\mathfrak{A}_T|}(x^*_{|\mathfrak{A}_T|})\mu_T(x^*_T)}{\mu_V(x^*_V)}$$

and when $\mu_V(x^*_V) = 0$ set $\mu_{|\mathfrak{A}|}(x) = 0$. For any $X \subset S_{|\mathfrak{A}|}$ set

$$\mu_{|\mathfrak{A}|}(X) = \Sigma_{x\in X}\mu_{|\mathfrak{A}|}(x)$$

Examples of non-extendable families include the complex G_n of all proper subsets of a set of n elements for $n \geq 3$, and the complex Z_n consisting of skeletons of the form $\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{n-1}, a_n\}, \{a_n, a_1\}$ for $n \geq 3$.

In particular, note that taking a triple of probability measures and then considering correlations involving pairs of the individual event spaces of the form $(a_1, a_2), (a_2, a_3), (a_3, a_1)$ runs afoul of this theorem. This in turn has serious consequences for the Bell Inequalities which are based upon correlations of this type. Bell assumes the existence of the product rule pairwise. However, he requires the use of a triple of particles and under such a condition the product rule generally fails. Palmer showed that it could fail for constructive reasons [286]. Khrennikov showed that it could fail because of contextual reasons [207] dynamics of the generation of the primitive events, so that each individual event may be generated using only causally local information and yet global correlations may appear because multiple events may be generated by a *single* causal process.

Probabilistic incompatibility is a problem in quantum mechanics due to the noncommutative nature of the set of self adjoint operators representing quantum measurements and the presence of interference terms arising from probabilities based on Born's rule. It is also not true of classical system in general, something that has been mostly ignored (Simpson's paradox in the social sciences is an example of this [78]). There may be many different reasons for the breakdown of Kolmogorov probability structure. As in quantum mechanics, it may simply be impossible to carry out the measurements simultaneously. This is certainly true for non-commuting observables but this can happen classically as well. In psychology, because of learning and other neurophysiological processes, one can never truly restore a subject to their initial state and so one can never truly ever repeat an experiment. One can only ensure that certain large scale conditions are preserved and make assumptions about the nature of the variations that are observed. But that can be very flawed. The probability distribution underlying the generating process may be non-stationary, that is, may change over time, so one is not dealing with a time series from a single collective but rather from many collectives. It may also happen that there are interference or interaction effects between different components of the system under study. It may simply be impossible to observe the system in sufficient detail in order to detect and denote the distinct states involved in these interactions. In such a case one must stick with the observational space that one has. Experimental probabilities are just that - experimental probabilities. In such cases, non-Kolmogorov effects may be observed. Nonlocality is not required in order to obtain those results. They are wholly dependent upon the nature of the observables being measured and whether or not a single Kolmogorov probability space can be constructed. These observations raise questions as to whether it is absolutely impossible to have local hidden variable models at the lowest levels.

In the following sections, a number of very simplistic examples are presented to illustrate these points. They are not meant to provide rigorous mathematical arguments, but merely to point out that if we stop thinking in a rigid, orthodox manner, then more possibilities present themselves, and the description of the probabilities from a non-Kolmogorov framework is much more tenable than from the standard Kolmogorov point of view.

B.1.1 Failure of Additivity

The rule of additivity is fundamental in Kolmogorov's formulation of the laws of probability, providing one of its axioms. It fails in quantum mechanics. Indeed suppose that a single quantum system can exist in one of a set of distinct energy eigenstates Ψ_i. These energy eigenstates form an orthonormal set of functions in some Hilbert space. Suppose that the system is now created in a linear superposition of these energy eigenstates as is permitted by the Schrodinger equation. The wave function for this superposition will take the form $\Psi = \sum_i w_i \Psi_i$ where the weights w_i are chosen so that $\sum_i |w|_i^2 = 1$ which ensures that Ψ can be interpreted in its own right as a probability distribution. This guarantees that $< \Psi^* | \Psi >= \sum_{ij} w_i^* w_j < \Psi_i^* | \Psi_j >= \sum_i |w|_i^2 = 1$.

When the energy of such a quantum system is measured it will yield a single value corresponding to one of these energy eigenstates. If the system is subjected to repeated measurements of its energy it will remain in the same energy eigenstate. This is considered due to the collapse of the wave function that occurs as a result of the measurement process. If multiple identically created copies of the system have their energies measured then these energies will be distributed according to the probability distribution given by $(w_1^2, \ldots, w_n^2, \ldots)$.

The expectation value of the energy is given by $\hat{E} = \sum_i w_i^2 E_i$. Suppose though that one asks a slightly different question, namely, fix some region of space, say R, and ask what is the expectation value of the energy over the region R. This is calculated as

$$\hat{E}_R = \int_R \sum_i \sum_j w_i^* w_j \Psi_i^* E_j \Psi_j dV = \sum_i \sum_j w_i^* w_j E_j \int_R \Psi_i^* \Psi_j dV.$$

Rewriting yields $\hat{E}_R = \sum_j E_j \sum_i w_i^* w_j \int_R \Psi_i^* \Psi_j dV$. One sees that the new probability associated to each energy E_i is no longer w_i^2 but rather $\sum_i w_i^* w_j \int_R \Psi_i^* \Psi_j dV$. This is due to the fact that the wave functions may overlap on R. It is only in the context of the entire space-time that the wave functions are orthogonal. There is no guarantee that

$\sum_i \sum_i w_i^* w_j \int_R \Psi_i^* \Psi_j dV = 1$ so these probabilities are not additive even though the events, namely the E_i cover the range of possible energy values.

Also note that in constructing a superposition state one is in essence constructing a sum of probabilities, for if λ_i is the eigenvalue associated with eigenstate Ψ_i then the probability based upon the wave function of a superposition becomes

$$\Psi^*\Psi = [\sum_i P(\Psi_i)P(\lambda_i|\Psi_i)] + \text{interference terms}$$

This problem is frequently considered to be a feature of quantum mechanics because the quantum mechanical formalism allows for the phenomenon of quantum interference. That it appears in the classical realm as well is illustrated by the following simple model. Most everyone is familiar with the Danish children's toy, LEGO. Typical LEGO pieces are blocks of plastic having tiny solid cylinders protruding on the top surface of the block and corresponding cylindrical tubes in place on the undersurface. There are plates that can be used for mounting LEGO block structures. Consider the following scenario. There is a 2×2 mounting block fixed inside a sealed box. Within the box is a bag containing a 1×1 block and a 2×2 block. There is dial on the outside of the box which reads 0,1,2. When the dial is set, a reading is taken of the plate and a light turns on corresponding to whether there is no block on the plate (0), a 1×1 block, whether alone or combined with a 2×2 block (1) or a 2×2 block again alone or in combination with a 1×1 block (2). The examiner cannot look in the box and in fact has no knowledge of the contents of the box. They can only switch the dial and note whether or not a light appears. In another room a researcher can remotely arrange whatever they like on the plate: no block, a 1×1, or a 2×2 block and they change the arrangement immediately following each observation of the examiner. Clearly the probabilities of no block, a 1×1 block or a 2×2 block are all 1/3. Therefore for the examiner the probabilities of obtaining a light for 0,1,2 are all $1/3$.

Now let us change the game slightly. The researcher is now permitted to take no action, place a 1×1 or a 2×2 block on the plate, or to couple the 1×1 block to the top of the 2×2 block and affix this to the plate. Setting the dial to 1 or 2 results in a light so long as the corresponding block is present regardless of whether it is alone or in combination. Note that it is impossible in this arrangement to measure for 1 and 2 simultaneously. Now what is the probability of there being a light on 1? This probability is $1/2$ because there is a $1/4$ probability of there being a single 1×1 block and a $1/4$ probability of there being a $1 \times 1 - 2 \times 2$ combination. The same holds for the probability of a light on 2, while the probability of a light on 0 remains $1/4$. Note that now $P(0) + P(1) + P(2) = 1/4 + 1/2 + 1/2 = 11/2$. As far as the examiner is concerned, the outcomes are disjoint but the sum is not additive to 1.

A standard argument to correct this problem is to assert that the space of alternatives has been incorrectly constructed. If the examiner is allowed to look at the blocks then they might argue that only the global configurations constitute allowable events and these decompose into four equal probabilities, and then the probabilities of occurrence of the individual smaller blocks can be determined using conditional probabilities as per the Kolmogorov scheme. In such a case the probability of a 1x1 block becomes: $P(1\times 1) = P(0)P(1\times 1|0)+P(1\times 1)P(1\times 1|1\times 1)+P(1\times 1+2\times 2)P(1\times 1|1\times 1+2\times 2)+P(2\times 2)P(1\times 1|2\times 2)$ =1/4× 0 + 1/4× 1 + 1/4× 1 + 1/4× 0 = 1/2, which is the result given above. But this is a mathematical cheat because it assumes knowledge that the examiner does not and cannot possess. From the point of view of the examiner the space of alternatives was correctly constructed and they are disjoint. However they must also accept the necessity to introduce an interaction term, or to accept a non-standard form for the calculation of the total probability, namely $P(total) = P(0)+P(1)+P(2)+I(0,1,2) = 1/4+1/2+1/2-1/4$.

Arguing that this scenario is contrived is also a cheat because this is precisely the situation for the experimental physicist. Measurement devices provide only the results of measurements, they do not yield the states of the systems being measured which cannot be directly observed. Recall Tumulka's comments about the inadequacies of the density matrix for distinguishing between distinct ontological states [403]. The idea of an particle being in a superposition comes out of theory, not direct observation. As in many cases experiments are contrived to create a collection of particles in a pre-determined state so that the examiner has some knowledge beforehand. If no such knowledge is obtainable, or if simultaneous measurements cannot be made it may not be possible to confirm the existence of such interaction states so as to expand the space of alternatives in such a manner so as to preserve the Kolmogorov property. The preservation of Kolmogorov probability appears to require that one begin with the most basic 'natural kinds' from which all other functions are derived, but if we do not know that combinations exist we can only deal with the event set in hand.

Interference creates a failure of the usual additivity rule in the quantum mechanical case and in this classical case as well. Thus one must accept that the Kolmogorov axioms may work well in many circumstances but there may be other situations in which they fail, and instead of denying the validity of these alternative situations, we should embrace the idea that, just as in the acceptance of non-Euclidean geometry, we should accept the existence of non-Kolmogorov probability theories. There is nothing a priori wrong with the question that the examiner asks, nor the interpretation made of the conditions under which the question is to be answered, unless one requires that the answer follow the conditions of Kolmogorov probability theory. Instead, this very simple example urges us to accept the existence of non-Kolmogorov probability theories, even in the classical setting, and in situations in which the basic elements of observations are derived from prior conditions that are able to interact or superpose in some manner. Such possibilities are abundant in quantum mechanics but also in the life and social sciences. Khrennikov has emphasized this point in his extensive writings on non-Kolmogorov probability theory [208].

The problem arises in this example because the Lego pieces are able to interact. The problem arises in quantum mechanics because in a superposition state the individual eigenstates interact. There is no fundamental difference between these two cases. Kolmogorov theory presumes that there is no interaction between individual events and that distinct events correspond to distinct natural kinds. This example might seem trivial yet it lies at the heart of the problem of measurement. When we consider an electron, for example, in a superposition of distinct energy states, what exactly do we mean? When we ask the question of its energy, we are asking exactly the question of the examiner above whether or not when we observe the electron do we observe one of its supposed constituent energy states. We do not think of the superposed electron as a different natural kind from the non-superposed electron. Rather we think of an electron in a particular state, and that very same electron can change state into one of the energy eigenstates or back into a different superposed energy state. The electron is the natural kind, not the state. Moreover an electron, to the best knowledge available today, does not appear to be composed of smaller natural kinds, it is a single whole.

The point to be made is that in any model providing a realist interpretation of quantum mechanics it is necessary to pay close attention to the subtle nature of interactions among the various elements that make up the model. One must be very careful not to project fundamental features of Kolmogorov probability theory onto non-Kolmogorov probability theories. These are subtle conceptual and logical errors which I suspect have arisen time and time again in our attempts to understand quantum mechanics. Over the past century we have become comfortable with non-Euclidean geometry and such logical errors no longer plague the field. Hopefully the same may one day be true of quantum mechanics. The most important consideration in constructing an alternative model of quantum mechanics is to ensure that the non-Kolmogorov nature of the probabilities be preserved in the model.

There are other issues at play besides the type of constitutional interference as noted above. The inability to construct a single space upon which all of the probability functions can be constructed is another feature that is frequently ignored, even in the classical application of Kolmogorov probability theory. This problem arises when one has a collection of distinct suitable state spaces upon which Kolmogorov probabilities are developed and then one attempts to combine these into a single space in order to calculate correlations and conditional probabilities and still expect the original individual probabilities to be derivable. Probability theorists have known for a century that such a construction is not always possible and yet time and again researchers proceed as if they can carry out such a construction.

In the model to be constructed below we shall utilize combinatorial games with tokens which frequently give rise to non-Kolmogorov probability structures.

B.1.2 Failure of Stationarity

Let us consider another classical example. Consider the following iterated function system, denoted ϕ. Consider a simple 2×2 block into which we place different numbers. For example one might configure the block as $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. There are two transformations α, β which can be applied to such a block. Transformation α interchanges the elements in the first row. Thus $\alpha \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ while transformation β interchanges the elements in the second row, so $\beta \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Now consider an iterative function system defined on the space of blocks $\{\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\}$. For simplicity denote this set as $\{a, b, c, d\}$ respectively.

Applying α, β to this set induces the following transformations where a move to the right represents an application of α and a move down represents an application of β.

$$\begin{array}{ccccc} a & \rightarrow & b & \rightarrow & a \\ \downarrow & & \downarrow & & \downarrow \\ c & \rightarrow & d & \rightarrow & c \\ \downarrow & & \downarrow & & \downarrow \\ a & \rightarrow & b & \rightarrow & a \end{array}$$

If we start with a and repeatedly apply α, β we will end up with a collection of possible sequences of blocks which can be represented in the form of a tree in which an arrow down and to the left means apply α and down and to the right means apply β.

$$
\begin{array}{ccccccccccccc}
 & & & & & & a & & & & & & \\
 & & & & & \swarrow & & \searrow & & & & & \\
 & & & & b & & & & c & & & & \\
 & & & \swarrow & & \searrow & & \swarrow & & \searrow & & & \\
 & & a & & & & d & & & & a & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & \\
b & & & & c & & & & b & & & & c
\end{array}
$$

Each layer represents a possible outcome after a fixed number n of iterations. In order to determine the probability of observing a particular outcome one must sum up the number of paths leading to said outcome and then divide by the total number of possible paths. Summing over the paths leading to each outcome leads to a tree diagram

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & \swarrow & & \searrow & & & & & \\
 & & & & 1 & & & & 1 & & & & \\
 & & & \swarrow & & \searrow & & \swarrow & & \searrow & & & \\
 & & 1 & & & & 2 & & & & 1 & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & \\
1 & & & & 3 & & & & 3 & & & & 1
\end{array}
$$

Dividing by the total number of paths at each level gives

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & \swarrow & & \searrow & & & & & \\
 & & & & 1/2 & & & & 1/2 & & & & \\
 & & & \swarrow & & \searrow & & \swarrow & & \searrow & & & \\
 & & 1/4 & & & & 1/2 & & & & 1/4 & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & \\
1/8 & & & & 3/8 & & & & 3/8 & & & & 1/8
\end{array}
$$

We now find the probability for a given outcome by summing over the probabilities for all paths leading to the outcome. This yields the following probability distributions:

Level $0 : f = (1,0,0,0)$, Level $1 : g = (0,1/2,1/2,0)$, Level $2 : h = (1/2,0,0,1/2)$, Level $3 : g = (0,1/2,1/2,0)$.

Thus as we successively iterate the system, the probability distributions at successive times oscillate $f, g, h, g, h, g \ldots$.

It is important to note that no probability distribution has been assigned a priori to the choices of the elements α, β. If a probability is pre-assigned then the above probabilities need to be modified by multiplying each path segment by the probability assigned to the particular path choice, either α or β.

The above model provides a simple, non-deterministic dynamical system which is entirely classical, where the probabilities are determined by a discrete version of real valued path integrals, and which yield temporally oscillating, spatially non-stationary probability distributions. The point of this example is to highlight the fact that quantum mechanical systems are not alone in having a probability structure that can be calculated utilizing path integrals. Combinatorial based classical systems such as the example described above and many combinatorial games possess this path integral structure. Whether or not there exists a limiting stationary probability distribution over the state space depends upon the tree structure induced by the dynamics of the combinatorial operations. Most iterated function systems involving actions on a continuous real space require some form of contraction so as to ensure that an invariant or stationary measure exists on the state space.

B.1.3 Failure of the Law of Total Probability

Let us stay with the block space. Consider a second iterated function system acting on the same space of blocks. Call it ρ. This time we have a single function γ acting on the block space. The action of γ on any block is to interchange the first and second columns. That is $\beta \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

The action of γ on the block space is simple: $a \to d$ and $b \to c$.

Again starting with block a and repeating the procedure of the previous section yields the following probability distributions:

Level $0 : f = (1, 0, 0, 0)$, Level $1 : j = (0, 0, 0, 1)$, Level $2 : i = (1, 0, 0, 0)$, Level $3 : j = (0, 0, 0, 1)$.

Note that the distribution for ρ is distinct from ϕ in the previous example and that these represent two distinct iterated function systems. Now let us consider the iterated function system σ on the block space generated by $\{\alpha, \beta, \gamma\}$. Again start with block a. Applying either of α, β or γ yields the outcomes b, d, c. Applying the maps to these outcomes yields outcomes $a, c, d, c, a, b, d, b, a$. Outcomes are repeated in the above listing as each represents a distinct path down the tree. Applying the maps once more yields the 27 outcomes .

$$b, d, c, d, b, a, c, a, b, d, b, a, b, d, c, a, c, d, c, a, b, a, c, d, b, d, c$$

The probability distributions are thus

Level 0 : $f = (1, 0, 0, 0)$, Level 1 : $k = (0, 1/3, 1/3, 1/3)$, Level 2 : $l = (1/3, 2/9, 2/9, 2/9)$, Level 3 : $m = (2/9, 7/27, 7/27, 7/27)$.

We may denote this iterated function system as $\sigma = (2/3)\phi + (1/3)\rho$. If we combine their probability distributions then we would obtain

Level 0 : $f = (1, 0, 0, 0)$, Level 1 : $2/3g + 1/3j = 2/3(0, 1/2, 1/2, 0) = 1/3(0, 1, 0, 0) = (0, 2/3, 1/3, 0)$, Level 2 : $2/3h + 1/3i = 2/3(1/2, 0, 0, 1/2) + 1/3(1, 0, 0, 0) = (2/3, 0, 0, 1/3)$, Level 3 : $2/3g + 1/3j = 2/3(0, 1/2, 1/2, 0) + 1/3(0, 1, 0, 0) = (0, 2/3, 1/3, 0)$.

Note that $k \neq 2/3g + 1/3j$, $l \neq 2/3h + 1/3i$ and $m \neq 2/3g + 1/3j$. Thus although these probabilities should add according to the usual notions of probability theory they do not because there is an interaction effect. In this case, although ϕ and ρ are distinct iterated functions systems and their superposition gives rise to a perfectly good iterated function system, the resulting probability distribution functions cannot be obtained from a simple weighted sum of the individual prior probability distributions because there is a structural dynamical constraint, namely $\alpha\beta = \gamma$, which gives the appearance of an interaction effect in the generation of events.

This is a simple example but it bears a formal similarity to the situation in quantum mechanics where one considers linear superpositions of eigenfunctions. In both cases, difficulties arise with the usual composition of probability distribution functions because of interaction effects, usually function overlap in the case of quantum mechanical systems, algebraic effects in the simple iterated function system discussed here. The significance of this example is that this demonstrates the failure of additivity even in the case of a classical system with real valued functions. Quantum mechanics is not necessary for such non-Kolmogorov effects to appear.

B.1.4 Failure of Bell's Theorem

Let us now consider the following pair of single player combinatorial games. They are not very interesting as games but they illustrate a feature of games which is that they can defeat the Bell inequality under certain conditions. For this example consider a pair of games played out on the previously defined 2×2 blocks a, b, c, d, one game using the transformation α and the other γ. We consider sequential game play, and we are interested in the outcome following every two steps of play. As described previously, Bell's original theorem involves relationships among three correlation functions based upon spin measurements on a pair of entangled particles. In this example we consider correlation functions based on trajectories defined by repeated game play, with different initial conditions replacing the different orientations of measurement.

We consider three initial conditions, a, b, d. In the correlations defined below, the first variable refers to the game generated by α and the second to that generated by γ. Measurements of a, b, c, d have defined values of $1, 1/2, -1/2, -1$ respectively.

Play using α or γ yields distinct trajectories. Nevertheless when we restrict ourselves to two play games we note that $\alpha\alpha = id$ and $\gamma\gamma = id$ so that we always obtain constant trajectories, namely just the initial condition.

Bell's inequality now takes the form

$$1 + E(\mathbf{b}, \mathbf{c}) \geqq |E(\mathbf{a}, \mathbf{b}) - E(\mathbf{a}, \mathbf{c})|$$

Note that only the expectations values are important, not the circumstances under which they were generated. It is only important that the measurement values in directions $\mathbf{a}, \mathbf{b}$ be ± 1. So long as we ensure that under two conditions the measurement values also be ± 1 then we meet the essential mathematical requirements of Bell's Theorem. Clearly in this simple example if we choose initial conditions a, d then we shall obtain measured expectation values of $1, -1$ respectively. Choose for the third initial condition the block b. Bell's inequality takes the form

$$1 + E(d, b) \geqq |E(a, d) - E(a, b)|$$

Since the play corresponds to simply applying the identity to the initial condition, the probability of observing each initial condition is 1 as is the probability of observing the pair of initial conditions, so that the expectation value of the measurement of the product becomes simply the product of the measurements. Therefore calculations of these correlations yields

$$1 + (-1)(1/2) \geqq |(1)(-1) - (1)(1/2)|$$

or

$$1 - 1/2 = 1/2 \geqq |-1 - 1/2| = |-3/2| = 3/2$$

which is clearly false. Thus we have a simple discrete, classical system which nevertheless exhibits correlations that violate the Bell inequality. The violation of the inequality holds for this special triple of initial conditions just as the violation of Bell's theorem in quantum mechanics occurs for certain measurement directions.

This model is intentionally simplistic. The point is to demonstrate that the assumption that a classical dynamical system *must* be describable by a Kolmogorov type probability theory is not actually correct. This example, while involving simple one player games, may also be understood as a deterministic dynamical system. As such it is deterministic and local and there is no interaction between the two systems. The coupling arises because of the choice of initial conditions and the processes themselves. The coupling is not at the level of the individual events but rather at the level of the dynamics generating those events. It also demonstrates that the conclusion from Bell's theorem that only a deterministic nonlocal hidden variable theory is capable of describing quantum mechanical phenomena is not necessarily true. This observation is in keeping with Palmer, who showed that an iterated function system may reproduce quantum mechanical spin statistics while still avoiding Bell's theorem. Palmer gets around Bell by showing that the necessary correlation functions fail to exist. This simple example shows that the theorem may be defeated directly. In situations in which the dynamics is generated by games (and possibly iterated function systems as well), the probability structure need not be Kolmogorov and consequently it may be possible to defeat the Bell inequality.

Khrennikov summed up these insights, stating "Violation of Bell's inequality is merely an exhibition of non-Kolmogorovness of quantum probability, i.e. the impossibility of representing all quantum correlations as correlations with respect to a single Kolmogorov probability space"[208, pg. 6]. Khrennikov has developed these ideas of non-Kolmogorov probability in his Växjö model of contextual probability theory. The details of this approach are not necessary here but it generalizes the addition formula for quantum probabilities and applies this to classical events. What is important is that the most fundamental assumption of Bell that classical events must follow the rules of Kolmogorov probability theory is not true in general and so the conclusions derived from Bell's theorem related to the necessity of nonlocality in any hidden variable model of quantum mechanics are also not universally valid.

B.1.5 Contextual Probability - Växjö model

The contextual probability theory of Khrennikov provides a coherent, non-Kolmogorov probability theory. It takes its inspiration from the early work of von Mises [411] on collectives (which generalize the notion of a finite sample of a random variable to the countable case) but corrects some of the defects in that early work related to notions of randomness. In the Växjö model, a context C is considered to be any complex of conditions, experimental, mental, sociological etc. A set of observables $\mathcal{O}$ is any collection that can be measured under the context C. To each observable $a \in \mathcal{O}$ is associated a measured value α and the set of all possible measured values is denoted X_a. A contextual probability space is a triple $(\mathcal{C}, \mathcal{O}, \pi)$ where $\mathcal{C}$ is a set of contexts, $\mathcal{O}$ a set of observables, and π is a set of probability distributions, $\{p_C^a\}$ where $C \in \mathcal{C}$, $a \in \mathcal{O}$ and for any $a \in X_a$, $p_C^a(\alpha) \equiv Prob(a = \alpha|C)$.

He requires

$$\sum_{\alpha \in X_a} Pr(a = \alpha|C) = 1, \; Pr(a = \alpha|C) \geq 0$$

For a given context C define the set of probabilities

$$W(\mathcal{O}, C) = \{Pr(a = \alpha|C) : a \in \mathcal{O}, \alpha \in X_a\}$$

Definition (Växjö model): A contextual probability model is a space $\mathcal{P}_{\text{cont}} = (\mathcal{C}, \mathcal{O}, \pi)$ containing a special family $\{C_\alpha^a\}_{a \in \mathcal{O}, \alpha \in X_a}$ of contexts (so-called $[a = \alpha]$ selection contexts) satisfying the condition $Pr(a = \alpha|C_\alpha^a) = 1$.

Given $a, b \in \mathcal{O}$, $\alpha \in X_a$, $\beta \in X_b$, define the contextual probability $p^{b|a}(\beta|\alpha) \equiv Pr(b = \beta|C_\alpha)$. This is the contextual version of the transition probability $Pr(b = \beta|a = \alpha)$. Denote the set of contextual probabilities for all pairs $a, b \in \mathcal{O}$ by $\mathcal{D}(\mathcal{O}, \mathcal{C})$. Taking the union over all contexts C gives the collection $\mathcal{D}(\mathcal{O}, \mathcal{C})$. A given model can be written as $(\mathcal{C}, \mathcal{O}, \mathcal{D}(\mathcal{O}, \mathcal{C}))$.

Khrennikov shows that for dichotomous observables $a, b \in \mathcal{O}$ and a context C such that

1. Not all elements of the matrix of $P^{b|a}$ are strictly positive
2. The context C is a-nondegenerate, i.e. $p^a(\alpha) \equiv Pr(a = \alpha|C) > 0$ for all $\alpha \in X_a$ then

$$p^b(\beta) = \sum_\alpha p^a(\alpha) p^{b|a}(\beta|\alpha) + 2\lambda(\beta|\alpha, C)\sqrt{\prod_\alpha p^a(\alpha) p^{b|a}(\beta|\alpha)}$$

If the terms $\lambda(\beta|\alpha, C) \leq 1$ for $\beta \in X_b$ then these terms may be rewritten in terms of probabilistic angles and put in the more familiar form

$$p^b(\beta) = \sum_{\alpha} p^a(\alpha)p^{b|a}(\beta|\alpha) + 2\cos\theta(\beta|\alpha, C)\sqrt{\prod_{\alpha} p^a(\alpha)p^{b|a}(\beta|\alpha)}$$

which is just the formula for the interference of probabilities.

Khrennikov provides many examples of models induced by Kolmogorov, NRQM, and von Mises. He also shows how the Växjö model may be used to construct probability spaces for models in psychology, game theory and economics [207, 208, 211, 215]. A description of these models is beyond the scope of this work but it suffices to demonstrate that contextual probability theory is a rigorous, logically consistent theory with many examples arising in classical settings as well as quantum mechanics. These demonstrate rather conclusively that the assumption that classical hidden variables must conform to the rules of Kolmogorov probability theory is simply false. They are not the privilege of quantum mechanics. Of course, as for any theory, Khrennikov's theory has its detractors [142]. Time will tell wether or not his ideas have merit.

B.1.6 Quantum Bayesianism

An alternative approach to contextual probability is provided by the quantum Bayesian program. Indeed, Fuchs and Schack write "A measurement does not merely "read off " the value but enacts or creates them by the process itself." [143]. The goal of the quantum Bayesian paradigm, according to Fuchs and Schack, is to eliminate the possibility of a hidden variable model of quantum mechanics. Drawing on the work of Peres [292], they state that "unperformed measurements have no outcomes" . Measurement values do not exist prior to a measurement being performed.

The quantum Bayesian viewpoint interprets probability solely from a personalist perspective. They do allow for the possibility of a reality that is "out there" , whose existence is independent of the observer, but they deny that such a reality comes endowed with pre-existing characteristics. They suggest that quantum mechanics is not a theory of the world itself, but rather a theory about how a user interacts with the world.

Fuchs and Schack view the Born rule as an assumption that must be added to the usual Bayesian structure to enable one to calculate the probabilities of outcomes of intended measurements based upon prior probabilities for and conditional probabilities consequent upon the imagined outcomes of a single special counterfactual reference measurement. They go on to suggest how this enables the construction of the probability structure of quantum mechanics without the need for a wave function. To do requires the use of a symmetric informationally complete (SIC) quantum measurement, which is a specific positive operator valued measurement (POVM). It is not yet clear whether this approach can be applied to all possible quantum systems. SICs have been confirmed to exist only for finite quantum system up to dimension 100.

No more will be said about quantum Bayesianism. It does stand as a viable alternative to the contextual probability approach of Khrennikov, but the point of Khrennikov's approach is that contextuality is a property of *all* situations involving uncertainty, whether at the classical level or at the quantum level. The discussion of probability in quantum mechanics, and in quantum information almost always assumes that contextuality is an aspect unique to the quantum situation, which enables the argument against any possibility of locality or hidden variables at the quantum or sub-quantum level. This seems based on the notion that contextuality exists at the lowest level and then somehow vanishes as one moves up to the classical level. Rather, the arguments of this section suggest that contexutality is a feature of emergent systems. One may have a non-contextual foundation but as emergent systems arise contextuality also emerges as a feature of their interactions.

B.2 Principle of Continuity

The concept of continuity appears in physics in two complementary ways. First, mathematical methods based upon the idea of continuity play a major role in the formulation of physical problems and in their analysis. Such usage can range from veridical, to effective, to merely demonstrative. A nice survey of these issues is provided by Lesne [231]. Second, continuity may be applied as an assertion about the ontological character of a physical entity or property - that is, the assertion that the physical entity is continuously distributed, is under continuous motion, or has properties that may take values from a continuous range.

The concept of continuity has a long tradition in philosophy, dating back at least to the time of Zeno, and in physics, at least from the time of Newton and Leibnitz and the mathematical formulation of the calculus. A proper mathematical description of the intuitive notion of continuity was not achieved until the mid-19th century with the formulation of the concept of the limit and of the continuum in set theory. These ideas have led to a remarkable wealth of powerful mathematical tools for analyzing systems whose descriptions are continuous in space and time. These include ordinary and partial differential equations, integral equations, geometry, real manifolds, complex analysis, complex manifolds, forms, infinite series, infinite products, Lie groups, Lie algebras, transfinite set theory to name a few. The requirement that a function be continuous and especially that the function be smooth, meaning that all of its derivatives are continuous, places considerable constraint on the form which that function may take. Requiring the function to satisfy a differential equation constrains the range of functions even more. Requiring that the function be smooth and hold over the space of complex variables leads to quite profound behaviour that again greatly aids in calculation.

Many problems involving continuous variables and functions may be reformulated into discrete and finite versions. In doing so, one often loses certain computational power but one may gain a richer variety of behaviour. For example, continuous methods may permit the discovery of closed form solutions to problems, something uncommon when working in the discrete case. Asymptotic approximations are generally best carried out on a continuous setting.

The shift from a continuous equation such as

$$\frac{dx}{dt} = \mu x(1-x)$$

to a discrete version such as given by an iterated map

$$x_n = f^n(x) \text{ where } f = \mu x(1-x)$$

results in a shift from wholly predictable to wholly unpredictable chaotic behaviour. This is quite profound. It is often the case that discrete and continuous versions of a system yield different dynamical behaviours, the discrete case usually providing the richer variety since it is often possible to, at least theoretically, reduce a differential equation to an iterated map using Poincaré sections of the phase space and studying the dynamics on the reduced space.

Continuous methods, while powerful, can nevertheless exhibit troublesome behaviour. For one thing, since calculations involving continuous variables or infinite cases must often be carried out by invoking the existence of various limits, it may happen that such limits fail to exist, because they oscillate, diverge or different subcalculations trend to different results. Calculations may yield finite results under some conditions but not under others, such as in conditional convergence when a calculation holds for the original values but fails to hold if the absolute values of those values are used. It may be that one is attempting to sum or integrate values that are too large, or there might be too many of them.

Discrete methods may also sometimes fail if one is asking for a calculation involving an infinite sum of values, but if there are only a finite number of values, however many, then there is always a finite solution. Sophisticated methods have been developed to eliminate or reduce these divergences or infinities and continuous theories are now often restricted to those that can be handled using these methods. One such class of theories is the class of renormalizable theories in quantum field theory. Unfortunately attempts to unify the standard model of the weak, strong and electromagnetic forces with gravity have so far failed, in part because gravity is a non-renormalizable theory [264]. It is interesting to question how nature tames these infinities if these continuous models are actually veridical and not merely effective.

Another interesting question has arisen from the application of computability theory to the study of dynamical systems. It has been realized that many dynamical systems correspond to NP problems, meaning that they are virtually impossible to calculate [262]. It is also possible that certain types of neural network models may be non-computable [324]. If this is the case, for example for the human brain, then how is it possible for nature to evolve the dynamics? The use of continuous models generally occurs in situations in which there are a vast number of components which exist at scales much smaller than that at which the system is being observed and where the details of the individual components can be ignored, and fluctuations are small and uncorrelated so that individual behaviour can be averaged away and replaced with mean behaviour. Continuous models assume the existence of a continuous space domain and a continuous time domain, and generally that the values that the various properties of the systems can be found in continuous spaces.

Problems arise within the quantum mechanical setting. It is often the case that at least some properties of the system, such as energy, angular momentum, spin, take on discrete values. Moreover, these values are generally bounded in reality by some upper limit even though the spectrum of the measurement operator giving rise to these values is unbounded. Calculations, however, must often assume the existence of all possible values, even when discrete. For example, to use Green's function methods for the solution of a partial differential equation one needs a complete set of eigenfunctions $\{\phi_n\}$ and eigenvalues $\{\lambda_n\}$ and these are usually infinite in number. The Green's function is then constructed as

$$G(x,x') = \sum_n \frac{\phi_n(x)\phi_n^*(x')}{\lambda_n}$$

In reality the values of these properties for actual systems are bounded. The same is true for positions and momenta. In such cases the models may be viewed as idealizations, yet it is somewhat disquieting that knowledge of these non-realized conditions is necessary to solve for the realized conditions.

Another problem arises as scales become infinitesimally small. A consequence of the infinite divisibility of continuous systems is that the basic entity of which a manifold is composed, the mathematical point, has no extension at all. This leads to infinities in the definitions of various potentials that are associated with these points. That is turn frequently leads to infinities in calculations. Problematic though is that attempts to provide fundamental particles such as the electron with spatial extension leads to predictions of some experiments, such as scattering experiments, that appear to disagree with what is actually observed. Electrons may behave like points but treating them as mathematical points leads to difficulties. Of course one might argue that the mathematics merely provides a useful metaphorical langauge for describing physical systems, but if it merely metaphor then why has it been so good at times at making predictions and yielding calculations that agree remarkably well with experiment?

The point of this preamble is to suggest that the story has not yet been carved in stone as to the most suitable mathematical language with which to describe physical systems. The archetypal dynamics approach described in the Appendix A suggests that no such ideal mathematical language may exist and instead different languages will be required for different domains, different phenomena, different spatio-temporal scales, and that the task of science will be to identify these diverse languages, to explore their ranges of applicability and their limitations, and to provide methods for translating one to the other at the boundaries and interfaces. The present work aims to explore the applicability of a new area of mathematical research, that of causal tapestries and their game theoretic generators, the reality games, to the problem of foundational questions in non-relativistic quantum mechanics. In particular the question as to whether is is truly impossible to have a realist hidden variable model utilizing (mostly) local information.

The second issue concerning continuity is the question of the principle of continuity, which according to Gisin [149] asserts that "everything in nature, mass, energy, information, momentum, propagates gradually and continuously in time and that there is no instantaneous transport, that is, no discontinuous jumps from place to place. In particular this implies the absence of action at a distance"(pg 9). This is a question of ontology, and its assertion must of necessity demand that both space and time be continuous. Closely related to it is the question of whether the fundamental particles of nature are themselves discrete, like mathematical points, or are extended objects and if so are they continuous. This latter questions enters into the territory of quantum field theory, string theory and brane theory and will not be discussed further here.

The question as to whether or not space and time are continuous has been argued for a very long time. The advent of general relativity and various field theories would seem to have settled it in favour of continuity, but research into quantum gravity has raised questions about whether this is truly necessary. String theories appear to require a continuous background space-time but several competing approaches - causal set theory [59], loop quantum gravity and causal dynamical triangulations [264] all suggest a universe in which space and time have discrete features at the smallest spatio-temporal scales and the continuous space-time that we envisage is an emergent entity arising from this lower level. Kempf [203] has suggested that the theory of interpolation can provide a bridge between a discrete fundamental space-time and the continuum space-time that we appear to observe empirically. This key insight will be used to motivate the models discussed in Part II.

In the simplest version, Kempf considers space-time coordinates provided by operators X^i, one for each space-time dimension, such that the standard deviation obeys a lower bound

$$\Delta X^i(\phi) =< (X^i(\phi) - < X^i(\phi) >)^2 >^{1/2} \geq \Delta X^i_{\mathrm{min}}(< \phi | X^i | \phi >)$$

where ϕ is any vector upon which X^i acts. The action of X^i on a field is assumed to be of the usual form $X^i \phi(x^i, y) = x^i \phi(x^i, y)$. Kempf defines a discretization of the coordinate x^i as a set of real numbers $\{x^i_n\}$ with $x_n < x_{n+1}$ such that every real number lies within some interval $[x_n, x_{n+1}]$. A partitioning is a smoothly parameterized family of discretizations $\{x^i_n(\alpha)\}$. He then assumes that to each coordinate X^i there corresponds a partitioning $\{x^i_n(\alpha)\}$ such that if a field is known on the discretization corresponding to some fixed α, that is, the values $\phi(x_n(\alpha))$ are known for all n, then the field is given as

$$\phi(x) = \sum_n G(x, x_n(\alpha)) \phi(x_n(\alpha))$$

The beauty of this approach is that the values attributed to ϕ using this reconstruction do not depend upon α. All expectation values and correlations are likewise independent of α.

This is the non-uniform version of the Shannon sampling theorem. The causal tapestry model defined below uses the uniform version for mathematical simplicity but also in order to ensure that the function G given above can be calculated based upon local information. As shown in Appendix D, there is a non-uniform version of the Shannon sampling theorem which works well for band limited functions but where the function G is defined at any point based upon information obtained over the entire lattice, whereas using the uniform

form it is possible to define G so that only local knowledge is required at any point, which is necessary to fulfill the goal of constructing a local model of NRQM. In Appendix D it is also shown that alternatives are still possible in the non-uniform case but the technical details would take us too far afield from the main point, which is a demonstration of the possibility of local realist models of NRQM.

In the process model, the partitioning is implicit, since the embedding of the lattice used for the construction is not specified and, indeed, the freedom to choose the embedding without affecting the interpolation constitutes a kind of gauge freedom.

Direct observation of virtually every physical system known to man has revealed that as the scale of observation gets progressively smaller, the apparent sense of continuity at higher scales gives way to, first of all, a loss of smoothness, then a loss of differentiability generally, then a loss of spatial contiguity, and finally a loss of continuity per se. Large entities break down into component entities that break down into conglomerates, then into molecules, then into atoms, then into fundamental particles. The same may be true of space and time itself.

The perception of continuity is a universal perceptual illusion, well studied by cognitive and perceptual psychologists. Unlike other supposed illusions of conception suggested to resolve fundamental problems in physics, the ability of the mind to generate illusions of continuity is an established psychological fact. Of course, if space and time are in fact discrete at the fundamental level, then the principle of continuity would have no ontological basis and could be discarded or at best viewed an as effective postulate but not an ontological one.

At the level of the fundamental particle continuity appears to reassert itself in the form of the wave function. The position vector of the classical particle is replaced by the probabilistic wave function. The uncertainty relations preclude the idea of a trajectory being ascribed to a fundamental particle so the continuity principle need not hold at that level. It is applied to the wave function itself though, whose evolution in time is given by a unitary operator derived from the Hamiltonian. Although the evolution of the wave function is determined by a form of wave equation, interactions, particularly measurement interactions, occur as if the particle were actually localized. In the two slit experiment for example, repeated measurements of the particles as they hit the recording screen result in a distribution of intensity that follows an interference pattern when two slits are open and a particle pattern when one slit is open. This is an ensemble effect, however. Each individual particle only leaves a record of itself at one location on the screen. There is no wave like effect as far as the spatial extension of hits on the screen is concerned. At one time the particle appears to occupy exactly one point, although the location of that point appears to follow a random distribution given by the square of the amplitude of the wave function.

A similar situation exists for a bound particle, say an electron bound to a proton in a hydrogen atom. In the ground state the electron wave function is stationary and the probability of locating the electron at some location nearby to the proton is determined by the wave function but quantum mechanics does not permit a determination of the pattern of successive position measurements

The principle of continuity breaks down if time and space are not continuous. Discrete time and space have frequently been used computationally to facilitate the approximate solution of differential equations lacking closed form solutions or in carrying out complex perturbative calculations as in lattice gas or lattice field theory models [279]. The use of discreteness there is heuristic, not ontological. There have been models in which space and time are postulated as being discrete ontologically, that is, in their actual nature. An early model of discrete time physics was proposed by Lee [226] but it predated the development of discrete calculus that has taken place in subsequent years and so did not proceed very far. In the 1980's, coincident with the early development of complex systems theory, there was a flurry of interest in cellular automata models [128]. These are based on a lattice structure representing a discrete space-time and information is passed between local sites (cells) according to some rule and the states of these sites were then updated time step by time step [429]. The patterns of states formed the emergent entities that were the object of study. These could be propagating patterns or periodically propagating patterns, or boundaries between regions of patterns having particular symmetries or dynamical properties. Rules varied according to whether local or non-local information was used in the updating of cells, whether cells were updated simultaneously, sequentially in a fixed order or randomly, and whether information was incorporated in a deterministic, non-deterministic or stochastic manner. The study of cellular automata models formed the basis of my own early research into emergent computation [356]. Unfortunately the initial enthusiasm for cellular automata gradually dissipated. Wolfram's synthesis [430] of much of this work and his advocacy using these as the basis for a new paradigm for science fell on largely deaf ears.

At around the same time, interest renewed for the study of quantum gravity, and the suggestion that space-time could be discrete was again raised. Ashtekar's reformulation of general relativity and the first appearance of ideas related to causal sets and loop quantum gravity took place during the 80's. Sorkin's development of causal sets [59] was a motivator for my own mathematical Ph.D. thesis work on order automata [343] and provided the initial inspiration for the causal tapestry idea. The process model to be presented in this work is a generalization of the causal set approach. A causal set is simply a locally finite ordered set. By an ordered set one means a set equipped with a relation $\prec$ between elements such that

1. If $x \prec y$ and $y \prec x$ then $x = y$ (transitivity)
2. If $x \prec y$ an d$y \prec z$ then $x \prec z$ (antisymmetry)

(In the mathematical literature one often sees the reflexive condition $x \prec x$ but that is actually somewhat redundant and not used in causal set theory)

By locally finite one means that for any pair of elements $x \prec y$ the subset $\{z|x \prec z \prec y\}$ is finite. This implies that the causal set is discrete and formed of links, which are pairs of elements $x \prec y$ such that there is no z such that $x \prec z \prec y$. The past of an element x, $C_x = \{y|y \prec x\}$.

Sorkin creates causal sets by means of a stochastic percolation dynamics, called causal sequential growth dynamics, in which elements are added to the causal set sequentially according to some fictitious time. Within the causal set, the time relation is given by the order relation. At each step, a new element is added to the existing causal set which extends the previous order, but does not link existing elements, so that it truly extends space-time. The generation process induces a 'birth ' order on the elements of the causal set. This birth order provides a linear extension of the original causal set order. Sorkin suggests a principle of general covariance as requiring that the generating dynamics be independent of birth order.

The collection of all possible causal sets can itself be considered to be an ordered set. Two causal sets A, B form a link $A \prec B$ if $B = A \cup \{b\}$ and for any $a \in A$, either $a \prec b$ or $a||b$ (read a is incompatible with b, meaning there is no order relation between them). Thus B is a strict extension of the order on A. Viewed in this way, the addition of an element to the causal set corresponds to a move up one link in the ordered set of all causal sets. The ordered set of all causal sets forms a tree and can be viewed as the tree diagram of a single player combinatorial game, where each link is a possible move by the player on a given step of game play.

The causal sequential growth dynamic specifies a way of generating paths in the ordered set of causal sets that satisfies certain general conditions, namely,the condition of internal temporality, the condition of discrete general covariance, Bell causality condition, and the Markov sum rule. It is equivalent to the basic percolation method for constructing random graphs.

In percolation models of random ordered graphs, the percolation algorithm is used to seed a set with links which are then completed via transitive closure to form an ordered set (possibly with some pruning to eliminate cycles). The object of study, however, is the completed graph, not the construction process. So too with causal sets. It is the final causal set that is the space-time, and it is treated as such. In the process approach, a game dynamics generates a succession of causal tapestries, but each tapestry represents a complete reality in itself, a complete representation of a transient now, and the succession of tapestries as generated by game play represents the flow of time according to a succession of transient nows. Space-time in the process approach is dynamic, ever in the process of becoming.

Sorkin has described an approach within quantum field theory based upon the concept of a quantal measure, which incorporates features of the path integral and decoherent histories approach [332, 333]. This approach seeks to eliminate the need for the wave function in determining the probability structure and so departs from the ideas presented below in which the wave function is accorded an ontological status as an expression of the local 'strength ' of the associated process. In the process approach, a central importance is ascribed to the path integral and propagator methods for constructing game strategies and the quantal measure approach may be useful in extending the formalism to quantum field theory, but that is beyond the scope of the present work.

B.2.1 Continuity and Non-locality

The principle of continuity is thus a powerful tool for constructing theories and models and for facilitating the analysis of such models. It can also be a source of many conundrums. Nature itself is rather coy when it comes to revealing whether or not physical entities are continuous in actuality or merely effectively described by continuous models. Indeed, as far back as the time of Lucretius there have have been models of reality posited on the existence of discrete elements. Lucretius called these 'atoms ' and it was only in the 19th century that the notion of the atom began to be widely accepted.

Our direct experience suggests that interactions between physical entities occurs either locally (direct causation) or though some intermediary (local common cause) which must act or propagate between them. In the case of a local common cause, the intermediary propagates according to the principle of continuity in a gradual and continuous manner. Moreover, according to the theory of special relativity, the speed of such propagation must be less than that of light. Newton's theory of gravity created conceptual problems because it appears to involve a force which acted instantaneously upon massive bodies, and so involved action at a distance. The mathematical forms of the electric and magnetic vector potentials suggested that they too might act at a distance, but later discoveries culminating in Maxwell's equations of electromagnetism showed that in fact electromagnetic effects propagated with finite speed c, and that the usual potential had to be replaced by the so-called retarded potential which takes into account the time delay in propagating the effect of the potential. The general theory of relativity eliminated the problem of action at a distance in gravity, and predicted the presence of gravitational waves, evidence for which is still being sought.

With the advent of quantum mechanics, the state of a particle as a point in position-momentum space was replaced with the wave function. The wave function itself is a continuous function on a manifold and it also evolves according to the principle of continuity in a gradual and continuous manner.

The motion of the electron itself became an unanswerable question. It is said that electrons do not follow trajectories, although it is apparent through direct observation that the *effect* of an electron is distributed in space and time. The experimental setup of an electron source and a detector apparatus already has implicit within it the notion that some effect of the electron (whether this is the electron itself or not) passes from the source to the detector. Moreover, if the source is weak so that only a single electron is generated at a given time, the detector only ever registers a single discrete event at any given time. These single events distribute themselves over the surface of the detector and the frequency of their detection follows a distribution similar to that of a wave passing through a slit. What then of the electron? An effect appears to propagate through space and time from the source to the detector, but it cannot do so in a gradual continuous manner on account of the uncertainty principle. This is because if its position were to be localized its momentum would need to spread out over a broader range. Indeed the principle of continuity applies only to the wave function, not to the effect that the electron propagates through space and time. The shift from position state to position wave function introduces an element of non-realism into the quantum formalism. Keeping the principle of continuity appears to require the existence of superluminal information transfer in order for the non-local correlations of entanglement to be upheld. In order to keep the relativistic constraint it would seem necessary to abandon the principle of continuity.

Entanglement provides another challenge to realism and the principle of continuity. When a pair of photons are created through parametric down conversion an entangled state of the form

$$\Psi = |0> |0> + |1> |1>$$

can appear. In such a state the polarization of the two photons is highly correlated, even though the photons may be space-like separated by quite large distances. Current experiments suggest that the speed with which an awareness that one photon has been observed in a specific polarization state propagates to the other photon so as to determine its polarization state is on the order of $50,000$c or greater. If the speed with which information travels is limited to c, then how is it that such correlations are possible? Note that it is not the photons themselves (or any other entangled particles) which travel at superluminal speeds. Rather, it is presumed that the information that sustains the correlation between the photons and which ensures that a measurement of one photon as being in a particular state guarantees that the other photon is in the correlated state travels at superluminal speeds.

As Gisin points out [149], to maintain a correlation between two distinct entities requires either a direct interaction between the two entities or the presence of an intermediary though which they are linked. If an intermediary can only move at speeds below that of c then that would appear to force the two entities to directly interact with one another, effectively implying the presence of action at a distance or non locality. The alternative is to assert that it is possible for some signals to travel at speeds greater than c.

Gisin [149] shows that it is possible for there to exist a universal reference frame where signals propagate at speeds much larger than c without contradicting special relativity and without leading to causal paradoxes. Gisin does this by deriving a Bell type inequality.

Gisin considers a two party scenario, Alice, and Bob, with measurement settings x, y and results a, b respectively.

If the conditional probability $p(a|x,y) \equiv \sum_b p(a,b|x,y)$ depends *explicitly* on y then it is said to be signalling (Shimony would call this parameter dependence). The no-signalling condition means that $p(a|x,y) = p(a,x)$ and $p(b|y,x) = p(b|y)$.

Assuming a pure local common cause, mediated by some hidden variables λ, Gisin constructs conditional probabilities

$$p(a,b|x,y) = \sum_{\lambda} p(\lambda)p(a|x,\lambda)p(b|y,\lambda)$$

Note that this is a classical Kolmogorov formulation of the total probability and therefore subject to the limitations and questions noted in the previous section.

To deal with the pure direct cause case, Gisin assumes that in the privileged reference frame, Alice makes her measurement first and then a signal is transmitted to the rest of the universe, in particular to Bob. There are two possibilities. If the speed of this signal is limited say, to some speed v (which may be much greater than c), then Bob must make his choice after the signal has presumably reached him, and so the conditional probability will be

$$p(a,b|x,y,\ \text{v-connected}) = \sum_{\lambda} p(\lambda)p(a|x,\lambda)p(b|y,\lambda,x,a)$$

In the second case, Bob must makes his choice prior to receiving the information and in such a case the conditional probability must be determined on the basis of the hidden variables alone. In that case

$$p(a,b|x,y,\ \text{v-not connected}) = \sum_{\lambda} p(\lambda)p(a|x,\lambda)p(b|y,\lambda)$$

Gisin then considers a 4-party scenario involving Alice, Bob, Charlie and Dave. Alice measures first and Dave measures at a time so that the hypothetical signal from Alice to Dave arrives on time. Bob and Charlie are not v-connected but measure in such a way that the hypothetical signals from both Alice and Dave arrive to them on time.

Bancal et.al. [30] were able to construct a theorem which is relevant here:

Theorem: Let $p(a,b,c,d|x,y,z,w)$ be a correlation, i.e. a conditional probability distribution with binary inputs $x,y,z,w \in \{0,1\}$ and outcomes $a,b,c,d \in \{+1,-1\}$. If

1. The correlation $p(a,b,c,d|x,y,z,w)$ is non-signalling and
2. $p(b,c|y,z,a,x,d,w)$ is local for all a,x,d,w (meaning that it satisfies the Clauser-Horne inequality, then $S \leq 7$ where

$$\begin{aligned}S = -3 < A_0 > - < B_0 > - < B_1 > - < C_0 > -3 < D_0 >\\ - < A_1B_0 > - < A_1B_1 > + < A_0C_0 >\\ + 2 < A_1C_0 > + < A_0D_0 > + < B_0D_1 >\\ - < B_1D_1 > - < C_0D_0 > -2 < C_1D_1 >\\ + < A_0B_0D_0 > + < A_0B_0D_1 > + < A_0B_1D_0 >\\ - < A_0B_1D_1 > - < A_1B_0D_0 > - < A_1B_1D_0 >\\ + < A_1C_0D_0 > +2 < A_1C_0D_0 > -2 < A_0C_1D_0 >\end{aligned}$$

where X_n means observer X chooses measurement $r = n$.

Gisin demonstrates a 4-qubit state for which $S \approx 7.2$.

It follows that

1. Assuming no-signalling, a violation implies that Bob and Charlie share non-local correlations (i.e. without common cause)
2. Assuming that Bob and Charlie are local (as in v-signalling), a violation implies that $p(b, c|y, z, a, x, d, w)$ is signalling.

Since quantum mechanics violates this inequality, at least in theory, it follows that either

1. quantum mechanics violates the principle of continuity (so that hidden influences are hidden forever) or
2. the hidden influence can be used to communicate at speeds at or below the speed of the hidden influence (v), meaning superluminal communication is possible

In the process approach, it is the principle of continuity which is assumed to be violated, but only at the level of fundamental reality. As will be seen, the process approach allows many 'either-or' situations to be replaced by 'both-and' or 'neither-nor'.

B.3 Unitarity of Dynamics

In this section I intend to address some particular issues related to the role of time in the process framework. The subject of time is vast, contentious and unresolved, and opinions range across the spectrum, often being held with great vigor. There are many excellent books on time, surveys [69, 73, 74, 187, 236, 426] and discussions of specific positions [34, 62, 110, 230, 254, 269, 277, 295, 319, 330, 392, 399, 437]. I particularly like Dainton's wonderful text "Time and Space" [103] and the discussion to follow is based largely upon his argument. I do not intend to enter deeply into these waters but merely to sketch out some current views, to situate physical attitudes towards the problem of time, and to argue for the direction taken in the process approach.

The issue of time is perhaps one of thorniest in physics. Nevertheless to a large extent it is no longer contentious since the majority of physicists have convinced themselves that it does not exist. There is space-time, but neither time nor space separately. A recent article in New Scientist, June 15-21, 2013, pp 34-37 even suggests that there is either time *or* space, but not both. These days any argument to the contrary seems to be met with derision and ad hominem arguments, talk of pseudoscience or psychological illusions (perhaps delusions?).

The study of the psychology of time is well beyond the scope of this thesis but it is a very subtle area of research. The argument that time is illusion is often bolstered by evidence that our ability to accurately perceive the *flow* of time is poor and our estimations of time flawed. Depending upon our cognitive and emotional states, we may perceive time as passing slower or faster, and thus our estimates of time passing vary [69]. People have difficulty determining causality in relation to internal events - for example failing to accurately note the timing of decisions to act. These arguments do not carry much weight, however. Human beings do not possess the ability to provide precise measurements of anything - not time, nor space, not mass, not momentum, not energy - not anything. I am aware of *no* physical construct that humans are capable of measuring with exact reliability or accuracy (in the absence of technology). That is the fundamental reason why scientists have developed so many methods for measurements and established standards for them. Humans can be easily fooled in every perceptual domain. So if our perception of time is an illusion so is our perception of everything else.

If this argument holds, then all of science must also be illusory. One cannot, with any intellectual honesty, cherry pick certain illusions over others to believe in. But this argument misses the point. Most of these illusions are contrived, and arise in situations in which humans are asked to perform tasks that are not ecologically salient, i.e. tasks which serve no purpose in the environment within which humans evolved and therefore did not evolve the capacity to carry out. The human eye can detect a single photon. It cannot detect every photon, but only those within a certain range, because only those are salient to its ecological niche. Humans can perceive time, space, mass, energy, momentum, angular momentum - they simply are not good at precisely measuring any of them.

One cannot have one's cake and eat it too. One cannot, following the usual Copenhagen interpretation of quantum mechanics, assert that quantum systems possess no properties unless an observer observes them, and then assert that said observer suffers from perceptual illusions and so none of what they observe can be taken to be real. In Chapter 2, the London and Bauer argument showed that in the usual formulation of the measurement problem it is necessary that there be a conscious observer to bring the measurement process to a close - but if everything that conscious observer perceives is held to be an illusion, then how does it close? Is it merely the act of reading a dial on a measurement apparatus? Presumably that dial or mark or whatever is there whether someone looks or not. And if looking is the key, well then the entire process is flawed because looking is precisely what humans are not perfectly good at.

Ad hominem arguments based on the failings and foibles of human observers do not make any case for or against the existence of aspects of reality. We must turn to other evidence.

The most commonly held view of time within the physics community is that time does not exist, the flow of time is an illusion, and the universe exists as a 'block ' universe, in which all events, past, present, future co-exist. In such a block universe there is no becoming - everything simply is. Space-time and its events can be construed as a tapestry. The role of physics is to describe the structures of the observed patterns, and relationships and correlations among its sub-patterns. But it cannot explain how the tapestry appeared in the first place. The appearance of the passage of time is somehow an illusion, although a remarkably persistent one. Let us consider some of the issues involved in the study of time in more detail.

Dainton points to two separate issues in the philosophy of time

1. The issue as to the nature of time in and of itself: static or dynamic
2. The issue of the terms and concepts needed to describe and to discuss time: tensed or tenseless

Discussions of A-lines and B-lines relates to the second of these two questions but will be discussed first.

A classic philosophical argument against a generative dynamic and in favor of a static mode was first posed by McTaggart [103]. He argued for the existence of three modes of description of the temporal relationships among events. A lines refer to events as being past, present or future. B lines refer to events being earlier, simultaneous or later. C lines describe a purely static block universe of events which are ordered but without reference to time at all. McTaggart argued that descriptions involving A lines require events to have descriptions that form an infinite regress, clearly an undesirable situation. Since events cannot possess all three properties, pastness, presentness and futurity simultaneously, they must possess them at different times, i.e. being present in the present, or future in the present, or present in the past. This leads from 3 first order relations to 9 second order relations, and a moments reflection reveals that this does not solve the problem because there is now the question of when these second order relations hold, leading to third order and so on. Thus McTaggart argued that A lines provide an inadequate description of temporality, and B lines are equally deemed unsatisfactory as they also require some notion of A line in order to account for descriptors like earlier than and later than. Thus he argues that all that is possible is an atemporal block universe, the C case. In so doing, he makes the case for the non-existence of a moving present.

One problem with McTaggart's argument, at least to my mind, is the notion that there exists some enduring object or objects to which this infinite regress of properties applies. The things talked about are discussed as if it is they that possess these various attributes of past, present or future. The block universe is already implicit in such an argument since the objects are accorded some existence in the future, in the present, in the past, which is the case only in a block universe. Having implicitly assumed the existence of such a block universe, McTaggart's argument proceeds to show that a notion of temporality does not provide a consistent description. He shows that either one has a block universe, or one has temporality, but not both.

In a generative world, prior to its coming into being, an object does not yet exist, except perhaps in the imagination of some being. There is no object, in actuality, to which a property of 'being in the future' applies. It might be conceived of in the imagination of some being, but not another, and so 'it' possesses properties in one mind but not another, but in neither case does it actually exist. There is no physical 'it' to which the property applies. When the object does come into existence it does so transiently and at that moment it can be said to exist. If it persists for some time, it does so because the generating process sees fit to manifest an object like its predecessor at subsequent generations. Information from the previous instantiation is propagated forward to maintain some continuity with the subsequent instantiation of the object, but in each moment that which is called the object moves out of existence and to be then viewed as part of the past, appearing now only as a memory in the mind of some being. It is only during the moment of instantiation that the 'object ' possesses reality - the rest of the time it is only as a fantasy or as a memory that we can relate to it. The main problem with McTaggart's argument lies with the notion of an object, an issue which I will address in Section 3.4.

Dainton points out that most modern views on time, especially those held within the physics community, are in accord with the notion of B-lines and reject that of A-lines (and perhaps of C-lines as well). It is one thing to dismiss the A-line view; it is another to explain why our perception of time is what it is. As noted above, dismissing it merely as illusion does not work. The illusion must be explained and the alternative made plausible or demonstrated to be true. One solution is that of bounded awareness. This is the idea that our awareness of time, at any point in time, is limited to a small region of time surrounding that point. Combined with the principle of continuity, this makes for an infinity of regions of awareness. If not, if awareness is bound only to certain points in time, then why those and not others? Another viewpoint, espoused by Mellor [254], is that the appearance of a flow of time is a reflection of how our memories are created. But memories are not static things. They are processes, and reconstructions as well, not simply reproductions. Memories arise in time. But this still does not explain why just these memories and not others. If one asserts that this is happening everywhere along the time line then there are an infinite number of copies of everyone all experiencing each moment simultaneously. So much for parsimony.

Dainton describes a number of arguments, many proposed by Mellor [254], based upon linguistic or logical considerations, distinguishing between many different forms and meanings of linguistic constructions and emphasizing token-type distinctions. Although much beloved by philosophers I am not strongly convinced by them since they appear to attribute too much significance to linguistic representations of reality and to my mind they highlight the complexity of the relationship between representation and realisation (see Appendix A for more on this) more than they do that of reality. Purely linguistic arguments hold weight only if it can be held true that any statement in language must be either true or false, so that a logical argument can be derived from the existence of such a statement. This point of view, however, has no basis in fact. Language is a universal structure. Anything can be stated. It is then necessary to verify in some manner whether or not a given statement is true or false. Unfortunately this is not possible in general. It is not possible in logic as Gödel showed [43]. It is not possible in common language either. The belief that it can be otherwise can only be held so on ideological grounds.

Explaining change is a problem for both A and B theories. McTaggart argued that change requires time. A spatial variation in a property does not constitute change. It simply shows that different parts possess different properties. Change, however, requires a part to possess a property and then to possess a different property, and the distinction between these two possessions must occur in time. As a consequence, change requires time and the existence of A theories. McTaggart argued that B theories essentially reduce change in time to the equivalent of a variation in space, and that is not change. As McTggart argued, A theories are untenable, and therefore change, and time, do not exist.

Mellor argues for a distinction between objects (like a person) and events (like a symphony performance), so that objects, being a whole, can change over time as a result of causal effects, while a symphony does not change over time since the different performances at different times merely constitute its different temporal parts (akin to spatial parts). Of course this fails to take note of the subtlety that an object such as a body is actually a dynamic entity, a process, and behaves more like a symphony than Mellor might wish to admit. The time scales might differ but that is beside the point. Other B theorists do treat the states of objects at different times as akin to the properties of objects at different spatial locations, Whether change now occurs in such a B theory depends upon whether or not one is willing to accept spatial variations as evidence of change.

Dainton [103, pg 41] briefly describes the possibility that time is an emergent phenomenon - not present at the fundamental levels of reality but emerging in some fashion at macroscopic scales. He argues that such a state of affairs does not diminish the importance of time. He writes:

> ... if it does turn out that time is absent from the equations of quantum gravity theory (or whatever theory in physics turns out to be fundamental), this will be an interesting and significant result....If this should prove to be the case, time will have the status of an *emergent* phenomenon, in somewhat the same way as macroscopic material things such as planets and animals (which "emerge" only when elementary particles are arranged in certain ways). but even if the status of time is thus diminished, its *existence* is scarcely threatened, for at the macroscopic level the distinction between spatial and temporal dimensions remains perfectly real. Cats are emergent entities; are they illusory? (Sic)(pg 42-43)

Moreover, even if it is the case that A theories are false, they nevertheless remain important

> If there are no A-facts why do we constantly think in A-terms? Mellor argues that the A-framework, in particular A-beliefs, are of vital practical importance even though there are no A-facts....the important point is that we can understand why the A-framework is indispensable for practical purposes without assuming that there are any A-facts. The account just sketched appeals only to B-facts and B-relationships. A-*beliefs* play an ineliminable role in successful timely actions, but as we have seen, B-facts can serve as truthmakers for A-beliefs. (Sic) (pg 35)

Lest one believe that the answer concerning A-lines has been given, Lowe has written a series of papers in which he shows that the arguments surrounding the internal inconsistency of the A-line approach are false. He provides a means of constructing tensed sentences that support an A-line description of events and yet remain internally consistent.

Past-ness and present-ness are not properties of events. He argues that the need for tenses is fundamental, and that some notions such as past-ness, present-ness, future-ness, may simply be fundamental irreducible notions. According to Dainton, he remains agnostic as to the question of whether past, present and future events are equally real, so whether the universe in actuality is a block or a developing universe.

Physicists are used to seeing the universe in static terms. Time and space are laid out in space-time diagrams. In the Newtonian framework, these have the form $\mathbb{R} \times \mathbb{R}^3$ with time singled out. In the Einsteinian framework, these have the form of a 4-dimensional curved manifold $\mathbb{M}^4$, with time and space given an equal footing, at least initially. The events that comprise observable reality are then plotted out on this space-time to produce a space-time depiction of reality.

In a sense there is no real causality in physics since in such a block universe nothing ever actually happens. There are correlations and symmetries which can be determined and measured but these simply *are*. That there are symmetries in nature is indisputable. Determining whether or not such symmetries are fundamental or emergent is another matter. Understanding why these symmetries tend to appear only in special situations also demands consideration. These space-time diagrams are more in the imagination than in reality. It is impossible to actually determine these space-time diagrams. Most of the information required to create such a diagram is not merely lacking, it is fundamentally inaccessible due to relativistic constraints. In an ideal universe it might be possible to completely describe the past light cone of the Earth say. Given our present technology it is impossible to describe its future light cone except for certain known periodic phenomena. We cannot know about non-periodic phenomena, nor can we know about phenomena that have not even begun to manifest. Moreover, since the time of Poincaré, it has been realized that the existence of chaos renders predictions of the infinite future and past untenable. An infinite degree of knowledge is required of every instant of time in order for such predictions to hold for all but a brief time.

A deep conceptual problem, as noted by Mermin [259], is the reification of mathematical structures. The space-time diagram may well be one such example. Its success as a conceptual and calculational tool can lead one to believe that it is not merely a representation of reality but a veridical replica of reality. That would be a serious conceptual error, an example perhaps of Whitehead's notion of misplaced concreteness.

One argument in favour of the block or static point of view is the presence of time reversal symmetry. In the context of quantum mechanics, it is the statement that the time evolution of a quantum mechanical system is described by a unitary operator. If the unitary operator U_t provides a solution to the Schrödinger equation, then so does U_{-t}.

In other words the direction of any supposed flow of time can be reversed and the result will still provide a solution to the equation. In the classical case this also means that one can traverse a trajectory in the phase space in either the time-forward or time-reversed directions and still obtain a solution to the dynamical equations. One important feature of a unitary dynamics is that it is norm preserving, so that the temporal evolution of a wave function ensures that it retains the essential features of a probability distribution. This essential feature forms an important aspect of the process theory of measurement, presented in Chapter 6. It is not, however, necessary that the full algebraic features of unitarity be preserved when operating on the space of processes.

Time reversal symmetry is a well observed local symmetry. But what of the global scale? It has long been known that the local differentiable structure of a manifold does not determine its global structure [61]. Could this be the case for time reversal symmetry as well? There seven recognized so-called arrows of time, most of which appear at global scales [103].

These arrows of time are

1. Thermodynamic - the inexorable increase of entropy
2. Cosmological - the origin in the Big bang, the end in heat death
3. Radiative - radiation moves outwards from its source
4. Causal - causes precede effects
5. Weak - neutral Kaon decay
6. Quantum - the measurement problem
7. Psychological - the experienced passage of time

Dainton also describes several time asymmetries that form asymmetries in the *contents* of time [103]. These are:

1. Entropic asymmetry - as above
2. Causal asymmetry - as above
3. Fork asymmetry - single causes, multiple effects. But some effects may also have many causes (illness states, quantum measurements, chess moves)

4. Explanatory asymmetry - later events explained by earlier events
5. Knowledge asymmetry - there is knowledge of the past but not of the future
6. Action asymmetry - deliberations orient towards the future
7. Experience asymmetry - our lives are experienced as unfolding past to future

The standard explanation of these arrows of time is generally attributed to the second law of thermodynamics but it does not easily account for all of these arrows, and there is still ongoing debate.

Such a space-time level symmetry does not necessarily inform about how the events that are related by the symmetry arose in the first place. The important information is not the time symmetry but rather the presence (or absence) of causal relationships between events. It is well known that the causal structure of a causal manifold is preserved by Lorentz transformations while the coordinate structure is not. If space-time were to be generated, then the causal relationships inherent in the generation process would be reflected in the causal structure of the causal manifold and the fact that they are preserved suggests that they possess a fundamental importance.

Time reversal symmetry arises in the consideration of simple free motion. If one can move form one spatial location A to another spatial location B one can equally well move from B back to A. It is a wholly different matter to assert that if one can move from a space-time location $(t, \mathbf{A})$ to a space-time location $(t', \mathbf{B})$ then one can equally move from $(t', \mathbf{B})$ to $(t, \mathbf{A})$. It is an even greater leap to assert that when motion is neither simple nor free that one may move from a complex dynamical state Ψ to a complex dynamical state Ψ' and vice-versa. By a complex state I do not simply refer to spatial positions and momenta but to interactions, correlations, causes, relations, contexts etc. Instantaneously a live state and a dead state may appear to be identical but they are not the same. There are two completely different processes generating each such state and these processes are *not* interchangeable since Live $\rightarrow$ Dead but Dead $\rightarrow$ Live. This point seems to be regularly ignored in the physics literature, which fails to note the distinction between inanimate and animate and between object and process.

Several proposals have been made for a dynamic time, that is, for a universe that includes some notion of becoming. The Process Theory of Whitehead is a leading early contender (Appendix A) as are the philosophies of Heraclitus and Buddha. Considering more contemporary views, Broad [63] proposed the idea of a growing block universe, in which future events do not exist, but present and past events exist. Past events retain all of the properties and relationships that they held while present but in addition acquire a new property of past-ness.

Tooley [392] has proposed a theory of a growing block universe which is still tenseless. He bases this using causation as a foundational concept. Dainton succinctly summarizes his argument as follows [103, pg 77]:

1. Events in our world are causally related
2. The causal relation is inherently asymmetrical. Effects depend on their causes in a way that causes do not depend on their events
3. This asymmetry is only possible if a cause's effects are not real as of the time of their cause
4. Causes occur before their effects: "X is earlier than Y" means (roughly) that some event simultaneous with X causes some event simultaneous with Y
5. Our universe must, therefore, be a growing block.

McCall [253] provides a model which Dainton describes as the thinning tree model. Basically the idea is that the future manifests like the branches of a tree. As the moving present passes a node in the tree it moves down one branch and the branches left untravelled disappear.

Dainton points out that these two models present problems for a growing block notion since both seem to capture the essence of the problem, but one posits a universe that gets bigger while the other a universe that gets smaller. How about a universe that grows or shrinks simultaneously at both ends? And if the past is somehow real like the present, how do we know that we are actually in the present and not in the past?

An alternative to the growing block model is that of *presentism*; the doctrine that nothing exists that is not present. Reality is viewed as a succession of presents. Several authors consider such models: Le Poidevin [230], Bigelow [52], Barbour [34]. Dainton describes several problems with the viewpoint of presentism:

1. How long is the present?
2. If the past is not real, how can there be definite facts about the past? If the past does not exist, then all truthmakers for statements about the past do not exist, hence all statements about the past cannot be true

3. If truthmakers for the past exist but only within the present, and presents change, then so must their pasts - the past becomes inconstant

4. If truthmakers for the past exist only in the present, then the past depends upon whatever evidence exists in the present, suggesting that the bulk of the past vanishes

There are two static forms of presentism: , which asserts that there is only and has only ever been just one present, this present, and *many-worlds presentism*, which posits the existence of a collection of temporally unrelated presents. Reality is the totality of these disparate presents. Dainton suggests thinking of a deck of cards that can be shuffled into different sequences, each corresponding to a different possible history of reality. Barbour [34] has proposed a form of many worlds presentism. Barbour suggests that for each point in the phase space of the Wheeler-DeWitt equation, there exists a momentary 3-dimensional world, essentially a distinct present, each unrelated to any other. These fragments may contain traces of past or future occurrences, and somehow these give rise to sense of time passing. Time as such does not exist in Barbour's conception.

The other viewpoint of presentism is *dynamic presentism*, of which this thesis is an example. The notion of dynamical presentism dates back at least to the philosophy of Buddha and of Heraclitus. The following quoting from Heraclitus is quite descriptive of this viewpoint:

> This world neither any god nor man made, but it always was and is and will be, an ever-living fire, kindling in measures and being extinguished in measures. [103, pg 89]

Dainton describes dynamic presentism as supporting four claims [103, pg 85]

1. Nothing exists that is not present

2. At any time only one present exists

3. Each present is followed by another

4. Successive presents are causally related

The assumption that successive presents are causally related creates problems if dealt with naïvely. Causality is a relation between two things - cause and effect. Bigelow writes [52]:

> a two-place relation can only be manifested when it holds between two things, and in order for this to be so there must be two things which stand in the relation. And in saying "there must be two things which stand in the relation" , one is really asserting that "there must exist" two things - one is committed to the existence of these things. The principle of the existence entailment of a relation is an a priori truth.

If the idea of existence entailment for relations is upheld, then this would seem to imply that both cause and effect must be in existence at one and the same time. But dynamical presentism asserts that the previous present is causally related to the subsequent present, and that only one present is ever real. Existence entailment implies however that both antecedent and consequent presents must exist, leading to a contradiction. According to Dainton, this means that either there are no causal relations between successive presents, so that dynamic presentism becomes equivalent to many worlds presentism, or there are causal relations between successive presents which, on account of existence entailment, force all presents to be real and one is back at a static block universe.

Dainton suggests a possible way out of this dilemma in the form of *compound presentism*. Instead of requiring that only one present be real at any time

> The sum total of reality consists of at least two coexisting very brief reality slices (each spatially three dimensional). Suppose A and B are two such, and that A exists at time t_1 and B at time t_2. One of these slices, A, is annihilated and a new slice of reality, C, comes into existence, and with it a new time t_3. Slice B is annihilated and D is created, along with t_4; and so it goes on. [103, pg 87]

This model is immune to the argument from relations described above but subject to another problem, that being whether the relation of coexistence is transitive or not. If coexistence is transitive, then existence entailment forces one back to the static model. But

> Coexistence is certainly *symmetrical*, but it needn't be transitive; to suppose otherwise is simply to deny the dynamic nature of time, which involves precisely the coming-into-being and departing-from-being of times and events. Coexistence is transitive in *space* (at a given time) but to impose transitivity on time amounts to an unjustifiable spatialization of the latter (Sic) [103, pg. 88]

Coexistence fails to be transitive precisely in a world in which events come into being then cease to exist. Thus it must be a feature of any dynamical presentist model. Dainton also points out that the compound presentist model is a time symmetrical model, since the description of the flow of time runs both forward and backward. Thus he concludes:

> Consequently, if time has a direction, in any significant sense of the term, this is due to the contents within time and their modes of connection, not time itself. [103, pg 92]

I explored this issue in more detail in my Ph.D. Thesis, 'Order Automata' (U.W.O. 1989). In my thesis, I explored the interplay between ordered sets and automata. An ordered set is a mathematical description of causal structure while an automaton is a particular form of discrete dynamics system acting under the influence of external inputs or forces. Details are provided in Appendix B but it suffices that:

Definition: A semigroup is a set X together with a binary operation that is associative, meaning that $x(yz) = (xy)z$. A monoid M is a semigroup X together with an element 1 such that for all elements, $x1 = 1x = x$. An M-automaton is a triple (M, X, f) where M is a monoid, X is a set and $f : X \times M \to X$ such that

1. $f(x, 1) = x$ for every $x \in X$
2.) $f(f(x, a), b) = f(x, ab)$ for every $x \in X$ and $a, b \in M$

Definition: An order automaton is a 4-tuple (M, X, R, η) where M is a monoid, (X, η) is an M-automaton, and R is a partial order on X such that $x <_R y$ if and only if there exists an $f \in M$ such that $\eta(x, f) = y$. It can be shown that every ordered set is generated by an order automaton.

The idea is that the automaton action η induces a partial order on the state space X of the automaton. Starting on a given element x of the state space X, the automaton action will generate a sequence $x, f(x, a), f(f(x, a), b), \ldots$. This will induce a partial order on the set X. It is quite possible for the functions $f(x, a)$ for each $a \in M$ to be invertible maps. That is, they may demonstrate a form of local time reversal symmetry while nevertheless generating a global causal structure on the set X. This demonstrates that the notion of the generation of space-time was compatible with a notion of time reversal symmetry.

As a simple example, consider $\mathbb{R} \times \mathbb{R}$ and let the monoid M consist of the positive cone of $\mathbb{R} \times \mathbb{R}$ under addition. In other words, $M = \{(\Delta, p) | \Delta, p \geq 0\}$ and $(\Delta, p) + (\Delta', p') =$

$(\Delta + \Delta', p + p')$. Define an action f of M on $\mathbb{R} \times \mathbb{R}$ by $f((t, x), (\Delta, p)) = (t + \Delta, x + p)$. This generates an order on $\mathbb{R} \times \mathbb{R}$ such that $(t, x) < (t', x')$ iff $t' = t + \Delta$ and $x' = x + p$. Denote this order as $<_M$. The negative cone of $\mathbb{R} \times \mathbb{R}$ is also a monoid under addition and under a similarly defined action f' induces an order on $\mathbb{R} \times \mathbb{R}$. Let $N = \{(\Delta, p) | \Delta, p \leq 0\}$ in the same manner. Denote its induced order by $<_N$. Note that if $(t, x) <_M (t', x')$ then $t' = t + \Delta$ and $x' = x + p$, hence $t = t' - \Delta$ and $x = x' - p$ so that $(t', x') <_N (t, x)$ by virtue of the action of $(-\Delta, -p)$, which is an element of N. This implies that $<_N = <_M^R$, the reverse order of $<_M$. Thus under the reversal symmetry $M \to N$ induced by $(\Delta, p) \to (-\Delta, -p)$, we have $<_M \to <_N$. This means that the reversal symmetry reverses the causal order on $\mathbb{R} \times \mathbb{R}$ which is analogous to time reversal symmetry. Here Δ is akin to a shift in time, p to a velocity induced increment in space.

This is a simplistic example but note that the maps induced by each element of M (likewise N) are invertible, and that the reversal symmetry induces an isomorphism on the order $<_M$ which reverses the global causal ordering just like the reversing of time does. However, the dynamic on $\mathbb{R} \times \mathbb{R}$ is considered to be given by the action of M on $\mathbb{R} \times \mathbb{R}$ and this action does induce a definite causal order. The existence of a symmetry does not imply that the generator of the symmetry also possess the symmetry. The generator of $<_M$ is M, not $M \cup N$, which does not induce an order.

To generate the usual Euclidean ordering $(t, x) < (t', x')$ iff $t < t'$ one could define the action to be $f((t, x), (\Delta, p)) = (t + \Delta, p)$.

The kind of present moment that I propose in the following model is a compound present (for details see Chapter 5). I think that too often the notion of moment is conflated with the mathematical notion of point, which is an abstract idealization that need have nothing to do with how reality actually is. From the archetypal dynamics perspective, mathematics provides one system of representation of reality, it is not reality, and the two should not be confused for one another. The present moment in the process framework consists of the complete play of a single reality game, from the ending of the creation of the current tapestry to the ending of the creation of the nascent tapestry and the dissolution of the previously current tapestry. It is process that is fundamental here, and the succession of processes gives rise to the succession of presents. Each such present is well defined, consisting from the start of game play to its termination. This game play occurs "outside of time" , since in effect it generates a moment of time. It is discrete, but that does *not* mean that the actual occasions that result are necessarily organized into discrete well ordered timings. That is true for the non-relativistic model being describe in this thesis but it need not be true in general; certainly not in the relativistic case. One can think of game play as occurring in some sense *in time*, but in that case on a time scale so small that it effectively appears to an observer as if it were a single instant.

The key to the model is that information from the past propagates into the future in the form of the content of actual occasions (informons of the causal tapestry)(for details see Part II and Appendix B). It is not the prior events themselves that enter into the creation of subsequent actual occasions, rather, it is their information which does so. The prior actual occasions themselves cease to exist at the end of game play but they continue to exert a causal influence so long as their information continues to propagate into future actual occasions.

Whitehead also addressed the problem of time in his metaphysics and described a compound present. According to Hansen [168], Whitehead's solution to the problem rests upon a single key idea, namely, the idea that a concrete temporal fact is not global but local. Hansen points out that in consideration of the question of becoming the problem is always posed "by its friends as well as its enemies, as if either there are global temporal facts or there are no temporal facts (pg 151)". Indeed I will argue below that many of the problems with the underpinnings of modern physical metaphysics rests on confusions between local and global properties.

Hasen writes:

> So, one central point in Whitehead's system is that processes are themselves active - the temporal modalities are a function of the happening of processes themselves - the basic realities in the world pass from potentiality through actuality into pastness. each process is a unit of becoming, and according to Whitehead it "becomes in solido" ; it is not some temporally extended entity placed on an axis of time along which it could be played, like an audiotape, one stage or movement after the other. The idea is not that of zero extension like a point. It is the more radical idea that it is something different from extension - something out of which extension is *manifested*... The other equally important element in Whitehead's processural and relational account of time are the relations to other processes. (Sic) [168, pg 153]

The game model presented in Chapters 4 and 5 reflects Whitehead's notion of how processes make events determinate (in the sense of Bunge [71]). Actions of processes become initial conditions for other processes.

Hasen states:

> What it means for something to have already happened - to terminate is to be determinate - but this result should not be understood substantially. "Result "

> means it is available to be part of the initial conditions for new processes. Or perhaps better, it *is taken in* by these processes - the terminated parent processes are indeed said to be there, in some sense and to some degree "repeated" within the actively new process. (Sic) [168, pg 153-154]

This act of being taken in forms an integral component of the definition of the basic elements of the process model, in particular, of the content of informons (see Appendix B, Chapter 4).

Thus the first element of Whitehead's solution to the problem of becoming is to assert that the actions of processes serve as the ground for the concept of extension; processes generate space-time through their interrelationships. The second element rests with Whitehead's consideration of the myriad ways in which processes relate to one another. In particular he focusses on the following features of inter-process relatedness: they are causal, they have an asymmetric internal character, and reside within a causal universe. Causality is viewed as the one-way transfer of information from the terminated to the nascent process. Each process leaves its mark on all of its branching child processes. This causal asymmetry precludes any form of backward causation or reversibility. There is a deep, fundamental transience to all of reality, no matter how long an object appears to endure. Whitehead also asserted that no two processes share the same universe (or past causal cone), meaning that they cannot possess the exact same collection of parent, grandparent, great grandparent etc processes. Two processes may share parts of their universes but never the whole.

Whitehead uses the term *principle of relativity* to refer to "the principle that the *being* of terminated entities is expressed in their potentiality for the becoming of new process entities - to be part of their specific universes (pg 156, [425]).". By this account, a fact is something that has a definite ending. It is a terminated process or collection of processes that is available to be incorporated into newly active processes. Such facts are not private; they do not merely reside within the interior of some process. He writes [425]:"On the contrary, the pastness of everything in the universe is "public" in the sense that each of its elements is available to "everybody" in some appropriate causal neighbourhood"(pg 156, [425]). Whitehead thus accounts for local temporal facts without recourse to a notion of simultaneity.

Much of the previous discussion appears to suffer from what I would term the "fallacy of misplaced omniscience" . By that I mean two things. First of all, equating ontology with epistemology. The second is equating epistemology with human awareness. Let me illustrate with an example. My great-grandfather died the year before I was born. I am a causal descendent from him, in that he contributed to the conception of my maternal grandmother, who in turn gave birth to my mother, who in turn gave birth to me. Let us suppose that all records of my great-grandfather's existence disappeared, including his grave site. All of my relatives who might know of him are deceased. Suppose I develop a dementia and forget that he existed. Does that mean that he never existed? Not in the least. No one else living today might know of his existence, but that in no way changes reality and erases him from existence. If so, that would erase me from existence as well, but, at least as of this writing I do exist. Staying with reproduction, an average woman is known to have around 450 ova, most of which will mature and be released during ovulation at some point in her life. Given current technology there is no way of knowing how many ova a woman begins her reproductive life with, nor how many will ripen, nor which will ripen, nor which will be fertilized. Does that imply that none of the non-fertilized ova existed. Of course not.

The idea of existence entailment by a relation seems to be far too strong an expectation. So is the requirement of knowledge of some event to make it real. One key issue for me is that of relevance and efficacy, and this applies not to reality but to parsimony in our descriptions of reality. The key idea is that an event in reality is relevant so long as it makes a difference. By that I mean that it contributes in an efficacious manner as a cause or influence of some subsequent event. Removing it from existence would alter the subsequent event, even if only in some minor fashion. If removing this event from existence would make no difference, if the subsequent course of events would be completely unchanged by its absence, then that event can be said to be irrelevant and it can be removed from any model or description of reality without changing the subsequent flow of reality in any detectable way. Irrelevant events are *not* non-existent events. They are fundamentally unknowable events, rather sad and pathetic events, since their existence is meaningless, but they can still exist. The fallacy of misplaced omniscience equates reality with that which is knowable by a human observer, and makes anything knowable to such an observer *real*, not merely real *then*, but real *period*, at least if the principle of existence entailment is to be upheld.

But is this really necessary. I can say that my great-grandfather has a causal influence on my birth without his being real now, just so long as I accept that he was real then, so that the chain of causation could take place. It will still be true even in the absence of any truthmakers beyond my own existence, unless one were to believe in virgin births or the spontaneous creation of human beings. Sadly *I* might not know which man was my great-grandfather but that does not change the fact that at that time he helped to set off the chain of events leading to my birth.

Another setting in which the fallacy of misplaced omniscience shows itself is relativity theory. The argument for the non-existence of a present moment rests on the argument that there is no notion of simultaneity in relativity. There is no preferred reference frame. More relevant is the relationship between relativity theory and non-locality.

Maudlin [250] argues about some of the implications of quantum non-locality on relativistic issues. In his Introduction he writes:

> many physicists and philosophers would agree that Relativity prohibits *something* from going faster than light but disagree over just what that something is. Among the candidates we may distinguish:
>
> Matter or energy cannot be transported faster than light.
>
> Signals cannot be sent faster than light.
>
> Causal processes cannot propagate faster than light.
>
> Information cannot be transmitted faster than light.
>
> Most of these prohibitions are easily seen to be non-equivalent....
>
> Yet another interpretation that Relativity requires only that
>
> Theories must be Lorentz invariant.
>
> This requirement is compatible with the violation of every one of the prohibitions listed above.(Sic) [250, pg2-3]

In order to be simultaneous there must exist a frame of reference for the causal manifold such that all of the events on this space-like hypersurface can be assigned the same temporal value. We know from relativity theory that this is not possible universally, that moving observers will construct reference frames in which these events will have contradictory temporal assignments. The usual Minkowski space is a 4-dimensional manifold with metric ρ having signature $(-,+,+,+)$. One can define an ordered structure based on the past and future light cones (see Appendix B for details of ordered sets). If $\alpha = (t,x,y,z)$ is a point in 4-dimensional Minkowski space then the null hypersurface at α consists of the paths of all light rays that pass through the point α, i.e. all points $\beta = (t',x',y',z')$ such that $\rho(\alpha,\beta)^2 = -c^2(t-t')^2 + (x-x')^2 + (y-y')^2 + (z-z')^2 = 0$. The interior of this cone consists of all points such that $\rho(\alpha,\beta) < 0$. These can be further subdivided into past and future regions. An order may be then imposed on the Minkowski space such that $x < y$ means that y lies is the future light cone of x. The null, past and future light cones are preserved under Lorentz transformations and so the causal order is preserved as well. Thus the causal structure is a relativistic invariant. Borchers and Sen [61] turned this around and begin with a causal structure that describes the order theoretic structure of the order described above and then explore the degree to which this constrains the topology of the resulting space.

The define their fundamental objects to be:

1. A non empty set of points M
2. A distinguished family of subsets of M, called *light rays*
3. A total order $<^l$ defined on every light ray l

Several axioms are required to create a proper mathematical structure. These are

Axioms (The order Axioms):

1. If $x, y \in l$ and $x \neq y$, then either $x <^l y$ or $y <^l x$; if $x <^l y$ and $y <^l x$ then $x = y$
2. If $x, z \in l$, $x <^{ll} z$ (meaning $x < z$ and $x \neq z$), then $\exists y \in l$ such that $x <^{ll} y <^{ll} z$
3. If $y \in l$, then $\exists x, z \in l$ such that $x <^{ll} y <^{ll} z$
4. If $x, y \in l^1 \cap l^2$, then $x <^{l^1} y \Leftrightarrow x <^{l^2} y$

Axiom (The Identification Axiom): If l and l' are distinct light rays and $a \in S \equiv l \cap l'$, then there exist $p, q \in l$ such that $p <^{ll} a <^{ll} q$ and $l(p,q) \cap S = \{a\}$ where $l(p,q) \equiv \{x | x, a, b \in l, a <^{ll} x <^{ll} b\}$, and similarly for l'

Definition: A subset $W \subset M$ is called *increasing* (resp. *decreasing*) if

$$x \in W, x <^l y \Rightarrow y \in W$$

respectively

$$x \in W, y <^l x \Rightarrow y \in W$$

Definition (Cones)

$$C_x^+ = \cap_{x \in W, W \text{ increasing}} W$$

$$C_x^- = \cap_{x \in W, W \text{ decreasing}} W$$

Axiom (The Cone Axiom):

$$C_x^+ \cap C_x^- = \{x\} \; \forall x \in M$$

Using this order relation it is possible to define a collection of special sets called D-sets that serve as a basis for an order topology on M. Such topologies can be densely embedded in space that are locally, but not necessarily globally, like differentiable manifolds.

The argument against the possibility of a transient now is based on the absence of a concept of simultaneity and the absence of a preferred frame of reference. The absence of simultaneity arises because under a Lorentz boost there can be a transformation of the time coordinates of a space-like hypersurface coordinatized by a single time coordinate according to one reference frame. Under a Lorentz boost the coordinates will transform and in some cases will not only acquire different values but even reverse values suggesting a reversal of causal order. There are problems with this interpretation however. First of all, the actual causal order given by the metric is always preserved under a Lorentz boost. It is said that it is impossible to know whether one is moving relative to another frame but a reversal of causality is a clear indication of relativistic motion relative to an unfolding of events. Second, it is not possible, either in practice or in principle, to directly observe events unless one is at the same space-time location (or very nearby). In every other case, if the observer is separated from the event being observed, especially if space-like separated, then it is necessary for a signal to pass from the observed event to the observer. What the observer detects is then not the event itself but the signal arising from the event. That these signals might get crossed in time does not indicate at all that the originating events have somehow reversed their causal relationships. There is no need for the signal to retroactively affect the event being signalled. There is no reason for every aspect of the signal to convey information about the event. Additional information might be introduced, some information might be lost. Moreover, at best one can do this only for events that are nearby to the observer. The further the observer is space-like separated from the events being observed the greater is the delay in time between the emission of the signal and its reception.

In reality then any frame of reference constructed in practice can at best be a local frame of reference. In fact any frame of reference established in reality can only involve events within the past light cone of the observer. In truth it is never possible to establish a frame of reference, in reality, except perhaps in a local neighbourhood of the observer. There are no global frames of reference available to any observer existing with the space-time manifold itself. At best an observer existing external to the manifold might be able to establish such a frame of reference. But this runs afoul of the fallacy of misplaced omniscience. Only an omniscient being has the ability to possess a frame of reference for the entire space-time. It is possible to imagine the existence of such a frame of reference, but no entity residing within the space-time can have access to such a frame. Such information is fundamentally inaccessible to any entity within the space-time. Relativity does not merely show that there is no preferred frame of reference, it shows that there are no global frames of reference at all. The best that one can achieve is a patchwork of local frames of reference, just like the charts that define a manifold in the first place.

The light that we perceive from the Andromeda galaxy left 2 million years ago. When we look up into the night sky we perceive our past light cone. We do not know whether the Andromeda galaxy is still there. It might have disappeared one million years ago, or yesterday, but we won't know this because the only information that we can experience is that propagated by light (and perhaps gravitational waves). And both propagate at finite speed and in finite time. Nevertheless, it is unreasonable to believe that in the moment that I exist right now, there are not other things existing in the universe, in reality, whether I know about them or not. The planets continue in their motion whether I observe them or not. Pluto exists now, somewhere, and at in that location forms a part of the present of this moment. That I cannot state unequivocably where it is (though mathematical models let me predict this to high accuracy and it can be later verified and so retroactively affirmed) does not mean that it is nowhere in this moment. If the Andromeda galaxy still exists, it exists regardless of whether or not I observe the light coming from it, just as I exist whether or not entities in the Andromeda galaxy observe light coming from me. To assert otherwise carries the observer problem of quantum measurement theory to an extreme which in my view is untenable. The viewpoint becomes just another version of solipsism. It is most reasonable to assume symmetry in the universe as regards existence, and to assume that just as I exist so does the rest of the universe, whether I observe it or know it or not. Existence is local. Moreover ontology is independent of epistemology. That is not necessarily true of properties, however, as will be discussed in Chapter 5 dealing with the process theory of measurement. The nature of properties is intimately tied to the nature of interactions between processes. Properties are separate from existence.

The universe functions perfectly well without our knowing most of what takes place. It is impossible for us to know these things, at least until such time as information from these events reaches us. This will only occur sometime in the future, indeed over many futures. The configuration of the present is only available as a reconstruction in the future. To know otherwise would require access to non-local information, in direct violation of relativity. This seems to be ignored in most tellings of relativity theory. The fallacy of omniscience allows the physicist to speak of structures that might exist in an ideal Platonic universe in which there is no space or time as if they in fact occur within the real space-time and moreover are accessible to physicists. The supposed existence of these structures is then used as evidence for the non-existence of certain other structures. Except these structures cannot actually exist. This leads to a logical inconsistency at best, a paradox or contradiction at worst. Regardless, they cannot form the basis for an argument against the existence of a simultaneous now. Such a now would be no more observable that any other reference frame.

Although there are many arguments against a process of becoming or of a dynamic present, in the end these arguments do not hold up under scrutiny. Indeed there may actually be no way to decide whether to treat space-time as a static block or as a dynamical present beyond ideological preference. At the very least though, a dynamic present cannot actually be denied. Certainly there is no reason to believe that everything that exists is knowable. There is no reason to believe an event cannot exist unless someone knows it. To assume that leads to solipsism. If one believes in the existence of non-terrestrial objects then one must believe that they exist even in the absence of our direct observation of them. So too does the present.

As will be seen later, in the process framework, there is no non-locality among the informons, the actual occasions. It is only the processes that exhibit features of non-locality and even then it is a limited non-locality. There is no transfer of information taking place in such a non-locality. There is no violation of either the spirit or the principle of relativity.

The idea that a present moment does not exist because we humans cannot observe it and catalogue it and specify it to whatever degree is an example of the fallacy of misplaced omniscience. It is sad, but not world ending, nor concept ending, just humbling.

Many of the linguistic arguments involved in the discussion of time also seem to run afoul of the fallacy of misplaced omniscience. That I make some statement does not mean that such a statement can subsequently be determined to be true or false. For the vast majority of statements their truth value cannot be determined in practice or in principle. So what. Gödel in his incompleteness theorems showed that there are true statements in any mathematical theory (language) capable of expressing arithmetic (and natural language certainly is capable of expressing arithmetic) which cannot be proven logically within the theory. Mathematics did not come to a screeching stop. In fact, mathematics has flourished since this discovery. In some respects, like the discovery of the independence of the continuum hypothesis and the axiom of choice, it has been liberating.

The point of all of the above is that it is far from proven that reality must be static and of block form. It is perfectly possible for reality to be a dynamic present (though not a simplistic naïve present. Indeed the process model to be presented in Chapter 4 shows that this is the case. The issue ultimately may come down to which model of reality we prefer, just as one must choose which model of set theory to work within, which model of logic to follow etc.

The usual argument for dismissing the possibility of a transient now derives from a confusion of a now with a notion of simultaneity. The now, properly, refers to the front of a generative process and to the locus of events that co-exist before fading into past memory. Now is less an object and more a process. In the causal tapestry - reality game framework described below, there is no explicit now - rather, there is a causal tapestry, then a new causal tapestry, then a new one, each time the prior tapestry becoming content for the new one. A causal tapestry might be thought of as representing a notion of a now, but as that is more of a heuristic device than an ontological statement. The causal tapestry embeds into a causal manifold into a space-like hypersurface. These events can also be thought of as constituting a now. That does not imply that they are necessarily viewed as being simultaneous.

Note, however, that these events lie on a space-like hypersurface, and thus have *no* causal relationship to one another. Within the process framework they acquire a causal relationship if and only if information passes from one to another so that the former influences the generation of the latter. This never happens in a process model. Causal relations are preserved by Lorentz transformations and the very fact that different reference frames result in different causal relationships means that these are spurious relationships, not causal relationships. They are artifacts of our point of view, much as the centrifugal force is an artifact of rotational motion.

In this were not the case, then every passing high energy cosmic ray would wreak havoc with the causal ordering of events on the surface of the earth - this doesn't happen. My motion does not affect another objects unless we directly interact or I send an effective signal to the other. My motion may affect the linking process. For example, Fuentes-Schuller and Mann [144] have suggested that when one member of an entangled pair is accelerated relative to the other, the entanglement breaks down.

In Appendix B I show that the generation of causal tapestries by reality games is compatible with Lorentz invariance and so there is no conflict between the apparent transient now of the process and the absence of simultaneity of relativity.

There have been a few additional papers which touch on aspects of the process approach, particularly as regards the sufficiency of past information. Markopoulou [245, 175, 246] described an internal description of a causal set by defining a functor which maps each element of a causal set to its causal past. This was an attempt to model the view of causality that an element within the causal set would possess, as opposed to the view of the causal set itself which would correspond to that of an external observer. The path functor is defined additionally to the causal set but in the causal tapestry structure this past information forms an essential part of the definition of an informon (see Appendix B). Moreover, while the path functor describes the entire causal past of an event in the casual set, the content of an informon describes only that portion of the causal path that informationally contributes to the generation of the informon. This restriction was chosen because in the process approach, an informon is generated based upon local information, a portion of that information consisting of information from the causal past that has propagated forward in time to the time at which the informon is generated. Not all information in the causal past propagates forward in time. For example it may be absorbed or scattered or transformed or destroyed by entities that intervene in the region between the past event and the current informon. Such information cannot contribute to the generation of the current informon since it is impossible for the informon to be aware of its prior existence.

Another relevant paper is that of Dowker and Herbauts who showed that, in the context of a (1+1) field on a light come lattice, a knowledge of the field in some region of the past suffices to determine its values in the present. This lends some support to the process approach which generates the present moment information entirely from information propagated forward from the past.

Fleming [137] provides a radical contextual model in his notion of hyperplane dependence, which resolves some of the conflict between Relativity and non-locality by linking observations to each hyperplane from which they are observed. Each hyperplane serves as a context for each observation when determining correlations among events. The model is explicitly relativistic and so not strictly germane to the present thesis, but it is interesting. As Maudlin points out [251] though, it is unclear why, if reality possesses such a strong form of contextuality in the form of hyperplane dependence, that scientific research carried out in laboratories using manifestly local apparatus and measurements should be as successful as it has been.

Although the current consensus among physicists is that time does not exist, there are dissenting voices. Elitzur [121] and most recently Smolin [330] have argued for a reconsideration of the concept of time, particularly notions of becoming and development. Most physicists believe that the static block universe model is not simply the preferred model of reality, it *is* reality. This is not true of the philosophical community where the debate is far from settled. There are many phenomenological observations, such as the multiple arrows of time, that do not have satisfying solutions in the static universe model. Moreover, asserting that the psychological experience of a flow of time is merely an illusion does not explain this illusion, it merely evades a decidedly uncomfortable problem. The answer may lie in whether or not it is possible to construct theories in which time does exist, in which the universe is not static, and yet the predictions of NRQM can still be obtained. To do so would demonstrate that the question of time remains open and deserves a serious reconsideration. That such a model exists for NRQM is the focus of this thesis.

At the end of his work, Maudlin summarizes his examination of the relativistic constraints and their possible violation by quantum non-locality. He writes [251]:

> For four of the proposed constraints I have argued that the results are unequivocal:
>
> Violation of Bell's inequality does not require superluminal matter or energy transport.
>
> Violation of Bell's inequality does not entail the possibility of superluminal signalling.
>
> Violation of Bell's inequality does not require superluminal causal connections.
>
> Violation of Bell's inequality can be accomplished only if there is superliminal information transmission.
>
> Our last topic, Lorentz invariance, yielded a tangle of unexpected proposals. Lorentz invariant theories which predict violations of the inequality may be formulable if one admits either explicit backwards causation or hyperplane dependence. More radically one could interpret the violations as indicating the existence of a single preferred reference frame , a frame undetectable by any empirical means. Even more radically, one could adopt the Many Minds theory and deny that there are any violations of Bell's inequality by events at space-like separation: the relevant correlations exist only in individual minds. All of these options become yet more bizarre when one shifts from Special to General relativity. (pg 240-241)

Pertinently he concludes by stating:

> Quantum theory and Relativity theory seem not to directly contradict one another, but neither can they be easily reconciled. Something has to give: either Relativity or some foundational element of our world-picture must be modified. (pg 242)

The present thesis aims to present a model in which the world is a great deal less bizarre than most authors would lead us to believe. Certain cherished assumptions of physics must be given up, but the loss is not all that great and recovered again at the emergent, observational level. Locality is (mostly) recovered, though reality remains stubbornly contextual. That too is not a great loss. Every human being has lived in a contextual world for millennia and that has not stopped the march of Progress or our evolution as individuals or as a species. It does, however, require a loss of hubris, but I think we can survive better without it.

Jacobson [197] has argued that the evolution of a quantum mechanical system must be via a unitary operator in order for causality to be preserved. However an important assumption for his result is that one begins with a fixed causal structure on the space-time. But in a generative model such as in a dynamical present model or a process model, such a fixed structure does not exist, at least not until the end of history, a not too realistic prerequisite for making observations.

The important point is that many of these relativistic arguments are straw man arguments since, by relativity itself, they are impossible to carry out. The most fundamental aspect of relativity is the constancy of the speed of light, which is an experimentally verified *fact*. As a consequence, no signal can travel faster than light. NO signal. That means that it is fundamentally impossible to determine the events that lie within a present moment, and issues related to whether or not causal orderings are maintained or reversed become irrelevant since they can *never* be measured in the first place. In fact there can be causal relationships between events making up a single present moment. Causality acts only from one present moment to the next present moment, never within a present moment. Causal relationships are preserved by relativity and form the basis for the study of causal manifolds, causal sets, and now causal tapestries. We can know what constituted this present moment by passing into the future and observing what information from the space-like separated members of this present propagates to us at different times. The present can then be reconstructed, but it can never be directly experienced by any observer existing within that moment. Processes stand outside of the present moment because they are the generators of the present moment. They do not propagate in time or in space. Processes exist only in one of two states: active or inactive. As will be seen later they can combine in myriad complex ways but otherwise they simply are, or are not.

B.4 The Ubiquity of Objects

A fundamental difference between the subject matter of the complexity sciences and the traditional sciences, beyond that between linear reductionism and nonlinear complexity, is between the concepts of object and of process. The notion of an object is derived from the study of inanimate classical matter. The notion of process, in contrast, is derived from the study of animate matter. For more than two millennia, science has followed a program of objectification of reality organized around the construct of the object. An object is a mental abstraction, which some aspects of reality reflect well, and which engenders an attitude of objectification, in which entities of reality are related to as if they were objects, whether or not the abstraction can be fairly applied to them. Frequently, those aspects of reality that are the focus of scientific study, are termed the objects of that study, and are treated as if they are objects according to the abstract conception. This resulting objectification of reality influences all subsequent relationships to the entities of reality regardless of whether or not those entities actually fulfill the conception of an object.

As an abstraction, an object is considered to have an existence and properties that are independent of the environment within which it is embedded. In particular its existence and properties are independent of any particular observer examining said object. An object endures, that is, it persists across time, sometimes to the point of being treated as if it were eternal. An object simply *is*. Its existence of a given, intrinsic to the object, even if certain properties may vary over the course of the objects existence. An object may consist of component objects but if so then this set of components is a fixed property of the object. This continuity of structure over time gives an object a property of individuality. This is not to be confused with identity, since it is quite possible for two individual objects to be indistinguishable by any conceivable means. An object may undergo change, but only in the spatial configuration of its constituents or in certain attributes, such as those that describe position and orientation. Note that some properties are considered intrinsic and independent whereas some attributes are changeable and extrinsic, and thus may be environmentally dependent. Nevertheless, attributes are thought of as being possessed by the object even if not wholly specified by the object. For example the capacity to hold a location in space or a duration in time is possessed by the object even if the precise specification of such may depend upon details of the embedding environment. An object does not act, it reacts. An object cannot set goals or pursue goals. An object does not create. An object can transmit information but neither creates nor interprets it. An object cannot intend, apprehend, prehend, or understand. The concept of an object explicitly and specifically precludes any aspects that could be considered subjective. An archetypal expression of the concept of object is a rock. It has a fixed form and constitution.

It has properties, like hardness, color and constitution that are intrinsic and fixed, and attributes, like temperature, orientation, position, momentum and angular momentum, that are variable and relative. It reacts to its environment but does not act. It has no intentions, no goals

The concept of object lies at the heart of virtually all mathematical and physical constructs. Indeed, mathematical entities are the only true, ideal objects because they can be considered to exist outside of reality within some universe of Platonic ideals. All actual entities that exist within reality are at best approximations to the concept of object. Even that archetype of object, the rock, begins to appear a little dubious when examined near the Planck scale. Nevertheless, at human scales many of the entities of reality are well approximated conceptually as objects. In fact it might be fair to think of physics as the study of those aspects of reality that are objective or can be approximated as objects. This objectification of reality makes possible its description through mathematical language, and for physics at least, this program has proven to be highly successful. Success, however, is not without its perils, a serious one being the tendency to then misattribute a physical reality to those ideal mathematical objects used to describe some apparent objective features of reality. This conceptual error can then become a hidden impediment to further study as explored by Mermin [259] in a recent article on the dangers inherent in this reification of mathematical structures.

In spite of the success of this program of objectification of reality, it really only works when applied to a relatively limited range of natural phenomena, primarily inanimate matter observed at macroscopic scales and isolated from its natural environment. When applied more generally, objectification leads to conceptual confusion, explanatory restrictions and paradox. Objectification carried down to fundamental levels results in a plethora of paradoxes such as the Schrödinger's cat paradox. Objectification of natural language, such as through mathematical logic, leads to paradoxes of self reference such as the Liar paradox or explanatory restrictions such as the incompleteness theorem. Objectification of developing systems leads to the conceptual confusion of development being represented by models in which there is no development. Objectification of space-time leads to a universe in which nothing happens. The appearance of conceptual confusion, explanatory restrictions, and paradox should lead one to question the universal applicability of the concept of object to the entities of reality and trigger a search for more veridical alternatives.

The most fundamental impediment to the objectification of reality is the existence of living organisms and the products of their actions such as societies, languages, cultures, minds, and organizations that fail to possess one or more fundamental characteristics of an object. In fact even the most fundamental entities in nature, the fundamental particles, fail to fulfill all of the conditions of an object. Complementarity, the uncertainty principle, spin, and the linear superposition of quantum states are all departures from the concept of object.

The entities that form the subject matter of the complexity sciences are all fundamentally transient in nature; they arise, they develop, and they fade away. They interact with their environment openly, exchanging matter and energy. They have neither fixed, stable components, nor a fixed form. Properties as well as attributes may depend upon context. Nevertheless they are still capable of individuality and spatiotemporal coherence. They are generators, carriers and transformers of information and meaning. They have agency, acting upon their environment and not merely reacting to it. They intend, prehend, apprehend, and comprehend. They create, seek and follow goals.

The process perspective turns the usual reductionist hierarchy on its head. It explores which features of complex systems can be fruitfully applied to the fundamental levels of reality and whether from such a viewpoint it is possible to eliminate some or much of the conceptual confusion that has plagued quantum mechanics. Process takes centre stage, and the usual objects of reality are viewed as arising in an emergent manner from lower level processes. As will be shown in subsequent chapters, the process framework eliminates many of the paradoxes of NRQM which arises as an effective theory in a suitable asymptotic limit. In discussing the application of category theory to physics, Baez and Stay write [90, pg 95]

> A category has *objects* and *morphisms*, which represent *things* and *ways to go between things*. In physics the objects are often *physical systems*, and the morphisms are *processes* turning a state of one physical system into a state of another system - perhaps the same one. In quantum physics we often formalize this by taking *Hilbert spaces* as objects, and *linear operators* as morphisms. (Sic)

This is the usual sense of process in physics, as altering the states of pre-existing objects. In the process framework, the notion of process is quite different and follows the ideas of Whitehead. Process is taken to be fundamental, and the action of process is to generate the actual occasions that comprise fundamental reality. Everything observable, whether microscopic or macroscopic, is viewed as being emergent from these actual occasions. Process is held to be a generator of objects, and to endow them with properties as a consequence of their generation. Processes generate time, space, fundamental particles, fields, and everything else. Organisms are manifestations of process, psychological experience is a manifestation of process. The beauty of a process perspective is that there is langauge to describe all of these diverse types of phenomena, and to eliminate the artificial divide between the physical and life and social sciences created by the usual mathematical formalism of physics. That formalism becomes a representation of processes in one particular domain of experience. A different representation might pertain in a different phenomenological domain, but underlying both is the unifying concept of process.

The notion of process is considered in more detail in Appendix A. Throughout this thesis, processes will be represented by combinatorial games for mathematical and conceptual reasons that will be discussed in future chapters. There have been several other attempts to introduce ideas of process into physics. Many are discussed in the book "Physics and Whitehead " by Eastman and Keeton [118]. Shimony [321] also considered the idea of process for a time, though he later fell out of favour with it. A few other authors have also considered process type notions [77, 85, 86, 120, 121].

There have been a few attempts over the years to develop process based models in physics. Not all of these are explicit expressions of Whitehead's process theory. An early group of computational models which are explicitly Whiteheadian in their construction even if not framed in those terms are the cellular automata models that appeared throughout the 80's and 90's. Wolfram [429] was a seminal figures in this regard. Cellular automata consist of a collection of cells which exist on a spatial grid. Each cell is assigned a dynamical law, which takes its current state and that of the cells in its neighbourhood as input and as output provides its new state. A single time step is considered to be completed when all of the cells are updated from there current condition and a new lattice of cells is formed. In most cellular automaton models, the space-time behaviour is generated but the result is usually depicted as a growing block universe, although a compound present involving the current and nascent cell layers is actually more accurate since it is only the current state of the cells themselves that is observed. Any information about past configurations of the cells requires the use of a recording device which must be independently provided an is ontologically separate from the cellular automaton. Wolfram presented a number of intriguing ideas about the relationships between cellular automata and notion of thermodynamics, irreversibility, space and time [429].

Coecke [90] presents a wide variety of novel mathematical structures for expressing theories in physics. Many of these techniques involve category theory, certain logics developed within computer science, domain theory, topos theory. Coecke [90] develops a domain theoretic model of information in classical and quantum systems. He attempts a semantics of physical information routed in the computer science model of domains. It is not a process model per se, although it does attempt to understand measurement processes in terms of an acquisition of meaningful (relative to the domain concept) information.

Hiley [90] (Chapter 12) provides a more explicit model based upon process theory. He considers an algebra of processes which is based on the idea of a Clifford algebra. He notes that Grassmann developed his notion of a Grassmann algebra out of an attempt to understand the concept of becoming. Hiley treats a process as a pair $[T_1, T_2]$ where T_1, T_2 represent the opposite distinguishable poles of an indivisible process. These processes are presumed to form a structure over the reals having three basic properties:

1. Multiplication by a real scalar denotes the strength of the process
2. The process is oriented, that is $[T_1, T_2] = -[T_2, T - 1]$.
3. There exists an inner multiplication defined by $[T_1, T_2] \bullet [T_2, T_3] = \pm[T_1, T_3]$, referred to as the order of succession.

Further conditions on then placed on the operation $\bullet$ that result in a variety of Clifford algebras. The two point definition of process given by Hiley bears a superficial similarity to the pairing of causal tapestries (current and nascent) that form a process generative situation in the process model described in Chapter 4, but the structure of the process space there is based upon the algebra of combinatorial games, which is much richer and more complex as it possesses at least 3 distinct forms of sum and of product. In Hiley's model, space-time features arise more directly from the process algebra while in the process model, space-time arises in an emergent manner from an interpolation process acting upon the generated space of informons, which is discrete. Finkelstein [118] has also proposed a quantum language based upon the notion of a Clifford algebra.

Noyes [276], Finkelstein [135] and Selesnick [316] present process theory based models of quantum mechanics centered around logical or computational representations of measurement processes. Their theories are quite complex and I cannot say that I understand them but they do not appear to offer a realist model of the fundamental levels of reality such as is provided by the process model, or indeed the Bohmian or GRW models. Cahill [77] also has developed an explicit process theory based model of quantum physics using a computational paradigm involving neural net type objects. A drawback of his model is a reliance upon ideas of absolute space and time, which are understandably rather contentious to say the least. These authors are mentioned, not because their work is used in the process framework, but to indicate that ideas from process theory have been seriously explored for their applications to fundamental physics and so the current model is not entirely coming form left field.

There have been a number of attempts to derive emergent models of NRQM, including Nelson [268], Adler [4], and perhaps Bohm [55]. I include two time models such as stochastic quantization [104] and Bars two time physics [38] in this group as well, since the observable phenomenology can be thought of as emerging from the second time dynamics. There are many models of emergent space-times in quantum gravity [264]. Wen [232] has developed an emergent model of particles. The emergentist approach taken in the process framework thus has a number of forebears.

B.5 The Physics of Process

As will be shown later, the key for understanding non-locality lies in our understanding of process. Actual occasions are the fundamental elements from which all observable phenomena - objects, events - emerge. They are created locally, based upon local information. Non-locality arises because processes need not generate only single actual occasions during game play and game play need not be contiguous. Indeed, just as in human games where a player can paly several games simultaneously, so a process may generate multiple actual occasions simultaneously. This seems at first to be a strange notion yet it is no different in quantum field theory where fields can generate different numbers of particle modes. Moreover, just in the play of normal games, different moves may take place on different sites. There is no reason to require moves to take place only on contiguous sites. Non-locality in quantum mechanics arises in two main ways - through entanglement or through interference between wave functions. In the process framework, entanglement will be seen as an expression of this multi-generativity aspect of process while interference arises from the non-Kolmogorov structure of probabilities associated to processes.

The subject of process is vast, embracing not just fundamental physical systems but systems across all spatio-temporal scales including organisms. Indeed, the very idea of process is an extension of the dynamical characteristics and causal interrelationships of organism to the world of inanimate matter. As a consequence there is no single unitary theory of process. The closest would be the various theories of complex adaptive systems. The notion of process does not replace the standard ideas of physics but rather expands these ideas and principles. In physics one can point to Anderson's famous phrase "More is Different" for inspiration [19].

The physics of process is not the main focus of this thesis. For the purposes of fundamental physics, and this thesis, processes are assumed to be governed by the same basic laws of NRQM. In particular, processes will be presumed to implement a dynamic based upon the Schrödinger equation and the Lagrangian of the system being represented (see Chapters 4 and 7). The Lagrangian is expressed through path integral analogues while the Schrödinger equation is expressed through its Green's function. Each process is identified with the properties that it generates, and in interaction with other processes, individual processes become inactive or active and the associated properties interrelate according to the various conservation laws. Dynamics in the process framework is about interactions between processes. Processes are themselves implementers of dynamics in that they are the dynamical generators of actual occasions. But these are two distinct dynamics and an algebraic structure applies only to the dynamics of interactions among processes, not to the generation of actual occasions. No more can be said without specifying particular quantum systems to be modelled and that is a project left for the future. In the next Chapter I turn to a description of the process model and its implementation.

B.6 A Last Word: Conservation Laws

The subjects discussed in the preceding sections all share one common feature - they are generally assumed in the construction of theories and models of physical reality but they have not been proven to hold in all cases. Indeed they are often motivated from theory rather than from empirical observations. Indeed one result of the process program is to demonstrate that they are in fact not essential to the construction of a viable theory of physics at the fundamental level, and that at best they are emergent features in the asymptotic regime.

The assumption that Kolmogorov probability theory is an essential feature of the classical realm is simply false. The principle of continuity, while useful, leads to paradox. The concept of time reversibility, if understood as other than an expression of a pattern symmetry, also leads to conflict with observed reality, and requires the addition of other ad hoc laws to compensate. Dispensing with these assumptions in a careful manner, will, as shown below, eliminate these problems while still leading to NRQM as an effective theory.

There is, however, one class of assumptions for which there is abundant evidence empirically and mathematically, and that is the conservation laws. These laws have been meticulously verified in a wide range of settings over three centuries and there is no reason to doubt that they are true expressions of deep relations in the physical world. In applying them to the process situation, care must be taken. At least at the level of theory developed here, these conservation laws are held to apply at the level of process. They constrain how processes interact and what transformations among processes are possible. At the level of the actual occasion, however, it is not at all clear that these conservation laws apply. The actual occasions themselves are essentially informational in character, not physical. They convey information about processes and interactions forward in time and space. As will be shown, they serve as generators of interactions that are understood at the classical level as constituting properties. They carry information about process in the form of the wave function, which will be shown to represent a kind of 'strength ' of process (more will be said about this in Chapter 5). They serve as generators of properties as revealed in the course of interactions. Nevertheless, it is at the level of process that these properties have meaning, and therefore at the process level that the conservation laws also have meaning. Stating so does not imply that these laws could be violated. Any physical entity is an emergent aspect of process, and entities interact between themselves at the level of process. The actual occasions that manifest such processes in reality are determined by their processes - they do not exist independent of process. They do influence how processes change during interactions but their behaviour cannot be specified directly and so any potential violation of the conservation laws manifested at the level of the actual occasions cannot be used to create violations at the emergent levels. At present this is an assumption of the model. It is possible that in future research it might be necessary to develop a physics of information flow at the level of the actual occasions but whether this will bear any resemblance to physics at the emergent levels is an open question. After all, in cellular automaton models there is little if any relationship between the behaviour of the cells generating the emergent phenomena and the behaviour of the phenomena in themselves.

In the case of fundamental quantum systems (particles and fields), I will assume that these properties are associated with elements of various Lie groups and algebras, representing the allowable transformations. For example, in simple terms, suppose that we have two processes generating systems with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ and energy E_1 and E_2. Then the allowable transformations will be to processes having momenta $\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_k$, energy $E'_1, E'_2, \ldots, E'_k$ such that

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}'_1 + \mathbf{p}'_2 + \cdots + \mathbf{p}'_k$$

and

$$E_1 + E_2 = E'_1 + E'_2 + \cdots + E'_k$$

The important point that I wish to make is that the laws of physics apply to the generating processes and *not* to the actual occasions that these processes generate. The actual occasions are consequences of the actions of process. Actual occasions do not act, do not react, do not interact. They are passive carriers of information that is generated, transmitted and transformed by process. They serve to mark these actions of process and to make such information available, in the moment, to other processes, and to influence the subsequent inactivation, activation or transformation of processes. Actual occasions interact with processes but never with one another. There are no laws of physics applicable to actual occasions beyond those that are inherited from the processes that generate them. This is a critically important point.

Appendix C

Archetypal Dynamics

C.1 Basics of Process Theory

The modern conception of object originates in the work of Aristotle and Plato two thousand years ago and has been the subject of constant elaboration and refinement even since. An alternative view of reality, however, had been proposed 200 years earlier by the Greek philosopher Heraclitus and the Indian psychologist/philosopher Siddhartha Gautama. Both proposed a metaphysics in which reality consisted of an ever changing flux of phenomena organized into coherence by some form of underlying subjectivity. These ideas flourished in the East but languished in Western thinking. The first Western philosopher to seriously revisit this metaphysics was Alfred North Whitehead, particularly in his difficult book Process and Reality [425]. Whitehead's philosophy has been described as a philosophy of organism and re-introduced a concept of subjectivity into metaphysics in the notion of prehension. Whitehead conceived of a process as a sequence of events having a coherent temporal structure in which relations between the events are considered more fundamental than the events themselves. Whitehead viewed process as being ontologically prior to substance and becoming to be a fundamental aspect of being. This stands in contradistinction to both modern physics and mathematics, which have no notion of process and no notion of becoming.[1]

Whitehead views reality as emerging out of a lower level of reality consisting of *actual occasions*. Fundamental physical entities are viewed as emergent configurations of actual occasions. An analog lies in attempts in the 1980's to model reality as a cellular automaton where particles appeared as patterns manifesting over time on the cellular automaton

[1]Most of this appendix comes from my paper [380]

lattice [429]. Process is a construct well recognized in psychology and biology, for example the emergence theories of Trofimova [395] and Varela [406]. In physics the notion of process generally refers to an interaction between entities that unfolds in time. Processes may change certain dynamical parameters associated with continuous symmetries such as energy, position or momentum. Examples of these include scattering, state transitions, capture and emission. Processes may change certain discrete parameters associated with intrinsic characteristics like charge, charm, strangeness, lepton or baryon number. Creation, annihilation and decay are examples of these. Conceptually, fundamental physical entities enter into or emerge from various processes but their very existence is not viewed as arising out of process.

Organisms clearly fulfill Whitehead's conception of a process as do the entities considered in the complexity sciences. Organisms arise through a process of development and they continue to develop and adapt throughout their existence. Organisms are open to their environments and there is a continual flux of energy and matter between the organism and its environment. An organism's material constituents are never the same from one time to another. The individual identity of an organism does not derive from its constitution but rather from its history. Time asymmetry is a fundamental feature of biological dynamics. Organisms possess agency, acting upon their environments, changing them and then reacting to the changes in feedback relations that generate emergent functionality. This is particularly characteristic of social insect colonies. Emergence is the order of the day, appearing at all levels with intricate, almost byzantine tangles of feedback influences acting among and between levels. These are all features of process, not of object.

From a process perspective all matter would be viewed as emergent, arising from the evolution of actual occasions. These actual occasions would not be accessible to material entities in much the way that mind is incapable of sensing the actions of individual neurons, even though mind is emergent from the actions of neurons. The laws governing the behaviour of actual occasions need not be those of quantum mechanics, though the behaviour of entities emerging at the lowest spatiotemporal scales should obey those laws. Likewise, entities emerging from these fundamental quantum entities need not obey the laws of quantum mechanics, or at least quantum mechanical laws need not be relevant for understanding their behaviour, just as the laws governing the action of mind are not the same as the laws governing the neurons giving rise to it. This is a common situation in the theory of emergence. Although lower level entities may give rise to higher level entities, the relationship between these two may be such that there is no one-one correspondence between the behaviours at one level and those at the other level, so that the laws governing the lower level become irrelevant for understanding behaviour at the higher level.

Becoming is fundamental to the notion of process, and underlying becoming is the notion of transience. In Whitehead's metaphysics events do not have an enduring existence. Events by their very nature have a transient existence; they come into being, exist briefly then fade away. In Whitehead's view, each event, or actual occasion, exists only long enough to prehend the realities of the previous events and to form a response to them, immediately passing out of existence, thereby becoming data for subsequent events. There is a subjective, meaning laden thread linking these events. Whitehead considered the basic events of reality to be inseparable, to be occasions-in-connection, giving to reality a holistic aspect. Prehension underlies, and underlines, the subjectivity inherent in process. The act of prehension does not imply that events must possess some form of mind. Rather, recognition of the act of prehension is an acknowledgement that information plays a fundamental role in the unfolding of reality, and that in particular, it is meaning that is necessary to give rise to coherence among events.

The concept of becoming, fundamental to process, is singularly lacking in both mathematics and physics, Bohm being a notable exception (Bohm, 1980). In mathematics, entities simply are: there is no becoming. In physics too, events are simply held to be; they do not arise or emerge into existence, nor do they fade from existence. The past, present and future are all given the same ontological status. The transient now and the flow of time are held to be delusions [34]. Time does not exist. Though there have been a few dissenting voices [321, 120] this remains the dominant view. There are many compelling arguments in favor of this view. The two most commonly cited are due to McTaggart [103] and to Einstein [258].

According to Whitehead, all of reality is a manifestation of process. It is process that creates the events that constitute us and the reality that we observe. There are many different processes in the universe. These processes themselves are expressions of process. They arise, exist for a time, and then fade away. While they exist they manifest distinct attributes and functionality. Processes interact with one another according to their attributes and functionalities and the actual occasions that they manifest. When processes interact their combination may express emergent functionality, resulting in a superordinate process [395]. The presence of consistent and stable attributes and functional relationships among processes permits the representation of processes by means of formal mathematical spaces, much as actual occasions are represented by means of geometry as state or phase spaces. A fuller appreciation of reality thus involves not merely a description of the actual occasions as they arise but also a description of the processes that give rise to them. Thus at least two representational spaces are required, one for actual occasions and one for process. The particular form of representation for actual occasions in the archetypal dynamics approach is that of the tapestry, whereas that for process is the reality game.

Process theories, particularly the theory of Whitehead [425], possess several essential features that need to be considered in creating a representational system that expresses them. We can summarize these features:

1. The basic elements of experience, actual occasions, have a richer character than is generally attributed to elements of reality. Actual occasions possess a dual character. On the one hand they form a fundamental component of the fabric of reality. On the other hand, they serve as information for the creation of subsequent actual occasions.

2. Process theory is a generative theory. The actual occasions that form the essence of reality come into existence through a process of prehension, in which the information residue of prior occasions is interpreted and new occasions generated creatively in a non-deterministic manner.

3. Actual occasions are transient in nature. They arise, linger briefly and then fade away. In contrast to current physical thinking, process theory asserts the existence of a transient 'now'.

4. Essential to process is the idea of becoming. This is subtly different from the notion of generation. For example, an iterated function system generates a trajectory by the repeated application of the function to a previous point: x, $f(x)$, $f(f(x))$, However, the space upon which this function acts exists a priori. A trajectory in the space is generated, the space itself is not. In process theory, the space itself does not exist a priori, indeed it does not exist at all except as an idealization in some mathematical universe. All that exists is a collection of actual occasions that are continually in the process of becoming. An actual occasion has no existence unless and until it is brought into existence through the action of prehension. It subsequently fades into non-existence and any future influence that it might have arises solely through its representation in some form of memory.

5. In process theory events are fundamentally discrete, being comprised of vast numbers of actual occasions. The perception of events as being continuous may well be a deeply ingrained illusion. Its abstraction in mathematical form has given rise to powerful analytical tools, so much so that continuity has been reified as a property of space-time and histories and entities. Process theory asserts that entities and their motion are actually discrete and their apparent continuity is again a consequence of the process of idealization inherent in the formation of an interpretation.

6. Actual occasions are held to be holistic entities. It may be convenient for conceptual, descriptive or analytical purposes to consider actual occasions as consisting of

individual 'parts', but this again constitutes an idealization or contrivance. Each actual occasion must be considered to be a whole unto itself and any information or influence attributed to an actual occasion must be attributed to the actual occasion as a whole and not to any of its supposed parts. Any such parts must be considered to be unobservable as must any presumed properties or characteristics of these parts. Properties and characteristics that may be observed by other actual occasions must be attributed solely to the actual occasion as a whole.

Process has an inherently non local character. A biological organism is an expression of a vast array of processes but these processes are local only in a naive and superficial sense. The entities that participate in these processes are distributed widely in space and time. Mental processes cannot even be localized to the brain as the body and the environment play important roles. Secondly, the entities that participate in process are often fungible. Although a whole organism may not be fungible, its constituent molecules most certainly are, and sometimes components are not even of the same species as the organism, particularly in the case of digestive processes. A board game such as Chess is a simple example of a process. Although individual moves of chess pieces are local, the choice of which piece to move on a given play is inherently non local, though certain game positions may favour local or non local choices. Chess is fungible, so long as any replacement respects the current arrangement of pieces. Chess can be played anywhere, at any time, with almost any objects, real or virtual, so long as a suitable correspondence is established between the objects and their movements and their roles as chess pieces. Processes in themselves exist in an abstract, aspatial and atemporal world, while the actual occasions that they generate manifest in space and time and bear specific relations to one another that are interpreted as properties.

C.2 Information and Meaning

Whitehead suggests that as an actual occasion passes out of existence it becomes a datum for succeeding occasions, and that as an actual occasion comes into existence it prehends the immediately prior occasions and forms a response to them. This response in turn becomes a datum in its own right. Therefore, even though actual occasions pass into and out of existence, some or all of the information associated with those occasions (and the content or meaning contained therein) persists, recursively encoded in the data residue left behind by actual occasions. Thus actual occasions are bound together coherently by virtue of information. This suggests that what endures in reality is information, so that the most significant focus for study is not the actual occasions or events but rather the information associated with them.

The form of information alluded to by Whitehead informs responses to actual occasions, and therefore influences the selection of a particular response from a collection of possible responses. In Whitehead's notion of process, responses are under-determined, which permits novelty and creativity, capacities singularly lacking in a deterministic Newtonian reality. The information residing in the data of prior actual occasions has content and meaning and it is this which influences the formation and responses of subsequent actual occasions. This process information is to be distinguished from the form of information developed by Shannon [320], which is prevalent in mathematics and physics today. Whitehead is concerned with information that informs action, and therefore information that either conveys or elicits meaning that becomes expressed in the response of an actual occasion. Shannon was concerned less with the actual content of information than with the capacity of some signaling process to convey information. An important issue for a signaling device is to determine how much information it can carry. Shannon's idea was that the presence of information should impose some order or structure on the signal, the net effect being to introduce some degree of predictability upon the signal. If a signal is composed of a string of symbols i, each occurring with probability p_i , then the information content of the signal is given as the sum $I(p) = \sum_i (p_i \ln p_i)$. This is the negative of the entropy associated with the symbol string of the signal, demonstrating the close relationship between Shannon information and ideas of randomness and order.

Order alone, however, does not imply meaning. Symbol strings having the same degree of order, and hence the same amount of information, need not convey the same content or meaning. The words uxpg and fire have the same amount of information but one is meaningless whereas the other has quite a robust meaning. Meaning arises in the process of interaction between information and observer and is ultimately expressed in some behavior in the observer, and possibly the environment as well. An observer attributes meaning to information, and this process depends upon the observer's temperament, history, mood, culture among other factors. Two observers exposed to the same information may derive quite different meanings from it. Any couples therapist is well aware how two individuals faced with the same apparent information generate divergent meaning. It is meaning, however, that influences behavior. Shannon information is meaningless information. Therefore the concept is of little value for organisms that are capable of generating meaning and encoding it as information to be passed on, or subsequently extracting or attributing meaning to information received. Organisms base decisions upon the meanings that they attribute to information, not merely to the quantity of available information.

Objects, being unable to intend, prehend, apprehend, or comprehend, are incapable of extracting or generating meaning or information and therefore merely serve as carriers of information. In these circumstances, the quantity of information that can be carried by an object or collection of objects becomes significant. It is on account of the object like behavior of many physical systems that Shannon information has gained such prominence in physics. This also accounts for the importance placed upon entropy because objects cannot generate information and create order, they can only either repeat information, or lose it. Therefore for situations composed of objects, entropy either stays constant or increases. Organisms, on the other hand, do create information, order, and meaning, and so organisms are capable of reducing entropy. Admittedly this is a local effect, and the overall entropy of the organism and its environment increases, but this latter effect is not relevant as far as the behavior of the organism is concerned.

Information, as order, does tell us something about the process that created it, namely that it was not purely random but exhibits some correlations. The mere presence of correlations, however, says nothing about whether this information possesses content, that is, whether some entity imposed this order so as to encode content that some other entity could later decode so as to retrieve the original content. Meaning and information are linked but are not identical. They describe distinct aspects of patterns. Information says something about form but meaning says something about action. Meaning serves to determine subsequent action in response to some event.

The form of meaning that can be associated to sentient organisms, in particular human beings, is too complex and specialized a phenomenon to be applicable to processes generally. Indeed, Whitehead speaks of actual occasions forming a response to prehensions of prior occasions, but it is the capacity to respond that is most salient. The concept of prehension falls short of apprehension and comprehension and so meaning in its fully developed form is too strong an expectation. Rather what is desired is merely the capacity of an actual occasion to select out of a plethora of possible responses only those responses that sustain the coherence of the information flow being prehended. Thus a kind of proto-meaning is probably sufficient for this purpose.

This kind of proto-meaning does not require a conscious, sentient observer endowed with a psychology for its prehension and expression. Transient induced global response stabilization (TIGoRS) is a phenomenon exhibited by a wide range of complex systems models including tempered neural networks, driven cellular automata and cocktail party automata [354, 356, 359]. TIGoRS is a phenomenon that occurs in open complex systems dynamically coupled to a complex environment. The environment generates dynamical transients that interact with the components of the complex system eliciting a transient response from the system. In the presence of TIGoRS, the complex system produces, in response to a fixed environmental transient and across a range of initial conditions, a collection of dynamical transient responses that are similar to one another. The degree of correspondence can be striking. For example, cocktail party automata are adaptive cellular automata having homogenous neighborhoods but a non-uniform and dynamic rule structure. A random pattern of stimulation can be applied to a mere 10% of the cells of a cocktail party automaton resulting in a 98% agreement based on Hamming distance among responses generated from a set of distinct initial state and rule configurations. The rule configurations present following completion of the stimulating transient differ even though the patterns generated show this high degree of matching [356, 359]. This TIGoRS effect depends upon the structure of the stimulating transient and the dynamical characteristics of the complex system. In the case of the cocktail party automaton, random patterns frequently induce a greater degree of TIGoRS than do structured patterns. As humans we are used to structured patterns inducing a greater similarity of response, hence the value of written and verbal language.

The example of the cocktail party automaton reveals that it is not the form of the stimulating transient per se that is important, but rather its capacity to induce TIGoRS. TIGoRS enables a complex system to parse the space of dynamical transients into distinct collections to which coherent dynamical transient responses can be associated. This creates an input-output relationship between the complex system and its environment that resembles that of a symbol processing system, or automaton. More generally, Sulis machines, by virtue of TIGoRS [238], are capable of parsing spatiotemporal patterns into distinct classes according to their induced responses, thus extracting, or more precisely imposing, meaning on these transients as expressed in their subsequent actions. This coupling between environment and complex system takes on a basic linguistic structure which is why it can be described as a proto-meaning process. TIGoRS appears in such a wide variety of complex systems that it may serve as a model for proto-meaning attribution generally. In any case, it is a feature of complex adaptive open systems and so is more a feature of process than of object. Its significance is that the kind of proto-meaning that seems to be required of actual occasions can be manifested by systems exhibiting TIGoRS in the absence of the complex mental apparatus that appears necessary for the more complex attribution of meaning. This would seem to avoid the mentalism or psychism that plagues Whitehead's metaphysics and provides at least one mechanism by means of which actual occasions might maintain a coherent flow of information and so generate a process.

The cocktail party model illustrates a subtlety in notions of cognition, particularly in the distinction between embedded and embodied cognition. The cocktail party automaton does not act based upon any form of internalized representation, and so in this sense it manifests an embedded cognitive style typical of non-representational systems [64]. On the other hand, the stabilization process depends upon the internal dynamics of the collective of the automaton and the form of the coupling between the automaton and its environment. This coupling provides a formal description of the embodiment of the automaton within its environment, and so in part there is an aspect of embodied cognition present here as well. Thus the cocktail party automaton exhibits an admixture of both embedded and embodied cognitive styles. In that it seems to express a style similar to that described by Trofimova as *projection through capacities*, whereby "a person registers only those aspects of objects or situations that he can properly react to and deal with in their own behavior" [396].

Objects following a deterministic Newtonian type dynamics have no need of information to determine their behavior. The present state is sufficient for that purpose. This is not the case for stochastic systems, for complex adaptive systems, and in particular, not for organisms. Organisms are capable of many different responses to the same stimulus, but their responses become organized over time into fairly stable, consistent patterns and this process requires the influence of meaning. This is also true of a process, which must maintain coherence among its actual occasions so as not to degenerate into a meaningless mess of random occasions.

C.3 Semantic Frames

A process consists of a sequence of actual occasions having a coherent temporal structure provided by some set of proto-meaning relations. These proto-meaning relations arise through the prehension of prior actual occasions by subsequent actual occasions. One mechanism through which such prehension might occur is TIGoRS. The entirety of the proto-meaning relations of a process is termed its semantic frame. The semantic frame is a primitive conceptual entity through which a process in interaction with its environment derives meaningful responses to the six fundamental epistemological questions: who, what, when, where, how and why. *Who* determines the relevant entities with which the process interacts. In so doing it sets the specific scale of the process. A process could arise at a fundamental level, a micro-level, a meso-level or a macro-level, or at finer levels in between. *What* determines the form and limitations of the interactions between the process and these entities. *When* determines the temporal structure of these interactions. *Where* determines their spatial geometry. *How* determines the mechanisms underlying these interactions. *Why* determines the specific goals and functionalities of these interactions. Not all semantic frames will address all of these questions. Those which serve at the level of proto-meaning should, at the very least, provide sufficient answers to the questions of who, what, when and where so as to enable the process to interact in a meaningfully salient and coherent relationship within itself and with its environment. Frames operating at the level of meaning should also provide answers to how, and perhaps to why as well.

The act of prehension implies that every process generates or expresses a semantic frame depending upon whether one wishes to assume that process is prior to (proto-) meaning or that (proto-) meaning is prior to process. This difference in perspective is not fundamental but does generate different opportunities for description and analysis. Process and (proto-) meaning appear to be inextricably intertwined, so that the distinction becomes a matter of convenience rather than of ontology. As the semantic frame determines the manner in which the process sees its environment it is termed an internal semantic frame. The environment itself is a process and therefore it too has an associated semantic frame. This frame is an internal semantic frame for the environment but it is an external semantic frame for the original process. It determines how the environment sees the process. In particular it determines the scale of description and interaction with the process. In general, internal and external semantic frames come in distinct pairs which maintain coherence among the descriptive scales. Thus, for every external semantic frame which describes the process at scale A, there should exist an internal semantic frame which describes the environment at the same scale. Since a given process may be described from the viewpoint of several different scales it may, in general, be associated with several internal and external semantic frames. For any given process one seeks a complete collection of these internal and external frames together with a theory or understanding of the varied relationships that exist between the frames in each scale pair and between frames in different scale pairs.

The concept of a semantic frame, like that of process, is a primitive construct. Thus, it cannot be given a specific form since it is a notion that can be associated with an endless variety of processes, and therefore the specific form of representation chosen will depend upon the particular process in question. What is significant is not the form but rather the idea of a semantic frame as an organizing principle for a process in interaction with its environment. The semantic frame determines in what manner the flux of experience accessible to the process is to be decomposed into specific entities and in what manner is the process to interact with and respond to those entities. Semantic frames have, for example, formal expression in the various theories that constitute modern physics such as thermodynamics, classical mechanics, non-relativistic quantum mechanics, relativistic quantum mechanics, quantum field theory, Newtonian gravitation, general relativity, MOND, loop quantum gravity, string theory.

C.4 Emergent Situations

Organisms are not unitary entities. They consist of a complex intertwining of processes occurring across multiple spatiotemporal scales and standing in an open relation to one another and to the larger environment. The causal influences acting among these processes are inextricably knotted, much unlike the picture that naïve reductionism might suggest. There are influences acting from the lower scales to the higher, from the higher scales to the lower, and within scales, all more or less simultaneously. Nowhere is this seen more clearly than in the mechanisms underlying neural representation [359, 371]. In rats, information about their spatial location appears to be encoded in the firing rates of specific cells in the hippocampus called place cells. When the animal is in a specific place, a specific set of cells maximizes its firing rate. In principle one should be able to determine the location of the animal simply by observing the firing rates of these place cells. It turns, however, that this encoding is metastable, and the association between spatial location and place cells can be altered by intervening experience. Without knowledge of the psychological and behavioral history of the animal it becomes impossible to accurately decode the information represented by these place cells. In spite of this metastability, the animal is nevertheless able to utilize this representational system with great accuracy. This shows that even though neurons are necessary for the embodiment and expression of mind, knowledge of their dynamics and behavior alone is insufficient to explain the behavior observed at the level of the organism.

The flow of information in such a situation requires several semantic frames for its description and explication. These semantic frames reference processes appearing at different spatiotemporal scales. Two important questions are to understand how the different processes interact and influence one another and what influence they have over the subsequent flow of information. In terms of semantic frames interest centers upon the nature of the formal relationships between these different frames.

In the simplest case, a semantic frame at a higher scale may result from a straightforward transformation of a frame appearing at a lower level. These transformations may have a categorical or a logical character, representing products or quotients or averages or the like. One semantic frame is said to be reducible to another semantic frame if some such categorical or logical transformation exists mapping the latter to the former. For example, in the standard statistical mechanical model of an ideal gas, macroscopic parameters such as temperature and pressure are understood as averages of energy and momentum transfer over the constituent particles of the gas. This suggests that the macroscopic properties of the gas are simple extrapolations upwards from the microscopic properties of the particles. However there is a subtlety in that these macroscopic parameters, especially pressure, have no physical meaning unless the gas is constrained within a macroscopic physical volume, so that they really arise in an emergent manner from the interaction between the gas and the environment, and not from the gas alone.

More interesting though is the case in which the various semantic frames are mutually irreducible. Such a case is termed an emergent situation. Any organism provides a living example of an emergent situation. The significance of emergent situations is that is it precisely in such cases that naïve reductionist ideas fail. An emergent situation is comprised of a hierarchy of processes in which the upper level processes fail to be simple extrapolations from those at lower levels. Epistemologically, an understanding of the behavior of the lower level processes is insufficient to enable an understanding of those at the higher level. This is not to say that the higher level processes could exist without the support of the lower level processes - far from it. But the semantic frames of the lower level processes do not possess sufficient explanatory power to determine the phenomena that appear in the higher level processes. As an analogy, although paint is necessary for a painting, knowledge of paint alone is insufficient to explain art. A more subtle example of an emergent situation is expressed in the relationship between classical and quantum mechanics. It is an article of faith that quantum mechanics provides a fundamental theory of everything (or almost everything) in spite of the fact that there are enormous mathematical, logical, and philosophical differences between quantum mechanics and classical mechanics, both in theory and in experimental observations. One strong reason to think that the quantum and classical domains form an emergent situation involves the problem of linear superpositions. In quantum mechanics, the states of quantum systems can form linear superpositions and these superpositions appear to be carried upwards as single systems combine into collectives. The problem is that at the macroscopic level, especially the level of general relativity, these superpositions do not occur (hence the Schrödinger cat paradox), which would seem to be clear evidence that macroscopic behavior is not reducible to microscopic behavior and therefore we have an emergent situation. It would seem that the failure to understand this as an emergent situation results in all manner of conceptual conundrums, which could be avoided by giving up the reductionist program.

Most approaches to emergence focus upon the mechanisms involved in specific cases of emergence. These mechanisms are legion, universal for some classes of emergence but not for others. Different authors like to emphasize different mechanisms such as self organization, complex adaptation, multidirectional feedback, bidirectional causation. Many different mechanisms occur simultaneously in the same situation, such as in the case of collective intelligence systems [360]. The value of the notion of an emergent situation is that it focuses attention on the larger problem of emergence without getting trapped attempting to overgeneralize a particular example. A similar situation pertains in the theory of nonlinear dynamical systems, which describes a wide range of universal behaviors supported by a variety of different mechanisms. It is the universal that is most important and the idea of the emergent situation is directed towards the search for such universal patterns. Underlying this search are two basic informational mechanisms: salience and irrelevance [360].

It is within emergent situations that novelty and creativity arise. It is within emergent situations that living organisms appear and play out their lives. Indeed it is very likely the case that every instance of a naturally occurring computational system [352] represents an emergent situation. It is important, however, to note that there are many examples of non-emergent situations, and understanding them is also important. Most products of mechanical engineering are non-emergent systems, being complicated but not complex, and certainly not adaptive. Some social processes are also non-emergent a small social group of friends comes to mind. The study of the mechanisms whereby complex systems of interacting agents fail to express emergent behavior may prove to be just as important as understanding those mechanisms through which they do give rise to emergence. Naïve reductionism seems doomed to failure but a more judicious and subtle application of reductionist techniques might prove to be quite beneficial in the long run. Again it is essential to know the boundaries of any theory.

C.5 Archetypal Dynamics

A semantic frame must be semantically coherent and logically consistent. Moreover, an important notion borrowed from physics is that semantic frames must also be effective. That is, they should not only give definition and meaning to the varied entities that comprise a system, they should also accurately describe the dynamics among those entities, capture and determine universal relations among those entities and be capable of making predictions about hitherto unobserved behaviors among those entities. In physics it is now well recognized that physical theories come with an effective range. Within that range the theory has high descriptive and predictive power, but as one moves outside that range this power diminishes until ultimately it must be supplanted by a new effective theory. Likewise each semantic frame is effective for only a specific range of real phenomena and must be studied within that context in order to be useful. A semantic frame that is coherent and consistent but ineffective in describing any aspect of reality is worthless. There are far too many examples in the world where semantic frames are applied inappropriately to phenomena to which they do not apply, usually with disastrous consequences. For example, the misapplication of an economic frame to the study and implementation of health care has led to the denial of health care to ever increasing numbers of people

Although it is possible to study semantic frames in isolation from the reality that they purport to describe, such as occurs in mathematical logic and postmodern philosophy, this is not the case in archetypal dynamics. Equal status is most emphatically not ascribed to all possible semantic frames. Rather, archetypal dynamics is interested in understanding the relationship between particular slices of reality and the semantic frames that effectively describe those slices, and how the dynamical and informational interplay between different reality splices plays out in the relationships between their corresponding semantic frames. Thus in what follows one is only considering coherent, consistent and effective semantic frames.

Now processes can be compared by form or by functionality. If we imagine a class of processes that are similar according to some comparator then we can expect that the semantic frames associated with each of these processes will themselves satisfy some comparator. One can then search for some exemplar for this class of semantic frames. Natural processes can be expected to depart from any ideal process, thereby generating fluctuations in behavior and response that lead to transient and irrelevant proto-meaningful associations. Given a process P and a semantic frame S it is useful to ask two questions that directly address the issue of effectiveness of a frame for a given process. To what degree does P express the relations described in S? To what degree does S describe the behaviors exhibited by P? Clearly for a process P there is a generated semantic frame S(P) such that P expresses everything in S(P) and S(P) describes everything in P. But what if S and P are not so obviously related? We say that P is complete with respect to S if P expresses every relation described in S (and perhaps more). We say that S is complete with respect to P if S describes every behavior of P (and perhaps more). Given a process P consider the class of semantic frames SP consisting of all frames in respect of which P is complete. Given a class of processes P we can consider a kind of categorical pullback or greatest lower bound A on the class SP of classes of frames. Every process in P is complete with respect to this frame A. The semantic frame A is then termed an archetype for the class P of processes.

One can work in the opposite direction as well. Given a process P one can consider the class of all semantic frames SCP that are complete with respect to P. Given a class of processes P one can consider the class of classes SCP and in particular, the lowest upper bound on this class. This will be a semantic frame T that is complete with respect to every process in P. This frame will be called an explicandum for the class P since it describes everything that happens in every P. Questions related to the precise meanings to be attributed to the terms greatest lower bound, least upper bound and the existence of such objects for specific classes of processes is a matter of future research. Clearly for a given process P, P is complete with respect to S(P) and S(P) is complete with respect to

P.

The archetype for a class of processes thus describes the features common to all of the processes, much like the notion of an archetype in Jungian psychology describes the features common to a particular human role such as that of mother. Note that in mathematical logic the relationship between a theory and its model is that of archetype and process. The explicandum for a class of processes describes all of the details of behavior expressed by all of the processes. A conjecture would be that the explicandum of P is a formal union of all archetypes of P.

Although processes are spatiotemporal sequences it is possible for the structure of prehension to change over a time scale that is long compared to that of the creation of actual occurrences. It such a case it is interesting to explore what happens to the class of archetypes being expressed by the pattern. A useful analogy is to consider the relationship between process and archetype as between a dynamical system and its order parameter. As the system changes with respect to some parameter, say temperature, its order parameter may also undergo change, sometimes so great as to induce a new order parameter. This gives rise to a phase transition in physics. In physics phase transitions can be described by regions of non-analyticity of free energies, by non zero values of some order parameter, by changes in the qualitative nature of some probability distribution. In the case of processes one might expect that a slow change in a process results in a situation in which the process departs gradually but to an increasing degree from some archetype, until the discrepancy becomes so great that a new archetypal association is required. It then becomes interesting to study the nature and description of the transition process. Indeed bifurcation theory in the study of nonlinear dynamical systems has the form of an archetypal description of transitions between archetypes of dynamical behavior.

The picture can be simplified in the case that one is dealing with objects, such as in the case in physical theory. Objects serve only as carriers in information. They react but do not act, and therefore they do not attribute meaning or proto-meaning to the information they encounter. In this case there is no internal semantic frame associated with the object. The only frame is the external frame of the environment. Thus the situation of an object and its environment is greatly simplified as one only has to deal with only one semantic frame, the external semantic frame. Indeed it is to this external frame that one commonly attributes the notion of objectivity. In the more general case one will require an internal semantic frame for each member process and an internal frame for the environment.

Archetypal dynamics [380] is the name given to an approach to the study of emergence based upon an analysis of (proto-) meaning laden information flows in processes using the concept of the semantic frame. Archetypal dynamics adopts the viewpoint of that semantic frames are prior to process. This is to facilitate the use of mathematical representations and methods of analysis and is not meant to make any assertions about the ontology of process. It is purely pragmatic.

The central idea of archetypal dynamics is expressed in the fundamental triad. The fundamental triad for a semantic frame consists of a realisation, an interpretation, and a representation. A realisation of a semantic frame is any process for which the frame serves as an external frame. The notion is similar to that in mathematical logic of a theory and a model. An interpretation of a semantic frame is any process for which the frame serves as an internal frame. The classical notion of an observer is an interpretation for a semantic frame. Finally a representation of a semantic frame is any symbolic structure to which a semantic frame and a process can be placed in a one-one correspondence with for the purposes of description, communication and analysis. In principle one does not work directly with a semantic frame or with a process but rather with some representation of them. In doing so, one must always recognize that the representation may possess properties, characteristics, and behaviors that are not shared by the representation. This recognition is necessary in order to avoid the problem of reification described by Mermin [259]. A representation is not equivalent to that which is represented, but it is a useful tool and in fact is the best that we can do when formulating theories of any kind, whether mathematical, linguistic, artistic, spiritual or whatever.

Since there may be many different representational schemes for a given semantic frame and process it is not possible to define archetypal dynamics in more specific terms. One may think of archetypal dynamics as an organizing scheme for studying information flows in processes, or as a program for research.

Appendix D

Causal Tapestries and the Reality Game

This Appendix provides background and formal definitions of several key concepts in the theory of causal tapestries[1]. This theory has a long history extending over several publications [357, 359, 366, 367, 370, 375, 376, 380, 381, 382] and the interested reader is referred there for further details.

In mathematical logic there is an extensive literature dealing with the relationships between a formal theory and its models. A formal theory is a logically consistent collection of (usually) first order sentences. A model is a specific mathematical structure, having specific elements and relations defined upon them. A model of a theory is a structure about which the theory makes true statements. Mathematical logic is both a representation of the relationships between reality and the languages used to describe and understand it, and a realisation (model) and interpretation (theory) in its own right. Generally speaking, in logic one seeks to determine the models for a given theory, and all of the theories for a given model.

In the tapestry setting we do not operate with the formal theory directly but rather with models of the theory. The two fundamental questions become reformulated as follows. Given a collection of bare informons together with a generating reality game, what classes of models can serve as interpretations of these informons? In other words, into what classes of models can one faithfully embed these informons. This corresponds to determining which semantic frames provide an interpretation of the realisation represented by these informons. Conversely, given a collection of empty informons we seek the classes of reality

[1]This material is taken from my paper NDPLS 16(2) 113-136 (2012)

games that will provide content and dynamic for the tapestry such that the informons can be faithfully embedded in the interpretation model. This corresponds to determining which reality games give rise to realisations of the semantic frame expressed in the model.

D.1 The Role of Causal Connections

Our common sense experience of the flow of events in space-time has been so shaped by Euclidean geometry and its representation by the Cartesian plane that events have become almost synonymous with their coordinate representation. As a result we come to think of the coordinatization of the world as being fundamental and build our models upon it. This was certainly the view of Newton who believed in an absolute space-time. But in point of fact coordinatizations are entirely arbitrary. The particular forms that we use exploit certain observed symmetries of the space-times that we choose to model together with features of the real number system. The choice of symmetry is arbitrary (translations give us Cartesian coordinates, rotations give us Polar coordinates) as is the choice of number system, there being representations based on real numbers, complex numbers, quaternions and spinors. The choice of coordinate system is based on convention and convenience as well as computational effectiveness, but a completely random choice is permissible too, if not very useful. Relativity showed that the problem is deeper still in that even if a convention is agreed upon by all observers it is impossible to find one choice of expression of that convention which works for all observers. There is no absolute space-time, only a relative space-time. Relativity demonstrated that the coordinatization of reality was not a fundamental attribute of reality but instead was an observer dependent construction. Nevertheless in an even deeper manner it revealed the existence of aspects of reality that truly are invariant and independent of the observer. The most important to be considered here is that of causality. The causal structure of the world (defined rigorously below) is an invariant feature of reality. It is the absolute underlying reality, while our superficial frames of reference are actually illusory. It is for this reason that the model for information flow described here is based upon notions of causality.

Causality is a philosophical term with a long, illustrious, and oftentimes contentious history. Nevertheless an important aspect of the scientific endeavour is the attempt to elucidate the causes of natural phenomena. In the past, causation was considered to be of a deterministic, linear character of the form if A causes B then whenever A occurs B is certain to follow. The idea of stochastic causation weakened this relation to if A causes B with probability p then whenever A occurs B will occur with probability p. In the past century the advent of systems thinking introduced the idea of feedback so that A could cause B but it was also possible for B to cause A. The idea of emergence attempted to causally relate events occurring at one spatiotemporal level with those occurring at another. Originally this was thought to act from the lower to the higher but we now understand that causation may occur from lower to higher, from higher to lower and within levels [18]. The concept of causality implies, implicitly or explicitly, some notion of agency. That in turn can only be demonstrated through multiple observations of correlations between these two kinds of events and the outcomes of interventions designed to modify the causal relationship [289]. However, in complex systems such manipulation may be impossible without destroying the system and intervening variables may be unobservable. Indeed the notion of causation has become so fraught with difficulty that a better notion is that of influence. An influence may be thought of as anything whose presence correlates with some particular configuration of attributes of something else while its absence correlates with a different configuration. In this sense, information may be construed as an influence on events.

Physical theory explains events on the basis of the interactions among particles and their fields, constrained by various conservation laws and symmetry principles. Physics utilizes theory and mathematical models to describe its interactions. Causality in both physics and psychology concerns itself with explicating and predicting the dynamics of the events under consideration. Here, however, the focus is upon a description of the data, the raw experience as it were, upon which the dynamics plays out. Just as the Cartesian plane is often the setting for describing the dynamics of physical systems, and time series (or correlational data) provide the raw material for the playing out of psychological dynamics, so the tapestry provides the stage upon which the dynamics of the reality game takes place. The reality game is a form of combinatorial game which is utilized to generate a succession of tapestries. Causal tapestries describe the flow of events, and of information, that takes places within a natural system. The use of the term causal here must be distinguished from causality and derives from relativity theory. There, two events are causally related if it is possible, at least theoretically, for a signal to pass from one to the other. In other words, two events are causally related if it is possible for information to pass from one event to the other. This notion is much more general than the assertion that the former event causes the latter event. Although at first glance one might think that invoking relativistic considerations is excessive, since living systems never approach relativistic energies or speeds, in point of fact relativity provides a fundamental constraint on the speed with which any signal may move within any physically realizable system. It is important that finite signal speeds are taken into account when studying information flow since information may only pass from one event to another by virtue of some form of signal.

Causal tapestries address the more primitive situation in which one denotes merely the events themselves and their succession, which by their own nature constitute a form of signalling from prior actual occasions to subsequent occasions. Since such information flow must occur within a real physical environment by means of real physical processes, it must be subject to the basic physical principles governing such flow, the most important being those of the theory of relativity, and therefore it must pass at no greater than the speed of light and so must constitute a causal process in the relativistic sense. Such a causal structure naturally takes the form of a partial order on the space of events [61].

Thus causal tapestries must come equipped with, at a minimum, an intrinsic order structure. Pearl's causal networks are also based upon an order structure [289], but this actually seems to be misleading. In complex systems there are multiple forms of feedback among the various entities and such feedback relationships would invariably, it seems, induce cyclical structures in the graphs of the associated causal networks. This would not be a problem if the graphs represent relationships between the dynamical processes generating the flow of events but it would be a problem if the graphs are meant also to represent that flow as well. Moreover, the explanatory power of causal networks seems to be limited to situations in which the probabilities governing these causal relationships are of the classical Kolmogorov type. Some recent work shows that causal networks are unable to reproduce the causal relationships in a quantum mechanical Bell's situation, possibly due to the inherent non-Kolmogorov nature of the underlying probabilities [335].

It thus seems important to make a distinction between the fundamental flow of events which must be causal in the relativistic sense, and causality as reflecting the interrelationships between the various processes giving rise to the dynamics expressed in the causal structure. The causal tapestry is thus presented as a general formal structure meant to express those fundamental causal relationships while the reality game expresses the dynamics generating those relationships and thereby its causality.

D.2 Basic Ordered Set Theory

Mathematically, causal relations are described by a partial order [303]. Given a set X, a *partial order* on X is a binary relation $\leq$ satisfying the following conditions:

1. (transitivity) If $x \leq y$ and $y \leq z$ then $x \leq z$
2. (antisymmetry) If $x \leq y$ and $y \leq x$ then $x = y$

3. (reflexivity) $x \leq x$

Many authors consider only strict partial orderings, $<$, in which reflexivity fails. It is easy to show that strict partial orderings and partial orderings contain essentially the same information. To get a strict ordering from a partial ordering, eliminate the relation $x \leq x$. To get a partial ordering from a strict ordering, simply define $x \leq y$ to mean either $x < y$ or $x = y$. In this paper, ordering will mean strict partial ordering.

Two elements x, y are *comparable* if either $x < y$ or $y < x$. Otherwise they are *incomparable*, denoted $x||y$.

A *chain* (or linear order) is a subset of an ordered set such that every pair of elements is comparable. An *antichain* is a subset of an ordered set such that every pair of elements is incomparable.

Given two orderings $<_1$ and $<_2$ on a set X, we can define their intersection $<$ to mean $x < y$ if and only if $x <_1 y$ and $x <_2 y$.

Given two orderings $<_1$ and $<_2$ on a set X, we say that $<_1$ is a *suborder* of $<_2$ if whenever $x <_1 y$ then $x <_2 y$. We also say that $<_2$ is an extension of $<_1$.

Given two orderings $<_1$ on a set X and $<_2$ on a set Y, a mapping $f : X \to Y$ is said to be an *order homomorphism* if $fx <_2 fy$ whenever $x <_1 y$. f is an isomorphism if it is both one-one and onto.

The study of the generation of ordered sets dates back to work on random graphs and random orders [337]. Those ideas have been applied to the generation of causal sets by means of the causal sequential growth model [302]. A more information based approach to the generation of ordered sets began with the theory of order automata [343]. These are ordered sets whose structure is generated by the actions of an automaton on the base set of the order. This work demonstrated that it was possible to have an invertible action give rise to an order structure, which showed that a time reversible dynamics was compatible with a time asymmetric causal order.

Definition: A *semigroup* is a set X together with a binary operation that is associative, meaning that $x(yz) = (xy)z$. A *monoid* M is a semigroup X together with an element 1 such that for all elements, $x1 = 1x = x$. An M-automaton is a triple (M, X, f) where M is a monoid, X is a set and $f : X \times M \to X$ such that

1. $f(x, 1) = x$ for every $x \in X$
2. $f(f(x, a), b) = f(x, ab)$ for every $x \in X$ and $a, b \in M$

Definition (Order Automaton): An *order automaton* is a 4-tuple (M, X, R, η) where M is a monoid, (X, η) is an M-automaton, and R is a partial order on X such that $x <_R y$ if and only if there exists an $f \in M$ such that $\eta(x, f) = y$. It can be shown that every ordered set is generated by an order automaton.

Definition: A *free monoid* on n generators $(x_1, \ldots, x_n)$ consists of all products of the form $y_1 \cdots y_m$ where m is arbitrary and each y_i corresponds to one of the generators. A basic result of monoid theory is that every monoid is the homomorphic image of some free monoid.

Definition: The *free dimension* of an ordered set X is the cardinality of the set of generators of the smallest free order automaton generating X. The free dimension is related to the number of processes generating the order.

A element y of an ordered set X is an *immediate successor* of x if $x < y$ and there is no z such that $x < z < y$. A basic result of order automaton theory is that the free dimension is bounded below by the maximal number of immediate successors of points in X or is countably infinite if there is an element with no immediate successor.

The free dimension is an intrinsic invariant property of an ordered set which can be used to classify ordered sets into distinct groupings.

The important interpretive or archetypal structure in physics for describing causal relationships in the causal space or causal manifold [61]. This is a generalization of Minkowski space, which is a real four manifold together with a metric of the form $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. In relation to any point s of the space, this metric defines three distinct regions. There is the light cone or null cone, being those points whose distance from s is 0, the time-like region being those points whose distance from s is positive, and the space-like region being those points whose distance from s is negative. Light travels within the light-cone, physical signals travel within the time-like region to reach s, while no physical signals can travel from any point in the space-like region to s. The time-like region is divided into two disjoint subregions, one termed the past causal cone region of s, $\mathbb{P}(s)$, and the other the future causal cone region of s, $\mathbb{F}(s)$, since it is only points within the past causal cone region that can, through signals, influence s and it is only to those points in the future causal cone region that s itself can send a signal and thereby exert an influence. The order is defined by $x \prec y$ iff $x \in \mathbb{P}(y)$.

Moreover, under the symmetry transformations of relativity, the Lorenz group, this metric remains invariant so that these regions in turn are invariant. Thus while the coordinates of points relative to s may change under a Lorenz transform, even reversing time coordinates, the causal ordering induced by these causal regions remains invariant. Causal relations are preserved in these causal spaces even though our arbitrary assignments of time and space coordinates are not.

D.3 Tapestries

Before we define a tapestry, recall that a *graph* G consists of a pair $(V(G), E(G))$ where $V(G)$ is a set of *vertices* and $E(G)$ is a set of *edges*, that is a subset of ordered pairs of vertices, i.e. a subset of $V(G) \times V(G)$. A cycle is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that each adjacent pair (v_i, v_{i+1}) is an edge in $E(G)$. A graph is acyclic if it does not possess a cycle. A multigraph admits multiple edges between any pair of vertices and a labelling assigns to each edge an element from some set of labels.

At its most basic a tapestry consists of a set of elements termed informons, together with a dynamic based upon the concept of a combinatorial game that is used to generate a new tapestry. More formally:

Definition (Tapestry): Let K be a set of cardinality κ. Let $\{\Omega_1, \ldots, \Omega_n\}$ denote a set of mathematical structures. Let M denote some formal conjunction of these structures. This might be a product or co-product or some other structure depending upon the circumstances. Let e be an identified label. A *tapestry* $\mathcal{T}$ is an M labeled directed multigraph whose vertex set consists of elements (informons) of the form $[i] < \alpha > \{G\}$ where i is an element of K, α is an element of M , and G is an M-labeled directed multigraph. If in a tapestry one has $[i] < \alpha > \{G\}$ and $[i] < \alpha > \{G\}$ then $\alpha = \alpha$ and $G = G$. We can thus refer to $[i] < \alpha > \{G\}$ as simply i. The *identifier* of $[i] < \alpha > \{G\}$ is i. The *content* of $[i] < \alpha > \{G\}$ is G and is denoted by $c(i)$. The *interpretation* is α and is denoted by $I(i)$.

There are two types of tapestry element. A *nexus* is an element $[i] < \alpha > \{G\}$ of $\mathcal{T}$ such that G is a subgraph of $\mathcal{T}$ such that

1. for every edge $b \in G$ there exists an α labeled edge $b \in \mathbb{T}$ having the same initial and terminal vertices.
2. if l is a locus in G then $l \in \mathcal{T}$.

A *locus* is an element $[i] < \alpha > \{G\}$ where G is an α-labeled directed graph such that if n is a nexus in G whose scope contains a locus l, then $l \in G$.

A locus represents ontologically manifesting events or states of entities. A relator conveys information concerning relationships between informons, which may manifest ontologically or be merely inferential or descriptive in nature.

The interpretation of $[n] < \alpha > \{G\}$ may also be denoted as $I([n] < \alpha > \{G\})$ and the content as $C([n] < \alpha > \{G\})$. The vertex set of G is denoted $V(G)$ and the edge set of G is denoted $E(G)$. A bare informon has the form $[n] <> \{G\}$ while an empty informon has the form $[n] <> \{\}$.

Informons such as $[n] < \alpha > \{G\}$ are conceived as carriers of information, the label n being used solely to distinguish among informons formally, the interpretation α providing meaning as an element of the mathematical structure $\mathbb{M}$ which models the semantic frame, and the content G represents the information held within or associated with the informon n as an organization of some collection of informons, which could represent the past history of n, or a collection of potential interactors, or some formal relation among informons.

D.4 Causal Tapestries

We subdivide the interpretation structure into three components, $\mathcal{M}$, $\mathcal{H}(\mathcal{M})$ and D, so that we have $M = \mathcal{M} \times \mathcal{H}(\mathcal{M}) \times D$. $\mathcal{M}$ is a causal manifold, meaning a manifold equipped with a causal ordering. This manifold could be $\mathbb{R}^n$, an n- dimensional Euclidean space, or it could be M_n , an n- dimensional Minkowski space, or it could be a more general causal manifold. One thinks of the causal manifold as a space of possibilities. $\mathcal{H}(\mathcal{M})$ is a Hilbert space over $\mathcal{M}$. D is a parameter space which carries additional invariant structure like angular separation and interval, energy, charge, color and the like. It may also carry local, relative or relational structure, like spin, angular momentum, linear momentum, field strengths, mood ratings or whatever may be associated to the events and to the processes generating them. The mathematical structure of D is left unspecified as it will be determined by the specifics of the system in question.

The construction of a causal tapestry is recursive with the ground set assumed but usually left unspecified for simplicity. Here we shall denote the ground set by I_p and its set of informons by L_p and refer to it as the collection of prior causal tapestries (prior referring to the construction process, not necessarily an indicator of any temporal relationship).

Definition (Causal Tapestry): Let K be an index set of cardinality κ, $M = \mathcal{M} \times \mathcal{H}(\mathcal{M}) \times D$. $\mathcal{M}$ is a causal manifold, $\mathcal{H}(\mathcal{M})$ is a Hilbert space over $\mathcal{M}$, D a set of properties, and I_p a collection of prior causal tapestries. Then a (strict) *causal tapestry* $\mathcal{I}$ is a 4-tuple (L, K, M, I_p) where L is a set of informons (loci) satisfying the following conditions:

1. Each informon in L has the form $[n] < \alpha > \{G\}$ where $n \in K$, $\alpha \in M$ and G is an acyclic directed graph whose vertex set is a subset of L_p

2. The union of all G such that $[n] < \alpha > \{G\} \in L$, i.e. $\cup\{G \mid [n] < \alpha > \{G\} \in L\}$ forms an acyclic directed graph.

3. $[n] < \alpha > \{G\}$ and $[n] < \alpha' > \{G'\}$ in L implies $G = G'$ and $\alpha = \alpha'$

4. $[n] < \alpha > \{G\}$ and $[n'] < \alpha > \{G\}$ in L implies that $n = n'$

5. $[n] < \alpha > \{G\}$ in L implies that $[n] < \alpha > \{G\}$ is not an element of G

6. $[n] < \alpha > \{G\}$ and $[n'] < \alpha' > \{G'\}$ in L implies that neither $[n] < \alpha > \{G\} \in G'$ nor $[n'] < \alpha > \{G'\} \in G$

7. $[n'] < \alpha' > \{G'\} \to [n''] < \alpha'' > \{G''\} \in G$ and $[n] < \alpha > \{G\} \in H$, then $[n'] < \alpha' > \{G'\} \in H$ implies that $[n''] < \alpha'' > \{G''\} \in H$ for some $[n'''] < \alpha''' > \{H\}$

8. The mapping $i : L_p \cup L \to M$ given by $i([n] < \alpha > \{G\})$ is a causal embedding, meaning that it should be an injective causal order preserving map.

The first condition states that the content of a informon in L is based on the prior informon set L_p . The second condition is necessary in order to ensure that the local causal structure expressed by each content graph extends in a consistent manner to a global causal structure on the entirety of L_p . The third and fourth conditions ensure that an index appears only once and that a specific locus is determined solely by its content and interpretation. Note that the same content may be given different interpretations. That is, it is possible for $[n] < \alpha > \{G\}$ and $[n'] < \alpha' > \{G\}$ to be in L where n does not equal n' and α does not equal α'. The fifth condition ensures an absence of self reference and the existence of a unique order on $L \cup L_p$. The sixth condition ensures that L has the structure of an antichain, that is all informons in L are causally independent of one another. The seventh condition asserts that while information may be lost in the extension of a tapestry, nevertheless if a subsequent event contains a prior event it must also contain those events that exist in a direct causal path between the current and the prior. That is one cannot side step causality. However one can have a prior event and lose all information prior to the prior. The eighth condition ensures that at a minimum the interpretation preserves the essential causal structural features of the causal tapestry, so that it truly does interpret that causal structure in terms of the mathematical structure. The order on $L_p \cup L$ is the topological order as defined below. The interpretation map thus embeds the informons of $L_p \cup L$ into $\mathcal{M}$, interpreting each topological order relation of $L_p \cup L$ as a causal order relation in $\mathcal{M}$. Additional structural features should be preserved by the interpretation as the situation demands.

A tapestry that satisfies all conditions except for condition 2 is called a *local causal tapestry.*

In general the informons of a causal tapestry are meant to represent physical events occurring in space-time that are manifested by actual physical entities and these entities will carry additional properties, such as mass, charge, color, strangeness, energy and the like. More generally we have:

Definition (Causal Tapestry with Embellishments): Let K be an index set of cardinality κ, $M = \mathcal{M} \times \mathcal{H}(\mathcal{M}) \times D$. $\mathcal{M}$ is a causal manifold, $\mathcal{H}(\mathcal{M})$ is a Hilbert space over $\mathcal{M}$, D a set of properties, and I_p a collection of prior causal tapestries with embellishments. $M = \mathcal{M} \times \mathcal{H}(\mathcal{M}) \times D$ where $\mathcal{M}$ is a causal manifold, $\mathcal{H}(\mathcal{M})$ is a Hilbert space over $\mathcal{M}$, D a set of properties, and I_p a collection of prior causal tapestries with embellishments. Then a colored *causal tapestry with embellishments* I is a 4-tuple $(L \cup R, K, \mathbb{M}, I_p)$ such that

1. (L, K, M, I_p) is a causal tapestry
2. R is a set of informons termed relators having the property that for any informon $[n] < \alpha > \{G\} \in R$, we have $\alpha \in P$ and G is a P- labelled directed multigraph whose vertex set $V(G)$ lies in $L \cup R$
3. If $[n] < \alpha > \{G\} \in L$, then $V(G)$ is a subset of $L_p \cup R_p$.

These additional conditions permit the existence of relators, which we understand conceptually as being given from outside the causal tapestry I. Our new loci convey causal relationships among prior causal loci and relators, and relators, as usual, describe multi-level relations on the informons of the tapestry. Since loci could potentially confuse the histories of loci and relators it is best to keep them separated leading to the following condition:

Definition: A causal tapestry is *logically separating* if for any locus $[n] < \alpha > \{G\}$ in L, $V(G)$ consists either entirely of loci in L_p or entirely of relators in L_p . Unless stated otherwise we shall assume that a causal tapestry (with or without color or embellishments) is logically separating.

The transitive closure operation is ubiquitous in the theory of causal tapestries and is stated formally here.

Definition (Transitive Closure): Given a binary relation R, define $T(R)$ to be the relation defined by $(x, y) \in T(R)$ iff either $(x, y) \in R$ or there exists a z such that (x, z) and (z, y) in R. Define the *transitive closure* of R to be the relation $T_c(R) = \cup_n T^n(R)$.

The connection between acyclic directed graphs and partial orders is given in the following lemma.

Lemma: Let G be an acyclic directed graph. Then the edge relation $E(G)$ on $V(G)$ can be uniquely extended to an order relation $<$ on $V(G)$. Conversely to every partial order $(X, <)$ there corresponds an acyclic directed graph.

Proof: Let $T_c(E(G))$ be the transitive closure of the edge relation on G. Set $x < y$ if and only if $(x, y) \in T_c(E(G))$. It is clearly transitive. Suppose that $x < y$ and $y < x$. Then by the definition of the transitive closure there must exist a finite sequence of pairs in $E(G)$ such that

$$(x, x_1), (x_1, x_2), \ldots, (x_n, y), (y, y_1), (y_1, y_2), \ldots, (y_n, x),$$

but this contradicts the acyclic nature of G. For the converse, define a graph having X as its vertex set by setting an edge (x, y) if and only if $x < y$. Acyclicity follows from the antisymmetry of the order relation.

The information represented in the informon structure reflects the informational history of the entity described by the informon. At the very least, such information should convey the coherence of the entity over its existence in spacetime. Thus there should be at least a thread of information that persists across the lifetime of an entity and which gives it coherence as the history of said entity. This motivates the following:

Definition (Trajectory): A *trajectory* is a graph G such that $T_c(G) = G$ and G is connected. A local trajectory is a trajectory G such that, in addition, G is complete, meaning that for every pair of vertices x, y, either $x \to y$ or $y \to x$. A local trajectory thus gives rise to a linear order. A history is a graph G such that $T_c(G) = G$. A history is a collection of trajectories, i.e. an interconnected collection of entities. A history represents the evolution of a complex entity, possibly the co-evolution of multiple lesser entities. A trajectory represents a generalized entity which could be field-like or particle-like. A local trajectory represents a classical particle.

There are two causal orders that can be defined on the informon sets.

The first order is defined on L_p alone. It is called the content order and is defined as follows:

Definition (Content Order): The *content order* $<_c$ for elements $x, y \in L_p$, is given by $x <_c y$ if and only if there exists some informon $[n] < \alpha > \{H\} \in L$ such that $x \to y$ in H.

Condition 2 in the definition of a causal tapestry ensures that this is indeed a proper order relation on the informon set L_p . In the case of a local causal tapestry such a global order relation need not occur and a proper order can be defined only on the individual content sets.

The second order relation is defined on the set $L_p \cup L$ and is called the topological order. It is defined as follows:

Definition (Topological Order): The *topological order* $<_t$ for elements $x, y \in L_p \cup L$ is given by $x <_t y$ if and only if $y = [n] < \alpha > \{H\}$ and $x \in V(H)$.

It is easy to see that $x <_c y$ is a suborder of $x <_t y$. Several cases can then be defined.

Definitions: A causal tapestry is

1. *information preserving* if the content order is the topological order
2. *information losing* if the content order is a nontrivial suborder of the topological order
3. *information annihilating* (or memoryless) if the content order is an antichain

Definition (Bare Tapestry): Suppose that $\mathcal{C} = (L, K, M, I_p)$ is a causal tapestry. Then the *bare tapestry* of $\mathcal{C}$ is the tapestry $B(\mathcal{C}) = (L, K, I_p)$ defined by the mapping $\mathcal{C} \to B(\mathcal{C})$ given by $[n] < \alpha > \{G\} \to [n] <> \{G\}$, i.e. simply take the set of informons of $\mathcal{C}$ and replace each content with the empty value. More generally, a bare tapestry is a tapestry in which the interpretation structure is merely the empty set.

The bare tapestry provides a representation of a realisation, with the interpretation being represented by the archetypal mathematical structure. There is thus induced a mapping I from $B(\mathcal{C})$ to M called the interpretation of $B(\mathcal{C})$ given by $I([n] <> \{G\}) = \alpha$ if and only if $[n] < \alpha > \{G\} \to [n] <> \{G\}$. The conditions on labels ensures that this function is well defined. Thus we have an equivalence between a causal tapestry and its bare tapestry with interpretation.

Definition (Causal Morphism): Let $\mathcal{C} = (L, K, M, I_p)$ and $\mathcal{C}' = (L', K', M', I'_p)$ be two causal tapestries. Then a *causal morphism* F is a quadruple of morphisms (f, g, h, F_p) where $f : L \to L', g : K \to K', h : M \to M'$ and $F_p : I_p \to I'_p$ such that

1. $f : L \to L'$ is a set morphism
2. $g : K \to K'$ is a set morphism
3. $h : M \to M'$ consists of morphisms on each component

4. $F_p : I_p \rightarrow I'_p$ is collection of causal tapestry morphisms (f_p, g_p, h_p, F_p)
5. $f([n] < \alpha > \{G\}) = [k(n)] < h(\alpha) > \{f_p(G)\}$
6. f_p induces a graph morphism on G, i.e. $f_p(G)$ is a graph homomorphic image of G
7. If e and e' are the associated interpretations to $B(\mathcal{C})$ and $B(\mathcal{C}')$ respectively, then h induces a causal manifold homomorphism of $B(G)$, i.e. $e'(B(f_p(G)))$ is homomorphic to $e(B(G))$.

We write $\mathcal{C} \approx \mathcal{C}'$ if the maps are isomorphisms, and say that $\mathcal{C}$ is isomorphic to $\mathcal{C}'$.

A *reinterpretation* of a causal tapestry is a causal morphism $F = (f, g, h, F_p) : \mathcal{C} = (L, K, M, I_p) \rightarrow \mathcal{C}' = (L', K, M', I'_p)$ where $f : L \rightarrow L', id : K \rightarrow K, h : M \rightarrow M', F_p : I_p \rightarrow I'_p$, $B(\mathcal{C}) = B(\mathcal{C}')$, and $f([n] < \alpha > \{G\}) = [n] < h(\alpha) > \{f_p(G)\}$. Essentially (f, g, h, F_p) provides a new collection of interpretations of the informons of C.

D.5 Lorentz Invariance and the Transient Now

We treat causal relations as being fundamental and look to see if the causal structure embodied in causal tapestries is preserved under Lorentz transformations. If so then we recover the idea of a transient now and of a Lorentz invariant transient dynamics, but the price we must pay to do so is to give up on notions of a global coordinatizability. As noted above, since we do not have this in practice there is no real need to have it in principle.

Theorem: Let $\mathcal{C} = (L, K, M, I_p)$ be a causal tapestry and Λ the set of causal automorphisms on $\mathcal{M}$. Then the causal tapestry $\mathcal{C}$ is invariant under causal automorphisms in the sense that Λ induces a family of reinterpretations of $\mathcal{C}$.

Proof. Let $\mathcal{C} = (L, K, M, I_p)$ and Λ a causal automorphism on $\mathcal{M}$. Then Λ naturally induces a reinterpretation of $\mathcal{C}$ as follows: For any locus of $\mathcal{C}$, $[n] < \alpha > \{G\} \rightarrow \Lambda([n] < \alpha > \{G\}) = [n] < \Lambda(\alpha) > \{\Lambda_p(G)\}$.

where Λ_p is the natural Λ-induced morphism on L_p. If Λ is a causal automorphism on $\mathcal{M}$, then Λ induces an order isomorphism of the causal order on $\mathcal{M}$. Since from the definition, for any locus $[n] < \alpha > \{G\}$ and interpretation map e, we have $e(\Lambda([n] < \alpha > \{G\})) = e([n] < \Lambda(\alpha) > \{\Lambda_p(G)\}) = \Lambda(\alpha) = \Lambda(e([n] < \alpha > \{G\}))$, hence $e\Lambda = \Lambda e$ (abusing notation slightly). It follows therefore that $e(\Lambda(G)) = \Lambda(e(G))$ and since $\Lambda(e(G))$ is order isomorphic to $e(G)$ it follows that $e(\Lambda(G))$ is order isomorphic to $e(G)$. Thus to each causal automorphism Λ there corresponds a causal reinterpretation $\Lambda = (\Lambda, id, \Lambda, \Lambda_p)$ (abusing notation slightly). Hence $\mathcal{C} \approx \Lambda(\mathcal{C})$ and causal tapestries are invariant under causal automorphism. The Lorentz boosts form a subgroup of the group of causal automorphisms [216] and so it follows that causal tapestries are (restricted) Lorentz invariant. Hence any reality game that generates causal tapestries must generate a succession of Lorentz invariant tapestries (since the causal contents must always embed into the causal manifold $\mathcal{M}$) and thus we have a dynamic in which these causal tapestries unfold as a succession of transient 'nows'.

Thus any dynamic that maps causal tapestries to causal tapestries will be Lorentz invariant and in particular the dynamic generated by a reality game which generates a succession of causal tapestries will correspond to the generation of a succession of transient nows, which, nevertheless, remain Lorentz invariant. Thus we can have a dynamic of the form required by process theory while still retaining Lorentz invariance. Note that only the causal structure is preserved and no statement about the effect of the dynamic upon any form of coordinatization is expressed or implied.

D.6 Event and Transition Tapestries

In order to properly describe reality from a process theory perspective it is necessary to consider separately the space of actual occasions and the processes that generate them. Two causal tapestries will be required to fully describe a process based reality. One tapestry will represent the actual occasions being generated while the other tapestry will keep track of the actions of individual processes that are responsible for such generation. A link between two informons implies that the succeeding informon was generated using information derived from the preceding informon, and such occurred through some action carried out by a process. The transition tapestry provides a record of these actions, usually in the form of some operator on the relevant information space, here either the causal manifold M or its Hilbert space $\mathcal{H}(\mathcal{M})$.

The tapestry corresponding to the space of actual occasions, the event tapestry is defined as follows:

Definition (Event Tapestry): An *event tapestry* Δ is a causal tapestry with attributes, that is, a 4-tuple (L, K, M, I_p) where K is an index set of cardinality κ, $M = \mathcal{M} \times \mathcal{F}(\mathcal{M}) \times D \times P(\mathcal{M}')$ a mathematical structure with $\mathcal{M}$ a causal space, $\mathcal{F}(\mathcal{M})$ a function (state) space, either Banach or Hilbert, D a space of descriptors (properties), $P(\mathcal{M}')$ either a Lie algebra or tangent space on a manifold $\mathcal{M}'$, I_p a union of event tapestries. The event tapestry serves as a generalization of the notion of a position space.

Likewise we may define a transition tapestry as follows:

Definition (Transition Tapestry): A *transition tapestry* Θ is a causal tapestry with attributes, that is, a 4-tuple (L', K', M', I'_p) where K' is an index set of cardinality κ, $M' = \mathcal{M}' \times \mathcal{F}(\mathcal{M}') \times D' \times P(\mathcal{M})$ a mathematical structure with $\mathcal{M}'$ a causal space, $\mathcal{F}(\mathcal{M}')$ a function (state) space, either Banach or Hilbert, D' a space of descriptors (properties), $P(\mathcal{M})$ either a Lie algebra or tangent space on a manifold $\mathcal{M}$, I'_p a union of transition tapestries. The transition tapestry serves as a generalization of the notion of a tangent space.

Together these tapestries form an interlinked dyad.

Thus the interpretation for each informon in the event tapestry Δ takes the form $(\mathbf{x}, \phi, \mathbf{d}, p)$ where x is an element of the causal manifold $\mathcal{M}$ (spatial location), ϕ is a function on the causal manifold (state or trajectory), $\mathbf{d}$ is a vector of descriptors such as mass, charge, and p is an allowable transition on the transition tapestry. A similar interpretation holds for each informon of the transition tapestry.

An important concept in both mathematics and physics is that of duality. Duality refers to two distinct mathematical structures whose properties are nevertheless intertwined by some mathematical operation (an involution) that serves as its own inverse. The state space and momentum space representations in quantum mechanics are dual to one another and related by means of the Fourier transform. Event and process tapestries also exhibit an aspect of duality, although it is less exact except in special cases.

Let Δ be an event tapestry $(L, K, \mathcal{M} \quad \times \mathcal{F}(\mathcal{M}) \times D \times P(\mathcal{M}'), I_p)$ and Θ a transition tapestry $(L', K', \mathcal{M}' \times \mathcal{F}(\mathcal{M}') \times D' \times P'(\mathcal{M}), I'_p)$ with their structures appropriately interlinked.

Let $C(\Delta)$ denote the graph formed by taking the union over all content sets of informons of Δ and $C(\Theta)$ denote the graph formed by taking the union over all content sets of informons of Θ. These are both acyclic directed graphs by virtue of the consistency conditions for causal tapestries.

Consider first $C(\Delta)$. Each vertex a serves as end vertex for a set of edges $E(, a)$, and initial vertex for a set of edges $E(a,)$. Each edge in the event tapestry represents a flow of information from the initial to the terminal vertex arising through the process of origination of the informon as a consequence of the play of the reality game. The process that enables this flow of information to take place is one that manifests in reality, and thus corresponds to a physically realizable transition or transformation. This transformation is represented in the archetype interpreting the causal tapestry as either a global symmetry operation or as a local tangent. This in turn should correspond to an informon in the process tapestry.

Thus we expect that there should be a mapping of the edge set of $C(\Delta)$ into the vertex set of $C(\Theta)$. Likewise, each edge in the process tapestry marks a flow from one transition to another transition and each such transition between transitions requires some form of manifestation of an actual occasion to express it. This reflects the fact that the presence of an actual occasion has a meaningful impact upon reality, and that impact manifests itself as an influence shaping the particular manner in which the next actual occasion will come into being. That process of coming into being, being marked by some transition, will therefore be altered by the manifesting of an actual occasion, and so some new process may become manifest. Thus it makes sense that each transition between transitions should correspond to some informon which manifests the information that influences the appearance of the next transition. Thus it is reasonable to expect that there should exist a mapping from the edge set of $C(\Delta)$ into the vertex set of $C(\Theta)$.

Unfortunately it will usually be the case that each vertex of either graph will have more than one edge for which it is terminal or initial. In order to get around this problem let us define a new graph called a covering graph. Let $G = \{V, E\}$ be a generic directed graph. For each vertex $v \in V$, define a new vertex set by $\{e(, v)\} \times \{e(v,)\}$. Let $V(CG(G))$ consist of the union of all such vertex sets for v in V together with all terminal vertices of V (a vertex is terminal if it is not the initial vertex of any edge). For any pair of such vertices in $V(CG(G))$, $e(a, v)e(v, b)$ and $e(c, v')e(v', d)$, define an edge $e(e(a, v)e(v, b), e(c, v')e(v', d))$ if and only if $b = c$. For any terminal vertex $v \in V$, define an edge $(v, \emptyset)$. Define the new edge set $E(CG(V))$ to be the union of all of the above edges taken over all of V.

Definition (Tableau Duality Postulate): Let Δ and Θ be event and transition tapestries respectively. Let $C(\Delta)$ and $C(\Theta)$ denote their respective content graphs. Then there exist subgraphs $C'(\Delta)$ and $C'(\Theta)$ of $CG(C(\Delta))$ and $CG(C(\Theta))$ respectively such that there exists an order isomorphism between $C'(\Delta)$ and $C'(\Theta)$ with $V(C'(\Delta))$ mapping to $E(C'(\Theta))$ and $E(C'(\Delta))$ mapping to $V(C'(\Theta))$.

Moreover, in general we require an additional criterion to be met.

Definition (Tableau Consistency Criterion): Let Δ and Θ be event and transition tapestries respectively which satisfy the tableau duality postulate. Let $[n] < \alpha > \{G\}$ and $[n'] < \alpha' > \{G'\}$ be informons within the content set of some informon of Δ and assume that $[n] < \alpha > \{G\} \to [n'] < \alpha' > \{G'\}$ in the content graph. By the tableau duality assumption there will exist an informon $[i] < \beta > \{H\}$ within the content set of some informon of Θ such that this edge maps to this informon. Now $\alpha = (\mathbf{m}, \phi, \mathbf{d}, g)$ and $\alpha' = (\mathbf{m}', \phi', \mathbf{d}', g')$ and $\beta = (\mathbf{s}, \rho, \mathbf{h}, l)$. Then we also require that $\mathbf{m}' = l(\mathbf{m})$. A similar result holds for Θ. This ensures that if each informon is interpreted as an element of the causal manifold, then each edge is interpreted as a transformation from one element to the next.

Definition (Reality Tableau): A *reality tableau* consists of an event tapestry Δ and a transition tapestry Θ that satisfy the tableau duality postulate together with the tableau consistency criterion.

The processes involved in the creation of actual occasions will now generate a succession of reality tableaux. This occurs through the repeated play of a reality game.

D.7 Reality Game

The reality game is based upon the idea of forcing in mathematical logic (Hodges, 2006). There the goal is to generate a mathematical structure which exhibits all of the properties of some formal logical theory. Key is a notion of consistency, which is a collection of formal statements specifying conditions that any such model must meet and which are logically consistent. In particular, statements referring to specific constant elements, termed witnesses, are introduced and these witnesses are used to build up a model of the theory. This build up is carried out by means of a forcing game, in which two players successively carry out a set of tasks, which results in the addition of new witnesses and new consistency statements and progressively builds a model of the theory. Similarly, in a reality game a pair of players successively carry out a set of tasks resulting in the creation of informons and interpretations maps while remaining consistent with some particular set of dynamical constraints, usually given by means of some group or semigroup of symmetries such as a differential equation.

The reality game generates a succession of reality tableaux. Let (Δ, Θ) denote a reality tableau. That is, Δ is an event tapestry $(L, K, \mathcal{M} \times \mathcal{F}(\mathcal{M}) \times D \times P(\mathcal{M}'), I_p)$ and Θ is a transition tapestry $(L', K', \mathcal{M}' \times \mathcal{F}(\mathcal{M}') \times D' \times P'(\mathcal{M}), I'_p)$ for which the tableau duality postulate and the tableau consistency criterion hold. By a reality game is meant a game which is given Δ and Θ and at the end of play has created new tapestries Δ' and Θ' where $\Delta' = (L'', K'', \mathcal{M} \times \mathcal{F}(\mathcal{M}) \times D \times P(\mathcal{M}'), \Delta \cup I_p)$ and $\Theta = (L''', K''', \mathcal{M}' \times \mathcal{F}(\mathcal{M}') \times D' \times P'(\mathcal{M}), \Theta \cup I'_p)$ such that (Δ', Θ') form a reality tableau. Reality tableaux are defined recursively and so require the assumption of some prior set of prior tableaux upon which the reality game is to be played. This prior set may be empty or may be any self consistent set of tableaux.

The effect of the game is to create new sets of informons and new sets of past tapestries but with the interpretation structures left intact. The interpretation structures remain invariant because, as discussed in the introduction, they provide the archetypal frame of meaning underlying all information flow in the system. In order that these frames of meaning remain consistent and coherent through the history of the system it is necessary that they remain invariant under the play of the game. A change in the interpretation structures is interpreted as a fundamental change in some aspect of the system that results in a significant difference in behaviour and properties, for example a change of phase, or a change of symmetry or a change of distinguishable constituents. It is true that in general, information may evolve in a such a manner as to force the emerge of a new semantic frame and thus new interpretation structures. As in the case of forcing games, the reality game requires the completion of a set of tasks.

Consider an event tapestry Δ first as the case for Θ is analogous. Assume that the index set K has sufficient cardinality to account for all possible informons, past, present and future. In a universe of discrete events, a countable K will usually suffice. Otherwise it can generally be taken to be 2^κ where κ is the cardinality of the informon set of the current tapestry.

First of all, a new set of informons must be generated. Each informon has the form $[n] < \alpha > \{G\}$ with $n \in K$, $\alpha \in M$ and G an acyclic directed graph whose vertex set is a subset of $L \cup L_p$.

First, select an unused element $n' \in K$ and form a bare informon $[n'] <> \{\}$. Next, a content must be selected. The elements of the new content set must lie in $L \cup L_p$.

In general, in order to ensure that the consistency conditions are properly met, each content set will consist of the union of an up-set of a prior content set together with one or more informons from the current tapestry included as maximal elements. An up-set $\hat{G}$ of an ordered set G is a subset of G such that if $x \in \hat{G}$ then every element $y \in G$ such that $x < y$ also lies in $\hat{G}$. The new content set may have only one current informon, which means a simple extension. It may have several current informons, which means an interaction is about to take place. It may also be the case that several new informons may possess the exact same content (though of course the labels and interpretations will differ), which corresponds to the case in which multiple processes originate from a given informon (such as occurs following an interaction).

If G is an ordered set, let $\hat{G}$ denote an up-set of G. If G and H are ordered sets, define the ordered sum $G + H$ to be the ordered set such that every element of H is greater than every element of G. Then the different possible outcomes take the form:

1. Given an informon $a = [n] < \alpha > \{G\}$, add an informon of the form $[n'] < \alpha' > \{\hat{G} + \{a\}\}$

2. Given a collection of informons $\{a_i = [n_i] < \alpha_i > \{G_i\}\}$, add an informon of the form $[n'] < \alpha' > \{\cup_i \hat{G}_i + \{a_i\}\}$

3. Given an informon $[n] < \alpha > \{G\}$, add a collection of informons $\{a_i = [n_i] < \alpha_i > \{G_i\}\}$ where $G_i = \hat{G} + \{a\}$ (note that $\hat{G}$ may be different for different G_i)

Each informon of Δ and must also be interpreted within its mathematical structures. These interpretations must meet additional criteria:

1. $[n] < \alpha > \{G\}$ and $[n] < \alpha' > \{G'\}$ in L implies $G = G'$ and $\alpha = \alpha'$

2. $[n] < \alpha > \{G\}$ and $[n'] < \alpha > \{G\}$ in L implies that $n = n'$

3. Let $[n] < \alpha > \{G\}$ and $[n'] < \alpha' > \{G'\}$ be informons within the content set of some informon of Δ and assume that $[n] < \alpha > \{G\} \to [n'] < \alpha' > \{G'\}$ in the content graph. There exists an informon $[i] < \beta > \{H\}$ within the content set of some informon of Θ such that if $\alpha = (\mathbf{m}, \phi, \mathbf{d}, g)$ and $\alpha' = (\mathbf{m}', \phi', \mathbf{d}', g')$ and $\beta = (\mathbf{s}, \rho, \mathbf{h}, l)$, then $\mathbf{m}' = l(\mathbf{m})$.

4. The mapping $i : L_p \cup L \to \mathcal{M}$ given by $i([n] < \alpha > \{G\}) = i((\mathbf{x}, \phi, \mathbf{d}, p)) = \mathbf{x}$ is a causal embedding, meaning that it should be an injective causal order preserving map.

5. Let the map $m : L_p \cup L \to \mathcal{F}(\mathcal{M})$ be given by $m([n] < \alpha > \{G\}) = m((\mathbf{x}, \phi, \mathbf{d}, p)) = \phi$. Define an interpretation of the content G, denoted $m(G)$ by setting $m(G) = \sum_{a \in G} m(a)$.

6. The game dynamics relative to any particular informon should depend upon its interpretation, the interpretation of its content (which renders its meaning in terms of the archetypal interpretation), as well as the interpretations of surrounding informons. The game dynamic may involve additional constraints such as conservation laws, symmetry rules, extremal principles as so on.

Conditions 4 and 5 provide the critical criteria which any reality game must ensure are satisfied.

Appendix E

Game Theory

E.1 Combinatorial Games

A combinatorial game is a mathematical abstraction of games that are commonly played in real life such as Tic-Tac-Toe, Dots and Boxes, Checkers, Chess, Go and so on. Combinatorial games involve players who carry out moves in an alternating manner in the absence of random elements and in the presence of perfect information. The end of play is generally heralded by an inability of the players to make a move, the last player able to move being declared the winner. Combinatorial games are to be distinguished from the games usually studied in economics and biology in which players may move simultaneously in the presence of complete or incomplete information, in which there may be random elements, where there is often some payoff or transfer of resources between players, and in which the end of play is measured relative to some optimality criterion applied across all possible game plays.

The formal theory of combinatorial games began with Sprague-Grundy in the 1930's but became a mature branch of mathematics in the 1970's with the work of John H. Conway and others [12, 13, 97, 162, 274, 275, 49, 50]. A close cousin, the Ehrenfeucht-Fraisse game, has been used extensively in mathematical logic and model theory to construct representations of formal systems. The focus here is on Conway's theory, which has its most developed expression in the study of short determinate two player partisan games, though research continues to expand the theory to long indeterminate multi-player games with generalized outcomes. Short determinate two player partisan games form a partially ordered Abelian group. Moreover, there exists a subgroup of such games that can be interpreted as numbers and constitute the expanded field of surreal numbers. The same group admits additional elements that have an interpretation as infinitesimals, extending the field to include elements of non-standard analysis.

In the combinatorial games discussed below, it is assumed that there are two players, Left and Right, who move alternately, are capable of effecting possible moves that are distinct from one another, and possess complete information about the state of the game during any play. Moreover, the nature of the game is such that play is guaranteed to end after a finite number of plays. There are no cycles, stalemates or ties. The state of the game at any step can be fully determined. The options for a given player to move from a particular state of the game are simply the set of all states of the game that can follow a single play of the game by that player. There will be a distinct set of Left options and of Right options. Go is an example of such a game. The definition of a combinatorial game does, however, include one additional assumption, which is that the last player to play wins. Many games do not satisfy these conditions but are still capable of analysis within this framework. This particular definition was chosen because of its generality and the depth of its mathematical results, but there have since been many generalizations to include transfinite play, loopy play, misère play, play with different outcome determinants, and multiple players.

The play of a game begins with some initial state (position or configuration). A player moves, resulting in a new position. The other player then moves, again resulting in a new position, and the process repeats. The complete play of the game is thus described as a (finite) sequence of such positions, terminating when no further play is possible.

For convenience we can catalogue all possible sequences of game play by constructing a *game tree*, which is an ordered set consisting of game positions. Denote the players as Left and Right. Starting from a particular position, we arrange below and to the left, all possible positions that can be achieved by a move on the part of Left. Similarly we arrange below and to the right, all possible positions that can be achieved by a move on the part of Right. The process is then repeated for this new level of game and so on until no more positions can be achieved. A particular complete play of the game will correspond to a path down the game tree, beginning with the initial position and then proceeding to successive positions by alternating along left and right steps. More formally,

Definition (Game Tree): A *game tree* is an ordered set $\mathfrak{T}_G$ (or an acyclic directed graph) which satisfies the following conditions:

1. $\mathfrak{T}_G$ is locally finite, meaning that for all $x, y \in \mathfrak{T}_G$, the set $\{z | x \leq z \leq y\}$ is finite
2. For any $x \in \mathfrak{T}_G$ the set of immediate predecessors, i.e. the set $p(x) = \{y \mid y < x \text{ such that there is no } z, y < z < x\}$ contains at most one element

3. The set of immediate successors, $s(x) = \{y | x < y$ such that there is no z, $x < z < y\}$ is partitioned into two sets, left and right.

Since each game begins with a game position we can define a game by that initial position. Play then becomes a sequence of games rather than a sequence of positions. Each therefore has associated with it a specific set of games obtainable by a move of Left, termed Left options, and another specific set of games obtainable by a move of Right, termed Right options. Since the subsequent play of the game will depend only upon these sets of options a game may just as well be equated with these two sets of options. Therefore we associate any game G with its set of Left options G_L and its set of Right options G_R. Hence we define a game $G = \{G_L | G_R\}$ where G_L and G_R are sets of games. A game has an alternative definition in terms of its game tree.

The fundamental theorem of combinatorial games states that given a game G with players L and R such that L moves first, either L can force a win moving first, or R can force a win moving second, but not both.

This results in four distinct outcome classes for games. These are:

1. Positive: L can force a win regardless of who goes first.

2. Negative: R can force a win regardless of who goes first.

3. Zero: The second player to play can force a win.

4. Fuzzy: The first player to play can force a win.

Positive, negative and zero games form the class of surreal numbers under addition as defined below. The fuzzy games form the class of infinitesimals. Positivity-negativity is a symmetry operation given by reversal of the roles of Left and Right. For non-partizan games (the Left and Right options are always the same), there are only two outcome classes, fuzzy and zero.

The formal definition of a combinatorial game is conceptually quite confusing at first but it possesses great generality. It is inherently recursive and most constructions in combinatorial game theory arise through some (implicit) form of bootstrapping or through top-down induction. The technique is powerful and worth the mental effort to master it.

The formal definitions are [97]:

1. A combinatorial game G is given as $\{G_L | G_R\}$ where G_L and G_R are sets of games.

2. The sum of games $G + H$ is defined as $\{G_L + H, G + H_L | G_R + H, G + H_R\}$.
3. The negative of a game G, is defined as $-G = \{-G_R| - G_L\}$.
4. For two games G, H, equality is defined by $G = H$ if for all games X, $G+X$ has the same outcome as $H + X$.
5. For two games, G, H, isomorphism is defined as $G \approx H$ if G and H have the same game tree.
6. For two games, G, H, we say that $G \geq H$ if for all games X, Left wins $G + X$ whenever Left wins $H + X$.
7. A game G is a number if all elements of G_L and G_R are numbers and $g_L < G < g_R$ for all $g_L \in G_L$ and $g_R \in G_R$.

For any integer n, a game in which Left has n free moves is assigned the number n, while any game in which Right has n free moves is assigned the number $-n$. In the case of short games, the number assigned will be a dyadic rational, i.e. an integer of the form $m/2n$ for some integers n, m. The number of the sum of two games that are numbers is the sum of the numbers of the individual games. Multiplication and division can be defined on games that are numbers in such a way that these games form a field, the surreal numbers. Surreal numbers that are non dyadic rationals arise through a consideration of games of transfinite length and include the rationals, the reals and the ordinals. They may be generated using techniques similar to that of Dedekind cuts for the creation of the reals.

Games can be generated recursively starting with the simplest game $0 = \{|\}$. This is the game with no options at all. Call this day 0. At day 1, one may construct four possible games, $\{|\}, \{|\{|\}\}, \{\{|\}|\}$, and $\{\{|\}|\{|\}\}$ denoted 0, -1, 1 and * respectively. There are 36 games at day 2, 1474 games at day 3 and at day 4 somewhere between 3×10^{12} and 10^{434} games [12].

E.2 Combinatorial Token Games

The idea of a combinatorial game with tokens is used extensively so a few words are in order. Most combinatorial games are played on some kind of board using pieces or some kinds of marks which distinguish moves. Typical examples would be Chess, Checkers, Go, Hackenbush, Tic-Tac-Toe. A token is simply some kind of object that is placed on the board and which conveys information relevant to the play of the game. For example, chess pieces, by their association to specific roles, determine the kinds of moves available to them. In all of these games, pieces have different colours or shapes which distinguish pieces that are accessible (or inaccessible) to each player. In many cases, the number of tokens in a token game is constrained to some fixed value independent of the length of play. This is certainly true of a game like Chess. In other cases the number of tokens is determined solely by the possible length of play.

Tokens may have mathematical or physical properties in their own right which can be useful to the play of the game. In specifying a combinatorial game with tokens we are considering situations in which tokens are either created or modified in the course of game play and the operations that may be performed upon these tokens may be invoked in the construction of individual games or combinations of games.

For example, suppose that the tokens come endowed with some attribute that can be represented by a real or complex value. This could be a colour, an intensity, an angle relative to some axis. Whatever it may, this value can be modified by the application of multiplicative factor. Suppose that we have a token game $\mathbb{P}$ and a complex number w. Then the token game $w\mathbb{P}$ is understood to mean that every token in $w\mathbb{P}$ has the form $w\phi$ for some token ϕ in $\mathbb{P}$. In other words, the token game $w\mathbb{P}$ is constructed by changing the values of all tokens $\phi \in \mathbb{P}$ to $w\phi$. This enables the sums and products defined above to be expanded into more complex algebraic forms.

Another possible interpretation of the meaning of a multiplier is to assume that in a game G formed as a sum of subgames G_i according to $G = \sum_i w_i G_i$, the term $w_i G_i$ means that the fraction of tokens assigned to the game G_i is $|w_i|^2 f(n)$ where $f(n)$ is the total number of tokens (as a function of length of game play n).

Tokens may be purely informational. Tokens may also provide a physical or material instantiation of a game. In the case of a game such as Chess, tokens may subserve both functions. Tokens may possess an intrinsic nature apart from their use within a particular game. For example, tokens may form a vector space or a Lie algebra or some other mathematical structure. The reality games to be described below are played out on a causal manifold and tokens take the form of certain functions or vectors. The various operations that may be defined on token games permit such games to serve as representations of other mathematical structures. This might prove useful in extending the process framework to QFT.

E.3 Game sums and products

One distinct advantage of combinatorial game theory is that there are many different ways in which games may be combined to form new games. This makes it possible to define several different sums and products. Suppose that one has two games G and H and consider a generic sum $G+H$. The key defining feature of a sum of games is that on any given play the player whose turn it is may play either in G or in H, but *not* both. For example, the combinatorial or disjoint sum defined in the previous section describes games that may be thought of as being played out on different boards, one at a time but in no specific order. Thus play alternates between the two games but not necessarily in a sequential manner. This is a free sum of G and H since a move in one game will have no effect upon play in the other game.

When the games are played out on two different boards a game position will be described as a pair of positions gh, where g is a position for G and h is a position for H. The game tree for $G + H$ is built up in a rather complicated manner. Let UG denote the set of all game positions for G. Likewise for UH. Then $U(G + H) = (UG)(UH)$. Next one adds edges as follows. For any given game position g, let $T_{GL}(g)$ denote the subtree consisting of all subsequent left moves in G with similar notation for right moves and for H. Then from any combined position gh the set of subsequent edges is given as $T_{GL}(g)h \cup gT_{HL}(h) \cup T_{GR}(g)h \cup gT_{HR}(h)$.

It is more relevant to consider games that are played out on the same board since the reality game is of this type. Let us assume, therefore, that the distinct games are being played out on the *same* board, albeit with different pieces or tokens. The free sum of G and H, denoted $G \oplus H$ means, as before, that on any individual play of the game, a player may chose to play either a move in G or in H, but not both. The assumption is that a play of G has no effect upon any play of H and vice-versa, even if play takes place upon the same site on the board. In physics an example of such a situation is in the dynamics of bosons, where multiple bosons may occupy the same spatio-temporal location without affecting one another in any manner. An example might be where tokens from different games may be applied to a single location on the board without affecting one another in any way. Another occurs when play of each individual game is localized to non-overlapping regions of the board.

Let us consider the game tree $\mathfrak{T}_{G\oplus H}$ of the free sum $G \oplus H$. Let the board B be represented as some fixed set, which could be a finite or infinite lattice of dimension n, such as $I^n, \mathbb{N}^n$ or $\mathbb{R}^n$ for finite I, integers $\mathbb{N}$ or real $\mathbb{R}$. Let the set of tokens of G be denoted T_G and those of H be denoted T_H. In the reality game it is generally the case that multiple tokens will be applied over game play to any given site so we assume that to be the case generally. Let $\mathcal{P}()$ denote the power set operator. A position $\mathfrak{p}$ of the game is a mapping $B \to \mathcal{P}(T_G \cup T_H)$. Note that any site may have tokens form G, H or both. Define $\mathfrak{p}_G$ to be

$$\mathfrak{p}_G(x) = \mathfrak{p}(x) \cap T_G$$

and likewise for $\mathfrak{p}_H$. $\mathfrak{p}_G$ and $\mathfrak{p}_H$ are just $\mathfrak{p}$ with the image restricted to the tokens of G and H respectively. Note that for any position $\mathfrak{p}$ in the game tree for $G \oplus H$, $\mathfrak{p}_G$ and $\mathfrak{p}_H$ are positions in the game trees for G, H respectively. Define $\mathfrak{g} + \mathfrak{h}$ by $\mathfrak{g}(x) + \mathfrak{h}(x) = \mathfrak{g}(x) \cup \mathfrak{h}(x)$ for all $x \in B$. Then it is obvious that $\mathfrak{p}_G + \mathfrak{p}_H = \mathfrak{p}$. The game tree, as defined above, is an ordering on the set of positions with the partitioning given by the functions $\mathfrak{p}_G$ and $\mathfrak{p}_H$. Let s denote the successor function on the game tree for $G \oplus H$ and s_G, s_H denote the successor functions on the game trees for G, H respectively. Set $s(\mathfrak{x})|_G = \{\mathfrak{p}_G | \mathfrak{p} \in s(\mathfrak{x})\}$ and $s(\mathfrak{x})|_H = \{\mathfrak{p}_H | \mathfrak{p} \in s(\mathfrak{x})\}$.

A move in the game tree is a pair of adjacent positions $\mathfrak{p} \to \mathfrak{p}'$ such that either

1. $\mathfrak{p}'_G = \mathfrak{p}_G$ and for every $x \in B$, $\mathfrak{p}_H(x) \to \mathfrak{p}'_H(x)$ is a move in the game tree for H or,
2. $\mathfrak{p}'_H = \mathfrak{p}_H$ and for every $x \in B$, $\mathfrak{p}_G(x) \to \mathfrak{p}'_G(x)$ is a move in the game tree for G.

In the free sum we also require that for any $\mathfrak{p} \in \mathfrak{T}_{G \oplus H}$ we have

1. $s(\mathfrak{p})|_G = s_G(\mathfrak{p}_G)$ and $s(\mathfrak{p})|_H = s_H(\mathfrak{p}_H)$ and
2. $s(\mathfrak{p}) = [s(\mathfrak{p}_G) + \mathfrak{p}_H] \cup [\mathfrak{p}_G + s(\mathfrak{p}_H)]$
3. $s^2(\mathfrak{p}) = [s^2(\mathfrak{p}_G) + \mathfrak{p}_H] \cup [s(\mathfrak{p}_G) + s(\mathfrak{p}_H)] \cup [\mathfrak{p}_G + s^2(\mathfrak{p}_H)]$ and so on.

A second sum closely related to the free sum is the exclusive sum of G, H denoted $G \hat{\oplus} H$. In the exclusive sum, no move of G may occur on a site occupied by a piece of H and vice-versa, otherwise there are no restrictions on game play. There is a weak kind of interaction present between these two games but play of one game is not determined by the other, merely constrained sometimes and at some locations. It is a rather passive kind of interaction and there is no real interchange of information between the two games. An example in physics of such a situation is in the dynamics of fermions, which are not allowed to occupy identical states.

In terms of the game tree, each position now takes the form of a map $B \to \mathcal{P}(T_G) \cup \mathcal{P}(T_H)$. Each site on the board is now occupied by tokens form either T_G or T_H but never both. As before a move in the game tree is a pair of adjacent positions $\mathfrak{p} \to \mathfrak{p}'$ such that either

1. $\mathfrak{p}'_G = \mathfrak{p}_G$ and for every $x \in B$, $\mathfrak{p}_H(x) \to \mathfrak{p}'_H(x)$ is a move in the game tree for H or,
2. $\mathfrak{p}'_H = \mathfrak{p}_H$ and for every $x \in B$, $\mathfrak{p}_G(x) \to \mathfrak{p}'_G(x)$ is a move in the game tree for G.

In the exclusive sum we must restrict allowable maps to those in which there is no overlap of G and H tokens on a site. This forces

1. $s(\mathfrak{p})|_G \subset s_G(\mathfrak{p}_G)$ and $s(\mathfrak{p})|_H \subset s_H(\mathfrak{p}_H)$ and
2. $s(\mathfrak{p}) = \{[s(\mathfrak{p}_G) + \mathfrak{p}_H] \cup [\mathfrak{p}_G + s(\mathfrak{p}_H)]\} \cap (\mathcal{P}(T_G) \cup \mathcal{P}(T_H))^B$
3. $s^2(\mathfrak{p}) = \{[s^2(\mathfrak{p}_G) + \mathfrak{p}_H] \cup [s(\mathfrak{p}_G) + s(\mathfrak{p}_H)] \cup [\mathfrak{p}_G + s^2(\mathfrak{p}_H)]\} \cap (\mathcal{P}(T_G) \cup \mathcal{P}(T_H))^B$ and so on.

The reduction by $(\mathcal{P}(T_G) \cup \mathcal{P}(T_H))^B$ ensures the exclusion condition is met. Note that it is *universally* applied at each stage of construction and that there are no limits otherwise on the relationship between G moves and H moves. Thus there is a maximum freedom in the exclusive sum, subject to the single constraint.

The third important sum is not a single construction but rather is shorthand for an entire collection of possible constructions. This is the interactive sum, which may be free, denoted $G \boxplus H$, or exclusive, denoted $G \hat{\boxplus} H$. The interactive sum describes any situation in which the moves of G influence the possible moves that can be made in H, and vice versa. In this case, information from the play of one game has an effect upon the subsequent play of the other game and so an exchange of relevant information indeed takes place.

In the interactive sum, there are correlations between the play of the two games that extends across the board. Play may or may not be exclusive. There is no general formula available to describe an interactive sum since the possible choices will depend upon the choices that have already been made. The most that one can state generally is that in the free interactive sum we have

1. $s(\mathfrak{p})|_G \subset s_G(\mathfrak{p}_G)$ and $s(\mathfrak{p})|_H \subset s_H(\mathfrak{p}_H)$ and
2. $s(\mathfrak{p}) \subset [s(\mathfrak{p}_G) + \mathfrak{p}_H] \cup [\mathfrak{p}_G + s(\mathfrak{p}_H)]$

3. $s^2(\mathfrak{p}) \subset [s^2(\mathfrak{p}_G) + \mathfrak{p}_H] \cup [s(\mathfrak{p}_G) + s(\mathfrak{p}_H)] \cup [\mathfrak{p}_G + s^2(\mathfrak{p}_H)]$ and so on.

and in the exclusive interactive sum we have

1. $s(\mathfrak{p})|_G \subset s_G(\mathfrak{p}_G)$ and $s(\mathfrak{p})|_H \subset s_H(\mathfrak{p}_H)$ and
2. $s(\mathfrak{p}) \subset \{[s(\mathfrak{p}_G) + \mathfrak{p}_H] \cup [\mathfrak{p}_G + s(\mathfrak{p}_H)]\} \cap (\mathcal{P}(T_G) \cup \mathcal{P}(T_H))^B$
3. $s^2(\mathfrak{p}) \subset \{[s^2(\mathfrak{p}_G) + \mathfrak{p}_H] \cup [s(\mathfrak{p}_G) + s(\mathfrak{p}_H)] \cup [\mathfrak{p}_G + s^2(\mathfrak{p}_H)]\} \cap (\mathcal{P}(T_G) \cup \mathcal{P}(T_H))^B$ and so on.

The use of the descriptor $G \boxplus H$ is generic but in any specific application the particular relationships between moves will need to be spelled out in detail.

Technically, the exclusive sum $G \hat{\oplus} H$ is an interactive sum $G \boxplus H$. It is singled out because of its ubiquity and the fact that it represents more of an avoidance of interaction than interaction. This is a subtlety, but an important one.

In addition we have several different notions for the product of two games. There is a combinatorial product $G \cdot H$ which corresponds to the product of surreal numbers in the case that the G, H are both numbers. In the setting of reality games a different collection of products are considered. The key defining feature of the product of G and H is that on any given move, the player must make a move in *both* G and H simultaneously. As for sums there are three types of product:

1. The direct product $G \otimes H$, means that G and H are played simultaneously and freely
2. The exclusive direct product, $G \hat{\otimes} H$, means that G and H are played simultaneously but moves must never occur on the same board site
3. The interactive product, $G \boxtimes H$, means that G and H are played simultaneously but there are correlations which permit or exclude certain combinations of moves.

The games trees for the products are somewhat simpler than those for sums. In particular the successor functions take the form

1. Free product: $s(\mathfrak{p}) = s(\mathfrak{p}_G) + s(\mathfrak{p}_H)$
2. Exclusive product: $s(\mathfrak{p}) = \{s(\mathfrak{p}_G) + s(\mathfrak{p}_H)\} \cap (\mathcal{P}(T_G) \cup \mathcal{P}(T_H))^B$

3. Interactive free product: $s(\mathfrak{p}) \subset s(\mathfrak{p}_G) + s(\mathfrak{p}_H)$
4. Exclusive product: $s(\mathfrak{p}) \subset \{s(\mathfrak{p}_G) + s(\mathfrak{p}_H)\} \cap (\mathcal{P}(T_G) \cup \mathcal{P}(T_H))^B$

Sums would appear to best describe the generation of particles in superpositions of eigenstates since we want only a single informon to manifest at any step of game play. Products would better describe the situation of multiple particles since they allow multiple games to be played simultaneously, corresponding to the manifesting of multiple particles simultaneously. However there may be situations in which the generation of particles must occur sequentially and in such cases sums must be used. There might be situation involving fermions where exclusive sums are more useful than exclusive products. Since no such constraint applies to multiple bosons, they presumably may be described by free products.

E.4 Ehrenfeucht-Fraissé Games

Games may be used to analyze the structure of mathematical theories and to compare structures within these theories. A brief digression to explore the idea of an Ehrenfeucht-Fraissé will set the stage for the use of games to generate structures. The Ehrenfeucht-Fraissé game [185] appears in the study of mathematical logic where it is used to determine whether two structures may be viewed as expressing the same set of properties from the perspective of a specific logical theory. Mathematical logic consists of a collection of formal sentences constructed according to specified rules from an alphabet consisting of constant symbols, variable symbols, relational symbols, quantifiers and logical connectives. A sentence in formal logic has a counterpart in natural language but its formal nature makes it amenable to mathematical analysis. There are in addition a collection of rules which determine how one may create new sentences out of a pre-existing collection of sentences and which ensure that the new collection remains logically consistent and coherent. These are formal analogues of the laws of deduction taught in courses in philosophy and reasoning.

A first order language L consists of a collection of symbols having different interpretations and formed into finite length strings according to a predetermined set of rules. The rules are designed to maintain consistency in the interpretation of these formulas or sentences. The basic symbols are constants $a, b, c, \dots,$ variables $x, y, z, \dots,$ functions $F, G, H, \dots,$ relations $R, S, T, \dots,$ $=$, and the logical quantifiers $\neg$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$. A term consists of a constant, variable, or function of constants and/or variables. A closed term has no variables. An atomic formula consists of $\mathrm{s} = \mathrm{t}$ where s and t are terms, or a relation of terms. A formula consists of a finite application of the logical quantifiers to a collection of atomic formulas. A variable is free if it is not within the scope of some quantifier. A sentence is a formula having no free variables. A theory is a collection of sentences. A model of a theory is a mathematical structure such that each constant in the theory corresponds to an element of the structure, each function and relation of the theory corresponds to a function and relation of the structure, and such that every sentence of the theory may be interpreted in the model and found to be true.

Often one wishes to understand the explanatory power of a theory. Does a theory, for example, describe everything about a particular model or are some features left unmentioned? Is the theory powerful enough to distinguish between specific models? An answer to the latter question can often be obtained through the play of an Ehrenfuecht-Fraissé game.

Suppose that one is given two mathematical structures A, B, and one wishes to determine whether or not these two structures can be distinguished using a theory expressed in the language of first order logic. Assume that there are two players I, II and furthermore assume that play occurs for exactly n moves, where n is fixed in advance.

The game play is extraordinarily simple. Player I moves first and is free to choose any element they like from either A or B. Player II then moves and may pick any element they like but only from the structure that Player I did not choose from. Play is repeated but with the caveat that at each step each player must choose an element that has not already been chosen. If there are no such elements to choose from then they simply forfeit their turn. Play continues in this way until a total of n steps have been played.

At the end of play one determines which of the two players has won the game. Let a_i be the element of structure A selected at the i-th move (whether by Player I or II) and let b_i be the element of structure B selected at the i-th move. One says that Player II wins the game if, whenever a relation R holds in A for a sequence of elements $a_i, a_j, \ldots, a_n$ then it also holds for the corresponding elements $b_i, b_j, \ldots, b_n$ of B. Otherwise one says that Player I wins.

A strategy is a systematic procedure which tells a player how to move following a particular series of game plays. For example, one could simply pick an element at random. Usually one is interested in strategies that are deterministic, meaning that given a particular sequence of points selected in previous game plays there is a unique point to be selected on the current play. Such a strategy is called deterministic since the choices are fixed in advance. If Player II possesses a deterministic strategy which guarantees a win in n plays against Player I no matter how Player I plays, then we say that the game is determined and write $A \approx_n B$.

Returning to theory, given a formal sentence ϕ and a model A, we write $A \vdash \phi$ if one can find elements, constants and relations in A corresponding to those in ϕ so that the precise relations expressed by ϕ are satisfied by these corresponding relations in A. If for every logical formula ϕ having at most n quantifiers $A \vdash \phi$ if and only if $B \vdash \phi$, then we write $A \equiv_n B$.

The power of Ehrenfeucht-Fraisse games arises from the fact that $A \approx_n B$ if and only if $A \equiv_n B$. The game is often much easier to use to solve the logic problem than are logic tools alone. Thus games may be used heuristically without any ontological attribution being made as to the nature and status of the players.

E.5 Generative Games and Forcing

Another important question facing logicians is to determine when a logical theory actually possesses a model and to exhibit such a structure. One of the most famous examples of this was the continuum hypothesis. This question concerns the sizes of sets and in particular whether the set of real numbers and the set of all subsets of natural numbers have the same size (cardinality). Cohen showed that is was possible to find models of set theory which extended the usual set theory, one of which satisfied the continuum hypothesis and one which did not. In this way he solved a long standing foundational problem in mathematical logic. The technique that he used to create such models involves a method called forcing [185].

The details are very technical but begin with the idea of a notion of consistency. A notion of consistency enables one to determine which theories actually possess models. Not all theories possess models. For example, the theory given by the single sentence $(a = b) \wedge \neg(a = b)$ has no model. Theories can be built up step by step provided that at each step one maintains consistency among the statements of the theory. This follows from the compactness theorem which states that if every finite subset of a theory T possesses a model then the theory T itself possesses a model. Building a theory step by step in this manner requires some notion of consistency. Formally, a notion of consistency N is a collection of sets of sentences of L which satisfy certain rules of logical consistency. For example, if $p \in N$ and t is any closed term in L, then $p \cup \{t = t\}$ is in N. As an example involving a sentence, suppose that $\neg(\phi \wedge \sigma)$ lies in some subset $p \subset N$. Then either $p \cup \{\phi\}$ or $p \cup \{\sigma\}$ lies in N, but not both. There are seventeen such rules whose details are not necessary here (see [185]). Each element $p \in N$ is called a condition. The idea is that each condition consists of a collection of formal sentences that are logically consistent. The important point is that if N is a notion of consistency and p is a condition of N, then p has a model.

This is proven by virtue of a game. Assume that there are two players, I and II. The number of plays of the game is fixed in advance and described by some infinite ordinal number. The players alternate in making a move, Player I playing first. The goal of the game is to construct an increasing set of conditions possessing a model at each stage, and then forcing the final union of all of these conditions to have a model as well. At each stage of the construction different tasks are assigned according to each of the seventeen rules and these tasks are performed in such a way that only a finite number of new elements are added to the previously constructed condition. For example, one task might be as follows: given some condition p constructed up to this point, one selects a closed term t, and if it is not already present in p, one adds $t = t$ to p. Similarly, suppose that p has already been constructed and that the formula $\neg(\phi \wedge \sigma)$ lies in p. Then this task might be to add either ϕ or σ, but not both. Whenever a limit ordinal is reached one simply assigns it the condition formed by taking the union over all previously constructed conditions. The tasks are each repeated a sufficient number of times to ensure that at the end of the construction no possible moves have been left undone. One possible strategy is to assume that a sufficient number of steps are carried out at each stage of the construction so as to parse at least once through the collection of all possible instances of all possible rules. Of course that will in general amount to a transfinite number of tasks to be performed at each stage of the construction and possibly a transfinite number of stages to complete the construction. That this procedure works is due to the recursive nature of the ordinals upon which this inductive process depends. Additional constraints may be placed on the choices made at each step. Finally some criterion is established which determines who wins the game. In other words the set of all possible sequences of play is partitioned into two disjoint subsets, one consisting of all wins for Player I and the other for all possible wins for Player II.

One begins with a particular first order language L and enlarges L to form a new language $L(W)$ by adding a set W of new constants, called witnesses. A notion of forcing for $L(W)$ is a notion of consistency N which satisfies the following two conditions:

1. If p is a condition in N and t a closed term (meaning no variables) in $L(W)$ and c is a witness which does not appear in either p or t, then $p \cup \{t = c\}$ lies in N
2. At most only finitely many witnesses appear in any $p \in N$

Let us restrict ourselves to games in which there are only a countable number of steps. Let P be some property that we would like our model to possess. One introduces witnesses and atomic formulae describing the expression of the property. We allow players I and II to alternate play as above, carrying out all of the necessary tasks and incorporating these witnessed formulae into the notion of consistency. If at the end of play the union of the chain of created conditions has property P then we say that Player II wins. If Player II has a strategy which enables them to win no matter how Player I plays, then the property P is said to be N-enforceable.

The importance of forcing is that it allows us to build up a structure step by step using a particular kind of game and ensure that it possesses a particular property. The game approach is not only simpler in many cases than the axiomatic approach but it possesses the generative character that we seek for any model based on process theory. A more general technique for constructing classes of mathematical structures using games is presented in Hirsch and Hodgkinson [184].

Appendix F

Interpolation Theory

F.1 Sinc Interpolation Theory

Kempf [203] appears to have been the first to suggest the use of interpolation theory to provide a bridge between discrete and continuous representations of space-time and quantum fields. Modern interpolation theory does not appear to be part of the usual mathematical toolkit used by physicists and so in this appendix I shall survey some results of interpolation theory required to understand this thesis [199, 247, 340, 436]. I have included more results than are actually used in the thesis in order to give the interested reader an brief overview of the power of the methods as well as to provide some direction for how the present theory might be extended to deal with more realistic situations.

In classical physics, the state of a system is represented as a vector in some space, usually a real $2k$-dimensional manifold, representing the position and either the velocity or momentum of the system. In quantum mechanics, these vectors become complex infinite dimensional (usually countable) vector spaces. These vectors are generally represented as complex valued functions over some n-dimensional manifold M, usually considered to be either the position space or the momentum space.

These vectors are equipped with an inner product

$$< \phi|\rho >= \int \phi^* \rho \, d^n x$$

and the class of functions is generally restricted to $L^2(M)$, being those functions such that

$$< \phi|\phi >= ||\phi||^2 = \int \phi^* \phi \, d^n x < \infty$$

The collection $L^2(M)$ of complex functions over M is called a Hilbert space, $\mathcal{H}(M)$.

As in the finite dimensional case, it may be possible to find a set of vectors $\{\Psi_i\}$ such that

1. $||\Psi_i|| = 1$
2. $< \Psi_n|\Psi_m >= \delta_{nm}$
3. For any Ψ there exist values w_i such that $\Psi = \sum_i w_i \Psi_i$ and $w_i = \int \Psi \Psi_i \, d^n x$

Note that in order to determine w_i it is necessary to know the values of Ψ over the entire manifold M.

There are situations in which one might possess some information about Ψ but not all information. In particular, it may be the case that, as a result of observations, one has knowledge of Ψ at a set X of discrete points $\mathbf{x}_n$. Under what circumstances does knowing Ψ on X suffice to determine Ψ on all of M? The answer, in many case, is provided by sampling and interpolation theory.

The idea is to determine under what circumstances there exists a collection of functions $G_n(x, x_n)$ such that we can write

$$\Psi(x) = \sum_n \Psi(x_n) G_n(x, x_n)$$

The simplest interpolation models are those in which the functions $G_k(x, x_k)$ are all derived from a single template function $G(x)$ through the use of a translation operator T_{x_k} so that

$$G_k(x, x_k) = T_{x_k} G(x) = G(x - x_k)$$

One may then write the original equation in the form

$$\Psi(x) = \sum_n \Psi(x_n) T_{x_n} G(x)$$

In the ideal case, assume that these points lie on some regular lattice embedded in M. That is, there exists a set $I = \{(x_1 + l_1 n_1, \ldots, x_k + l_k n_k) | n_i \text{ are integers}\}$, a subset $L \subset M$ and an isomorphism $f : I \to L$ such that

$$d(x_1 + l_1 n_1, \ldots, x_k + l_k n_k, x_1 + l_1 n_1', \ldots, x_k + l_k n_k')^2 = l_1^2 (n_1 - n_1')^2 + \ldots l_k^2 (n_k - n_k')^2$$

If M is a flat manifold then we may simply consider the lattice as being a subset of M.

The Whittaker-Shannon-Kotel'nikov theorem was the first to address the question of function reconstruction on such a lattice. The following is taken from Zayed [436].

In one dimension we have:

Theorem (Whittaker-Shannon-Kotel'nikov): If $f(t)$ is a signal (function) band limited to $[-\sigma, \sigma]$, i.e.

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} F(\omega) e^{it\omega} d\omega$$

for some $F \in L^2(-\sigma, \sigma)$, then it can be reconstructed from the sampled values at the points $t_n = n\pi/\sigma, n = 0, \pm 1, \pm 2, \ldots,$, via the formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{\sin \sigma(t - t_n)}{\sigma(t - t_n)} = \sum_{n=-\infty}^{\infty} f(\frac{\pi n}{\sigma}) \frac{\sin(\sigma t - n\pi)}{(\sigma t - n\pi)}, t \in \mathbb{R}$$

with the series being absolutely and uniformly convergent on compact sets.

The sampling frequency σ/π is known as the Nyquist rate. Put differently then, if the function is limited to the band from 0 to W cycles per second then the Nyquist rate is $2W$.

We define the sinc function as $sinc(x) = \sin x/x$. In addition, define $sinc_\sigma(x) = \frac{\sin(\pi x/\sigma)}{(\pi x/\sigma)}$, which shall also be referred to as a sinc function.

Rewriting,

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) T_{t_n} sinc_\sigma(t)$$

It sometimes helps to write the sampling series in the form

$$f(t) = \sin \sigma t \sum_{n=-\infty}^{\infty} f(t_n) \frac{(-1)^n}{\sigma(t - t_n)}$$

A multidimensional version of the WSK theorem was discovered by Parzen [436]. It states:

Theorem (Parzen): Let $f(t_1, \ldots, t_N)$ be a function band limited to the N-dimensional rectangle $B = \prod_{i=1}^{N}(-\sigma_i, \sigma_i)$, $\sigma_i > 0$, $i = 1, \ldots, N$ so that its Fourier transform $F(\omega_1, \ldots, \omega_N)$ is such that

$$\int_{-\sigma_1}^{\sigma_1} \cdots \int_{-\sigma_N}^{\sigma_N} |F(\omega_1, \ldots, \omega_N)|^2 d\omega_1 \cdots d\omega_N < \infty$$

and $F(\omega_1, \ldots, \omega_N) = 0$ for $|\omega_x| > \sigma_k$, $k = 1, \ldots, N$, then $f(t_1, \ldots, t_N) =$

$$\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} f\left(\frac{\pi k_1}{\sigma_1}, \ldots, \frac{\pi k_N}{\sigma_N}\right) sinc(\sigma_1 t_1 - \pi k_1) \cdots sinc(\sigma_N t_N - \pi k_N)$$

In terms of the lattice parameters $l_1, \ldots, l_N$ we may set

$$sinc_{l_1 \cdots l_N}(t_1, \ldots, t_N) = sinc\left(\frac{\pi t_1}{l_1}\right) \cdots sinc\left(\frac{\pi t_N}{l_N}\right).$$

Thus we may write $f(t_1, \ldots, t_N) =$

$$\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} f(l_1 k_1, \ldots, l_N k_N) sinc_{l_1 \cdots l_N}(t_1 - l_1 k_1, \ldots, t_N - l_N k_N)$$

and generalizing the translation operator we can write this as,

$$\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} f(l_1 k_1, \ldots, l_N k_N) T_{l_1 k_1 \cdots l_N k_N} sinc_{l_1 \cdots l_N}(t_1, \ldots, t_N)$$

It is important to note that the sampling series will converge provided that the sampling rates exceed the Nyquist rate. Oversampling does not alter the interpolation. Thus we have:

Theorem: Let $f(t_1, \ldots, t_N)$ be a function band limited to the N-dimensional rectangle $B = \prod_{i=1}^{N}(-\sigma_i, \sigma_i)$, $\sigma_i > 0$, $i = 1, \ldots, N$ and $\omega_i \geq \sigma_i$ for $i = 1, \ldots, N$. Then $f(t_1, \ldots, t_N) =$

$$\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} f(\frac{\pi k_1}{\omega_1}, \ldots, \frac{\pi k_N}{\omega_N}) sinc(\omega_1 t_1 - \pi k_1) \cdots sinc(\omega_N t_N - \pi k_N)$$

the value $r_i = \sigma_i/\omega_i$ is called the i-th sampling rate parameter. Note that $0 < r \leq 1$.

The class of functions that have a representation in terms of such a sinc sampling series is smaller than $L^2(M)$ (or $L^p(M)$ more generally).

Indeed for $\sigma \geq 0$ and $1 \leq p \leq \infty$, let B^p_σ denote the class of entire functions (on $\mathbb{C}$) of exponential type at most σ belonging to $L^p(\mathbb{R})$ when restricted to $\mathbb{R}$, that is, for $z = x + iy$,

1. $$|f(z)| \leq \sup_{x \in \mathbb{R}} |f(x)| \exp(\sigma |y|), z \in \mathcal{C}$$

2. $$\int_{-\infty}^{\infty} |f(x)|^p dx < \infty \text{ if } 1 \leq p < \infty$$

3. $$ess. \sup_{x \in \mathbb{R}} |f(x)| < \infty \text{ if } p = \infty$$

It follows that

$$|f(x + iy)| \leq C ||f||_p e^{\sigma |y|}$$

where the constant C depends only on p and σ and

$$||f||_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p}$$

is the usual L^p norm.

The following inclusions hold:

$$B^1_\sigma \subset B^p_\sigma \subset B^q_\sigma \subset B^\infty_\sigma \ (1 \leq p \leq q \leq \infty)$$

and if $f \in B^p_\sigma$ then $f^{(n)} \in B^p_\sigma, n = 0, 1, 2, \ldots$ and $|f^{(n)}|_p \leq \sigma^n |f|_p$. Thus the B^p_σ are closed under differentiation, unlike the class of $L^p(I)$ functions which is not closed under differentiation for any open interval I.

The significance of the classes B^p_σ is contained in the following theorem:

Theorem: Let $f \in B^p_\sigma$, $1 \leq p \leq \infty, \sigma > 0$. Then for $p < \infty$

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\sin \sigma(t - t_k)}{\sigma(t - t_k)}, t \in \mathbb{R}$$

with the series being absolutely and uniformly convergent on compact sets. In order to extend to the case where $p = \infty$ one must settle for the slightly weaker result that

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\sin \sigma'(t - t_k)}{\sigma'(t - t_k)}, t \in \mathbb{R}$$

with the series being absolutely and uniformly convergent on compact sets for $1 \leq p \leq \infty$ and $0 < \sigma < \sigma'$.

There is also the following theorem of Poussin [436]:

Theorem: If f is continuous and of bounded variation on $[a, b]$ and zero outside of this interval then

$$f(t) = \lim_{m \to \infty} \sum_{t_k \in [a,b)} f(t_k) \sin m(t - t_k)/m(t - t_k)$$

where $t_k = k\pi/m$, $k = 0, \pm 1, \pm 2, ..$ and $m = n$ or $m = n + 1/2, n = 1, 2, 3, \ldots$

The class of functions having sinc sampling representations (or variations of sinc sampling) has been extended even further but these results would take us beyond the needs of the present work.

It is important to note that an entire function cannot be both band limited and time or space limited, so that its domain in time or space cannot be bounded. This means that the function cannot have bounded support nor can its Fourier transform. This is a paradox in sampling theory since real signals are both band limited and bounded in time and space, but this will not be addressed here [436].

F.2 Properties of Sinc Functions

Stenger has written an excellent reference on sinc function methods, not just for sampling but for the solution of a wide range of numerical problems [340]. These techniques often surpass many of the more common solution methods and it is surprising that they are not widely known outside of the engineering literature.

Some basic properties of the sinc function will be described here. Here I shall use Stenger's notation so $S(k,h)(x) = sinc\ \pi(\frac{x}{h} - k)$. In terms of the previous notation, $S(k,h) = T_{kh} sinc_h(x)$, so in terms of lattice parameters, $h = l$.

The Fourier transform of a band limited function $f(t)$ is

$$\hat{f}(x) = \int_{\mathbb{R}} f(t) e^{ixt} dt = \begin{cases} h \sum_{k=-\infty}^{\infty} f(kh) e^{ikhx} & \text{if } |x| < \pi/h \\ 0 & \text{if } |x| >= \pi/h \end{cases}$$

If the sampling interval is decreased, the accuracy of the interpolation increases accordingly. Let

$$f_h(x) = \sum_{k=-\infty}^{\infty} f(kh) S(k,h)(x)$$

Then if $\sup_{t\in\mathbb{R}} |f(t) - f_h(t)| < \epsilon$ it follows that $\sup_{t\in\mathbb{R}} |f(t) - f_{h/2}(t)| < \epsilon^2$

Some additional properties are:

1. Let $\zeta \in \mathbb{C}$. Then $e^{i\zeta x} = \sum_{n\in\mathbb{Z}} S(n,h)(\zeta) e^{inhx}$ for $-\pi/h < x < \pi/h$
2. $\frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ix(\zeta - kh)} dx = S(k,h)(\zeta)$
3. $\int_{\mathbb{R}} S(k,h)(x) S(l,h)(x) dx = h\delta_{k-l}$
4. $S(k,h)(lh) = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$
5. $||f(t)||^2 = \int_{\mathbb{R}} |f(t)|^2 dt = h \sum_{n=-\infty}^{\infty} |f(nh)|^2$
6. Derivatives may also be calculated. Let $\delta_{m-n}^{(k)} = \left(\frac{d}{dx}^k\right) S(n,1)(x)|_{x=m}, k = 0, 1, 2, \ldots$. Then

$$f^k(x) = h^{-k} \sum_{n=-\infty}^{\infty} \left\{ \sum_{m=-\infty}^{\infty} f(mh)\delta_{m-n}^{(k)} \right\} S(n,h)(x)$$

Stenger provides some results for functions that are not band limited but which show specific decay rates towards $\pm\infty$. In particular let $f(t)$ be defined on $\mathbb{R}$ and suppose that for some positive constant d,

$$|\hat{f}(y)| = \mathcal{O}(e^{-d|y|}) \;\; y \to \pm\infty$$

Then as $h \to 0$,

$$||f(t) - \sum_{k=-\infty}^{\infty} f(kh)S(k,h)(t)|| = \mathcal{O}(e^{-\pi d/h})$$

Thus the approximation is quite good when one has an infinite number of sample points even when the sampled function is not band limited. If one only has a finite number of samples to work with then the approximation is worse. In certain cases however, it may still prove adequate.

Definition: Let $L_{\alpha,d}$ denote the family of functions such that

$$\begin{aligned} f(t) = &\quad \mathcal{O}(e^{-\alpha|t|}) \quad t \to \pm\infty \\ \hat{f}(x) = &\quad \mathcal{O}(e^{-d|x|}) \quad x \to \pm\infty \end{aligned}$$

Then

$$||f(t) - \sum_{k=-N}^{N} f(kh)S(k,h)(t)|| = \mathcal{O}(\epsilon_N)$$

where

$$\epsilon_N = N^{1/2} \exp\{-(\pi d\alpha N)^{1/2}\}$$

F.3 Sampling Errors

In the previous section two types of errors were discussed - those due to the function not being band limited, which leads to so-called aliasing errors, and those due to an insufficiency of sample values, which leads to truncation errors. A third source of error is due to inaccuracies in the sampling or generating process. A few comments about these errors will be presented.

The truncation error $T_N f(t)$ is defined as

$$T_N f(t) = f(t) - \sum_{n=-N}^{N} f(t_n) \frac{\sin \sigma(t - t_n)}{\sigma(t - t_n)}$$

An estimate that was discovered early in interpolation theory is due to Tsybakov and Iakovlev [398] and is given as follows:

Theorem (Tsybakov and Iakovlev): Let $-T \leq t \leq T$, $0 < \triangle t < (1/\sigma)$ and $E = \int_{\infty}^{\infty} |F(\omega)|^2 d\omega$ where $F(\omega)$ is the Fourier transform of $f(t)$. E is called the total energy of f. Then

$$|T_N f(t)| = (\sqrt{2}/\pi) E |sin(\pi t/\triangle t)| \sqrt{(T\triangle t)/(T^2 - t^2))}$$

Another estimate due to Helms and Thomas [179] states:

Theorem (Helms and Thomas): Let $f(t)$ be band limited with frequencies no greater than $r\sigma/2\pi$ cps, $0 < r \leq 1$. If $M = \max_{-\infty<t<\infty} |f(t)|$ then

$$|T_N(t)| \leq \frac{4M}{\pi^2 N(1-r)} = \frac{4M}{\pi^2 N q}, \quad -\infty < t < \infty$$

with $q = 1 - r$.

In the case that the truncation interval is asymmetrically placed relative to the time of evaluation one may define

$$T_{N_1,N_2} = f(t) - \sum_{k=K(t)-N_1}^{K(t)+N_2} f(t_k) \frac{\sin \sigma(t - t_n)}{\sigma(t - t_n)}$$

They also showed that

$$|T_{N_1,N_2}| \leq \frac{2M}{\pi^2 q}\left[\frac{1}{N_1}+\frac{1}{N_2}\right] \quad -\infty < t < \infty$$

Yao and Thomas further improved this [436] to

Theorem (Yao and Thomas):

$$|T_{N_1,N_2}| \leq \frac{M|\sin \sigma t|}{2\pi \cos(r\pi/2)}\left[\frac{1}{N_1}+\frac{1}{N_2}\right] \quad -\infty < t < \infty$$

In the multi-dimensional case they found:

Theorem: Let $f(t_1,\ldots,t_N)$ be a function band limited to the N-dimensional rectangle $B = \prod_{i=1}^{N}(-\sigma_i,\sigma_i)$, $\sigma_i > 0$, $i = 1,\ldots,N$ so that its Fourier transform $F(\omega_1,\ldots,\omega_N)$ is such that

$$\int_{-\sigma_1}^{\sigma_1}\cdots\int_{-\sigma_N}^{\sigma_N} |F(\omega_1,\ldots,\omega_N)|^2 d\omega_1\cdots d\omega_N < \infty$$

and $F(\omega_1,\ldots,\omega_N) = 0$ for $|\omega_k| > \sigma_k$, $k = 1,\ldots,N$. Set

$$T = f - \sum_{k_1=K_1^1}^{K_2^1}\cdots\sum_{k_N=K_1^N}^{K_2^N} f(t_{1,k_1},\cdots,t_{N,k_N})T_{t_{1,k_1}\cdots t_{N,k_N}}sinc_{\sigma_1\cdots\sigma_N}(t_1,\ldots,t_N)$$

Then

$$|T| \leq \frac{M}{(2\pi)^N}\frac{|\sin\sigma_1 t_1|}{\cos(\pi r_1/2)}\cdots\frac{|\sin\sigma_N t_N|}{\cos(\pi r_N/2)}\left[\frac{1}{K_1^1}+\frac{1}{K_2^1}\right]\cdots\left[\frac{1}{K_1^N}+\frac{1}{K_2^N}\right]$$

The aliasing error occurs if either the function is not band limited or it is limited to a band which is larger than $[-\sigma,\sigma]$. We define the aliasing error as

$$R_\sigma(f(t)) = f(t) - \sum_{k=-\infty}^{\infty} f(k\pi/\sigma)sinc\,(\sigma t - k\pi)$$

We have:

Theorem:

$$||R_\sigma(f(t)|| \leq \sqrt{\frac{2}{\pi}} \int_{|\omega|>\sigma} |\hat{f}(\omega)| d\omega$$

Amplitude errors arise when there are errors in the sampled values of the function, but not their timing. We define the amplitude error as:

$$A_\epsilon(f(t)) = \sum_{k=-\infty}^{\infty} [f(t_k) - \tilde{f}(t_k)] sinc\, \sigma(t - t_k)$$

where the $\tilde{f}(t_k)$ are the measured values of f.

Theorem: If E_ϵ is the total energy of $A_\epsilon(f(t))$ then the following estimate holds provided that f is band limited to $[-\sigma, \sigma]$, namely

$$|A_\epsilon(f(t))| \leq \sqrt{\frac{\sigma E_\epsilon}{\pi}}$$

Errors may also arise due to variations in the timing of samples, so-called time jitter errors, and also due to missing information but it is much more difficult to generate estimates in these cases.

Closely related to the problem of error is the issue of over-sampling. If a function is band limited to frequencies $[-\sigma, \sigma]$ then it can be considered band limited for any $\sigma' > \sigma$. Sampling at rates greater than the Nyquist rate results in an excess of information about the function.

If a function is over sampled then the interpolation can tolerate a rather large number of missing samples. In fact in an infinite interpolation the interpolation can tolerate an arbitrarily large but finite number of missing samples. Let $\mathcal{M}$ denote a set of M integers corresponding to the locations of M lost samples. Assume that the function f is over-sampled but that there are missing data. Using the known data set $\{f(n/2W)|n \notin \mathcal{M}\}$ it is possible to calculate the data set $\{f(n/2W)|n \in \mathcal{M}\}$. Write

$$f(t) = r \left[\sum_{n \in \mathcal{M}} + \sum_{n \notin \mathcal{M}} \right] f(n/2W) sinc(2Bt - rn)$$

With a little algebra one obtains

$$\sum_{n\in\mathcal{M}} f(n/2W)\{\delta[n-m] - rsinc[r(n-m)]\} = g(m/2W); m \in \mathcal{M}$$

where

$$g(t) = r\sum_{n\notin\mathcal{M}} f(n/2W)sinc(2Bt - rn)$$

This equation has the form $H\mathbf{x} = \mathbf{g}$ where $x_n = f(n/2W)$, $g_m = g(m/2W)$ and $H = I - S$ where S has elements

$$s_{nm} = s_{n-m} = rsinc[r(n-m)],\ (n,m) \in \mathcal{M} \times \mathcal{M}$$

One can directly interpolate from the samples as follows. Set

$$f(t) = \sum_{n\notin\mathcal{M}} f(n/2W)k_n^r(2Wt)$$

where the interpolation function is

$$k_n^r(t) = rsinc(t - rn) + r^2 \sum_{p\in\mathcal{M}}\sum_{q\in\mathcal{M}} a_{pq} sinc(r(n-p))sinc(t - rq)$$

where the elements a_{pq} are elements of H^{-1}.

F.4 Non-Uniform Sampling

A problem for the formulation of the WSK theorem is that time (and space) must be subdivided into a regular lattice whose spacings must relate to the frequencies of the sampling function. This suffices for an in-principle demonstration of the approach that is being presented here but in reality there is no reason to expect that actual occasions should come into being in precisely demarcated space-time intervals. It is more reasonable to assume the presence of at least a modest degree of variability in the relative positioning of occasions. This constitutes the case of non-uniform sampling. Extensive research has been devoted to studying non-uniform sampling, the effects of various kinds of information deficits, the use of different basis functions, the use of interpolation for solving various differential equations, and the use of interpolation for wider classes of functions[203, 436]. One complication of the use of non-uniform methods is that there are different requirements for sampling compared to interpolation [159, 222].A proper study of non-uniform sampling requires the use of Fechtinger-Gröchenig theory [436]. This and the above mentioned topics are beyond the scope of this paper but are important for the development of this approach into a proper alternative to standard quantum mechanics.

To provide a little taste of those results though, there is a theorem in the case that the function f is separable [436]. This is a very strong result:

Theorem (Paley-Weiner-Parzen): Let G_i be the entire function defined by

$$G_i(t_i) = (t_i - t_{i,0}) \prod_{k_i} (1 - t_i/t_{i,k_i})(1 - t_i/t_{i,-k_i}) \; i = 1, 2, ..N$$

where t_{i,k_i} are real numbers satisfying the estimate

$$\sup_{k_i} |t_{i,k_i} - k_i \pi/\sigma_i| < \pi/4\sigma_i \; i = 1, 2, ..N$$

Then for any signal in separable form $f(t_1, \ldots, t_N) = f_i(t_1) \cdots f_N(t_N)$ that is band limited to $\prod_i [-\sigma_i, \sigma_i]$ we have $f(t_1, \ldots, t_N) =$

$$\sum_1 \cdots \sum_N f(t_{1,k_1}, \ldots, t_{N,k_N}) \frac{G_1(t_{1,k_1})}{G'_1(t_{1,k_1})(t_1, -t_{1,k_1})} \cdots \frac{G_N(t_{N,k_N})}{G'_N(t_{N,k_N})(t_N, -t_{N,k_N})}$$

Unfortunately no such explicit series is known in the case that f is non-separable.

Although the above result shows that non-uniform sampling can provide a robust approximation in general, this particular form is not suitable for use in the process model because it is decidedly non-local in form, requiring a knowledge of the locations of all of the sampling times (or points) but this information will be unavailable to each informon. That would violate the goal of the model, which is to use only local information when constructing informons and their interpretations.

It is possible, however, to utilize non-uniform sampling with sinc functions, although in this case there is less information available to date concerning the sizes of the various possible errors.

Maymon and Oppenheim [252] studied the effect of non-uniform sampling in the context of sinc interpolation in one dimension. They assumed that in the ideal case the function would be sampled at equal intervals of length T, that is, for time values nT, n some integer. They assumed that the function is sampled at times that vary slightly from ideal due to a random fluctuation, ξ_n. The pdf for this fluctuation may or may not be known in advance. The sampling times are thus given by $t_n = nT + \xi_n$. The times assigned to the interpolation were also constructed, possibly deviating from ideal by a random fluctuation, ζ_n. Thus the times assigned in the interpolation take the form $\tilde{t}_n = nT + \zeta_n$. They then considered four different methods for constructing the interpolation.

These are

1. Randomized Sinc Interpolation: Let ζ_n, ξ_k be independent random variables such that ζ_n is independent of ξ_k for $n \neq k$.
2. Uniform Sinc Interpolation: $\zeta_n = 0$ for all n, i.e. act as if the values were sampled truly.
3. Nonuniform Sinc Interpolation: $\zeta_n = \xi_n$ for all n. Accept the sampling times as is.
4. Independent Sinc Interpolation: ζ_n and ξ_k are independent for all n, k.

Unfortunately they did not present absolute bounds on the errors generated by these different interpolation choices. Instead, they examined the statistics of the signal function and of the interpolation and showed that the mean squared error can be minimized most often by the choice of uniform sinc interpolation, and this method works very well when the size of the fluctuations is small relative to the lattice size. Qualitatively, simulations provide the best fit to the theoretical estimates when uniform sinc interpolation is used. They do note that the error associated with uniform sinc interpolation varies inversely with the bandwidth, since the smaller the bandwidth the slower the variations in the function and so the less sensitive it will be to variations in the sampling times. Nonuniform sinc interpolation appears to be of greatest value in situations in which the sampling rate is sub-Nyquist.

A proper study of the non-uniform sampling case requires the use of more sophisticated techniques using the theory of frames [167], reproducing Hilbert spaces, Fechtinger-Gröchenig theory [436] or Jorgensen theory [201] and is beyond the scope of this thesis.

Part III

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