Qualitative Theory of Switched Integro-differential Equations with Applications

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Switched systems, which are a type of hybrid system, evolve according to a mixture of continuous/discrete dynamics and experience abrupt changes based on a switching rule. Many real-world phenomena found in branches of applied math, computer science, and engineering are naturally modelled by hybrid systems. The main focus of the present thesis is on hybrid impulsive systems with distributed delays (HISD). That is, studying the qualitative behaviour of switched integro-differential systems with impulses. Important applications of impulsive systems can be found in stabilizing control (e.g. using impulsive control in combination with switching control) and epidemiology (e.g. pulse vaccination control strategies), both of which are studied in this work.

In order to ensure the models are well-posed, some fundamental theory is developed for systems with bounded or unbounded time-delays. Results on existence, uniqueness, and continuation of solutions are established. As solutions of HISD are generally not known explicitly, a stability analysis is performed by extending the current theoretical approaches in the switched systems literature (e.g. Halanay-like inequalities and Razumikhin-type conditions). Since a major field of research in hybrid systems theory involves applying hybrid control to problems, contributions are made by extending current results on stabilization by state-dependent switching and impulsive control for unstable systems of integro-differential equations.

The analytic results found are applied to epidemic models with time-varying parameters (e.g. due to changes in host behaviour). In particular, we propose a switched model of Chikungunya disease and study its long-term behaviour in order to develop threshold conditions guaranteeing disease eradication. As a sequel to this, we look at the stability of a more general vector-borne disease model under various vaccination schemes. Epidemic models with general nonlinear incidence rates and age-dependent population mixing are also investigated. Throughout the thesis, computational methods are used to illustrate the theoretical results found.

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Dedication

To my Parents and Julie.

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$\mathbb{R}^n$	Euclidean space of n-dimensions
$\mathbb{R}_+$	set of nonnegative real numbers
$\mathbb{Z}$	set of all integers
$\mathbb{N}$	set of positive integers
$\lambda_{\min}(A)$	minimum eigenvalue of a symmetric matrix A
$\lambda_{\max}(A)$	maximum eigenvalue of a symmetric matrix A
$cl(A)$	closure of the set A
∂A	boundary of the set A
$\\| \cdot \\|$	Euclidean norm: $\\|x\\| = \sqrt{x_1^2 + \ldots + x_n^2}$ for $x \in \mathbb{R}^n$
$\mathcal{B}_b(c)$	open-ball of radius $b > 0$ centred at $c \in \mathbb{R}^n$: $\{x \in \mathbb{R}^n : \\|x-c\\| < b\}$
$C(A, B)$	set of continuous functions mapping A to B
C	short form for $C([-\tau, 0], \mathbb{R}^n)$ where $\tau > 0$ is a constant
$C^1(A, B)$	set of continuously differentiable functions mapping A to B

$PC([a,b],D)$	set of piecewise continuous functions mapping the interval $[a,b]$ to the set D
PC	short form for $PC([-\tau,0],\mathbb{R}^n)$ where $\tau>0$ is a constant
$PCB([a,b],D)$	set of piecewise continuous bounded functions mapping the interval $[a,b]$ to the set D
PCB	short form for $PCB([\alpha,0],\mathbb{R}^n)$ where $-\infty\leq\alpha<0$
$\|\cdot\|_\tau$	sup norm: $\|\psi\|_\tau=\sup_{-\tau\leq s\leq 0}\|\psi(s)\|$ for $\psi\in PC$
$\|\cdot\|_{PCB}$	sup norm: $\|\psi\|_{PCB}=\sup_{\alpha\leq s\leq 0}\|\psi(s)\|$ for $\psi\in PCB$
$\mathcal{K}_0$	$\{w\in C(\mathbb{R}_+,\mathbb{R}_+):w(0)=0,w(s)>0 \text{ for } s>0\}$
$\mathcal{K}_1$	$\{w\in\mathcal{K}_0:w \text{ is nondecreasing in } s\}$
$\mathcal{K}$	$\{w\in C(\mathbb{R}_+,\mathbb{R}_+):w(0)=0 \text{ and } w \text{ is strictly increasing}\}$
$\mathcal{K}_\infty$	$\{w\in\mathcal{K}:w(s)\to\infty \text{ as } s\to\infty\}$
ν_0	class of locally Lipschitz and piecewise continuous functions mapping $\mathbb{R}_+\times\mathbb{R}^n$ to $\mathbb{R}_+$
ν_{PC}^*	class of locally Lipschitz and piecewise continuous functionals mapping $\mathbb{R}_+\times PC$ to $\mathbb{R}_+$
HISD	hybrid impulsive system with distributed delays
σ	switching rule
$\mathcal{P}$	finite index set $\{1,2,\ldots,m\}$ where $m>1$ is an integer
$\mathcal{P}_u$	set of all unstable modes ($\mathcal{P}_u\subseteq\mathcal{P}$)
$\mathcal{P}_s$	set of all stable modes ($\mathcal{P}_s\subseteq\mathcal{P}$)

Γ	impulsive set in $\mathbb{R}^{n+1}$
$\mathcal{I}$	set of all admissible impulsive sets
$\mathcal{S}$	set of all admissible switching rules
$\mathcal{S}_{\text{periodic}}$	set of periodic switching rules ($\mathcal{S}_{\text{periodic}} \subset \mathcal{S}$)
$T_i(t_0, t)$	total activation time of the i^{th} subsystem on $[t_0, t]$
$T^+(t_0, t)$	total activation time of the unstable subsystems on $[t_0, t]$
$T^-(t_0, t)$	total activation time of the stable subsystems on $[t_0, t]$
$N(t_0, t)$	total number of impulses applied on the interval $[t_0, t]$
$\Phi_i(t_0, t)$	number of switching times t_k which satisfy $t_k \in [t_0, t)$ and $\sigma(t_k) = i$ (i.e. total activations of i^{th} mode on interval)
$\Phi(t_0, t)$	number of switching times t_k which satisfy $t_k \in [t_0, t)$ and $\sigma(t_k) \in \mathcal{P}_s$ (i.e. total activations of stable modes on interval)
$\bar{\Upsilon}_i$, Ω_i, $\widehat{\Omega}_i$	switching regions associated with state-dependent switching
$\mathcal{D}_{k_0}(a)$	generalized algorithm cycle set: $\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : a^{k_0} < V(t, x) \leq a^{k_0+1}\}$ where $k_0 \in \mathbb{Z}$ and $a > 1$ is a constant
$\mathcal{R}_0$	basic reproduction number of an infectious disease
C_H^c	cumulative number of infected humans with control
C_H^0	cumulative number of infected humans without control
F_0	control strategy efficacy rating ($F_0 = 100C_H^c/C_H^0$)
Ψ	total number of vaccinations administered
χ	cost-benefit rating of the control scheme ($\chi = \Psi/(C_H^0 - C_H^c)$)

Chapter 1

Introduction

Integro-differential equations arise frequently in modelling a variety of physical and biological phenomena [85]. Examples can be found in biological population models, predator-prey models with a past hereditary influence, grazing systems, wave propagation, nuclear reactors, large-scale systems, heat flow, and chemical oscillations [28, 85]. Neural networks are another application since delays are inherent features of both biological and artificial neural networks [18]. Originally motivated by problems in mechanics, mathematical biology, and economics, the study of integro-differential and integral equations can be traced back to the works of Abel, Lotka, Fredholm, Mathlus, Verhulst, and Volterra [85]. In particular, the work of Volterra on the problem of competing species was vitally important for the development of work in this area; since then, the theory and applications of Volterra integro-differential equations with bounded and unbounded delays have emerged as an important area of research [85].

Switched systems, which are a type of hybrid system, model phenomena that combine continuous or discrete dynamics with logic-based switching (called a switching rule). Switched systems most commonly arise in two contexts [32]: a natural system with abruptly changing dynamics (e.g. due to environmental factors) or a continuous system being stabilized via switching techniques (e.g. switching and impulsive control). Real-world examples of switched systems exist in mechanical systems, the automotive industry, air traffic control, intelligent vehicle/highway systems, robotics, integrated circuit design, multimedia, power electronics, chaos generators, computer disk drives, high-level flexible manufacturing systems, job scheduling, and chemical processes [32, 46, 56, 102]. Switched systems exhibit interesting stability behaviour, such as the switched control of unstable subsystems that leads to a stable system [56, 57], and the instability of a switched system comprised solely of stable subsystems [102]. For a review of the hybrid and switched systems literature, see [14, 32, 33, 46, 99, 101–103, 166, 179, 199] and the references therein.

An important application of hybrid models of integro-differential equations is found in epidemiology. Since the influenza pandemic of 1918, where over 20 million people died worldwide, many new infectious diseases have emerged [69]: for example, Lyme disease (1975), Legionnaire's disease (1976), the human immunodeficiency virus (HIV) (1981), hepatitis C (1989), hepatitis E (1990), and hantavirus (1993). New antibiotic-resistant strains of pneumonia, gonorrhea, and tuberculosis have appeared while diseases such as malaria, dengue, and yellow fever have re-emerged and are spreading into new regions as a result of climate change [69]. More recently, SARS began in one region of China in 2003 and spread to most of China and other countries while the H1N1 influenza virus appeared in Mexico in April 2009 and spread globally due to the travel of infected individuals [122,197].

Mathematical models of infectious diseases are used to identify trends, build and test theories, assess quantitative conjectures, and answer qualitative questions [69]. Epidemic models are crucial in gaining knowledge of the underlying mechanisms driving an epidemic and for estimating the number of vaccinations needed to eradicated a disease [66, 143]. Comparing, implementing, evaluating, and optimizing control schemes can be done through mathematical modelling and numerical simulations [69]. Integro-differential equations provide the proper framework for modelling the spread of certain types of diseases. For example, when individuals in the population mix with age-dependency (e.g. see [157]) and when a disease is spread via a vector agent, such as a mosquito (e.g. see [176]).

Most developed countries employ cohort immunization programs where vaccinations are administered to the susceptible population continuously in time. Measles immunization strategies in many areas of the Western world recommend a vaccination dose at 15 months of age followed by a second dose at around 6 years of age [170]. There now exist vaccines for a wide range of infectious diseases such as polio, hepatitis B, parotitis, and encephalitis B [98]. Alternatively, a pulse vaccination scheme is based on the idea that an infectious disease can be more efficiently controlled by antagonizing the natural temporal process [1, 169]. More precisely, pulse vaccination is the control technique of vaccinating a portion of the susceptible population in a short period of time (relative to the time scale involved in the dynamics of the disease).

Theoretical results have shown that pulse vaccination strategies are able to achieve disease eradication at relatively low vaccination levels when compared to conventional strategies [1]. In recent years, pulse vaccination has gained in prominence due to its success in controlling measles and polyomyelitis throughout Central and South America [170], in preventing rabies and hepatitis B [150], and in achieving a mean coverage of 92% for measles and rubella in children aged five to 16 in the UK in 1994 [170]. The most prominent example of a successful control program was the World Health Organization's global initiative to eradicate smallpox in the 20th century. Beginning in 1967 with approximately 15 million cases per year, the control scheme led to worldwide eradication by 1977 [69].

Seasonal variations in the transmission of an infectious disease play an important role in its spread. Examples include changes in the survivability of pathogens (outside hosts), differences in host immunity, variations in host behaviour, differences in the abundance of vectors due to weather changes [39, 55]. Reports have found that many diseases show periodicity in their transmission, such as measles, chickenpox, mumps, rubella, poliomyelitis, diphtheria, pertussis, and influenza [70]. Depending on the particular disease of interest and population behaviour, the appropriate model of the disease's spread may be using term-time forcing where the model parameters change abruptly in time (for example, see [44, 76, 113, 115, 117, 163]). A hybrid impulsive system composed of integro-differential equations provides a natural framework to model the application of control strategies to combat the seasonal spread of certain types of diseases.

A number of stability methods for nonlinear integro-differential systems can be found in the delay differential equations literature, such as Lyapunov-Krasovskii functionals and Razumikhin-type theorems (for example, see [61, 82, 85]). Other techniques for integro-differential equations include linear and nonlinear variation of parameters, stability in variation, and the method of reduction (for a detailed account, see [28, 85]). Analytic methods have been developed in recent years to study the stability of switched systems. Methods include common Lyapunov functions and multiple Lyapunov functions [26, 27, 32, 101, 152], switched invariance principles [14, 63, 64], and switched systems with average dwell-time switching [64, 65, 101, 166]. Reports have been given for switched systems with subsystems that are triangularizable [101, 139], linear switched systems with commuting subsystems [101, 145], stabilization of switched systems using feedback control and control Lyapunov functions [142], and the instability of switched systems under arbitrary switching [167].

The focus of the present thesis is on studying the qualitative behaviour of switched integro-differential equations with impulsive effects. That is, analyzing hybrid impulsive system with distributed delays (**HISD**). The main objective is to extend the current literature on the basic theory and stability theory of HISD. This type of model is a useful tool in analyzing the spread of an infectious disease as it takes into account distributed time-delays (e.g. age-dependent population mixing), impulsive control (e.g. pulse vaccination schemes), and switched model parameters (e.g. seasonal variations in population behaviour). Since we are interested in practical applications such as those in epidemiology, it is important to know the mathematical model is well-posed. Motivated by this we develop fundamental theory of HISD with unbounded delay and generalized impulsive effects and switching rules. In this work, the impulsive times do not need to match the switching times, and both effects can be time-dependent, state-dependent, or a mixture of both. The main contribution is existence and uniqueness results for HISD which satisfy certain smoothness conditions and which have admissible switching and impulsive effects.

To determine the long-term behaviour of HISD, we extend the current literature by presenting new stability theory focusing on: (i) HISD composed of a mixture of stable and unstable subsystems; or (ii) HISD composed entirely of unstable subsystems. In the first case, we find sufficient conditions guaranteeing stability of the overall switched system so long as the amount of time spent in the stable subsystems is sufficient. Impulsive effects are considered which can either act as a stabilizing feature of the HISD or as a disturbance. For hybrid impulsive systems with unbounded delay, we develop new Razumikhin-type stability results. The stability properties found are applied to classes of weakly nonlinear HISD and easily verifiable conditions are found.

Next we shift our focus to HISD composed entirely of unstable subsystems to extend the current research on the subject of state-dependent switching stabilization. Algorithms are given which construct stabilizing state-dependent switching rules explicitly from the partitioning of the state-space into different switching regions. When a switching region boundary is crossed by the solution trajectory, the algorithm chooses a new mode to activate. Results are found for a class of HISD based on a Lyapunov functional. To avoid unwanted switching behaviour (such as chattering), the results are broadened by allowing for overlapping switching regions. The state-dependent switching stabilization of nonlinear HISD, with bounded or unbounded delay, is formulated and proved. The special algorithm to construct the switching rule is generalized to include a wandering time.

The fundamental and stability theory results are applied to epidemics modelled by HISD. In doing so, the current mathematical epidemiology literature on diseases spread by a vector agent (such as a mosquito) is extended. We present a case study of the Chikungunya virus, which is a new model of the disease's outbreak on Reunion Island in 2005-06. Control strategies are considered (mechanical destruction of mosquito breeding sites, contact rate reduction), accompanied by analytic and numerical investigations. Control efficacy rates are calculated and some conclusions are drawn. An alternative modelling approach to a vector-borne disease is formulated and studied. In this case, we give new theoretical results ensuring eradication by analyzing both time-constant and impulsive vaccination strategies, followed by a cost-benefit numerical analysis. HISD are formulated for disease models with general nonlinear incidence rate, multi-city transportation, and age-dependent population mixing. Throughout the thesis, examples are given to illustrate the results and are augmented with Matlab numerical simulations.

The structure of the thesis is outlined as follows: in Chapter 2, the necessary foundational material is presented. To ensure the mathematical models are well-posed, classical techniques are extended to establish fundamental theory of HISD in Chapter 3. Since analytic solutions of HISD cannot be found explicitly in general, stability theory is developed to uncover a system's qualitative behaviour in Chapter 4. In Chapter 5, the stabilization of an unstable system via hybrid control is studied. The focus is on state-dependent switching stabilization of HISD by constructing special minimum rule algorithms. In Chapter 6, the theoretical results found in the thesis are applied to models of infectious diseases in order to determine whether or not the disease will be eradicated under certain control schemes. Conclusions are drawn and future directions are given in Chapter 7.

1.1 Summary of Contributions

The author's research contributions in the present thesis are detailed below.

Fundamental Theory of HISD (Chapter 3): The results on local existence, uniqueness, extended existence, and global existence for HISD are contributions by the author. The approach here is to extend existing techniques to be able to develop the basic theory of switched integro-differential equations with infinite delay. The switching rule and types of impulses are constructed in a general way so that a number of different formulations are captured. These investigations are important as they lay the foundation for further studies on the asymptotic behaviour of solutions (e.g. stability theory). The well-posedness of a mathematical model is important for applications such as epidemiology.

Stability Theory of HISD (Chapter 4): Since solutions of HISD are not known in general, we develop new stability theory to understand the qualitative behaviour of these types of system. First, we analyze the stability of HISD composed of stable and unstable modes by giving new switching Halanay-like inequalities for general switching rules, periodic switching rules, and impulsive effects. The results are applied to a class of weakly nonlinear HISD motivated by . Next we provide new Razumikhin-like theorems for HISD with unbounded delay and find verifiable sufficient conditions for stability in the form of constraints on the switching rules and impulsive effects.

Hybrid Control (Chapter 5): In this part we extend the current literature by analyzing the stability of HISD composed entirely of unstable subsystems. Algorithms are constructed for the state-dependent switching stabilization of weakly nonlinear HISD using a Lyapunov functional method. A new Razumikhin-like approach is presented for the hybrid control of nonlinear switched integro-differential equations with unbounded delay. In this case, the state-space partitioning algorithms are enhanced to avoid unwanted switching behaviour from a practical point of view. The analysis done here has important applications in control theory as the combination of switching control and impulsive control is a powerful stabilization tool.

Applications in Epidemic Modelling (Chapter 6): The novel methods developed earlier in the thesis are applied to epidemic models in order to determine the long-term behaviour of a spreading infectious disease. Contributions here include a case study of a new seasonal model of Chikungunya disease followed by an analysis of a more general vector-borne disease model. In both these investigations, we analyze various control strategies theoretically and numerically to determine their efficacy in eradicating the disease. This also allows us to draw some conclusions with regards to a response strategy in the face of an impending outbreak. Threshold criteria for disease eradication are also found for epidemic models with distributed delays and general nonlinear incidence rates. An infectious disease model with age-dependent population mixing and a latency period is studied. The potential impact of this work comes from the fact that the mathematical analysis of epidemic models is vital for the development and implementation of control schemes.

Chapter 2

Mathematical Background

This chapter provides background material necessary for the rest of the thesis. Some preliminaries and basic definitions are given for systems of ordinary differential equations in Section 2.1. Following this, standard results in functional differential equations are displayed in Section 2.2, including integro-differential equations. Section 2.3 is concerned with hybrid systems theory: impulsive systems are formulated in Section 2.3.1, followed by a brief overview of switched systems in Section 2.3.2.

2.1 Preliminaries and Basic Definitions

Unless otherwise specified, the material in this section is taken from [59]. For other references, the reader may refer to [77, 136, 151]. Let $\mathbb{R}^n$ denote the Euclidean space of n-dimensions equipped with the Euclidean norm $\|\cdot\|$. Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. Let t be a real scalar, let D be an open set in $\mathbb{R}^{n+1}$, let $f : D \to \mathbb{R}^n$ be continuous and let $\dot{x} = dx/dt$. Consider the following system of non-autonomous ordinary differential equations (ODE),

$$\dot{x}(t) = f(t, x(t)), \tag{2.1}$$

or, more briefly,

$$\dot{x} = f(t, x). \tag{2.2}$$

where $x = (x_1(t), \ldots, x_n(t))^T$ and $f(t, x) = (f_1(t, x_1, \ldots, x_n), \ldots, f_n(t, x_1, \ldots, x_n))^T$ is called a vector field on D. For a given $(t_0, x_0) \in D$, an initial value problem (IVP) for equation (2.2) consists of finding an interval $I \subset \mathbb{R}$ containing the initial time t_0 and a solution $x(t)$ of (2.2) satisfying $x(t_0) = x_0$. The IVP can be written as

$$\begin{cases} \dot{x} = f(t, x), & t \in I, \\ x(t_0) = x_0. \end{cases} \tag{2.3}$$

To use mathematical models to simulate real-world applications, is it important to ascertain the well-posedness of the IVP (2.3). For example, whether a solution exists and whether it is unique. A solution to the IVP with $t_0 \in I$ is a continuously differentiable function $x(t) = x(t; t_0, x_0)$ defined on I such that $(t, x(t)) \in D$ for $t \in I$; $(t_0, x(t_0)) = (t_0, x_0)$; and equation (2.2) is satisfied on I.

Theorem 2.1.1. (Existence)
If f is continuous in D then for any $(t_0, x_0) \in D$ there is at least one solution of the IVP (2.3) existing in an interval I.

To establish uniqueness of the solution, a stronger condition than continuity is required.

Definition 2.1.1. *A function $f(t, x)$ defined on a domain D in $\mathbb{R}^{n+1}$ is said to be locally Lipschitz in x if for any closed bounded set U in D, there exists a constant $L = L(U) \geq 0$ such that $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$ for $(t, x), (t, y) \in U$.*

Note that if $f(t, x)$ has continuous first partial derivatives with respect to x, then it follows that $f(t, x)$ is locally Lipschitz in x.

Theorem 2.1.2. (Uniqueness)
If $f(t, x)$ is continuous in D and locally Lipschitz with respect to x in D, then for any $(t_0, x_0) \in D$, there exists a unique solution of the IVP (2.3).

In the special case where system (2.3) is linear and autonomous, the initial time can be taken to be $t_0 = 0$ without loss of generality (simply define a new time variable $\tau = t - t_0$). The unique global solution can be given explicitly in terms of the matrix exponential.

Theorem 2.1.3. *If $f(t, x) \equiv Ax$ for all $(t, x) \in D$, where $A \in \mathbb{R}^{n \times n}$ is a constant matrix, then the IVP (2.3) has a unique solution for all time $t \in \mathbb{R}$, which is given by*

$$x(t) = e^{At} x_0, \tag{2.4}$$

where e^{At} is the matrix exponential, defined as follows:

$$e^{At} := \sum_{k=0}^{\infty} \frac{A^k t^k}{k!},$$

which converges for all $t \in \mathbb{R}$.

In general it is not possible to find an explicit solution of (2.3). However, important qualitative features of the system can be determined by a theoretical analysis. Important questions one may ask are: will the solution of the system converge to a point or a periodic function? If two solutions are initialized close to each other, will they remain close to each other? Many of these kinds of questions can be answered by analyzing the stability of the IVP (2.3), which is crucial in studying problems such as the synchronization of two systems, the stabilization of a system via hybrid control, and the long-term behaviour of a spreading disease.

Suppose that $\varphi(t)$ is a solution of (2.2). Let $z = x - \varphi$, then

$$\begin{aligned}\dot{z} =& \dot{x} - \dot{\varphi},\\ =& f(t,x) - f(t,\varphi(t)),\\ =& f(t, z+\varphi(t)) - f(t,\varphi(t)),\\ =& F(t,z),\end{aligned}$$

where $F(t,z) := f(t, z+\varphi(t)) - f(t,\varphi(t))$. Thus $x(t) = \varphi(t)$ is a solution of (2.2) if and only if $z(t) \equiv 0$ is a solution of $\dot{z} = F(t,z)$. Therefore, without loss of generality, assume that $f(t,0) \equiv 0$ for all $t \in \mathbb{R}$, i.e. the trivial solution $x \equiv 0$ is a solution of (2.2). The long-term behaviour of the IVP (2.3) can be characterized by studying the following stability properties of the trivial solution.

Definition 2.1.2. (Stability)
Let $x(t) = x(t;t_0,x_0)$ be the solution of the IVP (2.3) then the trivial solution $x = 0$ is said to be

(i) *stable if for all $\epsilon > 0$, $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0,\epsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \geq t_0$;*

(ii) *uniformly stable if δ in (i) is independent of t_0, that is, $\delta(t_0,\epsilon) = \delta(\epsilon)$;*

(iii) *asymptotically stable if (i) holds and there exists a $\beta > 0$ such that $\|x_0\| < \beta$ implies*

$$\lim_{t\to\infty} x(t) = 0;$$

(iv) *uniformly asymptotically stable if (ii) holds and there exists a $\beta > 0$, independent of t_0, such that $\|x_0\| < \beta$ implies that for all $\eta > 0$, there exists a $T = T(\eta) > 0$ such that for all $t_0 \in \mathbb{R}_+$, $\|x(t)\| < \eta$ if $t \geq t_0 + T(\eta)$;*

(v) *globally asymptotically stable if β in (iii) is arbitrary;*

(vi) globally uniformly asymptotically stable if β *in (iv) is arbitrary;*

(vii) exponentially stable if there exist constants $\beta, \gamma, C > 0$ *such that if* $\|x_0\| < \beta$ *then* $\|x(t)\| \leq C\|x_0\|e^{-\gamma(t-t_0)}$ *for all* $t \geq t_0$*;*

(viii) globally exponentially stable if β *in (vii) is arbitrary;*

(ix) unstable if (i) fails to hold.

Remark 2.1.1. *If* f *is autonomous (does not depend on* t *explicitly) then stability implies uniform stability. Also, exponential stability is a stronger condition than uniform asymptotic stability.*

In the late 19th century, A.M. Lyapunov developed some simple yet powerful geometric theorems for determining the stability of an equilibrium point of an ODE. For a constant a, suppose that $f : [a, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth to ensure a unique solution of (2.3) through every point (t_0, x_0) in $[a, \infty) \times \mathbb{R}^n$.

Definition 2.1.3. *Let* Ω *be an open set in* $\mathbb{R}^n$ *containing* 0*. A scalar function* $W(x)$ *is positive definite on* Ω *if it is continuous on* Ω*,* $W(0) = 0$*,* $W(x) > 0$ *for* $x \in \Omega \setminus \{0\}$*.*

Definition 2.1.4. *A Lyapunov function* $V : [a, \infty) \times \Omega \to \mathbb{R}$ *is said to be positive definite if* $V(t, 0) = 0$*,* V *is continuous in* t *and locally Lipschitz in* x*, and there exists a positive definite function* $W : \Omega \to \mathbb{R}$ *such that* $V(t, x) \geq W(x)$ *for all* $(t, x) \in [a, \infty) \times \Omega$*. The function* $V(t, x)$ *is said to be decrescent if there exists a positive definite function* $Z : \Omega \to \mathbb{R}$ *such that* $V(t, x) \leq Z(x)$ *for all* $(t, x) \in [a, \infty) \times \Omega$*.*

Definition 2.1.5. *The upper right-hand time-derivative of a function* $V(t, x)$ *that is continuous in* t *and locally Lipschitz in* x*, along the solution of (2.2), can be defined as*[1]

$$D^+V(t, x) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + hf) - V(t, x)].$$

When $V(t, x)$ *has continuous partial derivatives with respect to* t *and* x*, the upper right-hand derivative reduces to*

$$\dot{V}(t, x) = \frac{\partial V(t, x)}{\partial t} + \nabla V(t, x) \cdot f(t, x)$$

where $\nabla V(t, x)$ *is the gradient vector with respect to* x*.*

[1] The upper right-hand derivative is a Dini derivative, which is a generalized derivative. There are three other Dini derivatives: the upper left-hand, the lower right-hand, and the lower left-hand.

Theorem 2.1.4. *If $V : [a, \infty) \times \Omega \to \mathbb{R}$ is positive definite and $D^+V(t, x) \leq 0$ then the trivial solution of (2.3) is stable. If, in addition, V is decrescent, then the trivial solution is uniformly stable. Further, if $-D^+V(t, x)$ is positive definite then the trivial solution is uniformly asymptotically stable.*

The power of Lyapunov's direct method lies in the fact that explicit knowledge of the solution is not needed. Intuitively, $-D^+V(t, x)$ being positive definite implies that V is decreasing along orbits in $[a, \infty) \times \Omega$ and the orbit approaches the origin as $t \to \infty$.

2.2 Functional Differential Equations

Let $\tau > 0$ be a given real number and denote $C = C([-\tau, 0], \mathbb{R}^n)$ to be the set of continuous functions mapping $[-\tau, 0]$ into $\mathbb{R}^n$. For $\phi \in C$, consider the norm $\|\phi\|_\tau = \sup_{-\tau \leq s \leq 0} \|\phi(s)\|$. Then the space C is a Banach space[2]. If $x \in C([t_0 - \tau, t_0 + b], \mathbb{R}^n)$ for $t_0 \in \mathbb{R}$, $b \geq 0$, then we let $x_t \in C$ be defined by $x_t(s) = x(t + s)$ for $-\tau \leq s \leq 0$.

Let D be a subset of $\mathbb{R} \times C$ and let $f : D \to \mathbb{R}^n$ then a delay differential equation (DDE) on D is given by the relation

$$\dot{x} = f(t, x_t) \tag{2.5}$$

where $\dot{x}$ represents the right-hand time-derivative. Note that equation (2.5) is a general type of retarded functional differential equation which includes differential difference equations such as

$$\dot{x}(t) = f(t, x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_p(t)))$$

where $0 \leq \tau_j(t) \leq \tau$ for $j = 1, 2, \ldots, p$, and integro-differential equations such as

$$\dot{x}(t) = \int_{-\tau}^{0} g(t, s, x(t + s))ds.$$

Given $t_0 \in \mathbb{R}$ and an initial function $\phi_0 \in C$, the IVP associated with (2.5) is given by,

$$\begin{cases} \dot{x} = f(t, x_t), \\ x_{t_0} = \phi_0. \end{cases} \tag{2.6}$$

A function $x(t) = x(t; t_0, \phi_0)$ is said to be a solution to the IVP (2.6) on $[t_0 - \tau, t_0 + b)$ for $b > 0$ if $x \in C([t_0 - r, t_0 + b), \mathbb{R}^n)$, $(t, x_t) \in D$, $x(t)$ satisfies (2.6) for $t \in [t_0, t_0 + b)$, and $x(t_0 + s) = \phi_0(s)$ for $-\tau \leq s \leq 0$.

[2]Complete normed vector space.

Lemma 2.2.1. *If $t_0 \in \mathbb{R}$, $\phi_0 \in C$ are given and $f(t, \psi)$ is continuous, then finding a solution of (2.6) is equivalent to solving the integral equation*

$$\begin{cases} x_{t_0} = \phi_0, \\ x(t) = \phi_0(0) + \displaystyle\int_{t_0}^{t} f(s, x_s) ds, \qquad t \geq t_0. \end{cases} \tag{2.7}$$

For a DDE with finite delay as in (2.5), continuity of x ensures continuity of x_t. This is a useful property in establishing fundamental theory of (2.6).

Lemma 2.2.2. *Given a constant $\alpha > 0$, if $x \in C([t_0 - \tau, t_0 + \alpha], \mathbb{R}^n)$ then x_t is a continuous function of t for $t \in [t_0, t_0 + \alpha]$.*

Existence and uniqueness results can be given.

Theorem 2.2.3. (Existence)
If $f \in C(\Omega, \mathbb{R}^n)$ where Ω is an open set in $\mathbb{R} \times C$, then for any $(t_0, \phi_0) \in \Omega$ there is at least one solution of the IVP (2.6).

Definition 2.2.1. *A function $f(t, x)$ defined on a domain Ω in $\mathbb{R} \times C$ is said to be Lipschitz on Ω if there exists a constant $L = L(\Omega) \geq 0$ such that $\|f(t, \psi_1) - f(t, \psi_2)\| \leq L\|\psi_1 - \psi_2\|_\tau$ for $(t, \psi_1), (t, \psi_2) \in \Omega$.*

Theorem 2.2.4. (Uniqueness)
If $f \in C(\Omega, \mathbb{R}^n)$ where Ω is an open set in $\mathbb{R} \times C$ and $f(t, \psi)$ is Lipschitz in ψ in each compact set in Ω, then for any $(t_0, \phi_0) \in \Omega$ there is a unique solution of the IVP (2.6).

Assume that $f(t, 0) \equiv 0$ for all $t \in \mathbb{R}$ and assume that f is sufficiently smooth to have a unique solution. Stability concepts for the DDE (2.5) are analogous[3] to those from ODE theory in Definition 2.1.2. It is possible to investigate the stability of the nonlinear DDE (2.6) using a Lyapunov functional (an extension to Lyapunov function stability in ODE theory). Define the following $\mathcal{K}$-class functions.

$$\begin{aligned} \mathcal{K}_0 &= \{w \in C(\mathbb{R}_+, \mathbb{R}_+) : w(0) = 0, w(s) > 0 \text{ for } s > 0\}, \\ \mathcal{K}_1 &= \{w \in \mathcal{K}_0 : w \text{ is nondecreasing in } s\}, \\ \mathcal{K} &= \{w \in C(\mathbb{R}_+, \mathbb{R}_+) : w(0) = 0 \text{ and } w \text{ is strictly increasing}\}, \\ \mathcal{K}_\infty &= \{w \in \mathcal{K} : w(s) \to \infty \text{ as } s \to \infty\}. \end{aligned}$$

[3]See page 130 in [61].

Let $V : \mathbb{R} \times C \to \mathbb{R}$ be continuous in its first variable and locally Lipschitz in its second variable. Let $x(t) = x(t; t_0, \phi_0)$ be a solution of (2.6). Define the upper right-hand time-derivative of a functional V along the solution of (2.5) as

$$D^+V(t, \psi) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x_{t+h}) - V(t, x_t)]$$

where $\psi = x_t$.

Theorem 2.2.5. *Assume that $f : \mathbb{R} \times C \to \mathbb{R}^n$ takes $\mathbb{R} \times$ (bounded sets of C) into bounded sets of $\mathbb{R}^n$. Let $c_1, c_2 \in \mathcal{K}_1$ and $c_3 : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function. If there is a function $V : \mathbb{R} \times C \to \mathbb{R}$ which is continuous in its first variable, locally Lipschitz in its second variable, and satisfies*

(i) $c_1(\|\psi(0)\|) \le V(t, \psi) \le c_2(\|\psi\|_\tau)$ *for all* $(t, \psi) \in \mathbb{R} \times C$*;*

(ii) $D^+V(t, \psi) \le -c_3(\|\psi(0)\|)$*;*

then the trivial solution of (2.6) is uniformly stable. If $c_3(s) > 0$ for $s > 0$, then the trivial solution is uniformly asymptotically stable.

It is also possible to prove stability of the trivial solution of a DDE using Lyapunov functions, rather than functionals, which is the main idea behind Razumikhin theorems. The intuitive idea is explained as follows: if a solution of the DDE IVP (2.6) is initially in a ball and eventually leaves at some time t^*, then $\|x(t^* + s)\| \le \|x(t^*)\|$ for all $s \in [-\tau, 0]$ and $\frac{d}{dt}[\|x(t^*)\|] \ge 0$. Consequently we only need to consider initial data satisfying this property and the aim is to control $x(t) = x_t(0)$. Define the upper right-hand time-derivative of a function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ which is continuous in its first variable and locally Lipschitz in its second variable, along the solution of (2.5), as

$$D^+V(t, \psi(0)) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, \psi(0) + hf(t, \psi)) - V(t, \psi(0))]$$

where $\psi(0) = x(t)$.

Theorem 2.2.6. *Assume that $f : \mathbb{R} \times C \to \mathbb{R}^n$ takes $\mathbb{R} \times$ (bounded sets of C) into bounded sets of $\mathbb{R}^n$. Let $c_1 \in \mathcal{K}_1$, $c_2 \in \mathcal{K}$, and $c_3 : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function such that $c_3(s) > 0$ for $s > 0$. Let $q : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function such that $q(s) > 0$ for $s > 0$. If there is a function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ which is continuous in t, locally Lipschitz in x, and satisfies* *(i)* $c_1(\|x\|) \leq V(t,x) \leq c_2(\|x\|)$ *for* $(t,x) \in \mathbb{R} \times \mathbb{R}^n$*;*

(ii) $D^+V(t,\psi(0)) \leq -c_3(\|\psi(0)\|)$ *if* $V(t+s,\psi(s)) < q(V(t,\psi(0)))$ *for* $s \in [-\tau, 0]$*;*

then the trivial solution of (2.6) *is uniformly asymptotically stable.*

For more background on delay differential equations, the reader is referred to [61] which is where the above results were taken from. Integro-differential equations are a certain type of delay differential equation and can be most often classified into three types [85]: The first, sometimes called a Volterra integro-differential equation, is given by

$$\begin{cases} \dot{x} = f(t,x) + \displaystyle\int_{t_0}^{t} g(t,s,x(s))ds, \\ x(t_0) = x_0. \end{cases} \tag{2.8}$$

The second classification is

$$\begin{cases} \dot{x} = f(t,x) + \displaystyle\int_{t-\tau}^{t} g(t,s,x(s))ds, \\ x(t_0+s) = \phi_0(s), \quad s \in [-\tau, 0], \end{cases} \tag{2.9}$$

where $\tau > 0$ is an upper bound on the distribution of delays. The third classification is an equation with unbounded delay:

$$\begin{cases} \dot{x} = f(t,x) + \displaystyle\int_{-\infty}^{t} g(t,s,x(s))ds, \\ x(t_0+s) = \phi_0(s), \quad s \in (-\infty, 0]. \end{cases} \tag{2.10}$$

Motivated by the applications considered in the present thesis, the focus here is on the second and third classifications. The second classification (2.9) can be modelled by (2.6), however this is not true for the third classification with unbounded delay (2.10). For basic theory of integro-differential equations see Chapter 1 of [85], for linear analysis and Lyapunov stability see Chapters 2 and 3 of [85].

2.3 Hybrid Systems

Hybrid systems are systems in which continuous and discrete dynamics interact to generate the evolution of the system state. Impulsive systems are a focal point of the present thesis and are detailed in Section 2.3.1. Background on switched systems, also a type of hybrid system, is established in Section 2.3.2.

2.3.1 Impulsive Systems

The background in this section is taken from [83], unless otherwise specified. An impulsive differential equation (IDE) is a natural way to model the evolution of a system which experiences instantaneous changes in the system state, called impulsive effects. Consider the following general control system

$$\begin{cases} \dot{x} = f(t, x) + u, \\ x(t_0) = x_0, \end{cases} \tag{2.11}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^n$ is control input constructed using the generalized Dirac delta function, $\delta(t)$, by letting

$$u(t) = \sum_{k=1}^{\infty} g_k(x(t))\delta(t - T_k^-)$$

where g_k are the impulsive effects and $x(t^-) := \lim_{h \to 0^+} x(t - h)$. The sequence of times $\{T_k\}_{k=1}^{\infty}$ are the impulsive times (also called impulsive moments) which satisfy $t_0 < T_1 < T_2 < \ldots < T_k < \ldots \to \infty$ as $k \to \infty$. When $t \neq T_k$ the system evolves as an ODE and when $t = T_k$ an impulsive effect is applied to the system:

$$\begin{aligned} \lim_{h \to 0^+} x(T_k) - x(T_k - h) &= \lim_{h \to 0^+} \int_{T_k - h}^{T_k} \left[f(s) + \sum_{k=1}^{\infty} g_k(x(s))\delta(s - T_k^-) \right] ds, \\ &= g_k(x(T_k^-)) \end{aligned}$$

since $\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0)$ and $\int_{-\infty}^{\infty} g(t)\delta(t - a)dt = g(a)$. Then (2.11) can be re-written as the IDE IVP:

$$\begin{cases} \dot{x} = f(t, x), & t \neq T_k, \\ \Delta x = g_k(x(t^-)), & t = T_k, \\ x(t_0) = x_0, & k \in \mathbb{N}, \end{cases} \tag{2.12}$$

where $\mathbb{N}$ is the set of positive integers and $\Delta x := x(t) - x(t^-)$. Let $f : \mathbb{R} \times D \to \mathbb{R}^n$ where D is an open set in $\mathbb{R}^n$. A solution of (2.12) is a function $x(t) = x(t; t_0, x_0)$ on the interval I containing t_0 which satisfies the following [17]:

(i) $(t, x(t)) \in \mathbb{R} \times D$ for $t \in I$, and $x(t_0) = x_0$ where $(t_0, x_0) \in \mathbb{R} \times D$.

(ii) For $t \in I$, $t \neq T_k$, $x(t)$ satisfies $\dot{x}(t) = f(t, x(t))$.

(iii) $x(t)$ is continuous from the right and if $T_k \in I$ then $x(T_k) = x(T_k^-) + g_k(x(T_k^-))$.

Existence and uniqueness of the IDE IVP (2.12) can be established.

Theorem 2.3.1. (Existence and Uniqueness) [17]
Assume $f \in C^1(\mathbb{R} \times D, \mathbb{R}^n)$ and $x + g_k(x) \in D$ for each $k \in \mathbb{N}$ and $x \in D$. Then for each $(t_0, x_0) \in \mathbb{R} \times D$ there exists a unique solution of the IVP (2.12).

In the IDE IVP (2.12), impulses are applied at the fixed times $t = T_k$, which need not be the case. Consider the following IDE IVP with variable impulsive times

$$\begin{cases} \dot{x} = f(t, x), & t \neq T_k(x), \\ \Delta x = g_k(x(t^-)), & t = T_k(x), \\ x(t_0) = x_0, & k \in \mathbb{N}, \end{cases} \tag{2.13}$$

where $T_k(x) < T_{k+1}(x)$ and $\lim_{k\to\infty} T_k(x) = \infty$. The moments of impulsive effect, T_k, depend on the solution state and so solutions initialized at different points may have different points of discontinuity. A solution may hit the same surface several times and different solutions may coincide after some time. For more details on impulsive differential equations, including systems with variable impulse times, global existence, stability, and Lyapunov function methods, see [17, 83].

2.3.2 Switched Systems

A switched system, which is another type of hybrid system, evolves according to mode-dependent continuous/discrete dynamics and experiences abrupt transitions between modes triggered by a logic-based switching rule [166]. Switched systems most often arise in two contexts [32]: (i) a natural system which exhibits sudden changes in its dynamics based on, for example, environmental factors; (ii) when switching control is used to stabilize a continuous system. As discussed in Chapter 1, practical applications are wide-ranging. Unless otherwise specified, the results in this section are from [101].

Example 2.3.1. (Multi-controller architecture)
Given a process to manipulate, a continuous feedback control which achieves some desired behaviour may not exist. However, it may be possible to control the process by switching among a family of controllers, each of which is designed for a particular task in the implementation. As the system evolves, a decision maker determines which controller should be active in the closed-loop system. This is an example of switching via a logic-based supervisor and leads to a switched system architecture (see Figure 2.1).

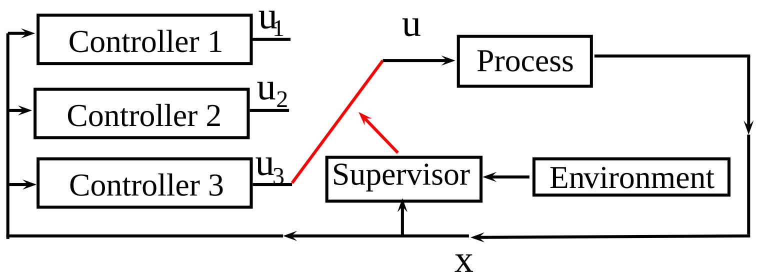


Figure 2.1: Desired behaviour is achieved via supervisory switching control.

Example 2.3.2. (Epidemic model)
The spread of an infectious disease can be modelled using a system of ordinary differential equations. Crucial model components include the transmission rate of the disease and the population behaviour. An important factor in the transmission of a disease is seasonal changes in its spread, which can be modelled by term-time forced parameters (piecewise constant) which abruptly change in time. For example, a student's school schedule causes sudden changes in their day-to-day pattern of contacts with other individuals. Hence, the system can be modelled as a switched system. See Figure 2.2 for the flow of individuals in the population: S represents individuals in the population that are susceptible; E represents individuals that have been exposed but are not yet infectious; I represents infected individuals that are infectious; and R represents recovered individuals.

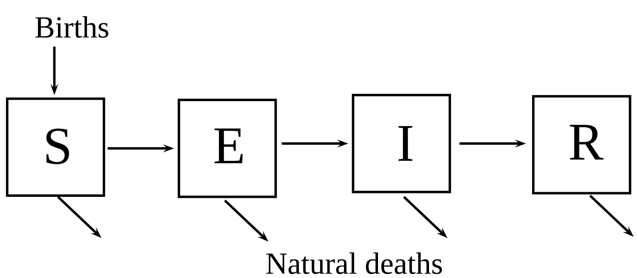


Figure 2.2: Flow diagram of SEIR epidemic model.

Example 2.3.3. (Air conditioner) [46]
A home climate-control system can be modelled naturally as a switched system. When the

temperature increases above some threshold level, the AC is turned on which causes a drop in the temperature. Once a different lower threshold is reached, the AC is turned off and the temperature may rise again to the ambient temperature. See Figure 2.3.

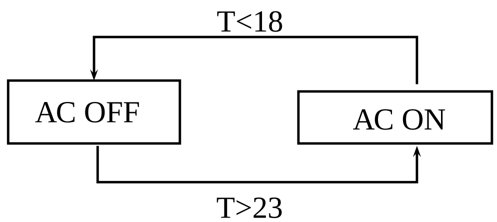


Figure 2.3: Air conditioner as a switched system.

Example 2.3.4. (Manual transmission) [166]
Consider a vehicle with manual transmission travelling along a fixed path. Its motion can be characterized by its position $x(t)$ and velocity $v(t)$. In a simplified model, the system has two control inputs: the current angle of the throttle, (denoted by u), and the current engaged gear (denoted by g). Each gear represents a mode of the system and changing gears (an abrupt action triggered by the driver) represents switching between different modes. See Figure 2.4 for an illustration.

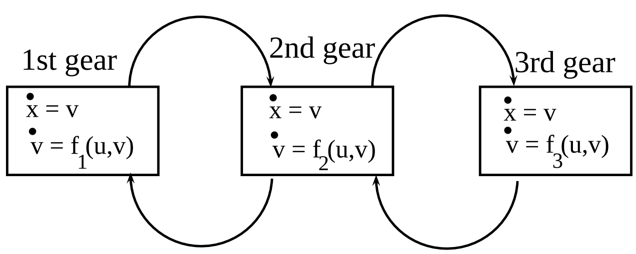


Figure 2.4: A hybrid model of a vehicle with manual transmission.

Here we consider a switched system as a dynamical system consisting of continuous-time subsystems (or modes) and a logical rule that orchestrates switching between them.

Consider the family of time-invariant ODEs

$$\dot{x} = f_i(x), \tag{2.14}$$

where $\{f_i : i \in \mathcal{P}\}$ is a family of sufficiently smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ parameterized by an index set $\mathcal{P}$ and a piecewise constant function $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$ which is assumed to be right-continuous. In the present thesis, the index set is assumed to be finite: $\mathcal{P} = \{1, 2, \ldots, m\}$ for some positive integer m. The function σ is called a switching signal or switching rule and is assumed to be deterministic. The switching times $\{t_k\}_{k=0}^{\infty}$ are assumed to satisfy $0 < t_1 < \ldots < t_k < \ldots$ with $t_k \to \infty$ as $k \to \infty$. The switching times can be time-dependent, state-dependent ($t_k = t_k(x)$), or a mixture of both[4].

Under this formulation the index i is chosen according to the switching rule and the system evolves according to the dynamics of the active subsystem. At the switching time t_k, the active subsystem changes from $\sigma(t_k^-) := \lim_{h \to 0^+} \sigma(t_k - h)$ to $\sigma(t_k)$. The solution evolves according to $\dot{x} = f_{\sigma(t_{k-1})}(x)$ for $t \in [t_{k-1}, t_k)$ and then according to $\dot{x} = f_{\sigma(t_k)}(x)$ for $t \in [t_k, t_{k+1})$. Since σ is piecewise constant, $\sigma(t_{k-1}) = \sigma(t_k^-)$. For an illustration of a simple switching rule, see Figure 2.5.

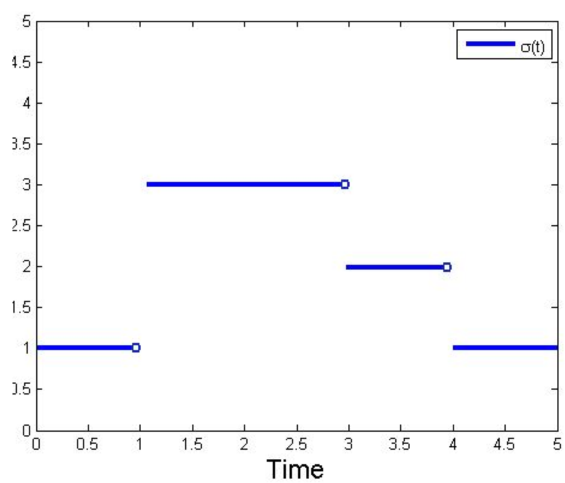


Figure 2.5: Example of a switching rule σ with switch times $t_k = 1, 3, 4$ and $\mathcal{P} = \{1, 2, 3\}$.

With a switching rule σ and an initial condition x_0, the family of systems (2.14) can

[4]For other types of switching rules, such as Markovian switching, see [102].

be written as a switched system

$$\begin{cases} \dot{x} = f_{i_k}(x), & t \in [t_{k-1}, t_k), \\ x(0) = x_0 & k \in \mathbb{N}, \end{cases} \tag{2.15}$$

where $i_k \in \mathcal{P}$ follows the switching rule σ. For a particular choice of the index $p \in \mathcal{P}$, the system $\dot{x} = f_p(x)$ is called the p^{th} subsystem or mode of the switched system (2.15). Assume that the initial time is $t_0 = 0$ since if this is not the case it is possible to shift the time by defining a new time variable $\tau = t - t_0$ and new switching times $h_k = t_k - t_0$.

The switched system (2.15) can be re-written in a more compact form:

$$\begin{cases} \dot{x} = f_{\sigma}(x), \\ x(0) = x_0. \end{cases} \tag{2.16}$$

System (2.16) admits a family of solutions that is parameterized both by the initial condition and the switching signal σ, which is unlike the ODE IVP (2.3) that admits a family of solutions parameterized solely by the initial condition [64]. A solution of the switched system (2.16) is a continuous function $x(t; x_0) : \mathbb{R}_+ \to \mathbb{R}^n$ which satisfies the following [14]: there exists a switching sequence $\{t_k\}_{k=1}^{\infty}$ and indices $i_1, i_2, i_3, \ldots$, with $i_k \in \mathcal{P}$, associated with a switching rule σ such that $x(t; x_0)$ is an integral curve of the vector field $f_{i_k}(x)$ for $t \neq t_k$ and $x(0; x_0) = x_0$.

Remark 2.3.1. *Since a solution of (2.16) is parameterized by the switching rule, we could write $x(t) = x(t; x_0, \sigma)$ to show this dependency but we choose the more compact form with this understanding in mind.*

The switched system (2.16) has an equilibrium point $\bar{x}$ (sometimes called a common equilibrium point) if $f_i(\bar{x}) = 0$ for all $i \in \mathcal{P}$. Since it is possible to shift such a point to the origin by setting $y = x - \bar{x}$, assume that $f_i(0) \equiv 0$ for all $i \in \mathcal{P}$. Then the definitions of stability of the trivial solution of (2.16) are analogous[5] to those in the classical theory of ODEs.

Since analytic solutions of the switched system (2.16) cannot be found explicitly in general, most of the switched systems literature can be categorized into one of the following problems [102]:

1. Find conditions guaranteeing asymptotic stability of the trivial solution for arbitrary switching rules.

[5] For example, see [14].

2. Identify classes of switching rules under which the trivial solution is asymptotically stable.

3. Construct switching rules such that the trivial solution is asymptotically stable.

We detail each of these problems below.

Problem 1: Stability under arbitrary switching

Preserving stability under arbitrary switching is of particular importance in switching feedback control. If the j^{th} subsystem of a switched system evolves according to an unstable mode then the switching rule $\sigma(t) = j$ leads to instability. Therefore, for stability under arbitrary switching to be possible, a necessary condition is that all subsystems must be stable. However, this is not a sufficient condition for stability. Switching between two stable subsystems can lead to instability, illustrated in the following example.

Example 2.3.5. *Consider (2.16) with $\mathcal{P} = \{1, 2\}$ and $f_1(x) = A_1 x$, $f_2(x) = A_2 x$ with*

$$A_1 = \begin{pmatrix} -0.1 & 1 \\ -2 & -0.1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -0.1 & 2 \\ -1 & -0.1 \end{pmatrix}.$$

Both A_1 and A_2 are Hurwitz[6] *matrices and so the origin is exponentially stable for each subsystem. Consider the following state-dependent switching rule: if $x_1 x_2 < 0$ choose subsystem 1 to be active, otherwise choose subsystem 2 to be active. See Figure 2.6 for an illustration, where the origin of the switched system is unstable.*

A sufficient condition for the asymptotic stability of the origin of (2.16) involves the existence of a so-called common Lyapunov function.

Theorem 2.3.2. *Let $V \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ and let $W \in C(\mathbb{R}^n, \mathbb{R}_+)$ be a positive definite and radially unbounded*[7] *function. If*

$$\nabla V(x) \cdot f_i(x) \leq -W(x) \tag{2.17}$$

for all $x \in \mathbb{R}^n$ and for all $i \in \mathcal{P}$ then the origin of the switched system (2.16) is globally asymptotically stable for arbitrary switching.

The main idea is that the rate of decrease of V along solutions is unaffected by the switching and asymptotic stability is uniform with respect to the switching rule σ. For more details regarding stability under arbitrary switching, see Chapter 2 of [101].

[6] All eigenvalues have negative real part.

[7] $W(x) \to \infty$ as $\|x\| \to \infty$.

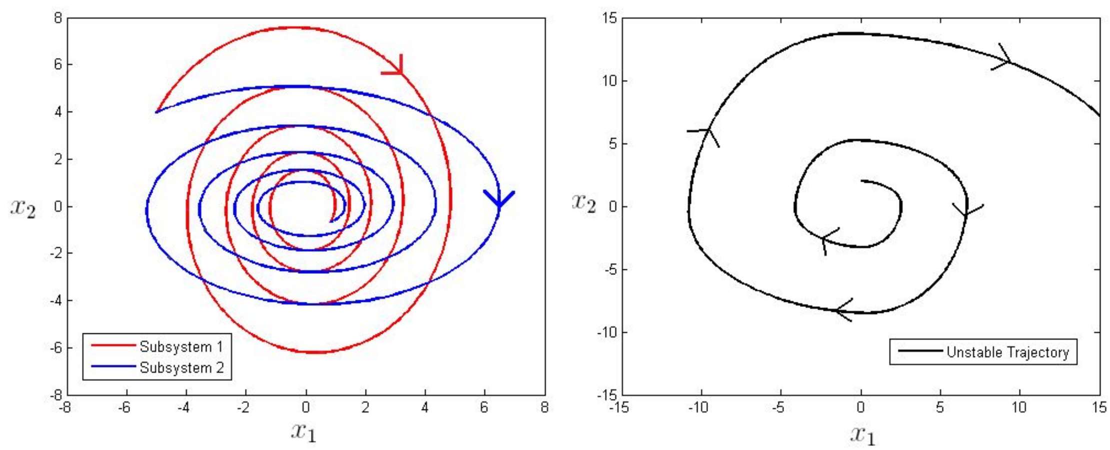


Figure 2.6: The subsystems are stable (left), however, the overall switched system is unstable (right).

Problem 2: Stability under constrained switching

Motivated by instability of a switched system composed entirely of stable subsystems, we seek classes of switching rules which avoid this unwanted behaviour. Again we consider a switched system (2.16) composed entirely of stable subsystems. This problem can be solved by putting restrictions on how fast the system can switch modes, which leads to the concept of stability under slow switching or dwell-time switching [102].

One way to guarantee the switching is sufficiently slow is the existence of multiple Lyapunov functions. If each subsystem is stable, then each subsystem has a Lyapunov function which decreases along solutions. If the Lyapunov functions satisfy appropriate conditions at the switching times then asymptotic stability of the switched system can be achieved.

Theorem 2.3.3. [63]
Let $D \subset \mathbb{R}^n$ be an open set and suppose that $f_i : D \to \mathbb{R}^n$ for all $i \in \mathcal{P}$. Assume there exist $V_i \in C^1(D, \mathbb{R}_+)$ for $i \in \mathcal{P}$ which satisfy $\nabla V_i(x) \cdot f_i(x) < 0$ for all $x \in D \setminus \{0\}$. Assume that

$$V_{i_{k+1}}(x(t_k)) \le V_{i_k}(x(t_k)) \tag{2.18}$$

at every switching time t_k. Then the trivial solution of the switched system (2.16) is asymptotically stable.

Since the Lyapunov functions do not increase at the switch times, the switching Lyapunov function V_σ is always decreasing along solutions of the switched system (2.16). In fact, stability can be guaranteed if the Lyapunov functions form a decreasing sequence at the switching times.

Theorem 2.3.4. *Let $D \subset \mathbb{R}^n$ be an open set and suppose that $f_i : D \to \mathbb{R}^n$ for all $i \in \mathcal{P}$. Assume there exist $V_i \in C^1(D, \mathbb{R}_+)$ for $i \in \mathcal{P}$ which satisfy $\nabla V_i(x) \cdot f_i(x) < 0$ for all $x \in D \setminus \{0\}$. Assume there exist positive definite continuous functions W_i, $i \in \mathcal{P}$ such that*

$$V_p(x(t_j)) - V_p(x(t_i)) \leq -W_p(x(t_i)), \tag{2.19}$$

for every pair of switching times (t_i, t_j), $i < j$ such that $\sigma(t_i) = \sigma(t_j) = p \in \mathcal{P}$ and $\sigma(t_k) \neq p$ for any t_k satisfying $t_i < t_k < t_j$. Then the trivial solution of the switched system (2.16) is asymptotically stable.

See Figure 2.7 for an illustration of condition (2.19).

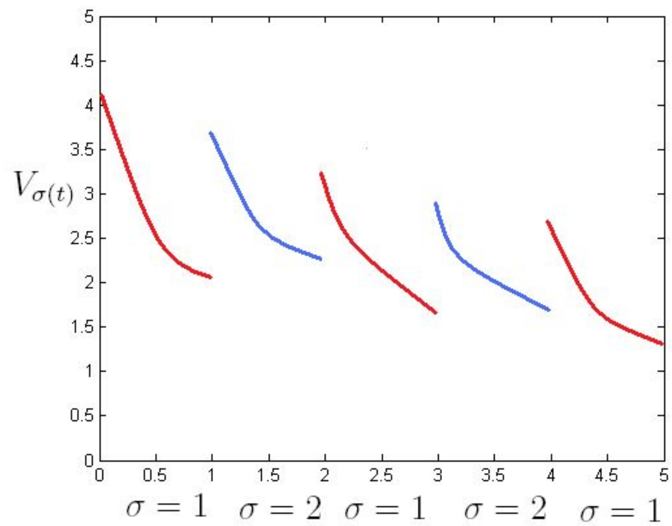


Figure 2.7: Two Lyapunov functions which satisfy (2.19). The red line corresponds to the first mode being active while the blue line corresponds to the second mode being active.

Remark 2.3.2. *If $f_i(x) \equiv A_i x$ for all $i \in \mathcal{P}$ where $A_i \in \mathbb{R}^{n\times n}$ are Hurwitz matrices, then it is straightforward to calculate a Lyapunov function for each subsystem as $V_i = x^T P_i x$ where*

P_i are positive definite matrices satisfying the Lyapunov equations $A_i^T P_i + P_i A_i^T = -Q_i$ for any positive definite matrix Q_i.

Unfortunately, the energy conditions at the switching times in equations (2.18) and (2.19) require explicit knowledge of the solution trajectory at the switching points t_k [152]. This might seem unreasonably strict, but it is often the case that the multiple Lyapunov function switching conditions are trivially satisfied or that the switching signal is constructed precisely with these conditions in mind.

An alternative approach to the problem of sufficiently slow switching is to restrict the set of admissible switching rules (which is especially convenient when the switching signals are trajectory dependent [64]). The switching rule of system (2.16) is said to have a dwell-time if there exists a constant $\eta > 0$ such that $\inf_{k \in \mathbb{N}}(t_k - t_{k-1}) \geq \eta$. Stability can be established based on a lower bound on η.

Theorem 2.3.5. *Consider (2.16) with $\mathcal{P} = \{1, 2\}$. Assume that there exist functions $V_1, V_2 \in C^1(\mathbb{R}^n, \mathbb{R})$ and positive constants a_1, a_2, b_1, b_2, c_1, and c_2 such that*

(i) $a_i \|x\|^2 \leq V_i(x) \leq b_i \|x\|^2$ for all $x \in \mathbb{R}^n$;

(ii) $\nabla V_i(x) \cdot f_i(x) \leq -c_i \|x\|^2$ for all $x \in \mathbb{R}^n$.

Then the trivial solution of (2.16) is asymptotically stable if

$$\eta > \left(\frac{c_1}{b_1} + \frac{c_2}{b_2}\right) \ln\left(\frac{b_1 b_2}{a_1 a_2}\right).$$

Dwell-time switching such as in Theorem 2.3.5 can be too restrictive for certain physical applications. For example, if the switching rule selects subsystems according to optimizing a particular behaviour, it may be possible that the performance deteriorates (due to, for example, system failure) to an unacceptable level before the required dwell-time has passed. Average dwell-time switching can alleviate this problem [65]. If there exist two positive constants N_0 and τ_a such that

$$\widetilde{N}_\sigma(t_0, t_1) \leq N_0 + \frac{t_1 - t_0}{\tau_a}, \quad \text{for all } t_0 \leq t \leq t_1, \tag{2.20}$$

where $\widetilde{N}_\sigma(t_0, t_1)$ is defined to be the number of discontinuities of the switching rule σ on the interval (t_0, t_1), then the switching signal σ is said to have an average dwell-time τ_a.

Theorem 2.3.6. *Consider* (2.16) *with* $\mathcal{P} = \{1, 2\}$*. Assume that there exist Lyapunov functions* $V_1, V_2 \in C^1(\mathbb{R}^n, \mathbb{R})$ *and* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and constants* $\mu > 0$, $\lambda > 0$ *such that for* $i = 1, 2$,

(i) $\alpha_1(\|x\|) \leq V_i(x) \leq \alpha_2(\|x\|)$ *for all* $x \in \mathbb{R}^n$;

(ii) $\nabla V_i(x) \cdot f_i(x) \leq -\lambda V_i(x)$ *for all* $x \in \mathbb{R}^n$;

(iii) $V_p(x) \leq \mu V_q(x)$ *for all* $p, q \in \mathcal{P}$.

Then the trivial solution of the switched system (2.16) *is globally asymptotically stable for any switching rule with average dwell-time satisfying* $\tau_a > \ln(\mu)/\lambda$ *where* N_0 *is arbitrary.*

Average dwell-time switching allows the possibility of fast switching on certain intervals but compensates for it by demanding sufficiently slow switching later on. For more background on the stability of switched systems with dwell-time and average dwell-time, see [65, 101, 166]. For examples of some other classes of switching signals, see [64].

Problem 3: Switching control

The third problem can be viewed as a control problem where switching control is used to stabilize an unstable continuous system. This may be required if continuous control is not suitable (due to the nature of the problem), cannot be found (due to model uncertainty), or cannot be implemented (due to sensor and/or actuator limitations). Not only can switching control stabilize an unstable system, switching between controllers in a certain way can also improve performance over a fixed continuous controller [32]. It can also prove to be easier to find a switching controller to perform a desired task versus finding a continuous one [46].

Consider the following example in which the solution trajectory in either of the subsystems grows over time, but not monotonically. The switching rule is constructed to take advantage of this fact.

Example 2.3.6. *Consider the switched system* (2.16) *with* $\mathcal{P} = \{1, 2\}$, $f_1(x) = A_1 x$, $f_2(x) = A_2 x$,

$$A_1 = \begin{pmatrix} 0.1 & -1 \\ 2 & 0.1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.1 & -2 \\ 1 & 0.1 \end{pmatrix}.$$

The eigenvalues of both of these matrices have positive real parts, and so each subsystem is unstable. It is possible to construct a stabilizing switching rule as follows: if $x_1 x_2 < 0$

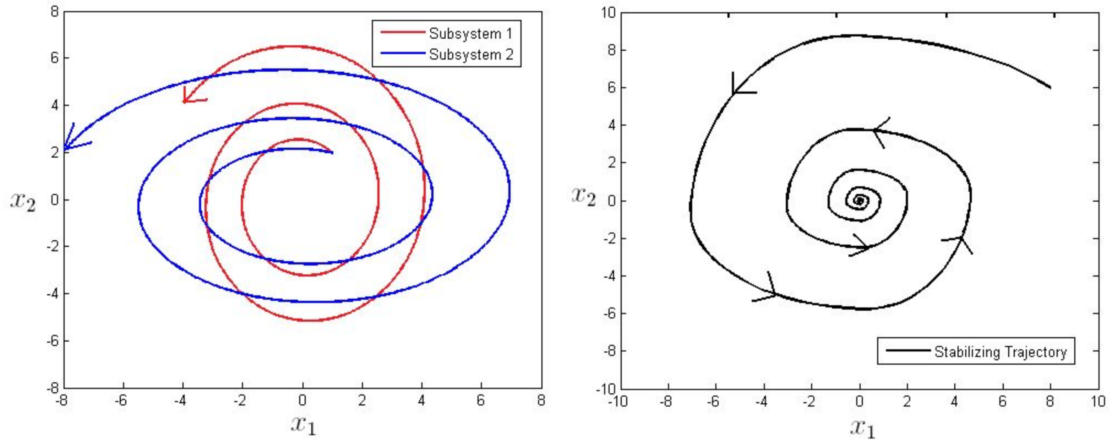


Figure 2.8: Both subsystems are unstable (left figure). The right figure shows a trajectory of the switched system under the stabilizing switching rule outlined above.

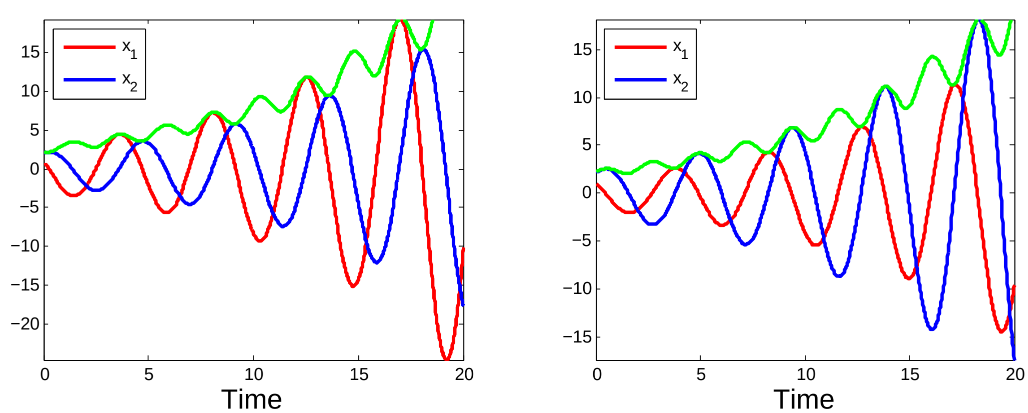


Figure 2.9: The solution trajectories of subsystem 1 (left figure) and subsystem 2 (right figure). Both are unstable systems, but the norms (green curves) do not increase monotonically.

choose subsystem 1 to be active, otherwise choose subsystem 2 to be active. See Figures 2.8 and 2.9.

In [191], Wicks et al. first constructed a stabilizing state-dependent switching rule for a linear switched system. In short: if there exists a scalar $0 < \alpha < 1$ such that the convex combination $\widetilde{A} := \alpha A_1 + (1-\alpha)A_2$ is Hurwitz, where $A_1, A_2 \in \mathbb{R}^{n\times n}$, then there is a stabilizing switching rule for the switched system $\dot{x} = A_\sigma x$ where $\sigma : [t_{k-1}, t_k) \to \{1, 2\}$. The switching rule can be constructed by partitioning the state space into $\Omega_1 = \{x \in \mathbb{R}^n : x^T(A_1^T P + PA_1^T)x < 0\}$ and $\Omega_2 = \{x \in \mathbb{R}^n : x^T(A_2^T P + PA_2^T)x < 0\}$ where P is a positive definite matrix which solves the Lyapunov equation $\widetilde{A}^T P + P\widetilde{A} = -Q$ for some positive definite matrix Q. The Lyapunov function $V = x^T Px$ decreases along solutions of the first system ($\dot{x} = A_1 x$) in the region Ω_1 and decreases along solutions of the second system ($\dot{x} = A_2 x$) in the region Ω_2. The switching rule takes the form:

$$\sigma = \begin{cases} 1 & \text{if } x \in \Omega_1, \\ 2 & \text{if } x \in \Omega_2. \end{cases} \tag{2.21}$$

The state-dependent switching rule is extendable to a linear switched system with m subsystems if there exist constants $\alpha_i > 0$ satisfying $\sum_{i=1}^m \alpha_i = 1$ such that the convex combination matrix $\widetilde{A} := \sum_{i=1}^m \alpha_i A_i$ is Hurwitz. For a more detailed account of the linear case, see the book Chapter 3 in [101] and the survey paper [103], where Lin and Antsaklis detailed results regarding the switching stabilization of linear systems. The nonlinear case is detailed in Section 5.1.2 of the present thesis. Graphically, the state-space is subdivided into switching regions and the current active mode depends on the location of the state trajectory (see Figure 2.10).

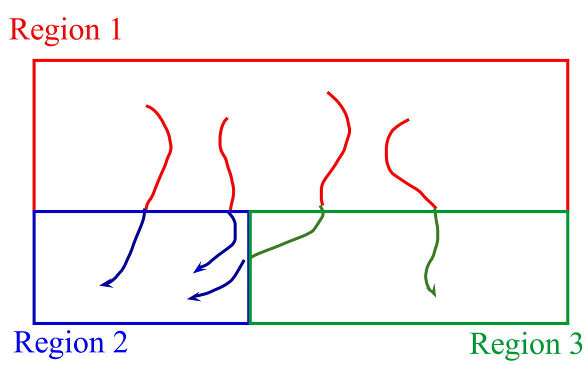


Figure 2.10: Solution trajectories for a switched system with a state-dependent switching rule. A switch occurs whenever the state trajectory crosses a switching region boundary.

Remark 2.3.3. *The state-dependent switching rule in* (2.21) *raises some concerns over the well-posedness of a state-dependent switching rule. For example, if the system crosses a boundary and switches, the trajectory could then immediately cross over the same boundary, forcing another switch. This raises the possibility of infinitely fast switching, or chattering, which is undesirable practically as it results in excessive equipment wear. See Figure 2.11 for an illustration.*

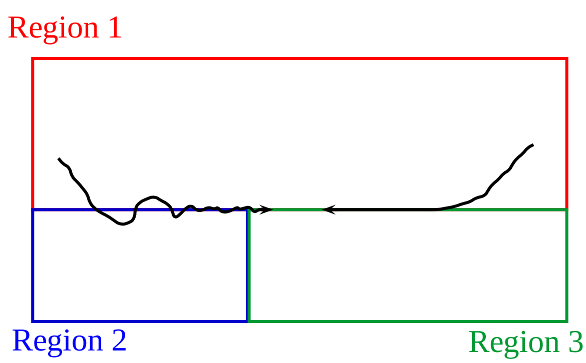


Figure 2.11: Possible chattering behaviour in a state-dependent switching rule.

If a switched system is composed entirely of unstable subsystems, it may also be possible to construct a purely time-dependent stabilizing switching rule. The main idea here is that since each subsystem is unstable, the switching strategy should be a high-frequency switching rule. If the switching rule dwells in any one subsystem for too long, instability occurs. This is the opposite of the dwell-time approach in the previous section where each subsystem is stable and stability is achieved as long as the switching does not occur too frequently. Sun et al. [174] detailed the idea of fast-switching stabilization via periodic time-dependent switching rules for linear systems. Time-dependent switching stabilization of nonlinear systems is presented later in Section 5.1.1 of this thesis. Consider the following linear example.

Example 2.3.7. *Consider the switched system* (2.16) *with* $\mathcal{P} = \{1, 2\}$*,* $f_1(x) = A_1 x$*,* $f_2(x) = A_2 x$*,*

$$A_1 = \begin{bmatrix} -9 & 1 \\ 3 & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -1 \\ 3 & -8 \end{bmatrix}.$$

Both matrices have eigenvalues with positive real part and the matrix $\widetilde{A} := 0.5A_1 + 0.5A_2$ *is Hurwitz. See Figure 2.12 for a simulation with periodic switching (every* 0.05 *time units)* *where it is observed that the solution trajectories of the switched system converges to the origin.*

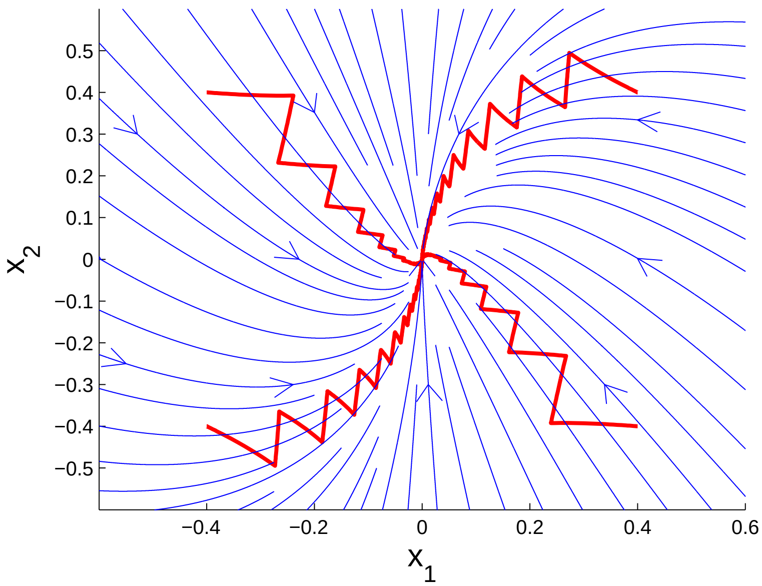


Figure 2.12: Simulation of Example 2.3.7. The blue lines are solution trajectories of the convex combination system $\dot{x} = \widetilde{A}x$. The red lines are solution trajectories of the switched system under periodic high-frequency switching.

Chapter 3

Fundamental Theory of HISD

To simulate real-world problems using mathematical models, it is important to investigate the well-posedness of said models. For applications in epidemic modelling, the physical problem certainly exists and given the same initial disease profile we should expect the same outcome. We hope then that the mathematical model exhibits these properties. Namely, the mathematical model has at least one solution (existence) and the model has at most one solution given the same initial data (uniqueness). Significant work has been done on fundamental theory for impulsive systems (for example, see [83]) and for functional differential equations (for example, see [61]).

Attention has also been given to functional differential equations with infinite delay. For example, the work by Hale and Kato in [60] on phase spaces for delay differential equations with unbounded delay. This work has been studied further in [8, 58, 73, 100, 162, 165]. Less work has been devoted to the investigation of basic theory for classical solutions of impulsive systems with finite delay: for example, existence and uniqueness results were given by Ballinger and Liu in [19] for impulsive systems with delay. Liu and Ballinger [111] extended these results for finite delay systems with state-dependent impulses. Work has been done on stochastic impulsive systems with finite delay (for example, see [4]). Investigations into the global existence of classical solutions for impulsive systems with finite delay have been completed (for example, see [111, 129]).

The results in the reports discussed above do not apply to switched systems with infinite delay and impulses. This is the focus of the present chapter where we develop results for the existence, uniqueness, and continuation of solutions to hybrid impulsive systems with distributed delays (**HISD**), including unbounded delay. The main results on existence and uniqueness are proved by adjusting classical techniques in order to deal with impulsive effects, infinite delay, and switching behaviour. The results found here are applicable to systems with finite delay and to systems where the impulsive times and switch times do not necessarily coincide. The HISD is formulated so that each subsystem can have a different domain of definition. The switching rule and impulsive effects are both dependent on the time and/or state and are constructed in a general way. The main contributions are results on the local existence, uniqueness, extended existence, and global existence of classical solutions to HISD with infinite delay and generalized impulsive sets, which, to the best of the author's knowledge, are extensions of the current literature. The material in this chapter formed the basis for [116].

3.1 Choice of Phase Space

For a delay differential equation with finite delay (such as equation (2.6)), the phase space chosen for the initial function is not qualitatively important [61]: after one delay interval, the history of the state belongs to the space of continuous functions. If the system has impulsive effects then even if the initial function contains no discontinuities, once the first impulsive effect is applied, the history contains a discontinuity. The space of piecewise continuous functions is an obvious choice for impulsive systems with finite delay, constructed as follows: given the constants a and b satisfying $a < b$ and the open set $D \subset \mathbb{R}^n$, define the following classes of piecewise continuous functions (for example, see [124])

$$\begin{aligned}
PC([a,b],D) = \{x : [a,b] \to D \mid\ & x(t) = x(t^+) \text{ for all } t \in [a,b); \\
& x(t^-) \text{ exists in } D \ \text{ for all } t \in (a,b]; \\
& x(t^-) = x(t) \text{ for all but at most a finite number} \\
& \text{ of points } t \in (a,b]\}, \\
PC([a,b),D) = \{x : [a,b) \to D \mid\ & x(t) = x(t^+) \text{ for all } t \in [a,b); \\
& x(t^-) \text{ exists in } D \ \text{ for all } t \in (a,b); \\
& x(t^-) = x(t) \text{ for all but at most a finite number} \\
& \text{ of points } t \in (a,b)\},
\end{aligned}$$

which can be extended to infinite intervals as

$$
\begin{aligned}
PC([a,\infty),D) &= \{x:[a,\infty)\to D \mid \text{ for all } c>a, x|_{[a,c]}\in PC([a,c],D)\},\\
PC((-\infty,b],D) &= \{x:(-\infty,b]\to D \mid x(t^+)=x(t) \text{ for all } t\in(-\infty,b);\\
&\qquad x(t^+) \text{ exists in } D \text{ for all } t\in(-\infty,b];\\
&\qquad x(t^-)=x(t) \text{ for all but a countable infinite number}\\
&\qquad \text{of points } t\in(-\infty,b]\},\\
PC(\mathbb{R},D) &= \{x:\mathbb{R}\to D \mid \text{for all } b\in\mathbb{R}, x|_{(-\infty,b]}\in PC((-\infty,b],D)\}.
\end{aligned}
$$

Equip the space with the usual sup norm: for $\psi\in PC([-\tau,0],\mathbb{R}^n)$, let

$$\|\psi\|_\tau = \sup_{-\tau\le s\le 0}\|\psi(s)\|.$$

In the case of infinite delay, the phase space is important as the history of the state always contains the initial data. If the delay is unbounded and an impulsive effect is applied to the solution state, the history will always contain the discontinuity (regardless of the smoothness of the initial function ϕ_0). Since the interval $(-\infty,0]$ is not compact, if a set is closed and bounded in $PC((-\infty,0],D)$, the image of a solution map may not be compact (see [87] and the references therein). It is often the case that a phase space is not given explicitly or even discussed in the literature [48]. The development of appropriate phase spaces for unbounded delay began with the work by Hale and Kato [60] and was further refined by Hale in [58]. For more background on this topic, including other possible phase space choices, see [8, 48, 61].

There have been numerous studies on the stability of impulsive systems with infinite delay using the phase space PCB of piecewise continuous bounded functions. This has included works using Razumikhin techniques and Lyapunov functionals. For example, see the reports [48, 49, 93, 94, 96, 128, 130] (some of which are detailed in the analysis done in the next chapter). We proceed by constructing the class of piecewise continuous bounded functions as follows:

$$
\begin{aligned}
PCB([a,b],D) &= \{x:[a,b]\to D \mid x\in PC([a,b],D)\},\\
PCB([a,b),D) &= \{x:[a,b)\to D \mid x\in PC([a,b),D)\},\\
PCB([a,\infty),D) &= \{x:[a,\infty)\to D \mid \text{ for all } c>a, x|_{[a,c]}\in PCB([a,c],D)\},\\
PCB((-\infty,b],D) &= \{x:(-\infty,b]\to D \mid x\in PC((-\infty,b],D),\\
&\qquad x \text{ is bounded on } (-\infty,b] \text{ with respect to } \|\cdot\|_{PCB}\},\\
PCB(\mathbb{R},D) &= \{x:\mathbb{R}\to D \mid \text{for all } b\in\mathbb{R}, x|_{(-\infty,b]}\in PCB((-\infty,b],D)\},
\end{aligned}
$$

where the norm is given by

$$\|\psi\|_{PCB} = \sup_{\alpha \le s \le 0} \|\psi(s)\|,$$

for delay $-\infty \le \alpha < 0$, which is understood to be

$$\|\psi\|_{PCB} = \sup_{s \le 0} \|\psi(s)\|,$$

when the delay is infinite. Since the main focus of the present thesis is on the long-term qualitative behaviour of HISD, including those with unbounded delay, and motivated by the applications considered, we consider use of the phase space PCB.

3.2 Problem Formulation

Consider the following family of impulsive systems with time-delays:

$$\begin{cases} \dot{x} = f_i(t, x_t), & (t, x) \notin \Gamma, \\ \Delta x = g_i(t, x_{t^-}), & (t, x) \in \Gamma, \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state; $i \in \mathcal{P} := \{1, \dots, m\}$ where $m > 1$ is an integer; $\Delta x := x(t) - x(t^-)$; and x_t is defined by

$$x_t(s) := x(t + s), \qquad \alpha \le s \le 0$$

for $-\infty \le \alpha < 0$. It is understood that if $\alpha = -\infty$ then the interval becomes $(-\infty, 0]$ and

$$x_t(s) := x(t + s), \qquad s \le 0.$$

The family of vector fields $\{f_i \mid i \in \mathcal{P}\}$ satisfy $f_i : J \times PCB([\alpha, 0], D_i) \to \mathbb{R}^n$ where $J = [t_0, t_0 + b)$, $0 < b \le \infty$, and $D_i \subset \mathbb{R}^n$ is open for each $i \in \mathcal{P}$. The impulsive functions satisfy $g_i : J \times PCB([\alpha, 0], D_i) \to \mathbb{R}^n$, where $x_{t^-} \in PCB$ is defined by

$$x_{t^-}(s) := \begin{cases} x(t + s), & \text{for } \alpha \le s < 0, \\ x(t^-), & \text{for } s = 0. \end{cases}$$

The impulsive effects are applied at any time t such that the (t, x)-trajectory belongs to the set $\Gamma \subset \mathbb{R}^{n+1}$. Denote any such impulsive moment by T_k, then the impulsive moments necessarily satisfy $(T_k, x(T_k^-)) \in \Gamma$.

To introduce switching into the system, assume that there is a logic-based rule which dictates the vector field f_i that is currently engaged in system (3.1). More precisely, suppose that the index i changes values according to a switching rule $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$ where $\mathcal{P} = \{1, 2, \ldots, m\}$ for some integer $m > 1$. The switching rule determines which subsystem (also called a mode) of the switched system is currently active and determines when the system experiences a switch (called the switching times, denoted t_k). The switching rule (also called a switching signal) is assumed to be a deterministic function and is assumed to be piecewise constant and continuous from the right. When a switch occurs at $t = t_k$, the old subsystem $\sigma(t_k^-)$ is disengaged and the next subsystem $\sigma(t_k)$ is engaged.

Remark 3.2.1. *To emphasize the fact that the switching times can be time-dependent, state-dependent ($t_k = t_k(x)$), or a mixture of both, the switching rule is written as $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathcal{P}$ and $\sigma(t, x)$ at times in this chapter.*

Parameterized by a switching rule and an initial function, system (3.1) can be re-written as the following HISD

$$\dot{x} = f_{\sigma(t,x)}(t, x_t), \qquad (t, x) \notin \Gamma, \tag{3.2a}$$

$$\Delta x = g_{\sigma(t^-,x(t^-))}(t, x_{t^-}), \qquad (t, x) \in \Gamma, \tag{3.2b}$$

$$x_{t_0} = \phi_0, \tag{3.2c}$$

where $t_0 \in \mathbb{R}_+$ is the initial time. The initial function is $\phi_0 \in PCB([\alpha, 0], D_{\sigma(t_0,x(t_0))})$ where $\sigma(t_0, x(t_0))$ is the subsystem which is active on the first switching interval $[t_0, t_1)$. Hence $D_{\sigma(t_0,x(t_0))}$ is the domain of definition of the vector field $f_{\sigma(t_0,x(t_0))}$ which is active during the first switching interval.

The main focus of this chapter is to study the fundamental theory of system (3.2). Due to the switching, impulsive effects, and delay behaviour, system (3.2) may fail to be differentiable at certain points (including but not limited to switching points, see [19]) and fails to be continuous at impulsive moments. The precise meaning of a solution $x(t) = x(t; t_0, \phi_0)$ must be made clear. As in [19], the notion of a solution is weakened to permit a finite number of points on any closed interval where the solution is right-continuous with right-hand derivative but is not differentiable. A finite number of points on any closed interval where the solution is only right-continuous is also permitted (to allow for the impulses).

Definition 3.2.1. *A function $x \in PCB([t_0 + \alpha, t_0 + \gamma], \widetilde{D})$, where $\widetilde{D} := \bigcup_{i=1}^m D_i$ and $\gamma > 0$ and $[t_0, t_0 + \gamma] \subset J$, is said to be a solution of (3.2) if*

(i) x is continuous at each $t \neq T_k$ in $(t_0, t_0 + \gamma]$;

(ii) the derivative of x exists and is continuous at all but at most a finite number of points t in $(t_0, t_0+\gamma)$;

(iii) the right-hand derivative of x exists and satisfies the switched system (3.2a) for all $t \in [t_0, t_0+\gamma)$;

(iv) x satisfies the impulse equation (3.2b) at each $T_k \in (t_0, t_0+\gamma]$; and

(v) x satisfies the initial condition (3.2c).

Definition 3.2.2. *A function $x \in PCB([t_0+\alpha, t_0+\beta), \widetilde{D})$, where $0 < \beta \leq \infty$ and $[t_0, t_0+\beta) \subset J$ is said to be a solution of (3.2) if for each $0 < \gamma < \beta$ the restriction of x to $[t_0+\alpha, t_0+\gamma]$ is a solution of (3.2).*

Remark 3.2.2. *It is possible to weaken the notion of a solution further, so that it is integrable in the Lebesgue sense but not in the Riemann sense. There has been work done investigating these weaker solutions in the non-switched case where Carathéodory conditions are used (for example see [20, 149]). This type of solution is not considered here.*

3.3 Admissible Impulsive Sets and Switching Rules

Before establishing existence and uniqueness results, we discuss the admissibility of the impulsive sets and the switching rule.

Definition 3.3.1. (Admissible impulsive set)
The impulsive set $\Gamma \subset \mathbb{R}^{n+1}$ is said to be admissible for system (3.2) if there exists a constant $\delta > 0$ such that $[T_k, T_k+\delta] \subset J$ and $(t, x(t) + g_{\sigma(t,x(t))}(t, x_t)) \notin \Gamma$ for all $t \in (T_k, T_k+\delta]$, $T_k \in J$. Denote the set of all such admissible impulsive sets by $\mathcal{I}$.

Remark 3.3.1. *The impulsive set Γ is admissible if there exists a constant $\epsilon > 0$ such that*

$$(T_k, x + g_i(T_k, \psi)) \notin Z_\epsilon$$

for all $T_k \in J$, $i \in \mathcal{P}$, $x \in \widetilde{D}$, $\psi \in PCB([\alpha, 0], \widetilde{D})$ satisfying $\psi(0) = x$, where

$$Z_\epsilon = \{(t, x) \in \mathbb{R}^{n+1} : \|(t, x) - (\widetilde{t}, \widetilde{x})\| < \epsilon \text{ for all } (\widetilde{t}, \widetilde{x}) \in \Gamma\}.$$

Remark 3.3.2. *By definition, the sequence of impulsive moments $\{T_k\}_{k=1}^{N}$, $1 \leq N \leq \infty$, associated with an admissible impulsive set exhibits a dwell-time: there exists a constant $\eta > 0$ such that*

$$\inf_k (T_{k+1} - T_k) \geq \eta, \tag{3.3}$$

for all $k = 1, 2, \ldots, N-1$. Moreover, the impulsive moments satisfy $t_0 \leq T_1 < T_2 < \ldots < T_N < t_0 + b$.

Remark 3.3.3. *Equation (3.2b) is a generalized formulation of an impulsive system. Consider the following possibilities for the set Γ when $b = \infty$:*

(i) $\Gamma = \{(t,x) \in \mathbb{R}^{n+1} | \ (t,x) \in M(T_k)\}$ *where $M(t)$ represents a sequence of planes $t = T_k$ with the sequence $\{T_k\}$ satisfying $T_k \to \infty$ as $k \to \infty$, then (3.2) reduces to a switched system with impulses at fixed times (time-dependent impulses). For example,* $\Gamma = \{(t,x) \in \mathbb{R}^2 | \ t = 2k, k = 1, 2, \ldots\}$.

(ii) $\Gamma = \{(t,x) \in \mathbb{R}^{n+1} | \ t = T_k(x)\}$ *which has a countable infinite number of roots, where $T_k(x) < T_{k+1}(x)$ and $\lim_{k\to\infty} T_k(x) = \infty$, then (3.2) reduces to a switched system with impulses at variable times (state-dependent impulses). For example,* $\Gamma = \{(t,x) \in \mathbb{R}^2 | \ t = 2x + 2k, k = 1, 2, \ldots\}$.

(iii) $\Gamma = \{(t,x) \in \mathbb{R}^{n+1} \mid h(t,x) = 0\}$, *where $h \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$, then the impulsive set is the zero level set of a hypersurface in the (t,x)-plane. For example,* $\Gamma = \{(t,x) \in \mathbb{R}^2 | \ x^2 + t^2 = k, k = 1, 2, \ldots\}$.

See [83] for more details.

An admissible switching rule is defined as follows.

Definition 3.3.2. (Admissible switching rule)
A switching rule $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathcal{P}$ and associated switching times $\{t_k\}_{k=1}^N$ are said to be admissible for system (3.2) if the following conditions are satisfied for $1 \leq N \leq \infty$:

(i) σ is piecewise constant (continuous from the right);

(ii) there exists $\eta > 0$ such that for $k = 1, 2, \ldots, N-1$,

$$\inf_k (t_{k+1} - t_k) \geq \eta. \tag{3.4}$$

Denote the set of all such admissible switching rules by $\mathcal{S}$.

Remark 3.3.4. *Equation (3.4) is a nonvanishing dwell-time condition which guarantees that the switching times $\{t_k\}_{k=1}^N$, $1 \leq N \leq \infty$ satisfy $t_0 \leq t_1 < t_2 < \ldots < t_{k-1} < t_k < \ldots < t_N < t_0 + b$.*

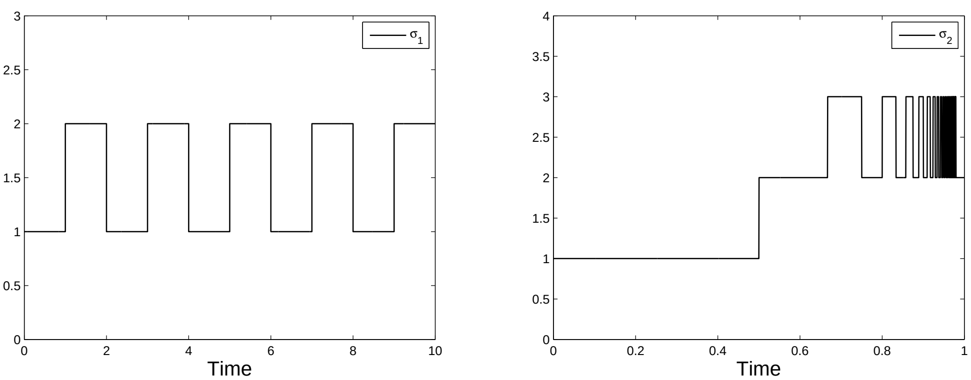
Example 3.3.1. *To illustrate the admissibility and inadmissibility of a switching rule according to the definition, consider the following two rules*

$$\sigma_1 = \begin{cases} 1, & t \in [2k-2, 2k-1), \\ 2, & t \in [2k-1, 2k), \qquad k \in \mathbb{N}, \end{cases} \tag{3.5}$$

and

$$\sigma_2 = \begin{cases} 1, & t \in [0, \frac{1}{2}), \\ 2, & t \in [1 - \frac{1}{2k}, 1 - \frac{1}{2k+1}), \\ 3, & t \in [1 - \frac{1}{2k+1}, 1 - \frac{1}{2k+2}), \qquad k \in \mathbb{N}. \end{cases} \tag{3.6}$$

It is clear that σ_1 satisfies condition (3.4) with $\eta = 1$ and is well-posed. However, it is not possible to choose a nonvanishing dwell-time $\eta > 0$ for σ_2, and hence this is not an admissible switching rule. The switching rule σ_2 exhibits an infinite number of switches in finite time. See Figure 3.1 for an illustration.



(a) $\sigma_1 \in \mathcal{S}$. (b) $\sigma_2 \notin \mathcal{S}$.

Figure 3.1: Example switching rules.

3.4 Local Existence of Solutions

Suppose that x is a function that maps $[t_0 - \tau, t_0 + b)$ to $\mathbb{R}^n$ for $b > 0$, $\tau > 0$. If x is continuous on $[t_0 - \tau, t_0 + b)$ then it follows that x_t is continuous on $[-\tau, 0]$ (see Lemma 2.2.2). However, if x is only piecewise continuous on $[t_0 - \tau, t_0 + b)$, then x_t is not necessarily piecewise continuous on $[-\tau, 0]$, in fact it may be discontinuous everywhere (see [19]). For the same reasons, if $x \in PCB((-\infty, t_0 + b), \widetilde{D})$ then it does not necessarily follow that $x_t \in PCB((-\infty, 0], \widetilde{D})$. This is an important problem that can arise in impulsive systems with time-delays and is problematic when considering classical solutions (functions that are discontinuous everywhere are not Riemann-integrable). This possibility is illustrated in the following example.

Example 3.4.1. *Consider the following function*

$$x(t) = sin(k), \qquad t \in [k, k+1), \qquad k = 0, \pm 1, \pm 2, \ldots$$

The function x is in $PCB(\mathbb{R}, [-1, 1])$. Suppose that $h_1, h_2 \in [0, 0.5]$ and $\delta > 0$ satisfies $\delta > h_2 - h_1 > 0$. Let $s = -h_2$ then

$$\|x(h_2 + s) - x(h_1 + s)\| = \|x(0) - x(h_2 - h_1)\| = |sin(-1)|,$$

and hence

$$\|x_{h_2} - x_{h_1}\|_{PCB} \geq |sin(-1)|.$$

Choose

$$\epsilon = \frac{|sin(-1)|}{2}$$

then for any $\delta > 0$, $|h_2 - h_1| < \delta$ implies $\|x_{h_2} - x_{h_1}\|_{PCB} \geq \epsilon$. Therefore x_t is discontinuous for all $t \in [0, 0.5]$. A similar procedure shows that x_t is discontinuous for all $t \in \mathbb{R}$.

Motivated by this observation, the authors Ballinger and Liu [19] introduced the composite-PC class of functions for impulsive non-switched systems with finite delay. A functional $f(t, x_t)$ is said to be composite-PC if x being piecewise continuous implies that the composite function $v(t) = f(t, x_t)$ is also piecewise continuous. For an idea of functionals that satisfy the composite-PC property, the reader is referred to [19]. We extend the notion to the class PCB as follows.

Definition 3.4.1. *A functional $f : J \times PCB([\alpha, 0], D) \to \mathbb{R}^n$ is composite-PCB if whenever $x \in PCB([t_0 + \alpha, t_0 + b], D)$ and x is continuous at each $t \neq T_k$ in $(t_0, t_0 + b]$ then the composite function $v(t) = f(t, x_t)$ satisfies $v \in PCB([t_0, t_0 + b], D)$.*

It is possible to re-formulate the solution of (3.2) in terms of an integral equation.

Lemma 3.4.1. *Assume that f_i is composite-PCB for each $i \in \mathcal{P}$, assume that $\sigma \in \mathcal{S}$ and assume that $\Gamma \in \mathcal{I}$. Consider a function $x \in PCB([t_0+\alpha, t_0+\gamma], D)$ where $\gamma > 0$ is a constant and $[t_0, t_0+\gamma] \subset J$. Then x is a solution of (3.2) if and only if x satisfies*

$$x(t) = \begin{cases} \phi_0(t-t_0), & \text{for } t \in [t_0+\alpha, t_0], \\ \phi_0(0) + \int_{t_0}^{t} f_{\sigma(s,x(s))}(s, x_s)ds & \\ + \sum_{\{k: t_0 \le T_k \le t\}} g_{\sigma(T_k^-, x(T_k^-))}(T_k, x_{T_k^-}), & \text{for } t \in (t_0, t_0+\gamma]. \end{cases} \tag{3.7}$$

In order to prove local existence of solutions, the following definitions are required.

Definition 3.4.2. *A functional $f : J \times PCB([\alpha, 0], D) \to \mathbb{R}^n$ is continuous in its second variable if for each $t \in J$, $f(t, \psi)$ is a continuous function of ψ in $PCB([\alpha, 0], D)$.*

Definition 3.4.3. *A functional $f : J \times PCB([\alpha, 0], D) \to \mathbb{R}^n$ is quasi-bounded if f is bounded on every set of the form $[t_0, t_0+\gamma] \times PCB([\alpha, 0], \Omega)$, where $\gamma > 0$, $[t_0, t_0+\gamma] \subset J$, and Ω is a closed and bounded subset of D.*

Since the proof method is to adjust classical techniques, we remind the reader of two definitions and the Ascoli-Arzelá lemma.

Definition 3.4.4. *The sequence of functions $\{x_n(t)\}$ defined on $[a, b]$ is uniformly bounded if there exists $N > 0$ such that $\|x_n(t)\| \le N$ for all n and for all $t \in [a, b]$.*

Definition 3.4.5. *The sequence of functions $\{x_n(t)\}$ defined on $[a, b]$ is equicontinuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in [a, b]$, $|t_1 - t_2| < \delta$ implies that $\|x_n(t_1) - x_n(t_2)\| < \epsilon$ for all n.*

Lemma 3.4.2. (Ascoli-Arzelá Lemma)
If $\{x_n(t)\}$ is a uniformly bounded and equicontinuous sequence of functions defined on $[a, b]$ then there exists a subsequence which converges uniformly on $[a, b]$.

We are now in a position to give the first existence result, which extends the work of Liu and Ballinger in [19] where the authors studied the non-switched finite delay case. The idea of the proof is to define a sequence of functions as in [19] and show the sequence has a converging subsequence which satisfies (3.7).

Theorem 3.4.3. (Local Existence)
Assume that $\sigma \in \mathcal{S}$ and $\Gamma \in \mathcal{I}$. Assume that f_i is composite-PCB, quasi-bounded, and continuous in each of its variables for all $i \in \mathcal{P}$. Assume that g_i is continuous in each of its variables for $i \in \mathcal{P}$. Then for each $t_0 \in J$ and $\phi_0 \in PCB([\alpha, 0], D_{\sigma(t_0, x(t_0))})$ there exists a solution of (3.2) on $[t_0+\alpha, t_0+\beta]$ for some $\beta > 0$.

Proof. Since each functional f_i is composite-PCB, Lemma 3.4.1 implies that a function $x \in PCB([t_0+\alpha, t_0+\gamma], \widetilde{D})$, where $\gamma > 0$ is a constant satisfying $[t_0, t_0+\gamma] \subset J$, that experiences discontinuities at the impulsive times $\{T_k\}_{k=1}^N$ with $t_0 \leq T_1 < \ldots < T_N < t_0+\gamma$ is a solution to (3.2) if and only if x satisfies (3.7). Define the following sequence of functions for $j = 1, 2, \ldots,$

$$x^{(j)}(t) = \begin{cases} \phi_0(t-t_0), & \text{for } t \in [t_0+\alpha, t_0], \\ \phi_0(0), & \text{for } t \in (t_0, t_0+\beta/j], \\ \phi_0(0) + \int_{t_0}^{t-\beta/j} f_{\sigma(s,x^{(j)}(s))}(s, x_s^{(j)})ds, & \text{for } t \in (t_0+\beta/j, t_0+\beta], \end{cases}$$

where the constant $\beta > 0$ satisfies

$$0 < \beta \leq \begin{cases} b, & \\ 0.5\min\{b, t_1-t_0, T_1-t_0\}, & t_1 \neq t_0, T_1 \neq t_0, \\ 0.5\min\{b, t_2-t_1, T_1-t_0\}, & t_1 = t_0, \\ 0.5\min\{b, t_1-t_0, T_2-T_1\}, & T_1 = t_0. \end{cases}$$

Next we prove a series of claims in order to show that the sequence $\{x^{(j)}\}$ contains a subsequence that converges to a piecewise continuous function satisfying (3.7) (and hence is a solution of (3.2)).

Claim: If $x^{(j)}$ is initialized on an impulsive set ($T_1 = t_0$) or switching hypersurface ($t_1 = t_0$), it immediately moves off it and there is a positive amount of time before the next switch and/or impulse is applied.

Proof of claim: If $(t_0, x^{(j)}(t_0)) \notin \Gamma$ then each $x^{(j)}(t_0)$ lies outside the impulsive set Γ and there must exist a positive constant δ such that $T_1 \notin (t_0, t_0+\delta]$. If $(t_0, x^{(j)}(t_0)) \in \Gamma$ then the solution is initialized in the impulsive set $\Gamma \in \mathcal{I}$ and there exists a constant $\delta > 0$ such that $(t, x^{(j)}(t)) \notin \Gamma$ for all $t \in (t_0, t_0+\delta]$ and $\|x(t) - \phi_0(0)\| < \delta_1$ for some $\delta_1 > 0$. That is, $x^{(j)}(t)$ cannot remain in the impulsive set for any positive amount of time past the initial time. Similarly, $\sigma \in \mathcal{S}$ implies that $x^{(j)}(t)$ cannot remain on a switching hypersurface for $t \in (t_0, t_0+\delta]$ for some constant $\delta > 0$. By choice of β, the solution does not reach the next switch or impulse in the interval $[t_0, t_0+\beta]$.

Claim: For each $j = 1, 2, \ldots,$ the function $x^{(j)}$ is in $PCB([t_0+\alpha, t_0+\beta], D_{\sigma(t_0,x(t_0))})$.

Proof of claim: For $t \in [t_0+\alpha, t_0+\beta/j]$ the function $x^{(j)}(t)$ is in $PCB([t_0+\alpha, t_0+\beta], D_{\sigma(t_0,x(t_0))})$. It follows from the composite-PCB property of $f_i(t, x_t)$ that the composition of functions $v^{(j)}(t) = f_{\sigma(t_0,x(t_0))}(t, x_t^{(j)})$ satisfies $v^{(j)} \in PCB[t_0+\alpha, t_0+\beta/j)$. Thus,

$$\widetilde{v}^{(j)}(t) = \int_{t_0}^{t-\beta/j} v^{(j)}(s)ds$$

is a continuous function on $(t_0+\beta/j, t_0+2\beta/j]$. If $x^{(j)} \in PCB([t_0+\alpha, t_0+l\beta/j], D_{\sigma(t_0,x(t_0))})$ for any $l \geq 1$ then $v^{(j)} \in PCB((t_0, t_0+l\beta/j, D_{\sigma(t_0,x(t_0))})$ so that $\widetilde{v}^{(j)}(t)$ is continuous on the interval $(t_0, t_0+(l+1)\beta/j]$. For any $\epsilon > 0$ and $j = 1, 2, \ldots$, there exists $l^* > 0$ such that $(l+1)\beta/j > \beta - \epsilon$ for all $l \geq l^*$. Therefore, $x^{(j)} \in PCB([t_0+\alpha, t_0+\beta], D_{\sigma(t_0,x(t_0))})$.

<u>Claim:</u> When restricted to the interval $[t_0, t_0+\beta]$, $x^{(j)}$ is continuous and uniformly bounded and the family of functions $\{x^{(j)}\}$ is equicontinuous.

<u>Proof of claim:</u> For any positive constants a_1, a_2, define

$$\begin{aligned} S(a_1, a_2) := \{ & y \in PCB([t_0+\alpha, t_0+a_1], D_{\sigma(t_0,x(t_0))}) \mid y_{t_0} = \phi_0, \\ & \text{y(t) is continuous and } \|y(t) - \phi_0(0)\| \leq a_2 \text{ for all } t_0 < t \leq t_0 + a_1 \}. \end{aligned}$$

We claim that $x^{(j)} \in S(a_1, a_2)$ for all $j = 1, 2, \ldots$, for some constants a_1, a_2 to be determined. To prove this claim, we require the quasi-boundedness property of $f_i(t, x_t)$: for $t \in [t_0+\alpha, t_0]$, the closure of the range of $\phi_0 : [\alpha, 0] \to D_{\sigma(t_0,x(t_0))}$, denoted Ω, is bounded since $\phi_0 \in PCB$ and is closed by definition. Hence Ω is a compact subset of $\mathbb{R}^n$. Since $f_i(t, x_t)$ is quasi-bounded, there exists a constant $M_1 > 0$ such that $\|f_i(t, x_t)\| \leq M_1$ for all $(t, \phi_0) \in [t_0, t_0+\beta] \times PCB([\alpha, 0], \Omega)$. It is possible to choose $h > 0$ sufficiently small so that

$$cl(\mathcal{B}_h(\phi_0(0))) = \{x \in \mathbb{R}^n : \|x - \phi_0(0)\| \leq h\}$$

is entirely contained in $D_{\sigma(t_0,x(t_0))}$. Since $cl(\mathcal{B}_h(\phi_0(0)))$ is compact, there exists a constant $M_2 > 0$ such that $\|f_i(t, \psi)\| \leq M_2$ for all $\psi \in PCB([\alpha, 0], cl(\mathcal{B}_h(\phi_0(0))) \bigcup \Omega)$ and $t \in [t_0, t_0+\beta]$.

Since $x^{(j)} \in PCB([t_0+\alpha, t_0+\beta], D_{\sigma(t_0,x(t_0))})$ and, in particular, $\bar{x}^{(j)} \in C([t_0, t_0+\beta], D_{\sigma(t_0,x(t_0))})$ where $\bar{x}^{(j)}$ is the restriction of $x^{(j)}$ to $[t_0, t_0+\beta]$, then $f_i(t, x_t)$ being composite-PCB for each $i \in \mathcal{P}$ implies that $v^{(j)}(t) = f_{\sigma(t_0,x(t_0))}(t, x_t^{(j)})$ satisfies $v^{(j)} \in PCB([t_0+\alpha, t_0+\beta], D_{\sigma(t_0,x(t_0))})$. Thus $\|f_{\sigma(t_0,x(t_0))}(t, x_t^{(j)})\| \leq M$, where $M = \max\{M_1, M_2\}$, for $t \in [t_0+\alpha, t_0+\beta]$. For all $j = 1, 2, \ldots$, and $t \in [t_0+\beta/j, t_0+\beta]$,

$$\begin{aligned} \|x^{(j)}(t) - \phi_0(0)\| &= \|\int_{t_0}^{t-\beta/j} f_{\sigma(t_0,x(t_0))}(s, x_s^{(j)}) ds\|, \\ &\leq \int_{t_0}^{t-\beta/j} \|f_{\sigma(t_0,x(t_0))}(s, x_s^{(j)})\| ds, \\ &\leq \int_{t_0}^{t_0+\beta} M ds, \\ &\leq M\beta. \end{aligned}$$

Thus $x^{(j)} \in S(\beta, M\beta)$ for all $j = 1, 2, \ldots$ and $\bar{x}^{(j)}$ is continuous. For $t \in [t_0, t_0 + \beta]$,

$$\begin{aligned}
\|\bar{x}^{(j)}(t)\| &\leq \|\phi_0(0) + \int_{t_0}^{t-\beta/j} f_{\sigma(t_0,x(t_0))}(s, x_s^{(j)})ds\|, \\
&\leq \|\phi_0(0)\| + \int_{t_0}^{t-\beta/j} \|f_{\sigma(t_0,x(t_0))}(s, x_s^{(j)})\|ds, \\
&\leq \|\phi_0(0)\| + \int_{t_0}^{t_0+\beta} M ds, \\
&= \|\phi_0(0)\| + M\beta
\end{aligned}$$

which proves that $\bar{x}^{(j)}$ is uniformly bounded. Finally, for all $h_1, h_2 \in [t_0, t_0 + \beta]$ and for all $j = 1, 2, \ldots$,

$$\|\bar{x}^{(j)}(h_2) - \bar{x}^{(j)}(h_1)\| \leq \| \int_{h_1-\beta/j}^{h_2-\beta/j} f_{\sigma(t_0,x(t_0))}(s, x_s^{(j)})ds\| \leq M|h_2 - h_1|$$

and hence $\{\bar{x}^{(j)}\}$ are equicontinuous on $[t_0, t_0 + \beta]$.

Claim: The sequence $\{\bar{x}^{(j)}\}$ has a uniformly convergent subsequence which converges to a function that satisfies the integral equation (3.7).

Proof of claim: Since the interval $[t_0, t_0 + \beta]$ is compact, it follows from the Ascoli-Arzelá Lemma that there exists a uniformly convergent subsequence, denoted by $\{\bar{x}^{(j_l)}\}$, on the interval $[t_0, t_0 + \beta]$ that converges to a continuous function (denoted by $\bar{x}(t)$) as $l \to \infty$.

We claim that the function

$$\widetilde{x}(t) = \begin{cases} \phi_0(t - t_0), & \text{for } t \in [t_0 + \alpha, t_0], \\ \bar{x}(t), & \text{for } t \in (t_0, t_0 + \beta], \end{cases}$$

is a solution to (3.7). By the arguments outlined above, $\|f_{\sigma(t_0,x(t_0))}(t, x_t^{(j_l)})\| \leq M$ for all $t \in [t_0, t_0 + \beta]$. Also, since $f_{\sigma(t_0,x(t_0))}(t, x_t)$ is continuous in its second variable for fixed t,

$$\lim_{l\to\infty} f_{\sigma(t_0,x(t_0))}(t, x_t^{(j_l)}) = f_{\sigma(t_0,x(t_0))}(t, \lim_{l\to\infty} x_t^{(j_l)}) = f_{\sigma(t_0,x(t_0))}(t, \widetilde{x}(t)).$$

Hence, for all $t \in [t_0, t_0 + \beta]$,

$$\lim_{l\to\infty} \int_{t_0}^{t} f_{\sigma(t_0,x(t_0))}(s, x_s^{(j_l)})ds = \int_{t_0}^{t} f_{\sigma(t_0,x(t_0))}(s, \widetilde{x}_s)ds$$

since the subsequence converges uniformly. Therefore, for all $t \in (t_0 + \beta/j, t_0 + \beta]$:

$$\begin{aligned}
\lim_{l\to\infty} x^{(j_l)}(t) &= \phi_0(0) + \lim_{l\to\infty} \int_{t_0}^{t-\frac{\beta}{j_l}} f_{\sigma(t_0,x(t_0))}(s, x_s^{(j_l)})ds, \\
&= \phi_0(0) + \lim_{l\to\infty} \int_{t_0}^{t} f_{\sigma(t_0,x(t_0))}(s, x_s^{(j_l)})ds \\
&\quad - \lim_{l\to\infty} \int_{t-\frac{\beta}{j_l}}^{t} f_{\sigma(t_0,x(t_0))}(s, x_s^{(j_l)})ds, \\
&= \phi_0(0) + \lim_{l\to\infty} \int_{t_0}^{t} f_{\sigma(t_0,x(t_0))}(s, x_s^{(j_l)})ds, \\
&= \phi_0(0) + \int_{t_0}^{t} f_{\sigma(t_0,x(t_0))}(s, \widetilde{x}_s)ds.
\end{aligned}$$

Since $\bar{x}(t) = \lim_{l\to\infty} x^{(j_l)}(t)$,

$$\widetilde{x}(t) = \begin{cases} \phi_0(t-t_0), & \text{for } t \in [t_0+\alpha, t_0], \\ \phi_0(0) + \int_{t_0}^{t} f_{\sigma(t_0,x(t_0))}(s, \widetilde{x}_s)ds, & \text{for } t \in (t_0, t_0+\beta]. \end{cases}$$

Thus $\widetilde{x}(t)$ satisfies (3.7) and hence is a solution of (3.2). □

Remark 3.4.1. *If the solution is initialized on an impulsive hypersurface or switching hypersurface, the solution immediately moves off it and there is a positive amount of time before the next switch and/or impulse. The sequence of functions $\{x^{(j)}(t)\}$ has a subsequence that converges to a continuous function satisfying (3.7), and hence is a local solution of (3.2). By construction of β in the proof, the local solution exists at least up until the first impulse or switching time. The boundedness property of the space PCB is used to ensure that the closure of the range of an element of PCB is bounded (from this the quasi-boundedness property of the functionals f_i can be employed).*

Motivated by the impulsive set described in (iii) of Remark 3.3.3 and the work of Liu and Ballinger on state-dependent impulses in [111], the following corollary is presented.

Corollary 3.4.4. *Assume that f_i is composite-PCB, quasi-bounded, and continuous in each of its variables for all $i \in \mathcal{P}$. Assume that g_i is continuous in each of its variables for $i \in \mathcal{P}$. Assume that $\sigma \in \mathcal{S}$ and assume that $\Gamma = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid h(t,x) = 0\}$, where $h \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$. Assume that for each $T_k \in J$ and $i \in \mathcal{P}$ there exists a constant $\delta > 0$ such that $[T_k, T_k + \delta] \subset J$ and*

$$\frac{\partial h(t,\psi(0))}{\partial t} + \nabla h(t,\psi(0)) \cdot f_i(t,\psi) \neq 0 \tag{3.8}$$

for all $t \in (T_k, T_k + \delta]$ *and* $\psi \in PCB([\alpha, 0], \widetilde{D})$. *Then for each* $t_0 \in J$ *and* $\phi_0 \in PCB([\alpha, 0], D_{\sigma(t_0, x(t_0))})$ *there exists a solution of* (3.2) *on* $[t_0 + \alpha, t_0 + \beta]$ *for some* $\beta > 0$.

Proof. If $h(t_0, x^{(j)}(t_0)) = 0$ then the solution is initialized on an impulsive hypersurface. Let $m(t) = h(t, x(t))$ for $t \in [t_0, t_0 + \beta]$ then, since h is continuously differentiable and f_i is composite-PCB for all i,

$$\frac{dm}{dt} = \frac{\partial h}{\partial t} + \nabla h \cdot \frac{dx}{dt} = \frac{\partial h}{\partial t} + \nabla h \cdot f_{\sigma(t_0, x(t_0))}(t, x_t).$$

Condition (3.8) implies that $m'(t) \neq 0$ in a small right neighbourhood of the initial time. This implies the existence of a constant $\delta > 0$ such that $m(t)$ is either strictly positive or strictly negative for all $t_0 < t < t_0 + \delta$. It follows that $h(t, x(t)) \neq 0$ for all $t \in (t_0, t_0 + \delta)$ and $\|x(t) - \phi_0(0)\| < \delta_1$ for some $\delta_1 > 0$. Therefore, $x^{(j)}(t)$ cannot remain on the impulsive hypersurface $h(t_0, x^{(j)}(t_0)) = 0$ for any positive amount of time past the initial time. If $h(t_0, x^{(j)}(t_0)) \neq 0$ then each $x^{(j)}(t_0)$ lies entirely between impulsive hypersurfaces. The rest of the proof follows the proof of Theorem 3.4.3. □

3.5 Uniqueness Result

For a solution to be unique, a stronger condition than continuity is required.

Definition 3.5.1. *A functional* $f : J \times PCB([\alpha, 0], D) \to \mathbb{R}^n$ *is Lipschitz in its second variable on* $H \subset J \times PCB([\alpha, 0], D)$ *if there exists* $L > 0$ *such that*

$$\|f(t, \psi_1) - f(t, \psi_2)\| \leq L\|\psi_1 - \psi_2\|_{PCB}$$

for $(t, \psi_1), (t, \psi_2) \in H$.

Definition 3.5.2. *A functional* $f : J \times PCB([\alpha, 0], D) \to \mathbb{R}^n$ *is locally Lipschitz in its second variable if for each* $t_0 \in J$ *and compact subset* Ω *of* D, *there exists* $\gamma > 0$ *such that* $[t_0, t_0 + \gamma] \subseteq J$ *and* f *is Lipschitz in its second variable on* $[t_0, t_0 + \gamma] \times PCB([\alpha, 0], \Omega)$.

Remark 3.5.1. *Two unique solutions of an impulsive switched system with infinite delay may intersect or even merge after some time* $t > t_0$. *This is possibly because of the switching dynamics (even with smooth vector fields* f_i*) or the impulsive effects (e.g. when* g_i *is not one-to-one). This behaviour was noted in [19] for the non-switched finite delay case and remains a possibility here.*

We are now in a position to give a uniqueness result.

Theorem 3.5.1. (Uniqueness)
Assume that $\sigma \in \mathcal{S}$ and $\Gamma \in \mathcal{I}$. Assume that f_i is composite-PCB, continuous in its first variable, and locally Lipschitz in its second variable for all $i \in \mathcal{P}$. Assume that g_i is continuous in each of its variables for $i \in \mathcal{P}$. Then for all $\beta \in (0, b - t_0]$ there exists at most one solution of (3.2) on $[t_0 + \alpha, t_0 + \beta)$.

Proof. Prove by contradiction: assume that there exist two solutions, denoted $x(t) = x(t; t_0, \phi_0)$ and $y(t) = y(t; t_0, \phi_0)$ which satisfy $x, y : [t_0 + \alpha, t_0 + \beta) \to \widetilde{D}$ for some positive constant β satisfying $\beta \le b$. As noted, two distinct solutions of (3.2) could intersect or merge. However, since $x(s) = y(s) = \phi_0(t_0 + s)$ for all $s \in [\alpha, 0]$, there must exist a time $t \in (t_0, t_0 + \beta)$ such that $x(t) \ne y(t)$ for x and y to be distinct solutions. That is, the set $\{t \in (t_0, t_0 + \beta) : x(t) \ne y(t)\}$ is non-empty. Let $h = \inf\{t \in (t_0, t_0 + \beta) : x(t) \ne y(t)\}$. It follows that $x(t) = y(t)$ for all $t \in [t_0 + \alpha, h)$. If $h = t_k$ for some k then $x(h) = x(h^-) = y(h^-) = y(h)$ since $x(t_k) = x(t_k^-)$ at the switching times. On the other hand, if $h = T_k$ for some k then

$$x(h) = x(h^-) + g_{\sigma(h^-, x(h^-))}(h, x_{h^-}) = y(h^-) + g_{\sigma(h^-, y(h^-))}(h, y_{h^-}) = y(h).$$

If $h \ne t_k$ and $h \ne T_k$ for all k, then $x(h) = x(h^-) = y(h^-) = y(h)$. Hence, $x(t) = y(t)$ for all $t \in [t_0 + \alpha, h]$.

For a positive constant a_1, define

$$\Omega(a_1) := \{\text{Range of } x(t), y(t) \mid t \in [t_0 + \alpha, h + a_1]\}.$$

Since $h \in [t_0, t_0 + \beta)$, $x_h \in PCB([\alpha, 0], \widetilde{D})$, $y_h \in PCB([\alpha, 0], \widetilde{D})$ and $x \equiv y$ for all $t \in [t_0 + \alpha, h]$ then it is possible to choose $\bar{a}_1 > 0$ sufficiently small such that $[h, h + \bar{a}_1] \subset J$, $t_k \notin (h, h + \bar{a}_1]$, and $T_k \notin (h, h + \bar{a}_1]$ for all k. Then $cl(\Omega(\bar{a}_1))$ is a compact subset of $\widetilde{D}$ and thus there exist $L_i > 0$ such that $\|f_i(t, \psi_1) - f_i(t, \psi_2)\| \le L_i \|\psi_1 - \psi_2\|_{PCB}$ for all $t \in [t_0 + \alpha, h + \bar{a}_1]$ and $\psi_1, \psi_2 \in PCB([\alpha, 0], cl(\Omega(\bar{a}_1)))$. Since x_t and y_t are in PCB for $t \in [h, h + \bar{a}_1]$ and since f_i is composite-PCB for each $i \in \mathcal{P}$, then $f_\sigma(t, x_t)$ and $f_\sigma(t, y_t)$ are in PCB for $t \in [h, h + \epsilon)$ where $\epsilon > 0$ is chosen sufficiently small so that $\epsilon < \bar{a}_1$ and $L\epsilon < 1$ where $L = \max_{i \in \mathcal{P}}\{L_i\}$. Then for $t \in [h, h + \epsilon]$,

$$\begin{aligned}\|x - y\| &= \|\int_{t_0}^{t} [f_\sigma(s, x_s) - f_\sigma(s, y_s)] ds\|, \\ &\le \int_{h}^{h+\epsilon} \|f_\sigma(s, x_s) - f_\sigma(s, y_s)\| ds,\end{aligned}$$

and so

$$\begin{aligned}\|x-y\| &\leq \int_h^{h+\epsilon} L\|x_s-y_s\|_{PCB}ds,\\ &= L\int_h^{h+\epsilon} \sup_{\alpha\leq\theta\leq 0}\|x(s+\theta)-y(s+\theta)\|ds,\\ &= L\int_h^{h+\epsilon} \sup_{\alpha\leq u\leq s}\|x(u)-y(u)\|ds,\\ &= L\int_h^{h+\epsilon} \sup_{h\leq u\leq s}\|x(u)-y(u)\|ds,\end{aligned}$$

and hence

$$\begin{aligned}\|x-y\| &\leq L\int_h^{h+\epsilon} \sup_{h\leq u\leq h+\epsilon}\|x(u)-y(u)\|ds,\\ &\leq L\epsilon \sup_{h\leq u\leq h+\epsilon}\|x(u)-y(u)\|,\end{aligned}$$

which holds for all $t \in [h, h+\epsilon]$. Therefore $x(t) = y(t)$ for all $t \in [t_0+\alpha, h+\epsilon]$, which is a contradiction to the definition of h.

□

Remark 3.5.2. *If f_i is locally Lipschitz and composite-PCB then it is necessarily continuous in its second variable and quasi-bounded (see [19]).*

3.6 Forward Continuation of Solutions

In this section we extend the local solution to a maximal interval of existence paying special attention to the possibility of an ill-defined switch or impulse (for example, an impulsive effect which sends the trajectory to outside the domain of definition of the active vector field).

Definition 3.6.1. *A switching time $t_k \in J$ is called a terminating switching time (or terminating switch) if $x(t_k) \notin D_{\sigma(t_k,x(t_k))}$.*

Definition 3.6.2. *An impulsive time $T_k \in J$ is called a terminating impulsive time (or terminating impulse) if*

$$x(T_k^-) + g_{\sigma(T_k^-,x(T_k^-))}(T_k, x_{T_k^-}) \notin D_{\sigma(T_k,x(T_k))}.$$

An extended existence result can be given which advances the work of Liu and Ballinger in [111].

Theorem 3.6.1. (Extended Existence)
Assume that $\sigma \in \mathcal{S}$ *and* $\Gamma \in \mathcal{I}$. *Assume that* f_i *is composite-PCB, quasi-bounded, and continuous in each of its variables for al* $i \in \mathcal{P}$. *Assume that* g_i *is continuous in each of its variables for* $i \in \mathcal{P}$. *Then for each* $t_0 \in \mathbb{R}_+$ *and each* $\phi_0 \in PCB([\alpha, 0], D_{\sigma(t_0, x(t_0))})$ *there exists a constant* $\bar{\beta} > 0$ *such that system* (3.2) *has a non-continuable solution on* $[t_0 + \alpha, t_0 + \bar{\beta})$. *If* $\bar{\beta} < b$ *then at least one of the following statements is true:*

(i) $t_0 + \bar{\beta}$ *is a terminating switching time;*

(ii) $t_0 + \bar{\beta}$ *is a terminating impulsive time;*

(iii) for every compact set $\Omega \subset D_{\sigma(t_0+\bar{\beta}^-, x(t_0+\bar{\beta}^-))}$ *there exists a time* $t \in (t_0, t_0 + \bar{\beta})$ *such that* $x(t) \notin \Omega$.

Proof. Since $\sigma \in \mathcal{S}$ and $\Gamma \in \mathcal{I}$, a solution cannot experience an infinite number of switches or impulses on any finite interval. That is, there exist constants $\eta_1 > 0$ and $\eta_2 > 0$ such that

$$\inf_{k=1,2,\ldots,N-1}(t_{k+1} - t_k) \geq \eta_1, \qquad \inf_{k=1,2,\ldots,N-1}(T_{k+1} - T_k) \geq \eta_2.$$

By Theorem 3.4.3, system (3.2) has a solution on the interval $[t_0 + \alpha, t_0 + \beta_1]$ for some constant $\beta_1 > 0$ where $t_0 + \beta_1 \neq t_k$ and $t_0 + \beta_1 \neq T_k$. Let $\bar{t}_0 = t_0 + \beta_1$ and $\bar{\phi}_0 = x_{t_0+\beta_1}$ then by Theorem 3.4.3, there exists a continuation of the original solution of (3.2) on the interval $[t_0 + \alpha, \bar{t}_0 + \beta_2] = [t_0 + \alpha, t_0 + \beta_1 + \beta_2]$ for some constant $\beta_2 > 0$. By adjusting the definition of β in Theorem 3.4.3, continue this process of extending the solution to an interval $[t_0 + \alpha, \widetilde{t}_0)$ where $\widetilde{t}_0 = t_0 + \beta_1 + \ldots + \beta_m$ and $(\widetilde{t}_0, x(\widetilde{t}_0^-)) \in \Gamma$. That is, $\widetilde{t}_0$ is an impulsive time. If the continuation of the solution satisfies

$$x(\widetilde{t}_0^-) + g_{\sigma(\widetilde{t}_0^-, x(\widetilde{t}_0^-))}(\widetilde{t}_0, x_{\widetilde{t}_0^-}) \notin D_{\sigma(\widetilde{t}_0, x(\widetilde{t}_0))}$$

then $\widetilde{t}_0$ is a terminating impulse and x is a non-continuable on $[t_0 + \alpha, t_0 + \bar{\beta})$ where $\bar{\beta} = \beta_1 + \ldots + \beta_m$. If

$$x(\widetilde{t}_0^-) + g_{\sigma(\widetilde{t}_0^-, x(\widetilde{t}_0^-))}(\widetilde{t}_0, x_{\widetilde{t}_0^-}) \in D_{\sigma(\widetilde{t}_0, x(\widetilde{t}_0))}$$

then $\widetilde{t}_0$ is not a terminating impulse and it is possible to extend the solution in the same manner using Theorem 3.4.3 with the solution initialized on an impulsive hypersurface.

On the other hand, if $\widetilde{t_0}$ is a switching time then there are two possibilities: either

$$x(\widetilde{t_0}) \in D_{\sigma(\widetilde{t_0}, x(\widetilde{t_0}))}$$

or

$$x(\widetilde{t_0}) \notin D_{\sigma(\widetilde{t_0}, x(\widetilde{t_0})).}$$

In the former case, the solution can be extended from $\widetilde{t_0}$ using Theorem 3.4.3 with the solution initialized on a switching hypersurface. In the latter case, $\widetilde{t_0}$ acts as a terminating switch and x is the non-continuable solution on $[t_0+\alpha, t_0+\bar{\beta})$ with $\bar{\beta} = \beta_1 + \ldots + \beta_m$. Thus, if a terminating impulse or switch is reached, the system has a non-continuable solution on a maximal interval of existence $[t_0 + \alpha, t_0 + \bar{\beta})$ with $t_0 + \bar{\beta} = t_k$ or $t_0 + \bar{\beta} = T_k$ for some k.

Next we consider the case where neither a terminating switch nor a terminating impulse occurs. Let X be the set consisting of the original solution x as well as all of its continuations, which are constructed as above. To show there is a solution that is a continuation of x and is itself non-continuable (a maximal element in X), we note that $x(t_k) = x(t_k^-)$ at the switching times and no terminating switch or impulse occurs. Then by replacing D by $\widetilde{D}$ and the space PC by PCB in the first part of the proof of Theorem 3.3 in [111], it follows that X has a maximal element and hence there exists $\bar{\beta} > 0$ and a non-continuable solution $x(t)$ of (3.2) on $[t_0+\alpha, t_0+\bar{\beta})$ with $t_0+\bar{\beta} \neq t_k$ and $t_0+\bar{\beta} \neq T_k$. Suppose $\bar{\beta} < b$ and suppose, by contradiction, that there exist a constant $\beta^* > 0$ and a compact set $\Omega_1 \subset \widetilde{D}$ such that $x(t) \in \Omega_1$ for all $t \in [t_0 + \beta^*, t_0 + \bar{\beta})$ where β^* is chosen so that $t_k, T_k \notin [t_0 + \beta^*, t_0 + \bar{\beta})$ for all k. Let

$$\Omega_2 = \{\text{Range of } x(t) \mid t \in [t_0 + \alpha, t_0 + \beta^*]\}.$$

Since the solution $x(t) \in PCB$, it follows that $cl(\Omega_2)$ is compact in the set $\widetilde{D}$. Therefore, for $t \in [t_0 + \alpha, t_0 + \bar{\beta})$, $x(t) \in \Omega = \Omega_1 \cup cl(\Omega_2)$ and Ω is a compact subset of $\widetilde{D}$.

Since each f_i is quasi-bounded, there exists a positive constant M such that $\|f_i(t, \psi)\| \leq M$ for all $i \in \mathcal{P}$ and $(t, \psi) \in (t_0, t_0 + \bar{\beta}) \times PCB([\alpha, 0], \Omega)$. This means that $\|\dot{x}\| \leq M$ for all $t \in [t_0, t_0 + \bar{\beta})$ and

$$w = \lim_{t \to (t_0 + \bar{\beta})^-} x(t)$$

exists and $w \in \Omega$. It follows that the solution $x(t)$ is defined for $[t_0 + \alpha, t_0 + \bar{\beta}]$ and $x(t) \in \Omega$ for all $t \in (-\infty, t_0 + \bar{\beta}]$. Let $\bar{t}_0 = t_0 + \bar{\beta}$ be the new initial time and $\bar{\phi}_0 = x_{t_0+\bar{\beta}}$ be the new initial function, then it is possible to extend the solution past $t_0 + \bar{\beta}$ by applying Theorem 3.4.3 to conclude the existence of $\delta > 0$ such that (3.2) has a solution on $[t_0 + \alpha, t_0 + \bar{\beta} + \delta)$, a contradiction to the assumption made above. It follows that $x(t) \notin \Omega$ for some $t \in (t_0, t_0 + \bar{\beta})$. □

If the switching times are purely time-dependent, $\sigma = \sigma(t) : J \to \mathcal{P}$, then some unique possibilities arise with regards to extended existence.

Definition 3.6.3. *The switching rule $\sigma : J \to \mathcal{P}$ is said to be cyclic if there exists a sequence of times $\{h_i\}_{i=1}^m$ such that $t_0 \le h_i < t_0 + b$ and $\sigma(h_i) = i$ for all $i \in \mathcal{P}$.*

If the switching rule is cyclic then each subsystem is activated at least once on J. If the domains of definition are not connected (i.e. $\bigcup_{i=1}^m D_i$ is not connected), then a terminating switch/impulse is reached or the solution must leave the domain (that is, $\bar{\beta} = \infty$ is not possible). This is captured in the following corollary, which follows directly from Theorem 3.6.1.

Corollary 3.6.2. *Assume that $\sigma \in \mathcal{S}$ and $\Gamma \in \mathcal{I}$. Assume that f_i is composite-PCB, quasi-bounded, and continuous in each of its variables for all $i \in \mathcal{P}$. Assume that g_i is continuous in each of its variables for $i \in \mathcal{P}$. Assume that $\sigma = \sigma(t)$ is cyclic and $\widetilde{D}$ is not connected. Then for each $t_0 \in \mathbb{R}_+$ and each $\phi_0 \in PCB([\alpha, 0], D_{\sigma(t_0, x(t_0))})$ there exists $\bar{\beta} > 0$ such that system (3.2) has a non-continuable solution on $[t_0 + \alpha, t_0 + \bar{\beta})$ and $\bar{\beta} < b$.*

If the domains of definition D_i are simply connected but share no overlapping region, then either each region D_i is visited by the solution trajectory of the switched system or the non-continuable solution terminates before the time $t_0 + b$ (due to a terminating switch or impulse).

Corollary 3.6.3. *Assume that $\sigma \in \mathcal{S}$ and $\Gamma \in \mathcal{I}$. Assume that f_i is composite-PCB, quasi-bounded, and continuous in each of its variables for all $i \in \mathcal{P}$. Assume that g_i is continuous in each of its variables for $i \in \mathcal{P}$. Assume that $\sigma = \sigma(t)$ is cyclic, $\widetilde{D}$ is simply connected, and $\bigcap_{i=1}^m D_i$ is the empty set. Then for each $t_0 \in \mathbb{R}_+$ and each $\phi_0 \in PCB([\alpha, 0], D_{\sigma(t_0, x(t_0))})$ there exists $\bar{\beta} > 0$ such that system (3.2) has a non-continuable solution on $[t_0 + \alpha, t_0 + \bar{\beta})$ and either*

(i) $\bar{\beta} < b$; or

(ii) for all $i \in \mathcal{P}$, there exists a time $h_i \in (t_0, t_0 + b)$ such that $x(h_i) \in D_i$.

If $D_i = \mathbb{R}^n$ for all $i \in \mathcal{P}$ then there is no possibility of a terminating switch or impulse and so an immediate consequence of Theorem 3.6.1 can be given.

Corollary 3.6.4. *Assume that $f_i : [t_0, \infty) \times PCB([\alpha, 0], \mathbb{R}^n) \to \mathbb{R}^n$ and assume that $g_i : [t_0, \infty) \times PCB([\alpha, 0], \mathbb{R}^n) \to \mathbb{R}^n$. Assume that f_i is composite-PCB, quasi-bounded,* *and continuous in each of its variables for all $i \in \mathcal{P}$. Assume that g_i is continuous in each of its variables for $i \in \mathcal{P}$. Assume that $\sigma \in \mathcal{S}$ and $\Gamma \in \mathcal{I}$. Then for each $t_0 \in \mathbb{R}_+$ and each initial function $\phi_0 \in PCB([\alpha, 0], \mathbb{R}^n)$ there exists $\bar{\beta} > 0$ such that system (3.2) has a non-continuable solution on $[t_0 + \alpha, t_0 + \bar{\beta})$ and either*

(i) $\bar{\beta} = \infty$; *or*

(ii) $\lim_{t \to t_0 + \bar{\beta}^-} \|x(t)\| = \infty$.

3.7 Global Existence

To prove a global existence result for HISD with unbounded delay, the Gronwall inequality for piecewise continuous functions (for example, see [111]) is used and a smoothness condition related to $\|x_t\|_{PCB}$ must be given.

Lemma 3.7.1. (Gronwall's Inequality)
Let $v, k \in PC([a, b], \mathbb{R}_+)$, $c \in \mathbb{R}_+$, and

$$v(t) \leq c + \int_a^t v(s)k(s)ds$$

for $t \in [a, b]$. Then it follows that

$$v(t) \leq c \exp\left[\int_a^t k(s)ds\right]$$

for $t \in [a, b]$.

Lemma 3.7.2. *Assume that $x \in PCB((-\infty, t_0 + b], \widetilde{D})$ and define $z(t) = \|x_t\|_{PCB}$ for $t \in [t_0, t_0 + b]$. Then $z \in PC([t_0, t_0 + b], \mathbb{R}_+)$ and the only possible points of discontinuity of $z(t)$ are at discontinuity points of $x(t)$.*

Proof. Let $h_1 \in [t_0, t_0 + b)$ and prove $z(h_1^+) = z(h_1)$. Since $x \in PCB$, it is right-continuous for all $t \in (-\infty, t_0 + b)$. For any $\epsilon > 0$, there exists $\delta \in (0, t_0 + b - h_1)$ such that if $t \in [h_1, h_1 + \delta]$, $\|x(t) - x(h_1)\| < \epsilon$. Then $\|x(t)\| < \|x(h_1)\| + \epsilon$ for all $t \in [h_1, h_2]$ where $h_2 \in (h_1, h_1 + \delta]$. It follows that

$$\|x(t)\| < \sup_{s \leq 0} \|x(h_1 + s)\| + \epsilon$$

for $t \in [h_1, h_2]$. Hence, $\|x(t)\| < z(h_1) + \epsilon$ for $t \in [h_1, h_2]$ which implies that $\|x(t)\| < z(h_1) + \epsilon$ for $t \in (-\infty, h_2]$. Therefore

$$\sup_{s \le 0} \|x(h_2 + s)\| < z(h_1) + \epsilon$$

so that $z(h_2) < z(h_1) + \epsilon$. Since $z(t)$ is a non-decreasing function, $z(h_1) < z(h_2) + \epsilon$ also holds. Thus, $\|z(h_1) - z(h_2)\| < \epsilon$ and z is right-continuous for all $t \in (-\infty, t_0 + b)$. By similar arguments, $z(h_1^-) = z(h_1)$ for all $h_1 \in [t_0, t_0 + b)$ except at any point of discontinuity of $x(t)$. □

Consider the following global existence result which extends the work of Liu and Ballinger in [111].

Theorem 3.7.3. (Global Existence)
Assume that $\sigma \in \mathcal{S}$ and $\Gamma \in \mathcal{I}$. Suppose that $f_i : [t_0, \infty) \times PCB([\alpha, 0], \mathbb{R}^n) \to \mathbb{R}^n$ and $g_i : [t_0, \infty) \times PCB([\alpha, 0], \mathbb{R}^n) \to \mathbb{R}^n$ for all $i \in \mathcal{P}$. Assume that f_i is composite-PCB, quasi-bounded, and continuous in each of its variables for $i \in \mathcal{P}$. Assume that g_i is continuous in each of its variables for $i \in \mathcal{P}$. Assume that there exist functions $h_1, h_2 \in PCB([t_0, \infty), \mathbb{R}_+)$ such that

$$\|f_i(t, \psi)\| \le h_1(t) + h_2(t)\|\psi\|_{PCB}$$

for all $i \in \mathcal{P}$ and $(t, \psi) \in J \times PCB([\alpha, 0], \mathbb{R}^n)$. Then for each $t_0 \in \mathbb{R}_+$ and $\phi_0 \in PCB([\alpha, 0], \mathbb{R}^n)$ there exists a solution of (3.2) on $[t_0 + \alpha, \infty)$.

Proof. System (3.2) has a non-continuable solution according to Theorem 3.6.1 without the possibility of termination by a switch or impulse because $D_i = \mathbb{R}^n$ for all $i \in \mathcal{P}$. Show by contradiction and suppose that the non-continuable solution exists on $[t_0 + \alpha, t_0 + \beta)$ for some $\beta \in (0, \infty)$. It follows from Corollary 3.6.4 that $\|x(t)\| \to \infty$ as $t \to (t_0 + \beta)^-$. Since $h_1, h_2 \in PCB$, there exist positive constants H_1 and H_2 so that $h_1(t) \le H_1$ and $h_2(t) \le H_2$ for all $t \in [t_0 + \alpha, t_0 + \beta]$.

Using the integral form of the solution, for $t \in [t_0, t_0 + \beta)$,

$$\begin{aligned}
\|x(t)\| &\le \|\phi_0(0)\| + \Big\| \sum_{\{k: t_0 \le T_k \le t\}} g_{\sigma(T_k^-, x(T_k^-))}(T_k, x_{T_k^-}) \Big\| + \Big\| \int_{t_0}^t f_{\sigma(s, x(s))}(s, x_s) ds \Big\|, \\
&\le \|\phi_0(0)\| + \sum_{\{k: t_0 \le T_k \le t\}} \|g_{\sigma(T_k^-, x(T_k^-))}(T_k, x_{T_k^-})\| + \int_{t_0}^t \|f_{\sigma(s, x(s))}(s, x_s)\| ds, \\
&\le \|\phi_0(0)\| + \sum_{\{k: t_0 \le T_k \le t\}} \|g_{\sigma(T_k^-, x(T_k^-))}(T_k, x_{T_k^-})\| + \int_{t_0}^t [h_1(s) + h_2(s)\|x_s\|_{PCB}] ds.
\end{aligned}$$

Therefore,

$$\|x_t\|_{PCB} \leq M + H_2 \int_{t_0}^{t} \|x_s\|_{PCB} ds$$

where

$$M = \|\phi_0(0)\| + \sum_{\{k: t_0 \leq T_k \leq t\}} g_{\sigma(T_k^-, x(T_k^-))}(T_k, x_{T_k^-}) + \beta H_1$$

is finite since there are a finite number of impulses in finite time. Let $z(t) = \|x_t\|_{PCB}$ for $t \in [t_0, t_0 + \beta)$. Restricted to the interval $[t_0, t_0 + \beta_1]$ where $0 < \beta_1 < \beta$, it follows that $z \in PC([t_0, t_0 + \beta_1], \mathbb{R}_+)$ by Lemma 3.7.2 if the delay is infinite or by Lemma 3.3 in [111] if the delay is finite. Gronwall's inequality implies $z(t) \leq Me^{H_2 \beta_1}$ and hence

$$\|x(t)\| \leq Me^{H_2 \beta_1}$$

for all $t \in [t_0 + \alpha, t_0 + \beta_1]$. This holds for β_1 arbitrarily close to β. It follows that the solution $\|x(t)\|$ is bounded as $t \to (t_0 + \beta)^-$, a contradiction. Hence the solution $x(t)$ is defined for all $t \geq t_0$. □

Chapter 4

Stability Theory of HISD

Two major areas of research in switched systems theory are finding conditions for stability under arbitrary switching and finding special classes of switching rules which guarantee stability (when stability under arbitrary switching is not possible). The former leads to the idea of a common Lyapunov function (one Lyapunov function that is common to all subsystems). In the second area of research, concepts of dwell-time switching and multiple Lyapunov functions are most often used for systems composed of stable subsystems where no common Lyapunov function can be found (for example, see [65, 101, 102]). The underlying idea is that if the switching is sufficiently slow, stability of the overall switched system can be shown. These notions are applicable to switched systems composed entirely of stable subsystems or those composed of a mixture of stable and unstable modes. In this latter scenario, close attention must be paid to the time spent in the stable subsystems versus the time spent in the unstable subsystems.

4.1 Introduction: Stability under Arbitrary Switching

To ensure a switched system of ODEs is stable under arbitrary switching, a common Lyapunov function approach can be used (as detailed in Section 2.3.2). The same idea holds for a switched system of DDEs if a common Lyapunov function or common Lyapunov functional can be found. Consider the following switched system with finite time-delays and impulses applied at pre-specified moments in time:

$$\begin{cases} \dot{x} = f_\sigma(t, x_t), & t \neq T_k, \\ \Delta x = g_k(t, x(t^-)), & t = T_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $t_0 \in \mathbb{R}_+$ and $\phi_0 \in PC([-\tau, 0], \mathbb{R}^n)$ is the initial function, $\tau > 0$ a constant. The functionals $f_i : \mathbb{R} \times PC([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ are assumed to be sufficiently smooth and satisfy $f_i(t, 0) \equiv 0$ for all $i \in \mathcal{P}$ and $t \in \mathbb{R}$. The switching rule $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$ where t_k are the switching times which satisfy $t_0 < t_1 < \ldots < t_k < \ldots$ with $t_k \to \infty$ as $k \to \infty$. The impulsive functions $g_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are assumed to satisfy $g_k(t, 0) \equiv 0$ for all $k \in \mathbb{N}$ and are assumed to be continuous in each variable. The impulsive times T_k are assumed to satisfy $t_0 < T_1 < \ldots < T_k < \ldots$ with $T_k \to \infty$ as $k \to \infty$. Note that there is no switch or impulse applied at the initial time in this formulation.

Definition 4.1.1. *Let $x(t) = x(t; t_0, x_0)$ be the solution of the switched system (4.1). Then the trivial solution $x = 0$ is said to be*

(i) stable if for all $\epsilon > 0$, $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|\phi_0\|_\tau < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \geq t_0$;

(ii) uniformly stable if δ in (i) is independent of t_0, that is, $\delta(t_0, \epsilon) = \delta(\epsilon)$;

(iii) asymptotically stable if (i) holds and there exists a $\beta > 0$ such that $\|\phi_0\|_\tau < \beta$ implies

$$\lim_{t \to \infty} x(t) = 0;$$

(iv) uniformly asymptotically stable if (ii) holds and there exists a $\beta > 0$, independent of t_0, such that $\|\phi_0\|_\tau < \beta$ implies that for all $\eta > 0$, there exists a $T = T(\eta) > 0$ such that for all $t_0 \in \mathbb{R}_+$, $\|x(t)\| < \eta$ if $t \geq t_0 + T(\eta)$;

(v) exponentially stable if there exist constants $\beta, \gamma, C > 0$ such that if $\|\phi_0\|_\tau < \beta$ then $\|x(t)\| \leq C\|\phi_0\|_\tau e^{-\gamma(t-t_0)}$ for all $t \geq t_0$;

(vi) globally exponentially stable if β in (vii) is arbitrary;

(vii) unstable if (i) fails to hold.

The stability properties in (iii), (iv), (v) are said to be global if they hold for arbitary β.

Note that the notions of uniform stability in Definition 4.1.1 are uniform with respect to the initial time, t_0, and not with respect to the switching rule, σ.

For (4.1) to be stable under arbitrary switching, a necessary condition is that each subsystem is stable. It is straightforward to give common Lyapunov functional and common Lyapunov function results for (4.1) by using results directly from the non-switched literature. Consider the following class of function (for example, see [124]).

Definition 4.1.2. *A function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is said to belong to the class ν_0 if*

(i) V is continuous in each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n$ and for each $x, y \in \mathbb{R}^n$, $t \in [t_{k-1}, t_k)$, $k = 1, 2, \ldots,$

$$\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$$

exists;

(ii) $V(t, x)$ is locally Lipschitzian in all $x \in \mathbb{R}^n$, and $V(t, 0) \equiv 0$ for all $t \geq t_0$.

Since V need not be differentiable, the upper right-hand derivative along the i^{th} subsystem is defined as follows.

Definition 4.1.3. *The upper right-hand derivative of a function $V \in \nu_0$ with respect to the i^{th} subsystem of (4.1) is defined by*

$$D^+V|_i(t, \psi(0)) = \limsup_{h\to 0^+} \frac{1}{h} \left[V(t + h, \psi(0) + hf_i(t, \psi)) - V(t, \psi(0))\right], \tag{4.2}$$

for $(t, \psi) \in \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n)$.

Remark 4.1.1. *If V has continuous partial derivatives, then (4.2) reduces to*

$$D^+V|_i(t, \psi(0)) = \frac{\partial V(t, \psi(0))}{\partial t} + \nabla V(t, \psi(0)) \cdot f_i(t, \psi).$$

A Razumikhin-type theorem can be established using a common Lyapunov function.

Theorem 4.1.1. [184]
Assume that there exist a function $V \in \nu_0$ and constants $p > 0$, $\lambda > 0$, $c_1 > 0$, $c_2 > 0$, $q \geq e^{\lambda\tau}$ and $d_k \geq 0$, $\delta_k \geq 0$ such that for $k \in \mathbb{N}$,

(i) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$;

(ii) along the solution of the i^{th} subsystem of (4.1) for $t \neq T_k$,

$$D^+V|_i(t, \psi(0)) \leq -\lambda V(t, \psi(0))$$

whenever $V(t+s, \psi(s)) \leq qV(t, \psi(0))$ for $s \in [-\tau, 0]$;

(iii) for all $\psi \in PC([-\tau, 0], \mathbb{R}^n)$ and $s \in [-\tau, 0]$,

$$V(T_k, \psi(0) + g_k(T_k, \psi(s))) \leq (1+\delta_k)V(T_k^-, \psi(0)) + d_k V(T_k^- + s, \psi(s));$$

(iv) $\sum_{k=1}^{\infty}(\delta_k + d_k e^{\lambda\tau}) < \infty$.

Then the trivial solution of system (4.1) is globally exponentially stable.

Proof. If the Razumikhin condition holds then $D^+V|_i(t, \psi(0)) \leq -\lambda V(t, \psi(0))$ holds for any subsystem and the Lyapunov function acts as a common Lyapunov function. The result follows immediately from the proof of Theorem 3.1 in [184]. □

Consider the following class of functionals (for example, see [124]).

Definition 4.1.4. *A functional $V : \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_+$ is said to belong to the class ν_{PC}^* if*

(i) V is continuous on $[t_{k-1}, t_k) \times PC([-\tau, 0], \mathbb{R}^n)$ and for all $\psi, \phi \in PC([-\tau, 0], \mathbb{R}^n)$, and $k = 1, 2, \ldots,$

$$\lim_{(t,\psi)\to(t_k^-,\phi)} V(t, \psi) = V(t_k^-, \phi)$$

exists;

(ii) $V(t, \psi)$ is locally Lipschitzian in ψ in each compact set in $PC([-\tau, 0], \mathbb{R}^n)$, and $V(t, 0) \equiv 0$ for all $t \geq t_0$;

(iii) for any $x \in PC([t_0 - \tau, \infty), \mathbb{R}^n)$, $V(t, x_t)$ is continuous for $t \geq t_0$.

A common Lyapunov functional theorem can be given for a system with stabilizing impulses.

Theorem 4.1.2. [124]
Assume that there exist $V_1 \in \nu_0$, $V_2 \in \nu_{PC}^*$, constants $p_1 > 0$, $p_2 > 0$ such that $p_1 \leq p_2$, $\lambda > 0$, $T > 0$, $\zeta > 0$, $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, constants $\delta_k \geq 0$ such that for $k \in \mathbb{N}$

(i) $c_1\|x\|^{p_1} \le V_1(t,x) \le c_2\|x\|^{p_1}$ *and* $0 \le V_2(t,\psi) \le c_3\|\psi\|_\tau^{p_2}$, *for all* $t \ge t_0$, $x \in \mathbb{R}^n$, $\psi \in PC([-\tau,0],\mathbb{R}^n)$*;*

(ii) along the solution of the i^{th} *subsystem of* (4.1) *for* $t \ne T_k$,

$$D^+V|_i(t,\psi) \le \lambda V(t,\psi)$$

where $V(t,x_t) = V_1(t,x) + V_2(t,x_t)$*;*

(iii) $V_1(T_k, x + g_k(T_k,x)) \le \delta_k V_1(T_k^-,x)$ *for all* $x \in \mathbb{R}^n$ *and each* T_k*;*

(iv) $\tau \le T_k - T_{k-1} \le T$ *and* $\ln(\delta_k + c_3/c_1) + \lambda T \le -\zeta T$.

Then the trivial solution of system (4.1) *is exponentially stable.*

Proof. Follows immediately from the proof of Theorem 3.1 in [124] since $D^+V|_i(t,\psi) \le \lambda V(t,\psi)$ holds for any subsystem. □

If each subsystem of a switched system of ordinary differential equations is stable then overall stability is achieved if the switching is not too frequent (see Section 2.3.2). This leads to the idea of dwell-time switching and average dwell-time switching where the switching rule satisfies a certain dwell-time condition to ensure the time spent in each subsystem is sufficiently long. This type of result has been extended to switched systems with time-delays. For example, see the report by Yan and Özbay in [193] and the work by Sun et al. in [173] where dwell-time based switching and average dwell-time based switching is used along with multiple Lyapunov-Razumikhin functions. In the rest of the present chapter, we focus on the constrained dwell-time switching stability of hybrid impulsive systems with distributed delays (HISD) where stability under arbitrary switching may not be possible.

4.2 HISD Composed of Stable and Unstable Modes

The focus of this section is on switched integro-differential equations with distributed delays that are composed of a mixture of stable and unstable subsystems. Both stabilizing impulses as well as disturbance impulsive effects are considered. The main contribution of this section is to extend the current literature by finding verifiable sufficient conditions for the stability of a class of nonlinear HISD composed of stable and unstable modes. In doing so, we use notions from dwell-time switching to ensure that the relationship between the time spent in the unstable modes versus the stable modes is such that the overall system remains stable when the impulsive effects are taken into account.

4.2.1 A Motivating Synchronization Problem

In the paper by Guan et al. [57], the authors considered the synchronization of nonlinear systems in view of potential applications in communication systems. Specifically, the authors considered a drive system

$$\dot{x} = Ax + f(t, x)$$

and a response system

$$\dot{y} = Ay + f(t, y) + u(t, x, y)$$

where $x \in \mathbb{R}^n$ is the state variable for the drive system, $y \in \mathbb{R}^n$ is the state variable for the response system, $A \in \mathbb{R}^{n\times n}$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth vector field $u(t, x, y)$ is the control input. The authors' goal was to use hybrid switching and impulsive control in order to synchronize the drive and response system so that

$$\lim_{t\to\infty} \|y(t) - x(t)\| = 0.$$

The authors constructed the control input as $u = u_1 + u_2$ with

$$\begin{aligned}
u_1(t) &= \sum_{k=1}^{\infty} B_{1k}[y(t) - x(t)]l_k(t), \\
&\qquad \text{where the indicator function } l_k(t) := \begin{cases} 1 & \text{if } t \in [t_{k-1}, t_k), \\ 0 & \text{otherwise,} \end{cases} \\
u_2(t) &= \sum_{k=1}^{\infty} B_{2k}[y(t) - x(t)]\delta(t - t_k^-),
\end{aligned}$$

where B_{1k} and B_{2k} are $n \times n$ constant matrices and $\delta(t)$ is the generalized Dirac delta function. The control $u_1(t)$ is a switching control, while $u_2(t)$ is an impulsive control. The closed-loop response system can be re-written as

$$\begin{cases} \dot{y} = Ay + f(t, y) + B_{1k}(y - x), & t \in [t_{k-1}, t_k), \\ \Delta y = B_{2k}(y(t^-) - x(t^-)), & t = t_k, \\ y(t_0) = y_0, & k \in \mathbb{N}, \end{cases}$$

where $\{t_k\}_{k=1}^{\infty}$ satisfies $t_0 < t_1 < \ldots < t_k < \ldots$ with $t_k \to \infty$ as $k \to \infty$. In this formulation, the switching times coincide with the impulsive times. The synchronization error, $e = y - x$, is governed by the system

$$\begin{cases} \dot{e} = [A + B_{1k}]e + f(t, y) - f(t, x), & t \in [t_{k-1}, t_k), \\ \Delta e = B_{2k} e(t^-), & t = t_k, \\ e(t_0) = e_0, & k \in \mathbb{N}. \end{cases}$$

The main objective is to determine a hybrid and switching control time sequence $\{t_k\}$, and control gain matrices $\{B_{1k}\}$ and $\{B_{2k}\}$ so that the two systems synchronize for large time, that is,

$$\lim_{t \to \infty} \|e(t)\| = 0.$$

Motivated by this problem, the authors Guan et al. analyzed the stability properties of the following nonlinear switched and impulsive system composed of stable and unstable modes:

$$\begin{cases} \dot{x} = A_{i_k} x + F_{i_k}(t, x), & t \in [t_{k-1}, t_k), \\ \Delta x = B_{i_k} x(t^-), & t = t_k, \\ x(t_0) = x_0, & k \in \mathbb{N}, \end{cases}$$

where the index i_k follows a switching rule $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$ where $\mathcal{P} = \{1, 2, \ldots, m\}$. Guan et al. used multiple Lyapunov functions (a different Lyapunov function for each subsystem): $V_i = x^T P_i x$, $i \in \mathcal{P}$, where P_i is a positive definite matrix. The authors assumed that the nonlinear functions $F_i(t, x)$ satisfy a certain condition and then found sufficient conditions for synchronization which depended on the rate of switching, the growth/decay rate of each subsystem, and the impulsive effects.

In the paper by Alwan and Liu [3], the authors analyzed the stability of a time-delay switched system made up of stable and unstable modes:

$$\begin{cases} \dot{x} = A_{i_k} x(t) + B_{i_k} x(t - r), & t \in [t_{k-1}, t_k), \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases}$$

where $r > 0$ is a discrete delay and $\phi_0 \in C([-r, 0], \mathbb{R}^n)$ is the initial function. The authors used Halanay's inequality and multiple Lyapunov functions $V_i = x^T P_i x$ (where each P_i is positive definite) to develop threshold criteria on the model's matrices A_i and B_i which dictated the long-term behaviour based on dwell-time conditions. If the amount of time spent in the stable modes is a particular multiple of the time spent in the unstable modes, stability is guaranteed.

In [194], Yang and Zhu used a switching Halanay-like inequality to study the stability properties of the switched and impulsive system with time-delay

$$\begin{cases} \dot{x} = A_{i_k}x + f_{i_k}(t, x(t-\tau(t))), & t \in [t_{k-1}, t_k), \\ \Delta x = B_k x(t^-), & t = t_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases}$$

where the delay satisfies $0 \leq \tau(t) \leq \tau$ for some constant $\tau > 0$, the initial function is $\phi_0 \in PC([-\tau, 0], \mathbb{R}^n)$, and f_i are sufficiently smooth and satisfy $f_i^T(t,x)y \leq L_i x^T y$ for all $x, y \in \mathbb{R}^n$ for some $L_i \geq 0$. The authors used a common Lyapunov function $V = x^T x$ and found sufficient dwell-time conditions.

Zhu [204] analyzed the following impulsive and switched system with delay

$$\begin{cases} \dot{x} = f_{i_k}(x(t), x(t-\tau_{i_k})), & t \in [t_{k-1}, t_k), \\ \Delta x = h_{i_k}(x(t^-)), & t = t_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases}$$

where $0 \leq \tau_i \leq \tau$ for $i \in \mathcal{P}$, $\phi_0 \in PC([-\tau, 0], \mathbb{R}^n)$, i_k follows a switching rule σ, and $f_i(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous vector-valued functions. Then Zhu proved a result using multiple Lyapunov functions $a(\|x\|) \leq V_i \leq b(\|x\|)$ where $a, b \in \mathcal{K}$. The author then applied the results to

$$\begin{cases} \dot{x} = A_{i_k}x(t) + B_{i_k}x(t-\tau_{i_k}), & t \in [t_{k-1}, t_k), \\ \Delta x = H_{i_k}x(t^-), & t = t_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases}$$

to establish easily verifiable sufficient conditions for uniform asymptotic stability using $V_i = x^T P_i x$.

Finally, Niamsup [147] and Zhang et al. [202] studied the stability of switched systems with time-delays using a generalized Halanay's inequality along with a variation of parameters approach. In particular, Niamsup investigated the system

$$\dot{x} = A_{i_k}(t)x(t) + B_{i_k}(t)x(t-\tau(t)) + \int_{t_0}^{t} f_{i_k}(t-s)x(s)ds,$$

for $t \in [t_{k-1}, t_k)$ where $x \in \mathbb{R}^n$, $A_i, B_i \in \mathbb{R}^{n \times n}$, $f_i \in C(\mathbb{R}_+, \mathbb{R}^{n \times n})$ and $0 \leq \tau(t) \leq \tau$ for some constant $\tau > 0$.

Here we extend the aforementioned reports by considering switched systems with distributed delays, nonlinear perturbations, and impulses. Consider the following HISD:

$$\begin{cases} \dot{x} = A_{i_k}x(t) + B_{i_k}x(t-r) + C_{i_k}\displaystyle\int_{t-\tau}^{t} x(s)ds + F_{i_k}(t, x_t), & t \in [t_{k-1}, t_k), \\ \Delta x = E_{i_k}x(t^-) + G_{i_k}\displaystyle\int_{t-\tau}^{t} x(s)ds, & t = t_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases} \tag{4.3}$$

where $x \in \mathbb{R}^n$ is the state; the index i_k follows the switching rule $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$; $x_t \in PC([-\bar{\tau}, 0], \mathbb{R}^n)$ is defined as $x_t(s) = x(t+s)$ for $s \in [-\bar{\tau}, 0]$ where $\bar{\tau} = \max\{r, \tau\}$; and where $\phi_0 \in PC([-\bar{\tau}, 0], \mathbb{R}^n)$ is the initial function. Note that PC is used since the delay is finite. The real $n \times n$ matrices A_i, B_i, C_i, and the family of functionals $F_i(t, x_t)$ are parameterized by the finite set $\mathcal{P}$. For each $i \in \mathcal{P}$, assume that F_i is composite-PC and locally Lipschitz, then it follows from Chapter 3 that each family in (4.3) has a unique solution. Each functional $F_i : \mathbb{R} \times PC([-\bar{\tau}, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is also assumed to satisfy $F_i(t, 0) \equiv 0$ for all $t \geq t_0$. The impulsive switching times are assumed to satisfy $t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ with $t_k \to \infty$ as $k \to \infty$.

Given the matrices A_i, B_i, C_i, E_i, G_i, and the nonlinear functionals F_i, the goal is to determine classes of switching/impulsive times $\{t_k\}$ so that the origin of (4.13) is asymptotically stable. To study this problem, switching Halanay-like inequalities are developed (Section 4.2.2) and are then applied to a set of Lyapunov functions $V_i = x^T P_i x$ (Section 4.2.3).

4.2.2 Switching Halanay-like Inequalities

In [204], Zhu used the following Halanay-like lemma to study switched system stability.

Lemma 4.2.1. [204]
Assume that $\beta, \alpha > 0$ and $u : [t_0 - \tau, \infty) \to \mathbb{R}_+$ satisfies the following delay differential inequality:

$$\dot{u}(t) \leq \beta \|u_t\|_\tau - \alpha u(t), \qquad t \geq t_0.$$

If $\beta - \alpha \geq 0$ then

$$u(t) \leq \|u_{t_0}\|_\tau e^{(\beta-\alpha)(t-t_0)}, \qquad t \geq t_0.$$

If $\beta - \alpha < 0$, then there exists a positive constant η satisfying $\eta + \beta e^{\eta\tau} - \alpha < 0$ such that

$$u(t) \leq \|u_{t_0}\|_\tau e^{-\eta(t-t_0)}, \qquad t \geq t_0.$$

Halanay-like inequalities have been generalized to include switching (for example, [194]), time-varying parameters (for example, [147, 202]), and impulsive effects (for example, [189, 192]). Here we extend the results for impulsive and switched differential inequalities.

Given a switched system composed of a mixture of stable and unstable modes, we denote $\mathcal{P}_s$ to be the set of modes that are stable and $\mathcal{P}_u$ the set of modes that are unstable. That is, $\mathcal{P} = \mathcal{P}_u \bigcup \mathcal{P}_s$ and $\mathcal{P}_u \subseteq \mathcal{P}$ and $\mathcal{P}_s \subseteq \mathcal{P}$. Consider the following dwell-time switching notions [56, 57]: let $T_i(t_0, t)$ be the Lebesgue measure[1] of the total activation time of the i^{th} subsystem on the interval $[t_0, t]$. Since there are m modes, it follows that $\bigcup_{i=1}^m T_i(t_0, t) = [t_0, t]$. Denote $\Phi_i(t_0, t)$ to be the number of switching times such that $\sigma(t_k) = i$ for $t_k \in [t_0, t)$ (i.e. the total number of activations of the i^{th} mode on the interval).

Example 4.2.1. *For the switching rule in Figure 4.1,*

$$T_1(0,5) = 2, \qquad T_1(0,4) = 1, \qquad T_2(3,3.5) = 0.5, \qquad T_3(0,5) = 2$$
$$\Phi_1(0,5) = 2, \qquad \Phi_1(0,4) = 1, \qquad \Phi_2(3,3.5) = 1, \qquad \Phi_3(0,5) = 1.$$

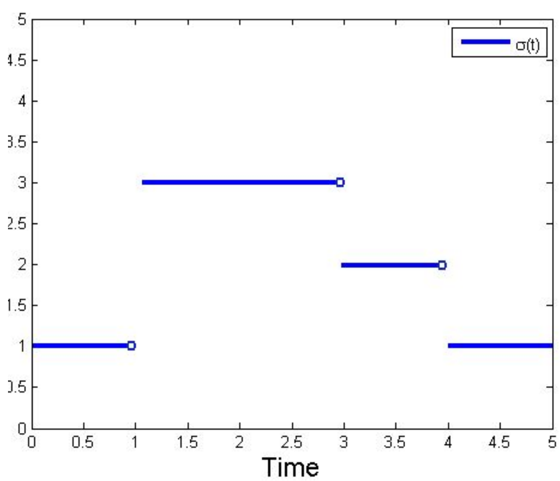


Figure 4.1: Example of a switching rule σ with switch times $t_k = 1, 3, 4$ and $\mathcal{P} = \{1, 2, 3\}$.

The Halanay-like inequality can be extended as follows.

Proposition 4.2.2. *Assume that $\beta_i \geq 0$ and $\alpha_i \geq 0$ for $i \in \mathcal{P}$. Assume that $u : [t_0 - \tau, \infty) \to \mathbb{R}_+$ satisfies the following switching delay differential inequality:*

$$\dot{u}(t) \leq \beta_\sigma \|u_t\|_\tau - \alpha_\sigma u(t).$$

[1] See page 582 of [146].

Let $\mathcal{P}_u = \{i \in \mathcal{P} : \lambda_i \geq 0\}$ *and* $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i < 0\}$ *where* $\lambda_i = \beta_i - \alpha_i$*, and for* $i \in \mathcal{P}_s$*,* $\eta_i > 0$ *are chosen so that*

$$\eta_i + \beta_i e^{\eta_i \tau} - \alpha_i < 0.$$

Then for $t \geq t_0$*,* $u(t)$ *satisfies*

$$u(t) \leq \|u_{t_0}\|_\tau \exp\left[\sum_{i\in\mathcal{P}_u} \lambda_i T_i(t_0,t) - \sum_{i\in\mathcal{P}_s} \eta_i (T_i(t_0,t) - \Phi_i(t_0,t)\tau)\right], \tag{4.4}$$

Proof. By Lemma 4.2.1 it follows that for $t \in [t_0, t_1)$,

$$u(t) \leq \begin{cases} \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t-t_0)}, & i_1 \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t-t_0)}, & i_1 \in \mathcal{P}_s, \end{cases}$$

where $\|u_{t_0}\|_\tau = \sup_{-\tau\leq s\leq 0} \|u(t_0+s)\| = \sup_{-\tau\leq s\leq 0} u(t_0+s)$. Hence,

$$u(t_1) \leq \begin{cases} \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)}, & i_1 \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-t_0)}, & i_1 \in \mathcal{P}_s, \end{cases}$$

and

$$\|u_{t_1}\|_\tau \leq \begin{cases} \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)}, & i_1 \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)}, & i_1 \in \mathcal{P}_s. \end{cases}$$

For $t \in [t_1, t_2)$, Lemma 4.2.1 implies that

$$u(t) \leq \begin{cases} \|u_{t_1}\|_\tau e^{\lambda_{i_1}(t-t_0)}, & i_2 \in \mathcal{P}_u, \\ \|u_{t_1}\|_\tau e^{-\eta_{i_1}(t-t_0)}, & i_2 \in \mathcal{P}_s. \end{cases}$$

so that

$$u(t) \leq \begin{cases} \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)} e^{\lambda_{i_2}(t-t_1)}, & i_1 \in \mathcal{P}_u, \quad i_2 \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)} e^{-\eta_{i_2}(t-t_1)}, & i_1 \in \mathcal{P}_u, \quad i_2 \in \mathcal{P}_s, \\ \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)} e^{\lambda_{i_2}(t-t_1)}, & i_1 \in \mathcal{P}_s, \quad i_2 \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)} e^{-\eta_{i_2}(t-t_1)}, & i_1 \in \mathcal{P}_s, \quad i_2 \in \mathcal{P}_s. \end{cases}$$

Therefore,

$$\|u_{t_2}\|_\tau \leq \begin{cases} \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)} e^{\lambda_{i_2}(t_2-t_1)}, & i_1 \in \mathcal{P}_u, \quad i_2 \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)} e^{-\eta_{i_2}(t_2-\tau-t_1)}, & i_1 \in \mathcal{P}_u, \quad i_2 \in \mathcal{P}_s, \\ \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)} e^{\lambda_{i_2}(t_2-t_1)}, & i_1 \in \mathcal{P}_s, \quad i_2 \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)} e^{-\eta_{i_2}(t_2-\tau-t_1)}, & i_1 \in \mathcal{P}_s, \quad i_2 \in \mathcal{P}_s. \end{cases}$$

Assume it holds on $[t_{k-1}, t_k)$:

$$u(t) \le \begin{cases} \|u_{t_0}\|_\tau \left(\prod_{i\in\mathcal{P}_u} e^{\lambda_i T_i(t_0,t_{k-1})} \prod_{i\in\mathcal{P}_s} e^{-\eta_i[T_i(t_0,t_{k-1})-\Phi_i(t_0,t_{k-1})\tau]}\right) e^{\lambda_{i_k}(t-t_{k-1})}, & i_k \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau \left(\prod_{i\in\mathcal{P}_u} e^{\lambda_i T_i(t_0,t_{k-1})} \prod_{i\in\mathcal{P}_s} e^{-\eta_i[T_i(t_0,t_{k-1})-\Phi_i(t_0,t_{k-1})\tau]}\right) e^{-\eta_{i_k}(t-t_{k-1})}, & i_k \in \mathcal{P}_s. \end{cases}$$

Then

$$\|u_{t_k}\|_\tau \le \begin{cases} \|u_{t_0}\|_\tau \left(\prod_{i\in\mathcal{P}_u} e^{\lambda_i T_i(t_0,t_{k-1})} \prod_{i\in\mathcal{P}_s} e^{-\eta_i[T_i(t_0,t_{k-1})-\Phi_i(t_0,t_{k-1})\tau]}\right) e^{\lambda_{i_k}(t_k-t_{k-1})}, & i_k \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau \left(\prod_{i\in\mathcal{P}_u} e^{\lambda_i T_i(t_0,t_{k-1})} \prod_{i\in\mathcal{P}_s} e^{-\eta_i[T_i(t_0,t_{k-1})-\Phi_i(t_0,t_{k-1})\tau]}\right) e^{-\eta_{i_k}(t_k-\tau-t_{k-1})}, & i_k \in \mathcal{P}_s. \end{cases}$$

And hence,

$$\|u_{t_k}\|_\tau \le \|u_{t_0}\|_\tau \left(\prod_{i\in\mathcal{P}_u} e^{\lambda_i T_i(t_0,t_k)} \prod_{i\in\mathcal{P}_s} e^{-\eta_i[T_i(t_0,t_k)-\Phi_i(t_0,t_k)\tau]} \right).$$

Finally, for $[t_k, t_{k+1})$, by Lemma 4.2.1,

$$\begin{aligned} u(t) &\le \begin{cases} \|u_{t_k}\|_\tau e^{\lambda_{i_{k+1}}(t-t_k)}, & i_{k+1} \in \mathcal{P}_u, \\ \|u_{t_k}\|_\tau e^{-\eta_{i_{k+1}}(t-t_k)}, & i_{k+1} \in \mathcal{P}_s. \end{cases} \\ &\le \begin{cases} \|u_{t_0}\|_\tau \left(\prod_{i\in\mathcal{P}_u} e^{\lambda_i T_i(t_0,t_k)} \prod_{i\in\mathcal{P}_s} e^{-\eta_i[T_i(t_0,t_k)-\Phi_i(t_0,t_k)\tau]}\right) e^{\lambda_{i_{k+1}}(t-t_k)}, & i_{k+1} \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau \left(\prod_{i\in\mathcal{P}_u} e^{\lambda_i T_i(t_0,t_k)} \prod_{i\in\mathcal{P}_s} e^{-\eta_i[T_i(t_0,t_k)-\Phi_i(t_0,t_k)\tau]}\right) e^{-\eta_{i_{k+1}}(t-t_k)}, & i_{k+1} \in \mathcal{P}_s, \end{cases} \\ &\le \begin{cases} \|u_{t_0}\|_\tau \prod_{i\in\mathcal{P}_u} e^{\lambda_i T_i(t_0,t)} \prod_{i\in\mathcal{P}_s} e^{-\eta_i[T_i(t_0,t)-\Phi_i(t_0,t)\tau]}, & i_{k+1} \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau \prod_{i\in\mathcal{P}_u} e^{\lambda_i T_i(t_0,t)} \prod_{i\in\mathcal{P}_s} e^{-\eta_i[T_i(t_0,t)-\Phi_i(t_0,t)\tau]}, & i_{k+1} \in \mathcal{P}_s. \end{cases} \end{aligned}$$

as required. □

Remark 4.2.1. *Note that if $\lambda_i = \beta_i - \alpha_i < 0$ for $i \in \mathcal{P}$ then it is always possible to choose $\eta_i > 0$ satisfying $\eta_i + \beta_i e^{\eta_i \tau} - \alpha_i < 0$. Let $F(\eta_i) = \eta_i + \beta_i e^{\eta_i \tau} - \alpha_i$, then $F(0) = \beta_i - \alpha_i < 0$ and $F'(\eta_i) = 1 + \beta_i \tau e^{\eta_i \tau} > 0$. By continuity of F, there exists $\eta_i^* > 0$ such that $F(\eta_i^*) = 0$ and η_i can be chosen as $0 < \eta_i < \eta_i^*$.*

Next we consider the case when the switching rule is periodic. Let $h_k = t_k - t_{k-1}$ and assume that $h_{k+m} = h_k$. Assume that $\beta_{i_k} = \beta_k$ and $\alpha_{i_k} = \alpha_k$ for $t \in [t_{k-1}, t_k)$. Assume that $\beta_k = \beta_{k+m}$ and $\alpha_k = \alpha_{k+m}$. Denote one period of the switching rule by $\omega = h_1 + \ldots + h_m$. Denote the set of all such periodic switching rules by $\mathcal{S}_{\text{periodic}} \subset \mathcal{S}$.

Proposition 4.2.3. *Assume that $\beta_i \ge 0$ and $\alpha_i \ge 0$ for $i \in \mathcal{P}$. Assume that $u : [t_0 - \tau, \infty) \to \mathbb{R}_+$ satisfies the following switching delay differential inequality:*

$$\dot{u}(t) \le \beta_\sigma \|u_t\|_\tau - \alpha_\sigma u(t).$$

Let $\mathcal{P}_u = \{i \in \mathcal{P} : \lambda_i \geq 0\}$ *and* $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i < 0\}$ *where* $\lambda_i = \beta_i - \alpha_i$, *and for* $i \in \mathcal{P}_s$, $\eta_i > 0$ *are chosen so that*

$$\eta_i + \beta_i e^{\eta_i \tau} - \alpha_i < 0.$$

If $\sigma \in \mathcal{S}_{periodic}$ *then* $u(t)$ *is bounded on any compact interval and*

$$u(t_0 + j\omega) \leq \|u_{t_0}\|_\tau \Lambda^j \tag{4.5}$$

for any positive integer j *where*

$$\Lambda = \exp\left[\sum_{i\in\mathcal{P}_u} \lambda_i h_i - \sum_{i\in\mathcal{P}_s} \eta_i (h_i - \tau)\right].$$

Proof. The boundedness of $u(t)$ on any compact interval follows immediately from Theorem 4.2.2. From equation (4.4), for $j = 1, 2, \ldots,$

$$\begin{aligned} u(t_0 + j\omega) \leq & \|u_{t_0}\|_\tau \exp[\sum_{i\in\mathcal{P}_u} \lambda_i T_i(t_0, t_0 + j\omega) - \sum_{i\in\mathcal{P}_s} \eta_i (T_i(t_0, t_0 + j\omega) - \Phi_i(t_0, t_0 + j\omega)\tau)], \\ = & \|u_{t_0}\|_\tau \exp[\sum_{i\in\mathcal{P}_u} \lambda_i j T_i(t_0, t_0 + \omega) - \sum_{i\in\mathcal{P}_s} \eta_i j (T_i(t_0, t_0 + \omega) - \Phi_i(t_0, t_0 + \omega)\tau)], \\ = & \|u_{t_0}\|_\tau \exp[j \sum_{i\in\mathcal{P}_u} \lambda_i h_i - j \sum_{i\in\mathcal{P}_s} \eta_i (h_i - \tau)], \\ = & \|u_{t_0}\|_\tau \Lambda^j, \end{aligned}$$

since $\sigma \in \mathcal{S}_{\text{periodic}}$ implies that $T_i(t_0, t_0 + j\omega) = jT_i(t_0, t_0 + \omega)$ and $\Phi_i(t_0, t_0 + j\omega) = j\Phi_i(t_0, t_0 + \omega)$. □

The Halanay-like inequality can be extended to switched impulsive systems. Denote $N(t_0, t)$ to be the total number of impulses of a system on the interval $[t_0, t]$.

Proposition 4.2.4. *Assume that* $\beta_i \geq 0$ *and* $\alpha_i \geq 0$ *for* $i \in \mathcal{P}$. *Assume that* $\delta_k \geq 0$ *and* $h_k \geq 0$ *for* $k \in \mathbb{N}$. *Assume that* $u : [t_0 - \tau, \infty) \to \mathbb{R}_+$ *satisfies the following switched impulsive delay differential inequality:*

$$\begin{cases} \dot{u}(t) \leq \beta_\sigma \|u_t\|_\tau - \alpha_\sigma u(t), & t \neq T_k, \\ u(t) \leq d_k u(t^-) + h_k \|u_t\|_\tau, & t = T_k, \quad k \in \mathbb{N}. \end{cases} \tag{4.6}$$

Let $\mathcal{P}_u = \{i \in \mathcal{P} : \lambda_i \geq 0\}$ *and* $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i < 0\}$ *where*

$$\lambda_i = \beta_i \sup_{k\in\mathbb{N}} \left\{\frac{1}{\delta_k}, 1\right\} - \alpha_i,$$

$$\delta_k = d_k + h_k e^{\xi\tau},$$

and $\xi = \max_{i\in\mathcal{P}_s}\{\xi_i\}$*, where* $\xi_i > 0$ *is chosen for* $i \in \mathcal{P}_s$ *such that* $\xi_i + \beta_i e^{\xi_i\tau} - \alpha_i < 0$*. For* $i \in \mathcal{P}_s$*, choose* $\eta_i > 0$ *such that*

$$\eta_i + \beta_i \sup_{k\in\mathbb{N}}\left\{\frac{1}{\delta_k}, 1\right\} e^{\eta_i\tau} - \alpha_i < 0.$$

Assume that $t_k - t_{k-1} \geq \tau$ *and* $T_k - T_{k-1} \geq \tau$ *for* $k \in \mathbb{N}$*. Then for* $t \geq t_0$*,* $u(t)$ *satisfies*

$$u(t) \leq \|u_{t_0}\|_\tau \left(\prod_{j=1}^{N(t_0,t)} \delta_j\right) \exp\left\{\sum_{i\in\mathcal{P}_u} \lambda_i T_i(t_0,t) - \sum_{i\in\mathcal{P}_s} \eta_i[T_i(t_0,t) - \Phi_i(t_0,t)\tau]\right\}. \tag{4.7}$$

Proof. First we consider the case when $t_k = T_k$ for $k \in \mathbb{N}$, that is, the impulse times are the same as the switching times. It immediately follows from Lemma 4.2.1 that for $t \in [t_0, t_1)$,

$$u(t) \leq \begin{cases} \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t-t_0)}, & i_1 \in \mathcal{P}_u, \\ \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t-t_0)}, & i_1 \in \mathcal{P}_s, \end{cases}$$

where $\|u_{t_0}\|_\tau = \sup_{-\tau\leq s\leq 0}\|u(t_0+s)\|$.

Suppose that $t \in [t_1, t_2)$ and suppose that $i_1 \in \mathcal{P}_u$ and $i_2 \in \mathcal{P}_s$. Then we claim that

$$u(t) \leq \delta_1 \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)-\eta_{i_2}(t-t_1)} =: w(t)$$

for all $t \in [t_1, t_2)$. If the claim is not true, then there exists a time $t^* \in [t_1, t_2)$ such that $u(t^*) = w(t^*)$, $u(t) \leq w(t)$ for all $t \in [t_1, t^*)$ and for any $\epsilon > 0$ there exists a time $t_\epsilon \in (t^*, t^*+\epsilon)$ such that $u(t_\epsilon) > w(t_\epsilon)$. Note that $\max_{i\in\mathcal{P}_s}\eta_i \leq \max_{i\in\mathcal{P}_s}\xi_i$ by construction of η_i and ξ_i. Thus,

$$\begin{aligned}
\dot{u}(t^*) \leq& \beta_{i_2} \sup_{-\tau\leq s\leq 0} u(t^*+s) - \alpha_{i_2} u(t^*), \\
\leq& \beta_{i_2} \sup_{k\in\mathbb{N}}(\delta_k, 1)\|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)-\eta_{i_2}(t^*-\tau-t_1)} - \alpha_{i_2}\delta_1\|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)-\eta_{i_2}(t^*-t_1)}, \\
\leq& \left[\beta_{i_2}\sup_{k\in\mathbb{N}}(1/\delta_k, 1)e^{\eta_{i_2}\tau} - \alpha_{i_2}\right]\delta_1\|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)-\eta_{i_2}(t^*-t_1)}, \\
\leq& -\eta_{i_2} w(t^*), \\
=& \dot{w}(t^*).
\end{aligned}$$

This is a contradiction as no such t_ϵ can exist, hence the claim holds.

If $i_1 \in \mathcal{P}_u$ and $i_2 \in \mathcal{P}_u$ then we claim that for all $t \in [t_1, t_2)$:

$$u(t) \leq \delta_1 \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)+\lambda_{i_2}(t-t_1)} =: w(t).$$

Assume that the claim is not true, then there exists a time $t^* \in [t_1, t_2)$ such that $u(t^*) = w(t^*)$, $u(t) \leq w(t)$ for all $t \in [t_1, t^*)$ and for any $\epsilon > 0$ there exists a time $t_\epsilon \in (t^*, t^* + \epsilon)$ such that $u(t_\epsilon) > w(t_\epsilon)$.

$$\begin{aligned}
\dot{u}(t^*) \leq& \beta_{i_2} \sup_{-\tau \leq s \leq 0} u(t^* + s) - \alpha_{i_2} u(t^*), \\
\leq& \beta_{i_2} \sup_{k \in \mathbb{N}} (\delta_k, 1) \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)+\lambda_{i_2}(t^*-t_1)} \\
&- \alpha_{i_2} \delta_1 \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)+\lambda_{i_2}(t^*-t_1)}, \\
\leq& \left[\beta_{i_2} \sup_{k \in \mathbb{N}} (1/\delta_k, 1) - \alpha_{i_2} \right] \delta_1 \|u_{t_0}\|_\tau e^{\lambda_{i_1}(t_1-t_0)+\lambda_{i_2}(t^*-t_1)}, \\
\leq& \lambda_{i_2} w(t^*), \\
=& \dot{w}(t^*),
\end{aligned}$$

which is a contradiction to the claim.

Suppose that $i_1 \in \mathcal{P}_s$ and $i_2 \in \mathcal{P}_s$ then we claim that for all $t \in [t_1, t_2)$,

$$u(t) \leq \delta_1 \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)-\eta_{i_2}(t-t_1)} =: w(t).$$

If the claim is not true, then there exists a time $t^* \in [t_1, t_2)$ such that $u(t^*) = w(t^*)$, $u(t) \leq w(t)$ for all $t \in [t_1, t^*)$ and for any $\epsilon > 0$ there exists a time $t_\epsilon \in (t^*, t^* + \epsilon)$ such that $u(t_\epsilon) > w(t_\epsilon)$.

$$\begin{aligned}
\dot{u}(t^*) \leq& \beta_{i_2} \sup_{-\tau \leq s \leq 0} u(t^* + s) - \alpha_{i_2} u(t^*), \\
\leq& \beta_{i_2} \sup_{k \in \mathbb{N}} (\delta_k, 1) \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)-\eta_{i_2}(t^*-\tau-t_1)} \\
&- \alpha_{i_2} \delta_1 \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)-\eta_{i_2}(t^*-t_1)}, \\
\leq& \left[\beta_{i_2} \sup_{k \in \mathbb{N}} (1/\delta_k, 1) e^{\eta_{i_2}\tau} - \alpha_{i_2} \right] \delta_1 \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)-\eta_{i_2}(t^*-t_1)}, \\
\leq& -\eta_{i_2} w(t^*), \\
=& \dot{w}(t^*).
\end{aligned}$$

This is a contradiction.

Finally, if $i_1 \in \mathcal{P}_s$ and $i_2 \in \mathcal{P}_u$ then we claim that for all $t \in [t_1, t_2)$,

$$u(t) \leq \delta_1 \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)+\lambda_{i_2}(t-t_1)} =: w(t).$$

If not, then there exists a time $t^* \in [t_1, t_2)$ such that $u(t^*) = w(t^*)$, $u(t) \leq w(t)$ for all $t \in [t_1, t^*)$ and for any $\epsilon > 0$ there exists a time $t_\epsilon \in (t^*, t^* + \epsilon)$ such that $u(t_\epsilon) > w(t_\epsilon)$.

$$\begin{aligned}
\dot{u}(t^*) \leq& \beta_{i_2} \sup_{-\tau \leq s \leq 0} u(t^* + s) - \alpha_{i_2} u(t^*), \\
\leq& \beta_{i_2} \sup_{k \in \mathbb{N}} (\delta_k, 1) \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)+\lambda_{i_2}(t^*-t_1)} \\
& - \alpha_{i_2} \delta_1 \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)+\lambda_{i_2}(t^*-t_1)}, \\
\leq& \left[\beta_{i_2} \sup_{k \in \mathbb{N}} (1/\delta_k, 1) - \alpha_{i_2} \right] \delta_1 \|u_{t_0}\|_\tau e^{-\eta_{i_1}(t_1-\tau-t_0)+\lambda_{i_2}(t^*-t_1)}, \\
\leq& \lambda_{i_2} w(t^*), \\
=& \dot{w}(t^*),
\end{aligned}$$

a contradiction.

Assume the result holds for $t \in [t_{k-1}, t_k)$. That is,

$$u(t) \leq \|u_{t_0}\|_\tau \left(\prod_{i=1}^{k-1} \delta_i \right) \exp \left\{ \sum_{i \in \mathcal{P}_u} \lambda_i T_i(t_0, t) - \sum_{i \in \mathcal{P}_s} \eta_i [T_i(t_0, t) - \Phi_i(t_0, t)\tau] \right\}.$$

Since $N(t_0, t) = k$ for $t \in [t_k, t_{k+1})$ if $t_k = T_k$, we aim to show it holds by claiming that $u(t) \leq w(t)$ for $t \in [t_k, t_{k+1})$ where

$$w(t) := \|u_{t_0}\|_\tau \left(\prod_{i=1}^{k} \delta_i \right) \exp \left\{ \sum_{i \in \mathcal{P}_u} \lambda_i T_i(t_0, t) - \sum_{i \in \mathcal{P}_s} \eta_i [T_i(t_0, t) - \Phi_i(t_0, t)\tau] \right\}. \tag{4.8}$$

If not, then there exists a time $t^* \in [t_k, t_{k+1})$ such that $u(t^*) = w(t^*)$, $u(t) \leq w(t)$ for all $t \in [t_k, t^*)$ and for any $\epsilon > 0$ there exists a time $t_\epsilon \in (t^*, t^* + \epsilon)$ such that $u(t_\epsilon) > w(t_\epsilon)$.

Let

$$\Psi(t_0, t) := \sum_{i \in \mathcal{P}_u} \lambda_i T_i(t_0, t) - \sum_{i \in \mathcal{P}_s} \eta_i [T_i(t_0, t) - \Phi_i(t_0, t)\tau]$$

and suppose that $i_{k+1} \in \mathcal{P}_s$, then

$$\begin{aligned}\dot{u}(t^*) \leq &\beta_{i_{k+1}} \sup_{-\tau \leq s \leq 0} u(t^* + s) - \alpha_{i_{k+1}} u(t^*),\\ \leq &\beta_{i_{k+1}} \sup_{k \in \mathbb{N}} (\delta_k, 1) \|u_{t_0}\|_\tau \left(\prod_{i=1}^{k-1} \delta_i \right) e^{\Psi(t_0,t_k)} e^{-\eta_{i_{k+1}}(t^* - \tau - t_k)}\\ &- \alpha_{i_{k+1}} \delta_k \|u_{t_0}\|_\tau \left(\prod_{i=1}^{k-1} \delta_i \right) e^{\Psi(t_0,t_k)} e^{-\eta_{i_{k+1}}(t^* - t_k)},\end{aligned}$$

and so

$$\begin{aligned}\dot{u}(t^*) \leq &\left[\beta_{i_{k+1}} \sup_{k \in \mathbb{N}} (1/\delta_k, 1) e^{\eta_{i_{k+1}} \tau} - \alpha_{i_{k+1}} \right] \|u_{t_0}\|_\tau \left(\prod_{i=1}^{k} \delta_i \right) e^{\Psi(t_0,t^*)},\\ \leq &- \eta_{i_{k+1}} w(t^*),\\ = &\dot{w}(t^*).\end{aligned}$$

No such t_ϵ can exist and therefore this is a contradiction. On the other hand, if $i_{k+1} \in \mathcal{P}_u$, then

$$\begin{aligned}\dot{u}(t^*) \leq &\beta_{i_{k+1}} \sup_{-\tau \leq s \leq 0} u(t^* + s) - \alpha_{i_{k+1}} v(t^*),\\ \leq &\beta_{i_{k+1}} \sup_{k \in \mathbb{N}} (\delta_k, 1) \|u_{t_0}\|_\tau \left(\prod_{i=1}^{k-1} \delta_i \right) e^{\Psi(t_0,t_k)} e^{\lambda_{i_{k+1}}(t^* - t_k)}\\ &- \alpha_{i_{k+1}} \delta_k \|u_{t_0}\|_\tau \left(\prod_{i=1}^{k-1} \delta_i \right) e^{\Psi(t_0,t_k)} e^{\lambda_{i_{k+1}}(t^* - t_k)},\end{aligned}$$

and so,

$$\begin{aligned}\dot{u}(t^*) \leq &\left[\beta_{i_{k+1}} \sup_{k \in \mathbb{N}} (1/\delta_k, 1) - \alpha_{i_{k+1}} \right] \|u_{t_0}\|_\tau \left(\prod_{i=1}^{k} \delta_i \right) e^{\Psi(t_0,t^*)},\\ \leq &\lambda_{i_{k+1}} w(t^*),\\ = &\dot{w}(t^*),\end{aligned}$$

a contradiction. To prove (4.7) holds for $t_k \neq T_k$, construct a new sequence of times $\{z_k\}_{k=1}^\infty$ by concatenating $\{t_k\}_{k=1}^\infty$ and $\{T_k\}_{k=1}^\infty$ so that $z_{k-1} < z_k$. That is, each element z_k in the new sequence is equal to either t_j or T_j for some $j \in \mathbb{N}$. If $t_j = T_j$ for some value of j, then only one associated element appears in $\{z_k\}$. For each $j \in \mathbb{N}$, if $z_j \notin \{t_k\}_{k=1}^{\infty}$ then z_j is an impulse time and $\sigma(z_j^-) = \sigma(z_j)$. If $z_j \notin \{T_k\}_{k=1}^{\infty}$ then z_j is a switching time ($\delta_j = 1$). The above arguments hold and hence for $[t_k, t_{k+1})$, $t_k \neq T_k$,

$$u(t) \leq \|u_{t_0}\|_\tau \left(\prod_{i=1}^{N(t_0,t)} \delta_i \right) \exp\left\{ \sum_{i \in \mathcal{P}_u} \lambda_i T_i(t_0, t) - \sum_{i \in \mathcal{P}_s} \eta_i [T_i(t_0, t) - \Phi_i(t_0, t)\tau] \right\}.$$

□

Remark 4.2.2. *The sets $\mathcal{P}_u \subseteq \mathcal{P}$ and $\mathcal{P}_s \subseteq \mathcal{P}$ represent the unstable and stable subsystems, respectively. Equation (4.7) gives an estimate for $u(t)$ based on the growth/decay rate estimates (λ_i and η_i, respectively), the time spent in the unstable versus stable modes, and the impulsive effects (captured by δ_i and $N(t_0, t)$).*

We illustrate how the switched Halanay-like inequality in Proposition 4.2.4 can be applied to a nonlinear switched system of integro-differential equations with the following example.

Example 4.2.2. *Consider the nonlinear HISD with $\mathcal{P} = \{1, 2\}$ and the following two subsystems:*

$$i = 1 : \begin{cases} \dot{x}_1 = -4.5x_1(t) + x_2(t) + x_2^5(t) + \displaystyle\int_{t-\tau}^{t} x_1(s)\sin(x_2(s))ds, \\ \dot{x}_2 = -4.5x_2(t) - x_1(t)x_2^4(t) + \displaystyle\int_{t-\tau}^{t} \frac{\pi x_2(s)}{\pi + \arctan(x_2(s))} ds, \end{cases}$$

and

$$i = 2 : \begin{cases} \dot{x}_1 = \dfrac{x_1(t-r)x_2(t-r)}{1 + |x_1(t)|}, \\ \dot{x}_2 = \displaystyle\int_{t-\tau}^{t} \frac{\sqrt{x_1^4(s) + x_2^4(s)}}{1 + x_2^2(s)} ds. \end{cases}$$

The switching rule is assumed to take the following form for $k = 1, 2, \ldots,$

$$\sigma = \begin{cases} 1, & t \in [t_{2k-2}, t_{2k-1}), \\ 2, & t \in [t_{2k-1}, t_{2k}), \end{cases} \tag{4.9}$$

where $t_0 = 0$ and

$$t_k = \begin{cases} t_{k-1} + 0.5 + 0.2k^2 e^{-k}, & k = 1, 3, 5, \ldots, \\ t_{k-1} + 0.2, & k = 2, 4, 6, \ldots, \end{cases} \tag{4.10}$$

which satisfies $t_k - t_{k-1} \geq 0.2$. *Suppose that at the impulsive moments* $t = T_k = k + 0.4\sin(k)$, $k = 1, 2, \ldots$,

$$\begin{cases} \Delta x_1 = -x_1(t^-) + \sqrt{(1 + 0.1/k)}\sin\left(\int_{t-\tau}^{t} x_2(s)ds\right), \\ \Delta x_2 = -x_2(t^-) + e^{-k}\int_{t-\tau}^{t} \sqrt{|x_1(s)x_2(s)|}ds. \end{cases} \tag{4.11}$$

Let $V(x) = (x_1^2 + x_2^2)/2$ *and take the time-derivative along solutions to subsystem* $i = 1$,

$$\begin{aligned} \frac{dV}{dt}\,|_{i=1} =& x_1(t)\left[-4.5x_1(t) + x_2(t) + x_2^5(t) + \int_{t-\tau}^{t} x_1(s)\sin(x_2(s))ds\right], \\ &+ x_2(t)\left[-4.5x_2(t) - x_1(t)x_2^4(t) + \int_{t-\tau}^{t} \frac{\pi x_2(s)}{\pi + \arctan(x_2(s))}ds\right], \\ \leq& -4.5(x_1^2(t) + x_2^2(t)) + x_1(t)x_2(t) + \tau \sup_{t-\tau\leq s\leq t} x_1^2(s) + 2\tau \sup_{t-\tau\leq s\leq t} x_2^2(s), \\ \leq& -9V(x(t)) + \frac{x_1^2(t) + x_2^2(t)}{2} + 3\tau \sup_{t-\tau\leq s\leq t}[x_1^2(s) + x_2^2(s)], \\ =& -8V(x(t)) + 6\tau \sup_{t-\tau\leq s\leq t} V(x(s)). \end{aligned}$$

Similarly, along the subsystem $i = 2$, *since* $x_1^4 + x_2^4 \leq (x_1^2 + x_2^2)^2$ *for all* $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned} \frac{dV}{dt}\,|_{i=2} =& x_1\left[\frac{x_1(t-r)x_2(t-r)}{1 + |x_1(t)|}\right] + x_2\left[\int_{t-\tau}^{t} \frac{\sqrt{x_1^4(s) + x_2^4(s)}}{1 + x_2^2(s)}ds\right], \\ \leq& x_1(t-r)x_2(t-r) + \int_{t-\tau}^{t}(x_1^2(s) + x_2^2(s))ds, \\ \leq& \sup_{t-\tau\leq s\leq t} \frac{x_1^2(s) + x_2^2(s)}{2} + \tau \sup_{t-\tau\leq s\leq t}[x_1^2(s) + x_2^2(s)], \\ =& (1 + 2\tau) \sup_{t-\tau\leq s\leq t} V(x(s)), \end{aligned}$$

where it is assumed that $\tau \geq r > 0$*. At the impulsive moments* $t = T_k$*,*

$$\begin{aligned}
V(x(T_k)) =& \frac{1}{2}\left[\sqrt{1+0.1/k}\sin\left(\int_{T_k-\tau}^{T_k} x_2(s)ds\right)\right]^2 \\
&+ \frac{1}{2}\left[e^{-k}\int_{T_k-\tau}^{T_k}\sqrt{|x_1(s)x_2(s)|}ds\right]^2, \\
\leq& \frac{1}{2}(1+0.1/k)\left(\int_{T_k-\tau}^{T_k} x_2(s)ds\right)^2 + \frac{\tau e^{-2k}}{2}\tau \sup_{T_k-\tau\leq s\leq T_k}|x_1(s)x_2(s)|, \\
\leq& \frac{1}{2}(1+0.1/k)\tau \sup_{T_k-\tau\leq s\leq T_k} x_2^2(s) + \frac{\tau e^{-2k}}{2}\tau \sup_{T_k-\tau\leq s\leq T_k}\frac{x_1^2(s)+x_2^2(s)}{2}, \\
\leq& (\tau(1+0.1/k) + \tau e^{-2k}/2)\sup_{T_k-\tau\leq s\leq T_k} V(x(s)).
\end{aligned}$$

Let $v(t) = V(x(t))$ *where* $x(t)$ *is a solution of the switched and impulsive system. Then* $v(t)$ *satisfies*

$$\begin{cases} \dot{v} \leq \beta_\sigma\|v_t\|_\tau - \alpha_\sigma v(t), & t \neq T_k, \\ v(t) \leq d_k v(t^-) + h_k\|v_t\|_\tau, & t = T_k, \quad k \in \mathbb{N}, \end{cases} \tag{4.12}$$

where $\alpha_1 = 8$*,* $\alpha_2 = 0$*,* $\beta_1 = 2\tau$*,* $\beta_2 = 1 + 2\tau$*,* $d_k = 0$*,* $h_k = \tau(1 + 0.1/k + e^{-2k}/2)$*. By Proposition 4.2.4,* $v(t)$ *satisfies*

$$v(t) \leq \|v_0\|_\tau \left(\prod_{i=1}^{N(0,t)} \delta_i\right) \exp\left[\lambda_2 T_2(0,t) - \eta_1(T_1(0,t) - \Phi_1(0,t)\tau)\right]$$

for $t \geq 0$ *where* $\lambda_2 = 6.89$ *and* $\eta_1 = 5.56$*. That is,* $\mathcal{P}_s = \{1\}$ *(first subsystem is stable) and* $\mathcal{P}_u = \{2\}$ *(second subsystem is unstable). From the switching rule,* $T_2(0,t) \leq 1 + 0.5[T_1(0,t) - \Phi_1(0,t)\tau]$*. Hence*

$$v(t) \leq \|v_0\|_\tau \exp\left[N(0,t)\ln\delta + (\lambda_2/2 - \eta_1)(T_1(0,t) - \Phi_1(0,t)\tau)\right]$$

where $\delta = \sup_{k\in\mathbb{N}}\delta_k = 0.174$*. Note that* $0.2 \leq T_k - T_{k-1} \leq 1.8$ *implies that* $t \leq 1.8[1 + N(0,t)]$*. Additionally,* $t \geq T_1(0,t) \geq T_1(0,t) - \Phi_1(0,t)\tau$ *and*

$$t = T_1(0,t) + T_2(0,t) \leq 2(T_1(0,t) - \Phi_1(0,t)\tau).$$

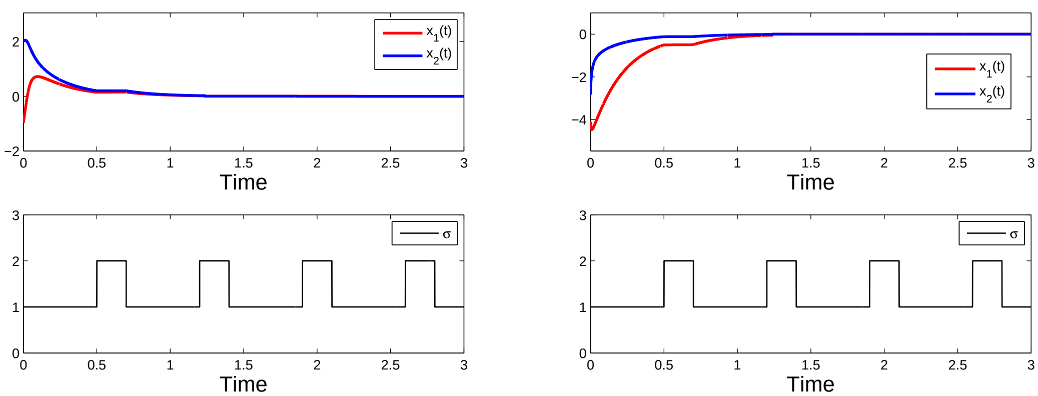
Thus,

$$\begin{aligned}
v(t) \leq & \frac{\|v_0\|_\tau}{\delta} \exp\left[\left(\frac{t}{1.8}\right)\ln\delta + (\lambda_2/2 - \eta_1)(T_1(0,t) - \Phi_1(0,t)\tau)\right], \\
\leq & \frac{\|v_0\|_\tau}{\delta} \exp\left[\left(\frac{\ln\delta}{1.8} + \frac{\lambda_2}{2} - \eta_1\right)(T_1(0,t) - \Phi_1(0,t)\tau)\right], \\
\leq & \frac{\|v_0\|_\tau}{\delta} \exp\left[\left(\frac{\ln\delta}{1.8} + \frac{\lambda_2}{2} - \eta_1\right)\left(\frac{t}{2}\right)\right], \\
\leq & \frac{\|v_0\|_\tau}{\delta} \exp\left[-1.54t\right],
\end{aligned}$$

for all $t \geq 0$. *Therefore for* $t \geq 0$

$$\|x(t)\|^2 \leq \frac{\|\phi_0\|_\tau^2}{\delta} \exp\left[-0.351t\right]$$

and hence the origin is globally exponentially stable. See Figure 4.2 for an illustration with $r = 0.05$ *and* $\tau = 0.1$.



(a) $\phi_0(s) = (-1, 2)$ for $-0.1 \leq s \leq 0$.

(b) $\phi_0(s) = (-4, -3)$ for $-0.1 \leq s \leq 0$.

Figure 4.2: Simulation of Example 4.2.2.

4.2.3 Application to a Class of Nonlinear HISD

We return to the HISD (4.3) in hopes of applying the switching Halanay-like inequalities to determine its stability. The system can be written more compactly as

$$\begin{cases} \dot{x} = A_\sigma x(t) + B_\sigma x(t-r) + C_\sigma \displaystyle\int_{t-\tau}^{t} x(s)ds + F_\sigma(t, x_t), & t \neq t_k, \\ \Delta x = E_\sigma x(t^-) + G_\sigma \displaystyle\int_{t-\tau}^{t} x(s)ds, & t = t_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases} \tag{4.13}$$

where $x \in \mathbb{R}^n$ is the state; $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$ is the switching rule; $\phi_0 \in PC([-\bar{\tau}, 0], \mathbb{R}^n)$ where $\bar{\tau} = \max\{r, \tau\}$. Each functional $F_i : \mathbb{R} \times PC([-\bar{\tau}, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is assumed to satisfy $F_i(t, 0) \equiv 0$ for all $t \geq t_0$ and the following nonlinearity assumption.

Assumption 4.2.1. *Assume that there exist non-negative constants $\vartheta_{1i}, \vartheta_{2i}$, and ϑ_{3i} for $i \in \mathcal{P}$ such that for $t \geq t_0$ and $\psi \in PC([-\bar{\tau}, 0], \mathbb{R}^n)$,*

$$\|F_i(t, \psi)\|^2 \leq \vartheta_{1i}\|\psi(0)\|^2 + \vartheta_{2i}\|\psi(-r)\|^2 + \vartheta_{3i} \int_{-\tau}^{0} \|\psi(s)\|^2 ds. \tag{4.14}$$

We also make use of the following lemmas in the theorems to follow.

Lemma 4.2.5. [72]
For a positive definite symmetric matrix W, a nonnegative scalar v and a vector function $w : [0, v] \to \mathbb{R}^n$,

$$\left(\int_0^v w(s)ds\right)^T W \left(\int_0^v w(s)ds\right) \leq v \int_0^v w^T(s) W w(s) ds.$$

Lemma 4.2.6. (Matrix Cauchy Inequality)
For any symmetric positive definite matrix $M \in \mathbb{R}^{n \times n}$ and $x, y \in \mathbb{R}^n$,

$$2x^T y \leq x^T M x + y^T M^{-1} y.$$

Denote $T^+(t_0, t)$ and $T^-(t_0, t)$ to be the total activation time $\sigma(t) \in \mathcal{P}_u$ and $\sigma(t) \in \mathcal{P}_s$ on $[t_0, t]$, respectively. Denote $\Phi(t_0, t)$ to be the number of switching times t_k such that $\sigma(t_k) \in \mathcal{P}_s$ and $t_k \in [t_0, t)$ (i.e. the total number of activations of a stable mode on the interval). We are now ready to present the first main result.

Theorem 4.2.7. *Suppose that Assumption 4.2.1 holds and assume that* $t_k - t_{k-1} \geq \bar{\tau}$ *for* $k \in \mathbb{N}$. *Assume that there exists a constant* $\nu \geq 0$ *such that* $T^+(t_0, t) \leq \nu(T^-(t_0, t) - \Phi(t_0, t)\bar{\tau}))$. *Assume that there exists a positive constant* M *such that*

$$\sup_{t \geq t_0} \frac{t - t_0}{T^-(t_0, t) - \Phi(t_0, t)\bar{\tau}} \leq M.$$

For a set of positive definite symmetric matrices P_i, $i \in \mathcal{P}$, *define the following constants*

$$\begin{aligned}
\beta_i =& \lambda_{max}(P_i^{-1})[\lambda_{max}(B_i^T P_i B_i) + \tau^2 \lambda_{max}(C_i^T P_i C_i) + \vartheta_{2i} + \tau\vartheta_{3i}],\\
\alpha_i =& -\lambda_{max}(P_i^{-1})[\lambda_{max}(A_i^T P + P_i A_i + I) + \vartheta_{1i} + 2\lambda_{min}(P_i)],\\
\mu_i =& \frac{\lambda_{max}(P_i)}{\lambda_{min}(P_i)}, \qquad \mu = \max_{i \in \mathcal{P}} \mu_i,\\
d_i =& \mu\lambda_{max}(P_i^{-1})\lambda_{max}[(I + E_i)^T P_i (I + E_i)],\\
h_i =& \mu\tau^2 \lambda_{max}(P_i^{-1})\lambda_{max}(G_i^T P_i G_i).
\end{aligned}$$

Using the definitions of λ_i, η_i, $\mathcal{P}_s$, *and* $\mathcal{P}_u$ *in Proposition 4.2.4, let* $\delta_i = d_i + h_i e^{\xi\bar{\tau}}$ *where* $\xi = \max_{i \in \mathcal{P}_s}\{\xi_i\}$ *with* $\xi_i > 0$ *chosen for* $i \in \mathcal{P}_s$ *such that* $\xi_i + \beta_i e^{\xi_i \bar{\tau}} - \alpha_i < 0$. *Let* $\lambda^+ = \max_{i \in \mathcal{P}_u} \lambda_i$ *and* $\lambda^- = \min_{i \in \mathcal{P}_s} \eta_i$.

(i) If $\delta = \max_{i \in \mathcal{P}} \delta_i < 1$ *and there exists a constant* $\chi > 0$ *such that* $\bar{\tau} \leq t_k - t_{k-1} \leq \chi$ *and*

$$\frac{\ln \delta}{\chi} + \nu\lambda^+ - \lambda^- < 0 \tag{4.15}$$

then the trivial solution of (4.13) *is globally exponentially stable.*

(ii) If $\delta = \max_{i \in \mathcal{P}} \delta_i > 1$ *and there exists a constant* $\zeta > 0$ *such that* $\bar{\tau} \leq \zeta \leq t_k - t_{k-1}$ *and*

$$\frac{\ln \delta}{\zeta} + \nu\lambda^+ - \lambda^- < 0 \tag{4.16}$$

then the trivial solution of (4.13) *is globally exponentially stable.*

Proof. Define $V_i = x^T P_i x$ for $i \in \mathcal{P}$. Then the time-derivative along the i^{th} mode satisfies

$$\begin{aligned}\dot{V}_i =& x^T(t)(A_i^T P_i + P_i A_i)x(t) + 2x^T(t)P_i B_i x(t-r)\\ &+ 2x^T(t)P_i C_i \int_{t-\tau}^{t} x(s)ds + 2x^T(t)P_i F_i(t, x_t),\\ \leq& x^T(t)(A_i^T P_i + P_i A_i)x(t) + 2x^T(t)P_i B_i x(t-r)\\ &+ 2x^T(t)P_i C_i \int_{t-\tau}^{t} x(s)ds + x^T(t)P_i^2 x(t) + \|F_i(t, x_t)\|^2.\end{aligned}$$

Using the nonlinearity assumption (4.14),

$$\begin{aligned}\dot{V}_i \leq& x^T(t)(A_i^T P_i + P_i A_i)x(t) + 2x^T(t)P_i B_i x(t-r) + 2x^T(t)P_i C_i \int_{t-\tau}^{t} x(s)ds\\ &+ x^T(t)P_i^2 x(t) + \vartheta_{1i} x^T(t)x(t) + \vartheta_{2i} x^T(t-r)x(t-r) + \vartheta_{3i} \int_{t-\tau}^{t} x^T(s)x(s)ds.\end{aligned}$$

Using Lemma 4.2.5 and Lemma 4.2.6,

$$\begin{aligned}&2(P_i x(t))^T \left(C_i \int_{t-\tau}^{t} x(s)ds\right)\\ \leq& (P_i x(t))^T P_i^{-1}(P_i x(t)) + \left(\int_{t-\tau}^{t} x(s)ds\right)^T C_i^T P_i C_i \left(\int_{t-\tau}^{t} x(s)ds\right),\\ \leq& V_i(x(t)) + \lambda_{\max}(C_i^T P_i C_i)\left(\int_{t-\tau}^{t} x(s)ds\right)^T \left(\int_{t-\tau}^{t} x(s)ds\right),\\ \leq& V_i(x(t)) + \lambda_{\max}(C_i^T P_i C_i)\left[\tau^2 \sup_{t-\bar{\tau}\leq s\leq t} \|x(s)\|^2\right],\\ \leq& V_i(x(t)) + \frac{\tau^2 \lambda_{\max}(C_i^T P_i C_i)}{\lambda_{\min}(P_i)} \sup_{t-\bar{\tau}\leq s\leq t} V_i(x(s)),\end{aligned}$$

Similarly,

$$\begin{aligned}
&2(P_i x(t))^T (B_i x(t-r)) \\
\leq &(P_i x(t))^T P_i^{-1} (P_i x(t)) + \big(B_i x(t-r)\big)^T P_i (B_i x(t-r)\big), \\
\leq &V_i(x(t)) + \lambda_{\max}(B_i^T P_i B_i) x^T(t-r) x(t-r), \\
\leq &V_i(x(t)) + \lambda_{\max}(B_i^T P_i B_i) \sup_{t-\bar{\tau}\leq s\leq t} \|x(s)\|^2, \\
\leq &V_i(x(t)) + \frac{\lambda_{\max}(B_i^T P_i B_i)}{\lambda_{\min}(P_i)} \sup_{t-\bar{\tau}\leq s\leq t} V_i(x(s)),
\end{aligned}$$

Let $v(t) = V_\sigma(x(t))$ where $x(t)$ is the solution of (4.13), then for $t \neq t_k$,

$$\begin{aligned}
\dot{v} \leq &\frac{\lambda_{\max}(A_\sigma^T P_\sigma + P_\sigma A_\sigma)}{\lambda_{\min}(P_\sigma)} v(t) + v(t) + \frac{\lambda_{\max}(B_\sigma^T P_\sigma B_\sigma)}{\lambda_{\min}(P_\sigma)} \|v_t\|_{\bar{\tau}}, \\
&+ v(t) + \frac{\tau^2 \lambda_{\max}(C_\sigma^T P_\sigma C_\sigma)}{\lambda_{\min}(P_\sigma)} \|v_t\|_{\bar{\tau}} + \frac{\lambda_{\max}(P_\sigma^2)}{\lambda_{\min}(P_\sigma)} v(t) \\
&+ \frac{\vartheta_{1\sigma}}{\lambda_{\min}(P_\sigma)} v(t) + \frac{\vartheta_{2\sigma}}{\lambda_{\min}(P_\sigma)} \|v_t\|_{\bar{\tau}} + \frac{\tau \vartheta_{3\sigma}}{\lambda_{\min}(P_\sigma)} \|v_t\|_{\bar{\tau}}, \\
= &-\alpha_\sigma v(t) + \beta_\sigma \|v_t\|_{\bar{\tau}}.
\end{aligned}$$

At the switching and impulsive moment $t = t_k$,

$$\begin{aligned}
V_{i_{k+1}}(x(t_k)) = &x^T(t_k) P_{i_{k+1}} x(t_k), \\
\leq &\mu x^T(t_k) P_{i_k} x^T(t_k), \\
= &\mu \left[(I + E_{i_k}) x(t_k^-) + G_{i_k} \int_{t_k-\tau}^{t_k} x(s) ds \right]^T P_{i_k} \\
&\times \left[(I + E_{i_k}) x(t_k^-) + G_{i_k} \int_{t_k-\tau}^{t_k} x(s) ds \right],
\end{aligned}$$

Hence,

$$\begin{aligned}
V_{i_{k+1}}(x(t_k)) \leq &\mu x^T(t_k^-) [(I + E_{i_k})^T P_{i_k} (I + E_{i_k})] x(t_k^-) \\
&+ \mu \left(\int_{t_k-\tau}^{t_k} x(s) ds \right)^T G_{i_k}^T P_{i_k} G_{i_k} \left(\int_{t_k-\tau}^{t_k} x(s) ds \right)^T \\
\leq &\mu \frac{\lambda_{\max}[(I + E_{i_k})^T P_{i_k} (I + E_{i_k})]}{\lambda_{\min}(P_{i_k})} V_{i_k}(t_k^-) \\
&+ \mu \frac{\tau^2 \lambda_{\max}(G_{i_k}^T P_{i_k} G_{i_k})}{\lambda_{\min}(P_{i_k})} \sup_{t_k-\tau\leq s\leq t_k} V_{i_k}(t)
\end{aligned}$$

Therefore, at the switching/impulsive times $v(t)$ satisfies

$$v(t_k) \leq \mu \left[\frac{\lambda_{\max}[(I+E_\sigma)^T P_\sigma (I+E_\sigma)]}{\lambda_{\min}(P_\sigma)} v(t_k^-) + \frac{\tau^2 \lambda_{\max}(G_\sigma^T P_\sigma G_\sigma)}{\lambda_{\min}(P_\sigma)} \|v_{t_k}\|_{\bar{\tau}} \right]$$

By Proposition 4.2.4, $v(t)$ satisfies

$$v(t) \leq \|v_{t_0}\|_{\bar{\tau}} \left(\prod_{i=1}^{k-1} \delta_i \right) \exp\left\{ \sum_{i\in\mathcal{P}_u} \lambda_i T_i(t_0,t) - \sum_{i\in\mathcal{P}_s} \eta_i [T_i(t_0,t) - \Phi_i(t_0,t)\bar{\tau}] \right\} \tag{4.17}$$

for all $t \geq t_0$ since $N(t_0,t) = k-1$ for $t \in [t_{k-1}, t_k)$. From $t_k - t_{k-1} \leq \chi$ it follows that $t - t_0 \leq k\chi$ for $t \in [t_{k-1}, t_k)$. Then from equation (4.17) and using $\delta < 1$,

$$\begin{aligned}
v(t) \leq & \|v_{t_0}\|_{\bar{\tau}} \exp\left[\sum_{j=1}^{k-1} \ln\delta_j + \lambda^+ \sum_{i\in\mathcal{P}_u} T_i(t_0,t) - \lambda^- \sum_{i\in\mathcal{P}_s} (T_i(t_0,t) - \Phi_i(t_0,t)\bar{\tau}) \right], \\
\leq & \|v_{t_0}\|_{\bar{\tau}} \exp\left[\sum_{j=1}^{k-1} \ln\delta_j + \lambda^+ T^+(t_0,t) - \lambda^-(T^-(t_0,t) - \Phi(t_0,t)\bar{\tau}) \right], \\
\leq & \|v_{t_0}\|_{\bar{\tau}} \exp\left[(k-1)\ln\delta + \lambda^+ T^+(t_0,t) - \lambda^-(T^-(t_0,t) - \Phi(t_0,t)\bar{\tau}) \right], \\
\leq & \|v_{t_0}\|_{\bar{\tau}} \frac{1}{\delta} \exp\left[\left(\frac{t-t_0}{\chi} \right) \ln\delta + \lambda^+ T^+(t_0,t) - \lambda^-(T^-(t_0,t) - \Phi(t_0,t)\bar{\tau}) \right],
\end{aligned}$$

and so,

$$\begin{aligned}
v(t) \leq & \|v_{t_0}\|_{\bar{\tau}} \frac{1}{\delta} \exp\left[(t-t_0)\left(\frac{\ln\delta}{\chi} \right) + (\nu\lambda^+ - \lambda^-)(T^-(t_0,t) - \Phi(t_0,t)\bar{\tau}) \right], \\
\leq & \|v_{t_0}\|_{\bar{\tau}} \frac{1}{\delta} \exp\left[(T^-(t_0,t) - \Phi(t_0,t)\bar{\tau}) \left(\frac{\ln\delta}{\chi} \right) \right] \\
& \times \exp\left[(\nu\lambda^+ - \lambda^-)(T^-(t_0,t) - \Phi(t_0,t)\bar{\tau}) \right], \\
\leq & \|v_{t_0}\|_{\bar{\tau}} \frac{1}{\delta} \exp\left[\left(\frac{\ln\delta}{\chi} + \nu\lambda^+ - \lambda^- \right) \frac{t-t_0}{M} \right].
\end{aligned}$$

The result follows and case (i) is proven. In order to prove case (ii), note that $\bar{\tau} \leq \zeta \leq$

$t_k - t_{k-1}$ implies that $t - t_0 \geq k\zeta$ for $t \in [t_{k-1}, t_k)$. Then, beginning from equation (4.17),

$$
\begin{aligned}
v(t) \leq & \|v_{t_0}\|_{\bar{\tau}} \exp\left[\sum_{j=1}^{k-1} \ln \delta_j + \lambda^+ \sum_{i \in \mathcal{P}_u} T_i(t_0, t) - \lambda^- \sum_{i \in \mathcal{P}_s} (T_i(t_0, t) - \Phi_i(t_0, t)\bar{\tau})\right], \\
\leq & \|v_{t_0}\|_{\bar{\tau}} \exp\left[\sum_{j=1}^{k-1} \ln \delta_j + \lambda^+ T^+(t_0, t) - \lambda^-(T^-(t_0, t) - \Phi(t_0, t)\bar{\tau})\right], \\
\leq & \|v_{t_0}\|_{\bar{\tau}} \exp\left[(k-1)\ln \delta + \lambda^+ T^+(t_0, t) - \lambda^-(T^-(t_0, t) - \Phi(t_0, t)\bar{\tau})\right], \\
\leq & \|v_{t_0}\|_{\bar{\tau}} \frac{1}{\delta} \exp\left[\left(\frac{t - t_0}{\zeta}\right) \ln \delta + \lambda^+ T^+(t_0, t) - \lambda^-(T^-(t_0, t) - \Phi(t_0, t)\bar{\tau})\right],
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
v(t) \leq & \|v_{t_0}\|_{\bar{\tau}} \frac{1}{\delta} \exp\left[(t - t_0)\left(\frac{\ln \delta}{\zeta}\right) + (\nu\lambda^+ - \lambda^-)(T^-(t_0, t) - \Phi(t_0, t)\bar{\tau})\right], \\
\leq & \|v_{t_0}\|_{\bar{\tau}} \frac{1}{\delta} \exp\left[(T^-(t_0, t) - \Phi(t_0, t)\bar{\tau})\left(\frac{\ln \delta}{\zeta}\right)\right] \\
& \times \exp\left[(\nu\lambda^+ - \lambda^-)(T^-(t_0, t) - \Phi(t_0, t)\bar{\tau})\right], \\
\leq & \|v_{t_0}\|_{\bar{\tau}} \frac{1}{\delta} \exp\left[\left(\frac{\ln \delta}{\zeta} + \nu\lambda^+ - \lambda^-\right) \frac{t - t_0}{M}\right].
\end{aligned}
$$

This proves case (ii). □

Remark 4.2.3. *The constant λ^+ is an estimate for the worst case growth rate for the unstable subsystems and λ^- represents the most conservative decay rate estimate for the stable subsystems. The condition $T^+(t_0, t) \leq \nu(T^-(t_0, t) - \Phi(t_0, t)\bar{\tau})$ gives a relationship between the time spent in the unstable subsystems and the time spent in the stable subsystems.*

Remark 4.2.4. *In case (i), the impulsive effects are stabilizing so an upper bound on the time between impulses is established via the constant χ. Equation (4.15) gives an explicit threshold for how strong the stabilizing impulses must be to achieve stability (the smaller δ is, the stronger the stabilizing impulsive effect must be):*

$$\delta < \exp[-\chi(\nu\lambda^+ - \lambda^-)].$$

The equation can be re-arranged to reveal conditions on the switching and impulsive times:

$$t_k - t_{k-1} \leq \chi < \frac{-\ln \delta}{\nu\lambda^+ - \lambda^-},$$

which means an impulse must be applied often enough to counter-act any growth during the switching portion of the system.

Remark 4.2.5. *In case (ii), the impulses act as disturbances and hence a lower bound, ζ, is imposed to guarantee the impulses are not applied too frequently. Again, (4.16) gives an explicit threshold for the maximum size of the impulses:*

$$\delta < \exp[-\zeta(\nu\lambda^{+} - \lambda^{-})]$$

which can be re-arranged to give that

$$t_k - t_{k-1} \geq \zeta > \frac{\ln \delta}{-(\nu\lambda^{+} - \lambda^{-})}$$

that is, the impulses must not be applied too often given a certain impulsive strength δ and the decay rate of the switching portion.

Example 4.2.3. *Consider a switching rule given by*

$$\sigma = \begin{cases} 1, & t \in [t_{2k-2}, t_{2k-1}), \\ 2, & t \in [t_{2k-1}, t_{2k}), \end{cases} \tag{4.18}$$

for $k \in \mathbb{N}$, where $t_0 = 0$ and

$$t_k - t_{k-1} = \begin{cases} 0.5 + 0.2k^2 e^{-k}, & k = 1, 3, 5, \ldots, \\ 0.2, & k = 2, 4, 6, \ldots \end{cases} \tag{4.19}$$

Suppose that $\mathcal{P}_s = \{1\}$ (first subsystem is stable) and $\mathcal{P}_u = \{2\}$ (second subsystem is unstable). If $\bar{\tau} = 0.1$ then $T^{+}(0,t) \leq \nu(T^{-}(0,t) - \Phi(0,t)\bar{\tau})$ is satisfied with $\nu = 0.5$ as the time spent in the stable subsystem is more than double the time spent in the unstable subsystem.

The following corollary can be given for when all the modes are either stable or unstable.

Corollary 4.2.8. *Suppose that Assumption 4.2.1 holds. For a set of positive definite symmetric matrices P_i, $i \in \mathcal{P}$, define β_i, α_i, μ, δ_i as in Theorem 4.2.7.*

(i) If $\mathcal{P}_u = \mathcal{P}$, there exists a constant $\chi > 0$ such that $\tau \leq t_k - t_{k-1} \leq \chi$ for $k = 1, 2, \ldots$, and there exists a constant $\gamma > 1$ such that $\ln(\gamma\delta_i) + \lambda_i\chi \leq 0$ for all $i \in \mathcal{P}$, then the trivial solution of (4.13) is globally asymptotically stable.

(ii) If $\mathcal{P}_s = \mathcal{P}$, there exists a constant $\zeta \geq \tau$ such that $t_k - t_{k-1} \geq \zeta$ for $k = 1, 2, \ldots$, and there exists a constant $\gamma > 1$ such that $\ln(\gamma\delta_i) - \eta_i(\zeta - \tau) \leq 0$, then the trivial solution of (4.13) is globally asymptotically stable.

Proof. First consider case (i). From equation (4.17)

$$\begin{aligned} v(t) \leq & \|v_{t_0}\|_{\bar{\tau}} \exp\left[\sum_{j=1}^{k-1} \ln \delta_{i_j} + \sum_{i\in\mathcal{P}} \lambda_i T_i(t_0,t)\right], \\ \leq & \|v_{t_0}\|_{\bar{\tau}} \exp\left[\ln \delta_{i_1} + \lambda_{i_1}\chi + \ln \delta_{i_2} + \lambda_{i_2}\chi + \ldots + \ln \delta_{i_{k-1}} + \lambda_{i_{k-1}}\chi + \lambda_{i_k}\chi\right], \\ \leq & \|v_{t_0}\|_{\bar{\tau}} \frac{1}{\gamma^k(\min_{i\in\mathcal{P}} \delta_i)} \gamma\delta_{i_1} e^{\lambda_{i_1}\chi} \cdots \gamma\delta_{i_k} e^{\lambda_{i_k}\chi}, \\ \leq & \left(\frac{\|v_{t_0}\|_{\bar{\tau}}}{\min_{i\in\mathcal{P}} \delta_i}\right) \frac{1}{\gamma^k}. \end{aligned}$$

Hence the origin is globally asymptotically stable. In case (ii),

$$\begin{aligned} v(t) \leq & \|v_{t_0}\|_{\bar{\tau}} \exp\left[\sum_{j=1}^{k-1} \ln \delta_{i_j} - \sum_{i\in\mathcal{P}} \eta_i(T_i(t_0,t) - \Phi_i(t_0,t)\tau)\right], \\ \leq & \|v_{t_0}\|_{\bar{\tau}} \exp\left[\ln \delta_{i_1} - \eta_{i_1}(\zeta - \tau) + \ldots + \ln \delta_{i_{k-1}} - \eta_{i_{k+1}}(\zeta - \tau) - \eta_{i_k}(\zeta - \tau)\right], \\ \leq & \|v_{t_0}\|_{\bar{\tau}} \frac{1}{\gamma^k(\min_{i\in\mathcal{P}} \delta_i)} \gamma\delta_{i_1} e^{\lambda_{i_1}\zeta} \cdots \gamma\delta_{i_k} e^{\lambda_{i_k}\zeta}, \\ \leq & \left(\frac{\|v_{t_0}\|_{\bar{\tau}}}{\min_{i\in\mathcal{P}} \delta_i}\right) \frac{1}{\gamma^k}, \end{aligned}$$

and the result holds. □

Suppose the switching rule is periodic: assume that the switching times satisfy $\rho_k = t_k - t_{k-1}$ and $\rho_{k+m} = \rho_k$. Assume the switching rule σ satisfies $i_k = k$ and $i_{k+m} = i_k$. Denote the period of the switching rule by $\omega = \rho_1 + \rho_2 + \ldots + \rho_m$. Denote the set of all such periodic switching rules by $\mathcal{S}_{\text{periodic}} \subset \mathcal{S}$.

Corollary 4.2.9. *Suppose that Assumption 4.2.1 holds. For a set of positive definite symmetric matrices P_i, $i \in \mathcal{P}$, define β_i, α_i, μ, δ_i as in Theorem 4.2.7. If $\sigma \in \mathcal{S}_{periodic}$ and*

$$\sum_{i=1}^{m} \ln \delta_i + \sum_{i\in\mathcal{P}_u} \lambda_i \rho_i - \sum_{i\in\mathcal{P}_s} \eta_i(\rho_i - \bar{\tau}) < 0$$

then the trivial solution of (4.13) is globally asymptotically stable.

Proof. From equation (4.17),

$$v(t) \leq \|v_{t_0}\|_{\bar{\tau}} \exp\left[\sum_{j=1}^{k-1} \ln \delta_{i_j} + \sum_{i\in\mathcal{P}_u} \lambda_i T_i(t_0,t) - \sum_{i\in\mathcal{P}_s} \eta_i (T_i(t_0,t) - \Phi_i(t_0,t)\bar{\tau})\right].$$

Since the switching rule and impulses are periodic, for $j = 1, 2, \dots,$

$$\begin{aligned}
v(t_0+j\omega) \leq& \|v_{t_0}\|_{\bar{\tau}} \exp\left[\sum_{i=1}^{jm} \ln \delta_i + \sum_{i\in\mathcal{P}_u} \lambda_i T_i(t_0, t_0+j\omega)\right] \\
&\times \exp\left[-\sum_{i\in\mathcal{P}_s} \eta_i (T_i(t_0,t_0+j\omega) - \Phi_i(t_0,t_0+j\omega)\bar{\tau})\right], \\
\leq& \|v_{t_0}\|_{\bar{\tau}} \exp\left[j\sum_{i=1}^{m} \ln \delta_i + j\sum_{i\in\mathcal{P}_u} \lambda_i T_i(t_0, t_0+\omega)\right] \\
&\times \exp\left[-j\sum_{i\in\mathcal{P}_s} \eta_i (T_i(t_0,t_0+\omega) - \Phi_i(t_0,t_0+\omega)\bar{\tau})\right], \\
\leq& \|v_{t_0}\|_{\bar{\tau}} \exp\left[j\sum_{i=1}^{m} \ln \delta_i + j\sum_{i\in\mathcal{P}_u} \lambda_i \rho_i - j\sum_{i\in\mathcal{P}_s} \eta_i(\rho_i - \bar{\tau})\right], \\
=& \|v_{t_0}\|_{\bar{\tau}} \Lambda^j.
\end{aligned}$$

where

$$\Lambda = \exp\left[\sum_{i=1}^{m} \ln \delta_i + \sum_{i\in\mathcal{P}_u} \lambda_i \rho_i - \sum_{i\in\mathcal{P}_s} \eta_i(\rho_i - \bar{\tau})\right]$$

satisfies $0 < \Lambda < 1$. Hence $\{v(t_0+j\omega)\}_{j=1}^{\infty}$ converges to zero. Since $v(t)$ is also bounded on any compact interval, global attractivity follows.

Suppose that $t \in [t_{k-1}, t_k)$ with $t_0 < t_k \leq t_0 + \omega$ and define

$$y(t) := \exp\left[\sum_{i=1}^{k-1} \ln \delta_i + \sum_{i\in\mathcal{P}_u} \lambda_i T_i(t_0,t) - \sum_{i\in\mathcal{P}_s} \eta_i (T_i(t_0,t) - \Phi_i(t_0,t)\bar{\tau})\right].$$

Then it follows that

$$\begin{aligned}
y(t+\omega) =& y(t)\exp\left[\ln\delta_k + \ln\delta_{k+1} + \ldots + \ln\delta_m\right] \\
&\times \exp\left[\sum_{i\in\mathcal{P}_u}\lambda_i T_i(t, t_0+\omega) - \sum_{i\in\mathcal{P}_s}\eta_i(T_i(t, t_0+\omega) - \Phi_i(t, t_0+\omega)\bar{\tau})\right] \\
&\times \exp\left[\ln\delta_{m+1} + \ldots + \ln\delta_{k-1+m}\right] \\
&\times \exp\left[\sum_{i\in\mathcal{P}_u}\lambda_i T_i(t_0+\omega, t+\omega)\right] \\
&\times \exp\left[-\sum_{i\in\mathcal{P}_s}\eta_i(T_i(t_0+\omega, t+\omega) - \Phi_i(t_0+\omega, t+\omega)\bar{\tau})\right],
\end{aligned}$$

and so,

$$\begin{aligned}
y(t+\omega) =& y(t)\exp\left[\ln\delta_k + \ldots + \ln\delta_m\right] \\
&\times \exp\left[\sum_{i\in\mathcal{P}_u}\lambda_i T_i(t, t_0+\omega) - \sum_{i\in\mathcal{P}_s}\eta_i(T_i(t, t_0+\omega) - \Phi_i(t, t_0+\omega)\bar{\tau})\right] \\
&\times \exp\left[\ln\delta_1 + \ldots + \ln\delta_{k-1}\right] \\
&\times \exp\left[\sum_{i\in\mathcal{P}_u}\lambda_i T_i(t_0, t) - \sum_{i\in\mathcal{P}_s}\eta_i(T_i(t_0, t) - \Phi_i(t_0, t)\bar{\tau})\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(t+\omega) =& y(t)\exp\left[\sum_{i=1}^{m}\ln\delta_i + \sum_{i\in\mathcal{P}_u}\lambda_i T_i(t_0, t_0+\omega)\right] \\
&\times \exp\left[-\sum_{i\in\mathcal{P}_s}\eta_i(T_i(t_0, t_0+\omega) - \Phi_i(t_0, t_0+\omega)\bar{\tau})\right], \\
=& y(t)\exp\left[\sum_{i=1}^{m}\ln\delta_i + \sum_{i\in\mathcal{P}_u}\lambda_i\rho_i - \sum_{i\in\mathcal{P}_s}\eta_i(\rho_i - \bar{\tau})\right], \\
=& y(t)\Lambda.
\end{aligned}$$

Hence, $y(t+\omega) < y(t)$ for all $t \in [t_{k-1}, t_k)$ such that $t_0 < t_k \leq t_0 + \omega$. Similarly, $y(t+j\omega) < y(t)$ for $j = 1, 2, \ldots,$ and $t \in [t_{k-1}, t_k)$ with $t_0 < t_k \leq t_0 + \omega$. This implies that

$y(t)$ achieves its maximum on $[t_0, t_0 + \omega]$. An upper bound is given by

$$A = \exp\left[\sum_{1\le i\le m:\delta_i>1} \ln \delta_i + \sum_{i\in\mathcal{P}_u} \lambda_i \rho_i\right].$$

For any $\epsilon > 0$ choose

$$\chi = \left[\frac{c_1}{2c_2 A}\right]^{\frac{1}{2}} \epsilon$$

where $c_1 = \lambda_{\min}(P)$ and $c_2 = \lambda_{\max}(P)$. Then $\|\phi_0\|_{\bar{\tau}} < \chi$ implies that $\|v_{t_0}\|_{\bar{\tau}} \le c_2\|\phi_0\|_{\bar{\tau}}^2 < c_2\chi^2 \le \frac{c_1\epsilon^2}{2A}$. It follows that

$$\|x(t)\| \le \left(\frac{\|v_{t_0}\|_{\bar{\tau}} A}{c_1}\right)^{\frac{1}{2}} < \epsilon$$

for all $t \ge t_0$. □

Example 4.2.4. *Consider the weakly nonlinear HISD (4.13) with $\mathcal{P} = \{1, 2, 3, 4\}$, $t_0 = 0$, distributed delay $\tau = 0.1$,*

$$A_1 = \begin{pmatrix} -3 & 2 \\ -2 & -3.8 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -22.1 & -11.3 \\ 11.3 & -22.3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1.8 & 0.1 \\ -0.1 & 1.6 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 3.1 & 0 \\ 0 & 3.1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0.12 & 0.13 \\ 0.3 & 0.1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0.4 & 0.4 \\ 0.3 & 0.1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1.05 & 1.1 \\ 1.11 & 1.04 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} -0.3 & -0.1 \\ -0.1 & -0.3 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -0.2 & 0 \\ 0 & -0.2 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -0.4 & 0 \\ 0.01 & -0.4 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0.1 & 0.1 \\ 0.2 & 0.1 \end{pmatrix},$$

$B_1 = B_2 = B_3 = B_4 = 0$, $F_2(t, x_t) = 0$, $F_4(t, x_t) = 0$, and

$$F_1(t, x_t) = \begin{pmatrix} 0.3sin(x_2(t)) \\ 0 \end{pmatrix}, \quad F_3(t, x_t) = \begin{pmatrix} 0 \\ 0.1\ln(cosh(x_1(t))) \end{pmatrix}.$$

The matrices A_1 and A_2 are Hurwitz while A_3 and A_4 have eigenvalues with positive real part. The switching rule is assumed to take the following form for $k = 1, 2, \ldots,$

$$\sigma = \begin{cases} 4, & t \in [t_{4k-4}, t_{4k-3}), \\ 1, & t \in [t_{4k-3}, t_{4k-2}), \\ 3, & t \in [t_{4k-2}, t_{4k-1}), \\ 2, & t \in [t_{4k-1}, t_{4k}), \end{cases} \tag{4.20}$$

where

$$t_k - t_{k-1} = \begin{cases} 0.4, & k = 1, 5, 9, \dots, \\ 1, & k = 2, 6, 10, \dots, \\ 0.4, & k = 3, 7, 11, \dots, \\ 0.2, & k = 4, 8, 12, \dots \end{cases} \tag{4.21}$$

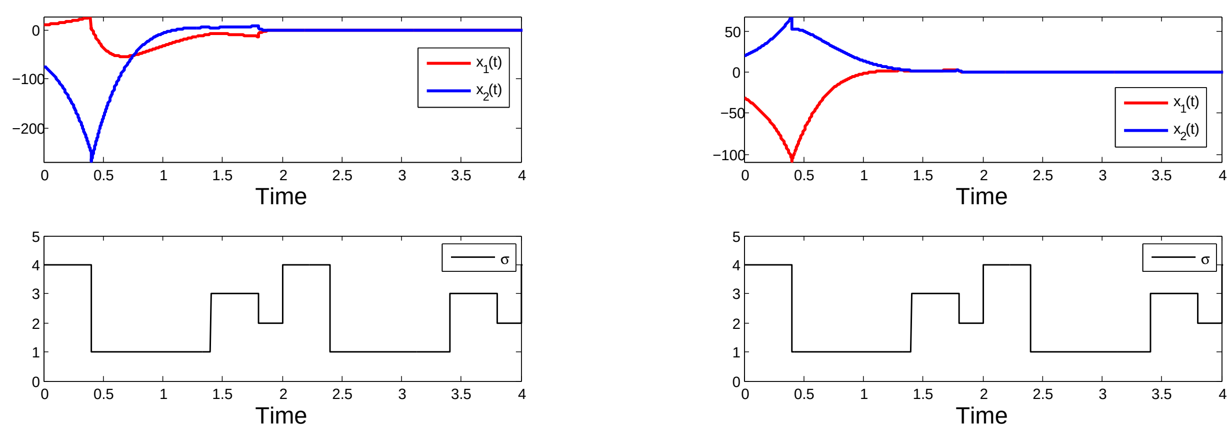
The switching rule is periodic with $\rho_1 = 0.4$, $\rho_2 = 1$, $\rho_3 = 0.4$, $\rho_4 = 0.2$ *and hence* $\omega = 2$. *Let* $\vartheta_{11} = (0.3)^2$, $\vartheta_{12} = 0$, $\vartheta_{13} = (0.1)^2$, $\vartheta_{14} = 0$, $\vartheta_{2i} = 0$, *and* $\vartheta_{3i} = 0$ *for* $i \in \mathcal{P}$. *Let*

$$P_1 = \begin{pmatrix} 0.808 & -0.0382 \\ -0.0382 & 0.678 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0.113 & 0.0002 \\ 0.0002 & 0.1122 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\beta_1 = 0.0008$, $\beta_2 = 0.0005$, $\beta_3 = 0.004$, $\beta_4 = 0.04$, $\alpha_1 = 2.85$, $\alpha_2 = 33.48$, $\alpha_3 = -6.64$, $\alpha_4 = -9.20$, $\mu = 1.23$, $\delta_1 = 0.925$, $\delta_2 = 0.791$, $\delta_3 = 0.449$, $\delta_4 = 1.92$. *Hence* $\mathcal{P}_s = \{1, 2\}$ *and* $\mathcal{P}_u = \{3, 4\}$. *Choose* $\eta_1 = 2.9$, $\eta_2 = 33$, $\lambda_3 = 6.64$, $\lambda_4 = 9.25$ *to get*

$$\sum_{i=1}^{4} \ln \delta_i + \sum_{i \in \mathcal{P}_u} \lambda_i \rho_i - \sum_{i \in \mathcal{P}_s} \eta_i (\rho_i - \tau) = -0.0285$$

and hence the origin is globally asymptotically stable by Corollary 4.2.9 (see Figure 4.3).



(a) $\phi_0(s) = (10, -70)$. (b) $\phi_0(s) = (-30, 20)$.

Figure 4.3: Simulation of Example 4.2.4.

4.3 Constrained Switching Stability for Systems with Infinite Delay

In the previous section the stability theory of switched integro-differential equations with distributed delays was investigated. The main approach was to use a Halanay-like switching inequality in order to establish sufficient conditions for stability. Under that technique, it is not possible to consider infinite delay since the method looks backwards on each switching interval and assumes $t_k - t_{k-1} \geq \tau$. Further, we focused on a class of linear HISD with nonlinear perturbations.

The main contribution of this section is to extend the current literature by finding verifiable sufficient conditions for the stability of impulsive nonlinear switched systems with infinite time-delay. In doing so, we develop dwell-time constraints on the switching times which guarantee overall stability. No restrictions are made on the total power of the impulses (such as requiring the power of the impulses to go to zero for large time).

4.3.1 Problem Formulation and Background Literature

The authors Luo and Shen investigated the stability of functional differential equations with infinite delays and impulsive effects in [126–128, 130]. In the paper [126], the authors analyzed Volterra-type equations and gave stability results using Lyapunov functional theorems and boundedness results using Lyapunov function theorems. Disturbance impulsive effects of finite total power were considered. In the report [127], the authors Luo and Shen proved some uniform asymptotic stability theorems using Razumikhin-type conditions. Luo and Shen [128] extended these results to systems with infinite delays and stabilizing impulsive effects where the continuous portion of the system is destabilizing. Luo and Shen [130] furthered the investigations by developing uniform asymptotic stability results applicable to systems with finite and infinite delay and perturbation impulses of finite total power.

Li [93] analyzed a class of impulsive delay differential equations and proved global uniform asymptotic stability using Razumikhin-type conditions under stabilizing continuous portion and stabilizing impulses. The authors' results are applicable to eventually stabilizing impulsive effects (which extends the results of [128]). Li also analyzed functional differential equations with infinite delays in [95] using Razumikhin-like theorems to prove uniform stability under impulsive perturbations of finite total power. In [96], Li et al. investigated uniform stability under impulsive stabilization or impulsive disturbances. In their paper, they considered impulsive effects that depend on past history of the system state.

Of particular importance to our work in this section are the papers by Li [94] and Li and Fu [97], where the authors considered the following system:

$$\begin{cases} \dot{x} = f(t, x_t), & t \neq T_k, \\ \Delta x = g_k(t, x(t^-)), & t = T_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases} \tag{4.22}$$

where $x \in \mathbb{R}^n$ is the state; $x_t \in PCB([\alpha, 0], \mathbb{R}^n)$ is given by $x_t(s) = x(t+s)$ for $s \in [\alpha, 0]$ where $-\infty \leq \alpha \leq 0$ and $[\alpha, 0]$ is understood to be $(-\infty, 0]$ when the delay is infinite. The initial function is $\phi_0 \in PCB([\alpha, 0], \mathbb{R}^n)$. In [94], the authors proved global exponential stability of the trivial solution of (4.22) under the case where eventually stabilizing impulsive effects could counteract a destabilizing continuous portion of the system in the following Razumikhin-like theorem.

Theorem 4.3.1. [94]
Assume that there exist functions $V \in \nu_0$, and constants $c_1 > 0$, $c_2 > 0$, $\lambda > 0$, $\delta_k \geq 0$, $q > 1$, $\gamma > 0$, such that for $k \in \mathbb{N}$,

(i) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$ *for all* $t \in [t_0 + \alpha, \infty)$ *and* $x \in \mathbb{R}^n$;

(ii) along the solution of (4.22) *for* $t \neq T_k$,

$$D^+V(t, \psi(0)) \leq \lambda V(t, \psi(0)),$$

whenever $e^{\gamma s}V(t+s, \psi(s)) \leq qV(t, \psi(0))$ *for* $s \in [\alpha, 0]$;

(iii) for each T_k *and for all* $x \in \mathbb{R}^n$,

$$V(T_k, x + g_k(T_k, x)) \leq \frac{1+\delta_k}{q} V(T_k^-, x)$$

with $\sum_{k=1}^{\infty} \delta_k < \infty$;

(iv) $\rho\lambda < \ln q$ *where* $\rho = \sup_{k\in\mathbb{N}}\{T_k - T_{k-1}\}$.

Then the trivial solution of (4.22) *is globally exponentially stable.*

In the report [97], Li and Fu found verifiable sufficient conditions for global exponential stability of the trivial solution of (4.22) for destabilizing impulsive effects of finite total power.

Theorem 4.3.2. [97]
Assume that there exist functions $V \in \nu_0$, constants $p > 0$, $c_1 > 0$, $c_2 > 0$, $\lambda > 0$, $q > 1$, $\gamma > 0$, and $\delta_k \geq 0$ such that

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$ *for all* $t \in [t_0 + \alpha, \infty)$ *and* $x \in \mathbb{R}^n$*;*

(ii) along the solution of (4.22) *for* $t \neq T_k$*,*

$$D^+V(t,\psi(0)) \leq -\lambda V(t,\psi(0)),$$

whenever $e^{\gamma s}V(t+s,\psi(s)) \leq qV(t,\psi(0))$ *for* $s \in [\alpha, 0]$*;*

(iii) for each T_k *and for all* $x \in \mathbb{R}^n$*,*

$$V(T_k, x + g_k(T_k, x)) \leq (1+\delta_k)V(T_k^-, x)$$

with $\sum_{k=1}^{\infty} \delta_k < \infty$*;*

(iv) $\mu\lambda > \ln q$ *where* $\mu = \inf_{k\in\mathbb{N}}\{T_k - T_{k-1}\}$.

Then the trivial solution of (4.22) *is globally exponentially stable.*

Remark 4.3.1. *The condition* $\sum_{k=1}^{\infty} \delta_k < \infty$ *in Theorem 4.3.2 necessarily requires the power of the disturbance impulses to go to zero for large time (that is,* $\delta_k \to 0$ *as* $k \to \infty$*).*

Motivated by this work, we consider the following family of impulsive subsystems with unbounded time-delay:

$$\begin{cases} \dot{x} = f_i(t, x_t), & t \neq T_k, \\ \Delta x = g_k(t, x(t^-)), & t = T_k, \quad k \in \mathbb{N}, \end{cases} \tag{4.23}$$

where the family $\{f_i : i \in \mathcal{P}\}$ is parameterized by the finite set $\mathcal{P} := \{1, \ldots, m\}$, where $m > 1$ is an integer. Assume that each functional $f_i : \mathbb{R} \times PCB([\alpha, 0], \mathbb{R}^n) \to \mathbb{R}^n$ satisfies $f_i(t, 0) \equiv 0$ for all $t \geq t_0$, $i \in \mathcal{P}$, and the impulsive functions $g_k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy $g_k(t, 0) \equiv 0$ for $t \geq t_0$, $k \in \mathbb{N}$. The impulsive moments are assumed to satisfy $t_0 < T_1 < T_2 < \ldots < T_k < \ldots$ with $T_k \to \infty$ as $k \to \infty$.

Parameterized by an initial function $\phi_0 \in PCB([\alpha, 0], \mathbb{R}^n)$ and a switching rule σ, system (4.23) is a nonlinear HISD with unbounded delay:

$$\begin{cases} \dot{x} = f_\sigma(t, x_t), & t \neq T_k, \\ \Delta x = g_k(t, x(t^-)), & t = T_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases} \tag{4.24}$$

where $t_0 \in \mathbb{R}$ is the initial time. For each $i \in \mathcal{P}$, assume that f_i is composite-PCB and locally Lipschitz. Assume that g_k is continuous in both variables for each $k \in \mathbb{N}$. Then the conditions imposed on $\{f_i\}$, σ, and $\{g_k\}$, guarantee the existence of a unique solution to system (4.24). The goal of the present section is to investigate the stability properties of system (4.24) as follows: given a set of stable and unstable vector fields $\{f_i\}$ and stabilizing and disturbance impulsive effects $\{g_k\}$, determine conditions on the switching rule, switching times and impulsive times such that the trivial solution of the switched system (4.24) is globally asymptotically stable.

Definition 4.3.1. *Let $x(t) = x(t; t_0, x_0)$ be the solution of the switched system (4.24). Then the trivial solution $x = 0$ is said to be*

(i) stable if for all $\epsilon > 0$, $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|\phi_0\|_{PCB} < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \geq t_0$;

(ii) uniformly stable if δ in (i) is independent of t_0, that is, $\delta(t_0, \epsilon) = \delta(\epsilon)$;

(iii) asymptotically stable if (i) holds and there exists a $\beta > 0$ such that $\|\phi_0\|_{PCB} < \beta$ implies

$$\lim_{t\to\infty} x(t) = 0;$$

(iv) uniformly asymptotically stable if (ii) holds and there exists a $\beta > 0$, independent of t_0, such that $\|\phi_0\|_{PCB} < \beta$ implies that for all $\eta > 0$, there exists a $T = T(\eta) > 0$ such that for all $t_0 \in \mathbb{R}_+$, $\|x(t)\| < \eta$ if $t \geq t_0 + T(\eta)$;

(v) exponentially stable if there exist constants $\beta, \gamma, C > 0$ such that if $\|\phi_0\|_{PCB} < \beta$ then $\|x(t)\| \leq C\|\phi_0\|_{PCB} e^{-\gamma(t-t_0)}$ for all $t \geq t_0$;

(vi) globally exponentially stable if β in (vii) is arbitrary;

(vii) unstable if (i) fails to hold.

The stability properties in (iii), (iv), (v) are said to be global if they hold for arbitary β.

4.3.2 Razumikhin-type Theorems

Recall the dwell-time notions defined earlier: $T_i(t_0, t)$ denotes the Lebesgue measure of the total activation time of the i^{th} subsystem on the interval $[t_0, t]$ (hence $\bigcup_{i=1}^{m} T_i(t_0, t) = [t_0, t]$) and it is understood that $T_i(t_0, t) = 0$ if $t \leq t_0$. Further, $N(t_0, t)$ represents the total number of impulses on the interval $[t_0, t]$. The first stability theorem can be given as follows, which extends the work in [94, 97].

Theorem 4.3.3. *Assume that there exist functions $V \in \nu_0$, $c_1, c_2 \in \mathcal{K}_\infty$, $p_i, w_i \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, and constants $\delta_k \geq 0$, $q > 1$, $\gamma > 0$, $\eta_i \geq 0$, such that for $i \in \mathcal{P}$ and $k \in \mathbb{N}$,*

(i) $c_1(\|x\|) \leq V(t, x) \leq c_2(\|x\|)$ *for all* $t \in [t_0 + \alpha, \infty)$ *and* $x \in \mathbb{R}^n$*;*

(ii) along the solution of the i^{th} subsystem of (4.23) for $t \neq T_k$,

$$D^+V|_i(t, \psi(0)) \leq -p_i(t) w_i(V(t, \psi(0))),$$

whenever $e^{\gamma s} V(t + s, \psi(s)) \leq qV(t, \psi(0))$ *for* $s \in [\alpha, 0]$*;*

(iii) for each T_k and for all $x \in \mathbb{R}^n$,

$$V(T_k, x + g_k(T_k, x)) \leq \frac{\delta_k}{q} V(T_k^-, x);$$

(iv) there exists a constant $c > 0$ such that for all $t \geq t_0$

$$\left(\prod_{j=1}^{N(t_0,t)} \delta_j \right) \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0, t) \right] \leq \exp[-c(t - t_0)],$$

where $\eta_i \leq \min\{\gamma, \rho_i\}$ *and* $\rho_i = \inf_{s \geq 0} p_i(s) \inf_{s > 0} w_i(s)/s$*.*

Then the trivial solution of (4.24) is globally exponentially stable.

Proof. Let $v(t) = V(t, x(t))$ where $x(t)$ is a solution of (4.24). First we consider the case where $t_k = T_k$ and aim to show that for $t \geq t_0$,

$$v(t) \leq qc_2(\|\phi_0\|_{PCB}) \left(\prod_{j=1}^{N(t_0,t)} \delta_j \right) \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0, t) \right]. \tag{4.25}$$

Consider the interval $[t_0, t_1)$, then we claim that

$$v(t) \leq qc_2(\|\phi_0\|_{PCB})e^{-\eta_{i_1}(t-t_0)}$$

for all $t \in [t_0, t_1)$. If the claim is not true, then there exists a time $t \in (t_0, t_1)$ such that $v(t) > qc_2(\|\phi_0\|_{PCB})e^{-\eta_{i_1}(t-t_0)}$. Denote the first such time by

$$t^* = \inf\{t \in (t_0, t_1) : v(t) > qc_2(\|\phi_0\|_{PCB})e^{-\eta_{i_1}(t-t_0)}\}.$$

Then $v(t^*) = qc_2(\|\phi_0\|_{PCB})e^{-\eta_{i_1}(t^*-t_0)}$ and $v(t) \leq qc_2(\|\phi_0\|_{PCB})e^{-\eta_{i_1}(t-t_0)}$ for $t \in [t_0, t^*)$. Since $v(t_0 + s) \leq c_2(\|\phi_0\|_{PCB}) < qc_2(\|\phi_0\|_{PCB})$ for $s \in [\alpha, 0]$, there exists another time t_* on the interval $[t_0, t^*)$ defined as

$$t_* = \sup\{t \in [t_0, t^*) : v(t) \leq c_2(\|\phi_0\|_{PCB})e^{-\eta_{i_1}(t-t_0)}\}.$$

Note that $t_* < t^*$ since $q > 1$. Observe that for $t \in [t_0, t^*)$ and $s \in [\alpha, 0]$,

$$v(t+s) \leq \begin{cases} qc_2(\|\phi_0\|_{PCB})e^{-\eta_{i_1}(t+s-t_0)}, & t+s \geq t_0, \\ qc_2(\|\phi_0\|_{PCB}), & t+s < t_0. \end{cases} \tag{4.26}$$

Since $T_i(t_0, t) = 0$ if $t \leq t_0$, equation (4.26) can be written as

$$v(t+s) \leq qc_2(\|\phi_0\|_{PCB})e^{-\eta_{i_1}T_{i_1}(t_0,t+s)}.$$

Then for $s \in [\alpha, 0]$ and $t \in [t_*, t^*]$,

$$\begin{aligned} e^{\gamma s}v(t+s) &\leq e^{\eta_{i_1}s}v(t+s), \\ &= e^{-\eta_{i_1}(t-t_0)}v(t+s)e^{\eta_{i_1}(t+s-t_0)}, \\ &\leq e^{-\eta_{i_1}T_{i_1}(t_0,t)}v(t+s)e^{\eta_{i_1}T_{i_1}(t_0,t+s)}, \\ &\leq e^{-\eta_{i_1}T_{i_1}(t_0,t)}qc_2(\|\phi_0\|_{PCB}), \\ &\leq e^{-\eta_{i_1}T_{i_1}(t_0,t)}qv(t)e^{\eta_{i_1}T_{i_1}(t_0,t)}, \\ &= qv(t). \end{aligned}$$

Thus $D^+v(t) \leq -p_{i_1}(t)w_{i_1}(v(t))$. For any $t \in [t_0 + \alpha, \infty)$, define

$$\mathcal{L}(t) := v(t)\exp\left[\sum_{i=1}^{m}\eta_i T_i(t_0, t)\right].$$

Then for $t \in [t_*, t^*]$, $\mathcal{L}(t) = v(t)\exp[\eta_{i_1}(t-t_0)]$ and $\mathcal{L}(t_*) = c_2(\|\phi_0\|_{PCB}) < \mathcal{L}(t^*) = qc_2(\|\phi_0\|_{PCB})$. Also,

$$\begin{aligned} D^+\mathcal{L}(t) &= \eta_{i_1} e^{\eta_{i_1}(t-t_0)} v(t) + e^{\eta_{i_1}(t-t_0)} D^+ v(t), \\ &\leq \mathcal{L}(t)\left[\eta_{i_1} - p_{i_1}(t)\frac{w_{i_1}(v(t))}{v(t)}\right], \\ &\leq \mathcal{L}(t)\left[\eta_{i_1} - \rho_{i_1}\right], \\ &\leq 0. \end{aligned}$$

This is a contradiction since $\mathcal{L}(t_*) < \mathcal{L}(t^*)$ by the definitions of t_* and t^*. Thus $v(t) \leq qc_2(\|\phi_0\|_{PCB})e^{-\eta_{i_1}(t-t_0)}$ for all $t \in [t_0, t_1)$.

Assume the result holds for $t \in [t_{k-1}, t_k)$, that is,

$$v(t) \leq qc_2(\|\phi_0\|_{PCB}) \left(\prod_{j=1}^{N(t_0,t)} \delta_j\right) \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t)\right].$$

To show the result holds for $t \in [t_k, t_{k+1})$, suppose that the claim is not true. Then there exists a time $t \in [t_k, t_{k+1})$ such that

$$v(t) > qc_2(\|\phi_0\|_{PCB})y(t)$$

where

$$y(t) := \left(\prod_{j=1}^{N(t_0,t)} \delta_j\right) \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t)\right]$$

Define

$$t^\Delta = \inf\{t \in (t_k, t_{k+1}) : v(t) > qc_2(\|\phi_0\|_{PCB})y(t)\}.$$

Then

$$v(t^\Delta) = qc_2(\|\phi_0\|_{PCB}) \left(\prod_{j=1}^{N(t_0,t^\Delta)} \delta_j\right) \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t^\Delta)\right]$$

and

$$v(t) \leq qc_2(\|\phi_0\|_{PCB}) \left(\prod_{j=1}^{N(t_0,t)} \delta_j\right) \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t)\right]$$

for $t \in [t_k, t^\Delta)$.

Since $\prod_{i=1}^{N(t_0,t)} \delta_i = 1$ if $N(t_0,t) = 0$ and $T_i(t_0,t) = 0$ if $t \le t_0$, it is also true that

$$v(t) \le qc_2(\|\phi_0\|_{PCB}) \left(\prod_{j=1}^{N(t_0,t)} \delta_j \right) \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t) \right]$$

for $t \in [t_0 + \alpha, t^\Delta)$. At the switching/impulsive time t_k,

$$\begin{aligned} v(t_k) &\le \frac{\delta_k}{q} v(t_k^-), \\ &\le \frac{\delta_k}{q} qc_2(\|\phi_0\|_{PCB}) \left(\prod_{j=1}^{N(t_0,t_{k-1})} \delta_j \right) \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t_k) \right], \\ &= c_2(\|\phi_0\|_{PCB}) \delta_1 \delta_2 \cdot \delta_k \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t_k) \right], \\ &= c_2(\|\phi_0\|_{PCB}) \left(\prod_{j=1}^{N(t_0,t_k)} \delta_j \right) \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t_k) \right]. \end{aligned}$$

Thus,

$$v(t_k) \exp\left[\sum_{i=1}^{m} \eta_i T_i(t_0,t_k) \right] < qc_2(\|\phi_0\|_{PCB}) \prod_{j=1}^{N(t_0,t_k)} \delta_j.$$

Therefore,

$$v(t_k) \le c_2(\|\phi_0\|_{PCB}) \delta_1 \delta_2 \cdots \delta_k \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t_k) \right]$$

and, since $q > 1$, there exists a time on the interval $[t_k, t^\Delta)$ defined as follows:

$$t_\Delta = \sup\left\{ t \in [t_k, t^\Delta) : v(t) \le c_2(\|\phi_0\|_{PCB}) y(t) \right\}.$$

Then

$$v(t_\Delta) = c_2(\|\phi_0\|_{PCB}) \delta_1 \delta_2 \cdots \delta_k \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t_\Delta) \right]$$

and for $t \in [t_\Delta, t^\Delta]$,

$$v(t) > c_2(\|\phi_0\|_{PCB}) \delta_1 \delta_2 \cdots \delta_k \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t) \right].$$

Since $\eta_i \leq \gamma$ for all $i \in \mathcal{P}$ then it follows that for $s \in [\alpha, 0]$ and $t \in [t_\Delta, t^\Delta]$,

$$\begin{aligned} e^{\gamma s} v(t+s) &\leq \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t+s,t)\right] v(t+s), \\ &\leq \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t)\right] v(t+s) \exp\left[\sum_{i=1}^{m} \eta_i T_i(t_0,t+s)\right], \end{aligned}$$

hence,

$$\begin{aligned} e^{\gamma s} v(t+s) &\leq \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t)\right] q c_2(\|\phi_0\|_{PCB}) \delta_1 \delta_2 \cdots \delta_k, \\ &\leq \exp\left[-\sum_{i=1}^{m} \eta_i T_i(t_0,t)\right] q v(t) \exp\left[\sum_{i=1}^{m} \eta_i T_i(t_0,t)\right], \\ &= q v(t). \end{aligned}$$

Then $D^+ v(t) \leq -p_{i_{k+1}}(t) w_{i_{k+1}}(v(t))$ and

$$\begin{aligned} D^+ \mathcal{L}(t) &= \exp\left[\sum_{i=1}^{m} \eta_i T_i(t_0,t)\right] \left[\eta_{i_{k+1}} v(t) + D^+ v(t)\right], \\ &\leq \mathcal{L}(t) \left[\eta_{i_{k+1}} - p_{i_{k+1}}(t) \frac{w_{i_{k+1}}(v(t))}{v(t)}\right], \\ &\leq \mathcal{L}(t) \left[\eta_{i_{k+1}} - \rho_{i_{k+1}}\right], \\ &\leq 0, \end{aligned}$$

which contradicts the fact that $\mathcal{L}(t_\Delta) < \mathcal{L}(t^\Delta)$. Thus the claim holds on $[t_k, t_{k+1})$.

To prove (4.25) holds for $t_k \neq T_k$, construct a new sequence of times $\{z_k\}_{k=1}^{\infty}$ by concatenating $\{t_k\}_{k=1}^{\infty}$ and $\{T_k\}_{k=1}^{\infty}$. Each element z_k is equal to either t_j or T_j for some $j \in \mathbb{N}$, and the sequence is properly ordered so that $z_{k-1} < z_k$. If $t_j = T_j$ for some value of j, then only one associated element appears in $\{z_k\}$. For each $j \in \mathbb{N}$, if $z_j \notin \{t_k\}_{k=1}^{\infty}$ then z_j is an impulse time and $\sigma(z_j^-) = \sigma(z_j)$. If $z_j \notin \{T_k\}_{k=1}^{\infty}$ then z_j is a switching time and $g_j(z_j, x(z_j^-)) \equiv 0$ (i.e. $\delta_j = 1$). The above arguments hold and hence for $t \geq t_0$,

$$\|x(t)\| \leq c_1^{-1} \left(q c_2(\|\phi_0\|_{PCB}) \exp[-c(t-t_0)]\right)$$

and the result follows. □

Remark 4.3.2. *In Theorem 4.3.3 the constants η_i represent an estimate for the decay rate of the system state while the i^{th} stable subsystem is active (all subsystems are stable). Condition (iv) ensures that the combination of the switching portion and any destabilizing impulses is such that the overall switched system is stable.*

The following corollary can be given for when impulses occur at the switching times.

Corollary 4.3.4. *Suppose that $t_k = T_k$ for $k \in \mathbb{N}$. Suppose that there exists a function $V \in \nu_0$, and constants $c_1 > 0$, $c_2 > 0$, $p > 0$, $p_i \geq 0$, $\delta_k \geq 0$, $\zeta_k > 0$, $\beta > 1$, $q > 1$, $\gamma > 0$, $\eta_i \geq 0$, such that for $i \in \mathcal{P}$ and $k \in \mathbb{N}$,*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$ *for all* $t \in [t_0+\alpha, \infty)$ *and* $x \in \mathbb{R}^n$;

(ii) along the solution of the i^{th} subsystem of (4.23) for $t \neq T_k$,

$$D^+V|_i(t,\psi(0)) \leq -p_i V(t,\psi(0)),$$

whenever $e^{\gamma s}V(t+s,\psi(s)) \leq qV(t,\psi(0))$ *for* $s \in [\alpha, 0]$;

(iii) for each t_k and for all $x \in \mathbb{R}^n$,

$$V(t_k, x + g_k(t_k,x)) \leq \frac{\delta_k}{q} V(t_k^-, x);$$

(iv) $t_k - t_{k-1} \geq \zeta_k$ *and* $\ln(\beta\delta_k) - \eta_{i_k}\zeta_k \leq 0$ *where* $\eta_i \leq \min\{\gamma, p_i\}$.

Then the trivial solution of (4.24) is globally asymptotically stable.

Proof. Note that $t_k = T_k$ implies that $N(t_0,t) = k-1$ for $t \in [t_{k-1}, t_k)$. From equation (4.25), for $t \in [t_{k-1}, t_k)$,

$$\begin{aligned}
v(t) \leq & qc_2\|\phi_0\|^p_{PCB} \exp\left[\sum_{i=1}^{k-1} \ln\delta_i - \sum_{i=1}^{m} \eta_i T_i(t_0,t)\right], \\
\leq & qc_2\|\phi_0\|^p_{PCB} \exp\left[\ln\delta_1 - \eta_{i_1}\zeta_1 + \ldots + \ln\delta_{k-1} - \eta_{i_{k-1}}\zeta_{k-1} - \eta_{i_k}\zeta_k\right], \\
\leq & qc_2\|\phi_0\|^p_{PCB} \frac{1}{\beta^k(\inf_{k\in\mathbb{N}}\delta_k)} \beta\delta_1 e^{-\eta_{i_1}\zeta_1} \cdots \beta\delta_k e^{-\eta_{i_k}\zeta_k}, \\
\leq & \left(\frac{qc_2\|\phi_0\|^p_{PCB}}{\inf_{k\in\mathbb{N}}\delta_k}\right)\frac{1}{\beta^k},
\end{aligned}$$

and the result follows. □

We can also apply the Razumikhin technique to nonlinear integro-differential equations with unbounded delay and periodic switching. Assume that the switching times satisfy $h_k = t_k - t_{k-1}$ and $h_{k+m} = h_k$. Assume the switching rule σ satisfies $i_k = k$ and $i_{k+m} = i_k$. Denote the period of the switching rule by $\omega = h_1 + h_2 + \ldots + h_m$. Denote the set of all such periodic switching rules by $\mathcal{S}_{\text{periodic}} \subset \mathcal{S}$.

Corollary 4.3.5. *Assume that $\sigma \in \mathcal{S}_{periodic}$ and suppose that there exists a function $V \in \nu_0$, and constants $c_1 > 0$, $c_2 > 0$, $p > 0$, $p_i \geq 0$, $\delta_k \geq 0$, $q > 1$, $\gamma > 0$, $\eta_i \geq 0$, such that for $i \in \mathcal{P}$ and $k \in \mathbb{N}$,*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$ *for all* $t \in [t_0 + \alpha, \infty)$ *and* $x \in \mathbb{R}^n$;

(ii) along the solution of the i^{th} subsystem of (4.23) for $t \neq T_k$,

$$D^+V|_i(t,\psi(0)) \leq -p_iV(t,\psi(0)),$$

whenever $e^{\gamma s}V(t+s,\psi(s)) \leq qV(t,\psi(0))$ for $s \in [\alpha, 0]$;

(iii) for each T_k and for all $x \in \mathbb{R}^n$,

$$V(T_k, x + g_k(T_k,x)) \leq \frac{\delta_k}{q}V(T_k^-, x);$$

(iv) $\alpha_k = T_k - T_{k-1}$ ($\alpha_1 = T_1 - t_0$) satisfy $\alpha_{k+N} = \alpha_k$ where $N(t_0, t_0+\alpha) = N$, $\alpha = \alpha_1 + \ldots + \alpha_N$, and $\delta_k = \delta_{k+N}$. The constants $\eta_i \leq \min\{\gamma, p_i\}$ satisfy

$$\frac{1}{\alpha}\sum_{i=1}^{N}\ln\delta_i - \frac{1}{\omega}\sum_{i=1}^{m}\eta_ih_i < 0.$$

Then the trivial solution of (4.24) is globally asymptotically stable.

Proof. From equation (4.25),

$$v(t) \leq qc_2\|\phi_0\|^p_{PCB}\exp\left[\sum_{i=1}^{N(t_0,t)}\ln\delta_i - \sum_{i=1}^{m}\eta_iT_i(t_0,t)\right].$$

Let $z = lcm(\omega, \alpha)$[2], then for $j = 1, 2, \ldots,$

$$\begin{aligned} v(t_0 + jz) \leq & qc_2 \|\phi_0\|^p_{PCB} \exp\left[\sum_{i=1}^{N(t_0, t_0+jz)} \ln\delta_i - \sum_{i=1}^{m} \eta_i T_i(t_0, t_0 + jz)\right], \\ \leq & qc_2 \|\phi_0\|^p_{PCB} \exp\left[\frac{jz}{\alpha}\sum_{i=1}^{N} \ln\delta_i - \frac{jz}{\omega}\sum_{i=1}^{m} \eta_i T_i(t_0, t_0 + \omega)\right], \\ \leq & qc_2 \|\phi_0\|^p_{PCB} \exp\left[\frac{jz}{\alpha}\sum_{i=1}^{N} \ln\delta_i - \frac{jz}{\omega}\sum_{i=1}^{m} \eta_i h_i\right], \\ = & qc_2 \|\phi_0\|^p_{PCB} \Lambda^j. \end{aligned}$$

where

$$\Lambda = \exp\left[\frac{z}{\alpha}\sum_{i=1}^{N} \ln\delta_i - \frac{z}{\omega}\sum_{i=1}^{m} \eta_i h_i\right]$$

satisfies $0 < \Lambda < 1$. Hence $\{v(t_0 + jz)\}_{j=1}^{\infty}$ converges to zero. Since $v(t)$ is also bounded on any compact interval, global attractivity follows.

Suppose that $t \in [t_{k-1}, t_k)$ with $t_0 < t_k \leq t_0 + z$ and define

$$y(t) := \exp\left[\sum_{i=1}^{N(t_0, t)} \ln\delta_i - \sum_{i=1}^{m} \eta_i T_i(t_0, t)\right].$$

Then it follows that

$$\begin{aligned} y(t + z) = & y(t) \exp\left[\ln\delta_k + \ln\delta_{k+1} + \ldots + \ln\delta_{zN/\alpha} - \sum_{i=1}^{m} \eta_i T_i(t, t_0 + z)\right] \\ & \times \exp\left[\ln\delta_{zN/\alpha+1} + \ldots + \ln\delta_{k-1+zN/\alpha} - \sum_{i=1}^{m} \eta_i T_i(t_0 + z, t + z)\right]. \end{aligned}$$

[2]Least common multiple of ω and α.

Since $\delta_k = \delta_{k+N}$,

$$\begin{aligned} y(t+z) =& y(t) \exp\left[\ln \delta_k + \ldots + \ln \delta_{zN/\alpha} - \sum_{i=1}^{m} \eta_i T_i(t, t_0+z)\right] \\ & \times \exp\left[\ln \delta_1 + \ldots + \ln \delta_{k-1} - \sum_{i=1}^{m} \eta_i T_i(t_0, t)\right], \\ =& y(t) \exp\left[\sum_{i=1}^{zN/\alpha} \ln \delta_i - \sum_{i=1}^{m} \eta_i T_i(t_0, t_0+z)\right], \\ =& y(t) \exp\left[\frac{z}{\alpha} \sum_{i=1}^{N} \ln \delta_i - \frac{z}{\omega} \sum_{i=1}^{m} \eta_i h_i\right], \\ =& y(t)\Lambda. \end{aligned}$$

Hence, $y(t+z) < y(t)$ for all $t \in [t_{k-1}, t_k)$ with $t_0 < t_k \leq t_0 + z$. Similarly, $y(t+jz) < y(t)$ for all $t \in [t_{k-1}, t_k)$ with $t_0 < t_k \leq t_0 + z$. This implies that $y(t)$ achieves its maximum on $[t_0, t_0+z]$. An upper bound is given by

$$A = \max\left\{1, \exp\left[\frac{z}{\alpha} \sum_{1 \leq i \leq N : \delta_i > 1} \ln \delta_i\right]\right\}.$$

For any $\epsilon > 0$ choose

$$\chi = \left(\frac{c_1}{c_2 A}\right)^{\frac{1}{p}} \frac{\epsilon}{2}$$

then $\|\phi_0\|_{PCB} < \chi$ implies that

$$v(t_0) \leq c_2 \|\phi_0\|_{PCB}^p < c_2 \chi^p \leq c_2 \left(\frac{c_1}{c_2 A}\right) \left(\frac{\epsilon}{2}\right)^p.$$

Since $c_1 \|x(t)\|^p \leq c_2 \|\phi_0\|_{PCB}^p A$, it follows that for all $t \geq t_0$

$$\|x(t)\| \leq \left(\frac{c_2 \left(\frac{c_1}{c_2 A}\right) \left(\frac{\epsilon}{2}\right)^p A}{c_1}\right)^{\frac{1}{p}} < \epsilon.$$

□

Remark 4.3.3. *There is no restrictive requirement on the long-term behaviour of the disturbance impulses in the results here (that is, $\delta_k \to 1$ as $k \to \infty$ is not required).*

4.3.3 Results for a Class of Nonlinear HISD with Infinite Delay

In this section we apply the Razumikhin-type theorems found in Section 4.3.2 to a class of nonlinear hybrid and impulsive systems with unbounded delay to develop easily verifiable sufficient conditions for stability. Consider the following class of HISD with unbounded delay:

$$\begin{cases} \dot{x} = A_\sigma x(t) + F_\sigma(t, x_t) + \int_{-\infty}^{0} \Psi_\sigma(t, s, x(t+s))ds, & t \neq t_k, \\ \Delta x = E_k x(t), & t = t_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases} \tag{4.27}$$

where $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$ is the switching rule, $F_i : \mathbb{R} \times PC([-\bar{\tau}, 0], \mathbb{R}^n)$ is a finite-delay term, $A_i \in \mathbb{R}^{n\times n}$ are constant real matrices for $i \in \mathcal{P}$, the vector-valued functionals Ψ_i are continuous on $\mathbb{R}_+ \times (-\infty, 0] \times \mathbb{R}^n$, and $E_k \in \mathbb{R}^{n\times n}$ are real constant matrices for $k \in \mathbb{N}$.

Corollary 4.3.6. *Assume that there exist constants $\tau > 0$, $r > 0$, $\vartheta_{1i} \geq 0$, $\vartheta_{2i} \geq 0$, $\vartheta_{3i} \geq 0$, $\eta_i \geq 0$, $p_i \geq 0$, $\beta > 1$, $q > 1$, $\gamma > 0$, $L_i > 0$, $m_i \in C((-\infty, 0], \mathbb{R}^n)$, and a positive definite symmetric matrix P such that for $i \in \mathcal{P}$,*

(i) $\|\Psi_i(t, s, v)\| \leq m_i(s)\|v\|$ *for all* $(t, s, v) \in \mathbb{R}_+ \times (-\infty, 0] \times \mathbb{R}^n$;

(ii) $\int_{-\infty}^{0} m_i(s) \exp[-\gamma s/2]ds \leq L_i$;

(iii) *for* $t \geq t_0$ *and* $\psi \in PC([-\bar{\tau}, 0], \mathbb{R}^n)$,

$$\|F_i(t, \psi)\|^2 \leq \vartheta_{1i}\|\psi(0)\|^2 + \vartheta_{2i}\|\psi(-r)\|^2 + \vartheta_{3i} \int_{-\tau}^{0} \|\psi(s)\|^2 ds,$$

where $\bar{\tau} = \max\{r, \tau\}$;

(iv) $t_k - t_{k-1} \geq \zeta_k$ *and* $\ln(\beta\delta_k) - \eta_{i_k}\zeta_k \leq 0$ *where* $\eta_i \leq \min\{\gamma, p_i\}$, $\delta_k = q\lambda_{max}[P^{-1}((I + E_k)^T P(I + E_k))]$, *and*

$$\lambda_{max}[P^{-1}(A_i^T P + PA_i + P^2 + \vartheta_{1i}I)] + \lambda_{max}(P^{-1}) \left[\vartheta_{2i} q e^{\gamma r} + \vartheta_{3i} q e^{\gamma s} \left(\frac{e^{\gamma\tau} - 1}{\gamma}\right) + 2\sqrt{q\lambda_{max}(P^T P)} L_i\right] \leq -p_i.$$

Then the trivial solution of (4.27) is globally asymptotically stable.

Proof. Let $V = x^T P x$ and take the time-derivative along the solution of the i^{th} subsystem of (4.27),

$$\begin{aligned}
\dot{V}|_i =& x^T(t)(A_i^T P + PA_i)x(t) + 2x^T(t)PF_i(t,x_t) + 2x^T(t)P\int_{-\infty}^{0}\Psi_i(t,s,x(t+s))ds, \\
\leq& x^T(t)(A_i^T P + PA_i + P^2)x(t) + \vartheta_{1i}x^T(t)x(t) + \vartheta_{2i}x^T(t-r)x(t-r) \\
&+ \vartheta_{3i}\int_{-\tau}^{0} x^T(t+s)x(t+s)ds + 2\|x\|\|P\|\int_{-\infty}^{0}\|\Psi_i\|ds, \\
\leq& \lambda_{\max}[P^{-1}(A_i^T P + PA_i + P^2 + \vartheta_{1i}I)]x^T(t)Px(t) + \vartheta_{2i}x^T(t-r)x(t-r) \\
&+ \vartheta_{3i}\int_{-\tau}^{0} x^T(t+s)x(t+s)ds + 2\|x\|\|P\|\int_{-\infty}^{0} m_i(s)\|x(t+s)\|ds
\end{aligned}$$

If $e^{\gamma s}V(t+s,x(t+s)) \leq qV(t,x(t))$ for all $s \in [\alpha, 0]$ then

$$\lambda_{\min}(P)x^T(t+s)x(t+s) \leq qe^{-\gamma s}x^T(t)Px(t)$$

and hence,

$$\begin{aligned}
x^T(t-r)x(t-r) \leq& \frac{qe^{\gamma r}}{\lambda_{\min}(P)}x^T(t)Px(t), \\
\|x(t+s)\| \leq& \frac{\sqrt{q}e^{-\gamma s/2}}{\sqrt{\lambda_{\min}(P)}}\sqrt{x^T(t)Px(t)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\dot{V}|_i \leq& \lambda_{\max}[P^{-1}(A_i^T P + PA_i + P^2 + \vartheta_{1i}I)]x^T(t)Px(t) + \frac{\vartheta_{2i}qe^{\gamma r}}{\lambda_{\min}(P)}x^T(t)Px(t) \\
&+ \vartheta_{3i}\int_{-\tau}^{0}\frac{qe^{\gamma s}}{\lambda_{\min}(P)}x^T(t)Px(t)ds \\
&+ 2\|x\|\|P\|\int_{-\infty}^{0} m_i(s)\frac{\sqrt{q}e^{-\gamma s/2}}{\sqrt{\lambda_{\min}(P)}}\sqrt{x^T(t)Px(t)}ds,
\end{aligned}$$

so that

$$
\begin{aligned}
\dot{V}|_i \leq & \lambda_{\max}[P^{-1}(A_i^T P + PA_i + P^2 + \vartheta_{1i} I)] x^T(t) P x(t) + \frac{\vartheta_{2i} q e^{\gamma r}}{\lambda_{\min}(P)} x^T(t) P x(t) \\
& + \frac{\vartheta_{3i} q e^{\gamma s}}{\lambda_{\min}(P)} \left(\frac{e^{\gamma\tau} - 1}{\gamma} \right) x^T(t) P x(t) \\
& + 2 \frac{\sqrt{q}\|P\|}{\sqrt{\lambda_{\min}(P)}} \|x\| \sqrt{x^T(t) P x(t)} \left[\int_{-\infty}^{0} m_i(s) e^{-\frac{\gamma s}{2}} ds \right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\dot{V}|_i \leq & \lambda_{\max}[P^{-1}(A_i^T P + PA_i + P^2 + \vartheta_{1i} I)] x^T(t) P x(t) + \frac{\vartheta_{2i} q e^{\gamma r}}{\lambda_{\min}(P)} x^T(t) P x(t) \\
& + \frac{\vartheta_{3i} q e^{\gamma s}}{\lambda_{\min}(P)} \left(\frac{e^{\gamma\tau} - 1}{\gamma} \right) x^T(t) P x(t) + 2 \frac{\sqrt{q}\|P\| L_i}{\lambda_{\min}(P)} x^T(t) P x(t).
\end{aligned}
$$

Hence $\dot{V}|_i \leq -p_i V(x(t))$. At the impulsive times $t = t_k$,

$$
\begin{aligned}
V(t_k) = & x^T(t_k) P x(t_k), \\
= & [(I + E_k) x(t_k^-)]^T P [(I + E_k) x(t_k^-)], \\
= & x^T(t_k^-) [(I + E_k)^T P (I + E_k)] x(t_k^-), \\
\leq & \lambda_{\max}[P^{-1}((I + E_k)^T P (I + E_k))] V(x(t_k^-)), \\
= & \frac{\delta_k}{q} V(x(t_k^-)).
\end{aligned}
$$

All the conditions of Corollary 4.3.4 are satisfied, and hence the trivial solution of (4.27) is globally asymptotically stable. □

The Razumikhin-type theorems are also easily applied to the following scalar system:

$$
\begin{cases}
\dot{x} = -a_\sigma x(t) + f_\sigma(t, x_t) + \displaystyle\int_{-\infty}^{0} \Psi_\sigma(t, s, x(t+s)) ds, & t \neq t_k \\
\Delta x = g_k(t, x(t^-)), & t = t_k, \\
x_{t_0} = \phi_0, & k \in \mathbb{N},
\end{cases}
\tag{4.28}
$$

where $a_i > 0$ are constants; the functionals $f_i : \mathbb{R}_+ \times PC([-\bar{\tau}, 0], \mathbb{R}) \to \mathbb{R}$ for some positive constant $\bar{\tau} > 0$ (finite delay portion of the system); Ψ_i is continuous on $\mathbb{R}_+ \times (-\infty, 0] \times \mathbb{R}$; $g_k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are the impulsive effects.

Corollary 4.3.7. *Assume that $\sigma \in \mathcal{S}_{periodic}$ and assume that there exist constants $\tau > 0$, $r > 0$, $\vartheta_{1i} \geq 0$, $\vartheta_{2i} \geq 0$, $\vartheta_{3i} \geq 0$, $\eta_i \geq 0$, $p_i \geq 0$, $\beta > 1$, $q > 1$, $\gamma > 0$, $m_i \in C((-\infty, 0], \mathbb{R})$ such that for $i \in \mathcal{P}$,*

(i) $|f_i(t,\psi)| \leq \vartheta_{1i}|\psi(0)| + \vartheta_{2i}|\psi(-r)| + \vartheta_{3i}\int_{-\tau}^{0}|\psi(s)|ds$ *for* $t \geq t_0$ *and* $\psi \in PC([-\bar{\tau}, 0], \mathbb{R})$, *where* $\bar{\tau} = \max\{r, \tau\}$*;*

(ii) $|\Psi_i(t,s,v)| \leq m_i(s)|v|$ *for* $t \geq t_0$, $s \leq 0$, $v \in \mathbb{R}$*;*

(iii) $-a_i + \vartheta_{1i} + q\left[\vartheta_{2i}e^{\gamma r} + \vartheta_{3i}\left(\frac{e^{\gamma\tau}-1}{\gamma}\right) + \int_{-\infty}^{0} m_i(s)e^{-\gamma s}ds\right] \leq -p_i$*;*

(iv) $|x + g_k(T_k, x)| \leq \frac{\delta_k}{q}|x|$ *for all* $x \in \mathbb{R}$*;*

(v) $\frac{1}{\alpha}\sum_{i=1}^{N}\ln\delta_i - \frac{1}{\omega}\sum_{i=1}^{m}\eta_i h_i < 0$ *where* $\alpha_k = T_k - T_{k-1}$ *(*$\alpha_1 = T_1 - t_0$*) satisfy* $\alpha_{k+N} = \alpha_k$ *with* $N(t_0, t_0 + \alpha) = N$, $\alpha = \alpha_1 + \ldots + \alpha_N$, $\delta_k = \delta_{k+N}$, *and* $\eta_i \leq \min\{\gamma, p_i\}$.

Then the trivial solution of (4.28) is globally asymptotically stable.

Proof. Let $V(x) = |x|$ then along the i^{th} subsystem of (4.28) for $t \neq t_k$ and $|x| \neq 0$,

$$\begin{aligned} D^+V\,|_i =& \frac{x(t)}{|x(t)|}\left[-a_i x(t) + f_i(t, x_t) + \int_{-\infty}^{0}\Psi_i(t,s,x(t+s))ds\right], \\ \leq& \frac{x(t)}{|x(t)|}\left[-a_i x(t) + \vartheta_{1i}|x(t)| + \vartheta_{2i}|x(t-r)| + \vartheta_{3i}\int_{-\tau}^{0}|x(t+s)|ds\right] \\ &+ \frac{x(t)}{|x(t)|}\left[\int_{-\infty}^{0} m_i(s)|x(t+s)|ds\right]. \end{aligned}$$

If the Razumikhin condition in Corollary 4.3.5 holds, then $|x(t+s)| \leq qe^{-\gamma s}|x(t)|$ for $s \leq 0$. Hence,

$$\begin{aligned} D^+V\,|_i \leq& |x(t)|\left[-a_i + \vartheta_{1i} + q\vartheta_{2i}e^{\gamma r} + q\vartheta_{3i}\int_{-\tau}^{0}e^{-\gamma s}ds + q\int_{-\infty}^{0}m_i(s)e^{-\gamma s}ds\right], \\ =& |x(t)|\left[-a_i + \vartheta_{1i} + q\vartheta_{2i}e^{\gamma r} + q\vartheta_{3i}\left(\frac{e^{\gamma\tau}-1}{\gamma}\right) + q\int_{-\infty}^{0}m_i(s)e^{-\gamma s}ds\right], \\ \leq& -p_i V(x(t)). \end{aligned}$$

At the impulsive moments, $V(x(T_k)) \leq \frac{\delta_k}{q}V(x(T_k^-))$. The result follows from Corollary 4.3.5. □

4.3.4 Examples

In this section we illustrate the main results found in Section 4.3.2 through some examples.

Example 4.3.1. *Consider the switched system (4.24) with $\mathcal{P} = \{1, 2\}$. Suppose that*

$$f_1(t, x_t) = \begin{pmatrix} -4.5x_1(t) + x_2(t) + x_2^2(t) + \int_{-\infty}^{t} e^{a(s-t)} x_1(s)\sin(x_2(s))ds \\ -4.5x_2(t) - x_1(t)x_2(t) + \int_{t-\tau}^{t} \frac{\pi x_2(s)}{\pi + \arctan(x_2(s))} ds \end{pmatrix}, \tag{4.29}$$

and

$$f_2(t, x_t) = \begin{pmatrix} -16x_1(t) + x_2(t-\tau)/(1+t^2) \\ -17x_2(t) + \int_{-\infty}^{t} e^{a(s-t)} \sqrt{x_1^4(s) + x_2^4(s)} ds/(cosh^2(x_2(t))) \end{pmatrix}. \tag{4.30}$$

where $\tau > 0$ is a finite delay and $a > 0$ is a constant. The switching rule is assumed to take the following form for $k = 1, 2, \ldots,$

$$\sigma = \begin{cases} 1, & t \in [t_{2k-2}, t_{2k-1}), \\ 2, & t \in [t_{2k-1}, t_{2k}), \end{cases} \tag{4.31}$$

where the switching times are

$$t_k = \begin{cases} t_{k-1} + 0.5 + 0.2k^2 e^{-k}, & k = 2, 4, 6, \ldots, \\ t_{k-1} + 0.2, & k = 1, 3, 5, \ldots, \end{cases} \tag{4.32}$$

with $t_0 = 0$, and satisfy $t_{2k} - t_{2k-1} \geq 0.5$ and $t_{2k-1} - t_{2k-2} \geq 0.2$. Assume that at the switching times an impulsive effect is applied:

$$\begin{cases} \Delta x_1(t_k) = -x_1(t_k^-) + \sqrt{1.1}\sin(x_1(t_k^-)), \\ \Delta x_2(t_k) = -x_2(t_k^-) + \sqrt{|x_1(t_k^-)x_2(t_k^-)|}. \end{cases} \tag{4.33}$$

Let $V(x) = (x_1^2 + x_2^2)/2$ and take the time-derivative along solutions to subsystem $i = 1$,

$$\begin{aligned} \frac{dV}{dt}\Big|_{i=1} =& x_1(t)\left[-4.5x_1(t) + x_2(t) + x_2^2(t) + \int_{-\infty}^{t} e^{a(s-t)} x_1(s)\sin(x_2(s))ds\right], \\ &+ x_2(t)\left[-4.5x_2(t) - x_1(t)x_2(t) + \int_{t-\tau}^{t} \frac{\pi x_2(s)}{\pi + \arctan(x_2(s))} ds\right], \\ \leq& -4.5(x_1^2(t) + x_2^2(t)) + x_1(t)x_2(t) + x_1(t)\int_{-\infty}^{t} e^{a(s-t)} x_1(s)\sin(x_2(s))ds \\ &+ x_2(t)\int_{t-\tau}^{t} \frac{\pi x_2(s)}{\pi + \arctan(x_2(s))} ds, \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dV}{dt}\,|_{i=1} \leq &-9V(x(t)) + \frac{x_1^2(t)+x_2^2(t)}{2} + x_1(t)\int_{-\infty}^{0} e^{as}x_1(t+s)\sin(x_2(t+s))ds \\ &+ x_2(t)\int_{-\tau}^{0}\frac{\pi x_2(t+s)}{\pi+\arctan(x_2(t+s))}ds, \\ \leq &-8V(x(t)) + |x_1(t)|\int_{-\infty}^{0} e^{as}|x_1(t+s)|ds + |x_2(t)|\int_{-\tau}^{0}|x_2(t+s)|ds,\end{aligned}$$

If the Razumikhin condition (ii) in Corollary 4.3.4 holds, then $x_1^2(t+s)+x_2^2(t+s) \leq qe^{-\gamma s}[x_1^2(t)+x_2^2(t)]$ *for* $s \leq 0$ *and so*

$$\begin{aligned}\frac{dV}{dt}\,|_{i=1} \leq &-8V(x(t)) + |x_1(t)|\int_{-\infty}^{0}\sqrt{q}e^{(a-\gamma/2)s}\sqrt{x_1^2(t)+x_2^2(t)}ds \\ &+ |x_2(t)|\int_{-\tau}^{0}\sqrt{q}e^{-\gamma s/2}\sqrt{x_1^2(t)+x_2^2(t)}ds, \\ = &-8V(x(t)) + |x_1(t)|\sqrt{x_1^2(t)+x_2^2(t)}\int_{-\infty}^{0}\sqrt{q}e^{(a-\gamma/2)s}ds \\ &+ |x_2(t)|\sqrt{x_1^2(t)+x_2^2(t)}\int_{-\tau}^{0}\sqrt{q}e^{-\gamma s/2}ds,\end{aligned}$$

Hence,

$$\begin{aligned}\frac{dV}{dt}\,|_{i=1} \leq &-8V(x(t)) + \frac{|x_1(t)|^2+x_1^2(t)+x_2^2(t)}{2}\left(\frac{\sqrt{q}}{a-\gamma/2}\right) \\ &+ \frac{|x_2(t)|^2+x_1^2(t)+x_2^2(t)}{2}\sqrt{q}\left(\frac{e^{\gamma\tau/2}-1}{\gamma/2}\right), \\ \leq &-8V(x(t)) + (x_1^2(t)+x_2^2(t))\left[\frac{\sqrt{q}}{a-\gamma/2}+\sqrt{q}\left(\frac{e^{\gamma\tau/2}-1}{\gamma/2}\right)\right] \\ = &\left[-8+2\sqrt{q}\left(\frac{1}{a-\gamma/2}+\frac{e^{\gamma\tau/2}-1}{\gamma/2}\right)\right]V(x(t)),\end{aligned}$$

provided that $a-\gamma/2>0$*. Similarly, along the subsystem* $i=2$*, if the Razumikhin condition* *holds,*

$$\begin{aligned}\frac{dV}{dt}\mid_{i=2} =& x_1(t)\left[-16x_1(t)+\frac{x_2(t-\tau)}{1+t^2}\right]\\ & x_2(t)\left[-17x_2(t)+\frac{1}{cosh^2(x_2(t))}\int_{-\infty}^{t}e^{a(s-t)}\sqrt{x_1^4(s)+x_2^4(s)}ds\right],\\ \leq& -8V(x(t))+|x_1(t)||x_2(t-\tau)|+\int_{-\infty}^{t}e^{a(s-t)}[x_1^2(s)+x_2^2(s)]ds,\\ \leq& -8V(x(t))+|x_1(t)|\sqrt{q}e^{\gamma\tau/2}\sqrt{x_1^2(t)+x_2^2(t)}+\int_{-\infty}^{0}e^{as}[x_1^2(t+s)+x_2^2(t+s)]ds,\\ \leq& -8V(x(t))+\sqrt{q}e^{\gamma\tau/2}(x_1^2(t)+x_1^2(t)+x_2^2(t))+\int_{-\infty}^{0}qe^{(a-\gamma)s}[x_1^2(t)+x_2^2(t)]ds,\end{aligned}$$

where we have used the fact that $x_1^4+x_2^4\leq(x_1^2+x_2^2)^2$ *for all* $x_1,x_2\in\mathbb{R}$. *Thus,*

$$\begin{aligned}\frac{dV}{dt}\mid_{i=2}\leq& -8V(x(t))+2\sqrt{q}e^{\gamma\tau/2}(x_1^2(t)+x_2^2(t))+q(x_1^2(t)+x_2^2(t))\int_{-\infty}^{0}e^{(a-\gamma)s}ds,\\ =& -8V(x(t))+(x_1^2(t)+x_2^2(t))\left[2\sqrt{q}e^{\gamma\tau/2}+\frac{q}{a-\gamma}\right],\\ =& \left[-8+4\sqrt{q}e^{\gamma\tau/2}+\frac{2q}{a-\gamma}\right]V(x(t)),\end{aligned}$$

provided $a-\gamma>0$. *At the impulsive moments* $t=t_k$,

$$\begin{aligned}V(x(t_k)) =& \frac{1}{2}\left[\sqrt{1.1}\sin(x_1(t_k^-))\right]^2+\frac{1}{2}\left[\sqrt{|x_1(t_k^-)x_2(t_k^-)|}\right]^2,\\ \leq& \frac{1.1}{2}x_1^2(t_k^-)+\frac{1}{2}|x_1(t_k^-)x_2(t_k^-)|,\\ \leq& 1.1\left(\frac{x_1^2(t_k^-)+x_2^2(t_k^-)}{2}\right)+\frac{1}{2}\left(\frac{x_1^2(t_k^-)+x_2^2(t_k^-)}{2}\right),\\ =& 1.6V(x(t_k^-)).\end{aligned}$$

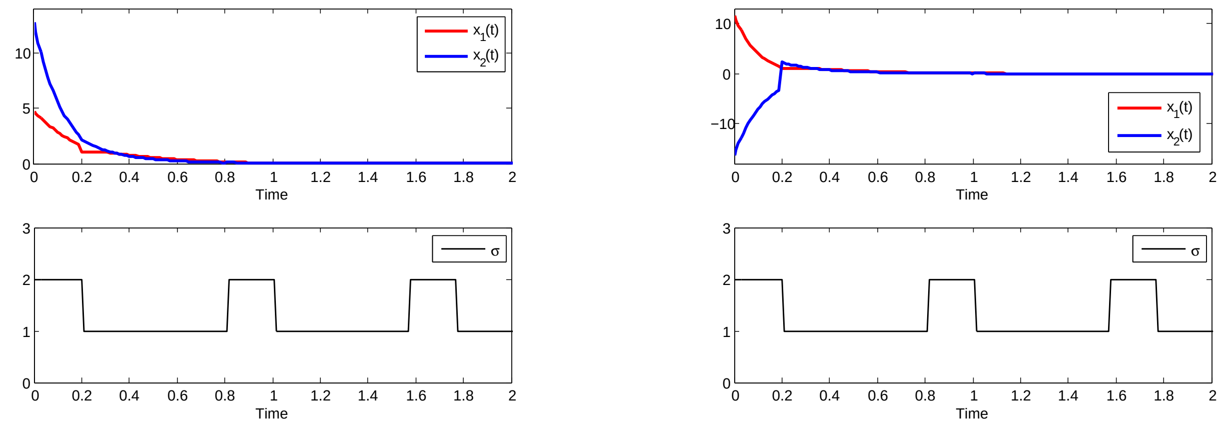
Suppose that $\tau=0.05$ *and* $a=8.5$. *Choose* $c_1=c_2=0.5$, $p=2$, $\gamma=5.5$, $q=1.1$, $\beta=1.01$, $\delta=\delta_k=1.6q=1.76$, $p_1=7.76$, $p_2=2.45$, $\zeta_{2k}=0.5$, $\zeta_{2k-1}=0.2$. *Take* $\eta_{i_{2k-1}}=5.5$, $\eta_{i_{2k}}=2.45$ *then*

$$\ln(\beta\delta)-\eta_{i_{2k-1}}\zeta_{2k-1}=-0.525$$

and

$$\ln(\beta\delta) - \eta_{i_{2k}}\zeta_{2k} = -0.651.$$

The trivial solution is globally asymptotically stable by Corollary 4.3.4. See Figure 4.4 for an illustration.



(a) $\phi_0(s) = (5, 14)$ for $s \le 0$.

(b) $\phi_0(s) = (13, -18)$ for $s \le 0$.

Figure 4.4: Simulation of Example 4.3.1.

Example 4.3.2. *Consider the switched system (4.24) with $\mathcal{P} = \{1, 2\}$. Suppose that*

$$f_1(t, x_t) = \begin{pmatrix} -1.4x_1(t) + x_2^4(t) + \int_{-\infty}^{t} e^{b(s-t)}[1 - \exp(-x_2^2(s))]^2 ds/(1 + |x_1(t)|) \\ -2x_2(t) - 4x_1^3(t)x_2(t) \end{pmatrix}, \quad (4.34)$$

and

$$f_2(t, x_t) = \begin{pmatrix} -2x_1(t) + x_2^4(t) \\ -1.5x_2(t) - 4x_1^3(t)x_2(t) + x_2(t)\exp[-|\cos(x_1(t))|] \end{pmatrix}. \quad (4.35)$$

The switching rule is assumed to take the following form for $k = 1, 2, \ldots,$

$$\sigma = \begin{cases} 1, & t \in [t_{2k-2}, t_{2k-1}), \\ 2, & t \in [t_{2k-1}, t_{2k}), \end{cases} \quad (4.36)$$

where

$$t_k = t_{k-1} + 0.2 + 0.1\arctan(k), \quad (4.37)$$

with $t_0 = 0$*. Suppose that the switch times coincide with the impulsive moments,* $t_k = T_k$*, with impulsive equations,*

$$\begin{cases} \left.\begin{array}{l} \Delta x_1 = -x_1(t^-) + d_1 x_1(t^-), \\ \Delta x_2 = -x_1(t^-) + d_2 x_2(t^-), \end{array}\right\} t = t_{2k-1}, \\ \left.\begin{array}{l} \Delta x_1 = -x_1(t^-) + d_3 x_2(t^-) e^{-x_1^2(t^-)}, \\ \Delta x_2 = -x_2(t^-) + d_4 sign(x_2(t^-)) x_1(t^-), \end{array}\right\} t = t_{2k}, \end{cases} \tag{4.38}$$

where $d_1, d_2, d_3, d_4 \geq 0$ *and where*

$$sign(y) := \begin{cases} 1, & for\ y > 0, \\ 0, & for\ y = 0, \\ -1, & for\ y < 0. \end{cases}$$

Note that $0.2 \leq t_k - t_{k-1} \leq 0.2 + \pi/20$*. The impulsive effects associated with the impulsive times* t_{2k-1} *are applied whenever the system switches from subsystem 1 to subsystem 2. Similarly,* t_{2k} *are associated with switching from subsystem 2 to subsystem 1.*

Let $V(x) = 4x_1^4 + x_2^4$ *and take the time-derivative along solutions to subsystem* $i = 1$ *for* $t \neq t_k$*,*

$$\begin{aligned} \frac{dV}{dt}\,|_{i=1} =& 16x_1^3(t)\left[-1.4x_1(t) + x_2^4(t) + \int_{-\infty}^{t} e^{b(s-t)}[1 - \exp(-x_2^2(s))]^2 ds/(1 + |x_1(t)|)\right] \\ & + 4x_2^3(t)\left[-2x_2(t) - 4x_1^3(t)x_2(t)\right], \\ \leq & -5.5V(x(t)) + \frac{16x_1^3(t)}{1 + |x_1(t)|}\int_{-\infty}^{t} e^{b(s-t)}[1 - \exp(-x_2^2(s))]^2 ds, \\ \leq & -5.5V(x(t)) + 16x_1^2(t)\int_{-\infty}^{t} e^{b(s-t)} x_2^2(s) ds, \\ = & -5.5V(x(t)) + 16x_1^2(t)\int_{-\infty}^{0} x_2^2(t+s) e^{bs} ds. \end{aligned}$$

If the Razumikhin-type condition (ii) holds then $4x_1^4(t+s) + x_2^4(t+s) \leq q e^{-\gamma s}[4x_1^4(t) + x_2^4(t)]$

for all $s \leq 0$ *which gives*

$$\begin{aligned}
\frac{dV}{dt}\,|_{i=1} \leq & -5.5V(x(t)) + 16x_1^2(t)\int_{-\infty}^{0}\sqrt{q}e^{-\gamma s/2}\sqrt{4x_1^4(t)+x_2^4(t)}e^{bs}ds, \\
= & -5.5V(x(t)) + [4x_1(t)]^2\sqrt{4x_1^4(t)+x_2^4(t)}\sqrt{q}\int_{-\infty}^{0}e^{(b-\gamma/2)s}ds, \\
\leq & -5.5V(x(t)) + \frac{16x_1^2(t)+4x_1^4(t)+x_2^4(t)}{2}\sqrt{q}\left(\frac{1}{b-\gamma/2}\right), \\
= & -5.5V(x(t)) + (10x_1^2(t)+0.5x_2^4(t))\sqrt{q}\left(\frac{1}{b-\gamma/2}\right), \\
\leq & -5.5V(x(t)) + (10x_1^2(t)+2.5x_2^4(t))\sqrt{q}\left(\frac{1}{b-\gamma/2}\right).
\end{aligned}$$

And so,

$$\begin{aligned}
\frac{dV}{dt}\,|_{i=1} \leq & -5.5V(x(t)) + 2.5(4x_1^2(t)+x_2^4(t))\sqrt{q}\left(\frac{1}{b-\gamma/2}\right), \\
= & \left[-5.5+\left(\frac{2.5\sqrt{q}}{b-\gamma/2}\right)\right]V(x(t)),
\end{aligned}$$

provided that $b-\gamma/2>0$. *Along the subsystem* $i=2$ *for* $t \neq t_k$,

$$\begin{aligned}
\frac{dV}{dt}\,|_{i=2} = & -32x_1^4(t) - 6x_2^4(t) + 4x_2^4(t)\exp[-|\cos(x_1(t))|], \\
\leq & -6V(x(t)) + 4x_2^4(t), \\
\leq & -2V(x(t)).
\end{aligned}$$

At the impulsive moments $t = t_{2k-1}$,

$$\begin{aligned}
V(x(t_{2k-1})) = & 4\left(d_1x_1(t_{2k-1}^-)\right)^4 + \left(d_2x_2(t_{2k-1}^-)\right)^4, \\
\leq & \max(d_1^4, d_2^4)V(x(t_{2k-1}^-)).
\end{aligned}$$

Similarly, whenever $t = t_{2k}$,

$$\begin{aligned}
V(x(t_{2k})) = & 4\left[d_3x_2(t_{2k}^-)cos(x_1^2(t_{2k}^-))\right]^4 + \left[d_4 sign(x_2(t_{2k}^-))x_1(t_{2k}^-)\right]^4, \\
= & 4d_3^4x_2^4(t_{2k}^-)cos^4(x_1^2(t_{2k}^-)) + d_4^4(sign(x_2(t_{2k}^-)))^4x_1^4(t_{2k}^-), \\
\leq & 4d_3^4x_2^4(t_{2k}^-) + d_4^4x_1^4(t_{2k}^-), \\
\leq & \max(d_3^4, d_4^4)V(x(t_{2k}^-)).
\end{aligned}$$

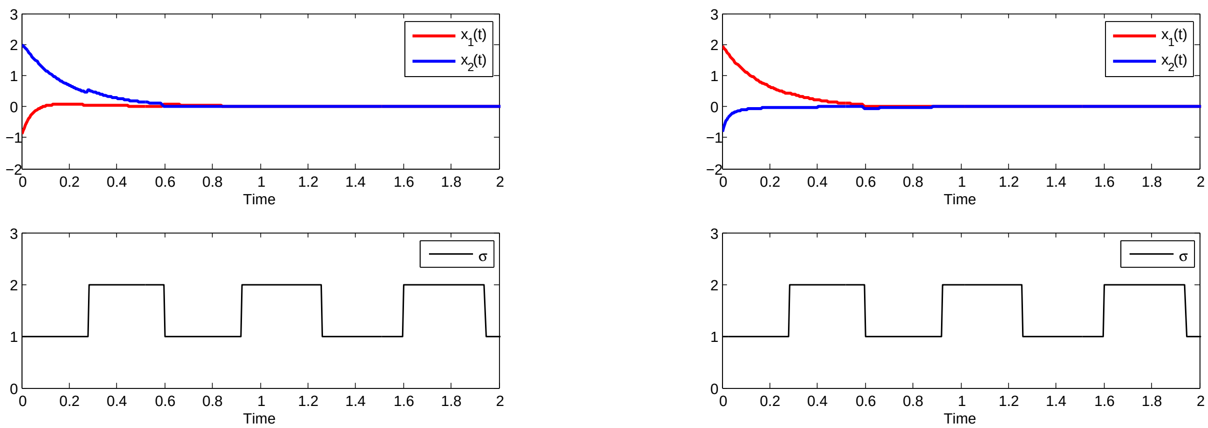
Suppose that $b = 5.5$, $d_1 = 1.05$, $d_2 = 1.2$, $d_3 = 0.9$, $d_4 = 1.05$. *Choose* $c_1 = 1$, $c_2 = 4$, $p = 4$, $q = 1.1$, $\gamma = 5.5$, $\beta = 1.01$. *Then* $p_1 = 4.55$, $p_2 = 2$, $\delta_{2k-1} = 2.28$, $\delta_{2k} = 1.34$. *Take* $\eta_{2k-1} = 4.55$, $\eta_{2k} = 2$, *and* $\zeta = 0.2$, *then*

$$\ln(\beta\delta_{2k-1}) - \eta_{i_{2k-1}}\zeta = -0.0748$$

and

$$\ln(\beta\delta_{2k}) - \eta_{i_{2k}}\zeta = -0.0996.$$

Thus the trivial solution of (4.24) *is globally asymptotically stable by Corollary 4.3.4. See Figure 4.5.*



(a) $\phi_0(s) = (-1, 2)$ for $s \leq 0$.

(b) $\phi_0(s) = (2, -1)$ for $s \leq 0$.

Figure 4.5: Simulation of Example 4.3.2.

Example 4.3.3. *Consider the following scalar HISD*

$$\begin{cases} \quad \dot{x} = -a_\sigma x(t) + d_\sigma \displaystyle\int_{-\infty}^{t} e^{c(s-t)} \arctan(|\sinh(x(s))|)ds, & t \neq T_k \\ \Delta x = vx(t^-), & t = T_k, \end{cases} \tag{4.39}$$

where $\mathcal{P} = \{1, 2\}$, a_1, a_2, d_1, d_2, *and* c *are positive constants. Assume that the switching rule is periodic and takes the form*

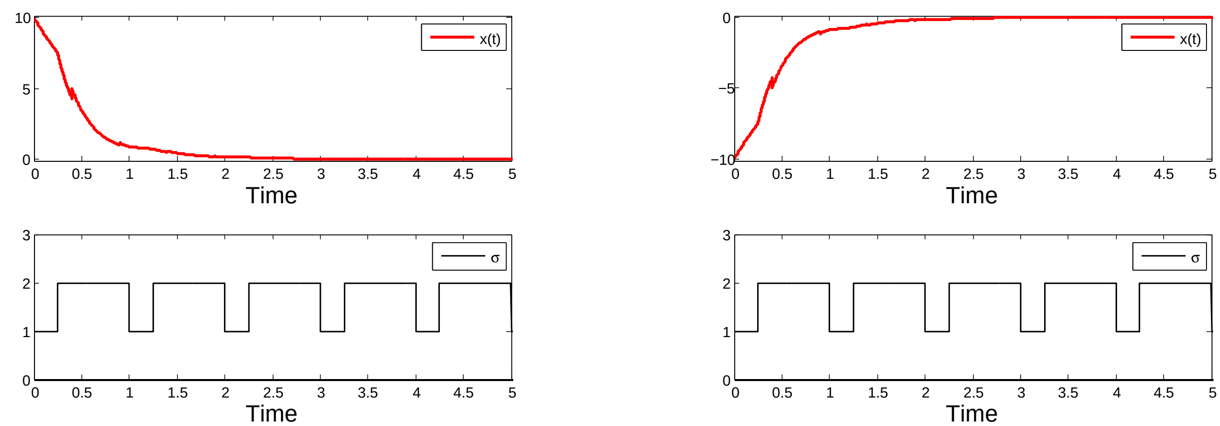
$$\sigma = \begin{cases} 1 & \textit{if } t \in [k, k + 0.25),\ k = 0, 1, 2, \ldots \\ 2 & \textit{if } t \in [k + 0.25, k + 1). \end{cases} \tag{4.40}$$

Suppose that an impulse is applied at each time $T_k = 2k$ for $k \in \mathbb{N}$. Then the impulses and switching are periodic with $\alpha = 2$, $N = 1$, $\omega = 1$, $h_1 = 0.25$, $h_2 = 0.75$.

Suppose that $a_1 = 1.2$, $a_2 = 4$, $d_1 = 6$, $d_2 = 3$, $v = 0.15$, and $c = 5.1$ for the model parameters. Note that $\arctan(|\sinh(u)|) \leq |u|$ for $u \in \mathbb{R}$. Take $m_i(s) = e^{5.1s}$, $q = 1.1$, $\gamma = 3.1$, , $\vartheta_{1i} = 0$, $\vartheta_{2i} = 0$, $\vartheta_{3i} = 0$. Let $p_1 = 0.1$, $p_2 = 0.7$ and choose $\eta_1 = 0.1$, $\eta_2 = 0.7$, $\delta = 1.265$. Then

$$\frac{\ln \delta}{\alpha} - \frac{1}{\omega}(\eta_1 h_1 + \eta_2 h_2) = -0.08$$

and thus the trivial solution is globally asymptotically stable by Corollary 4.3.7. See Figure 4.6.



(a) $\phi_0(s) = 10$ for $s \leq 0$.

(b) $\phi_0(s) = -10$ for $s \leq 0$.

Figure 4.6: Simulation of Example 4.3.2.

Chapter 5

Hybrid Control of Unstable Systems with Distributed Delays

In the previous chapter we investigated the stability of HISD where at least a portion of the subsystems were stable, and, overall stability could be guaranteed based on dwell-time conditions. A natural progression from those investigations is to study whether stability of an HISD composed entirely of unstable subsystems is possible, which is the goal of the present chapter.

The motivation for this area of research comes from a control perspective. One of the major categories in switched systems research is concerned with the construction of a switching rule to stabilize an unstable system using switching control. There are a number of reasons why switching control is desirable (or even required) over continuous control [101]: continuous control cannot be implemented because of sensor or actuator limitations; continuous control is not possible because of the nature of the problem; or continuous control cannot be found because of model uncertainty.

To stabilize switched systems composed entirely of unstable subsystems, there are two broad approaches:

(i) Time-dependent switching stabilization where the switching rule $\sigma(t)$ is found a priori and is pre-programmed into the data. The general idea in this strategy is to use high frequency switching so that the system does not dwell in any particular unstable subsystem for too long. If the problem is approached from a control perspective, this may be called stabilization via open-loop switching control.

(ii) State-dependent switching stabilization where the state-space is partitioned into switching regions which dictate the active mode. When the solution trajectory crosses between regions the switching rule disengages the current mode and engages a new one according to a special rule based on the time-derivative of a Lyapunov function/functional. In this construction, the switching rule, $\sigma(x)$, is state-dependent. From a control perspective this scheme uses switching state feedback and may be called stabilization via closed-loop switching control.

In Section 5.1, the problem is posed and a literature review is given on the current results for nonlinear systems without time-delay. The focus of the rest of this chapter is on the state-dependent switching approach: extensions are given to the state-dependent switching stabilization of a class of nonlinear HISD in Section 5.2. In Section 5.3, new results on the state-dependent switching stabilization of nonlinear HISD are presented, including results for both bounded and unbounded delay.

5.1 Literature Review

In order to frame the problem, we give a brief review of the current research efforts by detailing the literature on the stability of the following nonlinear switched system:

$$\begin{cases} \dot{x} = f_\sigma(x), \\ x(0) = x_0. \end{cases} \tag{5.1}$$

Since the objective is to construct switching rules which stabilize (5.1), it is assumed throughout this chapter that each subsystem $i = 1, \ldots, m$ is an unstable mode (otherwise, if j were a stable mode, simply setting $\sigma(t) = j$ for all $t \geq t_0$ would achieve stabilization). In Section 5.1.1, the problem is introduced by investigating the existence of a time-dependent switching rule to solve the aforementioned problem. Afterwards, the state-dependent switching approach, which is the focus of this chapter, is given for nonlinear switched systems in Section 5.1.2.

5.1.1 Time-dependent Switching Rule Approach

Switched systems can exhibit instability even when composed entirely of stable subsystems. The main source of this problem is switching too frequently: intuitively, if the system is allowed to dwell in each subsystem for a sufficient amount of time, the switched system should exhibit overall stability. This idea is the basis for dwell-time switching approaches where a constraint is placed on the switching rule to ensure the switching is not too frequent. On the other hand, the stabilization of a switched system composed entirely of unstable subsystems is also possible using a time-dependent switching rule so that the switched system experiences switches sufficiently often (see Example 2.3.7). If the switching between subsystems is sufficiently fast then the state remaining in one subsystem long enough to destabilize can be avoided. A high-frequency switching strategy for switched systems made up of unstable subsystems is the opposite of dwell-time switching for switched systems made up of stable subsystems.

Sun et al. [174] detailed the idea of fast-switching stabilization using the Campbell-Baker-Hausdorff formula in relation to linear systems. Bacciotti and Mazzi [15] analyzed the time-dependent switching stabilization of nonlinear switched systems with unstable subsystems. The authors Bacciotti and Mazzi continued to work in this area of research in [16] by analyzing eventually periodic switching rules for linear switched systems. Mancilla-Aguilar and Garcia [133] extended the open-loop switched control approach for nonlinear systems by investigating stabilization with respect to a compact set. In [172], we extended the nonlinear results to systems with impulsive effects. To gain a greater intuition of the time-dependent switching stabilization process, we highlight the results of [15] and [172].

Consider system (5.1) with the following goal in mind: given a set of vector fields $\{f_i\}_{i=1}^m$ and an initial condition x_0, find a time-dependent switching rule

$$\sigma = \sigma(t) : \mathbb{R}_+ \to \mathcal{P}$$

and associated switching sequence $\{t_k\}$ which stabilizes system (5.1). Bacciotti and Mazzi [15] were successful in finding conditions for the existence of such a time-dependent stabilizing switching rule. The authors' technique was to relate the state trajectory along the switched system (5.1) to the trajectory along a single smooth vector field via the Campbell-Baker-Hausdorff formula (see [24, 180]). Sufficient conditions for the existence of a stabilizing time-dependent switching rule (possibly dependent on the initial condition x_0) are given in the following theorem.

Theorem 5.1.1. [15]
Consider (5.1) and suppose that $f_i \in \mathcal{H}$ for each $i \in \mathcal{P}$ where $\mathcal{H}$ is the space of bounded, analytic vector fields on $\mathcal{B}_b(0)$ for some constant $b > 0$. Assume that there exist constants $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$ such that the trivial solution of

$$\begin{cases} \dot{x} = \sum_{i=1}^{m} \alpha_i f_i(x), \\ x(0) = x_0, \end{cases} \tag{5.2}$$

is asymptotically stable. Then there exists a time-dependent switching rule, possibly dependent on the initial condition, such that the trivial solution of system (5.1) is asymptotically stable.

Remark 5.1.1. *Unfortunately, explicit knowledge of the time-dependent switching rule is not possible (even if a Lyapunov function for system (5.2) is known). This is a product of the method of proof: in the construction of the time-dependent switching rule, some of the constants cannot be calculated explicitly and are only proven to exist.*

In [172], we considered how to extend the time-dependent switching stabilization results to include impulsive effects. The motivation is that with the addition of impulsive control, it may be possible to stabilize a system where high frequency time-dependent switching control is inadequate. Consider the following switched impulsive system

$$\begin{cases} \dot{x} = f_\sigma(x), & t \neq T_k, \\ \Delta x = g_k(x(t^-)), & t = T_k, \\ x(0) = x_0, & k \in \mathbb{N}, \end{cases} \tag{5.3}$$

where $\{f_i\}_{i=1}^m$ is a family of sufficiently smooth vector fields satisfying $f_i : D \to \mathbb{R}^n$ and $f_i(0) = 0$ for $i = 1, \dots, m$. The impulsive functions $\{g_k\}_{k=1}^\infty$ are assumed to satisfy $x + g_k(x) \in D$ for all $x \in D$ and $g_k(0) \equiv 0$ for all k. The goal of the open-loop switched control problem for system (5.3) is as follows: given a set of vector fields $\{f_i\}_{i=1}^m$ such that each subsystem $\dot{x} = f_i(x)$ is unstable, stabilizing or disturbance impulses $\{g_k\}_{k=1}^\infty$ with associated impulsive moments $\{T_k\}_{k=1}^\infty$, and initial condition x_0, find a time-dependent switching rule $\sigma(t)$ a priori such that the trivial solution of (5.3) is asymptotically stable.

The first result considers impulsive disturbances and is based on the existence of a stable convex combination of the subsystems.

Theorem 5.1.2. [172]
Consider system (5.3) and assume that there exist constants $\alpha_i > 0$ satisfying $\sum_{i=1}^m \alpha_i = 1$, constants $\lambda > 0, \delta_k > 0, \chi_k > 0$, $0 < \zeta_k < 1$, and functions $c_1, c_2 \in \mathcal{K}$, and $V \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ such that for $k \in \mathbb{N}$,

(i) $c_1(\|x\|) \le V(x) \le c_2(\|x\|)$ *for all* $x \in D$*;*

(ii) $\nabla V(x) \cdot (\sum_{i=1}^m \alpha_i f_i(x)) \le -\lambda V(x)$ *for all* $x \in D$*;*

(iii) $V(x + g_k(x)) \le (1 + \delta_k)V(x)$ *for all* $x \in D$*;*

(iv) $\ln(1+\delta_k) - \lambda(1-\chi_k)(T_k - T_{k-1}) < \ln \zeta_k$.

Then there exists a time-dependent switching rule $\sigma(t)$, possibly dependent on the initial condition, such that the trivial solution of (5.3) is asymptotically stable.

For stabilizing impulses, the following result was given.

Theorem 5.1.3. [172]
Consider system (5.3) and assume that there exist constants $\alpha_i > 0$ satisfying $\sum_{i=1}^{m} \alpha_i = 1$, constants $\lambda > 0, \delta_k > 0, \chi_k > 0$, $0 < \zeta_k < 1$, and functions $c_1, c_2 \in \mathcal{K}$, $V \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ such that for $k \in \mathbb{N}$

(i) $c_1(\|x\|) \leq V(x) \leq c_2(\|x\|)$ *for all* $x \in D$;

(ii) $\nabla V(x) \cdot (\sum_{i=1}^{m} \alpha_i f_i(x)) \leq \lambda V(x)$ *for all* $x \in D$;

(iii) $V(x + g_k(x)) \leq \delta_k V(x)$ *for all* $x \in D$;

(iv) $\ln \delta_k + \lambda(1+\chi_k)(T_k - T_{k-1}) < \ln \zeta_k$.

Then there exists a time-dependent switching rule $\sigma(t)$, possibly dependent on the initial condition, such that the trivial solution of (5.3) is asymptotically stable.

Example 5.1.1. [172]
Consider system (5.3) with $\mathcal{P} = \{1, 2\}$, impulsive moments $T_k = 2k$ for $k = 1, 2, \ldots$,

$$f_1(x_1, x_2) = \begin{pmatrix} 5x_1 + 2x_2^5 - x_2^2 e^{sin x_1} \\ -3x_2 - 2x_1 x_2^4 \end{pmatrix}, \quad f_2(x_1, x_2) = \begin{pmatrix} -6x_1 - x_2^5 \\ 2x_2 + x_1 x_2^4 + x_1 x_2 e^{sin x_1} \end{pmatrix},$$

$$g_{2k}(x_1, x_2) = \begin{pmatrix} sin(x_1)\sqrt{\left(1 + \frac{1}{e^{2k}}\right)(x_1^2 + x_2^2)} - x_1 \\ cos(x_1)\sqrt{\left(1 + \frac{1}{e^{2k}}\right)(x_1^2 + x_2^2)} - x_2 \end{pmatrix}, \quad g_{2k-1}(x_1, x_2) = \begin{pmatrix} 0.224\ x_1 \\ 0.224\ x_2 \end{pmatrix}.$$

Note that both $Df_1(0)$ and $Df_2(0)$ have eigenvalues with positive real part[1]*. Take $\alpha_1 = \alpha_2 = 0.5$ then*

$$\sum_{i=1}^{2} \alpha_i f_i(x_1, x_2) = \frac{1}{2}\begin{pmatrix} -x_1 + x_2^5 - x_2^2 e^{sin x_1} \\ -x_2 - x_1 x_2^4 + x_1 x_2 e^{sin x_1} \end{pmatrix}.$$

[1] $Df(x)$ is the Jacobian matrix of the vector field f evaluated at x.

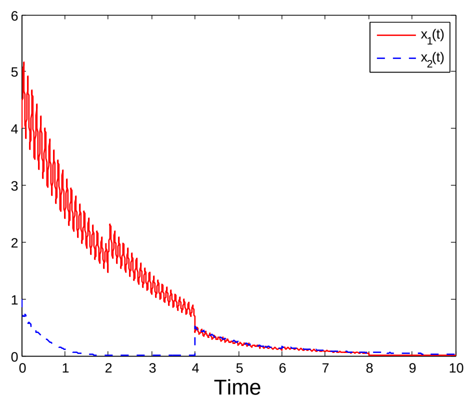
Consider the Lyapunov function $V = x_1^2 + x_2^2$, then along $\dot{x} = \alpha_1 f_1(x) + \alpha_2 f_2(x)$, $\dot{V} = -(x_1^2 + x_2^2) = -V$. At the impulsive times,

$$x_1(T_{2k})^2 + x_2(T_{2k})^2 \leq (1 + \frac{1}{e^{2k}})(x_1(T_{2k}^-)^2 + x_2(T_{2k}^-)^2)$$

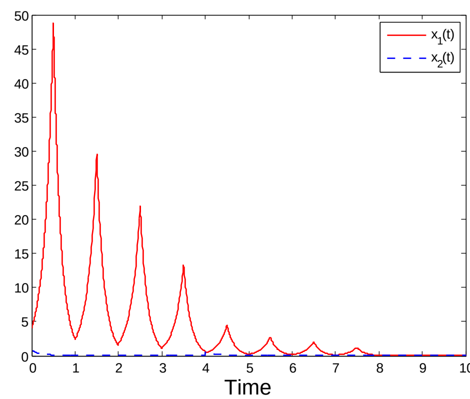
and

$$x_1(T_{2k-1})^2 + x_2(T_{2k-1})^2 \leq 1.5(x_1(T_{2k-1}^-)^2 + x_2(T_{2k-1}^-)^2).$$

Let $c_1(\|x\|) = c_2(\|x\|) = \|x\|^2$, $\lambda = 1$, $\chi_{2k} = \chi_{2k-1} = 0.01$, $\delta_{2k} = 1/e^2$, and $\delta_{2k-1} = 0.5$. Then the conditions of Theorem 5.1.2 are satisfied with $\zeta_k = 0.5$ and hence there exists a stabilizing time-dependent switching rule. From the simulations with $x_0 = (4, 1)$ (see Figure 5.1), the solution converges to the origin if the system is switched sufficiently fast.



(a) Switch every 0.05 time units (198 total switches).



(b) Switch every 0.5 time units (18 total switches).

Figure 5.1: Simulation of Example 5.1.1.

Remark 5.1.2. *To the best of the author's knowledge, there is currently no work done on the time-delay case for the open-loop approach as outlined above. Extensions of the current proof technique to delay differential equations and other infinite dimensional systems remains a direction for future work.*

5.1.2 State-dependent Switching Rule Approach

In this approach, the state-space is subdivided into switching regions (which may overlap) and the current active mode is determined based on which switching region the solution trajectory occupies. When the system trajectory crosses a switching region boundary, a special minimum rule is evoked in order to select the next subsystem. The motivating idea is that each switching region has at least one associated subsystem such that when active the time-derivative of a Lyapunov function is negative definite in that region of the state-space.

This avenue of research has been studied more extensively in the literature. Wicks et al. [191] first constructed a stabilizing switching rule for a linear system (the details were outlined in Problem 3 in Section 2.3.2). In [105], Liu et al. extended this line of research to the nonlinear switched system (5.1) by considering a state-dependent switching rule

$$\sigma = \sigma(x) : \mathbb{R}^n \to \mathcal{P}.$$

Continuing the common thread in this section, the authors Liu et al. showed that if there exists a convex combination of the family of vector fields $\{f_i\}_{i=1}^m$ that is stable then stabilization via a state-dependent switching rule is possible. More precisely, suppose that there exists a function $V \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ that is positive definite and radially unbounded and constants $\alpha_i > 0$ satisfying $\sum_{i=1}^m \alpha_i = 1$ such that for all $x \in \mathbb{R}^n$

$$\nabla V(x) \cdot \left(\sum_{i=1}^{m} \alpha_i f_i(x) \right) \leq -\lambda(\|x\|)$$

for some function $\lambda \in \mathcal{K}$. Then the domain can be partitioned into switching regions such that for any $x \neq 0$, there is at least one index $j \in \mathcal{P}$ and associated vector field f_j such that the state solution trajectory of (5.1) is being stabilized if mode j is active. This leads to the construction of the switching regions as

$$\widetilde{\Omega}_i = \{x \in \mathbb{R}^n : \nabla V(x) \cdot f_i(x) \leq -\lambda(\|x\|)\}\,.$$

If the solution trajectory of the switched system (5.1) is in the region $\widetilde{\Omega}_i$ then activate mode i. If the solution crosses a boundary into another region, switch to the appropriate mode.

Remark 5.1.3. *Since the switching rule outlined above is state-dependent, it may be possible for the solution state to cross a boundary, forcing a switch, and immediately cross back over the same boundary. This could lead to the possibility of chattering behaviour. As another example, it could be possible for the solution state to be initiated on a switching region boundary, or move down a switching region boundary (sliding motion behaviour). For more details, see [101, 105].*

With Remark 5.1.3 in mind, Liu et al. [105] extended the switching regions using a constant $\xi > 1$ so that they overlap in order to avoid chattering. This is a common approach in the state-dependent switching rule literature. Then the full strategy can be given in the following minimum rule algorithm.

Algorithm 5.1.1. (Minimum rule for nonlinear switched systems) [105]
Given a constant $\xi > 1$ and an initial state x_0, proceed as follows:

(MR1) *Choose the active mode according to the minimum rule*

$$\sigma(x_0) = \operatorname*{argmin}_{i\in\mathcal{P}} \nabla V(x_0) \cdot f_i(x_0).$$

(MR2) *Remain in the active mode as long as the solution trajectory $x(t)$ remains in the switching region*

$$\Omega_i = \left\{ x \in \mathbb{R}^n : \nabla V(x) \cdot f_i(x) \leq -\frac{\lambda(\|x\|)}{\xi} \right\}.$$

(MR3) *If $x(t)$ crosses the boundary of Ω_i at t_c, set $x_0 = x(t_c)$ and go to step (MR1).*

Sufficient conditions for the state-dependent switching rule stabilization are given in the following theorem.

Theorem 5.1.4. [105]
Assume that there exist constants $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$, $\lambda > 0$, functions $c_1, c_2 \in \mathcal{K}_\infty$, $V \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ such that

(i) $c_1(\|x\|) \leq V(x) \leq c_2(\|x\|)$ *for all* $x \in \mathbb{R}^n$*;*

(ii) $\nabla V(x) \cdot (\sum_{i=1}^m \alpha_i f_i(x)) \leq -\lambda(\|x\|)$ *for all* $x \in \mathbb{R}^n$*.*

Then the origin of system (5.1) is globally asymptotically stable under the state-dependent switching rule $\sigma(x)$ constructed according to Algorithm 5.1.1 for any $\xi > 1$ chosen beforehand.

Remark 5.1.4. *There is an apparent trade-off in the choice of the constant ξ: the larger the value of ξ, the less chattering behaviour there should be but the slower the rate of stabilization.*

Remark 5.1.5. *The authors Liu et al. [105] also investigated considerable complications to the above problem. Namely, the authors considered time-varying vector fields $f_i(t, x)$, two-measure stability, and extending the minimum rule to a generalized rule which is discussed in greater detail in Section 5.3. The underlying state-dependent switching approach using a minimum rule and switching regions remains unchanged for these obstacles.*

5.1.3 Comparison of the Closed-loop and Open-loop Approaches

There are practical reasons why a time-dependent switching rule may be desired over a state-dependent one [15]. For example, with a time-dependent approach, the switching rule is designed so that chattering behaviour along a switching boundary is avoided. Since this approach pre-programs the switching rule and switching times as data into the system, sensors are not needed (no state feedback control is required). However, there may also be disadvantages to implementing the open-loop approach. For example, the frequency of switching that is needed to stabilize the system may be unrealistically high. Further, there may be a cost associated with switching controllers, which could be a large drawback to this strategy. Motivated by an interest in comparing the two approaches numerically, we give the following example.

Example 5.1.2. *Consider the switched nonlinear system (5.1) and assume that $\mathcal{P} = \{1, 2\}$ with subsystems*

$$f_1(x_1, x_2) = \begin{pmatrix} 5x_1 + 2x_2^5 - x_2^2 e^{sinx_1} \\ -3x_2 - 2x_1x_2^4 \end{pmatrix},$$

$$f_2(x_1, x_2) = \begin{pmatrix} -6x_1 - x_2^5 \\ 2x_2 + x_1x_2^4 + x_1x_2e^{sinx_1} \end{pmatrix}.$$

Note that

$$Df_1(0) = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}, \qquad Df_2(0) = \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}$$

and hence each subsystem is unstable. Choose $\alpha_1 = \alpha_2 = 0.5$ so that

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) = \frac{1}{2} \begin{pmatrix} -x_1 + x_2^5 - x_2^2 e^{sinx_1} \\ -x_2 - x_1x_2^4 + x_1x_2e^{sinx_1} \end{pmatrix}.$$

Let $V = x_1^2 + x_2^2$ then for all $x \in \mathbb{R}^2$,

$$\nabla V(x) \cdot (\alpha_1 f_1(x) + \alpha_2 f_2(x)) = -(x_1^2 + x_2^2) = -\|x\|^2.$$

In the state-dependent switching approach, the conditions of Theorem 5.1.4 are satisfied with $c_1(s) = c_2(s) = s^2$ and $\lambda(s) = s^2$. Hence the switching rule $\sigma(x)$ constructed according to Algorithm 5.1.1 (choose $\xi = 2$) stabilizes the switched system with the overlapping switching regions

$$\Omega_1 = \{x \in \mathbb{R}^2 : 10x_1^2 - 6x_2^2 - 2x_1x_2^2e^{sinx_1} \leq -(x_1^2 + x_2^2)/2\},$$

$$\Omega_2 = \{x \in \mathbb{R}^2 : -12x_1^2 + 4x_2^2 + 2x_1x_2^2e^{sinx_1} \leq -(x_1^2 + x_2^2)/2\}.$$

In the time-dependent switching approach, the conditions of Theorem 5.1.1 are also satisfied which guarantees the existence of a purely time-dependent stabilizing switching rule $\sigma(t)$. From trial and error, we choose to switch periodically between the two subsystems every 0.05 time units. See Figure 5.2 for a simulation.

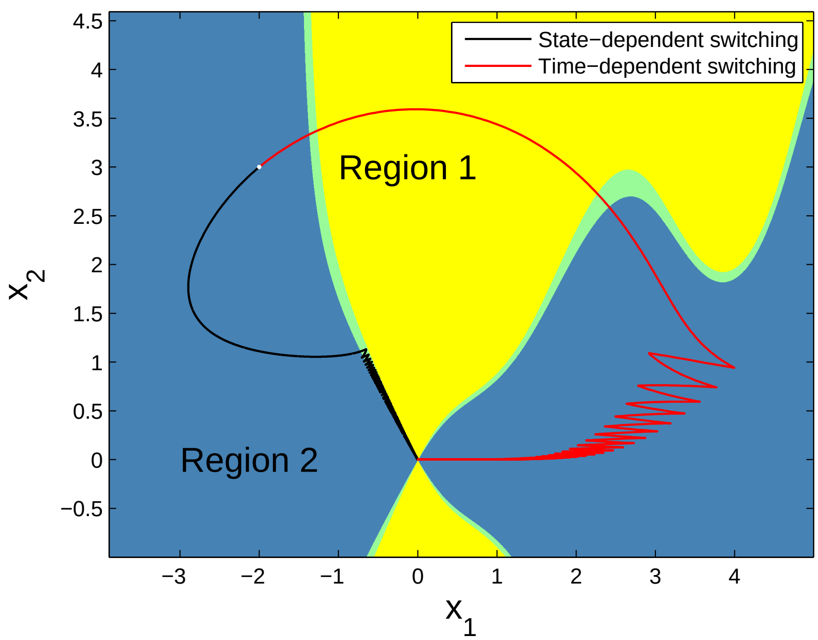


Figure 5.2: The yellow regions are Ω_1, the blue regions are Ω_2, and the green region is the overlapping region. The trajectories of the switched system are given in black (state-dependent switching approach) and red (time-dependent switching approach). The initial condition is $x_0 = (-2, 3)$.

For the switched system trajectory to satisfy $\|x(t)\| < 0.01$ given an initial condition of $x_0 = (-2, 3)$, the state-dependent switching rule requires 510 switches and takes a total time of 10.1. In contrast to this, the time-dependent switching rule requires only 229 switches, but takes a total time of 113.1. It seems that the state-dependent approach achieves stabilization faster, but requires more controller switching (which is disadvantageous practically). To further investigate these characteristics, we numerically solve Example 5.1.2 for one hundred different initial conditions and evaluate how long it takes the solution to converge to zero. In Table 5.1, the mean, minimum, maximum, and standard deviation is given for how long it takes the solution of the switched system to cross certain thresholds in its convergence to the origin (illustrated in Figure 5.3). The number of switches required (as well as minimum, maximum, and standard deviation) is also tallied.

The number of total switches required in the time-dependent approach depends entirely on the rate of switching that is chosen (e.g. every 0.05 time units in Figure 5.2). In the state-dependent approach, the number of switches is less, on average, but the variance is significantly higher. This makes sense intuitively as an initial condition near the overlapping region could require a vastly different number of switches from an initial condition far from a boundary. In fact, the minimum number of switches needed to approach the origin in the state-dependent approach can be zero. The variance in the time-dependent approach for different initial conditions is negligible since the switching rule is not state-dependent.

	State-dependent rule		Time-dependent rule	
	avg. (min, max)	std.	avg. (min, max)	std.
$\inf\{t : \|x\| < 0.5\|x_0\|\}$	0.051 (≈ 0, 0.29)	0.07	12.6 (12.1, 13.1)	.502
number of switches	0.147 (0, 5)	0.76	28.0 (27, 29)	1.00
$\inf\{t : \|x\| < 0.1\|x_0\|\}$	0.311 (≈ 0, 1.92)	0.58	44.6 (44.1, 45.1)	.502
number of switches	15.951 (0, 139)	43.9	92.0 (91, 93)	1.00
$\inf\{t : \|x\| < 10^{-3}\|x_0\|\}$	3.10 (≈ 0, 11.1)	4.46	136.7 (136, 137)	.499
number of switches	170.7 (0, 642)	241	276.1 (275, 277)	1.00

Table 5.1: Comparison of convergence times and number of switches for the state-dependent switching approach versus the time-dependent switching approach.

5.2 State-dependent Switching Rules for a Class of Nonlinear HISD

The state-dependent switching rule stabilization technique of partitioning the state-space into switching regions has been extended to systems with time-delays. For example, Kim et al. [78] were the first authors to extend the state-dependent switching rule stabilization technique to linear switched systems with discrete delay. The problem has also been analyzed for linear discrete-time systems with time-varying discrete delay by Phat and Ratchagit [154]. Liu studied the state-dependent switching stabilization of a linear system with mode-dependent time-varying delays [110]. This state-dependent approach has also

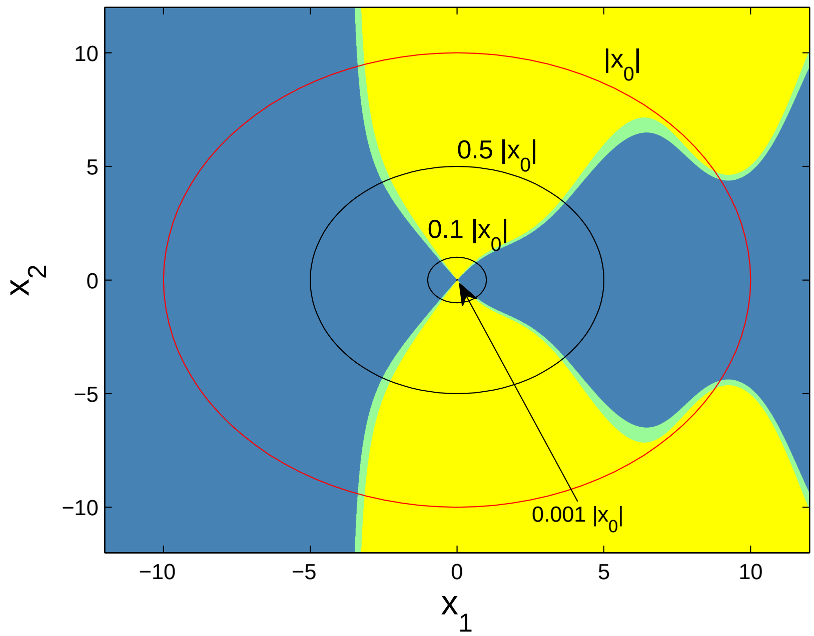


Figure 5.3: To construct Table 5.1, the trajectories of the switched system in Example 5.1.2 are initialized on the red circle ($x_1^2 + x_2^2 = 100$). The black circles represent the different convergence thresholds.

been extended to systems with distributed delays: in [50], Gao et al. extended the literature by considering linear switched systems with discrete and distributed delays. Li et al. [92] also focused on this problem by studying linear switched systems with mixed delays, including switching delays. The presence of uncertainties in the problem was considered by Hien et al. [71]. The approach in these reports is to use Lyapunov functionals to deal with the distributed delays and prove stability under a carefully constructed state-dependent switching rule.

Using switching proportional, delay, and integral feedback control to stabilize a linear switched system with distributed delays, Hien and Phat [72] analyzed the following switched system:

$$\begin{cases} \dot{x} =A_\sigma x(t) + B_\sigma x(t-\tau) + C_\sigma \displaystyle\int_{t-\tau}^{t} x(s)ds \\ \qquad + \bar{A}_\sigma u(t) + \bar{B}_\sigma u(t-r) + \bar{C}_\sigma \displaystyle\int_{t-r}^{t} u(s)ds, \\ x(s) =\phi_0(s), \quad s\in[-\tau^*,0], \quad \tau^* = \max\{\tau, r\}, \end{cases}$$

where $u\in\mathbb{R}^p$ is the control; $\tau > 0$, $r > 0$ are the time delays; and A_i, B_i, C_i, $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ are real constant matrices with appropriate dimensions for $i = 1,\dots,m$. Hien and Phat applied the memoryless state feedback controller $u(t) = K_\sigma x(t)$ and designed a state-dependent switching rule $\sigma = \sigma(x) : \mathbb{R}^n \to \mathcal{P}$ such that the closed-loop system is exponentially stable.

The reports mentioned above do not consider how impulsive control can be used in conjunction with state-dependent switching control to help achieve stabilization, which is important as the combination of switching control and impulsive control can increase the desired performance of a system [57]. In [112, 171], we considered how the addition of impulsive effects could help achieve stability but did not consider delay.

Here we investigate the state-dependent switching problem when applied to a class of nonlinear HISD using a Lyapunov functional approach. The material in this section formed the basis for [119] and some extensions are given here. The main contributions of this section are to further the current literature by providing sufficient conditions for the asymptotic or exponential stability of a class of HISD with distributed delays under state-dependent switching. A major concern in this work is how impulsive control, applied at pre-specified times or at the switching instances, affects the state-dependent switching stabilization of linear systems with nonlinear perturbations and distributed delays. Both stabilizing impulses as well as disturbance impulses are analyzed.

5.2.1 Problem Formulation from a Hybrid Control Perspective

Motivated by the work of Hien and Phat [72], consider the following general control system

$$\begin{aligned} \dot{x} =&Ax(t) + Bx(t-r) + C\int_{t-\tau}^{t} x(s)ds + F(t,x_t) \\ &+ u_1(t) + u_2(t-r) + \int_{t-\tau}^{t} u_3(s)ds + v(t), \end{aligned} \tag{5.4}$$

where the controllers $u_1, u_2, u_3, v \in \mathbb{R}^n$ are the switched proportional control, switched delay control, switched integral control, and impulsive control, respectively. The discrete delay is given by $r > 0$ and the distributed delay is given by $\tau > 0$. The matrices A, B and C are constant $n \times n$ matrices and the functional $F : \mathbb{R}_+ \times PC([-\tau^*, 0], \mathbb{R}^n) \to \mathbb{R}^n$, where $\tau^* = \max\{\tau, r\}$ and $x_t \in PC([-\tau^*, 0], \mathbb{R}^n)$ is defined by $x_t(s) = x(t+s)$ for $-\tau^* \leq s \leq 0$. It is assumed that $F(t, 0) \equiv 0$ for all $t \geq t_0$.

The switching controllers are constructed as follows (for example, see [46,56]): consider a collection of m linear feedback controllers $u = L_{1i}x(t)$, m nonlinear feedback controllers $u = J_{1i}(t, x)$, m linear delay controllers $u = L_{2i}x(t-r)$, m nonlinear delay controllers $u = J_{2i}(t, x(t-r))$, m linear integral controllers $u = L_{3i}\int_{t-\tau}^{t} x(s)ds$, and m nonlinear integral controllers $u = \int_{t-\tau}^{t} J_{3i}(s, x(s))ds$, where $i = 1, 2, \ldots, m$. Assume that the constant control gain matrices L_{1i}, L_{2i}, and L_{3i} are of corresponding dimension, and assume that the nonlinear functions $J_{1i}(t, x)$, $J_{2i}(t, x)$, and $J_{3i}(t, x)$ are piecewise continuous vector-valued functions that satisfy $J_{1i}(t, 0) \equiv J_{2i}(t, 0) \equiv J_{3i}(t, 0) \equiv 0$ for all $t \geq t_0$.

The switching control is incorporated into system (5.4) by setting

$$u_1(t) = \sum_{k=1}^{\infty} \left[L_{1i_k}x(t) + J_{1i_k}(t, x)\right] l_k(t),$$

$$\text{where the indicator function } l_k(t) := \begin{cases} 1 & \text{if } t \in [t_{k-1}, t_k), \\ 0 & \text{otherwise}, \end{cases}$$

$$u_2(t) = \sum_{k=1}^{\infty} \left[L_{2i_k}x(t-r) + J_{2i_k}(t, x(t-r))\right] l_k(t),$$

$$u_3(t) = \sum_{k=1}^{\infty} \left[\int_{t-\tau}^{t} (L_{3i_k}x(s) + J_{3i_k}(s, x(s)))ds\right] l_k(t).$$

The index $i_k \in \mathcal{P} = \{1, 2, \ldots, m\}$ follows a switching rule $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$. It is apparent from the definition of $l_k(t)$ that the controller u_1 switches its value at every instance $t = t_k$ so that $u_1(t)$ is a switching proportional controller. Similarly, u_2 and u_3 represent switching delay control and switching integral control, respectively.

To add impulsive control, assume that an impulse is applied to the system at the times $t = T_k$, $k = 1, 2, \ldots$, which necessarily satisfy $t_0 < T_1 < T_2 < \ldots < T_k < \ldots$ so that $T_k \to \infty$ as $k \to \infty$. Assume that the impulsive control can be broken down into two types: a linear impulsive control which takes the form $v_1 = E_k x(t)\delta(t - T_k^-)$, where each E_k is a constant control gain matrix and $\delta(t)$ is the Dirac delta generalized function; and a nonlinear impulsive control which takes the form $v_2 = Q_k(t, x)\delta(t - T_k^-)$, where $Q_k(t, x)$

are piecewise continuous vector-valued functions such that $Q_k(t,0) \equiv 0$ for all $t \geq t_0$. Add this control to the system by setting

$$v(t) = \sum_{k=1}^{\infty} v_1 + v_2 = \sum_{k=1}^{\infty} \left[E_k x(t) + Q_k(t,x)\right] \delta(t - T_k^-).$$

Then

$$\begin{aligned} x(T_k) - x(T_k - a) = &\int_{T_k - a}^{T_k} \left[Ax(s) + Bx(s-r) + C\int_{s-\tau}^{s} x(\theta)d\theta + F(s,x_s)\right] ds \\ &+ \int_{T_k - a}^{T_k} \left[u_1(s) + u_2(s-r) + \int_{s-\tau}^{s} u_3(\theta)d\theta + v(s)\right] ds, \end{aligned}$$

and in the limit as $a \to 0^+$,

$$x(T_k) - x(T_k^-) = E_k x(T_k^-) + Q_k(T_k, x(T_k^-)).$$

Thus there is a sudden jump in the state of the system at each time $t = T_k$ and v represents an impulsive controller. System (5.4) can be re-written as

$$\begin{cases} \dot{x} = A_\sigma x(t) + B_\sigma x(t-r) + C_\sigma \displaystyle\int_{t-\tau}^{t} x(s)ds + F_\sigma(t,x_t), & t \neq T_k, \\ \Delta x(t) = E_k x(t^-) + Q_k(t, x(t^-)), & t = T_k, \\ x(t_0 + s) = \phi_0(s), \quad s \in [-\tau^*, 0], & k \in \mathbb{N}, \end{cases} \tag{5.5}$$

where $A_i = A + L_{1i}$, $B_i = B + L_{2i}$, $C_i = C + L_{3i}$, and $F_i(t,x_t) = F(t,x_t) + J_{1i}(t,x) + J_{2i}(t, x(t-r)) + \int_{t-\tau}^{t} J_{3i}(s, x(s))ds$. The initial function is given by $\phi_0 \in PC([-\tau^*, 0], \mathbb{R}^n)$. Given a set of constant control matrices $\{L_{1i}\}$, $\{L_{2i}\}$, $\{L_{3i}\}$, $\{E_i\}$, and a set of nonlinear controllers $\{J_{1i}(t,x)\}$, $\{J_{2i}(t,x)\}$, $\{J_{3i}(t,x)\}$, $\{Q_i(t,x)\}$, the goal is to determine a switching rule σ and associated switching time sequence $\{t_k\}$, and an impulse time sequence $\{T_k\}$ such that the trivial solution of system (5.5) is asymptotically stable.

Remark 5.2.1. *The combination of impulsive control and switching control gives rise to the possibility of stabilization even if the switching control alone is inadequate (e.g. the switching control is only able to help reduce destabilization during the continuous portions of the system).*

Remark 5.2.2. *Although motivated by a hybrid control problem (where the hybrid control is a combination of state-dependent switching control and impulsive control), analyzing system (5.5) as it is posed can lead to results which are applicable to systems with impulsive perturbations.*

5.2.2 Stabilization under a Strict Completeness Condition

The first step towards constructing a state-dependent switching rule for (5.5) is partitioning the state-space into appropriate subregions. In order to do so, the following definition is needed.

Definition 5.2.1. [177] *The set of matrices* $\{\Phi_i\}_{i=1}^m$ *is said to be strictly complete if for every* $x \in \mathbb{R}^n \setminus \{0\}$*, there exists a* $j \in \mathcal{P}$ *such that* $x^T \Phi_j x < 0$*.*

Remark 5.2.3. *If there exist* $\alpha_i \geq 0$ *such that* $\sum_{i=1}^m \alpha_i = 1$ *and* $\sum_{i=1}^m \alpha_i \Phi_i$ *is negative definite*[2] *then the set* $\{\Phi_i\}$ *is strictly complete. If* $m = 2$ *then the condition is also necessary [177].*

Then we construct the switching regions as in, for example, [71, 72, 153]:

$$\Upsilon_i = \{x \in \mathbb{R}^n : x^T \Phi_i x < 0\}.$$

The main idea behind such a partition is that if the solution $x(t)$ of (5.5) is in Υ_i, then it is possible to show that the time-derivative of a Lyapunov functional along the solution is negative definite if the i^{th} subsystem is active. If this is true then the state-dependent switching rule associated with this partition should activate the i^{th} mode of (5.5) whenever the solution state is in the region Υ_i.

The strict completeness of the set $\{\Phi_i\}$ is sufficient and necessary to ensure that $\mathbb{R}^n = \cup_{i=1}^m \Upsilon_i$ [72]. That is, the union of the switching regions fully covers $\mathbb{R}^n$. However, there may be ambiguity here in how to switch the system if the regions Υ_i overlap. To deal with this, we follow the approach in the literature mentioned above by letting

$$\bar{\Upsilon}_1 = \Upsilon_1, \qquad \bar{\Upsilon}_i = \Upsilon_i \setminus \cup_{j=1}^{i-1} \bar{\Upsilon}_j, \quad i = 2, 3, \dots, m.$$

In this construction, the regions $\bar{\Upsilon}_i$ cannot overlap, and so we are in a position to construct the first state-dependent switching rule algorithm as follows.

Algorithm 5.2.1. (Strict completeness rule)
Given an initial state $x_0 = \phi_0(0)$*:*

(SC1) *Set* $\sigma(x_0) = i$ *where* i *is the index such that* $x_0 \in \bar{\Upsilon}_i$*.*

(SC2) *Remain in the* i^{th} *mode as long as the state* $x(t)$ *remains in* $\bar{\Upsilon}_i$*.*

[2] The symmetric matrix $A \in \mathbb{R}^{n \times n}$ is said to be negative definite if $x^T A x < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

(SC3) *If $x(t)$ crosses the boundary of $\bar{\Upsilon}_i$ at t_c, set $x_0 = x(t_c)$ and go to step (SC1).*

Remark 5.2.4. *In Algorithm 5.2.1, the initial mode is chosen by determining which of the switching regions $\bar{\Upsilon}_i$ the initial state lies in (this is unambiguous by construction of the regions). The state trajectory evolves according to the initial subsystem until a time t_c where $x(t_c^-) \in \bar{\Upsilon}_{\sigma(x_0)}$ and $x(t_c) \notin \bar{\Upsilon}_{\sigma(x_0)}$. This means the state trajectory has crossed the boundary (either by continuous dynamics or due to an impulse) and the next mode is chosen depending on which region the trajectory has entered. The process is repeated.*

Since the focus of this section is on linear systems with nonlinear perturbations, the following nonlinearity assumption is made on the functionals $F_i(t, x_t)$.

Assumption 5.2.1. *Assume that there exist nonnegative constants η_1, η_2, and η_3 such that for $t \geq t_0$ and $\psi \in PC([-\tau^*, 0], \mathbb{R}^n)$,*

$$\|F_i(t,\psi)\|^2 \leq \eta_1 \|\psi(0)\|^2 + \eta_2 \|\psi(-r)\|^2 + \eta_3 \int_{-\tau}^{0} \|\psi(s)\|^2 ds. \tag{5.6}$$

A lemma is required, which follows from the Matrix Cauchy Inequality (for example, see [72]).

Lemma 5.2.1. *For any $w, z \in \mathbb{R}^n$, the inequality $w^T z + z^T w \leq w^T w + z^T z$ always holds.*

We are now in position to state and prove the first result.

Theorem 5.2.2. *Suppose that Assumption 5.2.1 holds and suppose that there exist constants $\lambda > 0$, $\rho_k > 0$, $\alpha_i > 0$ satisfying $\sum_{i=1}^m \alpha_i = 1$, positive definite symmetric matrices P, R, S, and symmetric constant matrices D_k such that for $k \in \mathbb{N}$,*

(i) $\sum_{i=1}^m \alpha_i \Phi_i$ *is negative definite where*

$$\begin{aligned}\Phi_i =& A_i^T P + PA_i + \lambda P + P^2 + \eta_1 I + R + \tau S \\ &+ PB_i(e^{-\lambda r}R - \eta_2 I)^{-1} B_i^T P + \tau PC_i(e^{-\lambda\tau}S - \eta_3 I)^{-1} C_i^T P;\end{aligned} \tag{5.7}$$

(ii) $(e^{-\lambda r}R - \eta_2 I)$ *and* $(e^{-\lambda\tau}S - \eta_3 I)$ *are positive definite matrices;*

(iii) for each T_k and for all $x \in \mathbb{R}^n$,

$$2Q_k^T(T_k, x)P(I + E_k)x + Q_k^T(T_k, x)PQ_k(T_k, x) \leq x^T D_k x; \tag{5.8}$$

(iv) $T_k - T_{k-1} \geq \rho_k$ *and*

$$\ln \delta_k - \lambda \rho_k < 0 \tag{5.9}$$

where

$$\delta_k = \max\{1, \lambda_{\max}[P^{-1}((I + E_k)^T P(I + E_k) + D_k)]\}. \tag{5.10}$$

Then it follows that the trivial solution of system (5.5) is globally asymptotically stable under the state-dependent switching rule $\sigma(x)$ following Algorithm 5.2.1.

Proof. Define a Lyapunov functional $V(x_t) = V_1 + V_2 + V_3$ where,

$$\begin{aligned}
V_1 =& x^T(t)Px(t), \\
V_2 =& \int_{t-r}^{t} e^{-\lambda(t-s)} x^T(s)Rx(s)ds, \\
V_3 =& \int_{0}^{\tau} \int_{t-s}^{t} e^{-\lambda(t-\theta)} x^T(\theta)Sx(\theta)d\theta ds,
\end{aligned}$$

and note that $V(x_t)$ is positive for $x \neq 0$. Suppose that $\sigma = i_k$ on $[t_{k-1}, t_k)$ and take the time-derivative along solutions of (5.5) for $t \neq T_k$,

$$\begin{aligned}
\dot{V}_1 = & \left[A_{i_k}x(t) + B_{i_k}x(t-r) + \int_{t-\tau}^{t} C_{i_k}x(s)ds + F_{i_k}(t, x_t)\right]^T Px(t) \\
& + x^T(t)P\left[A_{i_k}x(t) + B_{i_k}x(t-r) + \int_{t-\tau}^{t} C_{i_k}x(s)ds + F_{i_k}(t, x_t)\right].
\end{aligned}$$

Also,

$$\begin{aligned}
\dot{V}_2 =& x^T(t)Rx(t) - e^{-\lambda r}x^T(t-r)Rx(t-r) - \lambda V_2, \\
\dot{V}_3 =& \int_{0}^{\tau} \left[x^T(t)Sx(t) - e^{-\lambda s}x^T(t-s)Sx(t-s)\right] ds \\
& - \lambda \int_{0}^{\tau} \int_{t-s}^{t} e^{-\lambda(t-\theta)} x^T(\theta)Sx(\theta)d\theta ds, \\
=& \tau x^T(t)Sx(t) - \int_{0}^{\tau} \left[e^{-\lambda s}x^T(t-s)Sx(t-s)\right] ds - \lambda V_3, \\
=& \tau x^T(t)Sx(t) - \int_{t-\tau}^{t} \left[e^{-\lambda(t-\theta)}x^T(\theta)Sx(\theta)\right] d\theta - \lambda V_3.
\end{aligned}$$

Hence,

$$\begin{aligned}\dot{V} &= x^T(t)(A_{i_k}^T P + PA_{i_k})x(t) + 2x^T(t)PB_{i_k}x(t-r) + 2F_{i_k}^T(t, x_t)Px(t) \\ &\quad + 2x^T(t)PC_{i_k}\int_{t-\tau}^{t} x(s)ds + x^T(t)Rx(t) - e^{-\lambda r}x^T(t-r)Rx(t-r) \\ &\quad + \tau x^T(t)Sx(t) - \int_{t-\tau}^{t} e^{-\lambda(t-\theta)}x^T(\theta)Sx(\theta)d\theta - \lambda(V_2 + V_3),\end{aligned}$$

and so,

$$\begin{aligned}\dot{V} &= x^T(t)(A_{i_k}^T P + PA_{i_k} + R + \tau S)x(t) - e^{-\lambda r}x^T(t-r)Rx(t-r) \\ &\quad + 2x^T(t)PB_{i_k}x(t-r) + 2F_{i_k}^T(t, x_t)Px(t) \\ &\quad - \int_{t-\tau}^{t} \left(e^{-\lambda(t-s)}x^T(s)Sx(s) - 2x^T(t)PC_{i_k}x(s)\right) ds - \lambda(V_2 + V_3).\end{aligned}$$

For a positive definite matrix S, constants $\tau > 0$ and $\lambda > 0$, and $t \geq t_0$,

$$\begin{aligned}-\textstyle\int_{t-\tau}^{t} e^{-\lambda(t-\theta)}x^T(\theta)Sx(\theta)d\theta &\leq -\int_{t-\tau}^{t} e^{-\lambda\tau}x^T(\theta)Sx(\theta)d\theta, \\ &= -e^{-\lambda\tau}\int_{t-\tau}^{t} x^T(\theta)Sx(\theta)d\theta.\end{aligned}$$

Using this fact along with Lemma 5.2.1,

$$\begin{aligned}\dot{V} &\leq x^T(t)(A_{i_k}^T P + PA_{i_k} + R + \tau S)x(t) - (e^{-\lambda r}x^T(t-r)Rx(t-r) \\ &\quad - 2x^T(t)PB_{i_k}x(t-r)) + F_{i_k}^T(t, x_t)F_{i_k}(t, x_t) + x^T(t)P^2x(t) \\ &\quad - \int_{t-\tau}^{t} (e^{-\lambda\tau}x^T(s)Sx(s) - 2x^T(t)PC_{i_k}x(s))ds - \lambda(V_2 + V_3).\end{aligned}$$

Applying the nonlinearity assumption in equation (5.6),

$$\begin{aligned}
\dot{V} \leq\ & x^T(t)(A_{i_k}^T P + PA_{i_k} + R + \tau S)x(t) - (e^{-\lambda h} x^T(t-r)Rx(t-r) \\
& - 2x^T(t)PB_{i_k}x(t-r)) + \eta_1 x^T(t)x(t) + \eta_2 x^T(t-r)x(t-r) \\
& + \eta_3 \int_{t-\tau}^{t} x^T(s)x(s)ds + x^T(t)P^2x(t) \\
& - \int_{t-\tau}^{t} (e^{-\lambda\tau} x^T(s)Sx(s) - 2x^T(t)PC_{i_k}x(s))ds - \lambda(V_2 + V_3), \\
=\ & x^T(t)(A_{i_k}^T P + PA_{i_k} + \lambda P + P^2 + \eta_1 I + R + \tau S)x(t) \\
& + x^T(t)[PB_{i_k}[e^{-\lambda r}R - \eta_2 I]^{-1}(PB_{i_k})^T \\
+\ & \tau PC_{i_k}[e^{-\lambda\tau}S - \eta_3 I]^{-1}(PC_{i_k})^T]x(t) \\
& - [(e^{-\lambda r}R - \eta_2 I)x(t-r) - (PB_{i_k})^T x(t)]^T [e^{-\lambda r}R - \eta_2 I]^{-1} \\
& \times [(e^{-\lambda r}R - \eta_2 I)x(t-r) - (PB_{i_k})^T x(t)] \\
& - \int_{t-\tau}^{t} [(e^{-\lambda\tau}S - \eta_3 I)x(s) - (PC_{i_k})^T x(t)]^T (e^{-\lambda\tau}S - \eta_3 I)^{-1} \\
& \times [(e^{-\lambda\tau}S - \eta_3 I)x(s) - (PC_{i_k})^T x(t)]ds - \lambda V.
\end{aligned}$$

Thus,

$$\begin{aligned}
\dot{V} \leq\ & x^T(t)(A_{i_k}^T P + PA_{i_k} + \lambda P + P^2 + \eta_1 I + R + \tau S)x(t) - \lambda V \\
& + x^T(t)[PB_{i_k}(e^{-\lambda r}R - \eta_2 I)^{-1}(PB_{i_k})^T]x(t) \\
& + x^T(t)[\tau PC_{i_k}(e^{-\lambda\tau}S - \eta_3 I)^{-1}(PC_{i_k})^T]x(t), \\
=\ & -\lambda V + x^T(t)\Phi_{i_k}x(t).
\end{aligned} \tag{5.11}$$

According to the state-dependent switching rule, the solution $x(t) \in \bar{\Upsilon}_{i_k}$ for $t \in [t_{k-1}, t_k)$, $t \neq T_k$. Therefore $x^T\Phi_{i_k}x < 0$ and so

$$\dot{V} \leq -\lambda V. \tag{5.12}$$

Define $m(t) = V(x_t)$, then for $t \in [T_k, T_{k+1})$,

$$m(t) \leq m(T_k)\exp[-\lambda(t - T_k)]. \tag{5.13}$$

For any symmetric matrix Q, $x^TQx \leq \lambda_{\max}(P^{-1}Q)x^TPx$ for any positive definite matrix

P. Using this fact, observe that immediately after an impulse is applied at $t = T_k$,

$$\begin{aligned} V_1(T_k) =& x^T(T_k^-)[(I+E_k)^T P(I+E_k)]x(T_k^-) \\ &+ Q_k^T(T_k, x(T_k^-))PQ(T_k, x(T_k^-)) \\ &+ Q_k^T(T_k, x(T_k^-))P(I+E_k)x(T_k^-) \\ &+ x^T(T_k^-)(I+E_k)^T PQ_k^T(T_k, x(T_k^-)), \\ \leq& x^T(T_k^-)[(I+E_k)^T P(I+E_k) + D_k]x(T_k^-) \\ \leq& \delta_k V_1(T_k^-), \end{aligned}$$

Further, by the continuity of V_2 and V_3, $V_2(T_k) = V_2(T_k^-)$ and $V_3(T_k) = V_3(T_k^-)$. Since $\delta_k \geq 1$, it follows that $m(T_k) \leq \delta_k m(T_k^-)$. Thus equation (5.13) implies

$$m(t) \leq m(T_k^-)\delta_k \exp[-\lambda(t - T_k)] \tag{5.14}$$

for $t \in [T_k, T_{k+1})$. Apply equation (5.14) successively on subintervals:

$$m(T_1^-) \leq m(t_0)\exp[-\lambda(T_1 - t_0)]$$

so that $m(T_1) \leq m(t_0)\delta_1 \exp[-\lambda(T_1 - t_0)]$. Similarly,

$$m(T_2) \leq m(t_0)\delta_1\delta_2 \exp[-\lambda(T_2 - t_0)] = m(t_0)C\delta_2 \exp[-\lambda(T_2 - T_1)],$$

where $C = \delta_1 \exp[-\lambda(T_1 - t_0)]$. In general, for $t \in [T_k, T_{k+1})$,

$$\begin{aligned} m(t) \leq& m(t_0)C\delta_2 \cdots \delta_k \exp[-\lambda(T_2 - T_1) - \ldots - \lambda(T_k - T_{k-1}) - \lambda(t - T_k)], \\ =& m(t_0)C \exp[\ln\delta_2 + \ldots + \ln\delta_k - \lambda(T_2 - T_1) - \ldots - \lambda(t - T_k)], \\ \leq& m(t_0)C \exp[\ln\delta_2 - \lambda\rho_2 + \ldots + \ln\delta_k - \lambda\rho_k]\exp[-\lambda\rho^*] \end{aligned}$$

where $\rho^* = \min\{p_k\}$. From equation (5.9), there exists a constant $\chi > 0$ such that

$$\ln\delta_k - \lambda\rho_k \leq -\chi.$$

Then,

$$m(t) \leq m(t_0)M\zeta^{k-1}, \tag{5.15}$$

where $M = C\exp[-\lambda\rho^*]$ and $\zeta = \exp[-\chi]$ satisfies $0 < \zeta < 1$. Evaluated along the initial function ϕ_0,

$$
\begin{aligned}
V(\phi_0) =&\phi_0(0)^T P\phi_0(0) + \int_{t_0-r}^{t_0} e^{-\lambda(t-s)}\phi_0^T(s)R\phi_0(s)ds \\
&+ \int_0^\tau \int_{t_0-s}^{t_0} e^{-\lambda(t-\theta)}\phi_0^T(\theta)S\phi_0(\theta)d\theta ds, \\
\leq&\lambda_{\max}(P)\|\phi_0\|_{\tau^*}^2 + \lambda_{\max}(R)\|\phi_0\|_{\tau^*}^2 \int_{t_0-r}^{t_0} e^{\lambda s}ds \\
&+ \lambda_{\max}(S)\|\phi_0\|_{\tau^*}^2 \int_0^\tau \int_{t_0-s}^{t_0} e^{\lambda\theta}d\theta ds, \\
=&(c_2+c_3)\|\phi_0\|_{\tau^*}^2
\end{aligned}
$$

where $c_2 = \lambda_{\max}(P)$ and

$$
c_3 = \left(\frac{1-e^{-\lambda r}}{\lambda}\right)\lambda_{\max}(R) + \left(\frac{\lambda\tau + e^{-\lambda\tau} - 1}{\lambda^2}\right)\lambda_{\max}(S) > 0. \tag{5.16}
$$

Also, $c_1\|\psi(0)\|^2 \leq V(\psi)$ for all $\psi \in PC([-\tau^*, 0], \mathbb{R}^n)$ where $c_1 = \lambda_{\min}(P)$. Hence equation (5.15) implies that for $t \in [T_k, T_{k+1})$,

$$
\|x(t)\|^2 \leq \frac{c_2 + c_3}{c_1}\|\phi_0\|_{\tau^*}^2 M\zeta^{k-1},
$$

where $0 < \zeta < 1$. Therefore, the trivial solution of system (5.5) is globally asymptotically stable. □

Remark 5.2.5. *Since $\delta_k \geq 1$, Theorem 5.2.2 can be interpreted as stabilizing state-dependent switching control that is robust to impulsive disturbances. From equation (5.9), a bound on the total disturbance of the impulses can be found:*

$$
1 \leq \delta_k < \exp[\lambda\rho_k].
$$

Another way to interpret this result is by observing that the time between successive impulses must be sufficiently large:

$$
T_k - T_{k-1} > \frac{\ln \delta_k}{\lambda}.
$$

Remark 5.2.6. *The switching rule $\sigma : \mathbb{R}^n \to \mathcal{P}$ constructed according to Algorithm 5.2.1 can be written as $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$ where the switching times $t_k = t_k(x)$ are state-dependent.*

To establish a result for the stabilizing impulsive case (where the switching control may not be adequate in stabilizing the system by itself), a lemma is required.

Lemma 5.2.3. [124]
Assume that there exist $V_1 \in \nu_0$, $V_2 \in \nu_{PC}^*$, positive constants $\lambda, T, \zeta, c_1, c_2, c_3$, and $\delta_k \geq 0$, such that for $k \in \mathbb{N}$,

(i) $c_1\|x\|^2 \leq V_1(t,x) \leq c_2\|x\|^2$, $0 \leq V_2(t,\psi) \leq c_3\|\psi\|_{\tau^*}^2$, *for all* $t \geq t_0$, $x \in \mathbb{R}^n$, $\psi \in PC([-\tau^*, 0], \mathbb{R}^n)$;

(ii) along solutions of (5.5) *for* $t \neq T_k$,

$$D^+V(t,\psi) \leq \lambda V(t,\psi)$$

where $V(t,x_t) = V_1(t,x) + V_2(t,x_t)$;

(iii) $V_1(T_k, (I+E_k)x + Q_k(T_k,x)) \leq \delta_k V_1(T_k^-, x)$ *for all* $x \in \mathbb{R}^n$;

(iv) $\tau^* \leq T_k - T_{k-1} \leq T$ *and* $\ln(\delta_k + c_3/c_1) + \lambda T \leq -\zeta T$.

Then the trivial solution of system (5.5) *is exponentially stable under the switching rule* σ.

Proof. Follows immediately from the proof of Theorem 3.1 in [124] with $g_k(t,x) = E_k x + Q_k(t,x)$ if the upper right-hand derivative satisfies $D^+V(t,\psi) \leq \lambda V(t,\psi)$ along the switching rule σ. □

We are now in a position to present the next result.

Theorem 5.2.4. *Suppose that Assumption 5.2.1 holds and suppose that there exist constants $\lambda > 0$, $\zeta > 0$, $T > 0$, $\alpha_i > 0$ satisfying $\sum_{i=1}^m \alpha_i = 1$, positive definite symmetric matrices P, R, S, and symmetric constant matrices D_k such that for $k \in \mathbb{N}$,*

(i) $\sum_{i=1}^m \alpha_i \Phi_i$ *is negative definite where*

$$\begin{aligned}\Phi_i =& A_i^T P + PA_i - \lambda P + P^2 + \eta_1 I + R + \tau S \\ &+ PB_i(e^{\lambda r}R - \eta_2 I)^{-1}B_i^T P + \tau PC_i(S - \eta_3 I)^{-1}C_i^T P;\end{aligned} \tag{5.17}$$

(ii) $(e^{\lambda r}R - \eta_2 I)$ *and* $(S - \eta_3 I)$ *are positive definite matrices;*

(iii) for each T_k and for all $x \in \mathbb{R}^n$,

$$2Q_k^T(T_k, x)P(I + E_k)x + Q_k^T(T_k, x)PQ_k(T_k, x) \leq x^T D_k x. \tag{5.18}$$

(iv) $\tau^* \leq T_k - T_{k-1} \leq T$ and

$$\ln\left(\delta_k + \frac{c_3}{c_1}\right) + \lambda T \leq -\zeta T, \tag{5.19}$$

where

$$\begin{aligned}
c_1 =& \lambda_{\min}(P), \\
c_3 =& \left(\frac{e^{\lambda r} - 1}{\lambda}\right)\lambda_{\max}(R) + \left(\frac{e^{\lambda\tau} - 1 - \lambda\tau}{\lambda^2}\right)\lambda_{\max}(S), \qquad (5.20)\\
\delta_k =& \lambda_{\max}[P^{-1}((I + E_k)^T P(I + E_k) + D_k)] \geq 0. \qquad (5.21)
\end{aligned}$$

Then the trivial solution of system (5.5) is exponentially stable under the state-dependent switching rule $\sigma(x)$ constructed from Algorithm 5.2.1.

Proof. Define a Lyapunov functional $V(x_t) = V_1 + V_2 + V_3$ where,

$$\begin{aligned}
V_1 =& x^T(t)Px(t), \\
V_2 =& \int_{t-r}^{t} e^{\lambda(t-s)}x^T(s)Rx(s)ds, \\
V_3 =& \int_0^{\tau}\int_{t-s}^{t} e^{\lambda(t-\theta)}x^T(\theta)Sx(\theta)d\theta ds,
\end{aligned}$$

and note that $V_1 \in \nu_0$, $V_2 \in \nu_{PC}^*$, and $V_3 \in \nu_{PC}^*$. Using Lemma 5.2.1, the nonlinearity assumption in equation (5.6), and the fact that for $\lambda > 0$, $\tau > 0$, and a positive definite matrix S,

$$-\int_{t-\tau}^{t} e^{\lambda(t-\theta)}x^T(\theta)Sx(\theta)d\theta \leq -\int_{t-\tau}^{t} x^T(\theta)Sx(\theta)d\theta,$$

then similar to the proof of Theorem 5.2.2 it is possible to show that along solutions of (5.5),

$$\begin{aligned}
\dot{V} \leq& x^T(t)(A_{i_k}^T P + PA_{i_k} - \lambda P + P^2 + \eta_1 I + R + \tau S)x(t) + \lambda V \\
&+ x^T(t)[PB_{i_k}(e^{\lambda r}R - \eta_2 I)^{-1}(PB_{i_k})^T]x(t) \\
&+ x^T(t)[\tau PC_{i_k}(S - \eta_3 I)^{-1}(PC_{i_k})^T]x(t), \\
=& \lambda V + x^T(t)\Phi_{i_k}x(t).
\end{aligned}$$

From the state-dependent switching rule construction, $x(t) \in \bar{\Upsilon}_{i_k}$ for $t \in [t_{k-1}, t_k)$. Thus, $x^T \Phi_{i_k} x < 0$ which means

$$\dot{V} \leq \lambda V. \tag{5.22}$$

Note that $c_1\|x\|^2 \leq V_1 \leq c_2\|x\|^2$ for all $x \in \mathbb{R}^n$ where $c_1 = \lambda_{\min}(P)$ and $c_2 = \lambda_{\max}(P)$. Also, $0 \leq V_2 + V_3 \leq c_3\|\psi\|_{\tau^*}^2$ for all $\psi \in PC([-\tau^*, 0], \mathbb{R}^n)$ where c_3 is given in (5.20). Further, $m(t) = V(x_t)$ satisfies $m'(t) \leq \lambda m(t)$ and $V_1(T_k^+) \leq \delta_k V_1(T_k)$ for $\delta_k \geq 0$. Finally, since $\tau^* \leq T_k - T_{k-1} \leq T$ and $\ln(\delta_k + c_3/c_1) + \lambda T \leq -\zeta T$, all the conditions of Lemma 5.2.3 are satisfied and hence the trivial solution of (5.5) is exponentially stable. □

5.2.3 A Minimum Rule for Overlapping Switching Regions

As discussed in Remark 5.1.3, the mathematical well-posedness and practical application of a state-dependent switching rule are of concern. Motivated by a desire to avoid chattering behaviour and sliding motions, we consider a different switching region partition for (5.5) following the literature (for example, see [50, 78, 92]). Suppose that there exist constants $\alpha_i > 0$ satisfying $\sum_{i=1}^m \alpha_i = 1$ such that $\sum_{i=1}^m \alpha_i A_i$ is a Hurwitz matrix. Then there exists a positive definite matrix P such that

$$\left(\sum_{i=1}^m \alpha_i A_i\right)^T P + P\left(\sum_{i=1}^m \alpha_i A_i\right) = -Q \tag{5.23}$$

for any positive definite matrix Q. For $x \neq 0$,

$$\begin{aligned} & x^T\left[\left(\sum_{i=1}^m \alpha_i A_i\right)^T P + P\left(\sum_{i=1}^m \alpha_i A_i\right)\right] x \\ & = \sum_{i=1}^m \alpha_i x^T (A_i^T P + P A_i) x, \\ & = -x^T Q x < 0. \end{aligned} \tag{5.24}$$

This inequality can be taken advantage of by defining the switching regions as

$$\widetilde{\Omega}_i = \{x \in \mathbb{R}^n : x^T(A_i^T P + P A_i)x \leq -x^T Q x\}.$$

The union of $\widetilde{\Omega}_i$ covers $\mathbb{R}^n$ and it is straightforward to prove (for example, see [92]): since $\alpha_i > 0$, equation (5.24) implies that for $x \neq 0$,

$$\alpha_i x^T (A_i^T P + P A_i) x < 0 \tag{5.25}$$

for at least one i. If this were not the case then for any α_i it must be true that $\alpha_i x^T(A_i^T P + PA_i)x \geq 0$ and hence $\sum_{i=1}^m \alpha_i x^T(A_i^T P + PA_i)x \geq 0$. This leads to $-x^T Q x \leq 0$ which is clearly a contradiction as Q is positive definite and therefore $\mathbb{R}^n = \cup_{i=1}^m \widetilde{\Omega}_i$. To avoid the chattering behaviour detailed above, extend the switching regions so that they overlap by re-defining the regions as

$$\Omega_i = \left\{ x \in \mathbb{R}^n : x^T(A_i^T P + PA_i)x \leq -\frac{1}{\xi} x^T Q x \right\}$$

for some $\xi > 1$ chosen beforehand.

Then similar to the algorithm in [92] for systems without impulses, a revised algorithm for state-dependent switching can be formulated by using a minimum rule to choose the current mode and by changing modes whenever a switching region boundary is crossed.

Algorithm 5.2.2. (Minimum rule)
Given an initial state $x_0 = \phi_0(0)$ and $\xi > 1$:

(MR1) *Choose the active mode using the minimum rule*

$$\sigma(x_0) = \operatorname*{argmin}_{i \in \mathcal{P}} x^T(A_i^T P + PA_i)x.$$

(MR2) *Remain in the active mode as long as the state $x(t)$ is in*

$$\Omega_i = \{x \in \mathbb{R}^n : x^T(A_i^T P + PA_i)x \leq -\frac{1}{\xi} x^T Q x\}.$$

(MR3) *If $x(t)$ crosses the boundary of Ω_i at t_c, set $x_0 = x(t_c)$ and go to step (MR1).*

Theorem 5.2.5. *Suppose that Assumption 5.2.1 holds and suppose that there exist constants $\lambda > 0$, $\xi > 1$, $\rho_k > 0$, $\alpha_i > 0$ satisfying $\sum_{i=1}^m \alpha_i = 1$, positive definite symmetric matrices Q, R, S, and symmetric constant matrices D_k such that for $k \in \mathbb{N}$,*

(i) $\sum_{i=1}^m \alpha_i A_i$ *is Hurwitz;*

(ii) for all $i \in \mathcal{P}$,

$$-\frac{\lambda_{min}(Q)}{\xi} + \lambda_{max}(\Lambda_i) < 0 \tag{5.26}$$

where

$$\begin{aligned}\Lambda_i =& \lambda P + P^2 + \eta_1 I + R + \tau S + PB_i(e^{-\lambda r}R - \eta_2 I)^{-1} B_i^T P \\ &+ \tau P C_i (e^{-\lambda\tau} S - \eta_3 I)^{-1} C_i^T P \end{aligned} \tag{5.27}$$

and P satisfies equation (5.23);

(iii) $(e^{-\lambda r}R - \eta_2 I)$ *and* $(e^{-\lambda\tau}S - \eta_3 I)$ *are positive definite matrices;*

(iv) for each T_k *and for all* $x \in \mathbb{R}^n$,

$$2Q_k^T(T_k, x)P(I + E_k)x + Q_k^T(T_k, x)PQ_k(T_k, x) \leq x^T D_k x; \tag{5.28}$$

(v) $T_k - T_{k-1} \geq \rho_k$ *and*

$$\ln \delta_k - \lambda\rho_k < 0 \tag{5.29}$$

where

$$\delta_k = \max\{1, \lambda_{\max}[P^{-1}((I + E_k)^T P(I + E_k) + D_k)]\}. \tag{5.30}$$

Then the trivial solution of system (5.5) is globally asymptotically stable under the state-dependent switching rule $\sigma(x)$ *constructed according to Algorithm 5.2.2.*

Proof. Use the same Lyapunov functional as in the proof of Theorem 5.2.2 and begin at equation (5.11) to get that along solutions of (5.5),

$$\begin{aligned}
\dot{V} &\leq x^T(t)(A_{i_k}^T P + PA_{i_k} + \lambda P + P^2 + \eta_1 I + R + \tau S)x(t) - \lambda V \\
&\quad + x^T(t)[PB_{i_k}(e^{-\lambda r}R - \eta_2 I)^{-1}(PB_{i_k})^T]x(t) \\
&\quad + x^T(t)[\tau PC_{i_k}(e^{-\lambda\tau}S - \eta_3 I)^{-1}(PC_{i_k})^T]x(t), \\
&= -\lambda V + x^T(t)(A_{i_k}^T P + PA_{i_k})x(t) + x^T(t)\Lambda_{i_k}x(t),
\end{aligned}$$

Under the state-dependent switching rule $x(t) \in \Omega_{i_k}$ on $[t_{k-1}, t_k)$, $t \neq T_k$, so that

$$\begin{aligned}
\dot{V} &\leq -\lambda V - \frac{1}{\xi}x^T(t)Qx(t) + \lambda_{\max}(\Lambda_{i_k})x^T(t)x(t), \\
&\leq -\lambda V + \left(-\frac{\lambda_{\min}(Q)}{\xi} + \lambda_{\max}(\Lambda_{i_k})\right)x^T(t)x(t),
\end{aligned}$$

Hence $\dot{V} \leq -\lambda V$ according to (5.26). The rest of the proof follows. □

Remark 5.2.7. *In Algorithm 5.2.2, the initial mode is chosen by evaluating* $\sigma(x_0)$ *according to the minimum rule, where* $x_0 = \phi_0(0)$. *The state trajectory evolves according to the initial subsystem until a time* t_c *where* $x(t_c^-) \in \Omega_{\sigma(x_0)}$ *and* $x(t_c) \notin \Omega_{\sigma(x_0)}$. *This means the state trajectory has crossed the boundary either by continuous dynamics or due to an impulse. The minimum rule is then applied to* $x(t_c)$ *to select the next appropriate mode and the process is repeated. See Figure 5.4 for an illustration.*

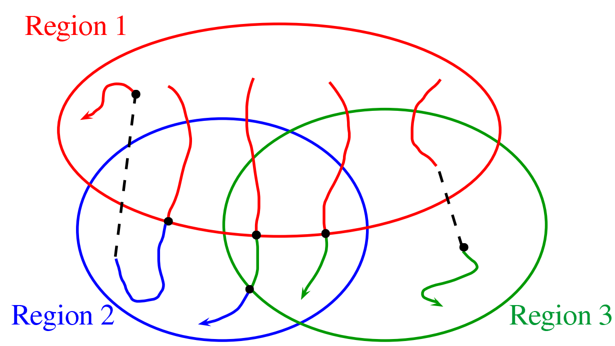


Figure 5.4: When a state trajectory crosses a switching region boundary, the system is switched according to the minimum rule. The dotted lines represent impulsive effects.

Remark 5.2.8. *Note that the time between impulses has a lower bound ($T_k - T_{k-1} \geq \rho_k > 0$) and the switching regions Ω_i are constructed so that they overlap near boundaries. Once the state switches to a new subsystem at the switching instant t_k, there is a possibility there is an impulse time arbitrarily close to t_k, sending the trajectory to another switching region and requiring another switch (a pathological case that is unlikely to occur in implementation). The constants ρ_k ensure that a period of time is then spent in the switching portion of the minimum rule algorithm with overlapping regions.*

5.2.4 Coinciding Switching and Impulsive Times

Next we consider the scenario when an impulse is applied at each switching time (i.e. $t_k = T_k$ for all $k \in \mathbb{N}$). Examples of switched systems that exhibit impulsive effects at the switching times include optimal control in economics, biological neural networks, and bursting rhythm models in pathology [56]. First we consider how the strict completeness rule can be extended to this case.

Theorem 5.2.6. *Suppose that $T_k = t_k$ for all $k \in \mathbb{N}$ and suppose that Assumption 5.2.1 holds. Assume that there exist constants $\lambda > 0$, $\alpha_i > 0$ satisfying $\sum_{i=1}^{m} \alpha_i = 1$, positive definite symmetric matrices P, R, S, and symmetric constant matrices D_k such that for $k \in \mathbb{N}$,* *(i)* $\sum_{i=1}^{m} \alpha_i \Phi_i$ *is negative definite where*

$$\begin{aligned}\Phi_i =& A_i^T P + P A_i + \lambda P + P^2 + \eta_1 I + R + \tau S \\ &+ P B_i (e^{-\lambda r} R - \eta_2 I)^{-1} B_i^T P + \tau P C_i (e^{-\lambda \tau} S - \eta_3 I)^{-1} C_i^T P;\end{aligned} \tag{5.31}$$

(ii) $(e^{-\lambda r} R - \eta_2 I)$ *and* $(e^{-\lambda \tau} S - \eta_3 I)$ *are positive definite matrices;*

(iii) for each T_k *and for all* $x \in \mathbb{R}^n$,

$$2Q_k^T(T_k, x) P (I + E_k) x + Q_k^T(T_k, x) P Q_k(T_k, x) \leq x^T D_k x; \tag{5.32}$$

(iv) there exists a constant $M > 0$ *such that* $\prod_{k=1}^{\infty} \max\{1, \delta_k\} < M$ *where*

$$\delta_k = \lambda_{\max}[P^{-1}((I + E_k)^T P (I + E_k) + D_k)] \geq 0. \tag{5.33}$$

Then the trivial solution of system (5.5) is globally exponentially stable under the state-dependent switching rule $\sigma(x)$ *outlined in Algorithm 5.2.1.*

Proof. Begin at equation (5.12) from the proof of Theorem 5.2.2. For $t \in [t_{k-1}, t_k)$,

$$m(t) \leq m(t_k) \exp[-\lambda (t - t_k)].$$

Further, $m(t_k) \leq \max\{1, \delta_k\} m(t_k^-)$. Therefore,

$$m(t) \leq m(t_k^-) \max\{1, \delta_k\} \exp[-\lambda (t - t_k)], \tag{5.34}$$

for $t \in [t_k, t_{k+1})$. Apply equation (5.34) successively on subintervals to get that

$$m(t) \leq m(t_0) \max\{1, \delta_1\} \max\{1, \delta_2\} \cdots \max\{1, \delta_k\} \exp[-\lambda (t - t_0)],$$

for $t \in [t_k, t_{k+1})$, which implies

$$\|x(t)\|^2 \leq \frac{(c_2 + c_3)}{c_1} \|\phi_0\|_{\tau^*}^2 M \exp[-\lambda (t - t_0)],$$

where $c_1 = \lambda_{\min}(P)$, $c_2 = \lambda_{\max}(P)$, and c_3 is given in equation (5.16). The result follows. □

Remark 5.2.9. *The condition* $\prod_{k=1}^{\infty} \max\{1, \delta_k\} < M$ *in Theorem 5.2.6 implies that disturbance impulsive effects must have finite total power (that is,* $\lim_{k \to \infty} \delta_k = 1$*). This is stricter than what is required in Theorem 5.2.2 and Theorem 5.2.4.*

The result can be extended to the case of overlapping switching regions.

Theorem 5.2.7. *Suppose that $t_k = T_k$ for all $k \in \mathbb{N}$ and suppose that Assumption 5.2.1 holds. Assume that there exist constants $\lambda > 0$, $\xi > 1$, $\alpha_i > 0$ satisfying $\sum_{i=1}^{m} \alpha_i = 1$, positive definite symmetric matrices Q, R, S, and symmetric constant matrices D_k such that for $k \in \mathbb{N}$,*

(i) $\sum_{i=1}^{m} \alpha_i A_i$ *is Hurwitz;*

(ii) for $i \in \mathcal{P}$,

$$-\frac{\lambda_{min}(Q)}{\xi} + \lambda_{max}(\Lambda_i) < 0 \tag{5.35}$$

where

$$\begin{aligned} \Lambda_i =& \lambda P + P^2 + \eta_1 I + R + \tau S + PB_i(e^{-\lambda r}R - \eta_2 I)^{-1}B_i^T P \\ &+ \tau PC_i(e^{-\lambda\tau}S - \eta_3 I)^{-1}C_i^T P \end{aligned} \tag{5.36}$$

and P is solved from equation (5.23);

(iii) $(e^{-\lambda r}R - \eta_2 I)$ *and* $(e^{-\lambda\tau}S - \eta_3 I)$ *are positive definite matrices;*

(iv) for each T_k and for all $x \in \mathbb{R}^n$,

$$2Q_k^T(T_k, x)P(I + E_k)x + Q_k^T(T_k, x)PQ_k(T_k, x) \le x^T D_k x; \tag{5.37}$$

(v) there exists a constant $M > 0$ such that $\prod_{k=1}^{\infty} \max\{1, \delta_k\} < M$ where

$$\delta_k = \lambda_{\max}[P^{-1}((I + E_k)^T P(I + E_k) + D_k)] \ge 0. \tag{5.38}$$

Then the trivial solution of system (5.5) is globally exponentially stable under the state-dependent switching rule following Algorithm 5.2.2.

Proof. The proof is similar to the proofs of Theorem 5.2.6 and Theorem 5.2.5. □

Remark 5.2.10. *According to the minimum rule Algorithm 5.2.2, if $t_k = T_k$ the initial mode is chosen by evaluating the minimum rule. Whenever the trajectory crosses the boundary of $\Omega_{\sigma(x_0)}$ at t_c, an impulse is applied and the minimum rule is used on $x(t_c)$ (the state after the impulse) to select the next appropriate mode. See Figure 5.5 for an illustration of this case.*

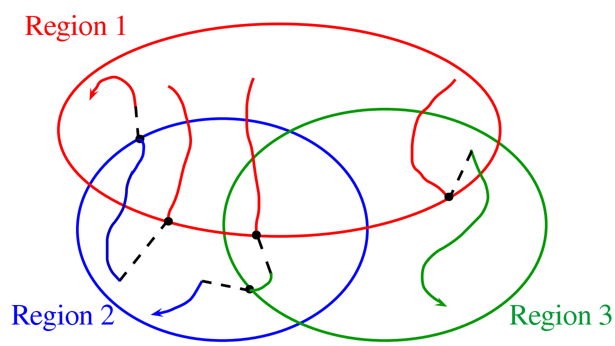


Figure 5.5: State trajectories under the revised state-dependent switching algorithm outlined in Algorithm 5.2.2. If $t_k = T_k$ then an impulse is applied whenever a boundary is crossed.

Remark 5.2.11. *Under Algorithm 5.2.2, if $t_k = T_k$ then the impulsive effects could continually send the state trajectory to a region boundary (leading to impulsive chattering/fast switching behaviour). It seems that this would be a pathological case and could be avoided by adjusting the switching in Algorithm 5.2.2 as follows: suppose that $x(t)$ is inside the region Ω_i, where $i \in \mathcal{P}$ is the current active subsystem, and hits the boundary at $t = t_k$ (which forces an impulse and switch at the current state $\bar{x} := x(t_k^-)$). The minimum rule is then applied to $x(t_k)$ and suppose that $j \in \mathcal{P}$ is the next appropriate mode chosen by the minimum rule. If after the impulse $x(t_k)$ lies on the boundary of Ω_j and the switching algorithm immediately sends $x(t_k)$ back to the point $\bar{x}$ on the boundary of Ω_i, then the switching regions must be adjusted by, for example, selecting a new ξ value to shift the boundaries.*

5.2.5 Numerical Simulations

Example 5.2.1. *Consider the HISD (5.5) with $\mathcal{P} = \{1, 2\}$, $t_0 = 0$, $T_k - T_{k-1} = 0.05$, discrete delay $r = 0.01$, distributed delay $\tau = 0.01$,*

$$A_1 = \begin{pmatrix} 10 & 1.3 \\ -0.3 & 0.3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.4 & -0.3 \\ 0.35 & 8.5 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.1 & 0.5 \\ 0.4 & 0.3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 0.3 & 0.1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} -1.2 & 1 \\ -0.8 & -0.1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1.2 & -1 \\ 0.8 & 1.2 \end{pmatrix},$$

$$F_1(t, x_t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad F_2(t, x_t) = \begin{pmatrix} 0.1\cos^2(t-r)\sqrt{x_1(t-r)^2 + x_2(t-r)^2} \\ 0.1\sin^2(t-r)\sqrt{x_1(t-r)^2 + x_2(t-r)^2} \end{pmatrix},$$

$$E_{2k-1} = \begin{pmatrix} -0.4 & -0.2 \\ 0 & -0.5 \end{pmatrix}, \quad E_{2k} = \begin{pmatrix} -0.8 & 0 \\ 0 & -0.6 \end{pmatrix},$$

$$Q_{2k-1}(t, x) = \begin{pmatrix} -0.25 sign(x_1)|x_2| \\ 0 \end{pmatrix}, \quad Q_{2k}(t, x) = 0,$$

where

$$sign(y) := \begin{cases} 1, & \text{for } y > 0, \\ 0, & \text{for } y = 0, \\ -1, & \text{for } y < 0. \end{cases}$$

The matrices A_1 and A_2 both have eigenvalues with positive real part. The nonlinear function Q_{2k-1} is taken from [112]. Choose $\lambda = 14$, $\eta_1 = 0$, $\eta_2 = 0.02$, $\eta_3 = 0$, $\delta_{2k-1} = 0.461$, $\delta_{2k} = 0.160$,

$$P = R = S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D_{2k} = 0, \quad D_{2k-1} = \begin{pmatrix} 0 & 0.150 \\ 0.150 & 0.1625 \end{pmatrix}.$$

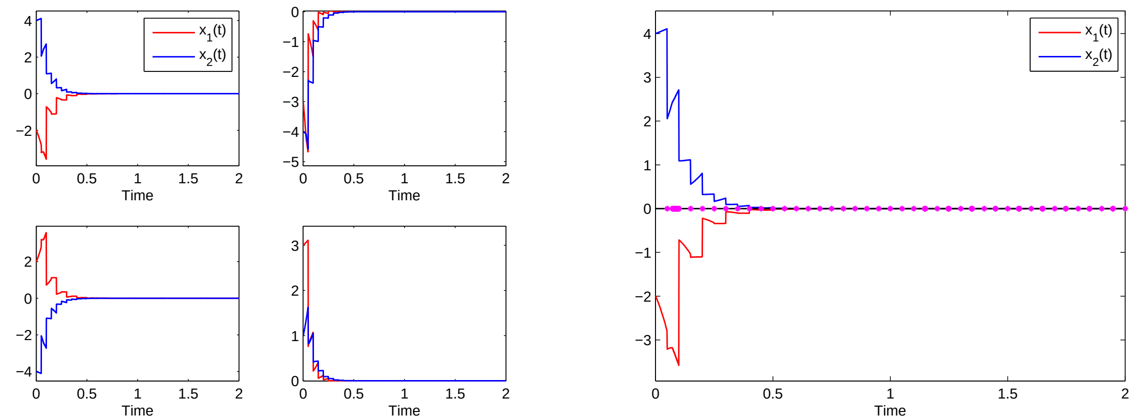
Then

$$\Phi_1 = \begin{pmatrix} 8.26 & 1.18 \\ 1.18 & -11.2 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} -10.3 & 0.136 \\ 0.136 & 5.12 \end{pmatrix}.$$

If $\alpha_1 = \alpha_2 = 0.5$ then $\alpha_1\Phi_1 + \alpha_2\Phi_2$ is negative definite. Here $c_1 = 1$, $c_3 = 0.0108$, $\rho_k = 0.05$, and equation (5.19) is satisfied with $\zeta = -0.0504$. All the conditions of Theorem 5.2.4 are satisfied and hence the system is exponentially stable under the switching rule constructed according to Algorithm 5.2.1. See Figure 5.6 for an illustration. Note that there are a total of 561 switches made in the simulation (the abundance of which occur towards the end of the simulation when the solution trajectory is very close to the origin). For a state-space illustration, see Figure 5.10. It is clear that the impulses act as a stabilizing feature for the system.

Example 5.2.2. *Consider the HISD (5.5) with $\mathcal{P} = \{1, 2\}$, $t_0 = 0$, discrete delay $r = 0.1$, and distributed delay $\tau = 0.1$,*

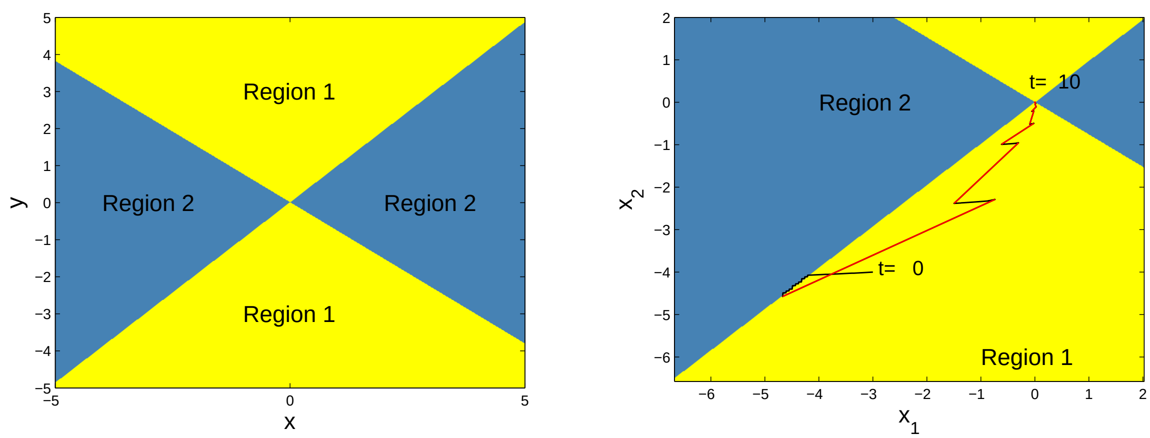
$$A_1 = \begin{pmatrix} 1 & -1.3 \\ -0.3 & -8.3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -12 & 1.3 \\ 1.5 & 0.5 \end{pmatrix},$$



(a) Simulations with different initial conditions. The green ticks on the t-axis mark impulsive moments.

(b) Initial function $\phi_0(s) = (-2, 4)$. The magenta ticks on the t-axis mark the switching times.

Figure 5.6: Simulation of Example 5.2.1.



(a) The yellow region represents $\bar{\Upsilon}_1$ and the blue region represents $\bar{\Upsilon}_2$.

(b) Initial function $\phi_0(s) = (-2, 4)$. The red lines represent impulsive effects.

Figure 5.7: Simulation of Example 5.2.1.

$$B_1 = \begin{pmatrix} 1.4 & 0.1 \\ 0.4 & 0.3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -0.4 & 0.1 \\ 0.3 & 0.7 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1.2 & 3 \\ 0.3 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0.2 & 0 \\ 0.1 & -0.4 \end{pmatrix},$$

$$F_1(t, x_t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad F_2(t, x_t) = \begin{pmatrix} 0.1\cos^2(t-r)\sqrt{x_1(t-r)^2 + x_2(t-r)^2} \\ 0.1\sin^2(t-r)\sqrt{x_1(t-r)^2 + x_2(t-r)^2} \end{pmatrix},$$

$$E_{2k-1} = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad E_{2k} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.4 \end{pmatrix},$$

$$Q_{2k-1}(t, x) = \begin{pmatrix} -0.25 sign(x_1)|x_2| \\ 0 \end{pmatrix}, \quad Q_{2k}(t, x) = 0.$$

Note that A_1 and A_2 both have an eigenvalue with positive real part. Choose $\lambda = 4$, $\delta_{2k-1} = 2.15$, $\delta_{2k} = 1.96$, $\eta_1 = 0$, $\eta_2 = 0.02$, $\eta_3 = 0$,

$$P = R = S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D_{2k} = 0, \quad D_{2k-1} = \begin{pmatrix} 0 & 0.350 \\ 0.350 & 0.0625 \end{pmatrix}.$$

Then

$$\Phi_1 = \begin{pmatrix} 12.7 & -0.192 \\ -0.192 & -9.95 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} -17.6 & 2.73 \\ 2.73 & 8.02 \end{pmatrix}.$$

If $\alpha_1 = \alpha_2 = 0.5$ then $\alpha_1\Phi_1 + \alpha_2\Phi_2$ is negative definite. Suppose that $T_k - T_{k-1} = 0.2$, then take $\rho_{2k-1} = \rho_{2k} = 0.2$ to get

$$\ln\delta_k - \lambda\rho_k \leq -0.0348$$

for all $k \in \mathbb{N}$. All the conditions of Theorem 5.2.2 are satisfied and so the trivial solution is globally asymptotically stable under the state-dependent switching rule constructed according to the strict completeness Algorithm 5.2.1. See Figure 5.8 for a simulation.

From a practical perspective, the number of switches required in stabilizing the system is of interest. Motivated by this, we illustrate the number of switches in Figure 5.9. Once the solution trajectory reaches the first switching region boundary, the trajectory bounces back and forth resulting in a high number of switches (8443 for the duration of the simulation). The impulsive effects send the trajectory away from the boundary which gives a brief pause in the switchings. For a state-space illustration, see Figure 5.10.

Motivated by the desire to avoid requiring an impractical number of switches, we consider applying the minimum rule Algorithm 5.2.2 to this example. The minimum rule

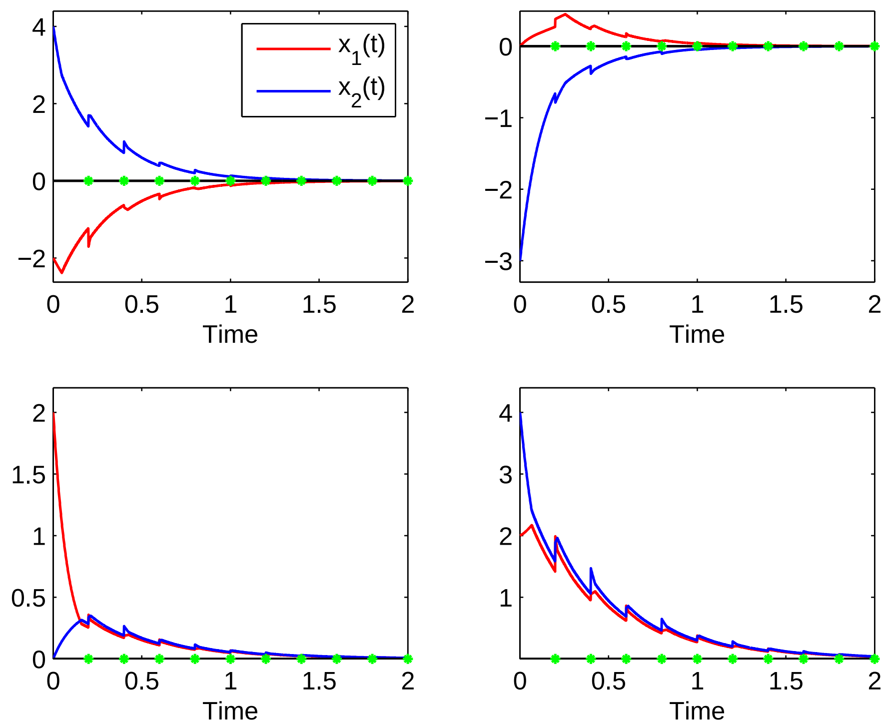


Figure 5.8: Simulation of Example 5.2.2 with $T_k - T_{k-1} = 0.2$ and different initial conditions. The green ticks on the t-axis mark impulsive moments

algorithm, where the switching regions overlap, is applicable here since $0.5A_2 + 0.5A_2$ is a Hurwitz matrix. Choose $Q = 10I$ then P can be solved from (5.23):

$$P = \begin{pmatrix} 0.909 & 0.058 \\ 0.058 & 1.29 \end{pmatrix}.$$

Choose $R = S = I$, $\delta_{2k-1} = 2.15$, $\delta_{2k} = 1.96$, $\eta_1 = 0$, $\eta_2 = 0.02$, $\eta_3 = 0$,

$$D_{2k} = 0, \quad D_{2k-1} = \begin{pmatrix} 0 & 0.3181 \\ 0.3181 & 0.0916 \end{pmatrix}.$$

If we consider $\lambda = 4$ then condition (5.26) fails to hold. Instead, we take $\lambda = 2$, $T_k - T_{k-1} = 0.4$, $\rho_{2k-1} = \rho_{2k} = 0.4$, then all the conditions of Theorem 5.2.5 are satisfied. The switching regions are given by $\Omega_i = \{x \in \mathbb{R}^2 : x^T(A_i^T P + PA_i)x \leq -\frac{10}{\xi}x^T x\}$ for $i = 1, 2$ since $Q = 10I$.

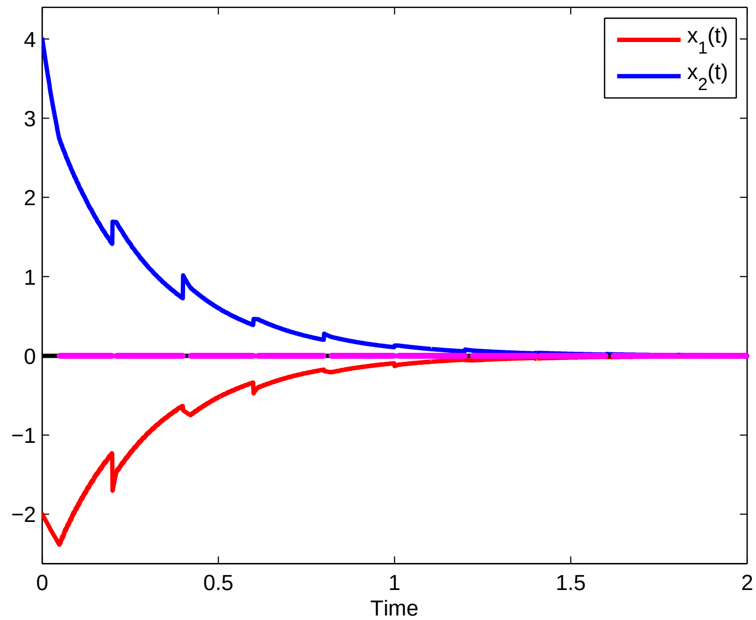


Figure 5.9: Simulation of Example 5.2.2 under the strict completeness algorithm with $T_k - T_{k-1} = 0.2$ and initial function $\phi_0(s) = (-2, 4)$ for $-0.1 \leq s \leq 0$. The magenta ticks on the t-axis mark the switching times.

See Figure 5.11. In this case the total number of switches are 599 (an order of magnitude less than the strict completeness algorithm).

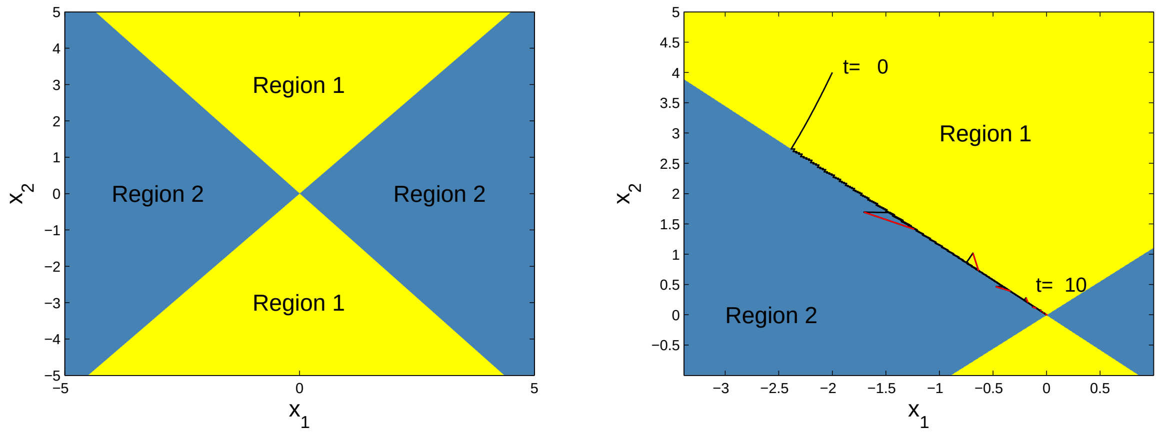
Increasing the constant ξ decreases the total number of switches required significantly (see Figure 5.12), however, the theorem conditions in Theorem 5.2.5 are no longer satisfied for these values of ξ with the above chosen constants and model parameters.

Example 5.2.3. *Consider the HISD* (5.5) *with* $\mathcal{P} = \{1, 2\}$, $t_0 = 0$, $T_k = t_k$, $r = 0.1$, $\tau = 0.1$,

$$A_1 = \begin{pmatrix} -3 & -0.3 \\ 6 & -5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & -0.3 \\ -1 & -4.5 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -1.4 & 1 \\ 0.4 & -8 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1.4 & 0 \\ 3.3 & -6 \end{pmatrix},$$

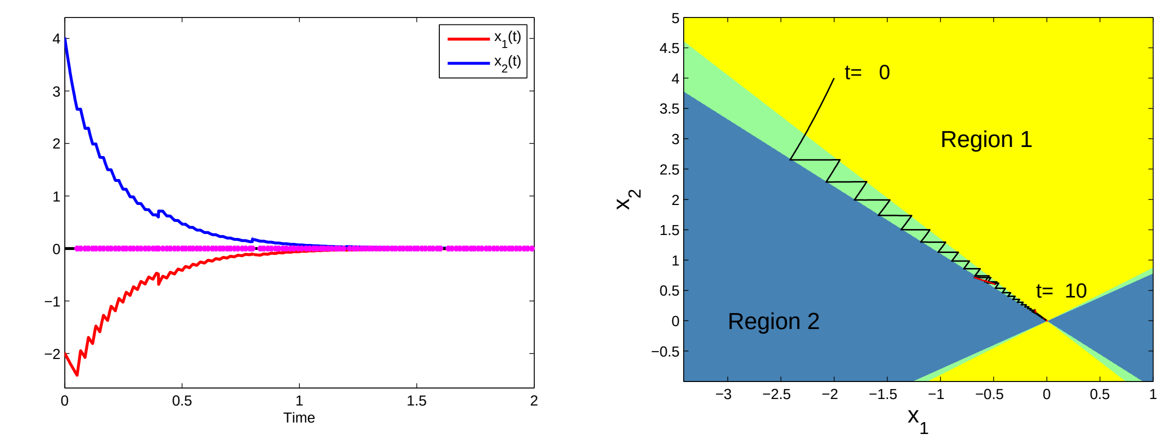
$$C_1 = \begin{pmatrix} 3 & 0 \\ 2.8 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -1.2 & -1 \\ -2.8 & -1.2 \end{pmatrix},$$



(a) The yellow region represents $\bar{\Upsilon}_1$ and the blue region represents $\bar{\Upsilon}_2$.

(b) The red lines represent impulsive effects.

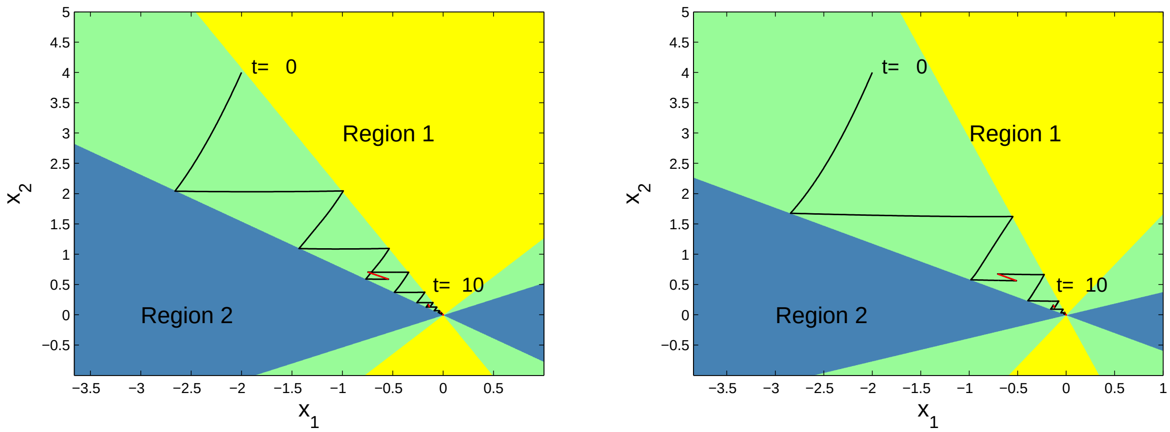
Figure 5.10: Simulation of Example 5.2.2 under the strict completeness algorithm with $T_k - T_{k-1} = 0.2$.



(a) Simulation with $\xi = 1.1$. The magenta ticks on the t-axis mark the switching times.

(b) The yellow region represents Ω_1 and the blue region represents Ω_2

Figure 5.11: Simulation of Example 5.2.2 under the minimum rule algorithm with $T_k - T_{k-1} = 0.4$.



(a) $\xi = 2$ results in 137 switches. (b) $\xi = 4$ results in 81 switches.

Figure 5.12: Simulation of Example 5.2.2 under the minimum rule algorithm with $T_k - T_{k-1} = 0.4$.

$$F_1(t, x_t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad F_2(t, x_t) = \begin{pmatrix} \cos^2(t-r)\sqrt{x_1(t-r)^2 + x_2(t-r)^2} \\ \sin^2(t-r)\sqrt{x_1(t-r)^2 + x_2(t-r)^2} \end{pmatrix},$$

$$E_{2k-1} = \begin{pmatrix} 1 - \sqrt{1 + 10\left(\frac{1}{2}\right)^{2k-1}} & 0 \\ 0 & -1 + \sqrt{1 + 10\left(\frac{9}{10}\right)^{2k-1}} \end{pmatrix},$$

$$E_{2k} = \begin{pmatrix} -1 + \sqrt{1 + 10\left(\frac{2}{3}\right)^{2k}} & 0 \\ 0 & -1 + \sqrt{1 + 10\frac{\left(\frac{2}{3}\right)^{2k}}{(2k)!}} \end{pmatrix},$$

and $Q_k(t, x) = 0$. Again A_1 and A_2 have eigenvalues with positive real part. Choose $\lambda = -1$, $D_1 = 0$, $D_2 = 0$, $\eta_1 = 0$, $\eta_2 = 2$, $\eta_3 = 0$, $\delta_{2k-1} = 1 + 10(0.9)^{2k-1}$, $\delta_{2k} = 1 + 10(2/3)^{2k}$,

$$P = R = S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\alpha_1 = \alpha_2 = 0.5$ then $\alpha_1\Phi_1 + \alpha_2\Phi_2$ is negative definite. Since $1 + a^k \leq \exp[a^k]$ for all $a \geq 0$ and $k \in \mathbb{N}$,

$$\prod_{k=1}^{\infty} \max\{1, \delta_k\} \leq \prod_{k=1}^{\infty}(1 + 10(0.9)^k) \leq \exp\left[\sum_{k=1}^{\infty} 10(0.9)^k\right] = \exp(90) = M.$$

Hence the trivial solution is globally exponentially stable by Theorem 5.2.6 under the switching rule following Algorithm 5.2.1. See Figure 5.13 for a simulation. The impulses are disturbances, however, they dissipate in time in total energy.

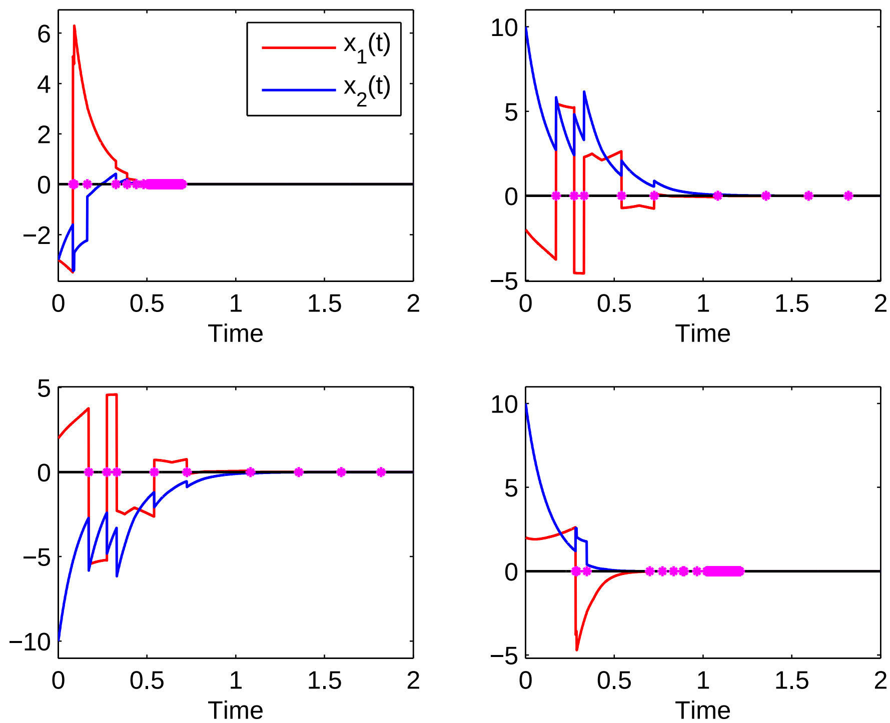
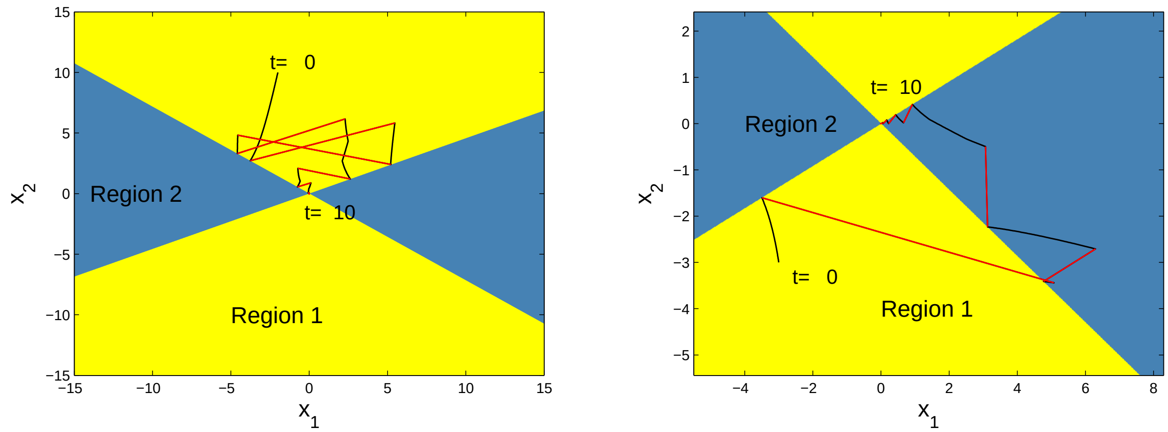


Figure 5.13: Simulation of Example 5.2.3 with $T_k = t_k$ and different initial functions. The magenta ticks on the t-axis mark the switching times (which coincide with the impulsive times).

For a state-space illustration, see Figure 5.14 (Figure 5.14a is the state-space representation of the top left image in Figure 5.13, while Figure 5.14b corresponds to the top right image in Figure 5.13). The total number of switches is significantly influenced by the initial state of the system $\phi_0(0)$.

Remark 5.2.12. *In the above examples, calculating η_i and the bounds ρ_k (or T) are straightforward, based on the system model. The next step is to choose candidates for the matrices S, R and P (in the case of overlapping regions, P is calculated via a Lyapunov equation after choosing Q). Once this is done, D_k and δ_k are straightforward to calculate.*



(a) 461 total switches required.

(b) 194 total switches required.

Figure 5.14: Simulation of Example 5.2.3 with $T_k = t_k$. Whenever a boundary is reached an impulse (red line) is applied

Then a candidate value for λ must be chosen, followed by a calculation of the matrices Φ_i (or Λ_i for the overlapping region case). In the case of the overlapping regions, the matrix P can be found from the Lyapunov equation, otherwise there does not seem to be a systematic method for choosing P, S, R, and λ. We began with $S = R = I$ and found candidate values of λ computationally by testing if the conditions on Φ_i (or Λ_i) were satisfied. The constants α_i can then be found by testing to see if $\sum_{i=1}^{m} \alpha_i \Phi_i$ is negative definite (or $\sum_{i=1}^{m} \alpha_i A_i$ is Hurwitz for the overlapping region theorems). The computational cost will depend on the number of subsystems and the dimension n of each subsystem. In the case of overlapping switching regions, a constant ξ must be chosen: a larger value means a larger region of overlap between the boundaries and hence potentially fewer switches (which may be advantageous from a practical perspective). The trade-off is that the rate of stabilization could be slower for large values of ξ.

5.3 State-dependent Switching Rules for Nonlinear HISD

Detailed in Section 5.1.2, the report by Liu et al. [105] investigated the state-dependent switching rule problem for nonlinear systems but did not consider distributed delays or impulses. The main objective of this section is to extend the state-dependent switching stabilization results to nonlinear systems with distributed delays and impulses.

5.3.1 Towards a Generalized Switching Rule Algorithm

Consider the following family of impulsive nonlinear systems with finite time-delay:

$$\begin{cases} \dot{x} = f_i(t,x) + h_i(t,x_t), & t \neq T_k, \\ \Delta x = g_k(t,x_{t^-}), & t = T_k, \end{cases} \tag{5.39}$$

where $k \in \mathbb{N}$, $i \in \mathcal{P}$, and the functional $x_{t^-} \in PC([-\tau,0],\mathbb{R}^n)$ is defined by

$$x_{t^-}(s) := \begin{cases} x(t+s), & \text{for } -\tau \leq s < 0, \\ x(t^-), & \text{for } s = 0. \end{cases}$$

Assume the functions $f_i(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy $f_i(t,0) \equiv 0$ and the functionals $h_i : \mathbb{R}_+ \times PC([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$ satisfy $h_i(t,0) \equiv 0$ for all $t \geq t_0$. Assume that f_i and h_i are sufficiently smooth so that a unique solution exists to each subsystem. Impulses are applied at the times $t = T_k$ which satisfy $t_0 < T_1 < T_2 < \ldots < T_k < \ldots$ with $T_k \to \infty$ as $k \to \infty$. The impulsive functionals g_k map $\mathbb{R} \times PC([-\tau,0],\mathbb{R}^n)$ to $\mathbb{R}^n$. That is, the impulsive effects can depend on the history of the solution trajectory.

Parameterized by a switching rule σ and an initial function, system (5.39) can be rewritten as

$$\begin{cases} \dot{x} = f_\sigma(t,x) + h_\sigma(t,x_t), & t \neq T_k, \\ \Delta x = g_k(t,x_{t^-}), & t = T_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases} \tag{5.40}$$

where $t_0 \in \mathbb{R}_+$ is the initial time and $\phi_0 \in PC([-\tau,0],\mathbb{R}^n)$ is the initial function. As in Section 5.2, system (5.40) can be derived from a control system perspective where impulsive and switching control are used on a nonlinear system with distributed delays. Again, since the objective is to determine switching rules which stabilize (5.40), assume that each subsystem $i = 1,\ldots,m$ is unstable.

Wang and Liu [183] considered impulsive effects at the switching times with a focus on a linear system with nonlinear perturbations and time-varying discrete delays. The authors used a Razumikhin-type approach to prove stability under state-dependent switching. Using a Lyapunov Razumikhin-type approach we extend the current literature to nonlinear HISD with distributed delays. In the first part, we turn our attention to stabilizing impulses. The switching portions of the system are destabilizing, even under the special switching rule, however, the impulses are strong enough to achieve stabilization. The key assumption which forms the basis for this stabilization approach is based on the behaviour of a convex combination of the unstable modes.

Assumption 5.3.1. *Suppose there exist $V \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, constants $p > 0$, $\lambda > 0$, $c_1 > 0$, $c_2 > 0$, and constants $d_k \geq 0$, $\delta_k \geq 0$, such that for $k \in \mathbb{N}$,*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$ *for all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$;

(ii) for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot \left(\sum_{i=1}^{m} \alpha_i f_i(t,x) \right) \leq \lambda V(t,x); \tag{5.41}$$

(iii) for each T_k *and for all* $\psi \in PC([-\tau,0],\mathbb{R}^n)$ *and* $s \in [-\tau,0]$,

$$V(T_k, \psi(0) + g_k(T_k, \psi(s))) \leq \delta_k V(T_k^-, \psi(0)) + d_k V(T_k^- + s, \psi(s)).$$

Equation (5.41) implies that λ is an estimate of the growth rate of the solution state of a convex combination system. Based on Assumption 5.3.1, we construct the switching regions as

$$\widehat{\Omega}_i = \left\{ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot f_i(t,x) \leq \xi \lambda V(t,x) \right\},$$

for some constant $\xi > 1$. Then we can show these regions cover the state-space by contradiction (similar method as in [105, 183]).

Proposition 5.3.1. *The switching regions $\widehat{\Omega}_i$ fully cover the state-space, that is, $\cup_{i=1}^m \widehat{\Omega}_i = \mathbb{R}_+ \times \mathbb{R}^n$.*

Proof. Suppose that $\cup_{i=1}^m \widehat{\Omega}_i = \mathbb{R}_+ \times \mathbb{R}^n$ is not true, then there exists a $(t^*, x^*) \in \mathbb{R}_+ \times \mathbb{R}^n$ such that

$$\frac{\partial V}{\partial t}(t^*, x^*) + \nabla V(t^*, x^*) \cdot f_i(t^*, x^*) > \xi \lambda V(t^*, x^*)$$

for all $i \in \mathcal{P}$. Since $\sum_{i=1}^{m} \alpha_i = 1$,

$$\begin{aligned}
&\sum_{i=1}^{m} \alpha_i \left(\frac{\partial V}{\partial t}(t^*, x^*) + \nabla V(t^*, x^*) \cdot f_i(t^*, x^*) \right) \\
&> \sum_{i=1}^{m} \alpha_i \xi \lambda V(t^*, x^*), \\
&= \xi \lambda V(t^*, x^*).
\end{aligned}$$

From equation (5.41), it is also true that

$$\begin{aligned}
&\sum_{i=1}^{m} \alpha_i \left(\frac{\partial V}{\partial t}(t^*, x^*) + \nabla V(t^*, x^*) \cdot f_i(t^*, x^*) \right) \\
&= \frac{\partial V}{\partial t}(t^*, x^*) \sum_{i=1}^{m} \alpha_i + \nabla V(t^*, x^*) \cdot \sum_{i=1}^{m} \alpha_i f_i(t^*, x^*), \\
&= \frac{\partial V}{\partial t}(t^*, x^*) + \nabla V(t^*, x^*) \cdot \sum_{i=1}^{m} \alpha_i f_i(t^*, x^*), \\
&\leq \lambda V(t^*, x^*).
\end{aligned}$$

This is a contradiction since $\xi > 1$. □

Then a state-dependent switching algorithm can be formulated according to the special minimum rule.

Algorithm 5.3.1. (Minimum rule)
Given $\xi > 1$ and the initial data t_0, $x_0 = \phi_0(0)$:

(MR1) *Choose the active mode according to*

$$\sigma(t_0, x_0) = \operatorname*{argmin}_{i \in \mathcal{P}} \frac{\partial V}{\partial t}(t_0, x_0) + \nabla V(t_0, x_0) \cdot f_i(t_0, x_0).$$

(MR2) *Remain in the active mode as long as the solution time-state trajectory $(t, x(t))$ is in the switching region*

$$\widehat{\Omega}_i = \left\{ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : \frac{\partial V}{\partial t}(t, x) + \nabla V(t, x) \cdot f_i(t, x) \leq \xi \lambda V(t, x) \right\}.$$

(MR3) *If the* $(t, x(t))$*-trajectory crosses the boundary of* $\widehat{\Omega}_i$*, denoted by* $\partial\widehat{\Omega}_i$*, at the time* t_c*, set* $t_0 = t_c$*,* $x_0 = x(t_c)$ *and go to step (MR1).*

A lemma is needed before the main theorem is presented.

Lemma 5.3.2. [184]
Assume that there exist a function $V \in \nu_0$*, constants* $\mu > \tau$*,* $p > 0$*,* $c_1 > 0$*,* $c_2 > 0$*,* $\lambda^* \geq \lambda > 0$*,* $q \geq e^{\lambda^*(2\mu+\tau)}$*, and constants* $d_k \geq 0$*,* $\delta_k \geq 0$*, such that for* $k \in \mathbb{N}$*,*

(i) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$ *for all* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$*;*

(ii) along solutions of (5.40) *for* $t \neq T_k$*,*

$$D^+V(t, \psi(0)) \leq \lambda V(t, \psi(0))$$

whenever $V(t + s, \psi(s)) \leq qV(t, \psi(0))$ *for* $s \in [-\tau, 0]$*;*

(iii) for each T_k *and for all* $\psi \in PC([-\tau, 0], \mathbb{R}^n)$ *and* $s \in [-\tau, 0]$*,*

$$V(T_k, \psi(0) + g_k(T_k, \psi(s))) \leq (1 + \delta_k)V(T_k^-, \psi(0)) + d_kV(T_k^- + s, \psi(s));$$

(iv) $0 < T_k - T_{k-1} < \mu$ *and*

$$\ln(\delta_k + d_ke^{\lambda^*\tau}) + \lambda^*(T_k - T_{k-1}) < -\mu\lambda^*.$$

Then for a solution $x(t)$ *of* (5.40) *under the switching rule* σ*,* $v(t) = V(t, x(t))$ *satisfies*

$$v(t) \leq M\|\phi_0\|_\tau^p e^{-\lambda^*(t-t_0)}$$

for any $t \in [T_{k-1}, T_k)$ *where* $M \geq 1$ *satisfies*

$$c_2 < Me^{-\lambda^*(T_1-t_0)}e^{-\mu\lambda} < Me^{-\lambda^*(T_1-t_0)} \leq qc_2.$$

Proof. Follows immediately from the proof of Theorem 3.3 in [184] since the upper right-hand derivative of the Lyapunov function V along the solution to the switched system satisfies $D^+V(t, \psi(0)) \leq \lambda V(t, \psi(0))$ for the specific switching rule σ. □

We extend the closed-loop switching stabilization results for nonlinear HISD with stabilizing impulses.

Theorem 5.3.3. *Suppose that Assumption 5.3.1 holds and suppose that there exist constants $c > 0$, $\mu > \tau$, $q \geq e^{(\xi\lambda+c)(2\mu+\tau)}$, such that for $\psi \in PC([-\tau, 0], \mathbb{R}^n)$, $t \geq t_0$, $i \in \mathcal{P}$,*

$$\nabla V(t, \psi(0)) \cdot h_i(t, \psi) \leq cV(t, \psi(0)) \tag{5.42}$$

whenever $V(t+s, \psi(s)) \leq qV(t, \psi(0))$ for $s \in [-\tau, 0]$. Assume that for $k \in \mathbb{N}$,

$$\ln(\delta_k + d_k e^{(\xi\lambda+c)\tau}) + (\xi\lambda + c)(T_k - T_{k-1}) < -\mu(\xi\lambda + c). \tag{5.43}$$

If σ is constructed according to Algorithm 5.3.1 then the trivial solution of (5.40) is globally exponentially stable.

Proof. Since $\bigcup_{i=1}^m \widehat{\Omega}_i = \mathbb{R}_+ \times \mathbb{R}^n$, the minimum rule algorithm $(MR1)-(MR3)$ implies that $\sigma = i_k$ for $t \in [t_{k-1}, t_k)$ such that the solution trajectory $(t, x(t)) \in \widehat{\Omega}_{i_k}$. By construction of $\widehat{\Omega}_i$ and using equation (5.42), the time-derivative of V along solutions of (5.40) is given by

$$\begin{aligned}\dot{V} =& \frac{\partial V}{\partial t}(t, x(t)) + \nabla V(t, x(t)) \cdot f_{i_k}(t, x(t)) + \nabla V(t, x(t)) \cdot h_{i_k}(t, x_t), \\ \leq& (\xi\lambda + c)V(t, x(t)),\end{aligned}$$

whenever $V(t+s, x(t+s)) \leq qV(t, x(t))$ for all $s \in [-\tau, 0]$. Lemma 5.3.2 then implies that

$$v(t) \leq M\|\phi_0\|_\tau^p e^{-(\xi\lambda+c)(t-t_0)}$$

for $t \neq T_k$. The result follows from condition (i) in Assumption 5.3.1. □

Next we consider the case where the impulses are perturbations and we consider how to extend the minimum rule algorithm to a generalized algorithm to avoid unwanted state-dependent switching behaviour.

Assumption 5.3.2. *Suppose that there exist $V \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, constants $p > 0$, $\lambda > 0$, $c_1 > 0$, $c_2 > 0$, $\alpha_i > 0$ satisfying $\sum_{i=1}^m \alpha_i = 1$, and constants $d_k \geq 0$, $\delta_k \geq 0$, such that for $k \in \mathbb{N}$,*

(i) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$ *for all* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$*;*

(ii) for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\frac{\partial V}{\partial t}(t, x) + \nabla V(t, x) \cdot \left(\sum_{i=1}^m \alpha_i f_i(t, x) \right) \leq -\lambda V(t, x);$$

(iii) for each T_k and for all $\psi \in PC([-\tau, 0], \mathbb{R}^n)$ and $s \in [-\tau, 0]$,

$$V(T_k, \psi(0) + g_k(T_k, \psi(s))) \leq (1 + \delta_k)V(T_k^-, \psi(0)) + d_k V(T_k^- + s, \psi(s)).$$

Since stabilization is achieved during the switching portions, the minimum rule strategy and switching regions are slightly adjusted.

Definition 5.3.1. (Minimum rule)
Given $\xi > 1$ and the initial data t_0, $x_0 = \phi_0(0)$:

(MR1) *Choose the active mode according to*

$$\sigma(t_0, x_0) = \operatorname*{argmin}_{i \in \mathcal{P}} \frac{\partial V}{\partial t}(t_0, x_0) + \nabla V(t_0, x_0) \cdot f_i(t_0, x_0).$$

(MR2) *Remain in the active mode as long as the solution time-state trajectory $(t, x(t))$ is in the switching region*

$$\Omega_i = \left\{ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : \frac{\partial V}{\partial t}(t, x) + \nabla V(t, x) \cdot f_i(t, x) \leq -\frac{\lambda V(t, x)}{\xi} \right\}.$$

(MR3) *If the $(t, x(t))$-trajectory crosses the boundary of Ω_i, denoted by $\partial\Omega_i$, at the time t_c, set $t_0 = t_c$, $x_0 = x(t_c)$ and go to step (MR1).*

As described in the minimum rule, the switching regions Ω_i are well-defined.

Proposition 5.3.4. [105] *Suppose that Assumption 5.3.2 holds. Then the switching regions Ω_i fully cover the time-state space, that is, $\bigcup_{i=1}^m \Omega_i = \mathbb{R}_+ \times \mathbb{R}^n$.*

As noted earlier, the overlapping of the switching regions arises from the constant $\xi > 1$. However, it does not exclude the possibility of Zeno behaviour where there is a finite accumulation point t^* for the switching times. In this scenario the switched system undergoes an infinite number of switchings in a finite interval of time. See [101, 105] for more details. In order to eliminate this possibility, the authors Liu et al. [105] considered a wandering rule (in conjunction with the minimum rule) where the system "wanders" in one particular mode for some positive amount of time (called the wandering time).

Definition 5.3.2. (Wandering rule)
Given the data (t_0, t_m, ω, D) where t_0 is the generalized rule initialization time; t_m is the wandering rule initialization time satisfying $t_m > t_0 \geq 0$; $\omega \geq 0$; and $D \subset \mathbb{R}^n$: maintain $\sigma(t_m)$ until $t - t_0 \geq \omega$ or $(t, x(t; t_0, \phi_0)) \in D$, whichever occurs first, and set $t_w = t$ (the terminal wandering time).

A general algorithm for the stabilization of (5.40) can then be formulated, by combining the minimum rule and the wandering rule as follows (motivated by the algorithm in [105]).

Algorithm 5.3.2. (Generalized rule)
Given $a > 1$, $\omega \geq 0$, $\xi > 1$, initial data (t_0, ϕ_0), proceed as follows:

(GR1) *Choose $k_0 \in \mathbb{Z}$ without loss of generality so that $(t_0, \phi_0(0)) \in \mathcal{D}_{k_0}(a)$ where*

$$\mathcal{D}_{k_0}(a) := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : a^{k_0} < V(t, x) \leq a^{k_0+1}\}.$$

Initiate and maintain $(MR1) - (MR3)$ with initial data (t_0, ϕ_0) and $\xi > 1$ until $(t, x(t)) \in cl(\mathcal{D}_{k_0-2})$. Let $t_m = t$ be the terminal minimum rule time and proceed to (GR2).

(GR2) *Calculate $t_m - t_0$:*

- *(i) If $t_m - t_0 \geq \omega$, let $t_0 = t_m$, $\phi_0 = x_{t_m}$ and go to (GR1).*
- *(ii) If $t_m - t_0 < \omega$, initiate the wandering rule with $(t_0, t_m, \omega, cl(\mathcal{D}_{k_0}))$ until the terminal wandering time, denoted t_w. Proceed to (GR3).*

(GR3) *Calculate $t_w - t_0$:*

- *(i) If $t_w - t_0 \geq \omega$, set $t_0 = t_w$, $\phi_0 = x_{t_w}$ and go to (GR1).*
- *(ii) If $(t_w, x(t_w)) \in cl(\mathcal{D}_{k_0})$, initiate the minimum rule with data (t_w, x_{t_w}) and $\xi > 1$ until $(t, x(t)) \in cl(\mathcal{D}_{k_0-2})$. Set $t_m = t$ and proceed to (GR2).*

Remark 5.3.1. *Following the notions in [105], each time the algorithm outlined above returns to (GR1) we say a cycle of the generalized switching rule algorithm is completed. Denote these times by the sequence $\{z_k\}_{k=1}^{\infty}$. Each full cycle lasts at least ω units of time, that is, $z_k - z_{k-1} \geq \omega$. If $\omega = 0$ then the generalized algorithm reduces to the familiar minimum rule algorithm.*

To guarantee the well-posedness of the generalized switching rule, a technical proposition is needed which details the workings of the minimum rule.

Proposition 5.3.5. [105]
Under Assumption 5.3.2, the sets

$$\partial\Omega_i \bigcap \Gamma_j \bigcap \partial\Omega_j \bigcap U(a_1, a_2, [t^1, t^2])$$

are empty for all $i, j \in \mathcal{P}$, $0 < a_1 \le a_2 < \infty$, $0 \le t^1 < t^2 < \infty$, *where*

$$U(a_1, a_2, [t^1, t^2]) = \{(t, x) \in [t^1, t^2] \times \mathbb{R}^n : a_1 \le V(t, x) \le a_2\},$$
$$\Gamma_j = \left\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : \operatorname*{argmin}_{i\in\mathcal{P}} \frac{\partial V}{\partial t}(t, x) + \nabla V(t, x) \cdot f_i(t, x) = j\right\},$$
$$\Omega_i = \left\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : \frac{\partial V}{\partial t}(t, x) + \nabla V(t, x) \cdot f_i(t, x) \le -\frac{\lambda V(t, x)}{\xi}\right\}.$$

Remark 5.3.2. *Note that since* $V \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ *and* $c_1\|x\|^p \le V(t, x) \le c_2\|x\|^p$ *for all* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$*, the set* U *is bounded, which is needed in the proof of the proposition.*

The following lemma follows directly from the proof of Proposition 3.3 in [105] (which studies the non-delay non-impulsive case) and is given here to show its workings.

Lemma 5.3.6. [105]
If Assumption 5.3.2 holds, the switching rule σ *is constructed according to Algorithm 5.3.2 with* $\omega > 0$*, and* $t^2 - t^1 < \inf(T_k - T_{k-1})$*, then there exists a positive constant* $\eta = \eta(U)$ *such that any two switching times of* σ *on* $[t^1, t^2]$ *are separated by at least* η *units of time.*

Proof. Assume that t^* is a switching time, that is, $(t^*, x(t^*)) \in \partial\Omega_i$ and choose a_1, a_2, t^1, t^2 such that $(t^*, x(t^*)) \in U(a_1, a_2, [t^1, t^2])$. Note that $(t^*, x^*) \in \Gamma_j$ where

$$j = \operatorname*{argmin}_{i\in\mathcal{P}} \frac{\partial V}{\partial t}(t^*, x^*) + \nabla V(t^*, x^*) \cdot f_i(t^*, x^*).$$

This implies the next mode of the subsystem should be j according to the minimum rule. From Proposition 5.3.5, the sets

$$\partial\Omega_i \bigcap \Gamma_j \bigcap \partial\Omega_j \bigcap U(a_1, a_2, [t^1, t^2]),$$

are empty so it follows that $(t^*, x^*) \in \partial\Omega_j$ which implies the existence of a constant $\eta = \eta(t^*) > 0$ such that the time-state trajectory $(t, x(t))$ does not leave Ω_j until an amount of time $t^* + \eta$ has passed, that is, $(t^* + \eta, x(t^* + \eta)) \in \partial\Omega_j$. Further, $\partial\Omega_i \bigcap \Gamma_j$ and $\partial\Omega_j$ are disjoint within U and are both closed sets. Also, f_i is bounded on the compact set. Finally, each $\partial\Omega_i = \bigcup_{i\in\mathcal{P}}(\partial\Omega_i \bigcap \Gamma_j)$ which implies that if $(t, x(t)) \in U$ and $t^2 - t^1 < \inf(T_k - T_{k-1})$ then there exists $\eta > 0$ such that any two successive switching times satisfy $t_k - t_{k-1} > \eta$. □

The following proposition can be given which establishes the well-posedness of the switching rule under the generalized switching rule in Algorithm 5.3.2. Here we adjust the notion of chattering behaviour slightly to deal with impulsive effects that are arbitrarily close to a switching time which could send the trajectory to a switching region boundary. Namely, since the impulses are separated by a positive amount of time, any three consecutive switching times must be separated by a positive amount of time.

Proposition 5.3.7. *Suppose that Assumption 5.3.2 holds. Then a switching rule σ constructed according to Algorithm 5.3.2 with $\omega > 0$ satisfies the following: there exists a positive constant $\eta = \eta(U)$ such that any two switching times of σ on $[t^1, t^2]$ are separated by at least η units of time if $t^2 - t^1 < \inf(T_k - T_{k-1})$. Any three switching times of σ on $[t^1, t^2]$ are separated by at least η units of time if $t^2 - t^1 \geq \inf(T_k - T_{k-1})$. That is, σ exhibits no chattering or Zeno behaviour and is well-defined.*

Proof. The case $t^2 - t^1 < \inf(T_k - T_{k-1})$ follows from Lemma 5.3.6. On the other hand, if $t^2 - t^1 \geq \inf(T_k - T_{k-1})$ then there is a possibility of a fast switch immediately after an impulse is applied. However, the next impulsive effect does not occur for $T_{k+1} - T_k$ time units, which means $t_{k+1} - t_k > \eta$ by the above arguments. Hence, three successive switching times must satisfy $t_{k+1} - t_{k-1} > \eta$. □

Remark 5.3.3. *The addition of the wandering time ω is the essential piece to avoid Zeno behaviour while the addition of the constant ξ in the switching regions Ω_i avoids chattering by introducing some overlapping area in the switching regions Ω_i while preserving the stabilizing properties of the minimum rule.*

Before giving the first main result, a lemma is needed which details the stability of an impulsive system with delays.

Lemma 5.3.8. [184]
Assume that there exist a function $V \in \nu_0$ and constants $p > 0$, $\lambda > 0$, $c_1 > 0$, $c_2 > 0$, $q \geq e^{\lambda\tau}$ and $d_k \geq 0$, $\delta_k \geq 0$ such that for $k \in \mathbb{N}$,

(i) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$ *for all* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$*;*

(ii) along solutions of (5.40) *for* $t \neq T_k$*,*

$$D^+V(t, \psi(0)) \leq -\lambda V(t, \psi(0))$$

whenever $V(t + s, \psi(s)) \leq qV(t, \psi(0))$ *for* $s \in [-\tau, 0]$*;*

(iii) for all $\psi \in PC([-\tau, 0], \mathbb{R}^n)$ *and* $s \in [-\tau, 0]$,

$$V(T_k, \psi(0) + g_k(T_k, \psi(s))) \leq (1 + \delta_k)V(T_k^-, \psi(0)) + d_k V(T_k^- + s, \psi(s)).$$

Then for a solution $x(t)$ *of* (5.40) *under the switching rule* σ, $v(t) = V(t, x(t))$ *satisfies*

$$v(t) \leq c_2 \|\phi_0\|_\tau^p \prod_{i=1}^{k-1} (1 + d_i + \delta_i e^{\lambda\tau}) e^{-\lambda(t-t_0)}$$

for any $t \in [T_{k-1}, T_k)$.

Proof. Follows immediately from the proof of Theorem 3.1 in [184] by taking the upper right-hand derivative of the Lyapunov function along the solution of the switched system under the switching rule σ. In the paper [184], the impulses are given by $g_k(x_{t^-})$ instead of $g_k(t, x_{t^-})$, however, the proof still holds. □

We are now in a position to extend state-dependent switching stabilization to nonlinear HISD under a generalized algorithm.

Theorem 5.3.9. *Suppose that Assumption 5.3.2 holds and suppose that there exist constants* $c > 0$, $\xi > 1$, $q \geq e^{(\frac{\lambda}{\xi} - c)\tau}$ *such that for* $\psi \in PC([-\tau, 0], \mathbb{R}^n)$, $t \geq t_0$, $i \in \mathcal{P}$,

$$\nabla V(t, \psi(0)) \cdot h_i(t, \psi) \leq cV(t, \psi(0)) \tag{5.44}$$

whenever $V(t + s, \psi(s)) \leq qV(t, \psi(0))$ *for* $s \in [-\tau, 0]$. *Assume that for* $k \in \mathbb{N}$

$$\ln(1 + \delta_k + d_k e^{\left(\frac{\lambda}{\xi} - c\right)\tau}) - \left(\frac{\lambda}{\xi} - c\right)(T_k - T_{k-1}) < 0. \tag{5.45}$$

If σ *is constructed according to Algorithm 5.3.2 with wandering time* $\omega \geq 0$,

$$a = \sup_{k \in \mathbb{N}} 1 + \delta_k + d_k \exp\left[\left(\frac{\lambda}{\xi} - c\right)\tau\right],$$

and $\xi > 1$ *then the trivial solution of* (5.40) *is globally asymptotically stable. Additionally, if* $\omega > 0$ *the switching rule is well-defined, while if* $\omega = 0$ *there is a possibility of a finite accumulation point* t^* *of the switching times with* $x(t) = 0$ *for all* $t \geq t^*$.

Proof. Let $v(t) = V(t, x(t))$ be the Lyapunov function from Assumption 5.3.2 evaluated along the solution of system (5.40). Note that $c_1\|x\|^p \leq V(t,x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and $v(t)$ is bounded on each set $\mathcal{D}_{k_0}$ and hence $\|x(t)\|$ is bounded in between cycle times. To prove global attractivity it suffices to show the existence of a constant $0 < \eta < 1$ such that $v(z_k) \leq \eta v(z_{k-1})$ where $\{z_k\}_{k=0}^{\infty}$ are the cycle times of Algorithm 5.3.2 (where we set $z_0 = t_0$) and the cycle times are all reached.

In step (GR1) of Algorithm 5.3.2, k_0 is chosen so that $(t_0, \phi_0(0)) \in \mathcal{D}_{k_0}(a)$ where

$$a = \sup_{k\in\mathbb{N}} \left(1 + \delta_k + d_k \exp\left[\left(\frac{\lambda}{\xi} - c\right)\tau\right]\right).$$

That is, $v(z_0) \in \mathcal{D}_{k_0}(a)$. The next step of the algorithm is to initiate the minimum rule (MR1)-(MR3) with (t_0, ϕ_0) for some $\xi > 1$ chosen beforehand. Since $\bigcup_{i=1}^m \Omega_i = \mathbb{R}_+ \times \mathbb{R}^n$, the minimum rule implies that while active $\sigma = i_k$ for $t \in [t_{k-1}, t_k)$ such that the solution trajectory $(t, x(t)) \in \Omega_{i_k}$. By construction of Ω_i and (5.44), the time-derivative of V along solutions of (5.40) is given by

$$\begin{aligned}\dot{V} =& \frac{\partial V}{\partial t}(t,x(t)) + \nabla V(t,x(t)) \cdot f_{i_k}(t,x(t)) + \nabla V(t,x(t)) \cdot h_{i_k}(t,x_t),\\ \leq& -\frac{\lambda}{\xi}V(t,x(t)) + cV(t,x(t)),\end{aligned}$$

whenever $V(t+s, x(t+s)) \leq qV(t,x(t))$ for all $s \in [-\tau, 0]$. Therefore,

$$\dot{V} \leq \left(-\frac{\lambda}{\xi} + c\right)V.$$

At any impulsive time reached during the minimum rule (it may be possible that none are reached), according to Assumption 5.3.2,

$$V(T_k, \psi(0) + g_k(T_k, \psi(s))) \leq (1+\delta_k)V(T_k^-, \psi(0)) + d_k V(T_k^- + s, \psi(s))$$

for all $\psi \in PC([-\tau, 0], \mathbb{R}^n)$, $s \in [-\tau, 0]$, $k \in \mathbb{N}$. Following Lemma 5.3.8, it follows that while the minimum rule is active,

$$v(t) \leq c_2\|\phi_0\|_\tau^p \prod_{t_0<T_k\leq t} \left(1 + \delta_k + d_k e^{(\frac{\lambda}{\xi}-c)\tau}\right) \exp\left[-\left(\frac{\lambda}{\epsilon} - c\right)(t - z_0)\right].$$

Noting that $z_0 = t_0$ (where no impulse is applied), equation (5.45) implies that for any impulsive time reached during the minimum rule,

$$v(T_k) \leq c_2\|\phi_0\|_\tau^p \chi^k$$

where $0 < \chi < 1$ is given by

$$\chi = \left(1 + d_k + \delta_k e^{(\frac{\lambda}{\xi} - c)\tau}\right) \exp\left[-\left(\frac{\lambda}{\epsilon} - c\right)(T_k - T_{k-1})\right].$$

It is clear that $v(t)$ is decreasing at the impulsive moments, that is, the sequence $\{v(T_k)\}$ is strictly decreasing. Also, $v(t) \leq v(z_0)$ for $t \geq z_0$ while the minimum rule is active. Therefore, there exists a terminal time, denoted t_m, of the minimum rule which satisfies

$$t_m = \inf\{t \geq t_0 : (t, x(t)) \in cl(\mathcal{D}_{k_0-2})\}$$

where $t_m \in [T_{l-1}, T_l)$ for some positive integer l. It is possible for the terminal time to satisfy $t_m < T_1$ if the state has been stabilized sufficiently so that $v(t) \in cl(\mathcal{D}_{k_0-2})$ before a disturbance impulse has been applied. This marks the end of the minimum rule step in (GR1) during the first cycle of the algorithm.

According to (GR2) of the algorithm, if $t_m - z_0 \geq \omega$ then the minimum rule was initiated for a sufficient amount of time and the algorithm returns to (GR1). This marks a complete cycle of the generalized switching algorithm with $z_1 = t_m$ and $v(z_1) \leq \eta v(z_0)$ where $0 < \eta = 1/a < 1$ by the definition of $\mathcal{D}_{k_0-2}$. The minimum rule is then re-activated with the new starting data $t_0 = t_m$ and $\phi_0 = x_{t_m}$.

However, if $t_m - z_0 < \omega$ then the wandering rule is initiated with generalized rule initialization time t_0, wandering rule initialization time t_m, domain $cl(\mathcal{D}_{k_0})$, and the constant ω. Under this rule, $\sigma(t_m)$ is maintained until either $t - t_0 \geq \omega$ or $(t, x(t)) \in cl(\mathcal{D}_{k_0})$, whichever occurs first. The terminal wandering time, which is always reached, is denoted by t_w.

In the next step, the algorithm continues according to (GR3) and $t_w - t_0$ is calculated. If $t_w - t_0 \geq \omega$ then $(t, x(t)) \in \mathcal{D}_{k_0-j}$ for some $j \geq 1$. That is, the time-state trajectory did not reach $cl(\mathcal{D}_{k_0})$ but instead the wandering rule ran out of time. This means $z_1 = t_w$ (marking end of cycle) with $v(z_1) \leq \chi v(z_0)$ for some $0 < \chi < 1$ since $v(z_0) \in \mathcal{D}_{k_0}$. It may be possible the time-state solution trajectory reaches $\mathcal{D}_{k_0-j}$ for $j \geq 2$ if the wandering mode, $\sigma(t_m)$, happens to be a stabilizing mode for the system along the trajectory.

If, on the other hand, $(t, x(t)) \in cl(\mathcal{D}_{k_0})$ occurs first in step (GR2), then according to (GR3) the minimum rule is re-initiated at $(t_w, x(t_w)) \in cl(\mathcal{D}_{k_0})$. If $(t, x(t)) \in \mathcal{D}_{k_0-1}$ during the minimum rule and an impulse is applied at T_k, then $(T_k, x(T_k)) \in \mathcal{D}_{k_0+j}$ is not possible for any $j \geq 1$ by the choice of a. Hence the minimum rule continues until $(t, x(t)) \in cl(\mathcal{D}_{k_0-2})$, which is achieved by the above arguments. Then, without loss of generality, the above process is repeated and eventually $v(z_1) \leq \eta v(z_0)$ where $\eta = \min\{1/a, \chi\}$ satisfies $0 < \eta < 1$.

Under any possible scenario in Algorithm 5.3.2, it is true that $v(z_1) \leq \eta v(z_0)$ at the end of the first cycle. Repeating the above procedure proves similarly that $v(z_2) \leq \eta v(z_1)$ for some constant $0 < \eta < 1$ and, in general, $v(z_k) \leq \eta^k v(z_0)$. This proves global attractivity of the trivial solution for any $\omega \geq 0$ with the possibility of a finite accumulation point t^* in the switching rule algorithm if $\omega = 0$.

For any $\epsilon > 0$, choose k_0 so that $cl(\mathcal{D}_{k_0})$ is entirely contained in $\mathcal{B}_\epsilon(0)$. Choose

$$\zeta = \frac{1}{2}\left(\frac{a^{k_0+1}}{c_2}\right)^{\frac{1}{p}} > 0$$

then $\|\phi_0(0)\| < \zeta$ implies that $c_2\|\phi_0(0)\|^p < a^{k_0+1}$ so that $v(t_0) \leq c_2\|\phi_0(0)\|^p < a^{k_0+1}$. That is, $V_0 \in cl(\mathcal{D}_{k_0}) \subset \mathcal{B}_\epsilon(0)$. Since we showed $v(t) \leq a^{k_0+1}$ for all $t \geq t_0$, it follows that $\|x(t)\| < \epsilon$ for all $t \geq t_0$. Hence the trivial solution is also stable.

The well-posedness of σ follows immediately from Proposition 5.3.4 and Proposition 5.3.7. If $\omega = 0$ then Proposition 5.3.4 still applies and the proof of Proposition 5.3.7 implies no chattering behaviour, however, there is the possibility of a finite accumulation point. □

Remark 5.3.4. *Contrasting Theorem 5.3.9 with Theorem 5.3.3, it is apparent that when the impulses are perturbations, successive impulsive moments must not occur too quickly. This can be observed from the threshold condition (5.45):*

$$T_k - T_{k-1} > \frac{\ln(1 + \delta_k + d_k \exp[(\frac{\lambda}{\xi} - c)\tau])}{\left(\frac{\lambda}{\xi} - c\right)}.$$

The decay rate from the switching portions of the system is estimated by $\lambda/\xi - c$, and so if $T_k - T_{k-1}$ is too small (pulsing too often), the threshold condition is not achieved. On the other hand, in Theorem 5.3.3, when impulses are stabilizing, the time between impulses must not be too great. This is captured in the threshold condition (5.43):

$$0 < T_k - T_{k-1} < \min\left\{\mu, \frac{-\ln\left(\delta_k + d_k \exp[(\xi\lambda + c)\tau]\right)}{\xi\lambda + c} - \mu\right\}.$$

The time between impulses must be sufficiently small because the impulses are stabilizing and must be applied often.

5.3.2 Extending the Generalized Rule to Systems with Infinite Delay

In the final part of this chapter's analysis we consider how to extend the state-dependent switching stabilization strategy to systems with unbounded delay. Consider the following HISD with infinite delay:

$$\begin{cases} \dot{x} = f_\sigma(t,x) + h_\sigma(t,x_t), & t \neq T_k, \\ \Delta x = g_k(t, x(t^-)), & t = T_k, \\ x_{t_0} = \phi_0, & k \in \mathbb{N}, \end{cases} \tag{5.46}$$

where the functional $x_t \in PCB([\alpha,0],\mathbb{R}^n)$ is defined by $x_t(s) = x(t+s)$ for $s \in [\alpha, 0]$ where $-\infty \le \alpha < 0$ and $[\alpha, 0]$ is understood to be $(-\infty, 0]$ when the delay is infinite. That is, $h_i : \mathbb{R}_+ \times PCB([\alpha,0],\mathbb{R}^n) \to \mathbb{R}^n$ for $i \in \mathcal{P}$ and satisfy $h_i(t,0) \equiv 0$ for all $t \ge t_0$. We consider how stabilizing impulsive effects can be used in conjunction with state-dependent switching in order to stabilize (5.46) using a lemma from [94].

Lemma 5.3.10. [94]
Assume that there exist functions $V \in \nu_0$, and constants $c_1 > 0$, $c_2 > 0$, $\lambda > 0$, $\delta_k \ge 0$, $q > 1$, $\gamma > 0$, such that for $k \in \mathbb{N}$,

(i) $c_1\|x\|^p \le V(t,x) \le c_2\|x\|^p$ *for all* $t \in [t_0 + \alpha, \infty)$ *and* $x \in \mathbb{R}^n$;

(ii) along solutions of (5.46) *for* $t \neq T_k$,

$$D^+V(t,\psi(0)) \le \lambda V(t,\psi(0)),$$

whenever $e^{\gamma s}V(t+s,\psi(s)) \le qV(t,\psi(0))$ *for* $s \in [\alpha, 0]$;

(iii) for each T_k *and for all* $x \in \mathbb{R}^n$,

$$V(T_k, x + g_k(T_k,x)) \le \frac{1+\delta_k}{q} V(T_k^-, x)$$

with $\sum_{k=1}^\infty \delta_k < \infty$;

(iv) $\rho\lambda < \ln q$ *where* $\rho = \sup_{k\in\mathbb{N}}\{T_k - T_{k-1}\}$.

Then for a solution $x(t)$ of (5.46) *under the switching rule σ, $v(t) = V(t,x(t))$ satisfies*

$$v(t) \le qc_2M\|\phi_0\|^p_{PCB}e^{-\eta(t-t_0)}$$

for $t \ge t_0$ where $M = \sum_{k=1}^\infty \delta_k < \infty$ and $\eta = \min\{\gamma, \frac{1}{2}\left(\frac{\ln q}{\rho} - \lambda\right)\}$.

Proof. Follows from Theorem 3.1 in [94] by taking the derivative of the Lyapunov function along the switched system parameterized by the particular switching rule σ. □

Then a state-dependent switching stabilization result for system (5.46) can be given as follows.

Theorem 5.3.11. *Assume that there exist $V \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, constants $p > 0$, $\lambda > 0$, $\gamma > 0$, $\rho > 0$, $c_1 > 0$, $c_2 > 0$, $c > 0$, $\xi > 1$, $q > 1$, $\alpha_i > 0$ satisfying $\sum_{i=1}^m \alpha_i = 1$, and constants $\delta_k \geq 0$ such that for $k \in \mathbb{N}$,*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$ *for all* $(t,x) \in [t_0 + \alpha, \infty) \times \mathbb{R}^n$;

(ii) *for* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot \left(\sum_{i=1}^m \alpha_i f_i(t,x) \right) \leq \lambda V(t,x);$$

(iii) *for* $\psi \in PCB([\alpha,0], \mathbb{R}^n)$ *and* $i \in \mathcal{P}$,

$$\nabla V(t,\psi(0)) \cdot h_i(t,\psi) \leq cV(t,\psi(0))$$

whenever $e^{\gamma s}V(t+s,\psi(s)) \leq qV(t,\psi(0))$ *for* $s \in [\alpha, 0]$;

(iv) *for each* $t = T_k$ *and for all* $x \in \mathbb{R}^n$,

$$V(T_k, x + g_k(T_k,x)) \leq \frac{1}{q}(1+\delta_k)V(T_k^-, x)$$

where $\sum_{k=1}^\infty \delta_k < \infty$;

(v) $T_k - T_{k-1} \leq \rho$ *and* $\rho(\xi\lambda + c) < \ln q$.

If σ is constructed according to Algorithm 5.3.1 then the trivial solution of (5.46) is globally exponentially stable.

Proof. Since $\bigcup_{i=1}^m \widehat{\Omega}_i = \mathbb{R}_+ \times \mathbb{R}^n$, the minimum rule $(MR1)-(MR3)$ in Algorithm 5.3.1 with the switching regions given as $\widehat{\Omega}_i$ implies that along solutions of (5.46), for $t \in [t_{k-1}, t_k)$, $t \neq T_k$,

$$\begin{aligned}\dot{V} =& \frac{\partial V}{\partial t}(t,x(t)) + \nabla V(t,x(t)) \cdot f_{i_k}(t,x(t)) + \nabla V(t,x(t)) \cdot h_{i_k}(t,x_t),\\ \leq& (\xi\lambda + c)V(t,x),\end{aligned}$$

whenever $V(t+s, x(t+s)) \leq qV(t, x(t))$ for all $s \in [\alpha, 0]$. Lemma 5.3.10 then implies that $v(t) = V(t, x(t))$ satisfies

$$v(t) \leq qc_2 M \|\phi_0\|_{PCB}^p e^{-\eta(t-t_0)}$$

for all $t \geq t_0$ where

$$\eta = \min\left\{\gamma, \frac{1}{2}\left(\frac{\ln q}{\rho} - (\xi\lambda + c)\right)\right\}.$$

The result follows. □

To study disturbance impulses for nonlinear HISD with unbounded delay, the following lemma is given.

Lemma 5.3.12. [97]
Assume that there exist functions $V \in \nu_0$, constants $p > 0$, $c_1 > 0$, $c_2 > 0$, $\lambda > 0$, $q > 1$, $\gamma > 0$, and $\delta_k \geq 0$ such that

(i) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$ *for all* $t \in [t_0 + \alpha, \infty)$ *and* $x \in \mathbb{R}^n$*;*

(ii) along solutions of (5.46) *for* $t \neq T_k$*,*

$$D^+V(t, \psi(0)) \leq -\lambda V(t, \psi(0)),$$

whenever $e^{\gamma s}V(t+s, \psi(s)) \leq qV(t, \psi(0))$ *for* $s \in [\alpha, 0]$*;*

(iii) for each T_k *and for all* $x \in \mathbb{R}^n$*,*

$$V(T_k, x + g_k(T_k, x)) \leq (1 + \delta_k)V(T_k^-, x);$$

(iv) $\mu\lambda > \ln q$ *where* $\mu = \inf_{k\in\mathbb{N}}\{T_k - T_{k-1}\}$*.*

Then for a solution $x(t)$ *of* (5.46) *with switching rule* σ*,* $v(t) = V(t, x(t))$ *satisfies*

$$v(t) \leq qw_2\|\phi_0\|_{PCB}^p \prod_{t_0\leq T_k\leq t}(1 + \delta_k)e^{-\eta(t-t_0)}$$

for $t \geq t_0$ *where* $\eta = \min\{\gamma, \frac{1}{2}\left(\lambda - \frac{\ln q}{\mu}\right)\}$*.*

Proof. Follows immediately from Corollary 3.1 in [97] by taking the time-derivative of the Lyapunov function along the HISD with switching rule σ. □

The final state-dependent switching stabilization theorem is given as follows, which considers the generalized algorithm for systems with unbounded delay.

Theorem 5.3.13. *Assume that there exist* $V \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, *constants* $p > 0$, $\lambda > 0$, $\gamma > 0$, $\mu > 0$, $c_1 > 0$, $c_2 > 0$, $c > 0$, $\xi > 1$, $q > 1$, $\alpha_i > 0$ *satisfying* $\sum_{i=1}^{m} \alpha_i = 1$, *and constants* $\delta_k \geq 0$ *such that for* $k \in \mathbb{N}$,

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$ *for all* $(t,x) \in [t_0 + \alpha, \infty) \times \mathbb{R}^n$;

(ii) for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot \left(\sum_{i=1}^{m} \alpha_i f_i(t,x) \right) \leq -\lambda V(t,x);$$

(iii) for $\psi \in PCB([\alpha, 0], \mathbb{R}^n)$ *and* $i \in \mathcal{P}$,

$$\nabla V(t, \psi(0)) \cdot h_i(t, \psi) \leq cV(t, \psi(0))$$

whenever $e^{\gamma s} V(t+s, \psi(s)) \leq qV(t, \psi(0))$ *for* $s \in [\alpha, 0]$;

(iv) for each T_k *and for all* $x \in \mathbb{R}^n$,

$$V(T_k, x + g_k(T_k, x)) \leq (1 + \delta_k) V(T_k^-, x);$$

(v) $T_k - T_{k-1} \geq \mu$ *and* $\mu(\lambda/\xi - c) > \ln q$ *and*

$$\ln(1 + \delta_k) - \eta\mu < 0 \tag{5.47}$$

where $\eta = \min\{\gamma, \frac{1}{2}\left(\frac{\lambda}{\xi} - c - \frac{\ln q}{\mu}\right)\}$.

If σ *is constructed according to Algorithm 5.3.2 with wandering time* $\omega \geq 0$, $a = \sup_{k \in \mathbb{N}} 1 + \delta_k$, *and switching regions*

$$\Omega_i = \left\{ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot f_i(t,x) \leq -\frac{\lambda V(t,x)}{\xi} \right\}$$

then the trivial solution of (5.46) *is globally asymptotically stable. Additionally, if* $\omega > 0$ *the switching rule is well-defined, while if* $\omega = 0$ *there is a possibility of a finite accumulation point* t^* *of the switching times with* $x(t) = 0$ *for all* $t \geq t^*$.

Proof. By construction of the switching regions Ω_i, the minimum rule implies that, while active, the time-derivative of V along solutions of (5.46) satisfies

$$\begin{aligned}\dot{V} =& \frac{\partial V}{\partial t}(t, x(t)) + \nabla V(t, x(t)) \cdot f_{i_k}(t, x(t)) + \nabla V(t, x(t)) \cdot h_{i_k}(t, x_t), \\ \leq& \left(-\frac{\lambda}{\xi} + c\right) V(t, x(t)),\end{aligned}$$

whenever $e^{\gamma s}V(t+s, x(t+s)) \leq qV(t, x(t+s))$ for all $s \in [\alpha, 0]$. At any impulsive time reached during the minimum rule (it may be possible that none are reached),

$$V(T_k, x + g_k(T_k, x)) \leq (1 + \delta_k)V(T_k^-, x).$$

From Lemma 5.3.12, it follows that, while the minimum rule is active, $v(t) = V(t, x(t))$ satisfies

$$v(t) \leq qc_2\|\phi_0\|_{PCB}^p \prod_{t_0 < T_k \leq t} (1 + \delta_k) \exp\left[-\eta(t - z_0)\right]$$

for $t \geq t_0$. Thus, for any impulses applied during the minimum rule,

$$v(T_k) \leq c_2\|\phi_0\|_\tau^p \chi^k$$

where $0 < \chi < 1$ is given by

$$\chi = (1 + \delta_k) \exp\left[-\eta(T_k - T_{k-1})\right].$$

The rest of the proof follows as the proof of Theorem 5.3.9. □

Remark 5.3.5. *If $f_i(t, x)$ and $h_i(t, x_t)$ in (5.46) are composite-PCB and locally Lipschitz and g_k are continuous in both variables, then it follows from Theorem 3.4.3 (existence) and Theorem 3.5.1 (uniqueness) in Chapter 3 that (5.46) has a unique solution when σ is constructed according to Algorithm 5.3.2 with $\omega > 0$.*

5.3.3 Numerical Simulations

Example 5.3.1. *Consider the nonlinear HISD with unbounded delay (5.46) with $t_0 = 0$ and the following two subsystems, that is, $\mathcal{P} = \{1, 2\}$:*

$$i = 1: \begin{cases} \dot{x}_1 = 5x_1(t) + 2x_2^5(t) - x_2^2(t)e^{\sin(x_1(t))} + \dfrac{1}{2}\displaystyle\int_{-\infty}^{t} e^{5(s-t)}x_1(s)sin(x_2(s))ds, \\ \dot{x}_2 = -3x_2(t) - 2x_1(t)x_2^4(t), \end{cases}$$

and

$$i=2:\begin{cases}\dot{x}_1=-6x_1(t)-x_2^5(t),\\ \dot{x}_2=2x_2(t)+x_1(t)x_2^4(t)+x_1(t)x_2(t)e^{sin(x_1(t))}+\displaystyle\int_{-\infty}^{t}\frac{e^{5(s-t)}\sqrt{x_1^4(s)+x_2^4(s)}}{2\cosh^2(x_2(t))}ds.\end{cases}$$

Suppose that the impulsive times are $T_k=1.2k+0.1\sin(k)$, $k\in\mathbb{N}$, *with impulsive effects given by*

$$\begin{cases}\Delta x_1(T_k)=-x_1(T_k^-)+\left(1.05+\dfrac{0.03}{k}\right)^{\frac{1}{2}}x_2(T_k^-),\\ \Delta x_2(T_k)=-x_1(T_k^-)+\sqrt{0.1}e^{-k}\sqrt{|x_1(T_k^-)x_2(T_k^-)|}.\end{cases}\tag{5.48}$$

Note that $1\le T_k-T_{k-1}\le 1.4$. *Let*

$$f_1(x)=\begin{pmatrix}5x_1+2x_2^5-x_2^2e^{\sin(x_1)}\\ -3x_2-2x_1x_2^4\end{pmatrix},$$

$$f_2(x)=\begin{pmatrix}-6x_1-x_2^5\\ 2x_2+x_1x_2^4+x_1x_2e^{\sin(x_1)}\end{pmatrix},$$

$$h_1(t,x_t)=\begin{pmatrix}\frac{1}{2}\int_{-\infty}^{t}e^{5(s-t)}x_1(s)\sin(x_2(s))ds\\ 0\end{pmatrix},$$

and

$$h_2(t,x_t)=\begin{pmatrix}0\\ \int_{-\infty}^{t}e^{5(s-t)}\sqrt{x_1^4(s)+x_2^4(s)}ds/(2cosh^2(x_2(t)))\end{pmatrix}.$$

Choose $\alpha_1=\alpha_2=0.5$ *and the Lyapunov function* $V(x)=x_1^2+x_2^2$. *Then*

$$\nabla V(x)\cdot(\alpha_1f_1(x)+\alpha_2f_2(x))=-(x_1^2+x_2^2)=-V(x).$$

Hence we choose $\lambda=1$. *Whenever* $e^{\gamma s}V(t+s,\psi(s))\le qV(t,\psi(0))$ *for* $s\le 0$, $x_1^2(t+s)+x_2^2(t+s)\le qe^{-\gamma s}[x_1^2(t)+x_2^2(t)]$ *for* $s\le 0$. *Hence,*

$$\begin{aligned}\nabla V\cdot h_1=&x_1(t)\int_{-\infty}^{t}e^{a(s-t)}x_1(s)\sin(x_2(s))ds,\\ \le&|x_1(t)|\int_{-\infty}^{0}e^{5\theta}|x_1(t+\theta)|d\theta,\end{aligned}$$

so that

$$\begin{aligned}
\nabla V \cdot h_1 \leq & |x_1(t)| \int_{-\infty}^{0} \sqrt{q} e^{(5-\gamma/2)\theta} \sqrt{x_1^2(t) + x_2^2(t)} d\theta \\
= & |x_1(t)| \sqrt{x_1^2(t) + x_2^2(t)} \int_{-\infty}^{0} \sqrt{q} e^{(5-\gamma/2)\theta} d\theta \\
\leq & \frac{|x_1(t)|^2 + x_1^2(t) + x_2^2(t)}{2} \left(\frac{\sqrt{q}}{5-\gamma/2} \right) \\
= & \left[\frac{\sqrt{q}}{5-\gamma/2} \right] V(x(t)),
\end{aligned}$$

provided $\gamma < 10$. *Similarly, along* $\dot{x} = h_2(t, x_t)$ *for* $t \neq T_k$*, using* $x_1^4 + x_2^4 \leq (x_1^2 + x_2^2)^2$ *and the Razumikhin condition*

$$\begin{aligned}
\nabla V \cdot h_2 = & \frac{x_2(t)}{cosh^2(x_2(t))} \int_{-\infty}^{t} e^{5(s-t)} \sqrt{x_1^4(s) + x_2^4(s)} ds, \\
\leq & \int_{-\infty}^{t} e^{5(s-t)} [x_1^2(s) + x_2^2(s)] ds, \\
= & \int_{-\infty}^{0} e^{5\theta} [x_1^2(t+\theta) + x_2^2(t+\theta)] d\theta,
\end{aligned}$$

then

$$\begin{aligned}
\nabla V \cdot h_2 \leq & \int_{-\infty}^{0} q e^{(5-\gamma)\theta} [x_1^2(t) + x_2^2(t)] d\theta, \\
= & q(x_1^2(t) + x_2^2(t)) \int_{-\infty}^{0} e^{(5-\gamma)\theta} d\theta, \\
= & \left[\frac{q}{5-\gamma} \right] V(x(t)),
\end{aligned}$$

provided $\gamma < 5$. *At the impulsive times* $t = T_k$,

$$\begin{aligned}
V(x(T_k)) = & \left[\sqrt{1.05 + 0.03/k} \sin(x_2(T_k^-)) \right]^2 + \left[\sqrt{0.1} e^{-k} \sqrt{|x_1(T_k^-) x_2(T_k^-)|} \right]^2, \\
\leq & (1.05 + 0.03/k) x_2^2(T_k^-) + 0.1 e^{-2k} |x_1(T_k^-) x_2(T_k^-)|, \\
\leq & (1.05 + 0.03/k) V(x(T_k^-)) + \frac{0.1 e^{-2k}}{2} V(x(T_k^-)), \\
= & \left[1.05 + \frac{0.03}{k} + \frac{0.1 e^{-2k}}{2} \right] V(x(T_k^-)).
\end{aligned}$$

Take $c = 0.183$, $\mu = 1$, $q = 1.1$, $\gamma = 2$, $\xi = 2$, $\delta_k = 0.05 + 0.03/k + 0.1e^{-2k}$ *(*$\sup 1 + \delta_k = 1.1$*). That is, set* $a = 1.1$. *Then*

$$\frac{\lambda}{\xi} - c > \frac{\ln q}{\mu}$$

and

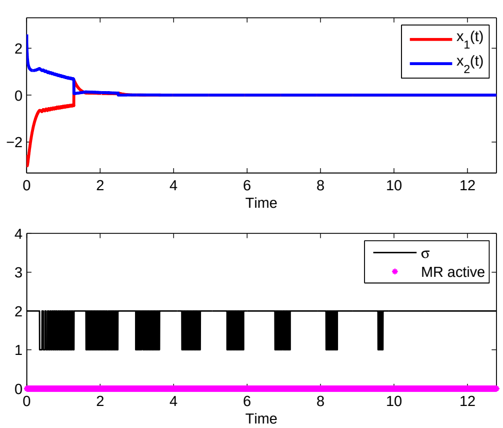
$$\ln(1 + \delta_k) - \eta\mu = -0.0154.$$

All the conditions of Theorem 5.3.13 are satisfied. The switches regions are given by

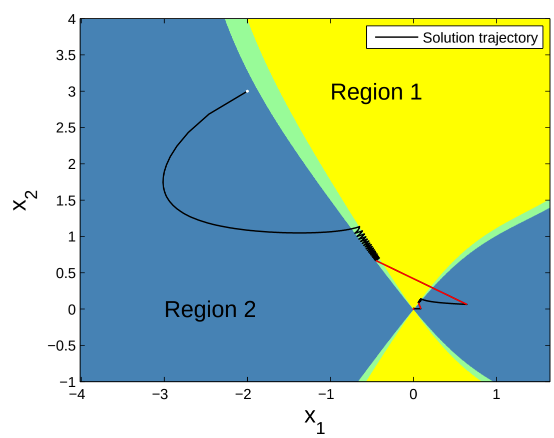
$$\Omega_1 = \{x \in \mathbb{R}^2 : 10x_1^2 - 6x_2^2 - 2x_1x_2^2e^{sinx_1} \le -(x_1^2 + x_2^2)/2\},$$

$$\Omega_2 = \{x \in \mathbb{R}^2 : -12x_1^2 + 4x_2^2 + 2x_1x_2^2e^{sinx_1} \le -(x_1^2 + x_2^2)/2\}.$$

as in Example 5.1.2. See Figure 5.15 for an illustration of the state trajectory under the generalized switching algorithm 5.3.2 with $\omega = 0$ *(no wandering time) and* $a = 1.1$. *Adding a wandering time of* $\omega = 0.5$ *reduces the number of switches needed drastically (see Figure 5.16) from* 208 *to* 12.



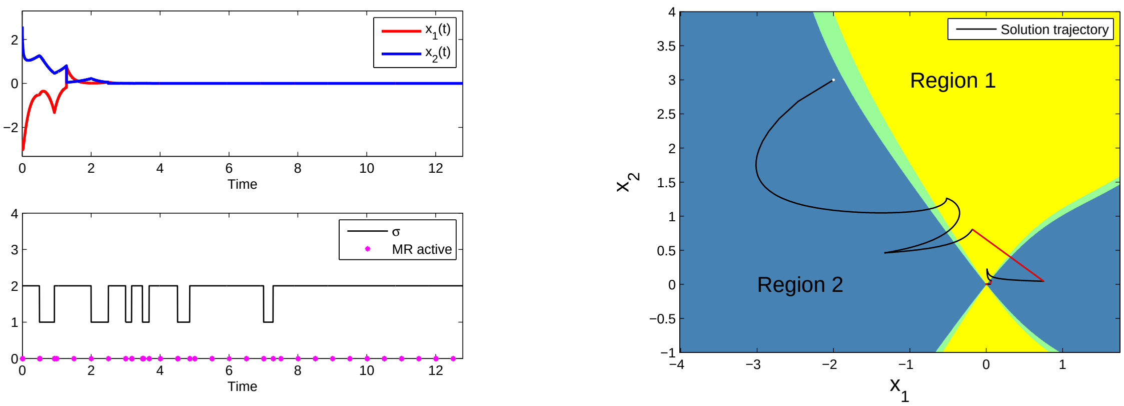
(a) The solution trajectories along with the switching rule σ. The minimum rule (MR) is active for the entire duration.



(b) Impulsive effects are marked in red.

Figure 5.15: Simulation of Example 5.3.1 with $\omega = 0$ (208 switches are required). The initial function is $\phi_0(s) = (-2, 3)$ for $s \le 0$.

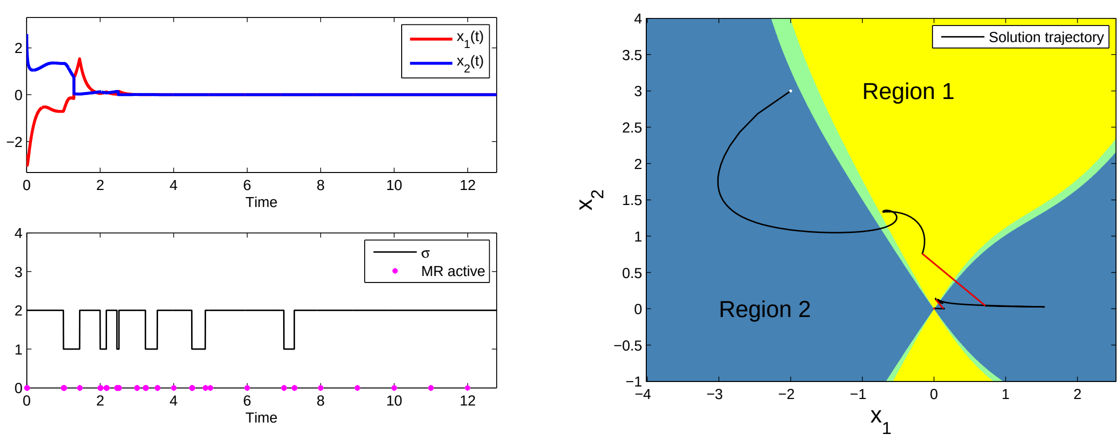
If the wandering time is increased to $\omega = 1$, *stabilization is still achieved (see Figure 5.17), however, the number of switches according to the algorithm remains unchanged from the case* $\omega = 0.5$ *(no reduction in switches).*



(a) The algorithm can be broken down into the minimum rule portion (the magenta ticks) and the wandering rule portion (no magenta ticks).

(b) Impulsive effects are marked in red.

Figure 5.16: Simulation of Example 5.3.1 with $\omega = 0.5$ (only 12 switches are needed).

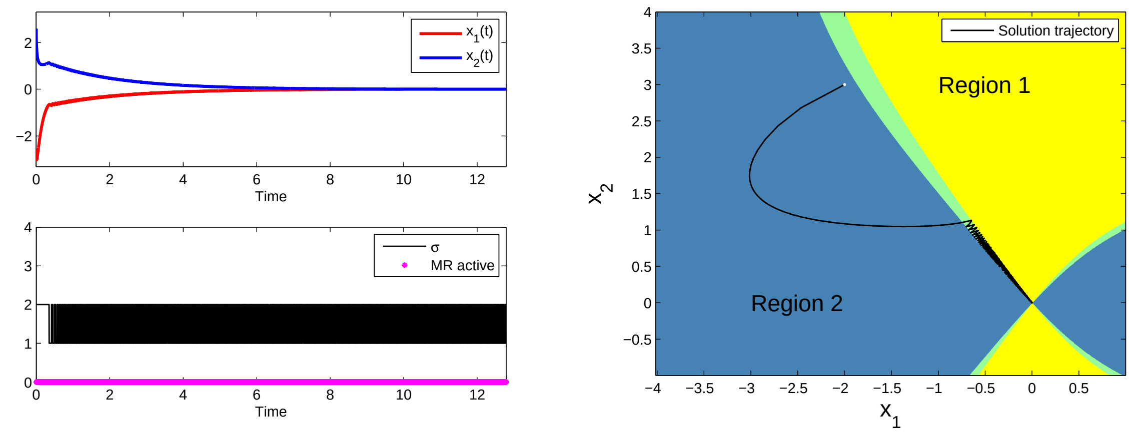


(a) The magenta ticks represent time spent in the minimum rule portion.

(b) Impulsive effects are marked in red.

Figure 5.17: Simulation of Example 5.3.1 with $\omega = 1$ (again 12 switches are needed).

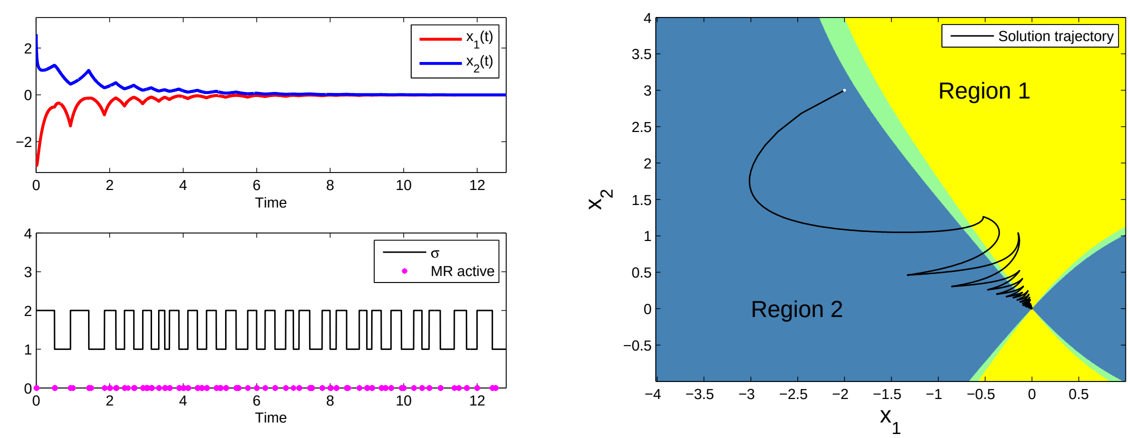
The wandering time becomes more crucial practically when there are no disturbance impulses present (see Figures 5.18 and 5.19). This is an interesting phenomenon: the perturbative impulses seem to help avoid chattering behaviour.



(a) The solution trajectories along with the switching rule σ.

(b) Impulsive effects are marked in red.

Figure 5.18: Simulation of Example 5.3.1 with no impulsive effects and $\omega = 0$ (599 switches are needed).



(a) The solution trajectories along with the switching rule σ.

(b) Impulsive effects are marked in red.

Figure 5.19: Simulation of Example 5.3.1 with no impulsive effects and $\omega = 0.5$ (only 45 switches are needed).

Chapter 6

Applications in Epidemiology

As mentioned in Chapter 1, the most famous example of a control scheme's successful application was the World Health Organization's initiative against smallpox, which began in 1967 when there were approximately 15 million cases per year and ended with worldwide eradication by 1977 [69]. The main objective of this chapter is to study mathematical models of acute communicable infectious diseases analytically and numerically to answer qualitative questions regarding their long-term behaviour. For example, determining whether or not there will be an outbreak, estimating its length and severity, analyzing how control schemes can be applied to eradicate the disease, etc. To do this, we formulate mathematical models and study their stability properties using the results found earlier in the present thesis.

6.1 An Introduction to Compartmental Epidemic Models: the SIR Model with Population Dynamics

To formulate the epidemic models in this chapter, we consider the continuous deterministic approach where the population is split into different groups where each group exhibits distinctive behaviour. For example, in the classic SIR model the population is broken into three compartments: those individuals who are susceptible to the disease, denoted by S_c; those individuals who have the disease and are infectious, denoted by I_c; and those who have recovered from the disease, denoted by R_c. The interaction between these groups determines how the disease spreads and is based on the population behaviour and the particular infectious disease being modelled. For example, if the disease of interest has a latency period of non-negligible duration, a group of individuals who have been exposed but are not yet infectious should be considered. For background on infectious disease mathematical modelling, see [6, 66, 69, 75, 143] and the references therein.

To give an introduction to this area of research, we formulate the classic SIR model with population dynamics by making the following assumptions on the epidemiological-demographic interactions:

(A1) The incubation period of the disease is negligible when compared to the other dynamics of the disease. When a susceptible individual is infected they immediately move to the infected class.

(A2) Individuals in the population mix homogeneously (i.e. any two individuals have an equal probability of coming into contact with one another). The average contact rate between individuals sufficient for disease transmission is given by $\beta > 0$.

(A3) The incidence rate of the disease, defined as the average number of new infections per unit time, is proportional to the number of infected and susceptible present, normalized by the total population (denoted by N). That is, the incidence rate takes on the standard (or proportionate mixing) form

$$\beta \frac{S_c I_c}{N}.$$

(A4) The rate of recovery from the infected class to the recovered class is proportional to the number of infected present, with proportionality constant $g > 0$. That is, the waiting time is exponentially distributed with an average infectious period of $1/g$.

(A5) The birth rate, $\mu > 0$, is equal to the natural death rate. All children are born healthy (no vertical transmission of the disease). The disease-induced mortality rate is negligible.

Under these assumptions, the model can be written as the following system of ordinary differential equations:

$$\begin{cases} \dot{S}_c = \mu N - \beta \dfrac{S_c I_c}{N} - \mu S_c, \\ \dot{I}_c = \beta \dfrac{S_c I_c}{N} - g I_c - \mu I_c, \\ \dot{R}_c = g I_c - \mu R_c, \end{cases}$$

where $S_c + I_c + R_c = N$. Note that $\dot{S}_c + \dot{I}_c + \dot{R}_c = 0$ so that the total population is constant. The variables can be normalized by letting $S = S_c/N$, $I = I_c/N$, $R = R_c/N$ to get the classic SIR model with population dynamics

$$\begin{cases} \dot{S} = \mu - \beta SI - \mu S, \\ \dot{I} = \beta SI - gI - \mu I, \\ \dot{R} = gI - \mu R, \end{cases} \tag{6.1}$$

with initial conditions $S(0) = S_0 > 0$, $I(0) = I_0 > 0$, $R(0) = R_0 \geq 0$. Each variable represents the fraction of the population that is a part of that group. The SIR model with population dynamics is a reasonable model for nonfatal diseases such as hepatitis B and measles [123] and has been studied extensively in the literature, for example see [67, 69, 75]. The initial conditions are assumed to satisfy $S_0 + I_0 + R_0 = 1$. Note that $\{\dot{S} + \dot{I} + \dot{R}\}_{S+I+R=1} = 0$, $\dot{S}\,|_{S=0} = \mu > 0$, $\dot{I}\,|_{I=0} = 0$, and $\dot{R}\,|_{R=0} = gI \geq 0$. The meaningful physical domain for this system is the plane $\Omega_{SIR} = \{(S, I, R) \in \mathbb{R}^3_+ |\ S + I + R = 1\}$, which is positively invariant to the system and the model is mathematically and epidemiologically well-posed [68]. The model is called an SIR model because individuals move between the compartments from S to I to R.

Remark 6.1.1. *The standard incidence rate $\beta S_c I_c/N$ is consistent with the known result that daily contact patterns are largely independent of community size but assumes homogeneous mixing and does not include a saturating effect which are poor assumptions [69, 80]. Since the incidence rate is critically dependent on the population behaviour and the disease being modelled, there are examples of many other incidence rates in the literature. For example, incidence rates with saturating effects, psychological effects, density dependency, and more (see [38, 51, 66, 80, 81, 89, 98, 108, 109, 138, 150]).*

Remark 6.1.2. *The usual simplifying assumption that the period of infection is a constant leads to an exponentially distributed infectious period [75]. For example, if infected individuals are removed linearly at the rate of recovery $g > 0$, then the fraction of infected who are still infected t time units after entering the class is given by $P(t) = e^{-gt}$ [69]. Then the rate of individuals that leave the infectious class is $-P'(t)$ and the average infectious period can be calculated as $\int_0^\infty t(-P'(t))dt = \int_0^\infty P(t)dt = 1/g$ [69]. See [69, 75] for more details, including other possible distributions.*

6.1.1 Threshold Criteria: The Basic Reproduction Number

The SIR model (6.1) has a disease-free equilibrium (where there are no infected present) given by $(S, I, R) = (1, 0, 0)$, and an endemic equilibrium (where the disease persists) given by

$$(S, I, R) = \left(\frac{1}{\mathcal{R}_0}, \frac{\mu}{\mu+g}\left(1-\frac{1}{\mathcal{R}_0}\right), \frac{g}{\mu+g}\left(1-\frac{1}{\mathcal{R}_0}\right)\right),$$

where

$$\mathcal{R}_0 = \frac{\beta}{\mu+g}. \tag{6.2}$$

The long-term behaviour of the model is completely determined by $\mathcal{R}_0$, which is the basic reproduction number of the disease, which is defined as the average number of secondary infections produced from a single infected individual introduced into a wholly susceptible population. Mathematically, it is the average contact rate, β, multiplied by the average death-adjusted infectious period $1/(g+\mu)$. If $\mathcal{R}_0 \leq 1$, then the disease-free equilibrium is globally asymptotically stable in the meaningful physical domain, while if $\mathcal{R}_0 > 1$ then the endemic equilibrium is globally asymptotically stable. For details, see, for example, [67,75]. See Figure 6.1 for phase plane portraits of the SIR model.

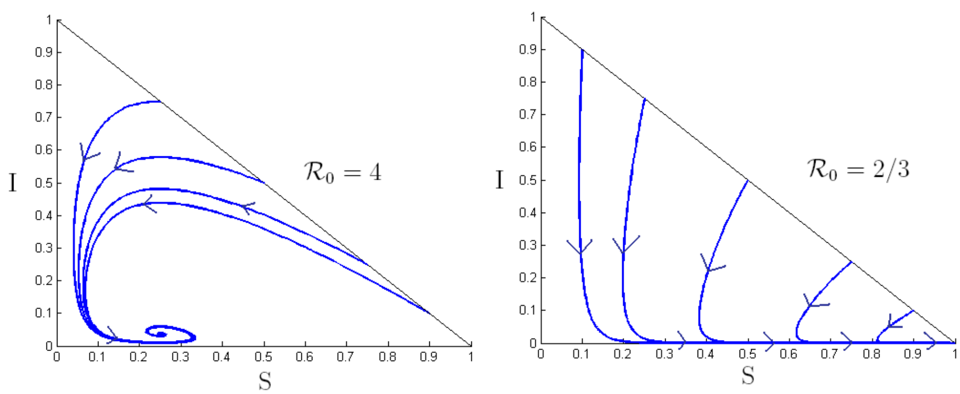


Figure 6.1: Phase plane portraits of the SIR model (6.1).

Threshold criteria involving the basic reproduction number are common in the epidemic modelling literature. It is typical for authors to prove that if a particular model's basic reproduction number is less than one, the disease will eventually be eradicated, while if it remains greater than one, the disease persists. See Table 6.1 for a list of the basic reproduction numbers of various infectious diseases from [5]. In [109], the authors give analytic expressions for the basic reproduction number for some of the classic mathematical epidemic models. For a general compartmental disease model, the basic reproduction number is the spectral radius of the so-called next generation matrix (for details, see [12,35,36,178,188]).

Disease	Infectious period (days)	Average age at infection (years)	$\mathcal{R}_0$
Measles	6 to 7	4.4 to 5.6	13.7 to 18.0
Whooping cough	21 to 23	4.1 to 5.9	14.3 to 17.1
Rubella	11 to 12	10.5	6.7
Chicken pox	10 to 11	6.7	9.0
Poliomyelitis	14 to 20	11.2	6.2

Table 6.1: Epidemiological data from [5].

6.1.2 Seasonal Changes in the Transmission of a Disease

Seasonal variations in the transmission of a disease play an important role how it spreads. Factors affected by seasonality include changes in host immunity, differences in host behaviour (e.g. school breaks for children), changes in the abundance of vectors (e.g. due to weather and temperature differences), changes in the survivability of pathogens, physiological changes in host susceptibility, and seasonal timing of reproduction [2,35,39,55,75]. Childhood infections peak at the start of the school year and decline in the summer months [75]. Other examples of diseases which display periodicity in their transmission include measles, chickenpox, mumps, rubella, poliomyelitis, diphtheria, pertussis, and influenza [70].

The contact rate, β, is traditionally assumed to be a constant in epidemic models [67,69,75,143], however, a more realistic formulation is to consider a time-varying contact rate. There are two main approaches studied in the literature and we detail them here. The first approach studied in the literature is to assume the contact rate is smoothly-varying:

$$\beta = \beta(t) = \beta_0(1 + \epsilon cos(2\pi t))$$

where $\beta_0 > 0$ is the base average contact rate and $\epsilon > 0$ captures seasonal variations in the contact rate. There has been an extensive amount of work done in this area of research. A pulse SIR model with sinusoidal forcing was considered by Shulgin et al. in [169]. Yang and Xiao studied an HIV model with periodicity in [198]. In [12], Bacaër and Souad analyzed a seasonal model of cutaneous leishmaniasis. A pulse control model with seasonality was investigated in [74] by Jin et al. In [131] Ma and Ma studied a seasonally forced SEIR model (susceptible, exposed, infected, recovered). Other examples can be seen in [11–13,54,55,75,104,107,144,160,169,198].

There have been cases where the transmission data is more accurately reflected in a term-time forcing model approach, where the contact rate changes abruptly in time [44]. For example [44]:

$$\beta = \beta(t) = \beta_0(1+\epsilon)^{Term(t)}$$

where

$$Term(t) = \begin{cases} +1 \text{ during school terms} \\ -1 \text{ during school breaks.} \end{cases}$$

This modelling approach was first considered by Schenzle in [163] and has since been studied in, for example, [44, 76, 113, 114, 117, 120]. Analytic methods for analyzing time-dependent contact rates are currently lacking, and, since relatively small variations in the contact rate can result in large amplitude fluctuations in a disease's incidence, this phenomenon warrants more attention [75]. This naturally leads to a switched systems modelling approach where the contact rate is a switching parameter:

$$\beta = \beta_{i_k}, \qquad t \in [t_{k-1}, t_k)$$

where the index i_k follows a switching rule $\sigma : [t_{k-1}, t_k) \to \{1, 2, \ldots, m\}$ for $k \in \mathbb{N}$ to model a piecewise-constant contact rate taking on values in the set $\{\beta_1, \beta_2, \ldots, \beta_m\}$ where each $\beta_i > 0$. The epidemic model becomes a switched system with switching times t_k which satisfy $t_0 = 0 < t_1 < t_2 < \ldots < t_{k-1} < t_k < \ldots$ with $t_k \to \infty$ as $k \to \infty$. Under this construction, $\beta = \beta_{i_k}$ on the interval $[t_{k-1}, t_k)$ and at the switch time t_k the contact rate changes to $\beta_{i_{k+1}}$. This direction of research was analyzed in [113] where we studied the application of control schemes to the switched SIR model given by

$$\begin{cases} \dot{S} = \mu - \beta_{i_k} SI - \mu S, \\ \dot{I} = \beta_{i_k} SI - gI - \mu I, \\ \dot{R} = gI - \mu R, \end{cases} \tag{6.3}$$

for $t \in [t_{k-1}, t_k)$. We applied both time-constant and pulse control strategies and gave threshold criteria for the long-term behaviour of the solution. In the report [120], we analyzed a switched SIS model whereby individuals who recover from the disease immediately become susceptible again (which is reasonable for many diseases that are not lethal and have low death rates such as the common cold and water pox [201]). For background on SIS modelling efforts, see [37, 98, 201]. The switched SIS model was given by

$$\begin{cases} \dot{S} = \mu_{i_k} - \beta_{i_k} SI + g_{i_k} I - \mu_{i_k} S, \\ \dot{I} = \beta_{i_k} SI - g_{i_k} I - \mu_{i_k} I, \end{cases} \tag{6.4}$$

where $i_k \in \{1, \ldots, m\}$ follows a switching rule so that β_{i_k}, g_{i_k}, and μ_{i_k} are all switching parameters. We extended the model by considering media coverage and the pattern of daily encounters in a local community. Vaccination strategies with waning immunity were considered as well as vertical transmission of the disease. For other examples of epidemic models with term-time forcing modelled by switching, see [115] where we analyzed an SIR epidemic model with general nonlinear incidence rate and seasonality, [114] where we considered the application of control schemes to an SIR with a term-time forced contact rate and stochasticity, and [117] where we investigated a multi-city SIR model with transported-related infections and abruptly-changing model parameters.

6.2 A Case Study: Application of Control Strategies to a Seasonal Model of Chikungunya Disease

In this section we analyze a new model of the vector-borne disease Chikungunya by considering time-varying switching parameters. In particular, the birth rate of the mosquito population varies between rainy season and dry season, and the contact rate between mosquito and human changes in time. Mechanical control of breeding sites and reduced contact rate strategies are studied. The material in this section formed the basis for [118].

The Chikungunya virus is a vector-borne viral disease which is transmitted primarily by mosquitoes of the Aedes genus: Aedes aegypti and Aedes albopictus [141, 155]. A Chikungunya infection is an acute illness generally described by the sudden onset of fever and incapacitating arthralgia (non-inflammatory joint pain), often accompanied by muscle pain, rash, and a headache (less frequent symptoms include nausea and vomiting) [155,158]. The first isolated cases of Chikungunya virus occurred in 1953 in Tanzania [158]. In the past decade, a series of outbreaks have occurred over a geographic area including African islands in the Indian Ocean and the Indian subcontinent: the first outbreak occurred in 2004 in Kenya, followed by outbreaks on the Comoros Islands in early 2005 and in India in 2005-06 where the World Health Organization reported an estimated 1.3 million cases [29, 155, 158, 159].

A confirmed case of Chikungunya virus was reported in Reunion Island (a French island located east of Madagascar) on April 29, 2005, imported from Grande-Comore. This led to an outbreak of Chikungunya virus on Reunion Island in 2005 and 2006, which consisted of two epidemic waves: the first wave occurred in May 2005 with 450 reported cases. The second wave began in December 2005 and was much larger, peaking in January and February 2006 with more than $47,000$ estimated cases [158]. In total, there were about

244,000 estimated cases during the outbreak [140, 158], which is approximately one third of the island's population [41]. The main focus of this section is studying the outbreak on Reunion Island.

The spread of Chikungunya is influenced by a number of factors such as the behaviour of the human and mosquito populations, as well as the environment in which it spreads [140]. Seasonal fluctuations in the environment play a crucial role in the spread of vector-borne diseases. For example, the transmission of Dengue (transmitted by Aedes aegypti) is high when the temperature is high, during wet and humid periods, while the transmission is low when the temperature is low [158, 195]. On Reunion Island, the 2005 outbreak appeared between March and June, which corresponds to the beginning of the winter season and end of the hot season (when the mosquito population is at a maximum) [42]. Another factor that played an important role in the outbreaks on Reunion Island is that two strains of the virus appeared [41]: the first strain was isolated in May 2005 during the first outbreak while the second strain, isolated in November 2005, had a higher rate of transmission from human to mosquito. It was shown that the probability of a mosquito contracting the infection by biting an infected human increased from 37% for the first strain to 95% for the second strain [41].

Currently the main strategies for preventing Chikungunya outbreaks involve the interruption of contact between humans and vectors (such as individual protection against mosquito bites) or the control of the mosquito population [141]. Measures to control the Aedes albopictus vector population were used on Reunion Island when the DRASS (an agency of the French government for disease prevention and vector control) conducted several interventions [41]: massive spraying of Deltamethrin (a chemical adulticide); localized treatment of a chemical larvicide BTI (Bacillus thuringensis israelensis); and the mechanical destruction of breeding sites by eliminating standing water in rain gutters, buckets, plastic covers, tires, tree holes, or any other potential breeding site for mosquitoes. In [41], Dumont and Chiroleu noted that larvicide treatments do not have a relatively large impact on a Chikungunya epidemic, compared to adulticide. The authors gave a potential explanation that this may be because only breeding sites are treated with the larvicide, which can be very localized. The use of adulticides can cause harm to the environment [41, 141, 158] and in some areas Aedes have rapidly developed resistances to adulticides (for example up to 60% resistances for Deltamethrin) [141]. Mechanical control requires the cooperation of the local population but is sustainable, relatively cheap, and can be effective depending on the duration and time of initiation [41]. Recently, a new technique called sterile insect technique has been proposed and studied where sterile male insects are periodically released into the wild to control the vector population [40, 43].

Motivated by the outbreaks in the last decade and since it is possible for Chikungunya virus to re-emerge after years or even decades of absence, there has been an increased interest in studying Chikungunya [158]. Dumont et al. [42] were the first authors to analyze a mathematical model based on the Chikungunya outbreak on Reunion Island. The authors computed the basic reproduction number of the disease, proved a necessary condition for eradication of the disease, and presented several simulations of the outbreak in different cities on the island. Dumont and Chiroleu [41] were the first authors to consider vector control for the outbreak on Reunion Island by analyzing and comparing the use of adulticide, larvicide, and mechanical control. In [43], Dumont and Tchuenche analyzed the use of sterile insects to help prevent the spread of Chikungunya disease by controlling the vector population. More recently, Dufourd and Dumont [40] studied the effects of periodic parameters on the temporal and spatio-temporal evolution of a vector population under the sterile insect technique. In [140], Moulay et al. investigated a Chikungunya model for the outbreak on Reunion Island with an embryonic, larvae, and adult stage for the vector population. The authors proved stability using the theory of competitive systems and Lyapunov function methods. Moulay et al. [141] studied optimal control of the Chikungunya disease by considering vector control (using larvicide, larvivore fish, and water traps), reducing the number of vector-host contacts, and treatment of individuals (such as by isolating infected patients in hospitals). The authors Bowong et al. [25] investigated a multi-city model for Chikungunya-like diseases where humans can travel between the cities.

In [9], Bacaër studied the Reunion Island outbreak by considering a periodic model of Chikungunya disease and approximating the basic reproduction number numerically. Bacaër noted that many Chikungunya models in the literature make the inappropriate assumption that the vector population is constant in time, but seasonality plays an important role in the spread of the disease [9]. From weather data on Reunion Island (e.g. see Figure 1 in [9]), rainfall and temperature both seem to achieve a maximum around February and a minimum around July. Therefore, it is reasonable to suppose that there is a peak in the vector population each year when rainfall is high [9]. Indeed, Dumont et al. [42] stated that one improvement to their work would be to add weather parameters, such as humidity and temperature. Dumont and Chiroleu [41] concluded that one possible improvement to their model would be to consider periodicity in some of the parameters in the mosquito population, due to changes in temperature and humidity. In [34], Delatte showed that the survival rate of adult Aedes albopictus is inversely correlated to the temperature. Hence, the vector is able to survive the dry season, which could be an explanation for why the Chikungunya virus survived from June to October 2005 on Reunion Island [41].

The aim of the present section is to improve the analysis of the spread of Chikungunya disease on Reunion Island by analyzing control schemes for a model of Chikungunya with time-varying parameters. More specifically, the birth rate of the vector population is assumed to be a switching parameter (to model a time-varying birth rate from wet season to dry season) and the transmission rate of the disease is assumed to change in time (due to shifts in the contact rate between humans and mosquitoes throughout the year). The possibility of a genetic mutation is also taken into account in the model as a switching parameter. To the best of the author's knowledge, there have been no studies in the literature on the mathematical analysis of a Chikungunya model with switching parameters. Mechanical control of breeding sites and a reduction of contact rates between humans and mosquitoes are considered. Hence, the contributions of this section are to further extend current knowledge on vector population control methods and human-mosquito interaction interruption methods.

6.2.1 Compartmental Model for Human-Mosquito Interaction

First we construct a stage structured compartmental model to simulate the dynamics of the vector population. In the case of Aedes albopictus, the vector stays in the area in which it was born, given that it has suitable conditions to develop and survive (such as blood and sugar meals), and has an expected adult life of about 10-11 days [41]. The life cycle of a vector consists of four stages: the embryonic stage, larvae stage, pupae stage, and adult stage. The first three stages require water while the final stage only requires air [140]. Motivated by the model in [140], assume that the vector population is broken into three distinct compartments: the embryonic stage, denoted by E, which consists of eggs; the aquatic stage, denoted by Q, which includes the larvae and pupae stages; and the adult stage, denoted by A.

Remark 6.2.1. *The main motivation for separating the embryonic stage from the aquatic stage is because mechanical control of breeding sites is not successful in destroying eggs, since they can cling to surfaces and are desiccation-resistant [140].*

Assume that the embryonic and aquatic life cycles of the vector population have a limited carrying capacity (due to the constraints on water levels and nutrients). This was first considered in [42] and also in [41, 140]. Assume that the carrying capacity of the habitat for the eggs is given by $\Gamma_E > 0$ and the carrying for the aquatic stages is given by $\Gamma_Q > 0$. Assume that the rate of transfer from embryonic stage to aquatic stage is $\eta_E > 0$ and the rate of transfer from aquatic stage to adult stage is $\eta_Q > 0$. Assume that the death rate of mosquitoes in the embryonic stage is $d_E > 0$ and the death rate of aquatic mosquitoes is $d_Q > 0$.

The authors Moulay et al. [140] noted that when considering a large mosquito population, it is reasonable to assume that the number of exposed vectors that are not yet infectious is a negligible part of the total population. Thus, consider two compartments for the adult mosquito population: the susceptible vector population, denoted by S_M, for adult mosquitoes that do not carry the virus but are able to contract the disease from biting an infected human; and the infected vector population, denoted by I_M, for adult mosquitoes that are infected with the virus and able to transmit it to healthy humans. The total adult population is given by $A = S_M + I_M$. Assume that the death rate of susceptible mosquitoes is given by $d_S > 0$ and the death rate of infected mosquitoes is given by $d_I > 0$. The average lifespan of infected mosquitoes is approximately five days, and, after contracting the disease, the vectors of Chikungunya remain infected until they die [9, 41]. Since the average lifespan of susceptible adult mosquitoes is approximately ten days (detailed above), assume that $1/d_S > 1/d_I$ so that $d_I > d_S$.

The rainy season lasts from November until March on Reunion Island, during which the climatic conditions lead to an increase in the number of breeding sites for Aedes albopictus (and hence a larger population of susceptible mosquitoes). This leads to an increase in the carrying capacity of the vector population [42]. In a study on the re-emergence of Chikungunya virus in 2001-2003 in Indonesia, Laras et al. [86] noted a negligible variability in average monthly 24-hour maximum-minimum ambient temperatures but significant seasonal fluctuations in rainfall. In August 2001, there was an increase in rainfall which corresponded to the beginning of an outbreak lasting from September to December 2001 [86]. The asian tiger mosquito's life-span is also strongly related to the temperature and humidity, which can vary greatly depending on the region [42].

Motivated by seasonal fluctuations in the vector population, consider a term-time forced vector birth rate b_{i_k} on the interval $[t_{k-1}, t_k)$, where $k = 1, 2, \ldots$, and the index $i_k \in \{1, 2, \ldots, m\}$ follows a switching rule $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$. That is, the birth rate is a switching parameter that is piecewise constant and takes on values in the set $\{b_1, b_2, \ldots, b_m\}$. The parameter changes values at the switching times t_k, which depend on changes from the rainy season to dry season and satisfy $t_0 = 0 < t_1 < \ldots < t_k \to \infty$ as $k \to \infty$. Throughout this chapter it is assumed that the switching rule is well-posed, that is, $\sigma \in \mathcal{S}$. Then, for $t \in [t_{k-1}, t_k)$, the dynamics of the vector population can be modelled as

$$\begin{cases} \dot{E} = b_{i_k}\left(1 - \dfrac{E}{\Gamma_E}\right)A - (\eta_E + d_E)E, \\ \dot{Q} = \eta_E\left(1 - \dfrac{Q}{\Gamma_Q}\right)E - (\eta_Q + d_Q)Q, \\ \dot{A} = \eta_Q Q - d_I I_M - d_S S_M. \end{cases} \tag{6.5}$$

In [41, 42, 140, 141], the authors made the following assumptions: the average number of contacts per day resulting in infection in a mosquito is constant in time, given by $\gamma > 0$. The per capita incidence rate among mosquitoes is modelled by

$$\gamma \frac{I_H}{N_H}$$

where I_H represents the infected human population (individuals who have been infected with the disease and are able to transmit it to a mosquito if they are bitten), and N_H is the total human population. This choice of incidence rate takes into account frequency of bites and encounters between susceptible mosquitoes and infectious humans. Here we extend this approach by considering an effective contact rate that may vary in time. Assume that the contact rate sufficient for transmission of the disease from human to mosquito is equal to $\gamma_{i_k} > 0$ on the interval $[t_{k-1}, t_k)$, where $k = 1, 2, \ldots$, and the index $i_k \in \{1, 2, \ldots, m\}$ follows the switching rule σ. Under this assumption, the contact rate is modelled as a term-time forced parameter.

As discussed above, a genetic mutation in the Chikungunya virus on Reunion Island resulted in a reduction in the extrinsic incubation period in Aedes albopictus to two days (from seven to twelve days) [41]. Further, the genetic mutation also affected the death rate of infected mosquitoes, as well as the transmission probability from human to mosquito [41]. This genetic mutation was first studied in the papers [41, 42]. Motivated by this, consider a mutation parameter that switches in time, δ_{i_k}, to reflect a possible genetic mutation in the virus causing a shift in the transmission rate. Then the adult vector population can be modelled as follows, for $t \in [t_{k-1}, t_k)$,

$$\begin{cases} \dot{S_M} = \eta_Q Q - \delta_{i_k}\gamma_{i_k}\dfrac{S_M I_H}{N_H} - d_S S_M, \\ \dot{I_M} = \delta_{i_k}\gamma_{i_k}\dfrac{S_M I_H}{N_H} - d_I I_M. \end{cases} \tag{6.6}$$

Next we consider the life cycle of the human population. Since vertical transmission of Chikungunya did not play a key role in the spread of the virus in Reunion Island [41], assume that all humans are born into the susceptible class, denoted by S_H. After a period of four to seven days, a human infected with the disease is able to transmit it to vectors [140]. By April 2006, 203 death certificates mentioned the Chikungunya infection as the direct or indirect cause of death, which means the overall mortality rate associated with Chikungunya virus in humans was approximately $0.3/1,000$ persons [158]. Hence, assume that the disease-induced mortality rate in humans is negligible.

Since most patients infected with Chikungunya recover quickly without any long-lasting chronic effects and acquire immunity against the virus [158], assume that once a human recovers from the disease, they move to the recovered class, denoted by R_H. Assume that humans who contract the disease recover naturally at a rate $g > 0$. Assume that the contact rate sufficient for transmission to a human is equal to $\beta_{i_k} > 0$ on the interval $[t_{k-1}, t_k)$ where the index $i_k \in \{1, 2, \ldots, m\}$ also follows the switching rule σ. The mosquito and human populations are governed by the switched equations, for $t \in [t_{k-1}, t_k)$:

$$\begin{cases} \dot{S_H} = \mu N_H - \beta_{i_k} \dfrac{S_H I_M}{N_H} - \mu S_H, \\ \dot{I_H} = \beta_{i_k} \dfrac{S_H I_M}{N_H} - (g + \mu) I_H, \\ \dot{R_H} = g I_H - \mu R_H, \end{cases} \tag{6.7}$$

where $N_H = S_H + I_H + R_H$ is constant in time.

Remark 6.2.2. *In the above formulation, the incidence rate is given by $\beta_{i_k} S_H I_M / N_H$ which is the switching form of the incidence rate $\beta S_H I_M / N_H$ used in [41, 42]. In [140, 141], the authors strayed from this incidence rate, instead opting for an incidence rate of the form $\beta S_H I_M / A$ which takes into account the vector dynamics with non-constant population size and a contact rate dependent on the size of the vector population [140].*

Before combining the equations to form the full Chikungunya model, first normalize equations (6.6) and (6.7) using $\bar{S}_M = S_M/A$ $\bar{I}_M = I_M/A$, $\bar{S}_H = S_H/N_H$, $\bar{I}_H = I_H/N_H$, $\bar{R}_H = R_H/N_H$. Observe that

$$\begin{aligned} \dot{\bar{I}}_M &= \frac{\dot{I}_M A - \dot{A} I_M}{A^2}, \\ &= \left[\delta_{i_k} \gamma_{i_k} \frac{S_M I_H}{N_H} - d_I I_M \right] \frac{1}{A} - [\eta_Q Q - d_I I_M - d_s S_M] \frac{I_M}{A^2}, \\ &= \delta_{i_k} \gamma_{i_k} \bar{S}_M \bar{I}_H - d_I \bar{I}_M - \eta_Q \frac{Q}{A} \bar{I}_M + d_I \bar{I}_M^2 + d_s \bar{S}_M \bar{I}_M, \\ &= \delta_{i_k} \gamma_{i_k} \bar{I}_H - \delta_{i_k} \gamma_{i_k} \bar{I}_M \bar{I}_H - \eta_Q \frac{Q}{A} \bar{I}_M - d_I \bar{I}_M + d_I \bar{I}_M^2 + d_S \bar{I}_M - d_S \bar{I}_M^2, \\ &= -\left(\eta_Q \frac{Q}{A} + \delta_{i_k} \gamma_{i_k} \bar{I}_H \right) \bar{I}_M + \delta_{i_k} \gamma_{i_k} \bar{I}_H + (d_S - d_I) \bar{I}_M + (d_I - d_S) \bar{I}_M^2, \end{aligned}$$

where the fact that $\bar{S}_M + \bar{I}_M = 1$ has been used. Similarly,

$$\begin{aligned}
\dot{\bar{S}}_M &= \left[\eta_Q Q - \delta_{i_k}\gamma_{i_k}\frac{S_M I_H}{N_H} - d_S S_M\right]\frac{1}{A} - [\eta_Q Q - d_I I_M - d_s S_M]\frac{S_M}{A^2},\\
&= \eta_Q \frac{Q}{A} - \delta_{i_k}\gamma_{i_k}\bar{S}_M\bar{I}_H - d_S\bar{S}_M - \eta_Q\frac{Q}{A}\bar{S}_M + d_S\bar{S}_M^2 + d_I\bar{S}_M\bar{I}_M,\\
&= \eta_Q\frac{Q}{A}\bar{I}_M - \delta_{i_k}\gamma_{i_k}(1-\bar{I}_M)\bar{I}_H - d_S\bar{S}_M + d_S\bar{S}_M^2 + d_I\bar{S}_M(1-\bar{S}_M),\\
&= \left(\eta_Q\frac{Q}{A} + \delta_{i_k}\gamma_{i_k}\bar{I}_H\right)\bar{I}_M - \delta_{i_k}\gamma_{i_k}\bar{I}_H + (d_I - d_S)\bar{S}_M + (d_S - d_I)\bar{S}_M^2.
\end{aligned}$$

Since $\dot{N}_H = 0$, the equations for $\bar{S}_H$, $\bar{I}_H$, $\bar{R}_H$ are straightforward to calculate. After dropping the bars, the dynamics of the full Chikungunya model is given by, for $t \in [t_{k-1}, t_k)$:

$$\begin{cases}
\dot{E} = b_{i_k}\left(1 - \dfrac{E}{\Gamma_E}\right)A - (\eta_E + d_E)E,\\
\dot{Q} = \eta_E\left(1 - \dfrac{Q}{\Gamma_Q}\right)E - (\eta_Q + d_Q)Q,\\
\dot{A} = \eta_Q Q - d_I I_M A - d_S S_M A,
\end{cases} \tag{6.8a}$$

$$\begin{cases}
\dot{S}_M = \left(\eta_Q\dfrac{Q}{A} + \delta_{i_k}\gamma_{i_k}I_H\right)I_M - \delta_{i_k}\gamma_{i_k}I_H + (d_I - d_S)(1 - S_M)S_M,\\
\dot{I}_M = -\left(\eta_Q\dfrac{Q}{A} + \delta_{i_k}\gamma_{i_k}I_H + (d_S - d_I)(1 - I_M)\right)I_M + \delta_{i_k}\gamma_{i_k}I_H,\\
\dot{S}_H = \mu - \beta_{i_k}\dfrac{S_H I_M A}{N_H} - \mu S_H,\\
\dot{I}_H = \beta_{i_k}\dfrac{S_H I_M A}{N_H} - (g + \mu)I_H,\\
\dot{R}_H = g I_H - \mu R_H.
\end{cases} \tag{6.8b}$$

where the initial conditions are nonnegative and are given by $E(0) = E_0$, $Q(0) = Q_0$, $A(0) = A_0$, $S_H(0) = S_{H0}$, $I_H(0) = I_{H0}$, $R_H(0) = R_{H0}$, $S_M(0) = S_{M0}$, and $I_M(0) = I_{M0}$, such that $S_{M0} + I_{M0} = 1$ and $S_{H0} + I_{H0} + R_{H0} = 1$. See Figure 6.2 for the flow of the model. The physical domain is given by

$$\begin{aligned}
\Omega_{\text{Chiku}} = \{&(E, Q, A, S_M, I_M, S_H, I_H, R_H) \in \mathbb{R}_+^8 \,|\, 0 \le E \le \Gamma_E,\ 0 \le Q \le \Gamma_Q,\\
&0 \le A \le \frac{\eta_Q \Gamma_Q}{d_S},\ S_M + I_M = 1,\ S_H + I_H + R_H = N_H\}.
\end{aligned}$$

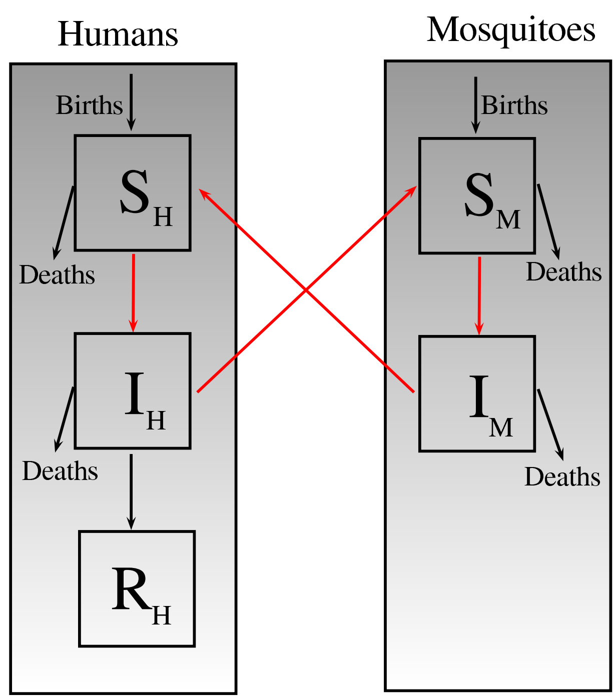


Figure 6.2: Flow diagram of model (6.8b) which shows the interaction between humans and mosquitoes. The red lines represent new infections.

Remark 6.2.3. *The domain Ω_{Chiku} is the region of biological interest. It is possible to show that if*

$$(E_0, Q_0, A_0, S_{M0}, I_{M0}, S_{H0}, I_{H0}, R_{H0}) \in \Omega_{Chiku}$$

then the solution remains in Ω_{Chiku}. For example, see [140] where the non-switched model is studied. In fact, if the initial conditions are nonnegative and

$$(E_0, Q_0, A_0, S_{M0}, I_{M0}, S_{H0}, I_{H0}, R_{H0}) \notin \Omega_{Chiku}$$

then solutions eventually enter and remain in Ω_{Chiku}.

Remark 6.2.4. *In equation (6.8), the spread of the Chikungunya virus is modelled by considering the interaction between the human and mosquito population. The main focus of the rest of the present section is studying the efficacy of different control strategies when applied to model (6.8). In Section 6.3, we will investigate the spread of a vector-borne* *disease (such as Chikungunya) by analyzing a switched epidemic model with distributed delays that only considers the dynamics of the human population. This will be possible because the human and mosquito populations evolve on two different time scales. The motivation for studying both approaches (non-delay in the present section and delay in the next section) is to properly frame the problem by building an intuition of both models' underlying dynamics and to be able to see the differences between the two modelling methods.*

6.2.2 Mechanical Destruction of Breeding Sites

The first control scheme considered is mechanical control of the mosquito breeding sites, which, as discussed at the beginning of Section 6.2, is a powerful tool to prevent the spread of Chikungunya. The main vector of Chikungunya, Aedes albopictus, is a container-inhabiting species that lays its eggs in any water-containing receptacle in urban, suburban, forest or rural area [140]. The development of immature Aedes albopictus depends vitally on the availability of water, as the mosquitoes rely on rainfall to raise water levels in containers so that the eggs may hatch [140]. Further, an increase in larval density or decrease in food or water can lead to a reduced number of adult mosquitoes [140]. Mechanical control consists of destruction of the breeding sites, and therefore, a reduction in the carrying capacity of the aquatic population. For example, it is recommended for people to check around their houses after a rainfall to clean or empty water containers where mosquitoes could lay eggs [141]. Assume in the model that the carrying capacity of the aquatic stage (larvae plus pupae) of the mosquito population is reduced to $\alpha\Gamma_Q$, where $0 < \alpha \leq 1$. Apply this mechanical control to model (6.8) to get, for $t \in [t_{k-1}, t_k)$,

$$\begin{cases} \dot{E} = b_{i_k}\left(1 - \dfrac{E}{\Gamma_E}\right)A - (\eta_E + d_E)E, \\ \dot{Q} = \eta_E\left(1 - \dfrac{Q}{\alpha\Gamma_Q}\right)E - (\eta_Q + d_Q)Q, \\ \dot{A} = \eta_Q Q - d_I I_M A - d_S S_M A, \end{cases} \tag{6.9a}$$

$$\begin{cases} \dot{S}_H = \mu - \beta_{i_k}\dfrac{S_H I_M A}{N_H} - \mu S_H, \\ \dot{I}_H = \beta_{i_k}\dfrac{S_H I_M A}{N_H} - (g+\mu)I_H, \\ \dot{R}_H = gI_H - \mu R_H, \\ \dot{I}_M = -\left(\eta_Q\dfrac{Q}{A} + \delta_{i_k}\gamma_{i_k}I_H + (d_S - d_I)(1 - I_M)\right)I_M + \delta_{i_k}\gamma_{i_k}I_H. \end{cases} \tag{6.9b}$$

The equation for S_M is omitted since $S_M + I_M = 1$. The initial conditions are given by $E(0) = E_0$, $Q(0) = Q_0$, $A(0) = A_0$, $S_H(0) = S_{H0}$, $I_H(0) = I_{H0}$, $R_H(0) = R_{H0}$, and $I_M(0) = I_{M0}$. The physical domain is given by

$$\Omega_{mechanical} = \{(E, Q, A, I_M, S_H, I_H, R_H) \in \mathbb{R}^7_+ |\ 0 \leq E \leq \Gamma_E,\ 0 \leq Q \leq \alpha\Gamma_Q,$$
$$0 \leq A \leq \frac{\alpha\eta_Q\Gamma_Q}{d_S},\ 0 \leq I_M \leq 1,\ S_H + I_H + R_H = N_H\}.$$

In order to analyze the long-term behaviour of model (6.9), we first focus on the dynamics of the vector population given by equation (6.9a). System (6.9a) has a common equilibrium $(E, Q, A) = (0, 0, 0)$, which is the mosquito-free equilibrium. Each subsystem $i_k = i$, $i = 1, 2, \ldots, m$, also has an endemic equilibrium

$$(E, Q, A) = (E_i^*, Q_i^*, A_i^*) = \left(1 - \frac{1}{r_i}\right)\left(\frac{\Gamma_E}{\nu_i}, \frac{\alpha\Gamma_Q}{\kappa_i}, \frac{\eta_Q}{d_I}\frac{\alpha\Gamma_Q}{\kappa_i}\right),$$

where

$$r_i = \frac{b_i}{\eta_E + d_E}\frac{\eta_E}{\eta_Q + d_Q}\frac{\eta_Q}{d_I} \tag{6.10}$$

and

$$\nu_i = 1 + \frac{(\eta_E + d_E)d_I\Gamma_E}{b_i\eta_Q\alpha\Gamma_Q}, \qquad \kappa_i = 1 + \frac{(\eta_Q + d_Q)\Gamma_Q}{b_i\eta_E\alpha\Gamma_E}.$$

Note that because of the switching it may be possible that the solution trajectory of the system moves between the endemic equilibria and does not converge to a particular one. In order to study the long-term dynamics of (6.9a), define the minimum and maximum vector birth rates as

$$b_{max} = \max_{i=1,\ldots,m} b_i, \quad b_{min} = \min_{i=1,\ldots,m} b_i.$$

Define the minimum and maximum endemic equilibria

$$E_{max} = \max_{i=1,\ldots,m} E_i^*, \quad E_{min} = \min_{i=1,\ldots,m} E_i^*,$$

and define Q_{max}, Q_{min}, A_{max}, and A_{min} similarly.

Proposition 6.2.1. *If $\bar{r} > 1$, where*

$$\bar{r} = \frac{b_{min}}{\eta_E + d_E}\frac{\eta_E}{\eta_Q + d_Q}\frac{\eta_Q}{d_I}, \tag{6.11}$$

then the solution of (6.9a) *converges to the set*

$$\Delta_{mechanical} = \{(E, Q, A) \in \mathbb{R}^3_+ |\ E_{min} \leq E \leq E_{max},\ Q_{min} \leq Q \leq Q_{max},$$
$$A_{min} \leq A \leq A_{max}\}. \tag{6.12}$$

Proof. Note that

$$\dot{E} \geq b_{min}\left(1 - \frac{E}{\Gamma_E}\right)A - (\eta_E + d_E)Q,$$
$$\dot{A} \geq \eta_Q Q - d_I I_M A - d_I S_M A = \eta_Q Q - d_I A.$$

Consider the comparison system

$$\begin{cases} \quad \dot{x} = b_{min}\left(1 - \dfrac{x}{\Gamma_E}\right)z - (\eta_E + d_E)x, \\ \quad \dot{y} = \eta_E\left(1 - \dfrac{y}{\Gamma_Q}\right)x - (\eta_Q + d_Q)y, \\ \quad \dot{z} = \eta_Q y - d_I z, \\ x(0) = E_0, \quad y(0) = Q_0, \quad z(0) = A_0. \end{cases} \tag{6.13}$$

Since $\bar{r} > 1$ then (6.13) converges to $(E_{min}, Q_{min}, A_{min})$ by Proposition 4.7 in [140]. Note that $b_{min}\left(1 - \frac{x}{\Gamma_E}\right)z \geq 0$ for $0 \leq x \leq \Gamma_E$ and $z \geq 0$, $\eta_E\left(1 - \frac{y}{\Gamma_Q}\right)x \geq 0$ for $0 \leq y \leq \Gamma_Q$ and $x \geq 0$, and $\eta_Q y \geq 0$ for $y \geq 0$. Then by the comparison theorem (see [84]) there exists $t^* > 0$ such that $E \geq E_{min} - \epsilon$, $Q \geq Q_{min} - \epsilon$ and $A \geq A_{min} - \epsilon$ for $t > t^*$. Similarly,

$$\dot{E} \leq b_{max}\left(1 - \frac{E}{\Gamma_E}\right)A - (\eta_E + d_E)E,$$
$$\dot{A} \leq \eta_Q Q - d_S I_M A - d_S S_M A = \eta_Q Q - d_S A.$$

Consider the comparison system

$$\begin{cases} \quad \dot{x} = b_{max}\left(1 - \dfrac{x}{\Gamma_E}\right)z - (\eta_E + d_E)x, \\ \quad \dot{y} = \eta_E\left(1 - \dfrac{y}{\Gamma_Q}\right)x - (\eta_Q + d_Q)y, \\ \quad \dot{z} = \eta_Q y - d_S z, \\ x(0) = E_0, \quad y(0) = Q_0, \quad z(0) = A_0. \end{cases} \tag{6.14}$$

Note that $\bar{r} > 1$ implies that $r_i > 1$ for each $i \in \mathcal{P}$, so that

$$\frac{b_{max}}{\eta_E + d_E}\frac{\eta_E}{\eta_Q + d_Q}\frac{\eta_Q}{d_I} > 1$$

and (6.14) converges to $(E_{max}, Q_{max}, A_{max})$ by Proposition 4.7 in [140]. Hence, there exists $\hat{t} > 0$ such that $E \leq E_{max} + \epsilon$, $Q \leq Q_{max} + \epsilon$ and $A \leq A_{max} + \epsilon$ for $t > \hat{t}$. Thus, for $t > \max\{t^*, \hat{t}\}$, $E_{min} - \epsilon \leq E \leq E_{max} + \epsilon$, $Q_{min} - \epsilon \leq Q \leq Q_{max} + \epsilon$ and $A_{min} - \epsilon \leq A \leq A_{max} + \epsilon$. It is clear that the solution of system (6.9a) converges to the set $\Delta_{mechanical}$. □

Remark 6.2.5. *In the special case that* $b_{i_k} \equiv b$, $d_S \equiv d_I \equiv d$, *system* (6.9a) *has two equilibria: the mosquito-free equilibrium* $(E, Q, A) = (0, 0, 0)$, *and the endemic equilibrium,*

$$(E, Q, A) = (E^*, Q^*, A^*) = \left(1 - \frac{1}{r}\right)\left(\frac{\Gamma_E}{\nu}, \frac{\alpha\Gamma_Q}{\kappa}, \frac{\eta_Q}{d}\right), \tag{6.15}$$

where

$$\nu = 1 + \frac{(\eta_E + d_E)d_I\Gamma_E}{b\eta_Q\alpha\Gamma_Q}, \qquad \kappa = 1 + \frac{(\eta_Q + d_Q)\Gamma_Q}{b\eta_E\alpha\Gamma_E},$$

and the basic offspring number is defined as

$$r = \frac{b}{\eta_E + d}\frac{\eta_E}{\eta_Q + d}\frac{\eta_Q}{d}, \tag{6.16}$$

which represents the average number of offspring per mosquito during an average lifetime. In [140], the authors showed that if $r < 1$ *then* $(0, 0, 0)$ *is globally asymptotically stable in the meaningful domain while if* $r > 1$ *then* (E^*, Q^*, A^*) *is globally asymptotically stable. Hence, the approximation* $\bar{r}$ *represents a lower bound for the average number of offspring from each mosquito when the mosquito birth rate is time-varying.*

Next we shift our focus to the long-term dynamics of the human population. Observe that $(S_H, I_H, R_H, I_M) = (1, 0, 0, 0)$ is a solution to the differential equations for S_H, I_H, R_H, and I_M in (6.9a). Motivated by this, define the set

$$\begin{aligned}\Psi_{mechanical} = \{(E, Q, A, I_M, S_H, I_H, R_H) \in \mathbb{R}^7_+ |\ (E, Q, A) \in \Delta_{mechanical},\\ S_H = 1,\ I_H = 0,\ R_H = 0,\ I_M = 0\}.\end{aligned} \tag{6.17}$$

For epidemic models of vector-borne diseases, the basic reproduction number $\mathcal{R}_0$ is defined as the average number of secondary cases produced by one primary infectious case by the vectors in a wholly susceptible population. For periodic models, the rate of infection changes based on the time of year and $\mathcal{R}_0$ can be interpreted as an asymptotic per generation growth rate of the epidemic model linearized about the disease-free equilibrium [10]. Depending on the model, the basic reproduction number can be given explicitly in terms of model parameters. However, since (6.9) has time-varying parameters and multiple infected compartments, the basic reproduction number can only be implicitly defined as the spectral radius of a next generation integral operator [11,12]. For background literature on the basic reproduction number of epidemic models with periodicity, see [12,188].

Remark 6.2.6. *When the model parameters in* (6.9) *are constant in time* ($b_{i_k} \equiv b$, $\gamma_{i_k} \equiv \gamma$, $\beta_{i_k} \equiv \beta$)*, there is no mutation factor* ($\delta_{i_k} \equiv 1$) *and the death rates of susceptible and infected mosquitoes are equal* ($d_S \equiv d_I \equiv d$) *the basic reproduction number of the disease can be given explicitly in terms of the model parameters [141]:*

$$\mathcal{R}_0^2 = \frac{\beta}{g+\mu}\frac{\gamma}{d}\frac{A^*}{N_H} \tag{6.18}$$

where A^* *is given in equation* (6.15)*. The physical interpretation is as follows [41, 42]: the fraction*

$$\mathcal{R}_{MH} = \frac{\beta}{g+\mu}$$

represents the rate of spread from mosquito to humans, while

$$\mathcal{R}_{HM} = \frac{\gamma}{d}\frac{A^*}{N_H}$$

represents the rate of spread from humans to mosquitoes. Hence the basic reproduction number of the overall system is $\mathcal{R}_0^2 = \mathcal{R}_{MH} \times \mathcal{R}_{HM}$*. Since* $0 = \eta_Q Q^* - dA^*$*, we can re-write* $\mathcal{R}_0^2$ *as*

$$\mathcal{R}_0^2 = \frac{\beta}{g+\mu}\frac{\gamma}{\eta_Q \frac{Q^*}{A^*}}\frac{A^*}{N_H} = \frac{\beta}{g+\mu}\frac{\gamma}{\eta_Q}\frac{(A^*)^2}{Q^* N_H}.$$

Motivated by Remark 6.2.6, consider system (6.9) and the following approximate basic reproduction numbers for each subsystem

$$\widetilde{\mathcal{R}}_i^2 = \frac{\beta_i \delta_i \gamma_i}{\eta_Q(g+\mu)}\frac{A_{max}^2}{Q_{min}N_H} \tag{6.19}$$

for $i = 1, \ldots, m$. There are several possibilities to control the disease: for example, it is possible to focus on controlling the mosquito population (e.g. sterile insect technique, larvicide, or adulticide to reduce $\bar{r}$, mechanical destruction to reduce $\widetilde{\mathcal{R}}_i^2$), or it is possible to control the human-mosquito interaction (e.g. reduced contact rates, which is studied in the next section).

In order to develop verifiable threshold conditions guaranteeing disease eradication under mechanical destruction, we remind the reader of some dwell-time switching notions. Let $T_i(0,t)$ denote the total activation time of the i^{th} subsystem during the interval $[0,t]$. Define the sets

$$\widetilde{\mathcal{P}}_s = \{i \in \mathcal{P} : \frac{\eta_Q Q_{min}}{A_{max}}\widetilde{\mathcal{R}}_i^2 + \delta_i\gamma_i < 1\}$$

and

$$\widetilde{\mathcal{P}}_u = \{i \in \mathcal{P} : \frac{\eta_Q Q_{min}}{A_{max}} \widetilde{\mathcal{R}}_i^2 + \delta_i \gamma_i \geq 1\}.$$

Let $T^-(0,t)$ and $T^+(0,t)$ be the total time such that $\sigma(t) \in \widetilde{\mathcal{P}}_s$ and $\sigma(t) \in \widetilde{\mathcal{P}}_u$ on the interval $[0,t]$, respectively. We are now in a position to prove the first eradication result.

Theorem 6.2.2. *Assume that there exists a constant $q \geq 0$ such that $T^+(0,t) \leq qT^-(0,t)$ for all $t \geq 0$. Assume that $\bar{r} > 1$ and*

$$q\lambda^+ - \lambda^- < 0, \tag{6.20}$$

where

$$\lambda_i = \frac{1 + \widetilde{\mathcal{R}}_i^2}{2\widetilde{\mathcal{R}}_i^2} \frac{1}{\left(1 + \frac{\eta_Q Q_{min}}{\delta_i \gamma_i A_{max}}\right)} \left[\frac{\eta_Q Q_{min}}{A_{max}} \widetilde{\mathcal{R}}_i^2 + \delta_i \gamma_i - 1\right], \tag{6.21}$$

$\lambda^- = \min_{i \in \widetilde{\mathcal{P}}_s} |\lambda_i|$ *and* $\lambda^+ = \max_{i \in \widetilde{\mathcal{P}}_u} \lambda_i$. *Then the solution of system (6.9) converges to the set $\Psi_{mechanical}$ and hence the disease is eradicated.*

Proof. Consider the set of Lyapunov functions $V_i = a_i I_M + b_i I_H$ for $i \in \mathcal{P}$ where

$$a_i = \frac{1 + \widetilde{\mathcal{R}}_i^2}{2\left(\delta_i \gamma_i + \eta_Q \frac{Q_{min}}{A_{max}}\right)} \left(\frac{\delta_i \gamma_i}{\widetilde{\mathcal{R}}_i^2} + \eta_Q \frac{Q_{min}}{A_{max}}\right), \quad b_i = \eta_Q \frac{Q_{min} N_H}{A_{max}^2} \frac{1 + \widetilde{\mathcal{R}}_i^2}{2\beta_i}.$$

Take the time-derivative of the i^{th} Lyapunov function along system (6.9) for $i_k = i$,

$$\begin{aligned}\dot{V}_i =& a_i \left[-\left(\eta_Q \frac{Q}{A} + \delta_i \gamma_i I_H\right) I_M + \delta_i \gamma_i I_H + (d_S - d_I) I_M + (d_I - d_S) I_M^2\right] \\ &+ b_i \left[\beta_i \frac{I_M S_H A}{N_H} - (g + \mu) I_H\right].\end{aligned}$$

Since $\bar{r} > 1$, for any $\epsilon > 0$ there exists a $t^* > 0$ such that $E \geq E_{min} - \epsilon$, $Q \geq Q_{min} - \epsilon$ and $A \leq A_{max} + \epsilon$ for $t > t^*$. Also, since $d_I > d_S$ and $0 \leq I_M \leq 1$,

$$(d_S - d_I) I_M + (d_I - d_S) I_M^2 \leq (d_S - d_I) I_M + (d_I - d_S) I_M = 0.$$

Then

$$
\begin{aligned}
\dot{V}_i \leq & a_i[-(\eta_Q \left(\frac{Q_{min}-\epsilon}{A_{max}+\epsilon}\right) + \delta_i\gamma_i I_H)I_M + \delta_i\gamma_i I_H] \\
& + b_i \left[\beta_i I_M S_H \left(\frac{A_{max}+\epsilon}{N_H}\right) - (g+\mu) I_H\right], \\
= & - a_i\eta_Q \left(\frac{Q_{min}-\epsilon}{A_{max}+\epsilon}\right) I_M + a_i\delta_i\gamma_i(1-I_M)I_H \\
& + b_i\beta_i I_M S_H \left(\frac{A_{max}+\epsilon}{N_H}\right) - b_i(\mu+g)I_H, \\
= & \left[a_i\delta_i\gamma_i(1-I_M) - \eta_Q \frac{Q_{min}N_H}{A_{max}^2}\left(\frac{1+\widetilde{\mathcal{R}}_i^2}{2}\right)\left(\frac{g+\mu}{\beta_i}\right)\right] I_H \\
& + \left[\eta_Q \frac{Q_{min}}{A_{max}}\left(\frac{1+\widetilde{\mathcal{R}}_i^2}{2}\right) S_H - a_i\eta_Q\left(\frac{Q_{min}-\epsilon}{A_{max}+\epsilon}\right) + b_i\beta_i\frac{S_H}{N_H}\epsilon\right] I_M.
\end{aligned}
$$

Note that

$$
\frac{Q_{min}}{A_{max}+\epsilon} = \frac{Q_{min}}{A_{max}}\left[\frac{1}{1+\frac{\epsilon}{A_{max}}}\right] = \frac{Q_{min}}{A_{max}}\left[1+f_1(\epsilon)\right],
$$

where

$$
f_1(\epsilon) = \frac{\frac{-\epsilon}{A_{max}}}{1+\frac{\epsilon}{A_{max}}}.
$$

Hence,

$$
\begin{aligned}
-\eta_Q\left(\frac{Q_{min}-\epsilon}{A_{max}+\epsilon}\right) = & - \eta_Q\frac{Q_{min}}{A_{max}}\left[1+f_1(\epsilon)\right] + \eta_Q\frac{\epsilon}{A_{max}+\epsilon}, \\
\leq & - \eta_Q\frac{Q_{min}}{A_{max}} + \eta_Q\frac{Q_{min}}{A_{max}}\left(\frac{\epsilon}{A_{max}}\right) + \frac{\eta_Q\epsilon}{A_{max}}, \\
= & - \eta_Q\frac{Q_{min}}{A_{max}} + \eta_Q\frac{Q_{min}}{A_{max}} f_2(\epsilon),
\end{aligned}
$$

where

$$
f_2(\epsilon) = \epsilon\left(\frac{1}{A_{max}} + \frac{1}{Q_{min}}\right).
$$

Thus

$$\begin{aligned}
\dot{V}_i \leq & \left[a_i(1-I_M) - \eta_Q \frac{Q_{min}N_H}{A_{max}^2}\frac{(\mu+g)}{\beta_i\delta_i\gamma_i}\left(\frac{1+\widetilde{\mathcal{R}}_i^2}{2}\right)\right]\delta_i\gamma_i I_H \\
& + \left[\eta_Q\frac{Q_{min}}{A_{max}}\left(\frac{1+\widetilde{\mathcal{R}}_i^2}{2}\right)S_H - a_i\eta_Q\frac{Q_{min}}{A_{max}} + a_i\eta_Q\frac{Q_{min}}{A_{max}}f_2(\epsilon) + \frac{b_i\beta_i}{N_H}\epsilon\right] I_M, \\
= & \left[a_i(1-I_M) - \frac{1}{\widetilde{\mathcal{R}}_i^2}\left(\frac{1+\widetilde{\mathcal{R}}_i^2}{2}\right)\right]\delta_i\gamma_i I_H \\
& + \left[\left(\frac{1+\widetilde{\mathcal{R}}_i^2}{2}\right)S_H - a_i + a_i f_2(\epsilon) + \frac{b_i\beta_i A_{max}}{\eta_Q Q_{min}N_H}\epsilon\right]\eta_Q\frac{Q_{min}}{A_{max}}I_M, \\
\leq & \left[a_i - \frac{1}{2}\left(\frac{1}{\widetilde{\mathcal{R}}_i^2}+1\right)\right]\delta_i\gamma_i I_H \\
& + \left[\left(\frac{1+\widetilde{\mathcal{R}}_i^2}{2}\right) - a_i + a_i f_2(\epsilon) + \frac{b_i\beta_i A_{max}}{\eta_Q Q_{min}N_H}\epsilon\right]\eta_Q\frac{Q_{min}}{A_{max}}I_M, \\
= & \lambda_i(I_M+I_H) + G_i(\epsilon)I_M.
\end{aligned}$$

where

$$G_i(\epsilon) = a_i\eta_Q\frac{Q_{min}}{A_{max}}\left[f_2(\epsilon) + \frac{b_i\beta_i}{N_H}\epsilon\right].$$

Define

$$c = \frac{1}{\min\{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m\}}.$$

Then,

$$\begin{aligned}
\frac{1}{c}\frac{d}{dt}(I_M+I_H) &\leq \frac{d}{dt}(a_iI_M + b_iI_H), \\
&\leq \lambda_i(I_M+I_H) + G_i(\epsilon)I_M, \\
&\leq (\lambda_i + G_i(\epsilon))(I_M+I_H).
\end{aligned} \tag{6.22}$$

Let $N > 1$ be the smallest integer such that $t_{N-1} > t^*$, then for $t \in [t_{N-1}, t_N)$,

$$I_H(t) + I_M(t) \leq c(I_H(t_{N-1}) + I_M(t_{N-1}))\exp\left[(\lambda_{i_N} + G_{i_N}(\epsilon))(t - t_{N-1})\right].$$

For $t \in [t_N, t_{N+1})$,

$$\begin{aligned} I_H(t) + I_M(t) \leq & c(I_H(t_N) + I_M(t_N)) \exp\left[(\lambda_{i_{N+1}} + G_{i_{N+1}}(\epsilon))(t - t_N)\right], \\ \leq & c(I_H(t_{N-1}) + I_M(t_{N-1})) \exp[(\lambda_{i_N} + G_{i_N}(\epsilon))(t_N - t_{N-1}) \\ & + (\lambda_{i_{N+1}} + G_{i_{N+1}}(\epsilon))(t - t_N)], \end{aligned}$$

Since $0 \leq I_M \leq 1$ and $0 \leq I_H \leq N_H$, $0 \leq I_H(t_{N-1}) + I_M(t_{N-1}) \leq M$ where $M = 1 + N_H$. Then, considering a general time interval $t \in [t_{N-1+j}, t_{N+j})$, $j = 1, 2, \ldots,$

$$\begin{aligned} I_H(t) + I_M(t) \leq & cM \exp\left[\sum_{l=1}^{j-1} (\lambda_{i_{N+l}} + G_{i_{N+l}}(\epsilon))(t_{N+l} - t_{N-1+l})\right] \times \\ & \exp[(\lambda_{i_{N+j}} + G_{i_{N+j}}(\epsilon))(t - t_{N-1+j})]. \end{aligned}$$

Thus,

$$\begin{aligned} I_H(t) + I_M(t) \leq & cM \exp\left[\sum_{i=1}^{m} (\lambda_i + G_i(\epsilon)) T_i(t_N, t)\right], \\ = & cM \exp\left[\sum_{i \in \widetilde{\mathcal{P}}_s} (\lambda_i + G_i(\epsilon)) T_i(t_N, t) + \sum_{i \in \widetilde{\mathcal{P}}_u} (\lambda_i + G_i(\epsilon)) T_i(t_N, t)\right], \end{aligned}$$

and so,

$$\begin{aligned} I_H(t) + I_M(t) \leq & cM \exp\left[\sum_{i \in \widetilde{\mathcal{P}}_s} (-\lambda^- + G_i(\epsilon)) T_i(t_N, t) + \sum_{i \in \widetilde{\mathcal{P}}_u} (\lambda^+ + G_i(\epsilon)) T_i(t_N, t)\right], \\ = & cM \exp\left[-\lambda^- T^-(t_N, t) + \lambda^+ T^+(t_N, t) + \sum_{i=1}^{m} G_i(\epsilon) T_i(t_N, t)\right], \\ \leq & cM \exp\left[-\lambda^- T^-(t_N, t) + q\lambda^+ T^-(t_N, t) + \epsilon G_{max}(t - T_N)\right], \\ = & cM \exp\left[(-\lambda^- + q\lambda^+) T^-(t_N, t) + \epsilon G_{max}(t - T_N)\right], \end{aligned}$$

where

$$G_{max} = \max_{i=1,2,\ldots,m} a_i \eta_Q \frac{Q_{min}}{A_{max}} \left(\frac{1}{A_{max}} + \frac{1}{Q_{min}} + \frac{b_i \beta_i}{N_H}\right).$$

Then, since $t - t_N = T^-(t_N, t) + T^+(t_N, t) \leq (1 + q) T^-(t_N, t)$,

$$\begin{aligned} I_H(t) + I_M(t) \leq & cM \exp\left[(-\lambda^- + q\lambda^+)\left(\frac{t - t_N}{1 + q}\right) + \epsilon G_{max}(t - T_N)\right], \\ = & cM \exp\left[(-\lambda^- + q\lambda^+ + \epsilon G_{max}(1 + q))\left(\frac{t - t_N}{1 + q}\right)\right]. \end{aligned}$$

The condition $-\lambda^- + q\lambda^+ < 0$ implies the existence of a positive constant χ such that $-\lambda^- + q\lambda^+ \leq -\chi$. Choose $0 \leq \epsilon \leq \frac{1}{2}\frac{\chi}{G_{max}(1+q)}$, then $-\lambda^- + q\lambda^+ + \epsilon G_{max}(1+q) \leq \frac{-\chi}{2} < 0$ and it follows that I_H and I_M converge to zero. Then from the reduced system with $I_H = I_M = 0$, it is clear that R_H converges to zero and $S_H = 1 - I_H - R_H$ implies that S_H converges to one. Therefore the solution converges to the set $\Psi_{mechanical}$. □

Remark 6.2.7. *From the definitions of λ^+ and λ^-, it is apparent that the set $\widetilde{\mathcal{P}}_u$ represents the set of contact rates such that the disease may be spreading (unstable subsystems) while the set $\widetilde{\mathcal{P}}_s$ represents subsystems where the disease is being eradicated (stable subsystems). The condition $T^+(0,t) \leq qT^-(0,t)$ gives a relationship between the time spent in the unstable subsystems versus the stable subsystems and ensures it is such that overall the disease is dying out. Note that the threshold condition (6.20) depends on the mechanical destruction rate α via A_{max}, Q_{min}, and $\widetilde{\mathcal{R}}_i^2$.*

Remark 6.2.8. *The approximate basic reproduction numbers for each subsystem, $\widetilde{\mathcal{R}}_i^2$, are analytic approximations to $\mathcal{R}_0^2$ in equation (6.18) (rather than a numerical approximation, for example see [9]). Note that the approximations are overestimates since*

$$\mathcal{R}_0^2 \leq \max_{i=1,\ldots,m} \widetilde{\mathcal{R}}_i^2.$$

Motivated by seasonal fluctuations in the vector population, consider a switching rule that satisfies $t_k - t_{k-1} = \tau_k$ such that $\tau_{k+m} = \tau_k$. Assume that $b_{i_k} = b_k$, $\beta_{i_k} = \beta_k$, $\delta_{i_k} = \delta_k$, and $\gamma_{i_k} = \gamma_k$ on $[t_{k-1}, t_k)$. Suppose that $b_k = b_{k+m}$, $\beta_k = \beta_{k+m}$, $\delta_k = \delta_{k+m}$, and $\gamma_k = \gamma_{k+m}$. Denote $\omega = \tau_1 + \ldots + \tau_m$ to be one period of the switching rule. Denote the set of periodic switching rules by $\mathcal{S}_{\text{periodic}}$.

Theorem 6.2.3. *Assume that $\sigma \in \mathcal{S}_{periodic}$, $\bar{r} > 1$, and*

$$\Lambda_{mechanical} = \sum_{i=1}^{m} \lambda_i \tau_i < 0 \tag{6.23}$$

where λ_i is given in equation (6.21). Then the solution of system (6.9) converges to the set $\Psi_{mechanical}$ and hence the disease is eradicated.

Proof. Begin from equation (6.22) and let $N > 1$ be the smallest integer such that $t_{N-1} > t^*$ and $mod(N, m) = 0$. Then, similarly to the proof of Theorem 6.2.2, it is true that

$$I_H(t_{N-1} + \omega) + I_M(t_{N-1} + \omega) \leq c(I_H(t_{N-1}) + I_M(t_{N-1})) \exp\left[\sum_{l=1}^{m} (\lambda_i + G_i(\epsilon))\tau_i\right].$$

By equation (6.23) and since $G_i(\epsilon) \to 0$ as $\epsilon \to 0$, it is possible to choose $\epsilon > 0$ sufficiently small so that

$$\exp\left[\sum_{l=1}^{m}(\lambda_i + G_i(\epsilon))\tau_i\right] < \chi$$

for some constant $0 < \chi < 1$. Since $0 \leq I_M \leq 1$ and $0 \leq I_H \leq N_H$, it follows that $0 \leq I_H(t_{N-1}) + I_M(t_{N-1}) \leq M$ where $M = 1 + N_H$. Hence

$$0 \leq I_H(t_{N-1} + \omega) + I_M(t_{N-1} + \omega) \leq cM\chi.$$

Similarly,

$$0 \leq I_H(t_{N-1} + 2\omega) + I_M(t_{N-1} + 2\omega) \leq cM\chi^2.$$

In general,

$$I_H(t_{N-1} + h\omega) + I_M(t_{N-1} + h\omega) \leq cM\chi^h,$$

and so the sequence $\{I_H(t_{N-1} + h\omega) + I_M(t_{N-1} + h\omega)\}_{h=1}^{\infty}$ converges to zero. Since the solutions do not blow up on any interval of the form $[t_{k-1}, t_k)$, it follows that I_H and I_M converge to zero. From the reduced system with $I_H = I_M = 0$, it is clear that the solution converges to the set $\Psi_{mechanical}$. □

6.2.3 Public Campaign for Reduction in Contact Rate Patterns

As discussed at the beginning of Section 6.2, another possible control effort involves interrupting the interaction between humans and mosquitoes. Suppose that there is a public health campaign to reduce the contact rate between humans and mosquitoes at certain key times in the spread of the disease. This can be achieved by, for example, staying indoors during peak mosquito hours, reducing skin exposure, and using mosquito nets. Assume that the contact rates β_{i_k} and γ_{i_k} are reduced by a control factor $0 \leq \theta_{i_k} \leq 1$ on the interval $[t_{k-1}, t_k)$. Under this construction, there is a term-time forced reduction in the contact rates. Applying this to model (6.8) gives, for $t \in [t_{k-1}, t_k)$,

$$\begin{cases} \dot{E} = b_{i_k}\left(1 - \dfrac{E}{\Gamma_E}\right)A - (\eta_E + d_E)E, \\ \dot{Q} = \eta_E\left(1 - \dfrac{Q}{\Gamma_Q}\right)E - (\eta_Q + d_Q)Q, \\ \dot{A} = \eta_Q Q - d_I I_M A - d_S S_M A, \end{cases} \tag{6.24a}$$

$$
\begin{cases}
\dot{S}_H = \mu - \theta_{i_k}\beta_{i_k}\dfrac{S_H I_M A}{N_H} - \mu S_H, \\
\dot{I}_H = \theta_{i_k}\beta_{i_k}\dfrac{S_H I_M A}{N_H} - (g+\mu) I_H, \\
\dot{R}_H = g I_H - \mu R_H, \\
\dot{I}_M = -\left(\eta_Q \dfrac{Q}{A} + \theta_{i_k}\delta_{i_k}\gamma_{i_k} I_H + (d_S - d_I)(1 - I_M)\right) I_M + \theta_{i_k}\delta_{i_k}\gamma_{i_k} I_H.
\end{cases}
\tag{6.24b}
$$

where $i_k \in \{1, 2, \ldots, m\}$ follows a switching rule and the initial conditions are $E(0) = E_0$, $Q(0) = Q_0$, $A(0) = A_0$, $S_H(0) = S_{H0}$, $I_H(0) = I_{H0}$, $R_H(0) = R_{H0}$, and $I_M(0) = I_{M0}$. The physical domain is given by $\Omega_{reduced} = \Omega_{\text{Chiku}}$. System (6.24a) has the mosquito-free equilibrium $(E, Q, A) = (0, 0, 0)$ and m endemic equilibria

$$
(E, Q, A) = (E_i^*, Q_i^*, A_i^*) = \left(1 - \frac{1}{r_i}\right)\left(\frac{\Gamma_E}{\nu_i}, \frac{\Gamma_Q}{\kappa_i}, \frac{\eta_Q}{d_I}\frac{\Gamma_Q}{\kappa_i}\right), \tag{6.25}
$$

where r_i is given in equation (6.10) and

$$
\nu_i = 1 + \frac{(\eta_E + d_E) d_I \Gamma_E}{b_i \eta_Q \Gamma_Q}, \qquad \kappa_i = 1 + \frac{(\eta_Q + d_Q)\Gamma_Q}{b_i \eta_E \Gamma_E}.
$$

Define the minimum and maximum endemic equilibria

$$
E_{max} = \max_{i=1,\ldots,m} E_i^*, \quad E_{min} = \min_{i=1,\ldots,m} E_i^*,
$$

with Q_{max}, Q_{min}, A_{max}, and A_{min} defined similarly. Then if $\bar{r} > 1$, with $\bar{r}$ defined in (6.11), it follows similarly to the proof of Proposition 6.2.1 that the solution of (6.24a) converges to the set

$$
\begin{aligned}
\Delta_{reduced} = \{(E, Q, A) \in \mathbb{R}_+^3 |\ & E_{min} \le E \le E_{max},\ Q_{min} \le Q \le Q_{max}, \\
& A_{min} \le A \le A_{max}\}.
\end{aligned}
\tag{6.26}
$$

Remark 6.2.9. *The values of E_{max}, E_{min}, Q_{max}, Q_{min}, A_{max}, and A_{min} defined above are different from those in Section 6.2.2 due to the fact that $\alpha = 1$ here (since there is no mechanical control applied in this scheme).*

Define the following approximate basic reproduction numbers for each subsystem of (6.24)

$$
\widehat{\mathcal{R}}_i^2 = \frac{\theta_i^2 \beta_i \delta_i \gamma_i}{\eta_Q (g + \mu)} \frac{A_{max}^2}{Q_{min} N_H}, \tag{6.27}
$$

for $i \in \mathcal{P}$. Define the sets

$$\widehat{\mathcal{P}}_s = \{i \in \mathcal{P} : \frac{\eta_Q Q_{min}}{A_{max}} \widehat{\mathcal{R}}_i^2 + \theta_i \delta_i \gamma_i < 1\}$$

and

$$\widehat{\mathcal{P}}_u = \{i \in \mathcal{P} : \frac{\eta_Q Q_{min}}{A_{max}} \widehat{\mathcal{R}}_i^2 + \theta_i \delta_i \gamma_i \geq 1\}.$$

Then the following result can be given.

Theorem 6.2.4. *Assume that there exists a constant $q \geq 0$ such that $T^+(0,t) \leq qT^-(0,t)$ for all $t \geq 0$. Assume that $\bar{r} > 1$ and*

$$q\lambda^+ - \lambda^- < 0, \tag{6.28}$$

where

$$\lambda_i = \frac{1 + \widehat{\mathcal{R}}_i^2}{2\widehat{\mathcal{R}}_i^2} \frac{1}{\left(1 + \frac{\eta_Q Q_{min}}{\theta_i \delta_i \gamma_i A_{max}}\right)} \left[\frac{\eta_Q Q_{min}}{A_{max}} \widehat{\mathcal{R}}_i^2 + \theta_i \delta_i \gamma_i - 1\right], \tag{6.29}$$

$\lambda^- = \min_{i \in \widehat{\mathcal{P}}_s} |\lambda_i|$ and $\lambda^+ = \max_{i \in \widehat{\mathcal{P}}_u} \lambda_i$. Then the solution of system (6.24) converges to the set

$$\begin{aligned} \Psi_{reduced} = \{(E, Q, A, I_M, S_H, I_H, R_H) \in \mathbb{R}_+^7 |\ (E, Q, A) \in \Delta_{reduced}, \\ S_H = 1,\ I_H = 0,\ R_H = 0,\ I_M = 0\}. \end{aligned} \tag{6.30}$$

and hence the disease is eradicated.

Proof. Similar to the proof of Theorem 6.2.2. □

In the case that the reduced contact rate is periodic, that is, $\theta_{i_k} = \theta_k$ on $[t_{k-1}, t_k)$, and $\theta_k = \theta_{k+m}$, the threshold condition for eradication is found as follows.

Theorem 6.2.5. *Assume that $\sigma \in \mathcal{S}_{periodic}$, $\bar{r} > 1$, and*

$$\Lambda_{reduced} = \sum_{i=1}^{m} \lambda_i \tau_i < 0 \tag{6.31}$$

where λ_i is given in equation (6.29). Then the solution of system (6.24) converges to the set $\Psi_{reduced}$ and hence the disease is eradicated.

Proof. Similar to the proof of Theorem 6.2.3. □

6.2.4 Control Strategy Efficacy Ratings

Here we compare and contrast the control strategies outlined above using a control efficacy measure introduced in [41]: let

$$F_0 = 100\frac{C_H^c}{C_H^0},$$

where C_H^c and C_H^0 are the cumulative number of infected humans with control and without control, respectively. F_0 measures the efficacy of the control scheme on suppressing the total number of humans infected as it represents how many fewer humans would be infected in an outbreak by using the particular control strategy. A low value of F_0 represents a very successful control method (with $F_0 = 0$ being perfect suppression), while a high value of F_0 means the scheme is less successful (with $F_0 = 100$ being total failure of the scheme).

For the simulations, the initial time, $t_0 = 0$, is assumed to coincide with the beginning of the dry season in March 2004 (approximately one year before the outbreak on Reunion Island). For the initial conditions we consider $E_0 = c_1 \times N_H$, $Q_0 = c_1 \times N_H$, $A_0 = c_2 \times N_H$, $S_{H0} = N_H$, $S_{M0} = c_2 \times N_H$ where $N_H = 136000$ is the population of the capital, Saint-Denis. The parameters are chosen to be $c_1 = 2$ and $c_2 = 5$ (as in [41], we focus on the outbreak in the capital).

Assume that the per capita number of eggs at each deposit (per day) is a switching parameter modelled by the switching rule $i_k = \sigma \in \{1, 2\}$ where

$$\sigma = \begin{cases} 1 & \text{if } t \in [365k, 365(k + \frac{7}{12})),\ k = 0, 1, 2, \ldots \\ 2 & \text{if } t \in [365(k + \frac{7}{12}), 365(k + 1)). \end{cases} \tag{6.32}$$

Note that the switching rule is periodic with $\tau_1 = \frac{7}{12} \times 365$ (dry season), $\tau_2 = \frac{5}{12} \times 365$ (rainy season) and $\omega = 365$ (one period). That is, $b = b_{dry}$ whenever $\sigma = 1$ (dry season) and $b = b_{rainy}$ when $\sigma = 2$ (rainy season). The entomological model parameters are given in Table 6.2. The time-average of the dry season and rainy season eggs per day is given by $b_{dry} \times \tau_1 + b_{rainy} \times \tau_2 = b$, hence the choice of b_{dry} and b_{rainy}.

Next we consider the epidemiological parameters of the models (see Table 6.3). Assume that the contact rates β and γ follow the seasonal switching rule (6.32). Note that $\beta_{dry} \times \tau_1 + \beta_{rainy} \times \tau_2 = \beta$ and $\gamma_{dry} \times \tau_1 + \gamma_{rainy} \times \tau_2 = \gamma$. As discussed earlier, the vector Aedes albopictus is sensitive to weather conditions (for example, temperature and humidity). Changes in seasonal weather patterns (such as dry season versus rainy season) also have an effect on the behaviour of the human population (for example, individuals are more active outside during the dry season in the late afternoon when Aedes albopictus is active,

Parameter	Description	Average value	Source
N_H	total human population in Saint-Denis	136000	[42]
b	per capita number of eggs at each deposit (per day)	6	[41]
b_{dry}	per capita number of eggs at each deposit (per day) in the dry season	3.27	
b_{rainy}	per capita number of eggs at each deposit (per day) in the rainy season	9.82	
Γ_Q	carrying capacity of aquatic mosquito population	$2 \times N_H$	[41]
Γ_E	carrying capacity of embryonic mosquito population	$2 \times N_H$	
η_Q	rate of maturation from embryonic to aquatic (per day)	0.1	[42]
η_E	rate of maturation from aquatic to adult (per day)	0.1	
d_Q	aquatic stage natural mortality rate (per day)	0.25	[42]
d_E	embryonic stage natural mortality rate (per day)	0.25	[42]
$1/\mu$	natural lifespan of human (days)	78×365	[41]

Table 6.2: Entomological and human demographic parameters.

skin exposure is higher during the dry season versus the rainy season due to clothing). Motivated by this, assume that $\beta_{dry} > \beta_{rainy}$ and $\gamma_{dry} > \gamma_{rainy}$ to model a higher pattern of average contacts during the dry season. We introduce the virus into the simulations around March 2005 (which occurs at $t = 365$ in our simulations). Further, as mentioned at the beginning of Section 6.2.1, there was a genetic mutation in the virus approximately 30 weeks after March 2005 ($t = 575$) which shifted γ from 0.375 to about 0.95 [41]. Hence, for the switching mutation parameter we assume that

$$\delta_{i_k} = \begin{cases} 1 & \text{if } t < 575, \\ 2.53 & \text{if } t \geq 575. \end{cases} \tag{6.33}$$

Then $\delta_{i_k}\gamma_{dry} \times \tau_1 + \delta_{i_k}\gamma_{rainy} \times \tau_2 = 0.375$ for $t < 575$ and $\delta_{i_k}\gamma_{dry} \times \tau_1 + \delta_{i_k}\gamma_{rainy} \times \tau_2 = 0.95$ for $t \geq 575$. We are now in a position to numerically analyze each control scheme in detail.

Parameter	Description	Average value	Source
d_S	natural death rate of susceptible adult mosquitoes (per day)	0.1	[41]
d_I	natural death rate of infected adult mosquitoes (per day)	0.2	[41]
β	contact rate resulting in human infection (per day)	0.375	[41]
β_1	contact rate resulting in human infection in dry season (per day)	0.4737	
β_2	contact rate resulting in human infection in rainy season (per day)	0.2368	
γ	contact rate resulting in mosquito infection (per day)	0.375	[41]
γ_1	contact rate resulting in mosquito infection in dry season (per day)	0.4737	
γ_2	contact rate resulting in mosquito infection in rainy season (per day)	0.2368	
g	human natural recovery rate (per day)	3	[41]

Table 6.3: Epidemiological parameters.

Assessment of Mechanical Destruction of Breeding Sites

Consider model (6.9) where mechanical destruction of breeding sites occurs from a public campaign to clean, remove, and destroy water receptacles. As in [41] we consider starting the scheme at different times, with respect to the timing of the outbreak, denoted by t_c, and for different durations, denoted by h. We also consider adjusting the rate of destruction of breeding sites by varying α.

Recall that the spread of the virus occurs around $t = 365$ days and begin by considering initiating the control scheme at $t_c = 250$ and continuing it for a duration of $h = 150$ (until $t = 400$). Consider $\alpha = 0.49$, so that about half of the breeding sites are destroyed. Note that $\bar{r} = 1.34$ in this case which implies persistence of the mosquito population by Proposition 6.2.1. After the genetic mutation and while the control scheme is being applied the thresholds are given by $\widehat{\mathcal{R}}_1^2 = 3.94$ (which corresponds to the dry season) and $\widehat{\mathcal{R}}_2^2 = 2.50$ (which corresponds to the rainy season). Then $\Lambda_{\text{mechanical}} = -78.54$ and hence the disease is eradicated by Theorem 6.2.3. See Figure 6.3 for a simulation of the model.

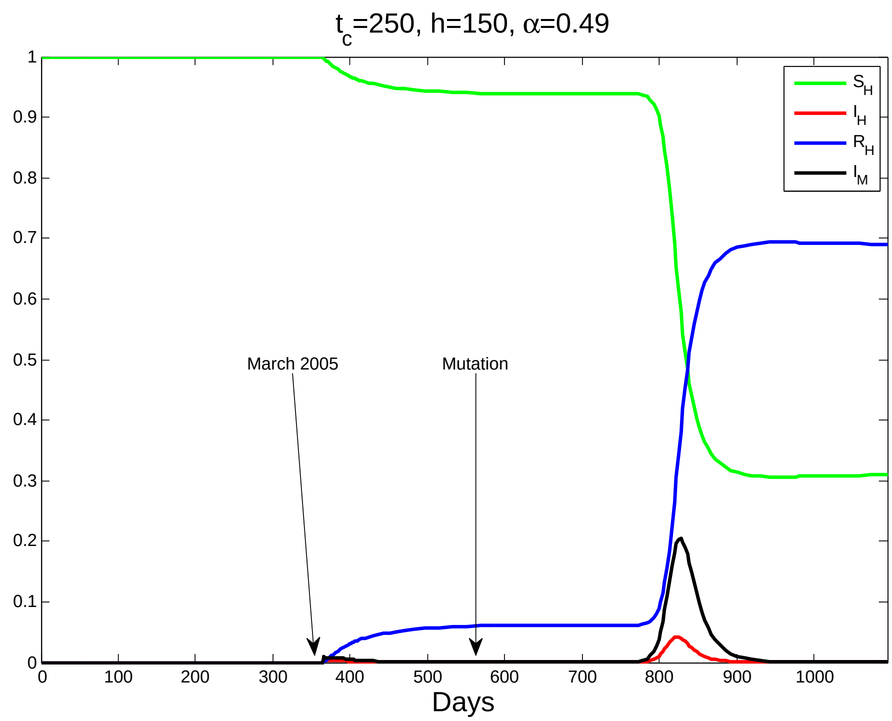


Figure 6.3: Mechanical destruction model (6.9).

The outbreak on Reunion began with the first wave in May 2005, followed by a much larger epidemic wave in January and February 2006. In their paper [41], Dumont and Chiroleu simulated a two-wave outbreak, separated by approximately 40 weeks, with the first wave being much smaller than the second wave. From Figure 6.4 it is apparent that after an initial small outbreak, there is a much stronger epidemic wave that forms later on (approximately 66 weeks later). The mechanical destruction, which decreases the number of aquatic mosquitoes, seems to cause a reduction in the strength of the second epidemic wave (peak value of approximately 5800, compared to about 13000 in the simulations in [41]). Further, it seems to also cause a delay between the two epidemic waves.

Remark 6.2.10. *In the modelling efforts here, we assumed that mechanical destruction only affects the aquatic stage, Q, and not the embryonic stage, E (see Remark 6.2.1). This is reflected in the fact that the carrying capacity Γ_Q is reduced to $\alpha\Gamma_Q$ while Γ_E remains unaffected by the mechanical control. However, since the females lay eggs in the containers, if the mechanical control for a particular container includes the actual destruction/removal*

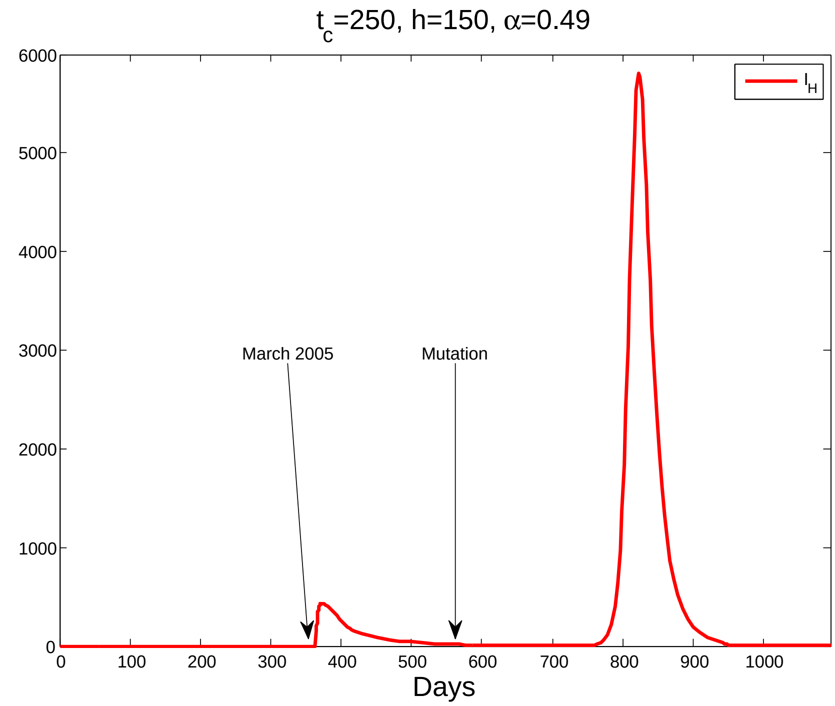
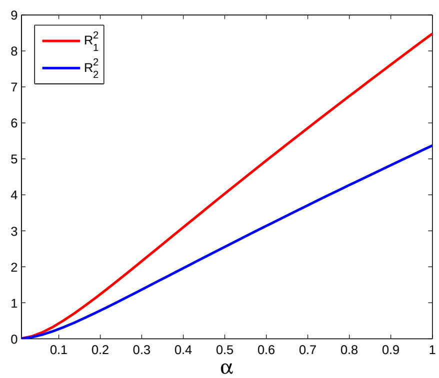


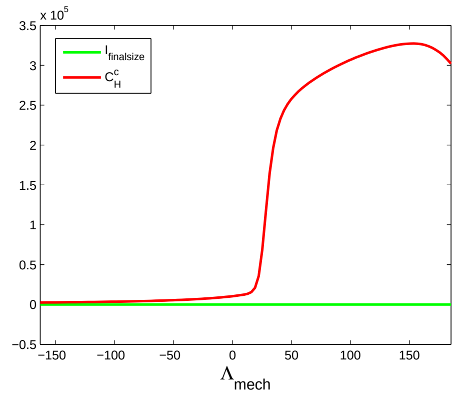
Figure 6.4: Total number of infected humans in model (6.9) for various control rates α.

of the container, this would also reduce capacity of any eggs deposited, and hence affect Γ_E*. For example, in the paper [41], the authors Dumont et al. only consider one compartment for the aquatic/embryonic stage, and hence assume that mechanical control affects both stages of life for the vector. This is also a possible explanation for the delay between the epidemic waves mentioned above.*

The relationship between the approximate basic reproduction numbers $\widetilde{\mathcal{R}}_i^2$ and α can be seen in Figure 6.5a. As expected, $\widetilde{\mathcal{R}}_1^2 > \widetilde{\mathcal{R}}_2^2$ for all values of α due to increased human-mosquito contact during the dry season. To further investigate the eradication condition in Theorem 6.2.3, consider Figure 6.5b, which illustrates the final size of the epidemic, $I_{\text{finalsize}}$, and the cumulative number of infected humans under mechanical control, C_H^c, for varying levels of $\Lambda_{\text{mechanical}}$. It is apparent that the disease is eradicated whenever $\Lambda_{\text{mechanical}} < 0$, however it also seems that the disease can still be eradicated when this condition does not hold (hence Theorem 6.2.3 is sufficient but not necessary). As $\Lambda_{\text{mechanical}}$ decreases, the total number of infected humans also decreases and there is a transition around $\Lambda_{\text{mechanical}} \approx 25$ where the cumulative infected humans decreases sharply.



(a) Values of $\widetilde{\mathcal{R}}_1^2$ (dry season) and $\widetilde{\mathcal{R}}_2^2$ (rainy season) for varying values of α.



(b) Final number of infected humans and cumulative infected humans.

Figure 6.5: Mechanical destruction model (6.9).

The timing (t_c), duration (h), and strength (α) of the destruction effort play an important role in the dynamics of the mosquito population (see Figure 6.6).

To illustrate the affects of adjusting the control parameters, we consider how they affect the total number of infected humans by calculating the control efficacy number F_0. From Figure 6.7, the importance of α, t_c, and h is apparent: if the duration is short, the scheme is not effective regardless of the start time. If the scheme is initiated too early, the only way the scheme is successful is if the duration is very long ($h = 365$). If the campaign is sufficiently long ($h = 150$ or $h = 365$) and is started immediately before the first outbreak ($t_c = 350$) or soon after ($t_c = 400$), then the control strategy is quite effective with $F_0 < 20$ for mechanical destruction rates of $\alpha < 0.4$. Note that there are sharp decreases in the efficacy rate F_0 at particular values of α for the above mentioned successful cases. This is important from a cost-benefit perspective since decreasing α slightly can cause a significant improvement in the efficacy rate.

Assessment of Reduced Contact Rates

Consider model (6.24) where the interaction between humans and mosquitoes is interrupted for a period of time. Consider the following possibilities:

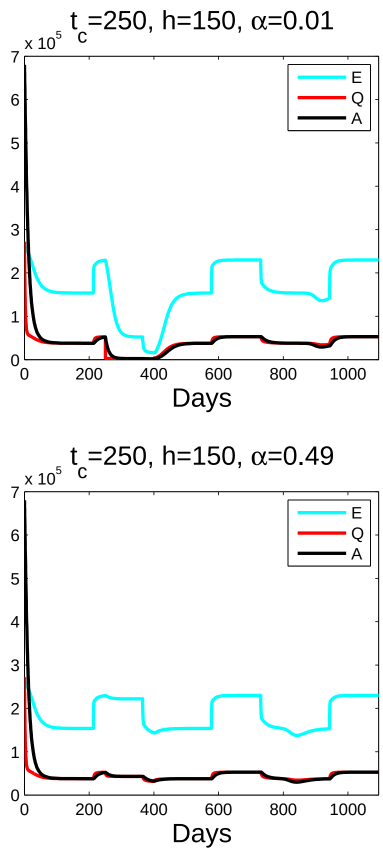


Figure 6.6: Dynamics of the mosquito population for the mechanical destruction model (6.9).

(i) different reduction values (varying θ_i);

(ii) different timings for commencement of the strategy (denoted by t_c); and

(iii) different durations for the period of reduction (denoted by h).

We consider altering t_c and h by assuming that θ_{i_k} follows the switching rule σ outlined as

$$\theta_{i_k} = \begin{cases} \theta_1 = 1 & \text{if } t < t_c \text{ or } t > t_c, \\ \theta_2 & \text{if } t_c \leq t \leq t_c + h. \end{cases} \tag{6.34}$$

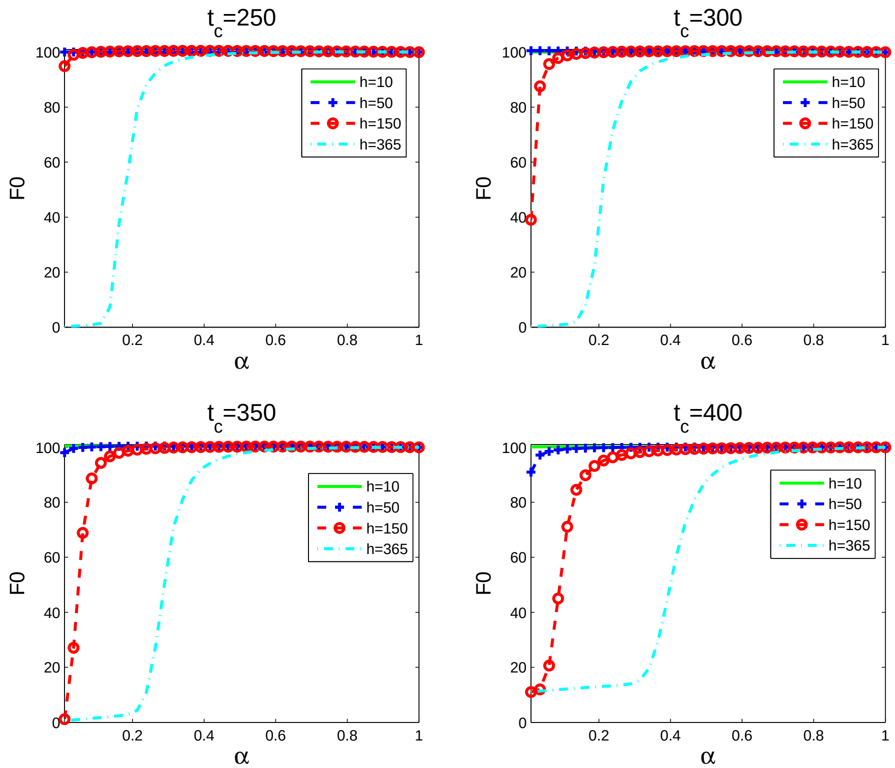


Figure 6.7: The efficacy measure F_0 for different values of the destruction rate α under the mechanical destruction control strategy (6.9).

If $\theta_2 = 0.64$ and if the genetic mutation has occurred then for the duration of the control scheme, the thresholds can be calculated as $\widehat{\mathcal{R}}_1^2 = 3.47$ and $\widehat{\mathcal{R}}_2^2 = 2.20$. Further, $\Lambda_{\text{reduced}} = -116.75$ and hence the disease is eradicated by Theorem 6.2.5. See Figure 6.8 for simulations of the cumulative infected humans for different values of θ_2.

To see how θ_2 affects the approximate basic reproduction numbers $\widehat{\mathcal{R}}_i^2$, see Figure 6.9a. The final size of the epidemic, $I_{\text{finalsize}}$, and the cumulative number of infected humans, C_H^c, for varying levels of Λ_{reduced} under the reduced contact rates strategy with $\theta_2 = 0.64$ can be seen in Figure 6.9b. It is apparent that the disease dies out when $\Lambda_{\text{mechanical}} < 0$, but the condition is sufficient and not necessary. Further, as $\Lambda_{\text{mechanical}}$ increases, the total number of infected humans increases which is undesirable.

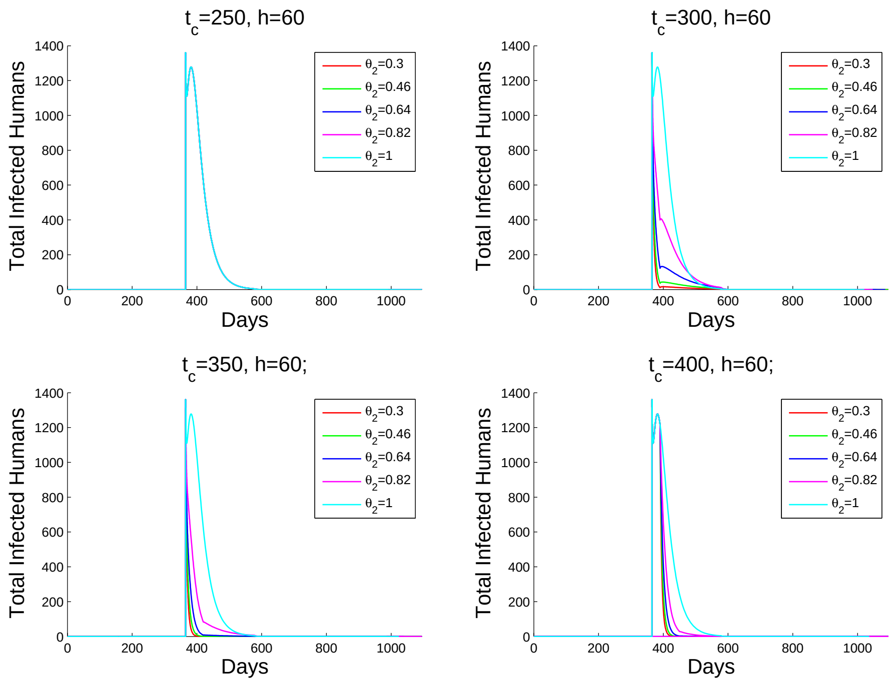
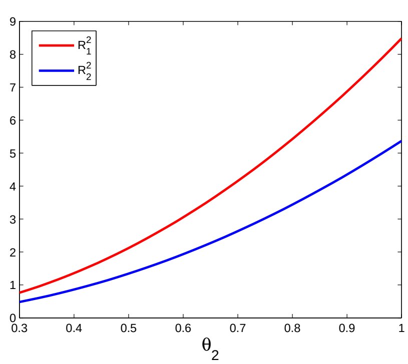
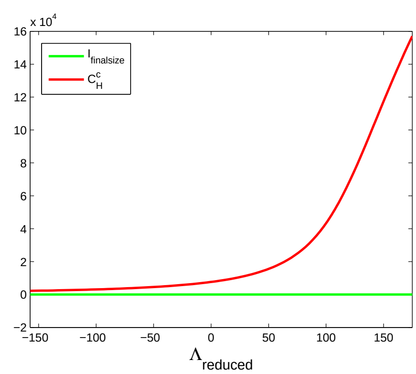


Figure 6.8: Total infected humans under the reduced contact rate strategy (6.24).



(a) Values of $\widehat{\mathcal{R}}_1^2$ (dry season) and $\widehat{\mathcal{R}}_2^2$ (rainy season) for varying values of θ_2.



(b) Final number of infected humans and cumulative infected humans for $\theta_2 = 0.64$.

Figure 6.9: Reduced contact rate model (6.24).

Unsurprisingly, the timing (t_c), duration (h), and strength (θ_2) play an important role in the dynamics of the disease spreading (see Figure 6.10). If the reduced contact strategy is initiated too early then the scheme is useless from an efficacy perspective unless the duration is sufficiently long. In fact, if $t_c = 250$ then only $h = 90$ is successful in controlling the disease, which may be unrealistically long for such an intrusive strategy. If the start time is after the outbreak, $t_c = 400$, then no strategy can achieve an efficacy rate below 40, but, importantly, the duration is not as vital. By far the most effective approach ($F_0 \approx 10$) is to initiate it immediately before the outbreak at $t_c = 350$ and for a duration of 30 days (60 and 90 achieve similar results). Unlike the mechanical destruction efficacy analysis, there are no sharp decreases in F_0 for small increases in the control rate, and so the best approach from a cost-benefit point of view is not as obvious.

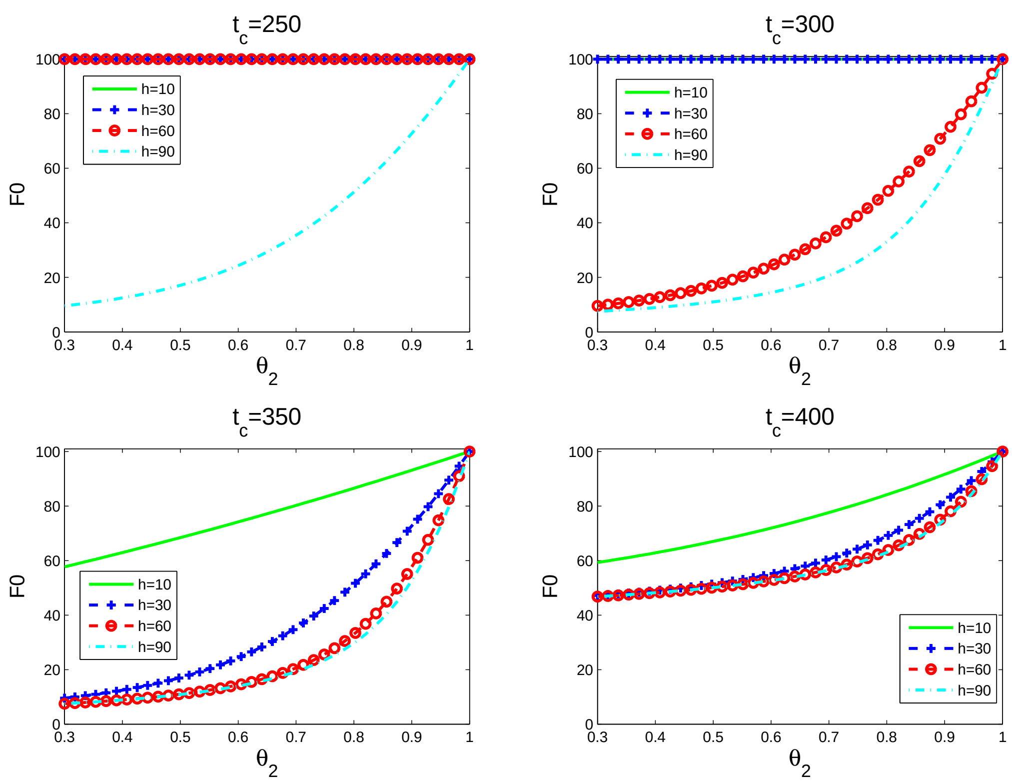


Figure 6.10: The efficacy measure F_0 for the reduced contact rate scheme.

6.2.5 Discussion

We are now in a position to make some observations and draw some conclusions regarding the control strategies outlined above.

1. Neither of the strategies result in a control efficacy greater than 100, which is expected (neither of the schemes result in more total infections than no strategy being applied at all).

2. If either the mechanical destruction scheme or the reduced contact scheme are initiated too early, then the other control parameters must be at the upper end of their ranges to compensate. For example, the mechanical scheme requires $h = 365$ and $\alpha < 0.2$ to achieve $F_0 < 50$ if $t_c = 250$, which might not be realistic (a public campaign of 80% breeding site destruction lasting a year). Similarly, the reduced contact strategy would require $h = 90$ for this starting time, which might also be unrealistic (three months of reducing human-mosquito interactions). If the durations are low, the schemes seem to be totally ineffective when initiated early.

3. If the mechanical destruction scheme is applied for a short duration ($h = 10$ or $h = 50$), the scheme is not successful at all ($F_0 \approx 100$) regardless of the destruction rate α. However, if the reduced contact rate strategy has a short duration, the scheme can still be effective if it is initiated near the epidemic outbreak ($t_c = 350$ or $t_c = 400$).

4. If contact rates are reduced before an outbreak ($t_c = 350$), excellent efficacy rates can be achieved (such as $F_0 < 20$) for reasonable control rates (such as $\theta_2 \approx 0.8$). Unfortunately, since this scheme's initiation would most likely be in response to an impending epidemic (and hence lag any outbreak in the mosquito population), $t_c = 400$ is more realistic. In this scenario some reasonable contact rate reduction levels (e.g. $\theta_2 \approx 0.8$) can lead to decent efficacy rates ($F_0 \approx 55$) while only requiring a duration of 30 days for the strategy.

5. In general, the mechanical destruction strategy requires the control rate α to be exceptionally low and the duration h to be large to achieve a desirable control efficacy (e.g. $F_0 < 50$). Although this seems quite undesirable, the comparatively low socio-economic cost of this strategy when compared to reduced contact rates might make up for this.

6. The observation that the mechanical strategy seems to do well when initiated after the outbreak ($t_c = 400$) if the duration is sufficiently long ($h = 365$) may be related to

the delay in the epidemic peak mentioned earlier. This warrants further investigation (possibly from an optimal control point of view).

7. None of the above analyses factor in the socio-economic cost of the control strategies. For example, as briefly mentioned above, mechanical destruction of breeding sites can be relatively cheap since it can be made up of a public-driven campaign. However, the reduced contact rate strategy may be quite restrictive and intrusive to the daily lives of the human population.

From these notes, it seems that the best course of action to combat future Chikungunya outbreaks on Reunion Island or other similar regions is to commence public campaigns of mechanical destruction of breeding sites in conjunction with a reduction in contact rate strategy in response to an outbreak. Since mechanical destruction may be comparatively cheap, the length and destruction rate should be made as high as realistically possible. In addition, a reduced contact rate strategy should be commenced immediately after an outbreak with a high reduction rate (low value of θ_2) for a short duration (e.g. $h = 10$ days), followed by a period of longer duration with a lower reduction rate (higher value of θ_2).

6.3 Vaccination Schemes for a Vector-borne Disease Model with Incubation

The Chikungunya virus is usually transmitted by Aedes aegypti, however, in more recent outbreaks it has been observed that the virus can also be transmitted by Aedes albopictus, as was the case in the Reunion Island epidemic [141]. This is notable since Aedes aegypti, which is also responsible for transmitting diseases such as Dengue, is a tropical and subtropical species, but the Aedes albopictus has developed capabilities to adapt to non-tropical regions in the last two decades and is now found in Southeast Asia, the Pacific and Indian Ocean islands, Japan, China, and more recently in Europe, USA, and Australia [41, 140, 141]. There was an outbreak recently in a region in Italy in 2007 [159], an example of the possible globalization of Chikungunya virus in temperate regions. There has been much focus in the literature on studying mathematical models for the spread of vector-borne diseases such as Dengue (for example, [195, 196]) and Chikungunya (see the reports discussed in the previous section).

The focus of this section is on a general epidemic model for a disease which spreads by vectors and displays a finite incubation time before becoming infectious (e.g. see

[21–23, 30, 53, 132, 135, 176]). Two important complications to the aforementioned modelling efforts are considered here: the first is the addition of seasonal effects to the model by considering model parameters that experience abrupt changes in time. The second complication we consider is the application of vaccination control strategies to suppress or eradicate the disease. Specifically, we analyze three control schemes: cohort immunization, time-dependent pulse vaccination, and state-dependent pulse vaccination. The first strategy entails a vaccination effort of susceptible individuals that is switching in time, while the second and third strategies involve vaccinating a significant fraction of the susceptible population in a short period of time. The pulse vaccination schemes considered have impulsive vaccination campaigns occurring at pre-specified times (time-dependent) or when the susceptible population reaches a critical threshold (state-dependent).

There have been numerous studies on time-constant control schemes (such as cohort immunization) in the epidemic literature, for example see [5, 75, 108, 123, 125] and the references therein. Time-dependent pulse vaccination, which has gained prominence in recent years for achieving disease eradication at lower vaccination levels than conventional time-constant control [1], was first proposed and studied mathematically by Shulgin et al. in [1]. Since then this approach has been further developed for many different types of models, for example, in [37, 51, 52, 137, 150, 169, 170, 203]. In contrast, state-dependent pulse vaccination strategies have been studied less extensively in the literature (for example, Nie et al. [148] proposed and analyzed such a control scheme for an SIR model).

To the best of the author's knowledge there has been no work done on the control of a vector-borne disease modelled with distributed delays and abruptly changing model parameters. Thus, the main aim of the present section is to extend the current research on this topic by studying the stability of a general vector-borne disease model. In doing so, critical thresholds for the control rates are found which guarantee eradication. From physical considerations, we also consider waning immunity and vaccine failure. By comparing the control schemes from an analytic and numerical perspective, we hope to be able to make some conjectures regarding an appropriate response to an impending epidemic from a cost-benefit perspective. The material in this section formed the basis for [121].

6.3.1 A Slow Time Scale Model Formulation

For the human population, we consider an SIR compartmental model (as in the Chikungunya model in the previous section). Assume that the vector population is split into the susceptible vectors, denoted by S_M, and the infected vectors, I_M. The following vital dynamic and epidemiological assumptions are made [22, 176]:

(A1) For the human vital dynamics, assume that the birth rate, $\mu_H > 0$, is equal to the natural death rate. Hence the total human population, $N = S + I + R$, is constant in time. Assume that the birth/death rate of the vector population is a constant $\mu_M > 0$ so that the total vector population $A = S_M + I_M$ is constant in time.

(A2) Let $\beta_H > 0$ denote the average number of contacts sufficient for transmission per unit time that susceptible humans make with infected vector agents. Similarly, denote by $\beta_M > 0$ to be the average number of contacts between infected humans and susceptible vectors.

(A3) Infected humans recover from the disease naturally at a rate $g_H > 0$ and move to the recovered class. Assume that once infected, a vector remains infected until it dies.

(A4) When infected, a susceptible vector exhibits a period of incubation, $u > 0$, before becoming infectious.

(A5) The time scale of vector vital dynamics is much faster than that of the human vital dynamics. More specifically, assume that $\epsilon = N/A \ll 1$ which implies that μ_M is significantly larger than μ_H.

Then the disease model can be written as follows:

$$\begin{cases} \dot{S} = \mu_H(N - S(t)) - \beta_H S(t) I_M(t), \\ \dot{I} = \beta_H S(t) I_M(t) - (g_H + \mu_H) I(t), \\ \dot{R} = g_H I(t) - \mu_H R(t), \\ \dot{S}_M = \mu A - \beta_M \exp(-\mu_M u) I(t-u) S_M(t-u) - \mu_M S_M(t), \\ \dot{I}_M = \beta_M \exp(-\mu_M u) I(t-u) S_M(t-u) - \mu_M I_M(t). \end{cases} \tag{6.35}$$

From Assumption $(A5)$, it is possible to consider two dimensionless time scales: a slow time scale associated with the human vital dynamics ($t_H = \beta_M N t$) and a fast time scale associated with the vector vital dynamics ($t_M = \beta_M A t$). By considering the evolution on the slow time scale, it is possible to re-write (6.35) as a system of delay differential equations as follows (see [176] for the details): introduce the dimensionless variables $s(t) = S(t)/N$, $i(t) = I(t)/N$, $r(t) = R(t)/N$, $s_M(t) = S_M(t)/A$, $i_M(t) = I_M(t)/A$. On the fast time scale the dimensionless vector equations become

$$\begin{aligned} \frac{ds_M}{dt_M} &= -\frac{di_M}{dt_M}, \\ \frac{di_M}{dt_M} &= \epsilon \left(\exp(-\mu_M u) i(t-u) s_M(t-u) - \frac{\mu_M}{\beta_M N} i_M(t) \right), \end{aligned} \tag{6.36}$$

where $s_M + i_M = 1$ and $s + i + r = 1$ for all $t \geq t_0$. From (6.36),

$$-\frac{\epsilon\mu_M}{\beta_M N} \leq \frac{di_M}{dt_M} \leq \epsilon \exp(-\mu_M u). \tag{6.37}$$

In the limit $\epsilon \to 0$ on the fast time scale,

$$\frac{ds_M}{dt_M} = -\frac{di_M}{dt_M} = 0$$

so that i_M and i_S attain their equilibrium values:

$$\begin{aligned} i_M(t) &= \frac{\beta_M N}{\mu_M}\exp(-\mu_M u)i(t-u)s_M(t-u), \\ s_M(t) &= 1 - i_M(t), \end{aligned} \tag{6.38}$$

and the vector fractions approach equilibrium since $i(t-u)$ can be regarded as a constant on the fast time scale. If $\frac{\beta_M N}{\mu_M}\exp(-\mu_M u) << 1$ then $s_M(t) \approx 1$. Hence,

$$i_M(t) \approx \frac{\beta_M N}{\mu_M}\exp(-\mu_M u)i(t-u)$$

so that $S_M(t) \approx N$ and

$$I_M(t) \approx \frac{\beta_M A \exp(-\mu_M u)}{\mu_M} I(t-u)$$

where $I(t-u)$ evolves on the slow time scale and can be viewed as a constant here. By omitting S_M and I_M since they no longer appear in the other equations, and normalizing the variables by the constant population, the disease model (6.35) can be re-written on the slow time scale as

$$\begin{cases} \dot{S} = \mu(1 - S(t)) - \beta S(t)I(t-u), \\ \dot{I} = \beta S(t)I(t-u) - (g+\mu)I(t), \\ \dot{R} = gI(t) - \mu R(t), \end{cases} \tag{6.39}$$

where

$$\beta = \frac{\beta_H A \exp(-\mu_M u)}{\mu_M}, \qquad g = \frac{g_H}{\beta_M N}, \qquad \mu = \frac{\mu_H}{\beta_M N}.$$

The force of infection in (6.39) is given by $\beta S(t)I(t-u)$. A more realistic assumption is that the period of incubation u follows a distribution, that is, $u \in [0, \tau]$, where $\tau > 0$ represents an upper bound for the incubation time in a vector [176]. Assume that after u

time units, a fraction $f(u)$ of the vector population becomes infectious. Then the force of infection is

$$\beta S(t)\int_0^\tau f(u)I(t-u)du.$$

The function $f(u)$ is assumed to be a nonnegative, square integrable function on $[0,\tau]$ which satisfies $\int_0^\tau f(u)du = 1$ (normalized) and $\int_0^\tau uf(u)du < \infty$ (finite average incubation time in the vector to become infectious) [23]. The general vector-borne disease model is a system of integro-differential equations,

$$\begin{cases} \dot{S} = \mu(1-S(t)) - \beta S(t)\displaystyle\int_0^\tau f(u)I(t-u)du, \\ \dot{I} = \beta S(t)\displaystyle\int_0^\tau f(u)I(t-u)du - (g+\mu)I(t), \\ \dot{R} = gI(t) - \mu R(t). \end{cases} \tag{6.40}$$

Cooke [30] first proposed a simple version of the general vector-borne disease model (6.40). More recently, Beretta and Takeuchi [22,23] analyzed the global stability properties of the disease-free equilibrium of general vector-borne disease models similar to (6.40). Takeuchi et al. [176] and Beretta et al. [21] extended this work by also considering stability of the endemic equilibrium. In [132], the authors Ma et al. analyzed the permanence of (6.40) with birth rate not equal to the death rate. Gao et al. [53] investigated a pulse vaccination scheme for an SIR vector-borne disease model with distributed delays. The work on global stability of the endemic equilibrium of (6.40), with birth rate unequal to death rate, was completed by McCluskey in [135].

Since seasonal changes are an important factor in how a vector-borne disease spreads in a population due to changes in the abundance of vectors and the host population behaviour, assume that the contact rate is a piecewise constant parameter which takes on the value β_{i_k} on the the interval $[t_{k-1}, t_k)$. Assume that there are a finite number of values for the contact rate, that is, $i_k \in \mathcal{P}$ follows a switching rule $\sigma : [t_{k-1}, t_k) \to \mathcal{P}$. The switched vector-borne epidemic model is given as

$$\begin{cases} \dot{S} = \mu(1-S(t)) - \beta_\sigma S(t)\displaystyle\int_0^\tau f(u)I(t-u)du, \\ \dot{I} = \beta_\sigma S(t)\displaystyle\int_0^\tau f(u)I(t-u)du - gI(t) - \mu I(t), \\ \dot{R} = gI(t) - \mu R(t). \end{cases} \tag{6.41}$$

The initial conditions are given by $S(0) = S_0 > 0$, $R(0) = R_0 \geq 0$, and $I(s) = I_0$ for $s \in [-\tau, 0]$ where $I_0 \in PC([-\tau, 0], \mathbb{R}_+)$. The main focus of this section is a stability investigation of (6.41) under various vaccination control strategies.

Remark 6.3.1. *Equation (6.41) is a switched model for a general vector-borne disease. The approach here is to use distributed delays to model the interaction between humans and mosquitoes, which is in contrast to the approach in the previous section (namely, model (6.8), where the full dynamics were modelled by considering both human and mosquito populations). In this approach we need not consider the dynamics of the mosquito population in the theoretical analysis, which can be an advantage. However, there are drawbacks to this approach such as possible theoretical complications arising from time-delays and an inability to formulate a strategy like mechanical destruction of breeding sites (since the mosquito population is excluded).*

6.3.2 Switching Cohort Immunization

Many developed countries have used cohort immunization (also known as time-constant vaccination) to control the spread of an infectious disease. For example, the strategy for measles immunization in many areas of the Western world recommends a vaccination dose at 15 months of age and a second dose at around 6 years of age [170]. There have been numerous studies in the literature on epidemic models with time-constant control programs (for example, see [5, 75, 125, 169] and the references therein). In the present section we consider a switching cohort scheme: assume that vaccinations are given continuously in time to susceptible individuals of the population, moving them to the vaccinated class, denoted by V. Assume that on the interval $[t_{k-1}, t_k)$, the susceptible population is being vaccinated at the rate $p_{i_k} > 0$. Similarly, assume that a fraction $0 \leq \rho_{i_k} \leq 1$ of all newborns are given a vaccination. Assume that the vaccine-induced immunity is temporary and that vaccinated individuals return to the susceptible class at a rate of $\theta > 0$. Under this formulation, the cohort immunization is constant on any switching interval but can be increased or decreased according to the switching rule.

An important complication which arises in real-world applications of a vaccine program is that the probability that a vaccinated individual can still become infected through transmission is reduced but is non-zero. For example, this is true in immunizing against measles [37]. This is incorporated into the model by assuming that vaccinated individuals become infected with the reduced transmission rate $\xi\beta_\sigma V(t)\int_0^\tau f(u)I(t-u)du$ where $0 \leq \xi \leq 1$ is a measure of the vaccine efficacy ($\xi = 1$ corresponds to total failure while $\xi = 0$ corresponds to perfect efficacy).

We also consider a switching treatment plan for infected individuals: assume that individuals who are exhibiting symptoms seek treatment from health services. Assume that the treatment rate per unit time is given by $c_{i_k} \geq 0$ on the interval $[t_{k-1}, t_k)$. Applied to (6.41), the cohort immunization model is given by

$$\begin{cases} \dot{S} = \mu(1-\rho_\sigma - S(t)) - \beta_\sigma S(t) \displaystyle\int_0^\tau f(u)I(t-u)du - p_\sigma S(t) + \theta V(t), \\ \dot{I} = \beta_\sigma (S(t) + \xi V(t)) \displaystyle\int_0^\tau f(u)I(t-u)du - (g+\mu+c_\sigma)I(t), \\ \dot{R} = gI(t) + c_\sigma I(t) - \mu R(t), \\ \dot{V} = \rho_\sigma \mu + p_\sigma S(t) - \xi\beta_\sigma V(t) \displaystyle\int_0^\tau f(u)I(t-u)du - (\theta+\mu)V(t). \end{cases} \tag{6.42}$$

The physical domain of interest for (6.42) is given by

$$\Omega_{\text{vaccination}} = \{(S, I, R, V) \in \mathbb{R}^4_+ : S + I + R + V = 1\}.$$

The initial condition for the vaccinated class is given by $V(0) = V_0 \geq 0$ and it is assumed that $(S_0, I_0(0), R_0, V_0) \in \Omega_{\text{vaccination}}$.

In order to perform the stability analysis, we seek the existence of a disease-free solution. Note that system (6.42) has m disease-free equilibria due to the time-varying vaccination rates

$$(S_i^*, I_i^*, R_i^*, V_i^*) = \left(\frac{\mu(1-\rho_i)(\theta+\mu) + \theta\mu\rho_i}{\mu + p_i(1-\theta)}, 0, 0, \frac{\mu\rho_i + p_i S_i^*}{\theta+\mu} \right).$$

In the absence of infectives, the solution trajectory moves between the infection-free equilibria since the vaccination rate is time-varying. This gives rise to the idea of convergence to a disease-free set. When $I(t) \equiv 0$, the fraction of individuals in the recovered class approaches zero and model (6.42) reduces to

$$\begin{cases} \dot{S} = \mu(1-\rho_\sigma - S) - p_\sigma S + \theta V, \\ \dot{V} = \rho_\sigma \mu + p_\sigma S - (\mu+\theta)V. \end{cases} \tag{6.43}$$

Let $p_{\min} = \min_{i \in \mathcal{P}} p_i$ and $p_{\max} = \max_{i\in\mathcal{P}} p_i$. Define $\rho_{\min}$ and $\rho_{\max}$ similarly. Since $S+V=1$,

$$\dot{S} \leq \mu(1-\rho_{\min}) - (\mu + p_{\min})S + \theta(1-S) = (\mu + p_{\min} + \theta)\left(\frac{\overline{S}_{\max}}{S} - 1\right) S,$$

so that $\dot{S} \leq 0$ if $1 \geq S \geq \overline{S}_{\max}$ where

$$\overline{S}_{\max} = \frac{\mu(1-\rho_{\min}) + \theta}{\mu + p_{\min} + \theta}.$$

Similarly, if $0 \leq S \leq \overline{S}_{\min} = \mu(1-\rho_{\max}+\theta)/(\mu+p_{\max}+\theta)$ then $\dot{S} \geq 0$ since

$$\dot{S} \geq \mu(1-\rho_{\max}) - (\mu+p_{\max})S + \theta(1-S) = (\mu+p_{\max}+\theta)\left(\frac{\overline{S}_{\min}}{S} - 1\right)S.$$

Further, $\dot{V} \leq 0$ whenever $1 \geq V \geq \overline{V}_{\max} = (\mu\rho_{\max}+p_{\max})/(\mu+p_{\max}+\theta)$ since

$$\dot{V} \leq \mu\rho_{\max} + p_{\max}(1-V) - (\mu+\theta)V = (\mu+p_{\max}+\theta)\left(\frac{\overline{V}_{\max}}{V} - 1\right)V.$$

Finally, if $0 \leq V \leq \overline{V}_{\min} = (\mu\rho_{\min}+p_{\min})/(\mu+p_{\min}+\theta)$ then $\dot{V} \geq 0$ since

$$\dot{V} \geq \mu\rho_{\min} + p_{\min}(1-V) - (\mu+\theta)V = (\mu+p_{\min}+\theta)\left(\frac{\overline{V}_{\min}}{V} - 1\right)V.$$

By similar arguments to the proof of Proposition 6.2.1, the solution trajectory of (6.43) converges to the set

$$\{(S,V) \in \mathbb{R}^2_+ : \overline{S}_{\min} \leq S \leq \overline{S}_{\max}, \overline{V}_{\min} \leq V \leq \overline{V}_{\max}\}.$$

Therefore, under the assumption that $I \equiv 0$, the solution of (6.42) converges to the disease-free convex set

$$\Psi_{\text{cohort}} = \{(S,I,R,V) \in \mathbb{R}^4_+ : \overline{S}_{\min} \leq S \leq \overline{S}_{\max}, I = 0, R = 0, \overline{V}_{\min} \leq V \leq \overline{V}_{\max}\}.$$

To prove threshold conditions for disease eradication, we focus on the set Ψ_{cohort} and use the switching Halanay-like Proposition 4.2.2. We remind the reader of the dwell-time switching notations from earlier: let $T^+(t_0,t)$ and $T^-(t_0,t)$ to be the total time $\sigma(t) \in \mathcal{P}_u$ and $\sigma(t) \in \mathcal{P}_s$ on $[t_0,t]$, respectively. Also, let $\Phi(t_0,t)$ be the number of switching times t_k such that $\sigma(t_k) \in \mathcal{P}_s$ for $t_k \in [t_0,t)$ (that is, the number of total activations of subsystems in the set $\mathcal{P}_s$ on the interval). We are now in a position to state and prove the first main eradication result.

Theorem 6.3.1. *Let $\lambda_i = \beta_i(\overline{S}_{max} + \xi\overline{V}_{max}) - (\mu+g+c_i)$ and let $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i < 0\}$, $\mathcal{P}_u = \{i \in \mathcal{P} : \lambda_i \geq 0\}$. For $i \in \mathcal{P}_s$, let the constants $\eta_i > 0$ satisfy*

$$\eta_i + \beta_i(\overline{S}_{max} + \xi\overline{V}_{max})e^{\eta_i\tau} - (\mu+g+c_i) < 0,$$

where

$$\overline{V}_{max} = \frac{\mu\rho_{max}+p_{max}}{\mu+p_{max}+\theta}$$

and

$$\overline{S}_{max} = \frac{\mu(1-\rho_{min})+\theta}{\mu + p_{min} + \theta}.$$

Define

$$\lambda^+ = \max_{i\in\mathcal{P}_u} \lambda_i, \qquad \lambda^- = \min_{i\in\mathcal{P}_s} \eta_i > 0.$$

Suppose that there exist $M > 0$, $\nu \geq 0$, and $\widetilde{t} > 0$ such that

$$\sup_{t\geq\widetilde{t}} \frac{t-\widetilde{t}}{T^-(\widetilde{t},t) - \Phi(\widetilde{t},t)\tau} \leq M, \tag{6.44}$$

$$T^+(\widetilde{t},t) \leq \nu(T^-(\widetilde{t},t) - \Phi(\widetilde{t},t)\tau)), \tag{6.45}$$

$$\nu\lambda^+ - \lambda^- < 0, \tag{6.46}$$

then the solution of (6.42) *converges to the disease-free set* Ψ_{cohort}.

Proof. Note that,

$$\begin{aligned}\dot{S} &= \mu - \beta_\sigma S(t)\int_0^\tau f(u)I(t-u)du - p_\sigma S(t) + \theta V(t),\\ &\leq \mu(1-S(t)) - p_\sigma S(t) + \theta V(t),\\ &\leq \mu(1-S(t)) - p_{\min} S(t) + \theta V(t),\\ &\leq \mu + \theta - (\mu+\theta+p_{min})S(t),\end{aligned}$$

since $V = 1 - S - I - R \leq 1 - S$. Similarly,

$$\begin{aligned}\dot{V} &= p_\sigma S(t) - \xi\beta_\sigma V(t)\int_0^\tau f(u)I(t-u)du - (\mu+\theta)V(t),\\ &\leq p_\sigma S(t) - (\mu+\theta)V(t),\\ &\leq p_{\max} S(t) - (\mu+\theta)V(t),\\ &\leq p_{\max}(1-V(t)) - (\mu+\theta)V(t),\\ &\leq p_{\max} - (p_{\max}+\mu+\theta)V(t).\end{aligned}$$

Then for any $\epsilon > 0$ there exists a time $t^* > 0$ such that $S(t) \leq \overline{S}_{\max} + \epsilon$ and $V(t) \leq \overline{V}_{\max} + \epsilon$ for all $t \geq t^*$. Let l be the smallest positive integer such that $t_l > \max\{\widetilde{t}, t^*\}$. On the interval $[0, t_l)$,

$$\begin{aligned}\dot{I} =& \beta_\sigma(S(t)+\xi V(t))\int_0^\tau f(u)I(t-u)du - (\mu+g+c_\sigma)I(t),\\ \leq& \beta_{\max}(1+\xi)\sup_{t-\tau\leq s\leq t} I(s) - (\mu+g+c_{\min})I(t).\end{aligned}$$

Then $I(t) \leq \|I_0\|_\tau e^{\eta t}$ for $t \in [0, t_l)$ where $\eta > 0$ satisfies $\eta + \beta_{\max}(1+\xi)e^{\eta\tau} - (\mu + g + c_{\min}) > 0$ by Lemma 4.2.1. For any $t \in [t_{k-1}, t_k)$ with $k - 1 \geq l$,

$$\dot{I} \leq \beta_\sigma[(\overline{S}_{\max} + \epsilon) + \xi(\overline{V}_{\max} + \epsilon)] \sup_{t-\tau \leq s \leq t} I(s) - (\mu + g + c_\sigma)I(t) \tag{6.47}$$

and $I_{t_l} \in PC([-\tau, 0], \mathbb{R}_+)$. Let

$$\lambda_{i,\epsilon} = \beta_i[(\overline{S}_{\max} + \epsilon) + \xi(\overline{V}_{\max} + \epsilon)] - (\mu + g + c_i)$$

and for $i \in \mathcal{P}_s$ let $\eta_{i,\epsilon} > 0$ satisfy

$$\eta_{i,\epsilon} + \beta_i[(\overline{S}_{\max} + \epsilon) + \xi(\overline{V}_{\max} + \epsilon)]e^{\eta_{i,\epsilon}\tau} - (\mu + g + c_i) < 0.$$

Then by Proposition 4.2.2

$$I(t) \leq C \exp\left[\sum_{i\in\mathcal{P}_u} \lambda_{i,\epsilon} T_i(t_l, t) - \sum_{i\in\mathcal{P}_s} \eta_{i,\epsilon}(T_i(t_l, t) - \Phi_i(t_l, t)\tau)\right] \tag{6.48}$$

for all $t \in [t_{k-1}, t_k)$, whenever $k - 1 \geq l$, where $C = \|I_0\|_\tau e^{\eta t_l}$.

Define $\lambda_\epsilon^+ = \max_{i\in\mathcal{P}_u} \lambda_{i,\epsilon}$ and $\lambda_\epsilon^- = \min_{i\in\mathcal{P}_s} \eta_{i,\epsilon}$. Then

$$\beta_i[(\overline{S}_{\max} + \epsilon) + \xi(\overline{V}_{\max} + \epsilon)]e^{\eta_{i,\epsilon}\tau} - (\mu + g + c_i) < -\eta_{i,\epsilon} \leq -\lambda_\epsilon^-,$$

which can be re-written as

$$\beta_i(\overline{S}_{\max} + \xi\overline{V}_{\max})e^{\eta_{i,\epsilon}\tau} - (\mu + g + c_i) + G_i\epsilon < -\eta_{i,\epsilon} \leq -\lambda_\epsilon^-$$

where

$$G_i = \beta_i(1 + \xi)e^{\eta_{i,\epsilon}\tau}.$$

Also,

$$\beta_i(\overline{S}_{\max} + \xi\overline{V}_{\max})e^{\eta_i\tau} - (\mu + g + c_i) < -\eta_i \leq -\lambda^-.$$

Therefore, there exists a constant F_1 such that $-\lambda_\epsilon^+ \leq -\lambda^- + F_1\epsilon$. Let $q = \text{argmax}_{i\in\mathcal{P}_u} \lambda_i$ then $\nu\lambda_\epsilon^+ \leq \nu\lambda^+ + F_2\epsilon$ where

$$F_2 = q\beta_q(\overline{S}_{\max} + \xi\overline{V}_{\max}) - (\mu + g + c_q).$$

Hence,

$$\nu\lambda_\epsilon^+ - \lambda_\epsilon^- \leq \nu\lambda^+ - \lambda^- + (F_1 + F_2)\epsilon.$$

Since $\nu\lambda^+ - \lambda^- < 0$ there exists a positive constant δ such that $\nu\lambda_\epsilon^+ - \lambda_\epsilon^- \leq -\frac{\delta}{2}$. Choose

$$0 < \epsilon \leq \frac{\delta(F_1 + F_2)}{2}$$

then $\nu\lambda_\epsilon^+ - \lambda_\epsilon^- \leq -\frac{\delta}{2}$.

It follows from equations (6.44), (6.45), and (6.48) that

$$\begin{aligned}
I(t) \leq& C \exp\left[\lambda_\epsilon^+ \sum_{i\in\mathcal{P}_u} T_i(t_l,t) - \lambda_\epsilon^- \sum_{i\in\mathcal{P}_s}(T_i(t_l,t) - \Phi_i(t_l,t)\tau)\right],\\
=& C\exp[\lambda_\epsilon^+ T^+(t_l,t) - \lambda_\epsilon^-(T^-(t_l,t) - \Phi(t_l,t)\tau)],\\
\leq& C\exp[\nu\lambda_\epsilon^+(T^-(t_l,t) - \Phi(t_l,t)\tau) - \lambda_\epsilon^-(T^-(t_l,t) - \Phi(t_l,t)\tau)],\\
=& C\exp[(\nu\lambda_\epsilon^+ - \lambda_\epsilon^-)(T^-(t_l,t) - \Phi(t_l,t)\tau)],\\
\leq& C\exp\left[(\nu\lambda_\epsilon^+ - \lambda_\epsilon^-)\frac{(t-t_l)}{M}\right].
\end{aligned}$$

Note that equation (6.45) guarantees that $T^-(t_l,t) - \Phi(t_l,t)\tau \geq 0$. Therefore, $I(t) \leq C\exp[-\frac{\delta}{2}(t-t_l)]$ for $t \geq t_l$. It follows that R converges to zero and (6.42) reduces to system (6.43) and hence the solution trajectory converges to the disease-free set Ψ_{cohort}. □

Remark 6.3.2. *Equation (6.45) gives that the amount of time spent in the unstable subsystems ($\mathcal{P}_u$), is some fraction $\nu \geq 0$ of the time spent in the stable subsystems ($\mathcal{P}_s$). The constant ν dictates the threshold values for the growth rate λ^+ (worst case scenario) and decay rate λ^- (conservative estimate) in equation (6.46).*

Remark 6.3.3. *If equation (6.46) holds then it must be true that*

$$\nu \max_{i\in\mathcal{P}_u}\{\beta_i(\overline{S}_{max} + \xi\overline{V}_{max}) - (\mu+g+c_i)]\} + \min_{i\in\mathcal{P}_s}\{\beta_i(\overline{S}_{max} + \xi\overline{V}_{max})e^{\eta_i\tau} - (\mu+g+c_i)\} < 0.$$

Let $q = \operatorname{argmax}_{i\in\mathcal{P}_u}\lambda_i$ and let $\zeta = \operatorname{argmin}_{i\in\mathcal{P}_u}\eta_i$. Then

$$\lambda^+ = \beta_q(\overline{S}_{max} + \xi\overline{V}_{max}) - (\mu+g+c_q)$$

and

$$\lambda^- = \beta_\zeta(\overline{S}_{max} + \xi\overline{V}_{max})e^{\eta_\zeta\tau} - (\mu+g+c_\zeta).$$

Hence (6.46) implies

$$\overline{\mathcal{R}}_{general} = \nu\frac{(\beta_q + \beta_\zeta e^{\eta_\zeta\tau})(\overline{S}_{max} + \xi\overline{V}_{max})}{2\mu+2g+c_q+c_\zeta} < 1, \tag{6.49}$$

which can be thought of as an approximate basic reproduction number of the disease. In fact, equation (6.46) implies (6.49) and hence (6.46) is a stricter condition on the model parameters.

Next we consider the case when the switching rule is periodic. Let $h_k = t_k - t_{k-1}$ and assume that $h_{k+m} = h_k$. Assume that $\beta_{i_k} = \beta_k$, $c_{i_k} = c_k$, $\rho_{i_k} = \rho_k$ and $p_{i_k} = p_k$ for $t \in [t_{k-1}, t_k)$. Assume that $\beta_k = \beta_{k+m}$, $c_k = c_{k+m}$, $\rho_k = \rho_{k+m}$ and $p_k = p_{k+m}$. Denote one period of the switching rule by $\omega = h_1 + \ldots + h_m$. Denote the set of all such periodic switching rules by $\mathcal{S}_{\text{periodic}}$. An eradication result can be given for this case as follows.

Theorem 6.3.2. *Let $\lambda_i = \beta_i(\overline{S}_{max} + \xi\overline{V}_{max}) - (\mu + g + c_i)$ and let $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i < 0\}$, $\mathcal{P}_u = \{i \in \mathcal{P} : \lambda_i \geq 0\}$. For $i \in \mathcal{P}_s$ let the constants $\eta_i > 0$ satisfy*

$$\eta_i + \beta_i(\overline{S}_{max} + \xi\overline{V}_{max})e^{\eta_i\tau} - (\mu + g + c_i) < 0,$$

where

$$\overline{V}_{max} = \frac{\mu\rho_{max} + p_{max}}{\mu + p_{max} + \theta}$$

and

$$\overline{S}_{max} = \frac{\mu(1 - \rho_{min}) + \theta}{\mu + p_{min} + \theta}.$$

Assume that $\sigma \in \mathcal{S}_{periodic}$ and

$$\Lambda_{cohort} = \sum_{i \in \mathcal{P}_u} \lambda_i h_i - \sum_{i \in \mathcal{P}_s} \eta_i(h_i - \tau) < 0 \tag{6.50}$$

then the solution of system (6.42) converges to the disease-free set Ψ_{cohort}.

Proof. Begin from equation (6.47) in the proof of Theorem 6.3.1 and choose the smallest positive integer z such that $z\omega > \max\{\widetilde{t}, t^*\}$. Then by Proposition 4.2.3, $I((z + j)\omega) \leq \|I_{z\omega}\|_\tau \delta^j$ for any $j = 1, 2, \ldots,$ where

$$\delta = \exp\left[\sum_{i \in \mathcal{P}_u} \lambda_{i,\epsilon} h_i - \sum_{i \in \mathcal{P}_s} \eta_{i,\epsilon}(h_i - \tau)\right]$$

and $\|I_{z\omega}\|_\tau \leq K$ for some constant K. From (6.50) and the arguments in the proof of Theorem 6.3.1, it is possible to choose $\epsilon > 0$ sufficiently small such that $0 < \delta < 1$. It follows that I converges to zero and, subsequently, R converges to zero. Then (6.42) becomes the reduced system (6.43) and hence the solution converges to Ψ_{cohort} as required. ∎

Remark 6.3.4. *Equation (6.50) implies that*

$$\begin{aligned}
&\sum_{i\in\mathcal{P}_u}[\beta_i(\overline{S}_{max}+\xi\overline{V}_{max})-(\mu+g+c_i)]h_i\\
&+\sum_{i\in\mathcal{P}_s}[\beta_i(\overline{S}_{max}+\xi\overline{V}_{max})e^{\eta_i\tau}-(\mu+g+c_i)](h_i-\tau),\\
&<\sum_{i\in\mathcal{P}_u}[\beta_i(\overline{S}_{max}+\xi\overline{V}_{max})-(\mu+g+c_i)]h_i+\sum_{i\in\mathcal{P}_s}(-\eta_i)(h_i-\tau),\\
&=\sum_{i\in\mathcal{P}_u}\lambda_i h_i-\sum_{i\in\mathcal{P}_s}\eta_i(h_i-\tau),\\
&<0.
\end{aligned}$$

That is, (6.50) implies that $\overline{\mathcal{R}}_{periodic}<1$ where

$$\overline{\mathcal{R}}_{periodic}=\frac{\sum_{i\in\mathcal{P}_u}\beta_i(\overline{S}_{max}+\xi\overline{V}_{max})h_i+\sum_{i\in\mathcal{P}_s}\beta_i e^{\eta_i\tau}(\overline{S}_{max}+\xi\overline{V}_{max})(h_i-\tau)}{\sum_{i\in\mathcal{P}_u}(\mu+g+c_i)h_i+\sum_{i\in\mathcal{P}_s}(\mu+g+c_i)(h_i-\tau)}. \tag{6.51}$$

$\overline{\mathcal{R}}_{periodic}$ may be viewed as an approximate basic reproduction number and it should be noted that the theorem condition is stricter than requiring $\overline{\mathcal{R}}_{periodic}<1$.

6.3.3 Time-dependent Pulse Vaccination

Assume that at the pre-specified times $t=T_k$, $k=1,2,\ldots$, a fraction $0\le v_k\le 1$ of the susceptible population is given a vaccination and immediately move to the vaccinated class. Since it is assumed that the period of time it takes to administer the vaccination is much shorter than the time scale of the disease spread, it is modelled as an impulsive effect. Assume that the pulse vaccination scheme is periodic: there exist a constant $T>0$ and positive integer N satisfying $T_k-T_{k-1}=\overline{T_k}$ with $\sum_{k=1}^{N}\overline{T_k}=T$ such that $T_{k+N}=T_k+T$ and $v_{k+N}=v_k$. Applied to (6.41), the time-dependent pulse vaccination model is given by

$$\begin{cases}\left.\begin{array}{l}\dot{S}=\mu(1-S(t))-\beta_\sigma S(t)\int_0^\tau f(u)I(t-u)du+\theta V(t),\\ \dot{I}=\beta_\sigma S(t)\int_0^\tau f(u)I(t-u)du-(\mu+g+c_\sigma)I(t),\\ \dot{R}=(g+c_\sigma)I(t)-\mu R(t),\\ \dot{V}=-\beta_\sigma\xi V(t)\int_0^\tau f(u)I(t-u)du-(\mu+\theta)V(t),\end{array}\right\}t\neq T_k,\\ \left.\begin{array}{l}S(t)=(1-v_k)S(t^-),\\ I(t)=I(t^-),\\ R(t)=R(t^-),\\ V(t)=V(t^-)+v_kS(t^-)\end{array}\right\}t=T_k,\end{cases} \tag{6.52}$$

with initial conditions $S(0) = S_0 > 0$, $R(0) = R_0 \geq 0$, $V(0) = V_0 \geq 0$, and $I(s) = I_0 \in PC([-\tau, 0], \mathbb{R}_+)$ for $s \in [-\tau, 0]$ satisfying $(S_0, I_0(0), R_0, V_0) \in \Omega_{\text{vaccination}}$.

In order to perform the stability analysis, we seek the existence of a disease-free solution to (6.52). Observe that $(S, I, R, V) = (1, 0, 0, 0)$ is not an equilibrium point of (6.52), but that $I(t) \equiv 0$ is an equilibrium solution for the differential and difference equations governing I. Under this assumption, it is apparent that the fraction of individuals in the recovered class approaches zero. More precisely, the set $\{(S, I, R, V) \in \mathbb{R}_+^4 | \ I = 0 \text{ and } R = 0\}$ is invariant to system (6.52). The reduced model is given by

$$\begin{cases} \dot{S} = \mu(1 - S) + \theta V, & t \neq T_k, \\ \dot{V} = -(\mu + \theta)V, & \\ S(t) = (1 - v_k)S(t^-), & t = T_k, \\ V(t) = V(t^-) + v_k S(t^-), & \end{cases} \tag{6.53}$$

where $S + V = 1$. Re-write the equation for S as $\dot{S} = (\mu + \theta)(1 - S)$ then it follows from Lemma 2.2 in [62] that (6.53) converges to the periodic disease-free solution $(\widetilde{S}(t), \widetilde{V}(t))$ where

$$\begin{cases} \widetilde{S}(t) = 1 + (\widetilde{S}_{j-1} - 1)e^{-(\mu+\theta)(t-kT-T_j)}, & t \in (kT + T_{j-1}, kT + T_j), \\ \widetilde{V}(t) = 1 - \widetilde{S}(t), & j = 1, 2, \ldots, N, \qquad k = 1, 2, \ldots, \end{cases} \tag{6.54}$$

with

$$\widetilde{S}_j = \sum_{l=1}^{j} \left\{ (1 - v_l)(1 - e^{-(\mu+\theta)\overline{T_l}}) \prod_{q=l+1}^{N} [(1 - v_q)e^{-(\mu+\theta)\overline{T_q}}] \right\} + \left\{ \prod_{l=1}^{j} [(1 - v_l)e^{-(\mu+\theta)\overline{T_l}} \right\} \widetilde{S}_0,$$

and

$$\widetilde{S}_0 = \frac{\sum_{l=1}^{N} \left\{ (1 - v_l)(1 - e^{-(\mu+\theta)\overline{T_l}}) \prod_{q=l+1}^{N} [(1 - v_q)e^{-(\mu+\theta)\overline{T_q}}] \right\}}{1 - e^{-(\mu+\theta)T} \prod_{l=1}^{N} (1 - v_l)}.$$

Therefore system (6.52) has the periodic disease-free solution $(\widetilde{S}(t), 0, 0, \widetilde{V}(t))$.

Theorem 6.3.3. *Let* $\widetilde{S}_{max} = \max_{0 \leq t \leq T} \widetilde{S}(t)$, $\widetilde{V}_{max} = \max_{0 \leq t \leq T} \widetilde{V}(t)$, $\lambda_i = \beta_i(\widetilde{S}_{max} + \xi\widetilde{V}_{max}) - (\mu + g + c_i)$, $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i < 0\}$, *and* $\mathcal{P}_u = \{i \in \mathcal{P} : \lambda_i \geq 0\}$. *For* $i \in \mathcal{P}_s$ *let the constants* $\eta_i > 0$ *satisfy*

$$\eta_i + \beta_i(\widetilde{S}_{max} + \xi\widetilde{V}_{max})e^{\eta_i \tau} - (\mu + g + c_i) < 0.$$

Define

$$\lambda^+ = \max_{i\in\mathcal{P}_u} \lambda_i, \qquad \lambda^- = \min_{i\in\mathcal{P}_s} \eta_i > 0.$$

Suppose that there exist $M > 0$, $\nu \geq 0$, and $\widetilde{t} > 0$ such that

$$\sup_{t\geq\widetilde{t}} \frac{t-\widetilde{t}}{T^-(\widetilde{t},t) - \Phi(\widetilde{t},t)\tau} \leq M, \tag{6.55}$$

$$T^+(\widetilde{t},t) \leq \nu(T^-(\widetilde{t},t) - \Phi(\widetilde{t},t)\tau)), \tag{6.56}$$

$$\nu\lambda^+ - \lambda^- < 0, \tag{6.57}$$

then the solution of system (6.52) *converges to the periodic disease-free solution* $(\widetilde{S}(t), 0, 0, \widetilde{V}(t))$ *in the meaningful domain.*

Proof. For $t \neq T_k$,

$$\begin{aligned}\dot{S} &= \mu(1 - S(t)) - \beta_\sigma S(t)\int_0^\tau f(u)I(t-u)du + \theta V(t),\\ &\leq \mu(1-S) + \theta(1 - S - I),\\ &\leq (\mu+\theta)(1-S),\end{aligned}$$

and

$$\dot{V} \leq -(\mu+\theta)V.$$

Consider the comparison system

$$\begin{cases} \dot{x} = (\mu+\theta)(1-x), & t \neq T_k,\\ \dot{y} = -(\mu+\theta)y, & \\ x(t) = (1-v_k)x(t^-), & t = T_k,\\ y(t) = y(t^-) + v_k x(t^-), & \\ x(0) = S_0, \quad y(0) = V_0, & k = 1,2,\ldots, \end{cases} \tag{6.58}$$

which converges to $(\widetilde{S}(t), \widetilde{V}(t))$. Since $S \leq x$ and $V \leq y$, then for any $\epsilon > 0$ there exists a time $t^* > 0$ such that $S(t) \leq \widetilde{S}(t) + \epsilon \leq \widetilde{S}_{\max} + \epsilon$ and $V(t) \leq \widetilde{V}(t) + \epsilon \leq \widetilde{V}_{\max} + \epsilon$ for $t \geq t^*$. Let l be the smallest integer such that $t_l \geq \max\{\widetilde{t}, t^*\}$. Then for any $t \in [t_{k-1}, t_k)$ with $k-1 \geq l$,

$$\dot{I} \leq \beta_\sigma[(\widetilde{S}_{\max} + \epsilon) + \xi(\widetilde{V}_{\max} + \epsilon)] \sup_{t-\tau\leq s\leq t} I(s) - (\mu + g + c_\sigma)I(t). \tag{6.59}$$

Further, $I_{t_l} \in PC([-\tau, 0], \mathbb{R}_+)$ and so I converges to zero by the same arguments as the proof of Theorem 6.3.1. Then it is apparent that R converges to zero and the system reduces to (6.53). Hence the solution converges to the periodic disease-free solution. □

When the vaccination inter-pulse period is constant (i.e. $T_k = kT$) and the vaccination effects are constant (i.e. $v_k = v$) then the periodic disease-free solution of (6.52) is given by $(\widetilde{S}(t), 0, 0, \widetilde{V}(t))$ where

$$\begin{cases} \widetilde{S}(t) = 1 - \dfrac{ve^{-(\mu+\theta)(t-(k-1)T)}}{1-(1-v)e^{-(\mu+\theta)T}}, \quad t \in [(k-1)T, kT), \\ \widetilde{V}(t) = 1 - \widetilde{S}(t). \end{cases} \tag{6.60}$$

This follows from, for example, Lemma 2.2 of [51]. A sufficient condition for eradication, which is more straightforward to calculate, can be given as follows.

Theorem 6.3.4. *Suppose that $T_k = kT$, $v_k = v$ and $\sigma \in \mathcal{S}_{periodic}$. Let $\lambda_i = \beta_i(\widetilde{S}_{max} + \xi\widetilde{V}_{max}) - (\mu + g + c_i)$ and $\eta_i > 0$ satisfy*

$$\eta_i + \beta_i(\widetilde{S}_{max} + \xi\widetilde{V}_{max})e^{\eta_i\tau} - (\mu + g + c_i) < 0,$$

for $i \in \mathcal{P}_s$, where

$$\widetilde{S}_{max} = 1 - \frac{ve^{-(\mu+\theta)T}}{1-(1-v)e^{-(\mu+\theta)T}}, \quad \widetilde{V}_{max} = \frac{v}{1-(1-v)e^{-(\mu+\theta)T}}.$$

If

$$\Lambda_{time\text{-}pulse} = \sum_{i\in\mathcal{P}_u} \lambda_i h_i - \sum_{i\in\mathcal{P}_s} \eta_i(h_i - \tau) < 0 \tag{6.61}$$

then the solution of system (6.52) converges to the periodic disease-free solution given in equation (6.60).

Proof. The proof is similar to the proof of Theorem 6.3.2 using the bound from equation (6.59). □

Remark 6.3.5. *Equation (6.61) implies $\widehat{\mathcal{R}}_{periodic} < 1$ where*

$$\widehat{\mathcal{R}}_{periodic} = \frac{\sum_{i\in\mathcal{P}_u} \beta_i(\widetilde{S}_{max} + \xi\widetilde{V}_{max})h_i + \sum_{i\in\mathcal{P}_s} \beta_i e^{\eta_i\tau}(\widetilde{S}_{max} + \xi\widetilde{V}_{max})(h_i - \tau)}{\sum_{i\in\mathcal{P}_u}(\mu + g + c_i)h_i + \sum_{i\in\mathcal{P}_s}(\mu + g + c_i)(h_i - \tau)} \tag{6.62}$$

and $\widehat{\mathcal{R}}_{periodic}$ can be viewed as an approximate basic reproduction number.

6.3.4 State-dependent Pulse Vaccination

The method of state-dependent pulse vaccination, which was detailed at the beginning of Section 6.3, is relatively new and has not been analyzed extensively. In this approach, if the susceptible population reaches a threshold value, denoted by $S_{\text{crit}} > 0$, then a portion of the susceptible population $0 \leq v \leq 1$ is given a vaccination and moved to the vaccinated class V. With the other assumptions of Section 6.3.3 (namely, waning immunity and a non-zero probability of the vaccinated class being infected), the model (6.41) becomes

$$\begin{cases} \left.\begin{array}{l} \dot{S} = \mu(1 - S(t)) - \beta_\sigma S(t) \int_0^\tau f(u)I(t-u)du + \theta V(t), \\ \dot{I} = \beta_\sigma(S(t) + \xi I(t)) \int_0^\tau f(u)I(t-u)du - (\mu + g + c_\sigma)I(t), \\ \dot{R} = gI(t) + c_\sigma I(t) - \mu R(t), \\ \dot{V} = -\beta_\sigma \xi V(t) \int_0^\tau f(u)I(t-u)du - (\mu + \theta)V(t), \end{array}\right\} S < S_{\text{crit}}, \\ \left.\begin{array}{l} S(t) = (1-v)S(t^-), \\ I(t) = I(t^-), \\ R(t) = I(t^-), \\ V(t) = V(t^-) + vS(t^-), \end{array}\right\} S \geq S_{\text{crit}}, \end{cases} \tag{6.63}$$

with initial conditions $S(0) = S_0 > 0$, $R(0) = R_0 \geq 0$, $V(0) = V_0 \geq 0$, and $I(s) = I_0 \in PC([-\tau, 0], \mathbb{R}_+)$ for $s \in [-\tau, 0]$ satisfying $(S_0, I_0(0), R_0, V_0) \in \Omega_{\text{vaccination}}$.

Remark 6.3.6. *If $S_0 > S_{crit}$ then the initial time is a critical time and it is possible that $S > S_{crit}$ after an impulse is applied depending on the value of v. If this is the case, then more impulses are applied until $S < S_{crit}$. Practically this means a single impulse is applied at the initial time so that $S < S_{crit}$. For $t > 0$, $S > S_{crit}$ is not possible under this scheme.*

The impulsive moments, $t = T_k(S)$, are dependent on the state of the susceptible population and are not known before hand. A dwell-time eradication condition can be established.

Theorem 6.3.5. *Let $\lambda_i = \beta_i(S_{crit} + \xi V_{crit}) - (\mu + g + c_i)$ and for $i \in \mathcal{P}_s$ let the constants $\eta_i > 0$ satisfy*

$$\eta_i + \beta_i(S_{crit} + \xi V_{crit})e^{\eta_i \tau} - (\mu + g + c_i) < 0,$$

where $V_{crit} = 1 - (1-v)S_{crit}$. Define

$$\lambda^+ = \max_{i \in \mathcal{P}_u} \lambda_i, \qquad \lambda^- = \min_{i \in \mathcal{P}_s} \eta_i > 0.$$

Suppose that there exist $M > 0$ and $\nu \geq 0$ such that

$$\sup_{t\geq 0} \frac{t}{T^-(0,t) - \Phi(0,t)\tau} \leq M, \tag{6.64}$$

$$T^+(0,t) \leq \nu(T^-(0,t) - \Phi(0,t)\tau)), \tag{6.65}$$

$$\nu\lambda^+ - \lambda^- < 0, \tag{6.66}$$

then the disease is eradicated.

Proof. By construction of the impulsive scheme, $S(t) \leq S_{\text{crit}}$ and $V(t) \leq V_{\text{crit}}$ for all $t > 0^+$. Then for any $t \in [t_{k-1}, t_k)$,

$$\dot{I} \leq \beta_\sigma[S_{\text{crit}} + \xi V_{\text{crit}}] \sup_{t-\tau\leq s\leq t} I(s) - (\mu + g + c_\sigma)I(t). \tag{6.67}$$

Hence I converges to zero by the same arguments as the proof of Theorem 6.3.1. □

Finally, a periodic switching rule result can be given for the state-dependent pulse vaccination case.

Theorem 6.3.6. *Assume that $\sigma \in \mathcal{S}_{periodic}$ and let $\lambda_i = \beta_i(S_{crit} + \xi V_{crit}) - (\mu + g + c_i)$ and $\eta_i > 0$ satisfy*

$$\eta_i + \beta_i(S_{crit} + \xi V_{crit})e^{\eta_i\tau} - (\mu + g + c_i) < 0,$$

for $i \in \mathcal{P}_s$, where $V_{crit} = 1 - (1 - v)S_{crit}$. If

$$\Lambda_{state\text{-}pulse} = \sum_{i\in\mathcal{P}_u} \lambda_i h_i - \sum_{i\in\mathcal{P}_s} \eta_i(h_i - \tau) < 0 \tag{6.68}$$

then the disease is eradicated.

Proof. See the proof of Theorem 6.3.4 using the bound (6.67). □

Remark 6.3.7. *From equation (6.68) it follows that $\widetilde{\mathcal{R}}_{periodic} < 1$ where*

$$\widetilde{\mathcal{R}}_{periodic} = \frac{\sum_{i\in\mathcal{P}_u} \beta_i(S_{crit} + \xi V_{crit})h_i + \sum_{i\in\mathcal{P}_s} \beta_i e^{\eta_i\tau}(S_{crit} + \xi V_{crit})(h_i - \tau)}{\sum_{i\in\mathcal{P}_u}(\mu + g + c_i)h_i + \sum_{i\in\mathcal{P}_s}(\mu + g + c_i)(h_i - \tau)} \tag{6.69}$$

and $\widetilde{\mathcal{R}}_{periodic}$ can be interpreted as an approximate basic reproduction number.

6.3.5 Cost-benefit Analysis

For the simulations here, assume that the initial conditions are given by $S_0 = 0.9$, $R_0 = V_0 = 0$, and $I_0(s) = 0.1$ for $-\tau \leq s \leq 0$. Assume that the switching rule σ takes the following form:

$$\sigma = \begin{cases} 1 & \text{if } t \in [k, k + \frac{3}{12}),\ k = 0, 1, 2, \ldots \\ 2 & \text{if } t \in [k + \frac{3}{12}, k + 1). \end{cases} \tag{6.70}$$

This is motivated from shifts in the model parameters between the seasons, as discussed in Section 6.2 and Section 6.1.2. Note that the switching rule is periodic with $h_1 = 3/12$ (can model a winter season or rainy season, depending on climate), $h_2 = 9/12$ (summer seasons or dry season) and $\omega = 1$. From [132], let

$$f(u) = \frac{e^{-u}}{1 - e^{-\tau}}.$$

Suppose the baseline model parameters are those found in Table 6.4.

Parameter	Description	Value
β_σ	average number of contacts per unit time	[8, 1.6]
μ	natural birth/death rate	1
g	recovery rate	1.5
τ	upper bound on the incubation time	0.1

Table 6.4: Epidemiological parameters. The parameter values given in brackets represent the switching value associated with $\sigma = 1$ and $\sigma = 2$, respectively.

First we investigate the effects of the epidemiological parameters on the uncontrolled model (6.41) by considering the effects of varying the switching contact rates β_σ, the recovery rate g, the birth/death rate μ, and the upper bound on the incubation period τ (while holding the other parameters constant). Recall that C_H^0 represents the cumulative number of humans infected during the entire duration of the epidemic. Since we are considering total number of individuals being infected, we consider a total constant population of $N_0 = 1000000$. See Figure 6.11 for simulations which illustrate the effects of each model parameter on the spread of the disease. The results are as expected: an increase in the number of contacts results in a significant increase in the disease spread while increasing the rate of recovery or natural death rate leads to a decrease in total number of infections (since the average infectious period is decreased). If the natural birth/death rate is sufficiently low then the total number of infective cases decreases due to a decreased influx of new susceptibles. Finally, as the incubation times are increased (by increasing the upper bound τ), the epidemic worsens.

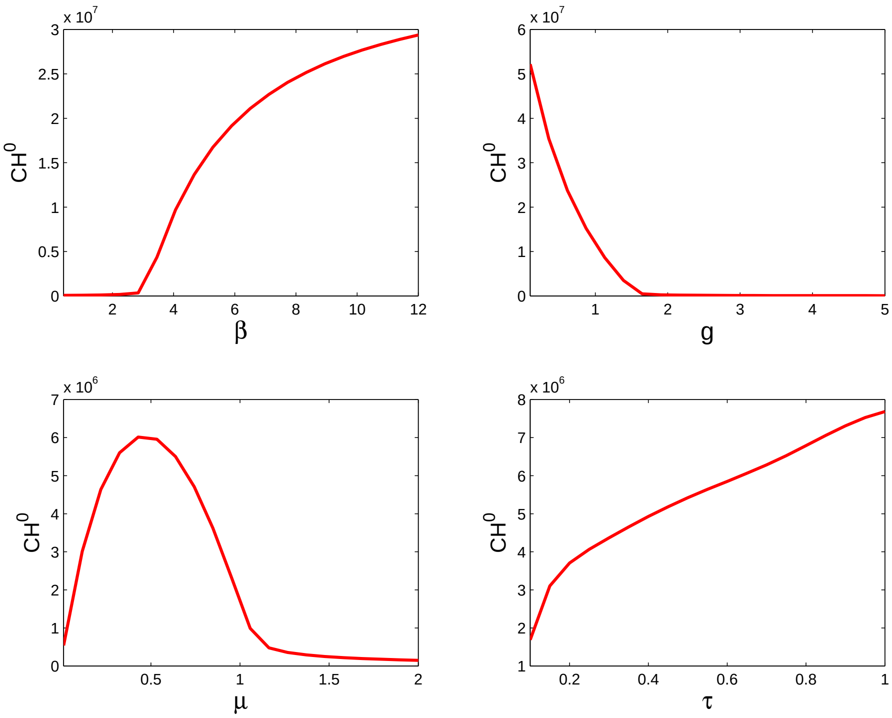


Figure 6.11: Cumulative number of infections, C_H^0, of system (6.41) as different model parameters are varied. Here β is the average contact rate over the period ω (i.e. $\beta = \beta_1 \times h_1 + \beta_2 \times h_2$).

Simulating Eradication under the Control Strategies

For the control schemes we differentiate between cohort immunization of the susceptible population ($p_\sigma > 0$, $\rho_\sigma \equiv 0$) and cohort immunization of newborns ($p_\sigma \equiv 0$, $\rho_\sigma > 0$). See Table 6.5 for the values of the control parameters (the epidemiological parameters in Table 6.4 are used again here).

For the susceptible cohort immunization program, the model parameters give $\overline{S}_{\max} = 0.3548$, $\overline{V}_{\max} = 0.7317$, and $\lambda_1 = 0.8948$ (that is, $\mathcal{P}_u = \{1\}$). Choosing $\eta_2 = 1$ (that is,

Control parameter	Description	Value
p_σ	immunization rate for the susceptible population	[3, 2]
θ	rate of waning immunity	0.1
ρ_σ	immunization rate for newborns	[0.6, 0.4]
c_σ	treatment rate	[2, 0.5]
ξ	vaccine failure	0.3
v_k	pulse vaccination rate	0.3
$T_k - T_{k-1}$	inter-pulse period	1
S_{crit}	critical threshold	0.3

Table 6.5: Control parameters. The parameter values given in brackets represent the switching value associated with $\sigma = 1$ and $\sigma = 2$, respectively.

$\mathcal{P}_s = \{2\}$) satisfies the necessary condition which implies that $\Lambda_{\text{cohort}} = -0.4263$. Thus by Theorem 6.3.2 the solution of (6.42) converges to the disease-free convex set Ψ_{cohort} and hence the disease is eradicated. For the newborn immunization program, $\overline{S}_{\max} = 0.6364$ and $\overline{V}_{\max} = 0.5455$ which gives $\lambda_1 = 2.7000$ and $\eta_2 = 1.75$. Hence $\Lambda_{\text{cohort}} = -0.4625$ for this scheme and the disease is eradicated by Theorem 6.3.2.

The maximum thresholds for the time-dependent pulse vaccination scheme are found to be $\widetilde{S}_{\max} = 0.8698$ and $\widetilde{V}_{\max} = 0.3911$ so that $\lambda_1 = 4.1971$ and $\eta_2 = 1.7$. Thus $\Lambda_{\text{time-pulse}} = -0.0557$ for this control strategy and the solution converges to the periodic disease-free solution by Theorem 6.3.4. Finally, for the state-dependent pulse vaccination scheme we have that $S_{\text{crit}} = 0.3$, $V_{\text{crit}} = 0.79$, $\lambda_1 = 0.5960$ and $\eta_2 = 0.25$. By Theorem 6.3.6, the disease is eradicated since $\Lambda_{\text{state-pulse}} = -0.0135$. For the simulations, see Figure 6.12.

Efficacy of the Control Schemes

To measure the efficacy rate of the control schemes numerically, consider the difference between how many infective cases there would be with control versus without. Recall the control efficacy rating [41]: let

$$F_0 = 100\frac{C_H^c}{C_H^0},$$

where C_H^c and C_H^0 are the cumulative number of humans infected with control and without control, respectively. A low value of F_0 corresponds to an effective control scheme

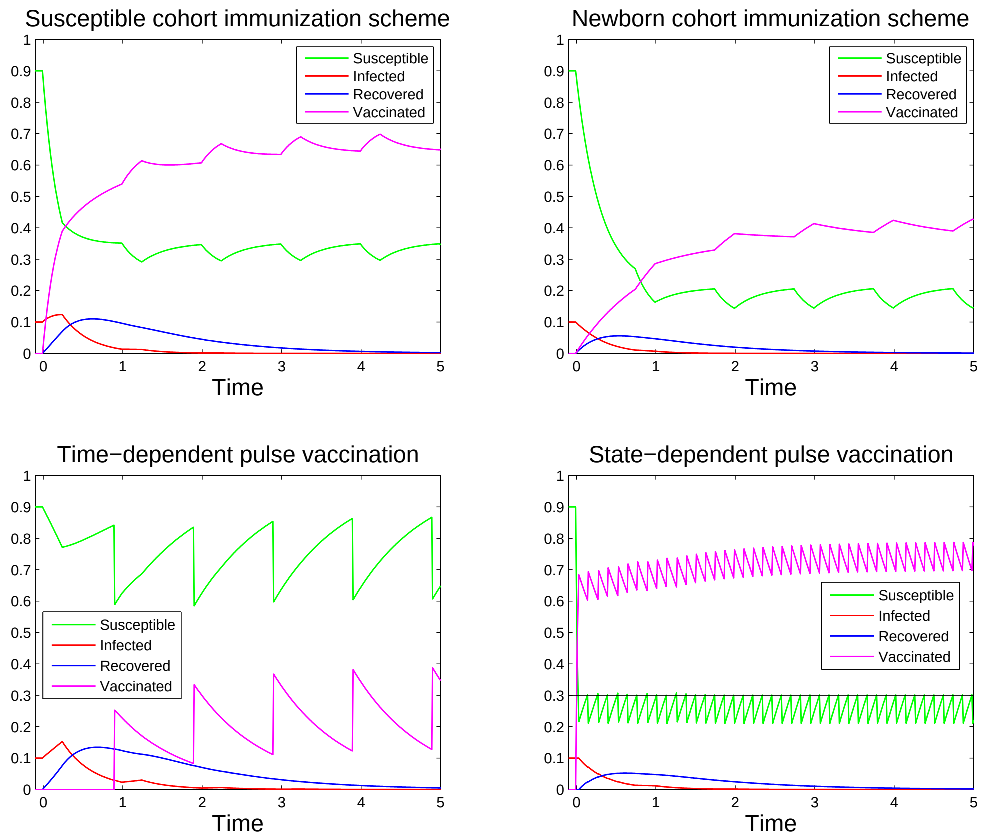


Figure 6.12: Simulations of the different control schemes with parameters given in Table 6.4 and Table 6.5. The horizontal black line in the state-dependent pulse vaccination simulation represents the critical threshold S_{crit}.

(significant reduction in infections), while a high value of F_0 corresponds to the failure of a scheme. For the simulations we use the epidemiological parameter values in Table 6.4 except with $\beta_1 = 30$ and $\beta_2 = 6$. For the cohort immunization schemes, we consider how varying the vaccination rates (p_σ and ρ_σ) and the duration of the strategy (denoted by t_c) affects the efficacy measure F_0. For the time-dependent pulse vaccination scheme, we consider different inter-pulse periods $T_k - T_{k-1} = T$ and different vaccination rates $v_k = v$. Finally, in the state-dependent pulse vaccination scheme, we analyze F_0 under varying vaccination rates $v_k = v$ and varying critical thresholds for the susceptible population, S_{crit}. See Figure 6.13 for the efficacy measures under these scenarios.

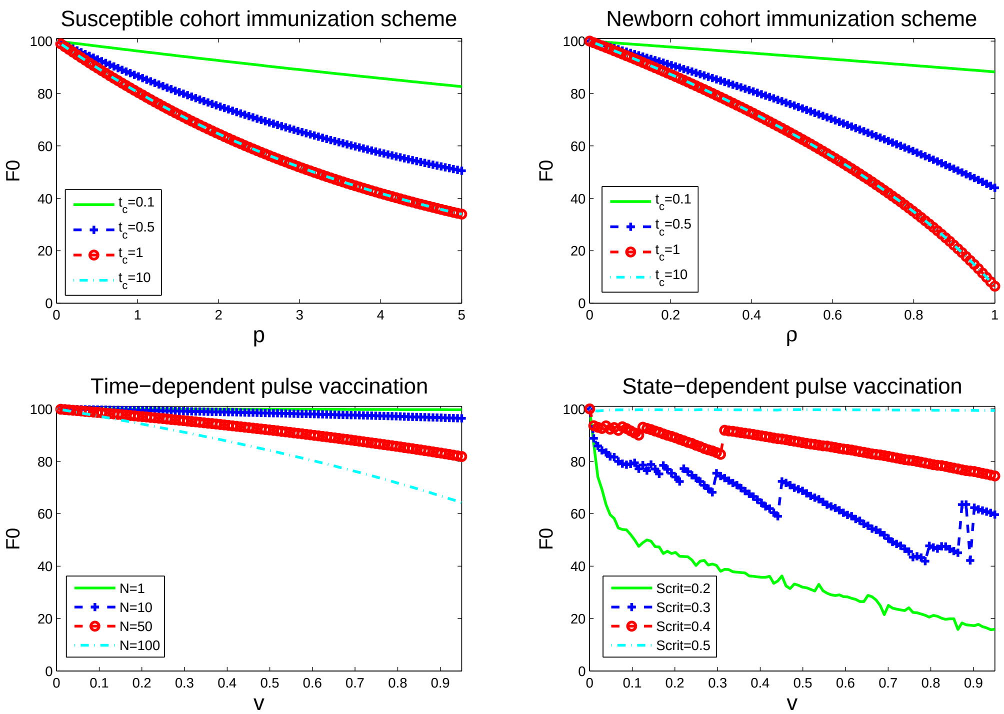


Figure 6.13: Control efficacy ratings for the different schemes. The parameters p, ρ, and v represent averages of p_σ, ρ_σ, and v_σ, respectively, over one period ω of the switching rule σ.

Cost-benefit Analysis

The control efficacy measure F_0 does not take any costs into account when analyzing the control schemes. Motivated by this, we construct a cost-benefit analysis by assuming that the cost of administering a vaccination to the susceptible population is the same whether it is through a cohort immunization program or a pulse vaccination campaign. For $C_H^0 \neq C_H^c$, let

$$\chi = \frac{\Psi}{C_H^0 - C_H^c}$$

where Ψ is the total number of vaccinations administered from the beginning of the control scheme to the end (which may be before the end of the simulation time for the cohort immunization schemes when t_c is small). Then χ can be viewed as the cost of the control scheme (measured in total vaccinations administered) normalized by the benefit gained (as measured by the number of individuals that do not contract the disease due to the control scheme). Therefore, the lower the value of χ, the better the scheme is from a cost-benefit point of view. See Figure 6.14 for the results.

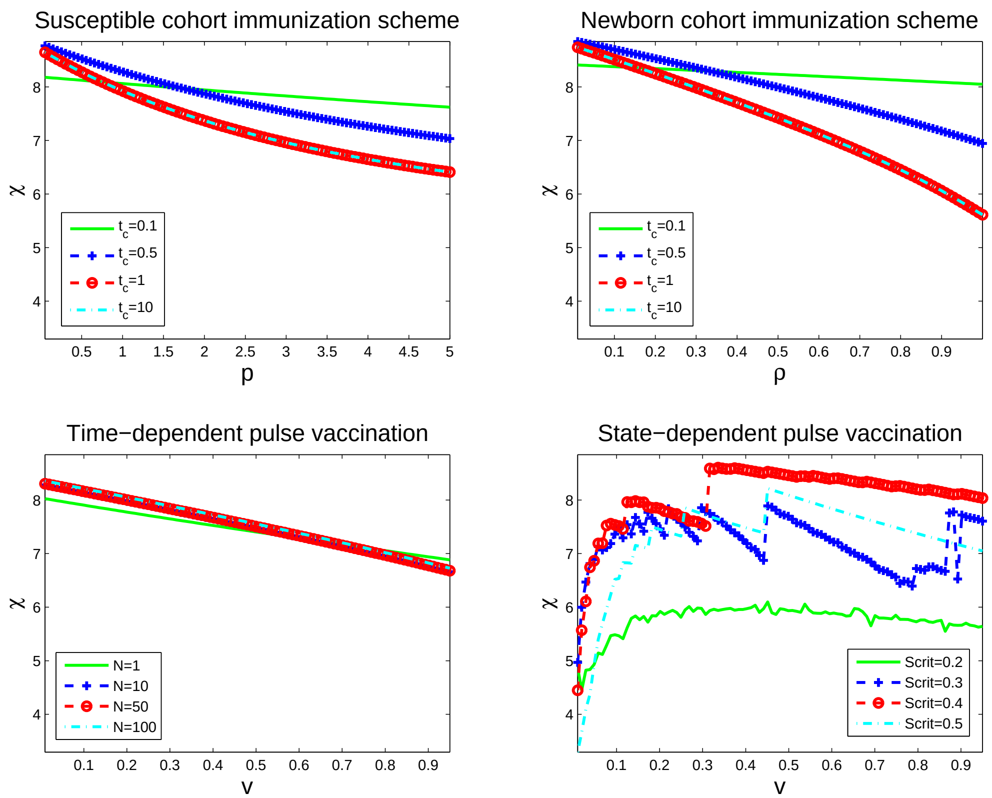


Figure 6.14: Cost-benefit analysis.

6.3.6 Discussion

We make the following observations on the control efficacy ratings illustrated in Figure 6.13:

1. Other than the state-dependent pulse vaccination scheme, the control strategies have an inverse relationship between F_0 and their control rate (p_σ, ρ_σ, v). In general,

as the vaccination rates increase, there should be a decrease in the total number of infected cases.

2. In the state-dependent pulse vaccination scheme, the relationship is roughly inverse but F_0 is not strictly decreasing as a function of v. This is an interesting phenomenon which is of practical interest from a cost perspective since greater control efficacy is achieved at lower rates of control.

3. In all four schemes, the efficacy increases (that is, F_0 decreases) by either increasing the duration of the scheme (in the case of cohort immunization), increasing the total number of impulses (in the case of time-dependent pulse vaccination), or decreasing the threshold pulse value S_{crit} (in the case of state-dependent pulse vaccination).

4. The cohort schemes seem to have a maximum efficient duration, above which increasing the duration has negligible effects. This could be a result of the fact that the response during the initial epidemic outbreak is the most important.

5. The lowest (and therefore best) control efficacy ratings are achieved in the cohort immunization of newborns and the state-dependent pulse vaccination strategy.

From Figure 6.14, we are in a position to make some observations and draw some conclusions regarding the costs versus the benefits of the control strategies outlined above:

1. From a cost-benefit perspective, state-dependent pulse vaccination performs the best.

2. Increasing the vaccination rate, which leads to fewer total infectives (as outlined above), is not necessarily the best course of action when costs are taken into account. In some cases, the costs can outweigh the benefits for high vaccination rates (this is particularly apparent in the state-dependent pulse scheme).

3. For the time-dependent pulse vaccination scheme, the benefit of increasing the number of total impulses N seems to be offset by the cost of increased vaccinations. On the other hand, increasing the vaccination rate v seems to have a non-negligible positive impact on the cost-benefit ratio of the scheme.

4. In the cohort programs, increasing the duration or the control rate outweigh the cost of additional vaccinations and are therefore beneficial for the population.

5. The state-dependent vaccination strategy cost-benefit ratios behave differently from the other strategies. In particular, the graphs are neither strictly increasing nor

strictly decreasing. Decreasing the critical threshold S_{crit} seems to increase the benefit (decreases χ) in general and it seems that the best possible strategy is to apply a pulse vaccination to a small fraction of the susceptible population with a small critical value for S_{crit}. The results on the state-dependent pulse vaccination scheme warrant further investigations.

Due to the effectiveness and versatility of the vaccination schemes outlined above, we tie these results back to the previous case study on Chikungunya disease by noting that there is currently no commercial vaccine for Chikungunya virus, but there are some candidate vaccines that have been tested in human beings and appear to be safe [155]. The US Army Medical Research Institute began vaccinations trials with some success [45], however, due to the emergence of potential terrorist biological weapons threats in 2003, the French National Institute of Health and Medical Research assumed control of the trials, with plans for a phase III trial of the candidate vaccine [155]. Recently there has been increased interest in the development of a Chikungunya vaccine and there are currently several vaccine candidates in the preclinical and clinical stages of development (see Table 1 in [190]). Based on the control efficacy and cost-benefit analysis above, it seems that more research and development should be devoted to developing a commercial vaccine for a vector-borne disease like Chikungunya. It has been noted that, in general, vaccines are cost-effective when compared to the cost of post-exposure treatment and disease management [190].

6.4 Epidemic Models with General Nonlinear Transmission

In the previous section we considered adding seasonality and various control strategies to the vector-borne infectious disease model:

$$\begin{cases} \dot{S} = \mu(1 - S(t)) - \beta S(t) \int_0^\tau f(u) I(t-u) du, \\ \dot{I} = \beta S(t) \int_0^\tau f(u) I(t-u) du - (g+\mu) I(t), \\ \dot{R} = g I(t) - \mu R(t). \end{cases} \tag{6.71}$$

In this section, we analyze the stability of the model under seasonality and two other important complications. The first complication is a generalized nonlinear transmission rate for the disease, which can more accurately reflect the spread of the disease and the population's response to an impending outbreak. The second complication is a transport model where individuals can travel between different geographic areas (e.g. cities).

6.4.1 Model Formulation

The transmission rate in a disease model is based on two crucial factors: the intrinsic infectivity of the disease and the population behaviour [38]. This is captured in the incidence rate of the disease, defined as the average number of new infected cases per unit time and given by $f_{i_k}(S, I) = \beta_{i_k} SI$ for a seasonally varying standard incidence rate. The choice of incidence rate in the problem formulation is vital as there have been reported cases in which disease eradication is not achieved under high vaccination levels (possibly due to the nonlinear dependence in the incidence rate not being correct) [38]. Non-standard incidence rates studied in the literature include the saturating effect incidence rate $f(S, I) = \beta S^p I^q$, where $0 < p < 1$. When the fraction of infected individuals is relatively high, exposure to the disease is virtually certain and the transmission rate can respond slower than linear to further increases in the number of infected [80, 81]. The psychological incidence rate $f(S, I) = \beta S I^p (1 - I)^{q-1}$ with $p > 1$, $q \geq 1$, accounts for shifts in population behaviour when knowledge of a severe epidemic becomes widespread and psychological effects cause susceptible individuals to go to extra measures to avoid infection (resulting in a decrease in the incidence rate as the number of infected individuals increases) [38]. For examples of compartmental epidemic models with a variety of different incidence rates, see [38, 66, 80, 89, 89, 98, 98, 108, 109, 109, 138, 138] and the references therein.

Prompted by this, we analyzed the following SIR epidemic model with general nonlinear incidence rate and seasonality in [115]:

$$\begin{cases} \dot{S} = \mu_{i_k}(t) - f_{i_k}(t, S, I) - \mu_{i_k}(t)S, \quad t \in [t_{k-1}, t_k), \\ \dot{I} = f_{i_k}(t, S, I) - g_{i_k}(t)I - \mu_{i_k}(t)I, \\ \dot{R} = g_{i_k}(t)I - \mu_{i_k}(t)R, \end{cases} \tag{6.72}$$

where i_k follows a switching rule σ. In this formulation, all the model parameters are switching time-varying functions of time. Moreover, the incidence rate is modelled by a general nonlinear function $f_{i_k}(t, S, I)$ which can change functional forms in time due to shifts in the population behaviour. We found eradication conditions under various control schemes based on the model parameters and functional forms $f_{i_k}(t, S, I)$. Some persistence results were also proved where the disease remains present in the population.

Motivated by this work, we incorporate a nonlinear switching incidence rate into (6.71) by assuming that the incidence rate is a general switching nonlinear function:

$$\begin{cases} \dot{S} = \mu(1 - S(t)) - \displaystyle\int_0^\tau f(u)\Psi_\sigma(S(t), I(t-u))du, \\ \dot{I} = \displaystyle\int_0^\tau f(u)\Psi_\sigma(S(t), I(t-u))du - (g+\mu)I(t), \\ \dot{R} = gI(t) - \mu R(t), \end{cases} \tag{6.73}$$

where σ is a switching rule which maps the switching intervals $[t_{k-1}, t_k)$ to $\mathcal{P}$, the initial conditions are given by $S(0) = S_0 > 0$, $R(0) = R_0 \geq 0$, and $I(s) = I_0$ for $s \in [-\tau, 0]$ where $I_0 \in PC([-\tau, 0], \mathbb{R}_+)$. From physical considerations, the functionals Ψ_i are assumed to satisfy $\Psi_i(u, v) > 0$ whenever $u > 0$ and $v > 0$; $\Psi_i(u, 0) \equiv 0$; and $\Psi_i(0, v) \equiv 0$ for all $i \in \mathcal{P}$. Further, Ψ_i are assumed to be sufficiently smooth so the model has a unique solution. From these conditions, (6.73) exhibits the disease-free equilibrium point $(S, I, R) = (1, 0, 0)$ whose stability we seek to determine. The physically meaningful domain is given by $\Omega_{\text{SIR}} = \{(S, I, R) \in \mathbb{R}^3_+ : S+I+R = 1\}$ and the initial conditions are assumed to satisfy $(S_0, I_0(0), R_0) \in \Omega_{\text{SIR}}$. The goal of the next section is to derive eradication conditions for model (6.73).

6.4.2 Disease Eradication under Pulse Treatment

We are interested in finding conditions on the incidence rates $\Psi_\sigma(S(t), I(t-u))$ under which the disease dies out. Since the main motivation for the switching here is due to seasonality, suppose the switching rule is periodic: assume that the switching times satisfy $h_k = t_k - t_{k-1}$ and $h_{k+m} = h_k$. Assume the switching rule σ satisfies $i_k = k$ and $i_{k+m} = i_k$. Denote the period of the switching rule by $\omega = h_1 + h_2 + \ldots + h_m$. The incidence rates outlined at the beginning of this section satisfy a weak nonlinearity property, namely $f(S, I) \leq \beta SI$. Motivated by this, we give the following eradication theorem.

Theorem 6.4.1. *Assume that there exist constants $\beta_i > 0$ such that $\Psi_i(u, v) \leq \beta_i uv$ for all $i \in \mathcal{P}$. Let $\lambda_i = \beta_i - g - \mu$ for $i \in \mathcal{P}$ and let $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i < 0\}$, $\mathcal{P}_u = \{i \in \mathcal{P} : \lambda_i \geq 0\}$. For $i \in \mathcal{P}_s$, choose $\eta_i > 0$ such that $\eta_i + \beta_i e^{\eta_i \tau} - (g+\mu) < 0$. If $\sigma \in \mathcal{S}_{periodic}$ and*

$$\Lambda_{generalized} = \sum_{i \in \mathcal{P}_u} \lambda_i h_i - \sum_{i \in \mathcal{P}_s} \eta_i (h_i - \tau) < 0$$

then the solution of (6.73) converges to the disease-free equilibrium.

Proof. For all $t \geq 0$,

$$\begin{aligned}\dot{I} =& \int_0^\tau f(u)\Psi_\sigma(S(t), I(t-u))du - (g+\mu)I(t), \\ \leq& \int_0^\tau f(u)\beta_\sigma S(t)I(t-u)du - (g+\mu)I(t), \\ \leq& \beta_\sigma \sup_{t-\tau\leq s\leq t} I(s) \int_0^\tau f(u)du - (g+\mu)I(t),\end{aligned}$$

since $0 \leq S \leq 1$. Then using the normalization condition on $f(u)$, it follows that

$$\begin{aligned}\dot{I} \leq& \beta_\sigma \sup_{t-\tau\leq s\leq t} I(s) - (g+\mu)I(t), \\ =& \beta_\sigma \|I_t\|_\tau - (g+\mu)I(t).\end{aligned}$$

By Proposition 4.2.3, $I(t)$ is bounded on any compact interval and for $j = 1, 2, \ldots,$

$$I(j\omega) \leq \|I_0\|_\tau \exp\left[j\Lambda_{\text{generalized}}\right]$$

which implies that I converges to zero. Then it follows that R converges to zero and $S = 1 - I - R$ implies that S converges to one. □

Remark 6.4.1. *The condition* $\Lambda_{generalized} < 0$ *implies that* $\overline{\mathcal{R}}_{generalized} < 1$ *where*

$$\overline{\mathcal{R}}_{generalized} = \frac{\sum_{i\in\mathcal{P}_u} \beta_i h_i + \sum_{i\in\mathcal{P}_s} \beta_i e^{\eta_i\tau}(h_i - \tau)}{\sum_{i\in\mathcal{P}_u}(\mu+g)h_i + \sum_{i\in\mathcal{P}_s}(\mu+g)(h_i-\tau)}. \tag{6.74}$$

which may be regarded as an approximate basic reproduction number.

If $\Lambda_{\text{generalized}} > 0$ then it is possible for the disease to persist. To combat the disease in this case, we consider a time-dependent pulse treatment scheme: suppose that a fraction of the infected population is treated periodically in time and assume that the treatment process is relatively short when compared to the time scale associated with the dynamics of the disease. More precisely, assume that a pulse treatment is applied every $\omega > 0$ time units to a fraction $0 \leq c \leq 1$ of the infected population, immediately moving them to the recovered class. Hence, the recovered class represents individuals who have recovered from the disease either naturally or by treatment. Note that $c = 0$ corresponds to the absence of control and $c = 1$ is unrealistic physically. Apply the pulse treatment strategy to (6.73) to get the control model

$$\begin{cases} \left.\begin{array}{l} \dot{S} = \mu(1 - S(t)) - \int_0^\tau f(u)\Psi_\sigma(S(t), I(t-u))du, \\ \dot{I} = \int_0^\tau f(u)\Psi_\sigma(S(t), I(t-u))du - (g+\mu)I(t), \\ \dot{R} = gI(t) - \mu R(t), \end{array}\right\} t \neq k\omega, \\ \left.\begin{array}{l} S(t) = S(t^-), \\ I(t) = (1-c)I(t^-), \\ R(t) = R(t^-) + cI(t^-), \end{array}\right\} t = k\omega, \end{cases} \tag{6.75}$$

where $k \in \mathbb{N}$. The treatment scheme aids in eradicating the disease, seen in the following result.

Theorem 6.4.2. *Assume that there exist constants $\beta_i > 0$ such that $\Psi_i(u, v) \leq \beta_i uv$ for all $i \in \mathcal{P}$. Let $\lambda_i = \beta_i - g - \mu$ for $i \in \mathcal{P}$ and let $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i < 0\}$, $\mathcal{P}_u = \{i \in \mathcal{P} : \lambda_i \geq 0\}$. For $i \in \mathcal{P}_s$, choose $\eta_i > 0$ such that $\eta_i + \beta_i e^{\eta_i \tau} - (g+\mu) < 0$. If $\sigma \in \mathcal{S}_{periodic}$ and*

$$\Lambda_{treatment} = \ln(1-c) + \sum_{i \in \mathcal{P}_u} \lambda_i h_i - \sum_{i \in \mathcal{P}_s} \eta_i (h_i - \tau) < 0 \tag{6.76}$$

then the solution of (6.73) converges to the disease-free equilibrium.

Proof. From the proof of Theorem 6.4.1, $\dot{I} \leq \beta_\sigma \|I_t\|_\tau - (g+\mu)I(t)$ for $t \neq k\omega$. Further, $I(k\omega) = (1-c)I(k\omega^-)$ at the impulsive times for $k \in \mathbb{N}$ and it follows from the proof of Corollary 4.2.9 that I converges to zero. Hence R converges to zero and S converges to one. □

Remark 6.4.2. *Condition (6.76) defines a critical pulse treatment rate c_{crit}*

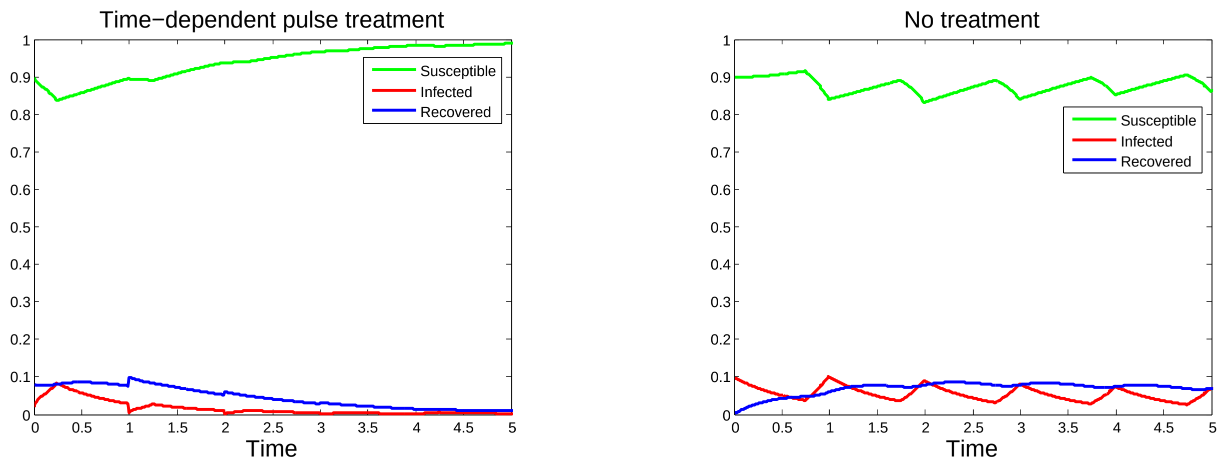
$$c_{crit} = 1 - \exp\left[-\sum_{i \in \mathcal{P}_u} \lambda_i h_i + \sum_{i \in \mathcal{P}_s} \eta_i (h_i - \tau)\right],$$

such that eradication is achieved if $c > c_{crit}$.

Example 6.4.1. *Consider system (6.78) with $\mathcal{P} = \{1, 2\}$ and initial conditions $S_0 = 0.9$, $I_0(s) = 0.1$ for $-0.1 \leq s \leq 0$, and $R_0 = 0$. Assume that the nonlinear incidence rates take the form $\Psi_i(u, v) = \beta_i uv(1-v)$, which take psychological effects into account (motivated by the incidence rate $f(S, I) = \beta S I^p (1-I)^{q-1}$ mentioned above with $p = 1$, $q = 2$). Suppose that the switching rule takes on the periodic form ($\omega = 1$):*

$$\sigma = \begin{cases} 1 & \text{if } t \in [k, k + \frac{3}{12}),\ k = 0, 1, 2, \dots \\ 2 & \text{if } t \in [k + \frac{3}{12}, k+1). \end{cases}$$

Suppose that $\beta_1 = 10$, $\beta_2 = 1$, $g = 1.5$, *and* $\mu = 1$. *Then* $\lambda_1 = 7.5$, $\lambda_2 = -1$, *and we can choose* $\eta_2 = 0.7$. *If* $c = 0$ *then* $\Lambda_{generalized} = 1.42$ *and the disease may persist, however, if* $c = 0.8$ *then* $\Lambda_{treatment} = -0.189$ *and disease eradication is guaranteed by Theorem 6.4.2. The critical treatment rate can be calculated to be* $c_{crit} = 0.758$. *See Figure 6.15 for an illustration.*



(a) $c = 0.8$. (b) $c = 0$.

Figure 6.15: Simulation of Example 6.4.1.

6.4.3 Transmission Between Cities

A multi-city switched epidemic model is used to model an infectious diseases that can spread to multiple cities via travelling individuals and transport-related infections. Examples of multi-city models in the epidemic literature can be found in [7,31,106,161,175,181,185–187]. Zhang and Zhao investigated a multi-city SIS model with general nonlinear birth-rate and periodic model parameters (including the contact rate) in [200]. In the paper [117], we investigated the following switched multi-city SIR model,

$$\begin{cases} \dot{S}_j = \mu_j N_j - f_{j\sigma}(t, S_j, I_j) - \mu_j S_j + \sum_{l=1}^{n} \alpha_{lj} S_j - \sum_{\substack{l=1 \\ l \neq j}}^{n} \alpha_{lj} h_{lj\sigma}(t, S_l, I_l), \\ \dot{I}_j = f_{j\sigma}(t, S_j, I_j) - g_{j\sigma} I_j - \mu_j I_j + \sum_{l=1}^{n} \alpha_{lj} I_j + \sum_{\substack{l=1 \\ l \neq j}}^{n} \alpha_{lj} h_{lj\sigma}(t, S_l, I_l), \\ \dot{R}_j = g_j I_j - \mu_j R_j + \sum_{l=1}^{n} \alpha_{lj} R_j, \end{cases} \tag{6.77}$$

with $i_k \in \{1, \dots, m\}$ following a switching rule and $j = 1, \dots, n$. Each city has population $N_j = S_j + I_j + R_j$ and $\sum_{j=1}^{n} N_j = N$, where the total population, N, is a constant. The birth rate in city j is given by $\mu_j > 0$, which is equal to the death rate and the recovery rate is $g_j > 0$. Individuals travel from city l to city j at a per capita rate $\alpha_{lj} \geq 0$, called the dispersal rate ($-\alpha_{jj} \geq 0$ is the emigration rate of individuals from city j to other cities). Individuals do not die, recover or give birth while travelling. The disease spreads in city j with the switching nonlinear incidence rate $f_{j\sigma}(t, S_j, I_j)$. The incidence rate of travelling infections for individuals travelling from city l to city j is given by the switching nonlinear function $h_{lj\sigma}(t, S_l, I_l)$. It is assumed that $h_{lji}(t, S_l, I_l) = h_{jli}(t, S_l, I_l)$, that is, the transportation method between city l and j is the same both ways (for example, by the same train system). Sufficient conditions were established guaranteeing global attractivity of the disease-free solution under a screening process and pulse control scheme with vaccine failure.

Here we consider extending (6.72) to include transport between cities. For a vector-borne disease, the incidence rate represents the transmission of the disease between humans and mosquitoes (i.e. an infected mosquito bites a susceptible human which results in a new human infection). This is a different mechanism driving the spread of the disease than was studied in the model (6.77) where susceptible and infected humans come into contact with each other resulting in a new case of infection. Motivated by this, we consider the following switched system of integro-differential equations:

$$\begin{cases} \dot{S}_j = \mu_j N_j(t) - \int_0^\tau f(u)\Psi_{j\sigma}(S_j(t), I_j(t-u))du - \mu_j S_j(t) + \sum_{l=1}^n \alpha_{lj} S_j(t), \\ \dot{I}_j = \int_0^\tau f(u)\Psi_{j\sigma}(S_j(t), I_j(t-u))du - (g_j + \mu_j)I_j(t) + \sum_{l=1}^n \alpha_{lj} I_j(t), \\ \dot{R}_j = g_j I_j(t) - \mu_j R_j(t) + \sum_{l=1}^n \alpha_{lj} R_j(t), \end{cases} \tag{6.78}$$

where $j = 1, \ldots, n$ and the initial conditions are $S_j(0) = S_{j,0} > 0$, $I_j(s) = I_{j,0}$ for $s \in [-\tau, 0]$ where $I_{j,0} \in PC([-\tau, 0], \mathbb{R}_+)$, and $R_j(0) = R_{j,0} \geq 0$. Each city has population $N_j = S_j + I_j + R_j$ and $\sum_{j=1}^n N_j = N$, where the total population, N, is a constant. The meaningful physical domain for system is

$$\Omega_{multi} = \left\{ (S, I, R) \in \mathbb{R}_+^{3n} | \sum_{j=1}^n S_j + I_j + R_j = N \right\}.$$

Theorem 6.4.3. *Assume that there exist constants $\beta_{ji} > 0$ such that $\Psi_{ji}(u, v) \leq \beta_{ji} uv/N_j$ for all $i \in \mathcal{P}$. Let*

$$\lambda_i = \max_{j=1,2,\ldots,n} \beta_{ji} - \min_{j=1,2,\ldots,n} (g_j + \mu_j)$$

for $i \in \mathcal{P}$ and let $\mathcal{P}_s = \{i \in \mathcal{P} : \lambda_i < 0\}$, $\mathcal{P}_u = \{i \in \mathcal{P} : \lambda_i \geq 0\}$. For $i \in \mathcal{P}_s$, choose $\eta_i > 0$ such that

$$\eta_i + \max_{j=1,2,\ldots,n} \beta_{ji} e^{\eta_i \tau} - \min_{j=1,2,\ldots,n} (g_j + \mu_j) < 0.$$

If $\sigma \in \mathcal{S}_{periodic}$ and

$$\Lambda_{multi} = \sum_{i \in \mathcal{P}_u} \lambda_i h_i - \sum_{i \in \mathcal{P}_s} \eta_i h_i < 0$$

then I_j converges to zero for $j = 1, 2, \ldots, n$ and hence the disease is eradicated in each city.

Proof. From the equation for the infected population in the j^{th} city, it follows that for $t \geq 0$:

$$\sum_{j=1}^n \dot{I}_j = \sum_{j=1}^n \left[\int_0^\tau f(u)\Psi_{j\sigma}(S_j(t), I_j(t-u))du - (g_j + \mu_j)I_j(t) + \sum_{l=1}^n \alpha_{lj} I_j(t) \right],$$

Since the emigration rate of individuals travelling from city j to city i must equal the immigration rate of individuals entering city i from city j, the sum of all immigration rates must equal the emigration rates:

$$\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq j}}^{n}\alpha_{lj}I_l = 0.$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^{n}\dot{I}_j &= \sum_{j=1}^{n}\left[\int_0^{\tau} f(u)\Psi_{j\sigma}(S_j(t), I_j(t-u))du - (g_j+\mu_j)I_j(t)\right],\\
&= \int_0^{\tau} f(u)\sum_{j=1}^{n}\Psi_{j\sigma}(S_j(t), I_j(t-u))du - \sum_{j=1}^{n}(g_j+\mu_j)I_j(t),\\
&\leq \int_0^{\tau} f(u)\sum_{j=1}^{n}\beta_{j\sigma}\frac{S_j(t)I_j(t-u)}{N_j}du - \sum_{j=1}^{n}(g_j+\mu_j)I_j(t),
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{j=1}^{n}\dot{I}_j &\leq \int_0^{\tau} f(u)\sup_{t-\tau\leq s\leq t}\sum_{j=1}^{n}\beta_{j\sigma}I_j(s)du - \sum_{j=1}^{n}(g_j+\mu_j)I_j(t),\\
&= \sup_{t-\tau\leq s\leq t}\sum_{j=1}^{n}\beta_{j\sigma}I_j(s)\int_0^{\tau} f(u)du - \sum_{j=1}^{n}(g_j+\mu_j)I_j(t),\\
&= \sup_{t-\tau\leq s\leq t}\sum_{j=1}^{n}\beta_{j\sigma}I_j(s) - \sum_{j=1}^{n}(g_j+\mu_j)I_j(t),\\
&\leq \max_{j=1,2,\ldots,n}\beta_{\sigma}\sup_{t-\tau\leq s\leq t}\sum_{j=1}^{n}I_j(s) - \min_{j=1,2,\ldots,n}(g_j+\mu_j)\sum_{j=1}^{n}I_j(t).
\end{aligned}$$

Then it follows from proposition 4.2.3 that $v(t) = \sum_{j=1}^{n} I_j(t)$ is bounded on any compact interval and

$$v(j\omega) \leq \|v_0\|_{\tau}\exp\left[j\Lambda_{\text{multi}}\right]$$

which implies that v converges to zero and hence each I_j must converge to zero. The disease is eradicated in each city. □

6.5 An SEIRV Model with Age-dependent Mixing

Many infectious diseases incubate inside the hosts for a non-negligible amount of time before the hosts become infectious, such as hepatitis B, Chagas' disease, HIV/AIDS, and tuberculosis (the last two having latent stages that may last years) [91, 131]. Motivated by this, we consider a so-called SEIR model formulation by assuming that the population is split into four compartments: the susceptible, S, who are healthy individuals able to contract the disease; the exposed, E, who have contracted the disease but are not yet infectious; the infected, I, who are infectious; and the recovered, R, who have recovered from the disease and have gained natural immunity.

Assume that the constant recruitment rate into the population is given by $\Lambda > 0$. Assume that the average contact rate between individuals (sufficient for transmission) is given by $\beta > 0$. Assume that the natural death rate of all individuals in the population is given by $\mu > 0$ and suppose that $d > 0$ is the disease-induced death rate. Suppose that the average latency period is $1/\gamma > 0$ and the average infectivity period is $1/g > 0$. Assume that the rate of increase of the exposed (and loss of susceptibles) is proportional to the number of infected and susceptible present.

If all individuals in the population mix homogeneously, then this leads to the classic formulation of the SEIR model (for example, see [75, 79, 88, 89, 131, 164]). However, homogeneous mixing is not a realistic assumption. The authors Röst and Wu considered age-dependent mixing [157]: let $i(t, a)$ be the number of individuals infected at age a and time t, then $I(t) = \int_0^\infty i(t, a)da$. Introduce the kernel function $0 \leq k(a) \leq 1$ where $k(a)$ represents the infectivity according to the age of infection. Then the SEIR model can be written as [157]:

$$\begin{cases} \dot{S} = \Lambda - \beta S(t) \displaystyle\int_0^\infty k(a)i(t, a)da - \mu S(t), \\ \dot{E} = \beta S(t) \displaystyle\int_0^\infty k(a)i(t, a)da - \gamma E(t) - \mu E(t), \\ \dot{I} = \gamma E(t) - gI(t) - dI(t) - \mu I(t), \\ \dot{R} = gI(t) - \mu R(t), \end{cases} \tag{6.79}$$

where the density $i(t, a)$ evolves by the partial differential equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) i(t, a) = -(g + d + \mu)i(t, a), \\ \qquad\qquad\qquad i(t, 0) = \gamma E(t), \end{cases} \tag{6.80}$$

which can be solved to get [157]:

$$i(t,a) = i(t-a,0)e^{-(g+d+\mu)a} = \gamma E(t-a)e^{-(g+d+\mu)a}.$$

Hence, system (6.81) can be written as

$$\begin{cases} \dot{S} = \Lambda - \beta\gamma S(t)\displaystyle\int_0^\infty k(a)E(t-a)e^{-(g+d+\mu)a}da - \mu S(t), \\ \dot{E} = \beta\gamma S(t)\displaystyle\int_0^\infty k(a)E(t-a)e^{-(g+d+\mu)a}da - \gamma E(t) - \mu E(t), \\ \dot{I} = \gamma E(t) - gI(t) - dI(t) - \mu I(t), \\ \dot{R} = gI(t) - \mu R(t). \end{cases} \tag{6.81}$$

The global stability properties of (6.81) were investigated in [157]. The basic reproduction number of (6.81) is given by

$$\mathcal{R}_0 = \frac{\beta\gamma}{g+\mu}\frac{\Lambda}{\mu}\int_0^\infty k(a)e^{-(g+d+\mu)a}da,$$

and the authors Rӧst and Wu proved global asymptotic stability of the disease-free equilibrium $(\Lambda/\mu, 0, 0, 0)$ if $\mathcal{R}_0 < 1$. The authors also gave a permanence result where the disease persists if $\mathcal{R}_0 > 1$. In [134], McCluskey resolved the endemic case and showed that $\mathcal{R}_0 > 1$ ensures global asymptotic stability of the endemic equilibrium using a Lyapunov functional.

In the report [156], Rӧst analyzed an SEI model with distributed delays and a death rate for the infected class that depends on the age of infection. A heterogeneous host population can be divided into homogeneous groups according to transmission characteristics (modes of transmission, contact patterns, geographic distributions, etc.) [90]. Motivated by this, a multi-group SEIR model with unbounded delay was studied in [90] by Li et al. to model within-group and inter-group interactions separately. The authors found global asymptotic stability results for the disease-free equilibrium and endemic equilibrium based on the spectral radius of the next generation matrix using Lyapunov functionals. These results were extended by Shu et al. in [168] to model generalized nonlinear transmission rates. Lyapunov functionals were used to give sufficient conditions for global asymptotic stability of the disease-free equilibrium and endemic equilibrium based on the basic reproduction number.

The authors Wang et al. [182] formulated and investigated the following SEIRV model

$$\begin{cases} \dot{S} = \Lambda - \beta\gamma S(t) \displaystyle\int_0^\infty k(a)E(t-a)e^{-(g+d+\mu)a}da - (\mu+v)S(t), \\ \dot{E} = \gamma[\beta S(t) + \beta_v V(t)] \displaystyle\int_0^\infty k(a)E(t-a)e^{-(g+d+\mu)a}da - \gamma E(t) - \mu E(t), \\ \dot{I} = \gamma E(t) - gI(t) - dI(t) - \mu I(t), \\ \dot{R} = gI(t) + g_v V(t) - \mu R(t), \\ \dot{V} = vS(t) - \beta_v \gamma V(t) \displaystyle\int_0^\infty k(a)E(t-a)e^{-(g+d+\mu)a}da - g_v V(t) - \mu V(t), \end{cases} \tag{6.82}$$

where v is the time-constant vaccination rate of susceptible individuals (moving them to the vaccinated class, V), β_v is the reduced contact rate ($\beta_v < \beta$) to represent the fact that vaccinated individuals have partial immunity during the vaccination process, and g_v is the average recovery rate for vaccinated individuals to obtain immunity. The basic reproduction number for (6.82) is adjusted to reflect these model changes:

$$\mathcal{R}_0 = \frac{[\beta S^* + \beta_v V^*]\gamma}{g+\mu} \int_0^\infty k(a)e^{-(g+d+\mu)a}da$$

where $S^* = \frac{\Lambda}{\mu+v}$ and $V^* = \frac{v\Lambda}{(\mu+v)(\mu+g_v)}$. Then Wang et al. showed that the disease-free equilibrium is globally asymptotically stable under the condition that $\mathcal{R}_0 < 1$. The authors also proved global asymptotic stability of an endemic equilibrium when $\mathcal{R}_0 > 1$ using a Lyapunov functional. In the next section, we extend the model by considering an impulsive control scheme and adding seasonality to the model in the form of term-time forced parameters.

6.5.1 Extending the Model

In the paper [148], Nie et al. proposed and analyzed a state-dependent impulsive vaccination strategy applied to an SIR model. Motivated by this, the first complication we consider for (6.81) is to consider the following vaccination strategy: if the infected population reaches a threshold value, $I = I_{\text{crit}}$, for some critical number of infected $I_{\text{crit}} > 0$, and the susceptible population is sufficiently high, $S \geq S_{\text{crit}}$ for some constant $S_{\text{crit}} > 0$, then a portion of the susceptible population is given a vaccination so that $S(T_k) = \epsilon S_{\text{crit}}$ where $0 < \epsilon < 1$ and T_k is an impulsive time. That is, the number of susceptible individuals after the impulse is less than the critical amount S_{crit}. Assume the time period of the vaccination is short compared to the time scale of the disease dynamics and that when the vaccination is given to a susceptible individual, they immediately move to the vaccinated class, denoted by V. Since the duration of vaccine-induced immunity is important, assume that individuals who have been vaccinated move back to the susceptible class at a rate $\theta > 0$. Similarly, assume that individuals who recover from the disease only do so temporarily for an average amount of time $1/\theta$. For example, herpes simplex tends to relapse after recovery and many other sexually transmitted diseases such as chlamydia and gonorrhea result in little to no acquired immunity [47, 91]. The second complication we consider for (6.81) is to assume that the contact rate is a piecewise constant switching parameter $\beta_i > 0$ with $i \in \{1, 2, \ldots, m\}$. Assume that the switching parameter is governed by a switching rule

$$\sigma(t) : [t_{k-1}, t_k) \to \mathcal{P},$$

for $k \in \mathbb{N}$, which is a piecewise constant.

Prompted by a seasonal increase in contact rate patterns, consider the addition of a pulse treatment strategy to help combat the spread of the disease in combination with the pulse vaccination scheme outlined above. Suppose that at certain points in time a portion of the infected class are given a treatment in a short period of time (with respect to the dynamics of the disease). That is, a public campaign is enacted whereby individuals experiencing symptoms travel to clinics to receive treatment during certain periods of time (perhaps every few months). Mathematically, assume that a portion $0 \leq p_{i_k} \leq 1$ of infected individuals receive treatment at the switching time t_k and immediately move to the recovered class. The constants $p_1, p_2, \ldots, p_m$ are the treatment rates with the idea being that the treatment rate becomes higher in response to a higher seasonal contact rate pattern. Hence if $\beta_2 > \beta_1$ then $p_2 > p_1$ in response to an increased incidence rate of the disease. Assume that the pulse treatment scheme does not apply to the exposed class, as they are not yet experiencing symptoms.

With these changes, (6.81) can be re-written as:

$$\begin{cases} \left.\begin{array}{l} \dot{S} = \Lambda - \beta_{\sigma(t)}\gamma S(t)\int_0^\infty k(a)E(t-a)e^{-(g+d+\mu)a}da - \mu S(t) + \theta V(t), \\ \dot{E} = \beta_{\sigma(t)}\gamma S(t)\int_0^\infty k(a)E(t-a)e^{-(g+d+\mu)a}da - (\gamma+\mu)E(t), \\ \dot{I} = \gamma E(t) - (g+d+\mu)I(t), \\ \dot{R} = gI(t) - \mu R(t), \\ \dot{V} = -(\theta+\mu)V(t), \end{array}\right\} x \notin \Gamma, \\ \left.\begin{array}{l} \Delta S = \epsilon S_{\text{crit}} - S(t^-), \\ \Delta E = 0, \\ \Delta I = -p_{\sigma(t)}I(t^-), \\ \Delta R = p_{\sigma(t)}I(t^-), \\ \Delta V = S(t^-) - \epsilon S_{\text{crit}}, \end{array}\right\} x \in \Gamma. \end{cases} \tag{6.83}$$

where $x = (S, E, I, R, V)^T$ and the initial conditions are $S(0) = S_0 > 0$, $I(0) = I_0 > 0$, $R(0) = R_0 \geq 0$, $V(0) = V_0 \geq 0$, and $E(s) = E_0$ for $s \leq 0$ where $E_0 \in PCB((-\infty, 0], \mathbb{R}_+)$. The impulsive set Γ is defined as

$$\Gamma = \{(S, E, I, R, V) \in \mathbb{R}_+^5 \mid S \geq S_{\text{crit}} \text{ and } I \geq I_{\text{crit}}\}.$$

A well-posedness analysis of a similar SEIR age-dependent mixing model formed the basis for a section of the report [116].

6.5.2 Well-posedness of the Model

In [157], the authors Röst and Wu analyzed (6.81) and considered the phase space UC_g of fading memory type, given by

$$UC_g = \left\{\psi : (-\infty, 0] \to \mathbb{R} \mid \frac{\psi}{g} \text{ is bounded and uniformly continuous } \right\},$$

where g satisfies the conditions [8, 48]:

(i) $g : (-\infty, 0] \to [1, \infty)$ is a non-increasing continuous function and $g(0) = 1$;

(ii) $\lim_{u \to 0^-} g(s+u)/g(s) = 1$ uniformly on $(-\infty, 0]$;

(iii) $g(s) \to \infty$ as $s \to -\infty$.

For example, these conditions are satisfied for $g(s) = e^{-\nu s}$ or $g(s) = (1+|s|)^\nu$, for $\nu \geq 0$ [8]. Then UC_g is a Banach space when equipped with the norm

$$\|\psi\| = \sup_{s\leq 0} \frac{|\psi(s)|}{g(s)}.$$

Then the authors chose the space

$$Y = \{\psi \in UC_g \mid \psi(s) \geq 0 \text{ for } s \leq 0\},$$

with $g(s) = e^{-\nu s}$, $0 < \nu < g + d + \mu$, for the history of the exposed class.

This particular choice of space is unreasonable for system (6.83) due to the impulsive discontinuities. The authors in [157] avoided the space BC of bounded continuous functions

$$BC = \left\{\psi \in C((-\infty, 0], \mathbb{R}^n) : \sup_{s\leq 0} \|\psi(s)\| < \infty\right\}$$

due to some undesirable qualitative properties with respect to functional differential equations with unbounded delay and referred the reader to [162]. However, the main problem with the space BC outlined in [162] is the fact that if x is in BC, it does not guarantee that x_t is in BC. This particular issue is resolved here by using the composite-PCB characteristic of the right-hand side of the integro-differential equations (6.83) and hence the choice of phase space here is PCB.

Proposition 6.5.1. *Assume that $\sigma \in \mathcal{S}$ and that $k(a) = e^{-qa}$ for some constant $q > 0$. Then for each $t_0 \in \mathbb{R}_+$, $S_0 \in \mathbb{R}_+$, $I_0 \in \mathbb{R}_+$, $R_0 \in \mathbb{R}_+$, $V_0 \in \mathbb{R}_+$, and $E_0 \in PCB((-\infty, 0], \mathbb{R}_+)$, there exists $0 < b \leq \infty$ such that system (6.83) has a unique non-continuable solution $(S(t), E(t), I(t), R(t), V(t))$ on $[t_0, t_0 + b)$ which satisfies $S(t) \geq 0$, $E(t) \geq 0$, $I(t) \geq 0$, $R(t) \geq 0$, $V(t) \geq 0$ for $t \geq 0$.*

Proof. System (6.83) can be written as

$$\begin{cases} x'(t) = f_{\sigma(t)}(x_t), & x \notin \Gamma, \\ \Delta x = g_{\sigma(t)}(x(t^-)), & x \in \Gamma, \end{cases}$$

where $x = (S, E, I, R, V)^T$,

$$g_i(x) = \begin{pmatrix} \epsilon S_{\text{crit}} - S \\ 0 \\ -p_i I \\ p_i I \\ S - \epsilon S_{\text{crit}} \end{pmatrix},$$

and

$$f_i(x_t) = \begin{pmatrix} \Lambda - \beta_i \gamma S(t) \int_0^\infty E(t-a) e^{-(\lambda+q)a} da - \mu S(t) + \theta V(t) \\ \beta_i \gamma S(t) \int_0^\infty E(t-a) e^{-(\lambda+q)a} da - (\gamma+\mu) E(t) \\ \gamma E(t) - \lambda I(t) \\ g I(t) - \mu R(t) \\ -(\theta+\mu) V(t) \end{pmatrix},$$

$$= \begin{pmatrix} \Lambda - \beta_i \gamma S(t) \int_{-\infty}^t E(s) e^{-(\lambda+q)(t-s)} ds - \mu S(t) + \theta V(t) \\ \beta_i \gamma S(t) \int_{-\infty}^t E(s) e^{-(\lambda+q)(t-s)} ds - (\gamma+\mu) E(t) \\ \gamma E(t) - \lambda I(t) \\ g I(t) - \mu R(t) \\ -(\theta+\mu) V(t) \end{pmatrix},$$

where $\lambda = g + d + \mu$. Consider the composition function

$$v(t) = \int_0^\infty e^{-(\lambda+q)a} E(t-a) da.$$

Suppose that $E \in PCB((-\infty, t_0 + b], \mathbb{R}_+)$ and note that $t - a$ is strictly increasing in t for any fixed a. Then $E(t-a)$ is a composition of a PCB-valued function with a strictly increasing continuous function of time. Hence it is an element of $PCB([t_0, t_0 + b], \mathbb{R}_+)$ and $\lambda, q, a > 0$ so that

$$|v(t)| \leq \int_0^\infty |e^{-(\lambda+q)a} E(t-a)| da$$

is finite for all time and, more importantly, $v \in PCB$ since $E(t-a) \in PCB$. Since the other components of f_i do not have delay, it follows that $f_i \in PCB$ for $i \in \mathbb{N}$.

Let

$$Z(E_t) = \int_0^\infty e^{-(\lambda+q)a} E(t-a) da.$$

Then for $\psi \in PCB((-\infty, 0], \mathbb{R}_+)$, we can write

$$Z(\psi) = \int_{-\infty}^0 e^{(\lambda+q)s} \psi_2(s) ds.$$

For any $\psi, \phi \in PCB((-\infty, 0], \mathbb{R}_+)$,

$$\begin{aligned}
\|Z(\psi) - Z(\phi)\| &= \left\| \int_{-\infty}^{0} e^{(\lambda+q)s}\psi(s)ds - \int_{-\infty}^{0} e^{(\lambda+q)s}\phi(s)ds \right\|, \\
&\leq \int_{-\infty}^{0} \|e^{(\lambda+q)s}\| \|\psi(s) - \phi(s)\| ds, \\
&\leq \sup_{s\leq 0} \|\psi(s) - \phi(s)\| \int_{-\infty}^{0} \|e^{(\lambda+q)s}\| ds, \\
&= \frac{1}{\lambda+q} \|\psi - \phi\|_{PCB}.
\end{aligned}$$

Write the functionals using Z to get

$$f_i(x_t) = \begin{pmatrix} \Lambda \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\mu S + \theta V \\ -(\gamma+\mu)E \\ \gamma E - \lambda I \\ gI - \mu R \\ -(\theta+\mu)V \end{pmatrix} + \begin{pmatrix} -\beta_i \gamma S Z \\ \beta_i \gamma S Z \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

then f_i is continuously differentiable with respect to S, Z, I, R, and V. For any compact set $\Omega \subset \mathbb{R}^5_+$, there exists a constant $C > 0$ such that for all $(S, Z, I, R, V), (\bar{S}, \bar{Z}, \bar{I}, \bar{R}, \bar{V}) \in \Omega$,

$$\begin{aligned}
&\|f_i(S, Z, I, R, V) - f_i(\bar{S}, \bar{Z}, \bar{I}, \bar{R}, \bar{V})\| \\
&\leq C \left(\|S - \bar{S}\| + \|Z - \bar{Z}\| + \|I - \bar{I}\| + \|R - \bar{R}\| + \|V - \bar{V}\| \right).
\end{aligned}$$

This means that for all $\psi, \phi \in PCB((-\infty, 0], \Omega)$,

$$\begin{aligned}
&\|f_i(\psi) - f_i(\phi)\| \\
&\leq C \left(\|\psi_1(0) - \phi_1(0)\| + \|Z(\psi_2) - Z(\phi_2)\| \right) \\
&\quad + C \left(\|\psi_3(0) - \phi_3(0)\| + \|\psi_4(0) - \phi_4(0)\| + \|\psi_5(0) - \phi_5(0)\| \right), \\
&\leq C \left(\|\psi_1(0) - \phi_1(0)\| + \frac{1}{\lambda+q} \sup_{s\leq 0} \|\psi_2(s) - \phi_2(s)\| \right) \\
&\quad + C \left(\|\psi_3(0) - \phi_3(0)\| + \|\psi_4(0) - \phi_4(0)\| + \|\psi_5(0) - \phi_5(0)\| \right), \\
&\leq C \sup_{s\leq 0} \left(\|\psi_1(s) - \phi_1(s)\| + \frac{1}{\lambda+q} \|\psi_2(s) - \phi_2(s)\| \right) \\
&\quad + C \sup_{s\leq 0} \left(\|\psi_3(s) - \phi_3(s)\| + \|\psi_4(s) - \phi_4(s)\| + \|\psi_5(s) - \phi_5(s)\| \right), \\
&\leq L \|\psi - \phi\|_{PCB}
\end{aligned}$$

where $L = C\max\{1, 1/(\lambda+q)\}$. Therefore f_i is locally Lipschitz for each $i \in \mathcal{P}$.

If the infected population is above the critical amount $I \geq I_{\text{crit}}$ and the susceptible population is above the threshold $S \geq S_{\text{crit}}$, an impulse is immediately applied. Denote the first impulsive moment to be T_1 (which may be the initial time). That is, $(S(T_1^-), E(T_1^-), I(T_1^-), R(T_1^-), V(T_1^-)) \in \Gamma$. From the impulsive equation for S, $S(T_1) = \epsilon S_{\text{crit}}$ which implies

$$(S(T_1), E(T_1), I(T_1), R(T_1), V(T_1)) \notin \Gamma$$

since $0 < \epsilon < 1$. Hence there must exist a constant $\delta_1 > 0$ such that

$$(S(T_1+t), E(T_1+t), I(T_1+t), R(T_1+t), V(T_1+t)) \notin \Gamma$$

for all $t \in (T_1, T_1+\delta_1]$. By similar arguments there exists a constant $\delta > 0$ such that $(S(T_k+t), E(T_k+t), I(T_k+t), R(T_k+t), V(T_k+t)) \notin \Gamma$ for all $t \in (T_k, T_k+\delta]$. Therefore, there exists a constant $\eta > 0$ such that

$$\inf_{k\in\mathbb{N}} T_{k+1} - T_k \geq \eta$$

and so the impulsive set $\Gamma \in \mathcal{I}$ is admissible. The conditions of Theorem 3.5.1 are satisfied and hence there exists a unique solution $x(t) = x(t; t_0, \phi_0)$ of (6.83) for $t \in [t_0, t_0+b)$ for some constant $b > 0$.

To prove non-negativity, note that for all $t \in \mathbb{R}_+$, $\psi \in PCB((-\infty, 0], \mathbb{R}_+^5)$, $i \in \mathcal{P}$, $j = 1, 2, 3, 4, 5$,

$$f_i^{(j)}(t, \psi)|_{\psi^{(j)}(0)=0} \geq 0 \tag{6.84}$$

where $f_i = (f_i^{(1)}, f_i^{(2)}, f_i^{(3)}, f_i^{(4)}, f_i^{(5)})^T$, since $E_0(s) : (-\infty, 0] \to \mathbb{R}_+$. Additionally, for all $t \in \mathbb{R}_+$, $\psi \in PCB((-\infty, 0], \mathbb{R}_+^5)$,

$$\psi^{(j)}(0) + g_i^{(j)}(t, \psi) \geq 0, \tag{6.85}$$

that is, $\psi(0) + g_i(T_k, \psi) \in \mathbb{R}_+^5$. Equation (6.84) implies that along any x_j-axis for $j = 1, 2, 3, 4, 5$, the differential equation in the switched system (3.2) satisfies $\dot{x}_j(t) \geq 0$, regardless of which subsystem is active. Also $x = 0$ is an equilibrium point of the system. The impulsive equation satisfies $x_j(T_k) \geq 0$ due to equation (6.85) so that the impulsive effects cannot result in the j^{th} component of the solution becoming negative. Therefore the solution $x(t)$ of (6.83) satisfies $x_j(t) \geq 0$ for $j = 1, 2, 3, 4, 5$. □

6.5.3 Numerical Simulations

For an illustration, assume that the switching rule takes on the following form

$$\sigma = \begin{cases} 1 & \text{if } t \in [k, k+0.25),\ k = 0, 1, 2, \dots \\ 2 & \text{if } t \in [k+0.25, k+1). \end{cases} \tag{6.86}$$

The contact rate experiences seasonal shifts between the winter season ($\beta_1 > 0$) and the other seasons of the year ($\beta_2 > 0$), where $\beta_1 > \beta_2 > 0$ since individuals come into contact more often in the winter season. Suppose that $p_1 > p_2$ so that after each winter period a stronger impulsive treatment is applied in response to a higher contact rate. In particular, we take $\beta_1 = 20$, $\beta_2 = 4$, $\Lambda = 100$, $\gamma = 0.1$, $\mu = 0.1$, $d = 0.1$, $g = 0.2$, $\theta = 2$, $p_1 = 0.75$, $p_2 = 0.25$. As is considered in [157], let $k(a) = e^{-qa}$ and we take $q = 1000$. The initial time is taken to be $t_0 = 0$, and the initial conditions are $S_0 = 800$, $I_0 = 100$, $R_0 = 0$, $V_0 = 0$, and $E_0(s) = 100$ for $s \leq 0$. See Figures 6.16, 6.17, and 6.18 for simulations.

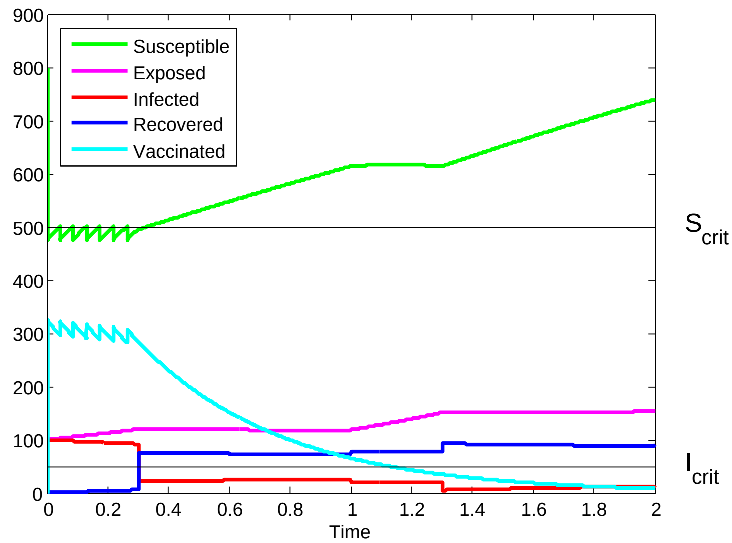


Figure 6.16: Simulation of Example 6.83 with $S_{\text{crit}} = 500$, $I_{\text{crit}} = 50$, $\epsilon = 0.95$.

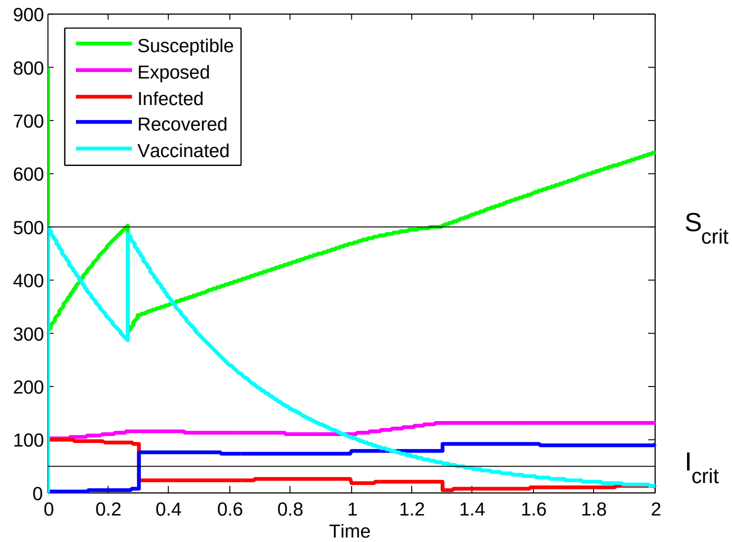


Figure 6.17: Simulation of Example 6.83 with $S_{\text{crit}} = 500$, $I_{\text{crit}} = 50$, $\epsilon = 0.6$.

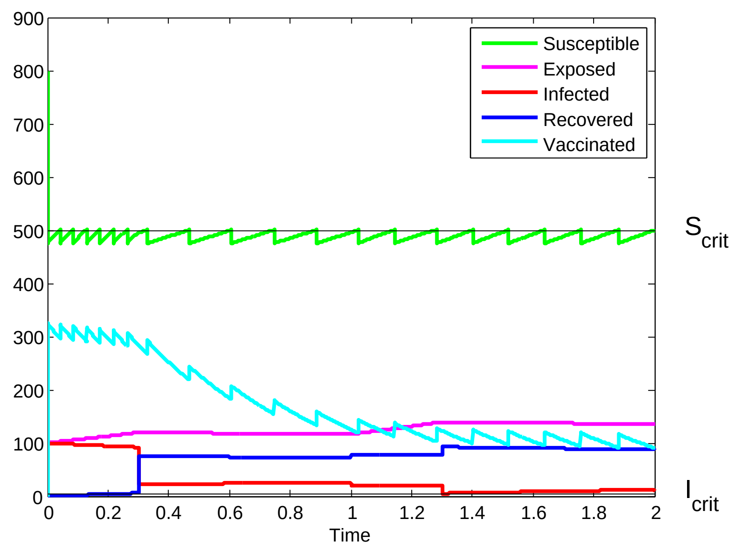


Figure 6.18: Simulation of Example 6.83 with $S_{\text{crit}} = 500$, $I_{\text{crit}} = 5$, $\epsilon = 0.95$.

Chapter 7

Concluding Remarks

7.1 Conclusions

In the present thesis we have investigated the qualitative behaviour of hybrid impulsive systems with distributed delays (HISD). The fundamental theory of HISD with infinite delay was studied in Chapter 3. Classic techniques were adjusted to account for the unbounded delay along with the switching and impulses to show local and global existence, uniqueness, and extended existence. The results apply to impulsive systems with state-dependent or time-dependent switching (where the switching times do not necessarily coincide with the impulsive times). The mathematical well-posedness of the models is important when drawing a connection to physical systems being modelled.

In Chapter 4, the stability of switched integro-differential systems with impulsive effects was analyzed in detail. First the focus was on HISD composed of stable and unstable subsystems. Verifiable sufficient conditions were developed which guarantee stability using Halanay-like switching inequalities. We followed up on this investigation by considering HISD with unbounded delay, where the Halanay-like techniques fail to work. Instead, we developed Razumikhin-type theorems which ensure stability under certain classes of switching rules. The work in this chapter has important implications in, for example, the synchronization of a driver and response system as well as disease modelling.

Hybrid control techniques were discussed in Chapter 5. Since switching and impulsive control can act as a powerful control method, this line of research has applications in control theory. We analyzed how using a combination of impulsive control and state-dependent switching control can stabilize an unstable integro-differential system with bounded or unbounded delay. Sufficient conditions guaranteeing stability were presented using Lyapunov functionals and Razumikhin-type approaches. This work extends current reports in the literature on state-dependent switching stabilization of systems with distributed delays by including nonlinearities and impulsive effects at pre-specified times or switching times. With practical implications in mind, we studied how overlapping regions and the addition of a wandering time to the stabilizing algorithms alleviates chattering-like behaviours and accumulation points. We also saw how disturbance impulses can help to avoid chattering.

The combination of expanding air travel globally and seaborne trade has removed geographic barriers to insect disease vectors [29]. With this in mind, we studied the spread of infectious diseases in Chapter 6. First, we analyzed a model of the Chikungunya virus in detail, including two possible control schemes: the mechanical destruction of breeding sites and a forced reduction in the contact rate pattern in the population. Theoretical and numerical results were given as well as an analysis of the control strategies' efficacy. Threshold criteria were presented for the eradication of the Chikungunya disease, both for periodic model parameters as well as general time-varying parameters. Multiple Lyapunov functions were used, from switched systems theory, in order to prove the results. This type of study has potential impact in designing a response strategy to an impending outbreak of the disease. We found that a reduction in the contact rates should begin immediately after an outbreak for a short period of time in conjunction with a lengthy mechanical destruction of breeding sites campaign.

We followed up these efforts with a study of vaccination schemes for a general vector-borne disease model, formulated by considering the time scales involved in the dynamics of the mosquito and human populations. We extended the current literature by analyzing switching cohort immunization, time-dependent pulse vaccination, and state-dependent pulse vaccination schemes for a vector-borne disease model with term-time forced model parameters. For each of the control schemes, eradication threshold conditions were proved. A numerical analysis was performed on the cost-benefit of the various control methods. From the control efficacy and cost-benefit studies, it seems that the best course of action to combat an impending epidemic is to administer pulse vaccinations to the public whenever the susceptible population reaches a critical threshold combined with cohort immunization programs. From the work here, we also concluded that more efforts should be spent in developing a commercial vaccine for Chikungunya.

Finally, we considered a theoretical analysis of an epidemic model with general nonlinear transmission rates and distributed delays and found eradication conditions based on the model parameters and functional form of the incidence rates. We also introduced an epidemic model for diseases with periods of latency and age-dependent population mixing. The model was extended by proposing a control strategy for eradication and by considering seasonality. We applied our fundamental theory results from earlier to find that the model is well-posed and solutions remain nonnegative.

7.2 Future Directions

One possible future direction is analyzing stochastic integro-differential equations. For example, investigating the basic theory and stability theory of HISD with unbounded delay and stochastic perturbations. Adjusting the Razumikhin-type approach in Chapter 4 for HISD with unbounded delay composed of stable and unstable modes is another avenue of possible work. Investigating conditions guaranteeing instability is an interesting and important future direction for research. This would help give insight into the conservativeness of the sufficient conditions for stability found in the present thesis by helping to narrow the gap between stability and instability.

For the hybrid control theory in Chapter 5, possible future work includes extending the stabilizing time-dependent switching approach to systems with distributed delays where currently there are no such results. Other possibilities would be to to consider optimal hybrid control for HISD with unbounded delay and models with uncertainty in the parameters. For the application in disease modelling in Chapter 6, there are many options for future research. For example, recently there has been an increased interest in sterile insect techniques to control Chikungunya, which was not considered here. A stability analysis of the SEIRV model studied in Section 6.5 is a potential line of future work. An optimal control analysis of the various control strategies studied in the disease model chapter also warrants future investigations.

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