On the local positivity of line bundles on algebraic surfaces

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Mathematics

Waterloo, Ontario, Canada, 2014

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Let X be a smooth variety and let $p \in X$. Given an effective line bundle L on X, we define

$$\gamma_p(L) = \sup\{t \geq 0 : \pi^* L - tE \text{ is effective } \}$$

where $\pi : \tilde{X} \to X$ denotes the blow-up of X at p with exceptional divisor E. This thesis develops the theory of γ_p, particularly on surfaces.

In chapter 2, after some calculations on curves and projective spaces, we specialize to the case of smooth, projective surfaces. We demonstrate a relationship between $\gamma_p(L)$ and $\epsilon_p(L)$, the Seshadri constant of L at p. We derive some general bounds on γ_p involving some Riemann-Roch type calculations, and we show that γ_p is linear on a finite collection of subcones of $\mathrm{Eff}(X)$, provided that $\mathrm{Nef}(\tilde{X})$ is finitely-generated.

In chapter 3, we specialize to the case where X is a smooth, complete, toric surface. We first show that $\gamma_p(L)$ is related to the number of copies of the two divisors corresonding to p that show up in L. Our main result, however, is that if $A, B \in \mathrm{Nef}(X)$ then we have that $\gamma_p(A + B) = \gamma_p(A) + \gamma_p(B)$. As a corollary we also obtain a result about which divisors show up in $\mathrm{Nef}(X)$, and answer a question about the product $\gamma_p(L)\epsilon_p(L)$ for a large class of toric X.

In chapter 4, we exhibit a surface X and a point p of X where $\gamma_p|_{\mathrm{Nef}(X)}$ is not linear. We calculate γ_p on several smooth toric 3-folds, and discuss future directions for this work.

Acknowledgements

This is a thesis. Rumour has it you thank people. Let's do that.

David has been my supervisor for about 6 years now, and I am extremely appreciative for his mentorship. His enthusiasm about mathematics was always transferred to me. He has always been quick to provide feedback on any notes I gave him, and has always been happy to meet as frequently as I wanted. He has always been happy to explain (repeatedly) concepts and ideas to me, and has provided fruitful guidance both inside and outside of my research.

It is quite appropriate that Rahim and Ruxandra are on my committee. Following David, the two of them have always been happy to entertain any questions I have had, and have always been available. For this I am grateful. They have both always been happy to write me letters for scholarship applications.

Thanks to Greg Smith and Eric Katz for being on my committee. I suspect that, for a faculty member, being asked to be on a thesis committee is a lot like a normal person asked to serve jury duty: you hope you don't get asked. If you do get asked, you look for a way out. And if you can't find a way out, then you console yourself by saying "it's the right thing to do". (For grad students, the appropriate analogue is a phenomenon called "pool proctoring".)

Thank you to friends. Carolyn, Jordan, and Timothy have been some of my closest friends over the past 4 years, and I greatly value their friendship, as well as Andrea. Faisal, Alejandra, Jordan, and Omar have provided me with many helpful discussions about math in general. I thank all other friends for their support as well.

My family has been very supportive throughout the years; a typical comment being "I'm sure you'll find a job after you graduate" or "How much longer do you have left?" I thank them for this - flying home for a week or two is always the high point of a year. It has been wonderful having grandparents and an uncle in the area. Entering the next stage of your life is not so dreadful when you have a strong support system in place.

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Chapter 1

Introduction

1.1 Toric Varieties

A toric variety is a normal variety X which contains a torus $T \cong (\mathbb{C}^*)^l$ as a dense open subset. The torus T is required to act (algebraicly) on X in a manner that extends the usual action $T \times T \to T$. These are a family of varieties very worthwhile of study due to their ease of calculations. For example, there are simple descriptions of Picard Groups, Intersection theories, and sheaf cohomology groups on toric varieties.

The simplest examples are the torus T itself, or $X = \mathbb{A}^n$ with the natural action. Another example is $X = \mathbb{P}^n$, where we write a point of X as $[a_1 : \cdots : a_n : a_{n+1}]$ with the usual homogeneous coordinates. The open subset $U \subseteq X$, defined by the first n coordinates being non-zero, is isomorphic to $(\mathbb{C}^*)^n$, and acts on X via $(t_1, ..., t_n) \bullet [b_1, ..., b_n, b_{n+1}] = [t_1 b_1, ..., t_n b_n, b_{n+1}]$. Another less trivial example, which is not obviously toric, is a Hirzebruch surface $\mathcal{H}_r = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1})$. Explicitly describing the torus action would require local coordinates; however, we note that $\mathcal{H}_r$ has one dense orbit, 4 1-dimensional orbits, and 4 T-invariant points.

There are singular examples too, such as a quadric cone $X = \{xy = z^2\} \subseteq \mathbb{A}^3$. Here, the open subset U of X is defined by the non-vanishing of all the coordinates. The isomorphism $(\mathbb{C}^*)^2 \to U$ is given by $(s, t) \mapsto (s^2, t^2, st)$, and we have the action $(s, t) \bullet (a, b, c) = (s^2 a, t^2 b, stc)$. In this thesis, we focus on smooth, compact examples.

Such a variety is rational; these therefore form a spase subset of the class of varieties: for example, the only compact toric curve is $\mathbb{P}^1$. Not every rational variety is toric; for example, the blow-up of $\mathbb{P}^2$ at 4 points in general position is not toric. Nonetheless, this family of varieties provides an excellent source of examples. In fact, there are only countably many isomorphism classes of toric varieties - this follows from Theorem 3.13 as there are countably many distinct fans.

The above definition does not properly illustrate why these varieties are a great source of examples and calculation. Another characterization of a toric variety is that it is a variety obtained by gluing the spectra of certain semigroup algebras. We tersely describe this construction at the beginning of chapter 3. The collection of semigroups, called a fan, is definited similarly to how a simplicial complex is defined in topology.

Just like in topology, properties of the variety may be readily deduced from properties of the fan. For example, it is simple to tell if a toric variety is complete (compact) or smooth, and the semigroup description tells us all the different orbits of the T-action. As well, it is straightforward to calculate the Picard group of such objects, and explicitly write down the intersection pairing.

Many conjectures in algebraic geometry are known to be true for toric varieties. For example, consider the coveted Fujita conjecture: this is especially worth mentioning since Seshadri constants were originally developed to help attack this conjecture:

Conjecture 1.1. *Let X be a smooth projective variety of dimension n over the complex numbers, and let A be an ample divisor on X. Let K_X denote the canonical divisor of X. Then*

- *If $\ell \geq n+1$ then $\mathcal{O}_X(\ell A + K_X)$ is basepoint free.*
- *If $\ell \geq n+2$ then $\mathcal{O}_X(\ell A + K_X)$ is very ample*

The proof of this, for smooth projective toric varieties, is quite simple. One first remarks that for these varieties, we have that A is basepoint free if and only if A is nef, and A is very ample if and only if A is ample. Then the conjecture follows from the Mori Cone Theorem (see, for example, theorem 7-2-1 of [15]). (An analogue of) the Fujita conjecture is even known for singular toric varieties [18].

Toric varieties were first introduced by DeMazure in 1970 in his paper [8]; he was interested in looking at certain subgroups of Cremona groups. This makes them a modern research area, postdating even the language of schemes. They have since become mainstream, and provide many bridges between algebraic geometry and combinatorics [4]. This class of varieties forms an important testing arena for new conjectures or theories.

One area of current research is the study of Cox rings; for a projective variety X, it is a ring that contains every homogeneous coordinate ring of X; it is defined to be

$$\bigoplus_{D \in \mathrm{Pic}(X)} \mathrm{H}^0(X, D)$$

The multiplication is defined by the tensor product of sections. These rings have been studied on many classes of varieties, such as for K3 surfaces; see [1] for example. These rings are well understood on a toric variety - see Cox's paper [6]. (He did not name them after himself.) The reason that these rings are well understood on toric varieties is because there is a simple description of the effective cone of such varieties.

1.2 Seshadri Constants

There are many ways to measure the "size" of a line bundle L on a variety X. One method is to look at the asymptotics of the sequence $(\mathrm{h}^0(X, L^{\otimes n}))_{n \geq 0}$. Seshadri constants take a different approach: we fix a point p on X, and study the family of curves $C \subseteq X$ which pass through p, and see how small the ratio $\frac{L.C}{\mathrm{ord}_p(C)}$ becomes. Thus we are studying the local behaviour of L. An introduction to Seshadri constants may be found in [13].

Let X be a smooth projective variety, and let $p \in X$. Recall that a divisor D on X is **numerically effective**, abbreviated **nef**, if $D.C \geq 0$ for all irreducible curves $C \subseteq X$. Let N be a nef divisor on X. The **Seshadri constant** of N at p, denoted $\epsilon_p(N)$, is defined to be the quantity

$$\sup\{t \geq 0 : \pi^* N - tE \text{ is nef}\} \tag{1.1}$$

where $\pi : \tilde{X} \to X$ denotes the blow-up of X at p. So $\epsilon_p(N)$ provides a measurement of how "positive" N is "at p". Note that this quantity may be defined at a possibly singular point by blowing up along the ideal sheaf $\mathcal{I}_p$ of $\mathcal{O}_X$. This quantity is also equal to

$$\inf_{p \in C \subseteq X} \frac{N.C}{\operatorname{ord}_p(C)}$$

where the infimum is taken over all irreducible curves C that contain X.

An excellent introduction and survey of Seshadri Constants is [2]. They were originally introduced by Damailly to prove the Fujita Conjecture. While this did not pan out, they have been realized to be an extremely interesting object of study in their own right. The Nagata Conjecture, a major open problem in algebraic geometry, may be formulated in the language of Seshadri constants.

In general, it is extremely difficult to precisely calculate $\epsilon_p(N)$. Bounds may sometimes be obtained. Giving an upper bound is not so bad: simply pick a curve $p \in C$ and voila: $\epsilon_p(N) \leq \frac{N.C}{\operatorname{ord}_p(C)}$. However, lower bounds are notoriously difficult. This involves showing that a divisor $\pi^* N - tE$ is nef, and even if we understand the structure of $\operatorname{Nef}(X)$, we may know very little about $\operatorname{Nef}(\tilde{X})$.

If N is very ample then we of course have that $\epsilon_p(N) \geq 1$ for all $p \in X$: in this situation we have that $X \subseteq \mathbb{P}^n$ and $N = \mathcal{O}_{\mathbb{P}^n}(1)|_X$. Then, for any curve C through p we have that $N.C = \deg(C) \geq \operatorname{ord}_p(C)$. Thus $\epsilon_p(A) > 0$ if A is ample.

There are many results giving lower bounds. For example, here is a result of Szemburg ([20]) concerning surfaces whose Picard rank is one:

Theorem 1.2. *Let S be a smooth projective surface with $\rho_S = 1$, and let L be an ample line bundle on S. Let $p \in X$.*

- *If S is not of general type, then $\epsilon_p(L) \geq 1$.*
- *If S is of general type, then $\epsilon_p(L) \geq \frac{1}{1+(K_S^2)^{1/4}}$.*

Both bounds are sharp.

Naturally, there has been much work trying to calculate Seshadri constants on toric varieties. At T-invariant points, the situation is quite simple, as will be shown for surfaces. This is because the blow-up of a toric variety at a T-invariant point is again a toric variety, and it is easy to tell if a divisor is nef on a toric variety. There are also results away from the T-invariant points, but these are harder, and usually manifest as bounds rather than exact values. For example, in [11], Ito starts with a surjective morphism $f : X \to Y$, a point p of X, and a pair of nef line bundles $L \to X$ and $M \to Y$ with some mild assumptions in place. He proves a bound involving $\epsilon_p(L)$ and $\epsilon_{f(p)}(M)$. This is then used to provide nice estimate of ϵ_p away from T-invariant points of toric varieties. He even studies some non-toric examples using toric resolutions.

Another question people ask about Seshadri constants is whether they are rational. If $\text{Nef}(\tilde{X})$ is finitely generated, then $\epsilon_p(N) \in \mathbb{Q}$ for all $N \in \text{Nef}(X)$. The connection is as follows: there exists a largest $a/b \in \mathbb{Q}$ so that $b\pi^*N - aE \in \text{Nef}(\tilde{X})$, and we have $\epsilon_p(N) = a/b$. There exist surfaces X so that $\text{Nef}(X)$ is finitely generated, but $\text{Nef}(\tilde{X})$ is not finitely-generated, which presents difficulties.

There are known examples where $\text{Nef}(\tilde{X})$ is not finitely generated, and yet we still obtain examples of $\epsilon_p(N) \in \mathbb{Q}$. For example, let X be a smooth projective K3 surface with $\text{rank}_{\mathbb{Z}}(\text{Pic}(X)) = 20$. It is shown in [19] that $\text{Aut}(X)$ is infinite, which implies that $\text{Nef}(X)$ is not finitely generated. Furthermore, every such X contains a line L. Let p belong to the line. Then $\epsilon_p(\mathcal{O}_{\mathbb{P}^n}(1)|_X) = 1$. This follows from the alternate equality

$$\epsilon_p(A) = \inf_{p \in C} \left\{ \frac{A.C}{\text{ord}_p(C)} \right\} \tag{1.2}$$

where the infimum is taken over all irreducible curves $C \subseteq X$ passing through p. Here, the witnessing curve is the line L itself, and we have that $\mathcal{O}_{\mathbb{P}^n}(1)|_X \bullet L = 1$ and $\text{ord}_p(L) = 1$.

It is conjectured that these constants are always rational, and this is indeed the case in every known example. Of course, they are rational provided that there is a witnessing curve (as in (1.2)). On a smooth complete toric surface, we always have that $\epsilon_p \in \mathbb{Z}$; this is proved in this thesis, though was previously known.

There are known examples where $\epsilon_p(L) \in \mathbb{Q} - \mathbb{Z}$. For example, Theorem 4.5 of [16] gives the following: let X be a smooth cubic surface in $\mathbb{P}^3$

$$\epsilon_p(-K_X) = \begin{cases} 1: & \text{if } x \text{ lies on one of the 27 lines} \\ \frac{3}{2}: & \text{if } x \text{ does not lie on any line.} \end{cases}$$

This also shows that ϵ does depend on the point p.

1.3 Effective Divisors and γ_p

As before, let X be a smooth projective variety, let $p \in X$. Let L be an effective divisor on X, meaning that $\text{h}^0(X, L) > 0$. Like the Seshadri constant, we define **gamma** of L at p, denoted by $\gamma_p(L)$, to be the quantity

$$\sup\{t \geq 0 : \pi^*L - tE \text{ is effective}\}. \tag{1.3}$$

The pullback of a nef divisor is always nef: this is not true for effective divisors. However, the pullback of an effective divisor via a dominant morphism is always effective, so π^*L is indeed effective. In the literature, γ_p has not been as intensively studied as ϵ_p. One example of its appearance is in [17]. In this paper, McKinnon and Roth investigate a relationship between $\epsilon_p(L)$ and another quantity $\alpha_p(L)$, which contains arithmetic information. They show that $\alpha_p(L)$ is sometimes well approximated by $\gamma_p(L)$.

Another paper where γ_p appears is [5]. In this paper, the symbol ς_p is used instead of γ_p, and the paper is called the **Nakayama constant**. However, we stick with the name gamma. As in shown in this paper, γ_p is related to other more common notions of volume, and the asymptotic behaviour of divisors. In this paper, Choi generalizes to a subvariety: given an effective divisor D and a subvariety $V \subseteq X$, he defines

$$\zeta(D, V) = \sup\{t > 0 : f^*D - tE \text{ is effective}\}$$

where $f : \tilde{X} \to X$ is the blow-up of X along the ideal sheaf $\mathcal{I}_{V \subseteq X}$. He proves that this number is related to the numerical Iitaka dimension of D, another construction that only depends on the numerical class $[D] \in \mathrm{NS}(X)$. He proceeds to show that these notions are related to Okounkov bodies, another construction used for measuring the positivity of line bundles.

There are very few explicit examples of γ_p on varieties. This arose to the question of investigating the behaviour of γ_p on smooth, complete, toric surfaces: the machinery of toric geometry is more than sufficient to calculate γ_p, at least if p is a T-invariant point of X.

This thesis is organized as follows. In chapter 2, we define the fundamental object of study: γ_p. We develop its elementary properties and then specialize to the case of (smooth, projective) surfaces. Almost immediately we get an elementary relationship between γ_p and ϵ_p:

Lemma 1.3. *Let X be a smooth, projective surface and let $p \in X$. Let $D \in \mathrm{Nef}(X) \cap \mathrm{Eff}(X)$. Then*

$$\gamma_p(D)\epsilon_p(D) \leq D^2$$

We prove some bounds on γ_p using sheaf cohomology and the Riemann-Roch Theorem. As an application, we obtain the following result:

Theorem 1.4. *Let X be a smooth projective surface with $\mathrm{Pic}(X) = \mathbb{Z}$, and suppose that we have a point $p \in X$ which satisfies $\mathrm{h}^0(\tilde{X}, K_{\tilde{X}}) = 0$. Let L be both nef and effective. Then*

$$\sqrt{L^2} + L.K_X \leq 2\chi(\mathcal{O}_X)$$

If these numbers are equal, then we in fact have

$$\gamma_p(L) = \epsilon_p(L) = \sqrt{L^2}$$

Besides $\mathbb{P}^2$, there are surfaces of general type that satisfy the hypotheses of the theorem. The result is a culmination of two separate bounds, both of which apply to a larger class of surfaces. We end chapter 2 with a theorem that shows that, if $\mathrm{Nef}(\tilde{X})$ is finitely-generated, then there exists a decomposition of $\mathrm{Eff}(X)$ into finitely many subcones so that γ_p is linear on each subcone. This theorem is how we calculate γ_p in explicit examples.

In chapter 3, we specialize to the case where X is a smooth, complete, toric surface. In this special case, we calculate some bounds on coefficients on nef divisors; these bounds are precisely what we need to prove the main theorem:

Theorem 1.5. *Let X be a smooth, complete, toric surface, and let p be a T-invariant point on X. Let $\pi : \tilde{X} \to X$ denote the blow-up of X at p. There exists a divisor $W \in \mathrm{Nef}(\tilde{X})$ so that*

$$\gamma_p(D) = W.\pi^*D$$

for all $D \in \mathrm{Nef}(X)$. In particular, $\gamma_p(A + B) = \gamma_p(A) + \gamma_p(B)$ for all $A, B \in \mathrm{Nef}(X)$ and $\gamma_p(D) \in \mathbb{N}$ for all $D \in \mathrm{Nef}(X)$.

Along the way to proving Theorem 1.5, we also prove an observation made by the author that arose over the course of this investigation. We later explain what the term “adjacent” means.

Theorem 1.6. *Let X be a smooth, complete, toric surface, and suppose we take a basis of adjacent divisors $A_1, ..., A_R$ for $\text{Pic}(X)$. Then*

$$\text{Nef}(X) \subseteq \bigoplus_{i=1}^{R} (\mathbb{R}^{\geq 0} A_i)$$

ie all the coefficients of a nef divisor are non-negative.

The question of how γ_p behaves on the entire effective cone $\text{Eff}(X)$ is also discussed. On every surface we investigated with $\text{rank}_{\mathbb{Z}}(\text{Pic}(X)) \geq 3$, we found that γ_p is not linear on the effective cone. Furthermore, it is not always integer valued:

Theorem 1.7. *There exists a smooth, projective surface X, a divisor $D \in \text{Eff}(X)$, and a point $p \in X$ so that $\gamma_p(D) \in \mathbb{Q} - \mathbb{Z}$.*

Finally, in the fourth chapter, we work out some examples of γ_p on varieties that are not smooth, complete, toric surfaces. We also raise some unsolved (by the author) questions, and comment on possible future directions to take this work. In the appendix, we provide explicit examples of γ_p on our toric surfaces, both on the effective cone and restricted to the nef cone.

Chapter 2

γ_p on Algebraic Surfaces

2.1 First properties of γ_p

Everything I know about Algebraic Geometry, particularly surfaces, may be found in [3] or the relatively unknown [10].

Let X be a smooth algebraic variety defined over the field $\mathbb{C}$ of complex numbers. The Picard Group of X, written $\mathrm{Pic}(X)$, is the group of divisors modulo linear equivalence. Given a curve C on X, there is a linear map $\mathrm{Int}_C : \mathrm{Pic}(X) \to \mathbb{Z}$ called **intersection with** C. This may be found, for example, in the appendix on Intersection Theory in [10] Here, by linear, we just mean that $\mathrm{Int}_C([A]+[B]) = \mathrm{Int}_C([A]) + \mathrm{Int}_C([B])$ for all divisor classes $[A],[B]$ and all curves C. Two divisor classes $[D_1],[D_2]$ are called **numerically equivalent** if $\mathrm{Int}_C([D_1]) = \mathrm{Int}_C([D_2])$ for all curves $C \subseteq X$. The **Neron-Severi** group of X, denoted by $\mathrm{NS}(X)$, is the quotient of $\mathrm{Pic}(X)$ by the subgroup of those $[D]$ which are numerically equivalent to $[0]$. From here on, we often omit the square brackets when talking about the class of a divisor inside $\mathrm{Pic}(X)$ or $\mathrm{NS}(X)$. It is a theorem that $\mathrm{NS}(X)$ is free, abelian, and finitely generated (i.e. $\mathrm{NS}(X) \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}$).

We say that a divisor $L \in \mathrm{Pic}(X)$ is **numerically effective**, abbreviated 'nef', if $L.C \geq 0$ for all irreducible curves $C \subseteq X$. We say that L is **effective** if $\mathrm{h}^0(X,L) \geq 1$. We say that L is **basepoint-free** if L is effective and the corresponding rational map to projective space is a morphism; in other words, $L = h^*\mathcal{O}_{\mathbb{P}^N}(1)$ for some morphism $h : X \to \mathbb{P}^N$. We say that L is **very ample** if L is basepoint-free and if the corresponding morphism is a closed embedding. Finally, we say that L is **ample** if kL is very ample for some $k \geq 1$. The **linear system** of L, denoted $|L|$, is the collection of effective divisors which are linearly equivalent to L. It carries the structure of a projective space whose dimension is one less than $\mathrm{h}^0(X,L)$. All notions above are defined on $\mathrm{Pic}(X)$; they are all well defined on the quotient $\mathrm{NS}(X)$. Thus we will use these definitions on elements of $\mathrm{NS}(X)$.

We may naturally view $\mathrm{NS}(X)$ as a lattice inside the $\mathbb{R}$-vector space $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. More precisely, denote this vector space by V. Define $g : \mathrm{NS}(X) \to V$ by $g(x) = x \otimes 1$. Then the map g embeds $\mathrm{NS}(X)$ into V; we view $\mathrm{NS}(X)$ as a subgroup of V via the map g. In the Neron-Severi group $\mathrm{NS}(X)$, we have the two semi-groups (with identity) $\mathrm{Nef}(X)$ and $\mathrm{Eff}(X)$ - the collection of all nef and effective divisor classes. We may also view them as convex semigroups inside V by taking convex hulls. More precisely, the convex hull of

$g(\text{Nef}(X))$ is a convex sub-semigroup of V, and $\text{Nef}(X)$ may be recovered as the inverse image under g of this semigroup. The same holds for $\text{Eff}(X)$. We will use these two objects interchangably, and likewise for $\text{Eff}(X)$.

A $\mathbb{Q}$**-divisor** is an element of $\text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and a $\mathbb{R}$**-divisor** is an element of V.

Given an non-degenerate bilinear pairing $< \bullet, \bullet >: W \times W \to \mathbb{R}$ on a real vector space W, and a closed convex additive sub-semigroup $\mathcal{G}$ of W, we define its **dual**, denoted $\mathcal{G}^\vee$, to be the set

$$\{x \in W :< g, x > \geq 0 \text{ for all } g \in \mathcal{G}\}.$$

On a surface X, since curves coincide with divisors, the intersection pairing is actually a bilinear map $\text{NS}(X) \times \text{NS}(X) \to \mathbb{Z}$, which is non-degenerate since we quotiented by all curves numerically equivalent to 0. It is a theorem that $\text{Nef}(X) = \overline{\text{Eff}(X)}^\vee$, where we are viewing both objects as convex semigroups in the space $V = \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Here, $\overline{\text{Eff}(X)}$ means the closure of $\text{Eff}(X)$ in the Euclidean topology on V.

We now define the main object of interest for this thesis. Given a (closed) point $p \in X$, consider the blow-up

$$\pi : \tilde{X} \to X$$

of X at p. Let E denote the exceptional divisor. If L is effective, then so is $\pi^* L$ since the map π is surjective. Our quantity of interest measures 'how effective' L is at p.

Definition 2.1. We define **gamma** of L at p to be the quantity

$$\gamma_p(L) = \sup\{t \in \mathbb{R}^{\geq 0} : \pi^* L - tE \text{ is effective}\}.$$

Remark 2.2. What does it mean to say that $\pi^* L - e^{\sqrt{2}} E$ is effective? There are two equivalent options. The first is that we actually define

$$\gamma_p(L) = \sup\left\{\frac{a}{b} \in \mathbb{Q}^{\geq 0} : b\pi^* L - aE \text{ is effective}\right\}$$

and never actually talk about $\mathbb{R}$-divisors. The above definition is purely in terms of $\mathbb{Z}$-divisors (i.e elements of $\text{NS}(X)$), even though it yields a rational number. Equivalently, we define an $\mathbb{R}$-divisor D to be effective if D belongs to the convex hull of $\text{Eff}(X)$ inside $\text{NS}(X)$.

Remark 2.3. Since $\pi_*(\pi^* L) = L$ (π is generically finite of degree 1) and $\pi_* \mathcal{O}_{\tilde{X}}(-E) = \mathcal{I}_{p \subseteq X}$ (this latter sheaf is the ideal sheaf of p in X.) we have that

$$\gamma_p(L) = \sup\left\{\frac{a}{b} \in \mathbb{Q}^{\geq 0} : \text{h}^0(X, L^{\otimes b} \otimes (\mathcal{I}_{p \subseteq X})^{\otimes a}) > 0\right\}.$$

Hence $\gamma_p(L)$ measures how much a section of some power of L can vanish at p, without making the power of L too big.

Here are the basic properties of γ_p.

Proposition 2.4. *Let $A, B \in \text{Eff}(X)$, then we have*

- $\gamma_p(lA) = l\gamma_p(A)$ *for* $l \in \mathbb{N}$.

- $\gamma_p(A+B) \geq \gamma_p(A) + \gamma_p(B)$.

Proof. For the second claim, let $\zeta \leq \gamma_p(A)$ and let $\sigma \leq \gamma_p(B)$, so that $\pi^*A - \zeta E$ and $\pi^*B - \sigma E$ are effective. In this case their sum

$$(\pi^*A - \zeta E) + (\pi^*B - \sigma E) = \pi^*(A+B) - (\zeta + \sigma)E$$

is also effective, and hence $\gamma_p(A+B) \geq \zeta + \sigma$. This inequality holds for any $\zeta \leq \gamma_p(A)$ and $\sigma \leq \gamma_p(B)$, which proves the inequality. The other claim is similar. □

As is pointed out on page 25 of [17], for any ample divisor A on $\tilde{X}$ and $D \in \mathrm{Eff}(X)$ we have that

$$\gamma_p(D) \leq \frac{(\pi^*D).A^{\dim(X)-1}}{E.A^{\dim(X)-1}}.$$

This follows immediately from the Nakai-Moishezon Criterion for ampleness. In particular, $\gamma_p(D)$ is finite.

Let's calculate γ on a curve.

Proposition 2.5. *Let C be a smooth projective curve of genus $g \geq 0$. Let $D \in \mathrm{Div}(C)$ be a divisor satisfying $d = \deg(D) > 0$. Then $\gamma_p(D) = d$ for any $p \in C$.*

Proof. The case $g = 0$ is covered in Example 2.6. Thus we assume that $g > 1$ for this proof.

Since points are divisors on curves, we use capital letters to denote the divisor P associated to the point p. Note that we are not insisting that D be effective, only that it's degree be positive. This is because (as we will see) the quantity $\gamma_p(D)$ is still well defined for any such divisor.

Since C is a smooth curve, we may take the blow-up $\pi : \tilde{C} \to C$ to be the identity map $C \to C$. Then the exceptional divisor E is just the point p. Thus we wish to calculate the quantity

$$\sup\left\{\frac{a}{b} \in \mathbb{Q}^{\geq 0} : bD - aP \text{ is effective}\right\}.$$

Observe that $\gamma_p(D) \leq d$: if $\frac{a}{b} > d$ then

$$\deg(bD - aP) = bd - a < 0$$

and such a divisor does not have a non-zero global section.

Conversely, for $n \in \mathbb{N}$, define the divisor $D_n = (n + 2g - 1)D - ndP$. Since $\deg(D_n) = d(n + 2g - 1) \geq 2g - 1$, the Riemann-Roch theorem gives us that

$$\mathrm{h}^0(C, D_n) = d(2g-1) + 1 - g \geq (2g-1) + 1 - g > 0.$$

Therefore we obtain that

$$\gamma_p(D) \geq \frac{nd}{n + 2g - 1}.$$

This ratio approaches d as $n \to \infty$. □

We can also calculate γ on a projective space.

Example 2.6. Let $X = \mathbb{P}^n$ for $n \geq 1$. Let $L = \mathcal{O}(a)$ be effective (i.e. $a \geq 0$). Then we have that, for any blow-up $\pi : \tilde{X} \to X$

$$\mathrm{Eff}(\tilde{X}) = \mathrm{Cone}(E, \pi^*\mathcal{O}(1) - E).$$

Of course the divisor $\pi^*\mathcal{O}(1) - E$ is effective, since it is the strict transform of a line through p. Also, we cannot have $b\pi^*\mathcal{O}(1) - aE$ effective for $\frac{a}{b} > 1$; this would correspond to a degree b curve which has multiplicity a at p, contraditing Bezout's Theorem.

Therefore, $\pi^*L - aE \in \mathrm{Eff}(\tilde{X})$, and clearly no larger value of a gives us this membership. Hence $\gamma_p(L) = a$.

2.2 Surfaces

We now assume that X is a smooth, algebraic, projective surface. We first obtain another expression for γ_p.

Lemma 2.7. *For a surface X and effective divisor L, we have that $\gamma_p(L) = \sup_{p \in C \in |kL|}\{\frac{\mathrm{ord}_p(C)}{k}\}$. Here, k ranges over all of $\mathbb{N} - \{0\}$.*

Proof. Recall that Remark 2.3 says

$$\gamma_p(L) = \sup\left\{\frac{a}{b} \in \mathbb{Q}^{\geq 0} : \mathrm{h}^0(X, L^{\otimes b} \otimes (\mathcal{I}_{p \subseteq X})^{\otimes a}) > 0\right\}.$$

We may view $L^{\otimes b} \otimes (\mathcal{I}_{p \subseteq X})^{\otimes a}$ as an $\mathcal{O}_X$-submodule of $L^{\otimes b}$ in a natural way: specifically, $L^{\otimes b} \otimes (\mathcal{I}_{p \subseteq X})^{\otimes a}$ may be considered as the sections of $L^{\otimes b}$ which vanish at p enough times. Hence $\mathrm{H}^0(X, L^{\otimes b} \otimes (\mathcal{I}_{p \subseteq X})^{\otimes a})$ may be viewed as a subspace of $\mathrm{H}^0(X, L^{\otimes b})$. In particular, it makes sense to talk about the divisor of a section of $L^{\otimes b} \otimes (\mathcal{I}_{p \subseteq X})^{\otimes a}$. On the one hand, if $0 \neq s \in \mathrm{H}^0(X, L^{\otimes b} \otimes (\mathcal{I}_{p \subseteq X})^{\otimes a})$, then $C := \mathrm{div}(s) \in |bL|$ and C satisfies $\mathrm{ord}_p(C) \geq a$. Conversely, given $C \in |nL|$ we have that $\mathrm{H}^0(X, L^{\otimes n} \otimes \mathcal{I}_{p \subseteq X}^{\otimes \mathrm{ord}_p(C)})) \neq 0$, whence $\gamma_p(L) \geq \frac{\mathrm{ord}_p(C)}{n}$ □

Definition 2.8. Given a nef divisor L on a smooth projective variety X, and a point p of X, we define the **Seshadri Constant** of L at p, denote $\epsilon_p(L)$, in a way analogous to γ_p:

$$\epsilon_p(L) = \sup\{t \geq 0 : \pi^*L - tE \text{ is nef}\}.$$

One may show that (see Chapter 5 of [13] for instance)

$$\epsilon_p(L) = \inf_{p \in C \subseteq X}\left\{\frac{L.C}{\mathrm{ord}_p(C)}\right\}$$

where C ranges over all irreducible curves on X which contain p.

Note that $\epsilon_p(L)$ is finite: we actually have that $\epsilon_p(L) \leq \sqrt{L^2}$: if $\pi^*L - tE$ is nef, then $(\pi^*L - tE)^2 \geq 0$, which says that $L^2 - t^2 \geq 0$. On surfaces, there is a basic relation between ϵ_p and γ_p.

Lemma 2.9. *If L be a divisor which is both nef and effective, then $\gamma_p(L)\epsilon_p(L) \leq L^2$.*

Proof. Define the **modified Seshadri constant** $\tilde{\epsilon}_p(L)$ to be

$$\inf_{p \in C \in |dL|, d \geq 1} \left\{ \frac{L.C}{\operatorname{ord}_p(C)} \right\}.$$

Since the modified infimum is taking over a subset of all curves through p, we have that

$$\epsilon_p(L) \leq \tilde{\epsilon}_p(L).$$

Next, observe that for $C \in |dL|$ we have that $L.C = dL^2$, and so

$$\tilde{\epsilon}_p(L) = L^2 \inf_{p \in C \in |dL|} \left\{ \frac{d}{\operatorname{ord}_p(C)} \right\}.$$

By Lemma 2.7, we thus see that the terms which $\tilde{\epsilon}_p$ are minimizing are exactly the reciprocals of the terms that γ_p is maximizing. Therefore, we obtain

$$\gamma_p(L)\epsilon_p(L) \leq \gamma_p(L)\tilde{\epsilon}_p(L) = L^2.$$

□

Remark 2.10. By Lemma 2.9, a lower bound on $\gamma_p(D)$ yields an upper bound on $\epsilon_p(D)$, and vice versa.

Does $\epsilon_p(L)\gamma_p(L) = L^2$ for some point p for all $L \in \operatorname{Eff}(X) \cap \operatorname{Nef}(X)$? By Lemma 2.9, this is equivalent to asking if $\epsilon_p(L) = \tilde{\epsilon}_p(L)$ for some point p and all $L \in \operatorname{Eff}(X) \cap \operatorname{Nef}(X)$.

Lemma 2.11. *Suppose that* $\operatorname{Pic}(X) \cong \mathbb{Z}$. *Then* $\gamma_p(L)\epsilon_p(L) = L^2$ *for any* L *both nef and effective.*

Proof. Let A be the ample generator of $\operatorname{Pic}(X)$. Then $\operatorname{Eff}(X) = \operatorname{Nef}(X) = \operatorname{Cone}(A)$. Also, every curve C through p lives inside $|lA|$ for some $l \geq 1$. Therefore $\gamma_p(nA)\epsilon_p(nA) = \gamma_p(nA)\tilde{\epsilon}_p(nA) = (nA)^2$. □

Theorem 2.12. *Let* X *be a smooth projective* $K3$ *surface with* $\operatorname{Pic}(X) = \mathbb{Z}L$ *for* L *ample. Let* $\beta = \lfloor \sqrt{L^2} \rfloor$. *There exists a point* p *on* X *so that either*

- $\gamma_p(L) \leq L^2/\beta$
- *or* $\gamma_p(L) \in \{\frac{L^2(\beta+1)}{\beta^2+\beta-2}, \frac{L^2(2\beta+1)}{2\beta^2+\beta-1}\}$.

Proof. This is just tacking on Lemma 2.11 to [12]. The unique theorem in [12] asserts that there exists a point p of X so that either

- $\epsilon_p(L) \geq \beta$.
- or $\epsilon_p(L) \in \{\beta - \frac{2}{\beta+1}, \beta - \frac{1}{2\beta+1}\}$.

from which our claim follows immediately. □

Let's look at some other special cases. We can sometimes use the Riemann-Roch theorem to obtain a bound on γ_p.

Lemma 2.13. *Let X be a surface, and let p be a point of X satisfying $\mathrm{h}^0(\tilde{X}, K_{\tilde{X}}) = 0$. If L is an effective divisor on X, then we have*

$$\gamma_p(L) \leq -\frac{1}{2} + \sqrt{2\chi(\mathcal{O}_X) + L^2 - L.K_X + 1/4}.$$

Proof. We use the Riemann-Roch theorem on the divisor $\pi^*L - tE$ on $\tilde{X}$. We know that $\chi(\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_X)$ and that $K_{\tilde{X}} = \pi^*K_X + E$. We thus obtain that

$$\chi(\pi^*L - tE) = \chi(\mathcal{O}_X) + \frac{1}{2}(L^2 - L.K_S - t(t+1)).$$

We are finding the largest t so that $\pi^*L - tE$ is effective. Since $K_{\tilde{X}}$ is not effective, it follows that for our optimal t we will have that $\mathrm{h}^2(\tilde{X}, \pi^*L - tE) = \mathrm{h}^0(\tilde{X}, K_{\tilde{X}} - (\pi^*L - tE)) = 0$. By dropping the h^1 term, we see that for our winning t

$$\mathrm{h}^0(\tilde{X}, \pi^*L - tE) \geq \chi(\mathcal{O}_X) + \frac{1}{2}(L^2 - L.K_S - t(t+1)).$$

To make the left hand side positive, it suffices to make the right hand side positive. Doing this yields that $\gamma_p(L)[\gamma_p(L) + 1] \leq 2\chi(\mathcal{O}_X) + L^2 - L.K_X$. Completing the square (!) gives the final answer. □

Theorem 2.14. *Let X be a surface with $\mathrm{Pic}(X) = \mathbb{Z}$ and $\mathrm{h}^0(\tilde{X}, K_{\tilde{X}}) = 0$ for some point $p \in X$. Let L be both nef and effective (i.e. a multiple of the minimal ample divisor). Then*

$$\sqrt{L^2} + L.K_X \leq 2\chi(\mathcal{O}_X).$$

Furthermore, if they are equal we have that

$$\gamma_p(L) = \epsilon_p(L) = \sqrt{L^2}.$$

Proof. Since $\epsilon_p(L) \leq \sqrt{L^2}$, we have that $\gamma_p(L) \geq \sqrt{L^2}$. This is because $\gamma_p(L)\epsilon_p(L) = L^2$ since $\mathrm{Pic}(X) \cong \mathbb{Z}$. Along with the previous lemma, we thus have the bounds

$$\sqrt{L^2} \leq \gamma_p(L) \leq \sqrt{2\chi(\mathcal{O}_X) + L^2 - L.K_X + 1/4} - 1/2.$$

The first statement is just from rearranging the inequality obtained by ignoring $\gamma_p(L)$. The second statement is when the upper bound equals the lower bound. □

Here is one possible way to establish that $\gamma_p(L) \geq 1$ in terms of a subvariety.

Lemma 2.15. *Let p be a point on a subvariety Z of X, and let L be an effective divisor on X. Suppose that $\mathrm{H}^1(X, L \otimes \mathcal{I}_{Z \subseteq X}) = 0$ and that $\mathrm{H}^0(Z, L|_Z \otimes \mathcal{I}_{\{p\} \subseteq Z}) \neq 0$. Then $\gamma_p(X, L) \geq 1$.*

Proof. Let $\iota : Z \to X$ denote the inclusion map. Then we have the restriction map $\mathcal{I}_{\{p\}\subseteq X} \to \iota_*\mathcal{I}_{\{p\}\subseteq Z}$ whose kernel is $\mathcal{I}_{Z\subseteq X}$. Tensoring by L, we have the short exact sequence

$$0 \to L \otimes \mathcal{I}_{Z\subseteq X} \to L \otimes \mathcal{I}_{\{p\}\subseteq X} \to L \otimes \iota_*\mathcal{I}_{\{p\}\subseteq Z} \to 0.$$

Taking cohomology yields a surjection

$$\mathrm{H}^0(X, L \otimes \mathcal{I}_{\{p\}\subseteq X}) \to \mathrm{H}^0(X, L \otimes \iota_*\mathcal{I}_{\{p\}\subseteq Z}) = \mathrm{H}^0(Z, L|_Z \otimes \mathcal{I}_{\{p\}\subseteq Z}).$$

Since the target space is non-zero, so is the image space. □

Here is another basic remark about γ_p. The definition of $\gamma_p(L)$ involves knowing that certain line bundles have a global section. If said line bundles have enough global sections, then we can obtain a better bound on γ_p.

Lemma 2.16. *Let $L \to X$ be effective and let $p \in X$, and let $\pi : \tilde{X} \to X$ denote the blow-up of X at p. Suppose that $\mathrm{h}^0(\tilde{X}, \pi^*L - \zeta E) \geq \zeta + 2$. Then $\gamma_p(L) \geq \zeta + 1$.*

Proof. On $\tilde{X}$, consider the sequence

$$0 \to \mathcal{O}_{\tilde{X}}(-E) \to \mathcal{O}_{\tilde{X}} \to \iota_*\mathcal{O}_E \to 0.$$

Tensor by $\pi^*L - \zeta E$ and take global sections to get the exact sequence

$$0 \to \mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\pi^*L-(\zeta+1)E)) \to \mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\pi^*L-\zeta E)) \xrightarrow{\psi} \mathrm{H}^0(\tilde{X}, (\iota_*\mathcal{O}_E)\otimes\pi^*L\otimes\mathcal{O}_{\tilde{X}}(-\zeta E)).$$

We wish to show that $\mathrm{h}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\pi^*L - (\zeta+1)E)) > 0$. This is equivalent to showing that ψ is not injective. One way to make sure that ψ is not injective is if the dimension of the domain is larger than the dimension of the codomain. Since

$$\begin{aligned} \mathrm{H}^0(\tilde{X}, (\iota_*\mathcal{O}_E) \otimes \pi^*L \otimes \mathcal{O}_{\tilde{X}}(-\zeta E)) &= \mathrm{H}^0(E, (\pi^*L - \zeta E)|_E) \\ &= \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\zeta)) \end{aligned}$$

has dimension $\zeta + 1$, the assumption ensures that ψ is not injective. □

We can repeatedly apply the technique of Lemma 2.16 to obtain the following lower bound, valid at any point.

Theorem 2.17. *Let $L \to X$ be an effective line bundle on X. Let $p \in X$. Let N satisfy $1 + 2 + 3 + \cdots + N < \mathrm{h}^0(X, L)$. Then $\gamma_p(L) \geq N$.*

Proof. Let $\pi : \tilde{X} \to X$ be the blow-up of X at p. Then we have a map of sections

$$\pi^* : \mathrm{H}^0(X, L) \to \mathrm{H}^0(\tilde{X}, \pi^*L)$$

which is injective since π is surjective. Furthermore, π^* is surjective: given a section $\sigma \in \mathrm{H}^0(\tilde{X}, \pi^*L)$, we may restrict to obtain a section $\sigma|_{\tilde{X}-E}$ of $\mathrm{H}^0(\tilde{X} - E, (\pi^*L)|_{\tilde{X}-E})$. The isomorphism $\tilde{X}-E \cong X-\{p\}$ induces, in a natural way, a section τ of $\mathrm{H}^0(X-\{p\}, L|_{X-\{p\}})$. Note that τ does not have any poles on p, since the divisor of τ is a union of curves. Thus

τ extends to a section $\tilde{\tau}$ of $\mathrm{H}^0(X, L)$, and the section $\tilde{\tau}$ maps to σ under the map π^*. Therefore we have that $\mathrm{h}^0(X, L) = \mathrm{h}^0(\tilde{X}, \pi^* L)$.

Write $\pi^* L = \mathcal{O}_{\tilde{X}}(D)$ for some divisor D. Consider the sheaf sequence

$$0 \to \mathcal{O}_{\tilde{X}}(D - E) \to \mathcal{O}_{\tilde{X}}(D) \to (\iota_* \mathcal{O}_E) \otimes \mathcal{O}_{\tilde{X}}(D) \to 0. \tag{2.1}$$

For $k \geq 0$, let Ω_k denote the image of the map $\mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(D - kE)) \to \mathrm{H}^0(E, (\mathcal{O}_{\tilde{X}}(D - kE))|_E)$, and let ω_k denote its dimension. Since $\mathrm{H}^0(E, (\mathcal{O}_{\tilde{X}}(D - kE))|_E) \cong \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$ we have that $\omega_k \leq k + 1$. Take global sections of (2.1) to obtain the short exact sequence

$$0 \to \mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(D - E)) \to \mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(D)) \to \Omega_0 \to 0$$

from which we count dimensions to obtain

$$\mathrm{h}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(D - E)) = \mathrm{h}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(D)) - \omega_0 \geq \mathrm{h}^0(X, L) - 1.$$

Provided that $\mathrm{h}^0(X, L) \geq 2$, this shows that $\gamma_p(L) \geq 1$. Multiply the sequence (2.1) by $-E$ and take sections to obtain the sequence

$$0 \to \mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(D - 2E) \to \mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(D - E) \to \Omega_1 \to 0$$

from which dimension counting gives

$$\mathrm{h}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(D - 2E)) = \mathrm{h}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(D - E)) - \omega_1 \geq \mathrm{h}^0(X, L) - 1 - 2.$$

Provided that $\mathrm{h}^0(X, L) - 1 - 2 > 0$, this shows that $\gamma_p(L) \geq 2$.

Suppose, for the sake of induction, that we have $1 + 2 + \cdots + \lambda + (\lambda + 1) < \mathrm{h}^0(X, L)$. By taking global sections of the appropriate sheaf sequence we obtain the short exact sequence

$$0 \to \mathrm{H}^0(\tilde{X}, D - (\lambda + 1)E) \to \mathrm{H}^0(\tilde{X}, D - \lambda E) \to \Omega_k \to 0.$$

This yields that

$$\begin{aligned} \mathrm{h}^0(\tilde{X}, D - (\lambda + 1)E) &= \mathrm{h}^0(\tilde{X}, D - \lambda E) - \omega_\lambda \\ &\geq \mathrm{h}^0(\tilde{X}, D - \lambda E) - (\lambda + 1) \\ &\geq \mathrm{h}^0(X, L) - 1 - 2 - \cdots - \lambda - (\lambda + 1) \end{aligned}$$

where the final inequality is the inductive step. Thus, if the final term is positive, so is $\mathrm{h}^0(\tilde{X}, D - (\lambda + 1)E)$, and so $\gamma_p(L) \geq \lambda + 1$.

□

Theorem 2.18. *Let $\pi : \tilde{X} \to X$ denote the blow-up of X at p. Suppose that* $\mathrm{Nef}(\tilde{X})$ *is finitely-generated. There exists subcones $C_1, ..., C_s$ of* $\mathrm{Eff}(X)$*, which cover* $\mathrm{Eff}(X)$*, so that γ_p is linear on each C_i.*

Proof. Let E be the exceptional curve of the blow-up. Let T be a finite set of generators of $\mathrm{Nef}(\tilde{X})$ and let $S \subseteq T$ be defined as the collection of $N \in T$ which satisfy $N.E > 0$. Let $D \in \mathrm{Eff}(X)$. We have that

$$\gamma_p(D) = \sup\{t \geq 0 : \pi^* D - tE \text{ is effective}\},$$

Now, $\pi^*D - tE$ is effective if and only if $(\pi^*D - tE).N \geq 0$ for all $N \in T$. If $N \notin S$ then $(\pi^*D - tE).N = \pi^*D.N \geq 0$, so the t only matters for those N which belong to S. For those N the condition $(\pi^*D - tE).N \geq 0$ is rewritten as

$$t \leq \frac{\pi^*D.N}{E.N}.$$

Therefore t is the largest number which satisfies the above inequality for all $N \in S$, ie

$$\gamma_p(D) = \min_{N \in S} \left\{ \frac{\pi^*D.N}{E.N} \right\}.$$

Label the elements of S as $N_1, ..., N_s$. Associated to N_k we define the set C_k to be the set of all $D \in \text{Eff}(X)$ which satisfy

$$\frac{\pi^*D.N_k}{E.N_k} \leq \frac{\pi^*D.N_j}{E.N_j}$$

for all $1 \leq j \leq s$. Since the expression $\frac{\pi^*D.N_k}{E.N_k}$ is linear in D, each C_k is a subcone of $\text{Eff}(X)$. It is also clear that each $L \in \text{Eff}(X)$ belongs to some C_l. □

Chapter 3

Smooth Complete Toric Surfaces

In this chapter, we say everything we can about γ on smooth, complete, toric surfaces.

3.1 Introduction to Toric Varieties

This section is a terse introduction to the theory of toric varieties. No proofs or examples are included; a good source of both these is [7] or [9]. Since we are working with finitely-generated $\mathbb{C}$-algebras, all occurences of the term 'Spec' are taken to mean 'variety' rather than 'scheme'.

Definition 3.1. Let σ be a subset of $\mathbb{R}^n$. We call σ a **cone** if there exist vectors $v_1, ..., v_k \in \mathbb{Z}^n \subseteq \mathbb{R}^n$ so that $\sigma = \sum_{i=1}^{k} \mathbb{R}^{\geq 0} v_i$. In other words, σ is a cone if it is a convex additive sub-semigroup of $\mathbb{R}^n$, finitely generated by integer points, which contains the origin.

For the rest of this section, σ denotes a cone in $\mathbb{R}^n$. The integer points of σ are also a semigroup. In fact, they are finitely generated: this is the content of Gordon's Lemma:

Lemma 3.2. *$\sigma \cap \mathbb{Z}^n$ is a finitely generated semigroup.*

Let $V = \mathbb{R}^n$ and $V^* = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$ be the dual vector space. Let $e_1, ..., e_n$ denote the standard basis of V. For $1 \leq j \leq n$, let e_j^* denote the corresponding dual functional: $e_j^*(e_i) = \delta_{ij}$. We let $N = \mathbb{Z}^n \subseteq V$ and $M = \oplus_{i=1}^{n} \mathbb{Z} e_i^* \subseteq V^*$. We call N a **lattice** in V, and M is the **dual lattice** in V^*.

Definition 3.3. The **dual cone** of σ, denoted $\sigma^\vee$, is defined to be the set

$$\{\psi \in V^* : \psi(x) \geq 0 \text{ for all } x \in \sigma\}.$$

The dual cone is a cone in V^*; this is the content of Farkas's Theorem:

Theorem 3.4. *There exist $\phi_1, ..., \phi_\ell \in M$ so that $\sigma^\vee = \sum_{i=1}^{\ell} \mathbb{R}^{\geq 0} \phi_i$.*

Lemma 3.2 and Theorem 3.4 imply that $\sigma^\vee \cap M$ is a finitely generated semigroup, which we denote by S_σ. Thus $\mathbb{C}[S_\sigma]$ is a finitely generated $\mathbb{C}$-algebra, and hence determines a complex algebraic variety.

Definition 3.5. The **affine toric variety** associated to σ is defined to be $\operatorname{Spec} \mathbb{C}[S_\sigma]$. We denote this by U_σ.

The association $\sigma \mapsto \sigma^\vee$ is order-reversing. Thus $\sigma^\vee$ is contained in $\{0\}^\vee = M$, and hence $\mathbb{C}[S_\sigma]$ is a $\mathbb{C}$-subalgebra of $\mathbb{C}[M]$. Since $\mathbb{C}[M] = \mathbb{C}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$, we thus have that U_σ is irreducible.

Definition 3.6. The n-dimensional **torus** is defined to be $\operatorname{Spec} \mathbb{C}[M]$. We denote it by T. It is isomorphic to $(\mathbb{C}^*)^n$.

T acts as an algebraic group on U_σ: this action corresponds to the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\sigma] \otimes_{\mathbb{C}} \mathbb{C}[M]$ given by $a \mapsto a \otimes a$.

Theorem 3.7. *U_σ is normal and T is a dense open subset of U_σ.*

Definition 3.8. Given $\lambda \in V^*$ we define $\lambda^\perp$ to be the subset $\{x \in V : \lambda(x) = 0\}$. Provided that $\lambda \neq 0$, it follows that $\lambda^\perp$ is a hyperplane in V. Let τ be a subcone of σ. We say that τ is a **face** of σ if $\tau = \sigma \cap \theta^\perp$ for some $\theta \in \sigma^\vee$.

For a cone σ in $\mathbb{R}^n$ we denote by A_σ the algebra $\mathbb{C}[S_\sigma]$. Let τ be a face of σ, realized by $\theta \in \sigma^\vee$. Then it may be shown that $S_\tau = S_\sigma + \mathbb{Z}\theta$. This shows that A_τ is the localisation of A_σ at θ, i.e. that $A_\tau = (A_\sigma)_\theta$. Therefore we obtain that

Lemma 3.9. *If τ is a face of σ, then U_τ is the principal open subset of U_σ defined by the non-vanishing of θ (viewed as an element of A_σ).*

A general toric variety is obtained by glueing together affine toric varieties. The relevant definition is that of a fan:

Definition 3.10. A **fan** in V is a finite collection Δ of cones (in $\mathbb{R}^n$) which satisfy the following two properties:

- If $\sigma \in \Delta$ and τ is a face of σ, then $\tau \in \Delta$.
- If $\sigma_1, \sigma_2 \in \Delta$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

For the rest of this section, Δ will denote a fan in V.

Definition 3.11. The **toric variety** X_Δ associated to Δ is defined to be the $(\bigsqcup_{\sigma \in \Delta} U_\sigma)/\sim$ where $\sim$ is defined by the glueing of U_σ and $U_{\sigma'}$ along $U_{\sigma \cap \sigma'}$ for $\sigma, \sigma' \in \Delta$.

The torus T acts on each U_σ, and the action agrees on overlap. Therefore T acts on X_Δ.

Theorem 3.12. *The variety X_Δ is normal, and is separated over* $\operatorname{Spec}(\mathbb{C})$. *Furthermore, X_Δ contains the torus T as a dense, open subset of X_Δ; there is an action of algebraic groups of T on X_Δ which extends the usual action of T on itself.*

Every variety satisfying the above hypothesis is of the above form:

Theorem 3.13. *Let Y be a variety satisfying the conditions of Theorem 3.12. Then there exists a fan Δ so that $Y = X_\Delta$. The fan Δ is unique modulo* $\mathrm{SL}(\mathbb{Z}^n)$.

Definition 3.14. The **minimal generators** of a cone σ are the smallest collection of vectors $v_1, ..., v_k$ so that $\sigma \cap \mathbb{Z}^n = \mathbb{N}v_1 + \cdots + \mathbb{N}v_k$. They are unique for any cone.

Definition 3.15. The **support** of Δ, denoted $|\Delta|$, is defined to be $\bigcup_{\sigma \in \Delta} \sigma \subseteq \mathbb{R}^n$. We say a cone σ is **smooth** if the minimal generators σ are part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. This means: denote the minimal generators of $\sigma \cap \mathbb{Z}^n$ by $v_1, ..., v_k$. Then we require that the v_j are $\mathbb{R}$-linearly independant, and may be extended to a $\mathbb{Z}$-spanning set of $\mathbb{Z}^n$.

Theorem 3.16. *The variety X_Δ is compact (i.e. the morphism $X_\Delta \to \mathrm{Spec}(\mathbb{C})$ is complete) if and only if $|\Delta| = \mathbb{R}^n$. The variety X_Δ is smooth if and only if each $\sigma \in \Delta$ is smooth.*

The following theorem, known as the Orbit-Cone Correspondence, gives a correspondence between elements of Δ and the orbits of the T action on X_Δ.

Theorem 3.17. *There is a bijection between Δ and the orbits of X_Δ. For a cone σ we denote its orbit by $\mathcal{O}_\sigma$. Each $\mathcal{O}_\sigma$ is a torus (not of full dimension) in X_Δ. In fact,* $\dim(\mathcal{O}_\sigma) = n - \dim(\sigma)$, *where* $\dim(\sigma)$ *is defined to be* $\dim_{\mathbb{R}}(\mathrm{Span}_{\mathbb{R}}(\sigma))$. *We have that*

$$U_\sigma = \bigcup_{\tau \subseteq \sigma, \tau \in \Delta} \mathcal{O}_\tau.$$

The closure of each orbit is a T-invariant subvariety of X_Δ, denoted by $V(\sigma)$ for the orbit $\mathcal{O}_\sigma$. We have that

$$V(\sigma) = \bigcup_{\sigma \subseteq \Sigma, \Sigma \in \Delta} \mathcal{O}_\Sigma$$

and that $\dim(V(\sigma)) = n - \dim(\sigma)$.

This leads into the theory of divisors on toric varieties. Theorem 3.17 shows that each one-dimensional cone ρ (called a **ray**) of Δ determines a T-invariant (since it is a union of orbits) subvariety D_ρ of codimension one. Here, T-invariant means that $T \bullet D_\rho = D_\rho$, not that each point is fixed by T. Conversely, given an (irreducible) T-invariant subvariety Y of X_Δ of codimension one, it must be a union of orbits, and hence must be one of the D_ρ by Theorem 3.17.

Denote by $\mathrm{Div}_T(X_\Delta)$ the group of T-invariant divisors of X_Δ. More precisely, let $\Delta(1)$ denote the one-dimensional cones of Δ. Then $\mathrm{Div}_T(X_\Delta) = \bigoplus_{\rho \in \Delta(1)} \mathbb{Z}D_\rho$.

There are some distinguished rational functions on X_Δ, namely the characters of T. More precisely, a **character** of a torus $T \cong (\mathbb{C}^*)^\ell$ is a group homomorphism $\chi : T \to \mathbb{C}^*$ which is a morphism of varieties. The characters of χ form an abelian group isomorphic to $\mathbb{Z}^\ell$. In fact, it can be shown that the characters of X_Δ naturally correspond to M, the lattice dual to $\mathbb{Z}^n$. That is, each dual linear functional $m \in M$ yields an element χ^m of $\mathbb{C}[M]$. Elements of $\mathbb{C}[M]$ correspond to regular maps $T = \mathrm{Spec}(\mathbb{C}[M]) \to \mathbb{C}$, and it may be verified that these χ^m are precisely the characters.

Thus we may ask for the divisor of a character. It may be verified that for a character χ^m we have that $\mathrm{div}(\chi^m) \in \mathrm{Div}_T(X_\Delta)$. More precisely: each ray $\rho \in \Delta(1)$ has a unique minimal generator $u_\rho \in \mathbb{Z}^n$, and the following lemma computes $\mathrm{div}(\chi^m)$.

Lemma 3.18. *We have that*

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Delta(1)} m(u_\rho) D_\rho$$

where $m(u_\rho)$ means the usual evaluation map $V^* \times V \to \mathbb{R}$.

Denote by $\mathrm{Cl}(X_\Delta)$ the group of Weil divisors on X_Δ, modulo linear equivalence. We get the following lovely method of computing $\mathrm{Cl}(X_\Delta)$.

Theorem 3.19. *The sequence*

$$0 \to M \to \mathrm{Div}_T(X_\Delta) \to \mathrm{Cl}(X_\Delta) \to 0$$

is exact, where the first map is $m \mapsto \operatorname{div}(\chi^m)$ and the second map is the standard projection.

From here on, we are going to assume that X_Δ is smooth and complete. In this case, the group $\mathrm{Cl}(X_\Delta)$ coincides with the group $\mathrm{Pic}(X)$ of line bundles modulo isomorphism. From now on, we denote a T-invariant (Weil) divisor by $D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho$; an arbitrary divisor is linearly equivalent to a T-invariant divisor by Theorem 3.19. For the sake of notation, we will simply write $D = \sum_\rho a_\rho D_\rho$. Also, let $\Delta(n)$ denote the n-dimensional cones of Δ (= the highest dimensional cones).

There is a convenient description of $\mathrm{h}^0(X_\Delta, D)$ as well. Begin by representing D as a T-invariant divisor, ie $D \sim \sum_\rho a_\rho D_\rho$. In V^* we form the polytope $P_D = P$ defined by

$$\{\phi \in V^* : \phi(u_\rho) \geq -a_\rho \text{ for all } \rho \in \Delta(1)\}.$$

On any smooth variety Y the space $\mathrm{H}^0(Y, D)$ may be realized as those rational functions $f \in \mathbb{C}(Y)$ which satisfy $\operatorname{div}(f) \geq -D$. Thus, by Lemma 3.18, P_D contains all characters χ which satisfy $\operatorname{div}(\chi) \geq -D$. These characters are in fact a basis of $\mathrm{H}^0(X_\Delta, D)$, and we therefore obtain:

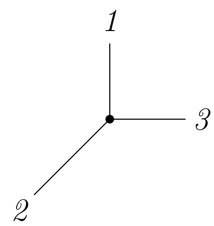
Proposition 3.20. *The dimension $\mathrm{h}^0(X_\Delta, D)$ is equal to $\#(P_D \cap M)$. (Recall that M is the dual lattice inside V^*, whose elements correspond to the characters of T.)*

From here on, we are going to assume that, in addition to X_Δ being smooth and complete, it is also a surface. Thus such a surface is specified by a fan in $\mathbb{R}^2$, whose support equals $\mathbb{R}^2$, and where each two-dimensional cone has two minimal generators which span $\mathbb{Z}^2$. Both these requirements follow from Theorem 3.16. There is a classification of such surfaces, which roughly says that each surface is obtained by finitely many blow-ups of $\mathbb{P}^2$ or a Hirzebruch Surface at T-invariant points.

In our illustrations of fans, it is understood that the fan includes all two-dimensional cones coming from adjacent vectors, all rays (coming from a single vector), and the zero cone.

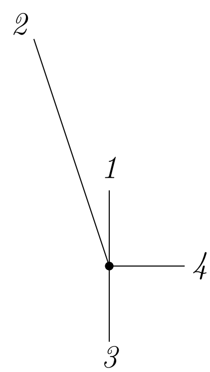
Theorem 3.21. *A smooth complete toric surface has the following structure. All fan descriptions are after possibly transforming by an element of $\mathrm{GL}(2, \mathbb{Z})$.*

- *If $r = 3$ then $X \cong \mathbb{P}^2$. Its fan is of the form*



with $u_1 = (0, 1)$, $u_2 = (-1, -1)$, and $u_3 = (1, 0)$.

- *If $r = 4$ then for some $n \geq 0$ we have $X \cong \mathcal{H}_n$, the n^{th} Hirzebruch Surface, which is defined to be $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(n))$. Its fan is of the form*



with $u_1 = (0, 1)$, $u_2 = (-1, n)$, $u_3 = (0, -1)$, and $u_4 = (1, 0)$.

- *If $r \geq 5$ then there exists some $1 \leq t \leq r$ and $\lambda \in \mathbb{Z}$ so that $\lambda u_t = u_{t-1} + u_{t+1}$. In this case, X is a blow-up of the toric variety whose fan is equal to the original fan minus u_t. In this case, D_t is the exceptional curve of the blow-up.*

Remark 3.22. The labeling conventions used above are unconvential. More precisely, it is standard to label the ray corresponding to $(1, 0)$ as the first ray, and move around counterclockwise. We label the ray corresponding to $(0, 1)$ as the first ray and move around counterclockwise. This is because the results are nicer to state with this labeling convention.

Definition 3.23. The **Cartier data** of $D = \sum_\rho a_\rho D_\rho$ is the collection $\{m_\sigma\}_{\sigma \in \Delta(n)}$ where $m_\sigma \in M$ is defined by $m_\sigma(u_\rho) = -a_\rho$ for $\rho \in \sigma(1)$.

This coincides with the notion of a Cartier divisor on a variety. More precisely, the collection $\{(U_\sigma, \chi^{m_\sigma})\}_{\sigma \in \Delta(n)}$ is Cartier data for the divisor D, in the sense of Proposition 6.11 of [10]. We mention the Cartier data because it is used to compute intersection numbers.

On a surface, a curve is a divisor, and therefore the intersection pairing is a non-degenerate bilinear map $\mathrm{NS}(X) \times \mathrm{NS}(X) \to \mathbb{Z}$. The T-invariant curves correspond to the rays. Each ray determines a pair of two-dimensional cones: namely the two cones whose intersection is the ray in consideration. The following result describes the computation of the intersection theory. We set the notation before stating the proposition:

Let $D = \sum_\rho a_\rho D_\rho$ have Cartier data $\{m_\sigma\}_{\sigma\in\Delta(2)}$. Let C be a curve, corresponding to two cones σ_1, σ_2 of $\Delta(2)$. Let u be the minimal generator of the ray corresponding to C. Pick $v \in \sigma_1 \cap \mathbb{Z}^2$ so that $\overline{v}$ is a generator of $\overline{\sigma_1}$ in $\mathbb{R}^2/\mathbb{R}u$.

Proposition 3.24. *We have that* $D.C = (m_{\sigma_1} - m_{\sigma_2})(v)$.

Proposition 3.24 actually holds on complete toric varieties of arbitrary dimension. When we are in the case of smooth, complete, toric surfaces, there is actually a much simpler description of the intersection pairing: let u_ρ denote the minimal generator of the ray ρ.

Theorem 3.25. *For each i there exists $\lambda_i \in \mathbb{Z}$ so that $\lambda_i u_i = u_{i-1} + u_{i+1}$. The intersection theory of X is given by*

$$D_i.D_j = \begin{cases} -\lambda_i : i = j \\ 1 : i \neq j, u_i \text{ is adjacent to } u_j \\ 0 : i \neq j, u_i \text{ is not adjacent to } u_j \end{cases}$$

This description comes from the fact that on a smooth, complete, toric surface, the Cartier data of a T-invariant curve is particularly simple to compute.

Corollary 3.26. $NS(X) = \mathrm{Pic}(X)$.

Proof. We must show that the only divisor numerically equivalent to 0 is the zero divisor. Since X is smooth, by Theorem 3.19 we have that

$$\mathrm{Pic}(X) = \bigoplus_{i=1}^{r-2} \mathbb{Z}D_i$$

where $r = \#\Delta(1)$ is the number of rays of the fan Δ. Let $D = \sum_{i=1}^{r-2} \delta_i D_i$ be a divisor which is numerically equivalent to 0. By Theorem 3.25, we have that $D.D_{r-1} = \delta_{r-2} = 0$. We then have that $D.D_{r-2} = \delta_{r-3} = 0$. Keep doing this to get that all $\delta_l = 0$. □

We also need to know about the nef cone $\mathrm{Nef}(X) \subseteq \mathrm{Pic}(X)$.

Theorem 3.27. *Let $D \in \mathrm{Pic}(X)$. The following are equivalent:*

- *D is nef.*
- *$D.D_i \geq 0$ for all $1 \leq i \leq r$*
- *D is basepoint free.*

In particular, $\mathrm{Nef}(X) \subseteq \mathrm{Eff}(X)$.

Proof. This is Theorem 6.3.12 of [7]. The "in particular" part: every basepoint free divisor is, of course, effective. □

Finally, we mention the toric description of blow-ups at T-invariant points. Let p be the point corresponding (Theorem 3.17) to the cone generated by the adjacent vectors u_i and u_{i+1}. Let $\tilde{X}$ denote the blow-up of X at p. Then the fan associated to $\tilde{X}$ is the same as the fan of X, except with the ray $u_i + u_{i+1}$ added in. This new ray corresponds (Theorem 3.17) to the exceptional curve of the blow-up.

Remark 3.28. There is a wonderful description of **toric morphisms**, that is morphisms $g : X \to Y$ of toric varieties which satisfy $g(T_X) \subseteq g(T_Y)$ and $g(t \bullet x) = g(t) \bullet g(x)$ for all $t \in T_X$ and $x \in X$, in terms of the fans of X and Y. However, we only use this once (Lemma 3.34) and as such do not include a description of these. See Chapter 3 of [7] for details.

3.2 Hirzebruch Surfaces

The Hirzebruch surface $\mathcal{H}_n$, whose fan is shown in Theorem 3.21, is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. In the fan, we have that A_2 and A_4 are fibres of the projection $\mathcal{H}_n \to \mathbb{P}^1$ (in fact, they are the fibres above the two T-invariant points on $\mathbb{P}^1$), A_1 is the unique irreducible curve which satisfies $A_1^2 = -n$, and A_3 is a section which satisfies $A_3^2 = n$. A proof of these facts may be found in chapter 1 of [9].

We work with the basis $\mathbb{Z}A_1 \oplus \mathbb{Z}A_2$ of $\mathrm{Pic}(X)$. Let $D = aA_1 + bA_2$. In this section, we write down a closed form expression for the number

$$\mathrm{h}^0(X, D)$$

in terms of r, a, and b. The author has never seen this formula written down in another source, and it seems like a nice example of some of the toric machinery. We assume that $n > 0$, since we already know global sections of $\mathcal{H}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Let's determine P_D, the polytope used to calculate $\mathrm{h}^0(X, D)$ by Proposition 3.20.

Let $e_1^* \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R})$ be defined by $e_1^*(1, 0) = 1$ and $e_1^*(0, 1) = 0$, and likewise for e_2^*. Then P_D is the collection of linear functionals $\phi = \alpha e_1^* + \beta e_2^*$ which satisfy

$$\begin{aligned}
\phi(u_1) &= \beta \geq -a \\
\phi(u_2) &= -\alpha + n\beta \geq -b \\
\phi(u_3) &= -\beta \geq 0 \\
\phi(u_4) &= \alpha \geq 0
\end{aligned}$$

It is easy to see that $P_D = \emptyset$ if either $a < 0$ or $b < 0$, so assume both a and b are non-negative. We count the number of lattice points of P_D. Along the $(\beta = 0)$-axis, we have the $b+1$ lattice points $(0, 0), (1, 0), ..., (b, 0)$. Along the $(\beta = -1)$-axis, we have the $b+1-n$ lattice points $(0, -1)$, $(1, -1)$, ..., $(b - n, -1)$. Continue summing the number of lattice points, counting along each row. We stop at either $-a$ or $\lfloor \frac{-b}{n} \rfloor$, whichever comes first. This yields the expression

$$\mathrm{h}^0(X = \mathcal{H}_n, D = aA_1 + bA_2) = \begin{cases} \sum_{k=0}^{\min(a, \lfloor \frac{b}{n} \rfloor)} (b + 1 - nk) & : a, b \geq 0 \\ 0 & : a < 0 \text{ or } b < 0 \end{cases}$$

Remark 3.29. The author has since been informed that this result has appeared in the literature using alernative methods. More precisely, this is Example 2.9 of [14].

3.3 NEF and Effective Divisors

We begin by describing the Picard group. Let r denote the number of rays on our fan.

Lemma 3.30. *$\mathrm{Pic}(X_\Delta)$ is free abelian of rank $r-2$. In particular, if A_i and A_{i+1} are adjacent divisors (possibly A_r and A_1), then $\mathrm{Pic}(X_\Delta)$ is the direct sum of the terms $\mathbb{Z}A_k$ as A_k ranges over all other divisors.*

Proof. Since X_Δ is smooth, after transforming by an element of $\mathrm{GL}(2,\mathbb{Z})$ we may assume that our adjacent divisors are A_1 and A_2 and that $u_1 = (1,0)$ and $u_2 = (0,1)$. By Theorem 3.19 we have that

$$A_1 + \sum_{i=3}^{r} a_i A_i \sim 0$$

$$A_2 + \sum_{i=3}^{r} b_i A_i \sim 0.$$

This implies that both A_1 and A_2 live inside $\bigoplus_{i=3}^{r} \mathbb{Z}A_i$; since there are no other relations (by Theorem 3.19) we have that $\mathrm{Pic}(X)$ equals this direct sum. □

Theorem 3.31. *Let D be a nef divisor on $X = X_\Delta$, and let p be the point corresponding to the two-dimensional cone generated by u_1 and u_2. Then $\epsilon_p(D) = \min\{D.A_1, D.A_2\}$.*

Proof. Let $\pi : \tilde{X} \to X$ be the blow-up of X at p, and let E be the corresponding exceptional curve. Recall that

$$\epsilon_p(D) = \sup\{t \geq 0 : \pi^*D - tE \text{ is nef}\}.$$

Now, a divisor is nef if and only if it lives inside $\overline{\mathrm{Eff}(X)}^\vee$. Since $\mathrm{Eff}(X)$ is finite-generated, it is already closed. Note that $(\pi^*D - tE).E = t \geq 0$. Letting $\tilde{A}_i$ being the strict transform of A_i under π we thus have that

$$\pi^*D - tE \text{ is nef} \leftrightarrow (\pi^*D - tE).\tilde{A}_i \geq 0 \text{ for all } 1 \leq i \leq r.$$

For all but $\tilde{A}_1$ and $\tilde{A}_2$ we have that $\tilde{A}_i.E = \pi^*A_i.E = 0$; since D is nef we thus have for these i that $(\pi^*D - tE).\tilde{A}_i = D.A_i \geq 0$. Therefore, we have that

$$\pi^*D - tE \text{ is nef} \leftrightarrow \pi^*D - tE.\tilde{A}_1 \geq 0 \text{ and } \pi^*D - tE.\tilde{A}_2 \geq 0.$$

So we are looking for the largest t so that

$$t \leq \frac{\pi^*D.\tilde{A}_1}{E.\tilde{A}_1} \text{ and } t \leq \frac{\pi^*D.\tilde{A}_2}{E.\tilde{A}_2}.$$

Observe that

$$\begin{aligned} \pi^*D.\tilde{A}_1 &= \pi^*D.(\pi^*A_1 - E) \\ &= D.A_1 \end{aligned}$$

and likewise that $\pi^*D.\tilde{A}_2 = D.A_2$. Since $E.\tilde{A}_1 = E.\tilde{A}_2 = 1$ we thus have that $\epsilon_p(D)$ is the largest t so that $t \leq D.A_1$ and $t \leq D.A_2$. We are done. □

Remark 3.32. On a toric surface, ϵ_p was so easy to calculate because we expressed it in terms of intersection theory on the effective cone of the blow-up; and it is easy to write down the generators of the effective cone. We will use the same strategy to calculate γ_p; it gets tricky because writing down generators of the nef cone is hard.

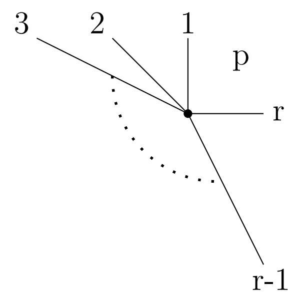
Here is a cute little numerical "result".

Corollary 3.33. *Let D be a nef divisor on our surface, and consider two adjacent divisors A_k, A_{k+1}. Then* $\min(D.A_k, D.A_{k+1}) \leq \sqrt{D^2}$.

Proof. For any point p we always have that $\epsilon_p(D) \leq \sqrt{D^2}$. □

3.4 γ_p on toric surfaces

In this sub-section, we always assume that $u_1 = (0,1)$ and $u_r = (1,0)$. Here is the picture.



We first show that on our toric surfaces, we (in principle) don't need to blow-up in order to calculate γ_p. However, we first include a lemma that computes the pullbacks of certain divisors under blow-ups.

Lemma 3.34. *Let B_i denote the strict transform of A_i under the blow-up $\pi : \tilde{X} \to X$ of X at p, with exceptional divisor E. Then we have that*

$$\pi^* A_i = \begin{cases} B_i & : \textit{if } 2 \leq i \leq r-1 \\ B_i + E & : \textit{if } i \in \{1, r\}. \end{cases}$$

Proof. By Theorem 3.17 the point p lives on both A_1 and A_r, and not on any of the other divisors. Since A_1 and A_r are both smooth curves (they are isomorphic to $\mathbb{P}^1$), we have that $\operatorname{ord}_p(A_1) = \operatorname{ord}_p(A_r) = 1$. That each B_i is the strict transform of A_i under π comes from the machinery of toric morphisms (Chapter 3 of [7]), and it easy enough to verify. Thus the lemma follows from the fact that for any curve C on a surface S, and a blow-up $\pi : \tilde{S} \to S$ at a point q, we have that

$$\pi^* C = C^{st} + \operatorname{ord}_q(C) E$$

where C^{st} denotes the strict transform of C under π. □

Theorem 3.35. *Let p be the point corresponding to* $\mathrm{Cone}(u_r, u_1)$*. If D is an effective divisor, then $\gamma_p(D)$ is the supremum $\zeta_p(D)$ of $s+t$ where s, t range over all non-negative rational numbers so that $D - sA_r - tA_1$ is effective.*

Proof. We first show that $\zeta_p(D) \leq \gamma_p(D)$. Suppose that $D - \alpha A_r - \beta A_1 \in \mathrm{Eff}(X)$; i.e. suppose that $\zeta_p(D) \geq \alpha + \beta$. Letting π denote the blow-up of X at p, we have that $\pi^*(D - \alpha A_r - \beta A_1)$ is also effective. But

$$\begin{aligned}\pi^*(D - \alpha A_r - \beta A_1) &= \pi^* D - \alpha \pi^* A_r - \beta \pi^* A_1 \\ &= \pi^* D - \alpha(E + B_r) - \beta(E + B_1) \\ &= \pi^* D - (\alpha + \beta)E - \alpha B_r - \beta B_1.\end{aligned}$$

Since this divisor is effective, so is the divisor $\pi^* D - (\alpha+\beta)E$, and therefore $\gamma_p(D) \geq \alpha+\beta$.

For the other inequality, suppose that $\pi^* D - \frac{a}{b}E$ is effective for some $a/b \in \mathbb{Q}$. What this really means is that the $\mathbb{Z}$-divisor $b\pi^* D - aE$ has a non-zero global section. Write $D = \sum_{i=1}^{r} \tau_i A_i$; since D is effective, we may assume that all the $\tau_i \geq 0$. Writing $u_i = (a_i, b_i)$ we have the relations

$$D_r \sim -\sum_{i=3}^{r-1} a_i A_i$$

$$D_1 \sim -\sum_{i=3}^{r-1} b_i A_i.$$

in $\mathrm{Pic}(X)$. Thus we have that

$$D \sim \sum_{i=2}^{r-1} (\tau_i - \tau_r a_i - \tau_1 b_i) A_i.$$

Since this divisor is supported away from A_1 and A_r, we have that

$$\pi^* D \sim \sum_{i=2}^{r-1} (\tau_i - \tau_r a_i - \tau_1 b_i) B_i.$$

So we are assuming that

$$\sum_{i=2}^{r-1} b(\tau_i - \tau_r a_i - \tau_1 b_i) B_i - aE$$

is effective. By Proposition 3.20, this is equivalent to saying that the polytope P associated to this divisor satisfies

$$P \cap \mathbb{Z}^2 \neq \emptyset.$$

By definition, P is the collection of all linear functionals $m = \alpha e_1^* + \beta e_2^*$ which satisfy $m(u_i) \geq -z_i$ where z_i is the coefficient of D_i. Let $\phi = Me_1^* + Ne_2^* \in P \cap \mathbb{Z}^2$. This gives us the inequalities

$$\begin{aligned}M, N &\geq 0 \\ Ma_i + Nb_i &\geq -b(\tau_i - \tau_r a_i - \tau_1 b_i) \text{ for } 2 \leq i \leq r-1 \\ M + N &\geq a.\end{aligned}$$

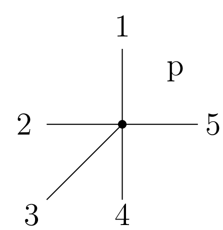
Multiplying the middle inequalities by -1 we thus obtain

$$\begin{aligned}
bD &= \sum_{i=2}^{r-1} b(\tau_i - \tau_r a_i - \tau_1 b_i) D_i \\
&\geq \sum_{i=2}^{r-1} (-Ma_i - Nb_i) D_i \\
&= M(-\sum_{i=2}^{r-1} a_i D_i) + N(-\sum_{i=2}^{r-1} b_i D_i) \\
&= MD_r + ND_1
\end{aligned}$$

where we write $A \geq B$ for divisors A and B to signify that each coefficient of A is greater than or equal to its B-counterpart. Therefore, the $\mathbb{Z}$-divisor $bD - MD_r - ND_1$ is effective, and hence the $\mathbb{Q}$-divisor $D - \frac{M}{b} D_r - \frac{N}{b} D_1$ is effective. It follows that $\zeta_p(D) \geq \frac{M+N}{b} \geq \frac{a}{b}$. $\square$

Remark 3.36. This theorem tells us that γ_p is always "witnessed" by a combination of D_r and D_1. In practice, this has not been useful for calculating γ_p. However, it has been useful for obtaining lower bounds for γ_p.

Example 3.37. In general, $\gamma_p : \mathrm{Eff}(X) \to \mathbb{R}^{\geq 0}$ is not linear. Let X be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a torus-invariant point. The fan Σ of X is pictured below



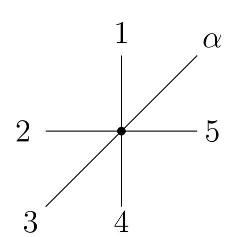
and we will calculate γ_p where p is the point corresponding to $\mathrm{Cone}(u_5, u_1)$. Here $u_1 = (0,1)$, $u_2 = (-1,0)$, $u_3 = (-1,-1)$, $u_4 = (0,-1)$, and $u_5 = (1,0)$. Let A_i be the curve (=divisor) corresponding to u_i. We use the basis

$$\mathbb{Z}A_2 \oplus \mathbb{Z}A_3 \oplus \mathbb{Z}A_4$$

for $\mathrm{Pic}(X)$. The intersection theory of X is $A_1^2 = 0$, $A_2^2 = -1$, $A_3^2 = -1$, $A_4^2 = -1$, and $A_5^2 = 0$. It is straightforward to verify that

$$\begin{aligned}
\mathrm{Eff}(X) &= \mathbb{N}A_2 \oplus \mathbb{N}A_3 \oplus \mathbb{N}A_4 \\
\mathrm{Nef}(X) &= \mathbb{N}(A_2 + A_3) \oplus \mathbb{N}(A_4 + A_3) \oplus \mathbb{N}(A_2 + A_3 + A_4).
\end{aligned}$$

We will use Theorem 2.18 to calculate γ_p. As indicated in the proof, this requires knowing generators for the nef cone of $\tilde{X}$. Pictured below is the fan for $\tilde{X}$:



where $u_\alpha = (1,1)$ gives the exceptional curve E. Let B_i be the curve (=divisor) corresponding to each ray. (Of course, B_i is the strict transform of A_i.) We have the new intersection theory

$$B_i^2 = -1 \text{ for all } 1 \leq i \leq 5$$
$$E^2 = -1.$$

On $\tilde{X}$, we use the basis

$$\mathbb{Z}B_1 \oplus \mathbb{Z}B_2 \oplus \mathbb{Z}B_3 \oplus \mathbb{Z}B_4$$

of $\mathrm{Pic}(\tilde{X})$. It is straightforward to verify that

$$\mathrm{Nef}(\tilde{X}) = \mathrm{Cone}(B_3 + B_4, B_2 + B_3, B_2 + B_3 + B_4, B_1 + B_2, B_1 + B_2 + B_3).$$

The first three are the pullbacks of generators of $\mathrm{Nef}(X)$, while the last two are the ones that intersect E. In particular, the generators N of $\mathrm{Nef}(\tilde{X})$ which satisfy $N.E > 0$ are

$$N_1 = B_1 + B_2$$
$$N_2 = B_1 + B_2 + B_3.$$

By Theorem 2.18 we thus have for $D = aA_2 + bA_3 + cA_4$ effective $(a, b, c \geq 0)$ that

$$\begin{aligned}\gamma_p(D) &= \min(\frac{\pi^* D.N_1}{E.N_1}, \frac{\pi^* D.N_2}{E.N_2}) \\ &= \min([aB_2 + bB_3 + cB_4].[B_1 + B_2], [aB_2 + bB_3 + cB_4][B_1 + B_2 + B_3]) \\ &= \min(b, a + c)\end{aligned}$$

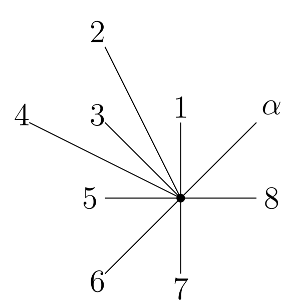
which is certainly not linear. However, for a nef divisor $N = x(A_2 + A_3) + y(A_4 + A_3) + z(A_2 + A_3 + A_4)$ (with $x, y, z \geq 0$) we have that

$$\begin{aligned}\gamma_p(N) &= \gamma_p((x + z)A_2 + (x + y + z)A_3 + (y + z)A_4) \\ &= \min(x + y + z, x + y + 2z) \\ &= x + y + z\end{aligned}$$

which shows that $\gamma_p|_{\mathrm{Nef}(X)}$ is linear. This is not a coincidence: our main result, which we now begin developing the machinery to prove, is that $\gamma_p|_{\mathrm{Nef}(X)}$ is linear.

In the above example, $\gamma_p(D)$ is expressed as the minimum of the intersection of D with two different divisors. So saying that $\gamma_p|_{\mathrm{Nef}(X)}$ is linear is saying that the intersection by one of these divisors (N_1) is always lower than the intersection with N_2. What makes the divisor N_1 more special than the divisor N_2? This is what we begin to investigate.

Example 3.38. In the previous example, N_1 satisfied some kind of "minimality" condition over N_2-it is the "smallest" nef divisor which intersects E positively. Consider the following fan:



where $u_1 = (0, 1)$, $u_2 = (-1, 2)$, $u_3 = (-1, 1)$, $u_4 = (-2, 1)$, $u_5 = (-1, 0)$, $u_6 = (-1, -1)$, $u_7 = (0, -1)$, $u_8 = (1, 0)$, and $u_\alpha = (1, 1)$. Let $\tilde{X}$ denote the corresponding variety, and let X denote the variety that $\tilde{X}$ is obtained from by adding the ray u_α. Letting E denote the exceptional divisor of this blow-up, we have that for $D \in \text{Eff}(X)$ that

$$\gamma_p(D) = \min_{N \in \text{Nef}(\tilde{X}), N.E > 0} \left\{ \frac{\pi^* D.N}{E.N} \right\}.$$

So we wish to find a nef divisor W that meets E positively, while having a small intersection number against the pullback of other effective divisors. The intersection theory of $\tilde{X}$ is

$$\begin{aligned} d_1 &= -3 \\ d_2 &= -1 \\ d_3 &= -3 \\ d_4 &= -1 \\ d_5 &= -3 \\ d_6 &= -1 \\ d_7 &= -1 \\ d_8 &= -1 \\ E^2 &= -1. \end{aligned}$$

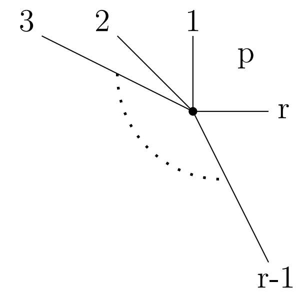
Work with the basis $D_1, D_2, D_3, D_4, D_5, D_6, D_7$ of $\text{Pic}(\tilde{X})$. We are looking for our divisor W. Where to start? Well, it had better satisfy $W.E > 0$. Let's choose to make it 1. The only (basis) divisor which satisfies $W.E > 0$ is D_1, so let's define $W = D_1$. Now, W is not nef because it intersects D_1 negatively ($W.D_1 = -3$), so let's add on another divisor to make our divisor intersect D_1 0 times. The only divisor that meets D_1 positively is D_2. Thus we redefine W to be $D_1 + 3D_2$. Well, now $W.D_1 = 0$ but we've created a new problem: $W.D_2 = -2$. We better modify W again. We don't want to add on more D_1's : that would just be going around in circles. So redefine W as $D_1 + 3D_2 + 2D_3$. We now have $W.D_3 = -3$, so let's redefine $W = D_1 + 3D_2 + 2D_3 + 3D_4$. Again, we have a problem at D_4 since $W.D_4 = -1$, so we redefine W to be $D_1 + 3D_2 + 2D_3 + 3D_4 + D_5$. Will this process ever end? You bet it does! Since $W.D_5 = 0$, we have a nef divisor:

$$W = D_1 + 3D_2 + 2D_3 + 3D_4 + D_5 \in \text{Nef}(\tilde{X})$$

You can check for yourself that $\gamma_p(N) = [\pi^* N].W$ for $N \in \text{Nef}(X)$.

Remarkably, the process described in the above example above always terminates, and the divisor you end up with is always the γ_p winner (for divisors in $\text{Nef}(X)$, not for $\text{Eff}(X)$). This is what we work towards proving.

In the process used, there is nothing special about E. Given any divisor D_l, we could attempt to find a "minimal" nef divisor that meets D_l exactly once by using a basis of adjacent divisors for $\text{Pic}(X)$. We thus temporarily forget about blow-ups, and consider the more general situation: use the basis $D_1, D_2, ..., D_{r-2}$ of $\text{Pic}(X)$, and find a nef divisor which intersects D_r exactly once. Since our "process" may be a bit vague, let's make it formal; as a reminder, here is our fan:



Algorithm to construct $W \in \text{Nef}(X)$ which satisfies $W.D_r = 1$:

- **Step 1**: Assign $W = D_1$, and assign $i = 1$.
- **Step 2**: If $i = r - 2$ then stop, and the algorithm is finished. If $W.D_i \geq 0$, then stop, and the algorithm is finished. Otherwise, assign $W := W + (-W.D_i)D_{i+1}$, and assign $i := i + 1$. Repeat step 2.

On the blown up surface, this algorithm may be used to construct a divisor which intersects E once. This divisor is going to be the 'winner' for γ_p on $\text{Nef}(X)$. It is not clear that the algorithm always gives us a nef divisor - if we get to the stage where $i = r - 2$, then we have no way of knowing if $W.D_i \geq 0$. However, the algorithm does indeed always stop: this is the content of the next lemma.

Let $1 \leq T < r$ be the (unique) integer which satisfies $b_T > 0$ and $b_{T+1} \leq 0$ (recall that b_i denotes the y-coordinate of u_i).

Lemma 3.39. *Define Ω to be the divisor $\sum_{i=1}^{T} b_i D_i$. Then $W = \Omega$. The divisor Ω has the following intersection theory:*

- $\Omega.D_r = 1$.
- $\Omega.D_T = -b_{T+1} \geq 0$.
- $\Omega.D_{T+1} = b_T \geq 0$.
- *For all $i \notin \{1, T, T+1\}$ we have that $\Omega.D_i = 0$.*

In particular, this divisor is nef.

Proof. The key point is that we always have the equality $u_{i-1} + (-D_i^2)u_i + u_{i+1}$ for three consecutive rays. In particular, $b_{i-1} + (-D_i^2)b_i + b_{i+1} = 0$.

We follow the algorithm to construct W, noting that at each stage the coefficient of D_i is in fact equal to b_i. Let μ_i denote the coefficient of D_i in W. Note that $\mu_1 = 1$ by step 1, and $b_1 = 1$ also. If $T = 1$, we are done: since $b_2 + (-D_1^2)b_1 + b_r = 0$ and $b_r = 0$ and $b_2 \leq 0$, we thus have that $W^2 = D_1^2 = -b_2 \geq 0$. So assume that $T > 1$, and proceed inductively. By definition of W, we have that $\mu_2 = -D_1^2 = b_2$ as was remarked in the previous sentence.

Suppose that we have shown that $\mu_k = b_k$ for all $1 \leq k < T$. So at this stage in the algorithm we have the divisor $W = \sum_{i=1}^{k} \mu_i D_i = \sum_{i=1}^{k} b_i D_i$. Since $k < T$ we have that $W.D_k < 0$; this is because $W.D_k = \mu_k D_k^2 + \mu_{k-1} = b_k D_k^2 + b_{k-1} = -b_{k+1} < 0$. In particular, we must add on the (positive) multiple b_{k+1} to D_{k+1}.

This proves that $W = \sum_{i=1}^{T} b_i D_i = \Omega$. The intersection claims are obvious. □

From now on, we denote this divisor by W. It is our candidate for the witness of the minimum divisor appearing in $\gamma_p|_{\mathrm{Nef}(X)}$. We now begin to develop the machinery to prove that it is indeed the witness. This will culminate in a series of estimations (Lemma 3.50 and Lemma 3.51), from which our main theorem (Theorem 3.55) will follow.

Definition 3.40. For $n \geq 0$ we define the n^{th} **toric surface polynomial** $P_n(x_1, ..., x_n) \in \mathbb{Z}[x_1, ...x_n]$ recursively as follows:

$$P_0 = 1$$

$$P_1 = -x_1$$

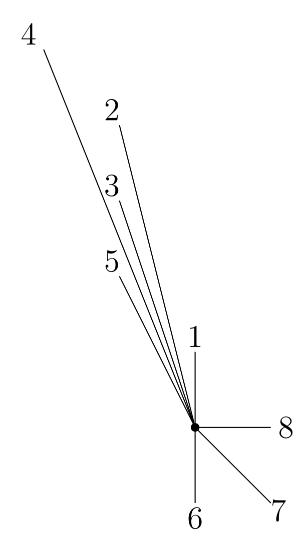
$$P_i + x_i P_{i-1} + P_{i-2} = 0 \text{ for } i \geq 2$$

Example 3.41.

$$P_2 = x_1x_2 - 1$$
$$P_3 = -x_1x_2x_3 + x_1 + x_3$$
$$P_4 = x_1x_2x_3x_4 - x_1x_2 - x_1x_4 - x_3x_4 + 1$$
$$P_5 = -x_1x_2x_3x_4x_5 + x_1x_2x_3 + x_1x_2x_5 + x_1x_4x_5 + x_3x_4x_5 - x_1 - x_3 - x_5$$

Why do we care about these polynomials? Let's start with an example.

Example 3.42. Consider the fan shown below



with ray vectors $u_1 = (0, 1)$, $u_2 = (-1, 4)$, $u_3 = (-1, 3)$, $u_4 = (-2, 5)$, $u_5 = (-1, 2)$,

$u_6 = (0, -1)$, $u_7 = (1, -1)$, and $u_8 = (1, 0)$. We write d_i for D_i^2 for the sake of notation; you will thank me later. It is readily verified that

$$d_1 = -4$$
$$d_2 = -1$$
$$d_3 = -3$$
$$d_4 = -1$$
$$d_5 = -2$$
$$d_6 = 1$$
$$d_7 = -1$$
$$d_8 = -1$$

Let's evaluate a sequence of consecutive d_k's into the P_l, starting with d_1:

$$P_0 = 1 = b_1$$

$$P_1(d_1) = -d_1 = 4 = b_2$$
$$P_2(d_1, d_2) = d_1 d_2 - 1 = 3 = b_3$$
$$P_3(d_1, d_2, d_3) = -d_1 d_2 d_3 + d_1 + d_3 = 5 = b_4.$$

That's kind of neat. What's going on? We are saying that with knowledge of $d_1, d_2, ..., d_k$, it is possible to determine u_{k+1}; this is how we do it. The lemma formalizes this pattern.

Lemma 3.43. *Let $d_i = D_i^2$. Then*

$$P_i(d_1, d_2, ..., d_{i-1}, d_i) = b_{i+1}.$$

Proof. We have that $P_0 = 1 = b_1$ and $P_1(d_1) = -d_1 = -d_1 b_1 = b_2 + b_0 = b_2$. Proceed by induction. By the definition of the TS-polynomials, for $n \geq 2$ we have that

$$\begin{aligned} P_n(d_1, ..., d_n) &= -d_n P_{n-1}(d_1, ...d_{n-1}) - P_{n-2}(d_1, ...d_{n-2}) \\ &= -d_n b_n - b_{n-1} \\ &= b_{n+1}. \end{aligned}$$

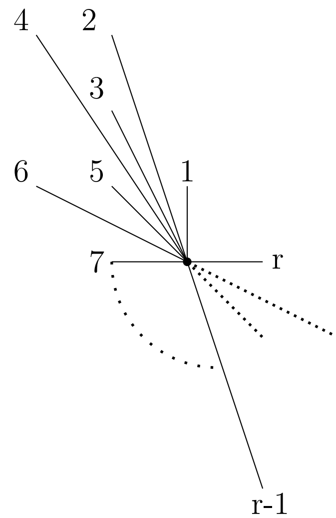
□

There is no reason why we should have to start at d_1, or move around anti-clockwise. We can plug in a consecutive sequence of the d_l's in the TS-polynomials. After all, we could always change basis to give ourselves the same setup of lemma 3.43. For what is coming, we are going to need to know that some of these terms (the P's evaluated at a sequence of consecutive d_i's) are non-negative. Here is the (easy) corollary.

Corollary 3.44. *Consider a sequence of j consecutive d_l's of the form $d_k, d_{k+1}, ..., d_{k+j-1}$. Then $P_j(d_k, d_{k+1}, ..., d_{k+j-1}) \geq 0$ if and only if u_{k+j} lies in the second quadrant of the plane obtained by performing the change of basis $u_{k-1} \mapsto (1, 0)$, $u_k \mapsto (0, 1)$. Likewise, for a sequence of the form $d_k, d_{k-1}, ..., d_{k-j+1}$, we have that $P_j(d_k, d_{k-1}, ..., d_{k-j+1}) \geq 0$ if and only if u_{k-j} lies in the second quadrant of the plane obtained by performing the change of basis $u_{k+1} \mapsto (1, 0)$, $u_k \mapsto (0, 1)$.*

Proof. Change basis, and use Lemma 3.43. □

Example 3.45. If you don't like abstract nonsense, here is an example of 3.44 in action. Consider a fan pictured below.



We are going to plug in a sequence of the d_l starting at d_5 and going back clockwise. We thus are going to change basis by sending $u_5 \mapsto (0,1)$ and $u_6 \mapsto (1,0)$. After changing basis, the first quadrant is equal to $\text{Cone}(u_6, u_5)$. We draw the negatives of u_5 and u_6 to see where the quadrants are. The second quadrant contains u_4, u_3, u_2, u_1, and u_r, while u_{r-1} lies in the fourth quadrant. We therefore have, by corollary 3.44 that

$$\begin{aligned} P_1(d_5) &> 0 \\ P_2(d_5, d_4) &> 0 \\ P_3(d_5, d_4, d_3) &> 0 \\ P_4(d_5, d_4.d_3, d_2) &> 0 \\ P_5(d_5, d_4, d_3, d_2, d_1) &> 0 \\ P_6(d_5, d_4, d_3, d_2, d_1, d_r) &< 0. \end{aligned}$$

This corollary formalizes the previous example.

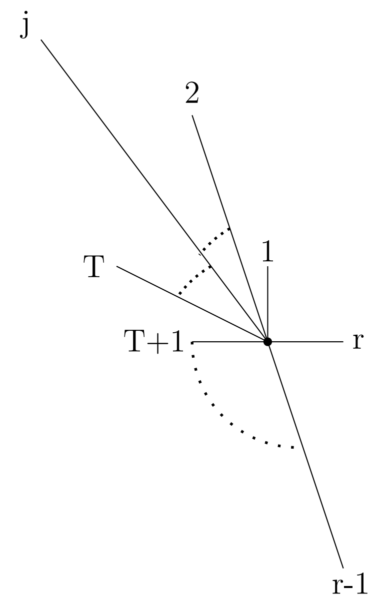
Corollary 3.46. *As before, we let T satisfy $b_T > 0$ and $b_{T+1} \leq 0$. Let $1 \leq j \leq T-1$. Then we have that*

$$\begin{aligned} P_1(d_j) &\geq 0 \\ P_2(d_j, d_{j-1}) &\geq 0 \\ P_3(d_j, d_{j-1}, d_{j-2}) &\geq 0 \\ &\vdots \\ P_j(d_j, d_{j-1}, d_{j-2}, ..., d_1) &\geq 0. \end{aligned}$$

Furthermore, if $T+2 \le j \le r-2$ then we have that Let $T+2 \le j \le r-2$. Then

$$P_1(d_j) \ge 0$$
$$P_2(d_j, d_{j+1}) \ge 0$$
$$\vdots$$
$$P_{r-j-1}(d_j, d_{j+1}, ..., d_{r-2}) \ge 0.$$

Proof. I hear proof by picture is a good thing - this picture is to use for the first set of inequalities.



It is clear that $u_{j-1}, u_{j-2}, ..., u_1, u_r$ all lie in the second quadrant of the transformation $u_j \mapsto (0,1)$, $u_{j+1} \mapsto (1,0)$. So use Corollary 3.44. A similar picture easily proves the second set of inequalities. □

Remark 3.47. If we evaluate the various P_l at sequences involving d_T or d_{T+1}, we may get a negative number. This reflects the fact (whose proof I have not put into this document) that a curve with positive self-intersection either appears at the spot r, $r-1$, T, or $T+1$. To prove our main result, we don't need such a (possibly negative) expression however.

Lemma 3.48. *For $n \ge 1$, define M_n to be the set of monomials $L = x_{i_1}x_{i_2}...x_{i_d}$, $1 \le i_1 < i_2 < ... < i_d \le n$, which satisfy:*

- $\deg(L) \le n$ *and* $\deg(L) \equiv n \mod 2$.
- $i_m \equiv m \mod 2$ *for all* m.

Then P_n is an integer linear combination of the monomials in M_n. Furthermore, P_n has a unique term of degree n with coefficient $(-1)^n$. Any term of degree $n-2k$ has coefficient $(-1)^{n+k}$.

Proof. Both P_0 and P_1 satisfy these claims. For $n \geq 2$, the definition $P_n(x_1, ..., x_n) = -x_n P_{n-1}(x_1, ..., x_{n-1}) - P_{n-2}(x_1, ..., x_{n-2})$ shows that each monomial has only linear powers of each x_l.

Assume that n is even, so that $n-1$ is odd and $n-2$ is even. The case with n odd is similar. Note that the recursive definition ensures that each monomial of P_n has even degree, and that the degree of each monomial is clearly at most n.

Every monomial of P_n is either a monomial from P_{n-2}, or the product of a monomial from P_{n-1} with x_n. Consider the monomials of P_{n-2}. Since $n-2$ is also even, by the induction hypothesis we have that these monomials satisfy both bullets. Consider a monomial $x_{i_1}...x_{i_d}$ of P_{n-1}. By the induction hypothesis, this monomial satifies both bullets (with n replaced by $n-1$). Thus the monomial $x_{i_1}...x_{i_d}x_n$ satisfies both bullets as well.

The statement about the sign of each monomial also follows by the inductive definition. □

Corollary 3.49. $P_n(x_1, x_2, ..., x_{n-1}, x_n) = P_n(x_n, x_{n-1}, ..., x_2, x_1)$

Proof. This follows from Lemma 3.48. It suffices to show that for $k \geq 0$ then degree $d := n - 2k$ piece of $P_n(x_1, x_2, ..., x_{n-1}, x_n)$ equals the degree d piece of $P_n(x_n, x_{n-1}, ..., x_2, x_1)$. A monomial in the degree d piece of $P_n(x_1, ..., x_n)$ is of the form

$$x_{i_1}x_{i_2}...x_{i_{d-1}}x_{i_d}$$

with $1 \leq i_1 \leq i_2 \leq ... \leq i_d \leq n$ and $i_l \equiv l \mod 2$. Since this monomial shows up in $P_n(x_1, ..., x_n)$, certainly the monomial

$$x_{i_d}x_{i_{d-1}}x_{i_2}x_{i_1}$$

shows up in $P_n(x_n, ..., x_1)$. But the variables are all commutative - to put it into the form of Lemma 3.48 we must switch the order again, since we want to write the indices from smallest to largest, and currently they are written largest to smallest. Thus the monomial

$$x_{i_1}x_{i_2}...x_{i_{d-1}}x_{i_d}$$

appears in $P_n(x_n, ..., x_1)$, and so they are equal. □

Here is the critical lemma. It is the reason why the properties of the P_l had to be developed.

Lemma 3.50. *Let $N = \sum_1^s n_i D_i$ be a nef divisor which satisfies $N.D_r > 0$. Let μ_i denote the quantity $n_i - b_i$. Then, for all $1 \leq i \leq T$ we have that $\mu_i \geq \mu_1 b_i$.*

Proof. Since $W.D_r = 1$ and $N.D_r = n_1 \geq 1$, we have that $\mu_1 \geq 0$. Furthermore, since $b_1 = 1$, we do have that $\mu_1 \geq b_1\mu_1$. Now intersect both W and N with D_1. We have that

$$W.D_1 = b_2 + b_1 d_1 = 0 \leq N.D_1 = n_2 + n_1 d_1$$

which yields that

$$\mu_2 \geq \mu_1(-d_1) = \mu_1 b_2. \tag{3.1}$$

For $3 \leq k \leq T$, since $W.D_{k-1} = 0$ and $N.D_{k-1} \geq 0$, we have the inequality

$$\mu_k \geq \mu_{k-1}(-d_{k-1}) + \mu_{k-2}(-1).$$

Now fix some value $3 \leq l \leq T$. Let's prove, by finite induction, that $\mu_l \geq \mu_1 b_l$. We will continually modify the previous inequalities by multiplying by various P_j's. Begin by considering the two inequalities

$$\mu_l \geq \mu_{l-1}(-d_{l-1}) + \mu_{l-2}(-1) \tag{3.2}$$

$$\mu_{l-1} \geq \mu_{l-2}(-d_{l-2}) + \mu_{l-3}(-1). \tag{3.3}$$

More suggestively of things to come, (3.2) may be written as

$$\mu_l \geq \mu_{l-1}P_1(d_{l-1}) + \mu_{l-2}(-P_0)$$

Multiply (3.3) by $-d_{l-1}$ to obtain

$$\mu_{l-1}(-d_{l-1}) \geq \mu_{l-2}(d_{l-2}d_{l-1}) + \mu_{l-3}(d_{l-1}).$$

Note that $-d_{l-1} > 0$ by Corollary 3.46. Substitute this back into 3.2 to obtain

$$\begin{aligned}\mu_l &\geq \mu_{l-2}(d_{l-2}d_{l-1}) + \mu_{l-3}(d_{l-1}) + \mu_{l-2}(-1)\\ &= \mu_{l-2}(d_{l-1}d_{l-2} - 1) + \mu_{l-3}(d_{l-1})\\ &= \mu_{l-2}P_2(d_{l-1}, d_{l-2}) + \mu_{l-3}(-P_1(d_{l-1})).\end{aligned}$$

This is the bound we prove by induction: that, for $2 \leq k \leq T-1$ we have that

$$\mu_l \geq \mu_{l-k}P_k(d_{l-1}, ..., d_{l-k}) + \mu_{l-k-1}(-P_{k-1}(d_{l-1}, ..., d_{l-k+1})). \tag{3.4}$$

The base case has been proved. Assume that 3.4 for some fixed k; we establish the above inequality for $k+1$. Multiply both sides of the inequality

$$\mu_{l-k} \geq \mu_{l-k-1}(-d_{l-k-1}) + \mu_{l-k-2}(-1)$$

by the number $P_k(d_{l-1}, d_{l-2}, ..., d_{l-k})$. Note that this number is positive by Corollary 3.46. We therefore obtain the inequality

$$\mu_{l-k}P_k(d_{l-1}, ..., d_{l-k}) \geq \mu_{l-k-1}(-d_{l-k-1}P_k(d_{l-1}, ..., d_{l-k})) + \mu_{l-k-2}(-P_k(d_{l-1}, ..., d_{l-k})).$$

Use this bound on (3.4). This yields that

$$\begin{aligned}\mu_l &\geq \mu_{l-k-1}(-d_{l-k-1}P_k(d_{l-1}, ..., d_{l-k})) + \mu_{l-k-2}(-P_k(d_{l-1}, ..., d_{l-k}))\\ &\quad + \mu_{l-k-1}(-P_{k-1}(d_{l-1}, ..., d_{l-k+1}))\\ &= \mu_{l-k-1}(-d_{l-k-1}P_k(d_{l-1}, ..., d_{l-k}) - P_{k-1}(d_{l-1}, ..., d_{l-k+1})) + \mu_{l-k-2}(-P_k(d_{l-1}, ..., d_{l-k}))\\ &= \mu_{l-k-1}P_{k+1}(d_{l-1}, ..., d_{l-k-1}) + \mu_{l-k-2}(-P_k(d_{l-1}, ..., d_{l-k})).\end{aligned}$$

This is exactly the desired bound where k has been replaced by $k+1$, provided that $k \leq l-3$. This gives the induction step. When we are at step $k = l-2$, we have the final step

$$\begin{aligned}\mu_l &\geq \mu_2 P_{l-2}(d_{l-1}, ..., d_2) + \mu_1(-P_{l-3}(d_{l-1}, ..., d_3)) \\ &\geq (-\mu_1 d_1) P_{l-2}(d_{l-1}, ..., d_2) + \mu_1(-P_{l-3}(d_{l-1}, ..., d_3)) \\ &= \mu_1(-d_1 P_{l-2}(d_{l-1}, ..., d_2) - P_{l-3}(d_{l-1}, ..., d_3)) \\ &= \mu_1 P_{l-1}(d_{l-1}, ..., d_1) \\ &= \mu_1 P_{l-1}(d_1, ..., d_{l-1}) \\ &= \mu_1 b_l.\end{aligned}$$

The second inequality is by (3.1). The second last equality is by Corollary 3.49 and the final equality is by Lemma 3.43. □

Lemma 3.51. *Let* $N = \sum_{i=1}^{r-2} n_i D_i$ *be a nef divisor on* X. *Then* $n_j \geq 0$ *for all* $T + 1 \leq j \leq R$.

Proof. For a general quality of life upgrade, define R to be $r-2$. We use a similar strategy to that of Lemma 3.50. Since N is nef, we have the inequalities

$$\begin{aligned} n_R &\geq 0 \\ n_{R-1} &\geq n_R(-d_R) \\ n_j &\geq n_{j+1}(-d_{j+1}) + n_{j+2}(-1) \text{ for } T+1 \leq j \leq R-2. \end{aligned}$$

Note that we may assume $-d_R \geq 0$. For if $d_R > 0$, then $R = T+1$, and there is nothing to worry about in the lower half of the fan anyway. So our first two terms (starting at $r-2$ and counting down) are non-negative. Fix some k satisfying $T+1 \leq k \leq R-2$.

Consider the inequalities

$$n_k \geq n_{k+1}(-d_{k+1}) + n_{k+2}(-1) \tag{3.5}$$

$$n_{k+1} \geq n_{k+2}(-d_{k+2}) + n_{k+3}(-1). \tag{3.6}$$

Multiply both sides of (3.6) by $-d_{k+1}$ to obtain $n_{k+1}(-d_{k+1}) \geq n_{k+2}(d_{k+2}d_{k+1}) + n_{k+3}(d_{k+1})$. Substitute this back into (3.5) to obtain that

$$\begin{aligned} n_k &\geq n_{k+2}(d_{k+2}d_{k+1}) + n_{k+3}(d_{k+1}) + n_{k+2}(-1) \\ &= n_{k+2}P_2(d_{k+1}, d_{k+2}) + n_{k+3}(-P_1(d_{k+1})). \end{aligned}$$

Proceed by finite induction. Suppose we have established that

$$n_k \geq n_{k+j}P_j(d_{k+1}, d_{k+2}, ..., d_{k+j}) + n_{k+j+1}(-P_{j-1}(d_{k+1}, d_{k+2}, .., d_{k+j-1})). \tag{3.7}$$

From the nef-ness of N we have the inequality

$$n_{k+j} \geq n_{k+j+1}(-d_{k+j+1}) + n_{k+j+2}(-1). \tag{3.8}$$

Multiply (3.8) by $P_j(d_{k+1}, d_{k+2}, ..., d_{k+j})$ - this number is non-negative by Corollary 3.46 - to obtain

$$n_{k+j}P_j(d_{k+1}, d_{k+2}, ..., d_{k+j}) \geq n_{k+j+1}(-d_{k+j+1}P_j(d_{k+1}, d_{k+2}, ..., d_{k+j})) + n_{k+j+2}(-P_j(d_{k+1}, d_{k+2}, ..., d_{k+j})) \tag{3.9}$$

and substitute (3.9) back into (3.7) to obtain that

$$
\begin{aligned}
n_k &\geq n_{k+j+1}(-d_{k+j+1}P_j(d_{k+1}, d_{k+2}, ..., d_{k+j})) + n_{k+j+2}(-P_j(d_{k+1}, d_{k+2}, ..., d_{k+j})) \\
&\quad + n_{k+j+1}(-P_{j-1}(d_{k+1}, d_{k+2}, .., d_{k+j-1})) \\
&= n_{k+j+1}P_{j+1}(d_{k+1}, d_{k+2}, ..., d_{k+j+1}) + n_{k+j+2}(-P_j(d_{k+1}, d_{k+2}, ..., d_{k+j})).
\end{aligned}
$$

This is the conclusion of our inductive step: it is the next step (ie. j has been replaced with $j+1$) for the induction hypothesis (3.7). Before the final step, by induction, we will have established that

$$
n_k \geq n_{R-1}P_{R-k-1}(d_{k+1}, d_{k+2}, .., d_{R-1}) + n_R(-P_{R-k-2}(d_{k+1}, d_{k+2}, ..., d_{R-2})). \tag{3.10}
$$

Use the bound $n_{R-1} \geq n_R(-d_R)$ (which is just the statement that $N.D_R \geq 0$) on (3.10) to obtain that

$$
\begin{aligned}
n_k &\geq n_R(-d_R)P_{R-k-1}(d_{k+1}, d_{k+2}, .., d_{R-1}) + n_R(-P_{R-k-2}(d_{k+1}, d_{k+2}, ..., d_{R-2})) \\
&= n_R[-d_R P_{R-k-1}(d_{k+1}, d_{k+2}, .., d_{R-1}) - P_{R-k-2}(d_{k+1}, d_{k+2}, ..., d_{R-2})] \\
&= n_R[P_{R-k}(d_{k+1}, d_{k+2}, ..., d_R)].
\end{aligned}
$$

In particular, $n_k \geq 0$. □

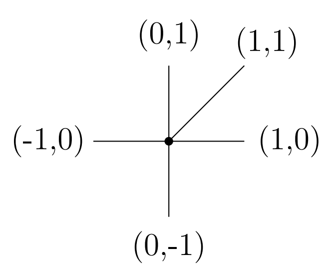
Remark 3.52. The actual bound we get for Lemma 3.51 is very similar to that of Lemma 3.50. However, for the main theorem, coming soon to a thesis near you, we only need the weaker inequality that the lower n_k are non-negative.

Corollary 3.53. *Use a basis of* $\operatorname{Pic}(X)$ *coming from* $r-2$ *adjacent rays (say* $D_1, D_2, ..., D_{r-2}$*). Then*

$$
\operatorname{Nef}(X) \cap \operatorname{Pic}(X) \subseteq \bigoplus_{i=1}^{r-2} \mathbb{N}D_i.
$$

Proof. Combine Lemma 3.50 and Lemma 3.51. □

Remark 3.54. If we do not use a basis of $\operatorname{Pic}(X)$ coming from adjacent divisors, then the coefficients of a nef divisor are not necessarily non-negative. For example, consider the fan Σ below.



Starting from $(1,0)$ and going around anti-clockwise, label the rays as $u_1, u_\alpha, u_2, u_3, u_4$. The variety $X = X_\Sigma$ is equal to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at a torus-invariant point. Label E as the exceptional curve in this blow-up, ie as the curve corresponding to the ray u_α. Since $D_1 + E \sim D_3$ and $D_2 + E \sim D_4$, we have that

$$
\mathbb{Z}D_3 \oplus \mathbb{Z}D_4 \oplus \mathbb{Z}E
$$

is a basis of Pic(X). In this basis, it is easily checked that

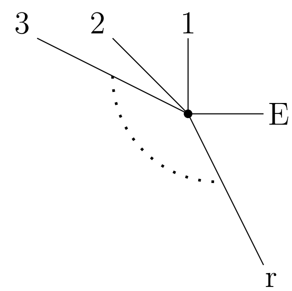
$$\text{Nef}(X) = \text{Cone}(D_3, D_4, D_3 + D_4 - E)$$

which does not lie inside $\mathbb{N}D_3 + \mathbb{N}D_4 + \mathbb{N}E$.

Recall that the natural number T is defined to be the unique number $1 \leq T < r$ which satisfies $b_T > 0$ and $b_{T+1} \leq 0$ (recall that b_i denotes the y-coordinate of u_i). We now state and prove our main theorem:

Theorem 3.55. *If $A, B \in \text{Nef}(X)$ then $\gamma_p(A + B) = \gamma_p(A) + \gamma_p(B)$.*

Proof. Change basis of our fan so that $u_E = (1, 0)$ and $u_1 = (0, 1)$. Here is the picture of the new fan:



Let $W = \sum_{i=1}^{T} b_i D_i$. We claim that $\gamma_p(D) = \pi^* D.W$ for any $D \in \text{Nef}(X)$. Let T be the set of generators N of Nef($\tilde{X}$) which satisfy $N.E > 0$. By Theorem 2.18, the claim is equivalent to showing that

$$\frac{\pi^* D.W}{E.W} \leq \frac{\pi^* D.N}{E.N} \text{ for all } N \in T.$$

We have that $E.W = 1$. Let such a N be written as $\sum_{i=1}^{r-1} n_i D_i$. Then $E.N = n_1 > 0$. So we must show that

$$\pi^* D.[N - n_1 W] \geq 0.$$

Well, we have that

$$\begin{aligned} \pi^* D.[N - n_1 W] &= \pi^* D.[\sum_{i=1}^{r-1} n_i D_i - n_1 \sum_{i=1}^{T} b_i D_i] \\ &= \sum_{i=1}^{T} (n_i - n_1 b_i)[\pi^* D].D_i + \sum_{i=T+1}^{r-1} n_i [\pi^* D].D_i. \end{aligned}$$

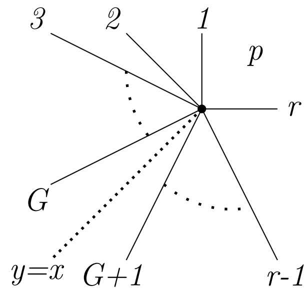
Now, $\pi^* D$ is nef since D is, and so each term $\pi^* D.D_i$ is non-negative. By Lemma 3.50 we therefore have that the first sum is non-negative. By Lemma 3.51 we also have that the second sum is non-negative. Therefore we indeed have that $\pi^* D.[N - n_1 W] \geq 0$. □

Corollary 3.56. *If $D \in \text{Nef}(X)$ then $\gamma_p(D) \in \mathbb{N}$.*

Proof. We showed that $\gamma_p(D) = \pi^* D.W$, which is a natural number. □

We can describe γ_p without using pullbacks also.

Corollary 3.57. *Let p be our point shown, and define G to be the unique integer so that $b_G - a_G > 0$ while $b_G - a_G \leq 0$.*



Then $\gamma_p(D) = D.\sum_{i-1}^{G}(b_i - a_i).D_i$.

Proof. In the proof of Theorem 3.55, we changed basis for the blown up fan to make results more convenient. That resulted in leaving u_1 alone, while we sent $u_E = (1,1)$ to $(1,0)$. Therefore, if we didn't change basis, we would have the divisor $\sum_{i=1}^{G}(b_i - a_i)D_i$. Since

$$\pi^*[\sum_{i=1}^{G}(b_i - a_i)D_i] = \sum_{i=1}^{G}(b_i - a_i)D_i + E$$

it follows that $\gamma_p(D) = \pi^*D.W = D.\sum_{i=1}^{G}(b_i - a_i)D_i$. □

3.5 More toric surface stuff

We proved earlier than for any divisor D which is nef and effective we always have that $\gamma_p(D)\epsilon_p(D) \leq D^2$. We return to the question of asking whether or not equality can always hold.

Theorem 3.58. *Let Δ be a fan for a smooth toric variety X which satisfies $b_{T+1} \neq 0$ and $T + 1 < r - 1$. There exists a divisor $D \in \mathrm{Nef}(X) = (\mathrm{Nef}(X) \cap \mathrm{Eff}(X))$ and a point p of X which satisfy*

$$\gamma_p(D)\epsilon_p(D) < D^2.$$

Proof. Let $D = \sum_{i=1}^{T} b_i D_i$, and let p be the point corresponding to $\mathrm{Cone}(u_{r-1}, u_r)$. Then $\epsilon_p(D) = \min\{D.D_{r-1}, D.D_r\} = \min\{0, 1\} = 0$; we have $D.D_{r-1} = 0$ since $T + 1 < r - 1$. On the other hand

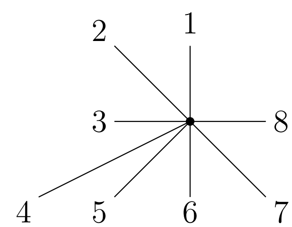
$$D^2 = \sum_{i=1}^{T} b_i(D_i.D) = b_T(D_T.D) = -b_T b_{T+1} > 0$$

and so $\gamma_p(D)\epsilon_p(D) < D^2$. □

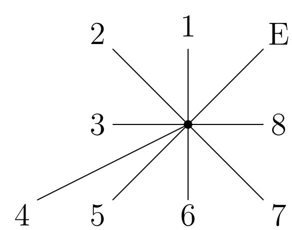
Another question was whether or not we have $\gamma_p(D) \in \mathbb{N}$ for $D \in \mathrm{Eff}(X)$. This question has a negative answer:

Theorem 3.59. *There exists a smooth complete toric surface X and a divisor $D \in \mathrm{Eff}(X)$ so that $\gamma_p(D) \in \mathbb{Q} - \mathbb{Z}$.*

Proof. This is Example A.4 of the Appendix. We flesh out some of the details here. Let X be the variety coming from the fan



where $u_1 = (0, 1)$, $u_2 = (-1, 1)$, $u_3 = (-1, 0)$, $u_4 = (-2, -1)$, $u_5 = (-1, -1)$, $u_6 = (0, -1)$, $u_7 = (1, -1)$, and $u_8 = (1, 0)$. Let p be the point corresponding to $\mathrm{Cone}(u_8, u_1)$, and let D be the divisor $A_2 + A_3 + A_4 + A_5 + A_6$. Pictured below is the fan for the blow-up:



Since D is supported away from A_1 and A_2, we have that $\pi^*D = B_2 + B_3 + B_4 + B_5 + B_6$. The intersection theory of $\tilde{X}$ is $E^2 = -1$, $b_1 = -2$, $b_2 = -1$, $b_3 = -3$, $b_4 = -1$, $b_5 = -2$, $b_6 = -2$, $b_7 = -1$, and $b_8 = -2$. Out of the 18 generators of $\mathrm{Nef}(\tilde{X})$, 7 of them intersect E positively. They are

$$\begin{aligned}
N_1 &= B_1 + 2B_2 + B_3 + B_4\\
N_2 &= B_1 + 2B_2 + B_3 + B_4 + B_5 + B_6 + B_7\\
N_3 &= 3B_1 + 6B_2 + 3B_3 + 3B_4 + 2B_5 + B_6\\
N_4 &= B_1 + 2B_2 + 2B_3 + 4B_4 + 2B_5\\
N_5 &= 2B_1 + 4B_2 + 2B_3 + 2B_4 + B_5\\
N_6 &= B_1 + 2B_2 + B_3 + 3B_4 + 2B_5 + B_6\\
N_7 &= B_1 + 2B_2 + B_3 + 2B_4 + B_5.
\end{aligned}$$

This gives the intersections

$$\begin{aligned}
\pi^*D.N_1 &= 1\\
\pi^*D.N_2 &= 1\\
\pi^*D.N_3 &= 2\\
\pi^*D.N_4 &= 3\\
\pi^*D.N_5 &= 2\\
\pi^*D.N_6 &= 2\\
\pi^*D.N_7 &= 2.
\end{aligned}$$

We thus have

$$\gamma_p(D) = \min_{1\leq l\leq 7}\left\{\frac{\pi^*D.N_l}{E.N_l}\right\} = \min(1, 1, \frac{2}{3}, 3, 1, 2, 2) = 2/3.$$

□

Chapter 4

Examples and Future Work

4.1 $\gamma_p|_{\mathrm{Nef}(X)\cap\mathrm{Eff}(X)}$ is not, in general, linear

We are going to prove that in general, $\gamma_p|_{\mathrm{Nef}(X)}$ (and actually $\gamma_p|_{\mathrm{Bpf}(X)}$) is not linear. Here we use $\mathrm{Bpf}(X)$ to denote the semigroup of basepoint-free divisors in $\mathrm{NS}(X)$. The comment in parathentheses is because, on our smooth complete toric varieties, we have that $\mathrm{Nef}(X) = \mathrm{Bpf}(X)$. Thus it would be reasonable to ask if γ_p is linear on the smaller cone $\mathrm{Bpf}(X)$, even if it is not linear on $\mathrm{Nef}(X)$. The example will use the following lemma.

Lemma 4.1. *Let S be a smooth projective surface and let $F \in \mathrm{Bpf}(S)$ satisfy $\mathrm{h}^0(S, F) = 2$. Then for most points p of S we have that $\gamma_p(F) = 1$.*

Proof. The hypotheses tell us that F corresponds to a morphism $f : S \to \mathbb{P}^1$. Then generic fibre of f is a smooth curve; let C be such a smooth curve and let $p \in C$. Let $\pi : \tilde{S} \to S$ denote the blow-up of S at p with exceptional curve E.

Since $C \sim F$ we have that $F^2 = 0$. Let F_λ denote the divisor $\pi^*F - \lambda E$. Since C is smooth at p we have that $\pi^*F = C^{st} + E$, and thus $F_\lambda = C^{st} + [1 - \lambda]E$. The smoothness of p of C also tells us that $(C^{st})^2 = C^2 - 1 = -1$.

Certainly $\gamma_p(F) \geq 1$ since $F_1 = C^{st}$ is effective. Suppose that $\gamma_p(F) > 1$ for an eventual contradiction. Under this assumption, there exists $\beta > 1$ so that $F_\beta = C^{st} + (1 - \beta)E$ is effective. Observe that $F_\beta.C^{st} = (C^{st})^2 + (1 - \beta)E.C^{st} = -\beta < 0$. Since F_β is effective we therefore have that C^{st} must be a component of F_β. So F_β is linearly equivalent to a divisor of the form $C^{st} + \sum_{i=1}^{\ell} n_i A_i$ with all $n_i \geq 0$ for some (irreducible, reduced) curves $A_i \subseteq \tilde{S}$. It follows that $(1 - \beta)E$ is linearly equivalent to the effective divisor, hence is effective. But this is impossible: if D is a non-zero divisor, we cannot have both a positive and negative multiple of it being effective. This is our contradiction.

□

Example 4.2. This example follows from certain calculations done in [16]. Let X be a smooth cubic surface in $\mathbb{P}^3$, and let p belong to a line of X. It is shown that $\gamma_p(-K_X) = 2$, and hence $\gamma_p(2(-K_X)) = 4$. However, we are able to write $-2K_X = F_1 + F_2 + F_3$ with F_i certain nef and effective divisors that satisfy $\gamma_p(F_i) = 1$; F_1 belongs to $\Gamma(L_1)$, while F_2 and F_3 belong to $\Gamma(h)$. For most choices of p we have $\gamma_p(F_i) = 1$ by Lemma 4.1. Thus $\gamma_p(F_1 + F_2 + F_3) > \gamma_p(F_1) + \gamma_p(F_2) + \gamma_p(F_3)$ since $4 > 3$.

Remark 4.3. It is worth noting that all of F_1, F_2, and F_3 are basepoint free (they all determine morphisms to $\mathbb{P}^1$). For our toric varieties, a divisor is nef if and only if it is basepoint free. Thus a natural question to ask would be if $\gamma_p|_{\mathrm{Bpf}(X)}$ is in general linear. The example shows that it is not.

4.2 Dimension 3

Remark 4.4. What about dimensions three or higher? One issue that immediately comes to mind is that if $\dim(X) \geq 3$, we no longer have that $\mathrm{Nef}(X) = \overline{\mathrm{Eff}(X)}^{\vee}$. Indeed, we no longer have a bilinear form $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$. However, $\gamma_p : \mathrm{Eff}(X) \to \mathbb{R}^{\geq 0}$ is still well-defined, where $\mathrm{Eff}(X) = \{D \in \mathrm{NS}(X) : \mathrm{h}^0(X, D) > 0\}$.

We work out a couple of three-dimensional examples.

Example 4.5. Let X be the variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ blown up at a point p. We will calculate γ_p at a point away from the exceptional divisor. Consider the following vectors in $\mathbb{Z}^3$:

$$\begin{aligned}
u_1 &= (1, 0, 0)\\
u_2 &= (-1, 0, 0)\\
u_3 &= (0, 1, 0)\\
u_4 &= (0, -1, 0)\\
u_5 &= (0, 0, 1)\\
u_6 &= (0, 0, -1)\\
u_7 &= (-1, -1, -1) = u_2 + u_4 + u_6.
\end{aligned}$$

We denote by C_i, C_{ij}, and C_{ijk} the cones $\mathbb{N}u_i$, $\mathbb{N}u_i + \mathbb{N}u_j$, and $\mathbb{N}u_i + \mathbb{N}u_j + \mathbb{N}u_k$ respectively. The C_i correspond to hypersurfaces, the C_{ij} correspond to curves, and the C_{ijk} correspond to points. Let A_i denote the divisor corresponding to C_i. We have the relations

$$\begin{aligned}
A_1 &\sim A_2 + A_7\\
A_3 &\sim A_4 + A_7\\
A_5 &\sim A_6 + A_7
\end{aligned}$$

in $\mathrm{Pic}(X)$, and we use the basis $\mathbb{Z}A_2 \oplus \mathbb{Z}A_4 \oplus \mathbb{Z}A_6 \oplus \mathbb{Z}A_7$ of $\mathrm{Pic}(X)$. Let $D = aA_2 + bA_4 + cA_6 + dA_7 \in \mathrm{Pic}(X)$.

There are 10 top dimensional cones: they are the cones

$$C_{135}, C_{136}, C_{145}, C_{146}, C_{235}, C_{236}, C_{245}, C_{247}, C_{267}, C_{467}.$$

The Cartier data of D, as in Theorem 4.2.8(d) of [7], are the elements $m_{ijk} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^3, \mathbb{Z})$ associated to C_{ijk} defined by $m_{ijk}(u_i) = -\zeta_i$ where ζ_i is the coefficient of a_i for D. These are given by

$$
\begin{aligned}
m_{135} &= 0\\
m_{136} &= ce_3^*\\
m_{145} &= be_2^*\\
m_{146} &= be_2^* + ce_3^*\\
m_{235} &= ae_1^*\\
m_{236} &= ae_1^* + ce_3^*\\
m_{245} &= ae_1^* + be_2^*\\
m_{247} &= ae_1^* + be_2^* + (d-a-b)e_3^*\\
m_{267} &= ae_1^* + (d-a-c)e_2^* + ce_3^*\\
m_{467} &= (d-b-c)e_1^* + be_2^* + ce_3^*.
\end{aligned}
$$

The T-invariant curves of X are

$$C_{13}, C_{14}, C_{15}, C_{16}, C_{23}, C_{24}, C_{25}, C_{26}.C_{27}, C_{35}, C_{36}, C_{45}, C_{46}, C_{47}, C_{67}.$$

With the Cartier data of D in hand, we use Proposition 6.3.8 of [7] to calculate $D.C_{ij}$. We obtain that

$$
\begin{aligned}
D.C_{13} &= c\\
D.C_{14} &= c\\
D.C_{15} &= b\\
D.C_{16} &= b\\
D.C_{23} &= c\\
D.C_{24} &= d-a-b\\
D.C_{25} &= b\\
D.C_{26} &= d-a-c\\
D.C_{27} &= a+b+c-d\\
D.C_{35} &= a\\
D.C_{36} &= a\\
D.C_{45} &= a\\
D.C_{46} &= d-b-c\\
D.C_{47} &= a+b+c-d\\
D.C_{67} &= a+b+c-d.
\end{aligned}
$$

From the intersection data, it follows that $\text{Nef}(X)$ is generated by

$$
\begin{aligned}
N_1 &= A_2 + A_7\\
N_2 &= A_4 + A_7\\
N_3 &= A_6 + A_7\\
N_4 &= A_2 + A_4 + A_6 + 2A_7.
\end{aligned}
$$

The divisors N_1, N_2, and N_3 are the pullbacks of the usual three generators of $\text{Nef}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. On X, let p be the point corresponding to C_{135}, and let $\pi : Y \to X$ be the blow-up of X at p with exceptional divisor E. Write B_i for the ray of A_i in the fan of Y, ie B_i corresponds to the strict transform of A_i. We use the basis $\mathbb{Z}B_2 \oplus \mathbb{Z}B_4 \oplus \mathbb{Z}B_6 \oplus \mathbb{Z}B_7 \oplus \mathbb{Z}E$ of $\text{Pic}(Y)$.

To calculate γ_p we can no longer use that the $\text{Nef}(Y)$ is dual to $\text{Eff}(Y)$, and so instead we resort to counting the size of polytopes. Let $N = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + \alpha_4 N_4$ be an arbitrary nef divisor (so $\alpha_i \in \mathbb{N}$). For $\mu \in \mathbb{R}$ define N_μ to be the divisor $\pi^* N - \mu E$: we have that

$$N_\mu = (\alpha_1 + \alpha_4)B_2 + (\alpha_2 + \alpha_4)B_4 + (\alpha_3 + \alpha_4)B_6 + (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4)B_7 - \mu E.$$

The polytope P_{N_μ} associated to N_μ is the collection of $\phi = re_1^* + se_2^* + te_3^* \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R})$ which satisfy

$$\begin{aligned} &0 \le r \le \alpha_1 + \alpha_4 \\ &0 \le s \le \alpha_2 + \alpha_4 \\ &0 \le t \le \alpha_3 + \alpha_4 \\ &\mu \le r + s + t \le \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4. \end{aligned}$$

We want to make μ as large as possible while still having these inequalities satisfied. Since the first three inequalities sum to say $r+s+t \le \alpha_1+\alpha_2+\alpha_3+3\alpha_4$, then fourth inequality is the one that matters: in particular, we can have $\mu = r + s + t = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4$. Therefore $\gamma_p(N) = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4$, so γ_p is linear on the nef cone.

Example 4.6. Perhaps the previous example wasn't complicated enough! We take the previous variety X and blow up at a point on A_7. Consider the following vectors in $\mathbb{Z}^3$:

$$\begin{aligned} u_1 &= (1, 0, 0) \\ u_2 &= (-1, 0, 0) \\ u_3 &= (0, 1, 0) \\ u_4 &= (0, -1, 0) \\ u_5 &= (0, 0, 1) \\ u_6 &= (0, 0, -1) \\ u_7 &= (-1, -1, -1) = u_2 + u_4 + u_6 \\ u_8 &= (-2, -2, -1) = u_2 + u_4 + u_7. \end{aligned}$$

As before, we use the notation C_i, C_{ij}, and C_{ijk}, and let A_i be the divisor corresponding to C_i. We have the relations

$$\begin{aligned} A_1 &\sim A_2 + A_7 + 2A_8 \\ A_3 &\sim A_4 + A_7 + 2A_8 \\ A_5 &\sim A_6 + A_7 + A_8. \end{aligned}$$

We use the basis $\mathbb{Z}A_2 \oplus \mathbb{Z}A_4 \oplus \mathbb{Z}A_6 \oplus \mathbb{Z}A_7 \oplus \mathbb{Z}A_8$ of $\text{Pic}(X)$. There are 12 top dimensional cones: they are the cones

$$C_{135}, C_{136}, C_{145}, C_{146}, C_{235}, C_{236}, C_{245}, C_{248}, C_{267}, C_{278}, C_{467}, C_{478}.$$

Let $D = aA_2 + bA_4 + cA_6 + dA_7 + eA_8$; we work out the Cartier data of D:

$$\begin{aligned}
m_{135} &= 0 \\
m_{136} &= ce_3^* \\
m_{145} &= be_2^* \\
m_{146} &= be_2^* + ce_3^* \\
m_{235} &= ae_1^* \\
m_{236} &= ae_1^* + ce_3^* \\
m_{245} &= ae_1^* + be_2^* \\
m_{248} &= ae_1^* + be_2^* + (e - 2a - 2b)e_3^* \\
m_{267} &= ae_1^* + (d - a - c)e_2^* + ce_3^* \\
m_{278} &= ae_1^* + (e - d - a)e_2^* + (2d - e)e_3^* \\
m_{467} &= (d - b - c)e_1^* + be_2^* + ce_3^* \\
m_{478} &= (e - d - b)e_1^* + be_2^* + (2d - e)e_3^*.
\end{aligned}$$

The T-invariant curves are

$$C_{13}, C_{14}, C_{15}, C_{16}, C_{23}, C_{24}, C_{25}, C_{26}.C_{27}, C_{28}, C_{35}, C_{36}, C_{45}, C_{46}, C_{47}, C_{48}, C_{67}, C_{78}.$$

Using the Cartier data, we calculate $D.C_{ij}$:

$$\begin{aligned}
D.C_{13} &= c \\
D.C_{14} &= c \\
D.C_{15} &= b \\
D.C_{16} &= b \\
D.C_{23} &= c \\
D.C_{24} &= e - 2a - 2b \\
D.C_{25} &= b \\
D.C_{26} &= d - a - c \\
D.C_{27} &= e + c - 2d \\
D.C_{28} &= a + b + d - e \\
D.C_{35} &= a \\
D.C_{36} &= a \\
D.C_{45} &= a \\
D.C_{46} &= d - b - c \\
D.C_{47} &= c + e - 2b \\
D.C_{48} &= a + b + d - e \\
D.C_{67} &= a + b + c - d \\
D.C_{78} &= a + b + d - e.
\end{aligned}$$

It follows that $\mathrm{Nef}(X)$ is generated by

$$\begin{aligned}
N_1 &= A_6 + A_7 + A_8 \\
N_2 &= A_4 + A_7 + 2A_8 \\
N_3 &= A_2 + A_7 + 2A_8 \\
N_4 &= A_2 + A_4 + 2A_6 + 3A_7 + 4A_8 \\
N_5 &= A_2 + A_4 + A_6 + 2A_7 + 4A_8.
\end{aligned}$$

On X, let p be the point corresponding to C_{135}, and let $\pi : Y \to X$ be the blow-up of X at p with exceptional divisor E. Write B_i for the ray of A_i in the fan of Y, ie B_i corresponds to the strict transform of A_i. We use the basis $\mathbb{Z}B_2 \oplus \mathbb{Z}B_4 \oplus \mathbb{Z}B_6 \oplus \mathbb{Z}B_7 \oplus \mathbb{Z}B_8 \oplus \mathbb{Z}E$ of $\mathrm{Pic}(Y)$.

Let $N = \sum_{i=1}^{5} \alpha_i N_i$ be an arbitrary nef divisor (so all $\alpha_l \geq 0$). For $\mu \in \mathbb{R}$, let N_μ denote the divisor $\pi^* N - \mu E$. We have that

$$\begin{aligned}
N_\mu = {} & (\alpha_3 + \alpha_4 + \alpha_5)B_2 + (\alpha_2 + \alpha_4 + \alpha_5)B_4 + (\alpha_1 + 2\alpha_4 + \alpha_5)B_6 \\
& + (\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 + 2\alpha_5)B_7 + (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 4\alpha_5)B_8 - \mu E.
\end{aligned}$$

The polytope P_μ associated to N_μ is the collection of $\phi = re_1^* + se_2^* + te_3^* \in \mathrm{Hom}^3(\mathbb{R}^3, \mathbb{R})$ which satisfy

$$\begin{aligned}
0 \leq r &\leq \alpha_3 + \alpha_4 + \alpha_5 \\
0 \leq s &\leq \alpha_2 + \alpha_4 + \alpha_5 \\
0 \leq t &\leq \alpha_1 + 2\alpha_4 + \alpha_5 \\
\mu \leq r + s + t &\leq \alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 + 2\alpha_5 \\
2r + 2s + t &\leq \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 4\alpha_5.
\end{aligned}$$

From this it follows that $\gamma_p(\sum_{i=1}^{5} \alpha_i N_i) = \alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 + 2\alpha_5$.

4.3 Future Work

There are many directions that this work could be continued. In view of the examples on three-folds in the previous section, we raise the following question:

Question 4.7. *Let X be a smooth, complete, toric variety, and let $p \in X$ be a T-invariant point. Is $\gamma_p|_{\mathrm{Nef}(X)}$ linear?*

In view of the examples of section 4.2, as well as Corollary 3.53, we also make a conjecture about the positivity of coefficients of nef divisors. Note that the smoothness assumed below means that any top dimensional cone has $\dim(X)$ generators. Since the idea of 'adjacent' rays no longer make sense, we use the appropriate analogue, which generalizes the two adjacent divisors for surfaces.

Conjecture 4.8. *Let X be a smooth, complete, toric variety coming from a fan Δ. Let $\Delta(1)$ denote the collection of rays (1-dimensional fans of Δ). Let σ be a top-dimensional cone of Δ. It follows that $\mathrm{Pic}(X)$ is free abelian of rank $\#\Delta(1) - \dim(X)$. Choose a basis of*

Pic(X) *coming from all the rays which aren't part of σ. Denote the divisors by $A_1, ..., A_k$. Then*

$$\text{Nef}(X) \subseteq \bigoplus_{i-1}^{k} \mathbb{N}A_i.$$

In this thesis, we focused on the case of smooth varieties. It would be interesting to do some calculations on singular varieties, especially surfaces. The constant γ_p could yield valuable information for the general program of resolution of singularities.

Furthermore, we focused on calculating γ_p for p a T-invariant point of X. Another avenue of study would be to follow the spirit of Ito ([11]), and study the situation at other points on our toric varieties (ie on the orbits that are bigger than one element). One would expect to only obtain bounds on γ_p, rather than explicit values.

Furthermore, following the setup of [5], it would be interesting to replace our T-invariant point p with a T-invariant subvariety V of X. We may then form the blow-up $\pi : \tilde{X} \to X$ of X along the ideal sheaf $\mathcal{I}_{V \subseteq X}$ of V in X. Let E denote the exceptional divisor. Given an effective divisor D on X, we then define

$$\gamma_V(L) = \sup\{t \geq 0 : \pi^* D - tE \text{ is effective}\}.$$

Since V is T-invariant, the blow-up $\tilde{X}$ will again be toric, and explicit calculations should be possible.

Another question the author was unable to answer, for smooth surfaces, concerns the product $\gamma_p(L)\epsilon_p(L)$ for $L \in \text{Nef}(X) \cap \text{Eff}(X)$. In every example worked out which has Picard rank at least 2, we are able to find an L which satisfies $\gamma_p(L)\epsilon_p(L) < L^2$. Furthermore, Theorem 3.58 shows that such an L exists for a large class of the toric surfaces. We thus make the following conjecture.

Conjecture 4.9. *Let X be a smooth, complete, surface with* $\text{rank}_{\mathbb{Z}}(\text{NS}(X)) \geq 2$. *Let $p \in X$. There exists $L \in \text{Nef}(X) \cap \text{Eff}(X)$ so that $\gamma_p(L)\epsilon_p(L) < L^2$.*

To the best of the author's knowledge, apart from the examples provided in this document, there are no other varieties X for which γ_p is well understood. The theory of this invariant should be studied on other classes of surfaces, such as $K3$ surfaces or abelian varieties.

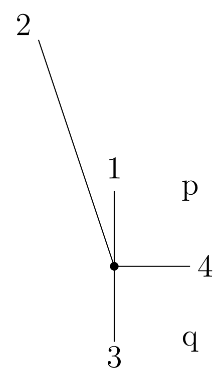
APPENDICES

Appendix A

Some examples

In this appendix, we give examples of closed form expressions for γ_p on many smooth, complete, toric surface, for both the effective cone and nef cone. Code to generate this data was written in Maple. A_i refers to the divisor corresponding to the i^{th} ray of our fan. We use E_l for generators of the effective cone, and N_l for generators of the nef cone. Unless said otherwise, we are always calculating γ_p at the point corresponding to Cone((1,0),(0,1)).

Example A.1. Let n be a non-negative integer. Let X be the variety coming from the fan



where $u_1 = (0,1)$, $u_2 = (-1,n)$, $u_3 = (0,-1)$, and $u_4 = (1,0)$. In this example In fact, $X = \mathcal{H}_n$ is a Hirzebruch surface, and our point p lives on A_1, the unique irredicuble curve which satisfies $A_1^2 = -n$, while q is not supported on A_1. The effective cone Eff(X) is generated by

$$E_1 = A_1$$
$$E_2 = A_2$$

and Nef(X) is generated by

$$N_1 = A_2$$
$$N_2 = A_1 + nA_2$$

We have that

$$\gamma_p(e_1E_1 + e_2E_2) = e_1 + e_2$$

and that

$$\gamma_p(n_1N_1 + n_2N_2) = n_1 + (n+1)n_2$$

For the point q, we continue using the basis $\mathbb{Z}A_1 \oplus \mathbb{Z}A_2$ of $\mathrm{Pic}(X)$. We have that

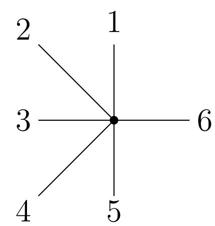
$$\gamma_q(e_1A_1 + e_2A_2) = e_2$$

and that

$$\gamma_q(n_1N_1 + n_2N_2) = n_1 + nn_2$$

This example illustrates that the point we work with does indeed matter. We see that A_1 does not contribute to γ_q: this is not surprising, in accordance with theorem 3.35: the divisor A_1 cannot be moved to have q in its support.

Example A.2. Let X be the variety coming from the fan



where $u_1 = (0,1)$, $u_2 = (-1,1)$, $u_3 = (-1,0)$, $u_4 = (-1,-1)$, $u_5 = (0,-1)$, and $u_6 = (1,0)$. The effective cone $\mathrm{Eff}(X)$ is generated by

$$\begin{aligned} E_1 &= A_1 \\ E_2 &= A_2 \\ E_3 &= A_3 \\ E_4 &= A_4 \\ E_5 &= A_1 + A_2 - A_4 \end{aligned}$$

and $\mathrm{Nef}(X)$ is generated by

$$\begin{aligned} N_1 &= A_2 + A_3 + A_4 \\ N_2 &= A_1 + A_2 \\ N_3 &= A_1 + A_2 + A_3 + A_4 \\ N_4 &= 2A_1 + 2A_2 + A_3 \\ N_5 &= A_1 + 2A_2 + A_3 \end{aligned}$$

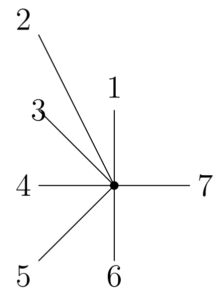
We have that

$$\gamma_p(\sum_{j=1}^{5} q_iN_i) = q_1 + q_2 + 2q_3 + 2q_4 + q_5$$

and that

$$\gamma_p(\sum_{k=1}^{5} e_iE_i) = e_1 + \min(e_4, e_3 + e_5, e_2 + 2e_5)$$

Example A.3. Let X be the variety coming from the fan



where $u_1 = (0,1)$, $u_2 = (-1,2)$, $u_3 = (-1,1)$, $u_4 = (-1,0)$, $u_5 = (-1,-1)$, $u_6 = (0,-1)$, and $u_7 = (1,0)$. The effective cone $\mathrm{Eff}(X)$ is generated by

$$\begin{aligned} E_1 &= A_1 \\ E_2 &= A_2 \\ E_3 &= A_3 \\ E_4 &= A_4 \\ E_5 &= A_5 \\ E_6 &= A_1 + 2A_2 + A_3 - A_5 \\ E_7 &= A_2 + A_3 + A_4 + A_5 \end{aligned}$$

The nef cone $\mathrm{Nef}(X)$ is generated by

$$\begin{aligned} N_1 &= A_2 + A_3 + A_4 + A_5 \\ N_2 &= A_1 + 2A_2 + A_3 \\ N_3 &= A_1 + 2A_2 + A_3 + A_4 + A_5 \\ N_4 &= 2A_1 + 4A_2 + 2A_3 + A_4 \\ N_5 &= A_1 + 2A_2 + 2A_3 + 2A_4 + 2A_5 \\ N_6 &= 3A_1 + 6A_2 + 4A_3 + 2A_4 \\ N_7 &= A_1 + 3A_2 + 2A_3 + A_4 \end{aligned}$$

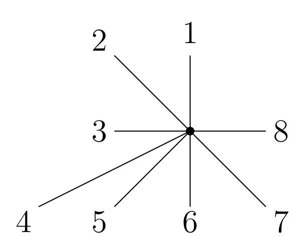
We have that

$$\gamma_p(\sum_{i=1}^{7} q_i N_i) = q_1 + q_2 + 2q_3 + 2q_4 + 3q_5 + 3q_6 + q_7$$

and that

$$\gamma_p(\sum_{i=1}^{7} e_i E_i) = e_1 + e_7 + \min(e_5, e_4 + e_6, e_3 + 2e_6, e_2 + 3e_6)$$

Example A.4. Let X be the variety coming from the fan



where $u_1 = (0, 1)$, $u_2 = (-1, 1)$, $u_3 = (-1, 0)$, $u_4 = (-2, -1)$, $u_5 = (-1, -1)$, $u_6 = (0, -1)$, $u_7 = (1, -1)$, and $u_8 = (1, 0)$. The effective cone $\mathrm{Eff}(X)$ is generated by

$$
\begin{aligned}
E_1 &= A_1 \\
E_2 &= A_2 \\
E_3 &= A_3 \\
E_4 &= A_4 \\
E_5 &= A_5 \\
E_6 &= A_6 \\
E_7 &= A_1 + A_2 - A_4 - A_5 - A_6 \\
E_8 &= -A_1 + A_3 + 3A_4 + 2A_5 + A_6
\end{aligned}
$$

and the nef cone $\mathrm{Nef}(X)$ is generated by

$$
\begin{aligned}
N_1 &= 2A_1 + 2A_2 + A_3 + A_4 \\
N_2 &= 3A_1 + 3A_2 + A_3 \\
N_3 &= A_1 + A_2 \\
N_4 &= A_2 + A_3 + 2A_4 + A_5 \\
N_5 &= A_1 + 2A_2 + A_3 + A_4 \\
N_6 &= 2A_1 + 3A_2 + A_3 \\
N_7 &= A_3 + 3A_4 + 2A_5 + A_6 \\
N_8 &= 3A_2 + 3A_3 + 6A_4 + 4A_5 + 2A_6 \\
N_9 &= A_2 + A_3 + 3A_4 + 2A_5 + A_6 \\
N_{10} &= A_1 + A_2 + A_3 + 2A_4 + A_5 \\
N_{11} &= 2A_2 + 2A_3 + 4A_4 + 2A_5 + A_6
\end{aligned}
$$

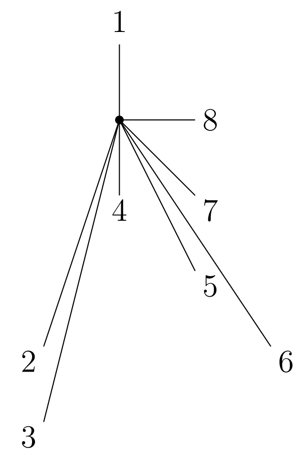
We have that

$$\gamma_p(\sum_{i=1}^{11} q_i N_i) = 2q_1 + 3q_2 + q_3 + q_4 + q_5 + 2q_6 + 2q_7 + 4q_8 + 2q_9 + 2q_{10} + 2q_{11}$$

and that

$$\gamma_p(\sum_{i=1}^{8} e_i E_i) = e_1 + e_8 + \min(e_5, e_4, \frac{1}{3}(2e_4 + e_7), \frac{1}{2}(e_4 + e_6), 2e_3 + e_7, e_3 + e_6, e_2 + 2e_6)$$

Example A.5. Let X be the variety coming from the fan



where $u_1 = (0, 1)$, $u_2 = (-1, -3)$, $u_3 = (-1, -4)$, $u_4 = (0, -1)$, $u_5 = (1, -2)$, $u_6 = (2, -3)$, $u_7 = (1, -1)$, and $u_8 = (1, 0)$. The effective cone $\mathrm{Eff}(X)$ is generated by

$$E_i = A_i \text{ for } 1 \leq i \leq 6$$
$$E_7 = A_1 - 3A_2 - 4A_3 - A_4 - 2A_5 - 3A_6$$
$$E_8 = -A_1 + 4A_2 + 5A_3 + A_4 + A_5 + A_6$$

and the nef cone $\mathrm{Nef}(X)$ is generated by

$$N_1 = A_1$$
$$N_2 = A_1 + A_2$$
$$N_3 = A_2 + A_3$$
$$N_4 = 5A_2 + 5A_3 + A_4 + A_5 + A_6$$
$$N_5 = 11A_2 + 11A_3 + 2A_4 + A_5$$
$$N_6 = 6A_2 + 6A_3 + A_4$$
$$N_7 = 4A_2 + 5A_3 + A_4 + A_5 + A_6$$
$$N_8 = 9A_2 + 11A_3 + 2A_4 + A_5$$
$$N_9 = 5A_2 + 6A_3 + A_4$$

We have that

$$\gamma_p(\sum_{i=1}^{9} q_i N_i) = 3q_1 + 4q_2 + q_3 + 5q_4 + 11q_5 + 6q_6 + 4q_7 + 9q_8 + 5q_9$$

and that

$$\gamma_p(\sum_{i=1}^{8} e_i E_i) = 3e_1 + e_8 + \min(e_2, e_1 + e_3)$$

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