Enumerative Applications of Integrable Hierarchies

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Countably infinite families of partial differential equations such as the Kadomtzev - Petviashvili (KP) hierarchy and the B-type KP (BKP) hierarchy have received much interest in the mathematical and theoretical physics community for over forty years. Recently there has been much interest in the application of these families of partial differential equations to a variety of problems in enumerative combinatorics.

For example, the generating series for monotone Hurwitz numbers, studied by Goulden, Guay-Paquet and Novak [44, 45], is known to be a solution to the KP hierarchy. Using this fact along with some additional constraints we may find a second order, quadratic differential equation for the generating series for simple monotone Hurwitz numbers (a specialization of the problem considered in [44, 45] corresponding to factorizations of the identity). In addition, asymptotic analysis can be performed and it may be shown that the asymptotic behaviour of the simple monotone Hurwitz numbers is governed by the map asymptotics constants studied by Bender, Canfield and Gao.

In their enumerative study of various families of maps, Bender, Canfield and Gao [6, 39] proved that for maps embedded in an orientable surface the asymptotic behaviour could be completely determined up to some constant depending only on genus and that similarly for maps embedded on a non-orientable surface the asymptotic behaviour could be determined up to a constant depending on the Euler characteristic of the surface. However, the only known way of computing these constants was via a highly non-linear recursion, making the determination of these constants very difficult.

Using the integrable hierarchy approach to enumerative problems, Goulden and Jackson [50] derived a quadratic recurrence for the number of rooted triangulations on an orientable surface of fixed genus. This result was then used by Bender, Richmond and Gao to show that the generating series for the orientable map asymptotics constants was given by a solution to a nonlinear differential equation called the Painlevé I equation. This gave a method for computing the orientable map asymptotics constants which was significantly simpler than any previously known method.

A remaining open problem was whether a suitable integrable hierarchy could be found which could be applied to the corresponding problems in the non-orientable case. Using the BKP hierarchy of partial differential equations applied to the enumeration of rooted triangulations on all surfaces (orientable or non-orientable) we find a cubic recursion for the number of such triangulations and, as a result, we find a nonlinear differential equation which determines the non-orientable map asymptotics constants.

In this thesis we provide a detailed development of both the KP and BKP hierarchies. We also discuss three different applications of these hierarchies, the two mentioned above (monotone Hurwitz numbers and rooted triangulations on locally orientable surfaces) as well as orientable bipartite quadrangulations.

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Finally, I would like to thank Catherine for the support that she has given me throughout the years. I cannot imagine having done any of this without you.

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$R[[x, y, z, \ldots]]$	Series with coefficients in R	18
$\lambda \downarrow j$	Switching the jth $\mathbf{U}$ to an $\mathbf{R}$	23
$\lambda \uparrow i$	Switching the ith $\mathbf{R}$ to a $\mathbf{U}$	22
$u_i(\lambda)$	The number of $\mathbf{U}$s that follow the ith $\mathbf{R}$	23
$B(\mathbf{p}; t)$	Bernstein operator	24
$B^{\perp}(\mathbf{p}; t)$	Adjoint Bernstein operator	24
$\mathfrak{H}(\phi; f, g)$	Hirota differential operator	32
$\phi_\lambda^{(i)}$	Polynomials appearing in the Hirota operator form of various integrable hierarchies	33
$\langle f(\mathbf{x}) \rangle_n$	Iterated integral operator related to KP hierarchy	42
$\langle\!\langle f(\mathbf{x}) \rangle\!\rangle_n$	Iterated integral operator related to the BKP hierarchy	53
$H_\alpha(n, v, k)$	Number of rooted bipartite maps with n edges and v vertices, k of which are white, where the half face degrees are given by the parts of α	58
$H(z, w, x; \mathbf{p})$	Generating function for the $H_\alpha(n, v, k)$	58
Q_g^n	Number of rooted bipartite quadrangulations of genus g with n faces	56
$Q_g^n(x)$	Generating polynomial of maps with genus g and n edges where x marks the number of faces	57
$Q_g^{n,f}$	Number of rooted maps of genus g with n edges and f faces	57
$M_g^{i,j}$	Number of rooted maps of genus g with i vertices and j faces	58
$T_g(n)$	Number of n-edged rooted maps on an orientable surface of genus g	68
$P_g(n)$	Number of n-edged rooted maps on a non-orientable surface of genus g	68
t_g	Orientable map asymptotics constant	68
p_g	Non-orientable map asymptotics constant	68
$\ell_{k,\alpha}$	Number of maps (not necessarily orientable) with k faces and vertex partition given by α	73
$L(t; y)$	Generating function for $\ell_{k,\alpha}$	73
$L^{(3)}(x; w)$	Generating series for cubic maps where x marks vertices and w marks Euler genus	73

$L^{(3)}(x,y)$	Generating series for cubic maps where x marks vertices and y marks faces	73
$L_g^{(3)}(z)$	Generating series for cubic maps with Euler genus g where the exponent of z records half the number of vertices	76
$T_g(z)$	A transformed $L_g^{(3)}(z)$	76
ψ_i	A family of algebraic series related to the structure of $L_g^{(3)}(z)$	79
$\mu_g(i)$	Coefficient of ψ_i in $L_g^{(3)}$	81
$\chi(i,j)$	Scaling factor appearing in the multiplication rule for the ψ_i series	80
α_g	Leading coefficient of $L_g^{(3)}(z)$ when expanded in the ψ_i basis	81
β_g	Rescaling of α_g	81
$\beta(z)$	Generating series for the β_g	83
$\Phi_n(f)$	Plancherel average of the function f	85
$\mathcal{H}$	Generating series for the Plancherel average of complete symmetric functions	86
γ	Scaled logarithm of the series $\mathcal{H}$	86
γ_g	Genus g component of the series γ	92
$\overrightarrow{\mathcal{H}}$	Generating series for the class expansion coefficients of the complete symmetric functions	87
Γ	Logarithm of the series $\overrightarrow{\mathcal{H}}$	87
Γ_g	Genus g component of the series Γ	88
J_k	Jucys-Murphy element	87
$\overrightarrow{h}_g(n)$	Simple monotone Hurwitz number	105

Chapter 1

Introduction and Background

1.1 Introduction

The relationship between integrable hierarchies and combinatorial enumeration has been a topic of great interest over the last fifty years, beginning with the statement and resolution of Witten's conjecture concerning the generating function for certain intersection numbers on the moduli space of marked curves. It is the purpose of this thesis to provide a clear description of two different integrable hierarchies (the KP hierarchy and the BKP hierarchy) in a way which is accessible to a combinatorial audience. In addition we discuss three new applications of these hierarchies to enumerative problems. In the remainder of this section we will briefly give an account of the KP hierarchy and its applications in enumerative combinatorics.

1.1.1 The Hurwitz Problem

One of the earliest enumerative applications of the KP hierarchy is the problem of counting equivalence classes of ramified coverings of the sphere with prescribed branching data, originally studied by Hurwitz [59] and refered to as the *Hurwitz problem*. For complete details concerning the Hurwitz enumeration problem we refer the reader to Lando and Zvonkin [74].

Suppose X is a genus $g \geq 0$ Riemann surface and that $f : X \to S^2$ is a continuous map from X to the sphere S^2. Let $\{z_1, \dots, z_k\} \subseteq S^2$ be a finite set of points on S^2 and suppose that for some fixed positive integer d and for any $y \in S^2 \backslash \{z_1, \dots, z_k\}$ there exists some neighborhood V such that $f^{-1}(V)$ is homeomorphic to $V \times T$ where T is a discrete set of size d. Moreover, $\{z_1, \dots, z_k\}$ do not satisfy this property. We say that f is a degree d ramified covering of the sphere with branch points $\{z_1, \dots, z_k\}$.

If $f_1 : X_1 \to S^2$ and $f_2 : X_2 \to S^2$ are degree d ramified coverings of S^2 then we say that they are equivalent if there exists homeomorphisms

$$\begin{aligned} \sigma &: X_1 \to X_2, \\ \rho &: S^2 \to S^2, \end{aligned}$$

such that $\rho \circ f_1 = f_2 \circ \sigma$. The Hurwitz problem is then to enumerate the equivalence classes of degree d ramified covers of S^2.

Hurwitz [59] proved that the equivalence classes of degree d ramified covers could be encoded combinatorially as permutation factorizations. A brief sketch of this argument proceeds as follows. Suppose $f : X \to S^2$ is a degree d ramified cover of S^2 with branch points $\{z_1, \dots, z_k\}$ and let D be an open disc such that the branch points are on the boundary of D. By definition there are d connected components in $f^{-1}(D)$ which we may label from 1 to d. We call the connected components in $f^{-1}(D)$ the sheets of the cover. If we look at a small neighborhood of z_i, beginning on sheet s and going around z_i counter clockwise, we will arrive at a point on another sheet, say $\pi^{(i)}(s)$. In doing so we construct a permutation $\pi^{(i)}$ for each branch point. If the permutation corresponding to a branch point is a transposition then we say that the branch point is simple.

If we begin at some point $x \in S^2 \backslash \{z_1, \dots, z_k\}$ and walk around each branch point as described above, then if we begin on a sheet s, we must end on sheet s since the corresponding loop on $S^2 \backslash \{z_1, \dots, z_k\}$ is contractible to a point. This means that

$$\pi^{(1)} \cdots \pi^{(k)} = 1,$$

where 1 is the identity permutation. This is called the monodromy condition. Also, since X is connected, we must be able to move from one sheet to any other and so the subgroup generated by $\pi^{(1)}, \dots, \pi^{(k)}$ must act transitively on $\{1, 2, \dots, d\}$. This is called the transitivity condition. Hurwitz [59] proved that the number of k-tuples $(\pi^{(1)}, \dots, \pi^{(k)})$ which satisfy the monodromy and transitivity conditions are in bijection with the number of equivalence classes of degree d ramified covers of S^2. Furthermore, the cycle type of the permutations corresponds to the ramification profile of the corresponding branch points. Thus, if we let $Cov_d(\lambda^1, \dots, \lambda^k)$ be the number of k-tuples $(\pi_1, \dots, \pi_k)$ where each π_i is a permutation of $\{1, \dots, d\}$ and $(\pi_1, \dots, \pi_k)$ satisfies the conditions

1. π_i has cycle type λ^i for all i,
2. $\pi_1 \cdots \pi_k = 1$, and
3. the subgroup generated by $\pi_1, \dots, \pi_k$ is transitive,

then the Hurwitz problem is equivalent to determining $Cov_d(\lambda^1, \dots, \lambda^k)$.

In what follows we will focus on the case that one of the branch points has arbitrary ramification and the remaining branch points are simple (has ramification profile given by 21^{d-2}). In other words, we will consider the numbers

$$h_k(\alpha) = Cov_d(\alpha, 21^{d-2}, \dots, 21^{d-2}),$$

where $\alpha \vdash d$ is a partition of d and there are k copies of the partition 21^{d-2}. Further, we will define

$$H(z, \mathbf{p}) = H(z, p_1, p_2, \dots) = \sum_{d \geq 1} \frac{1}{d!} \sum_{k=1}^{\infty} \sum_{\alpha \vdash d} h_k(\alpha) \frac{z^k}{k!} p_{\alpha_1} p_{\alpha_2} \cdots.$$

Using the class algebra of the symmetric group, the generating series $H(z,\mathbf{p})$ can be written in a slightly different way that makes it more amenable to the tools which we will be developing in this thesis. For any partition λ let C_λ be the element in the class algebra corresponding to permutations with cycle type λ. That is, C_λ is the sum of all permutations with cycle type λ. Then for partitions $\lambda^1, \lambda^2, \ldots, \lambda^k$ of d we see that

$$Cov_d(\lambda^1, \ldots, \lambda^k) = [C_{1^d}]C_{\lambda^1} \cdots C_{\lambda^k} = [C_{\lambda^1}]C_{\lambda^2} \cdots C_{\lambda^k},$$

since a permutation and its inverse have the same cycle type. In particular, this means that

$$h_k(\alpha) = [C_\alpha]C_{21^{d-2}}^k.$$

Using the results in Chapter 5, Section 5.2 this allows us to write the generating series $H(z,\mathbf{p})$ as

$$\begin{aligned} H(z,\mathbf{p}) &= \sum_{d\geq 1} \frac{1}{d!} \sum_{\alpha \vdash d} [C_\alpha] \exp\left(C_{21^{d-2}} z\right) p_{\alpha_1} p_{\alpha_2} \cdots, \\ &= \sum_{d\geq 1} \frac{1}{d!} \sum_{\alpha \vdash d} \prod_{(i,j)\in\alpha} e^{(j-i)z} s_\alpha, \end{aligned} \tag{1.1}$$

where the product is over all cells (i,j) in α and s_α is the Schur polynomial indexed by α. In other words, (1.1) says that $H(z,\mathbf{p})$ is a content-type series (see Chapter 2, Section 2.4.2 for more details) and so satisfies the KP hierarchy. In particular, $h(z,\mathbf{p}) = \log H(z,\mathbf{p})$ satisfies the KP equation,

$$\frac{1}{12}\frac{\partial^4 h(z,\mathbf{p})}{\partial p_1^4} - \frac{\partial^2 h(z,\mathbf{p})}{\partial p_1 \partial p_3} + \frac{\partial^2 h(z,\mathbf{p})}{\partial p_2^2} + \frac{1}{2}\left(\frac{\partial^2 h(z,\mathbf{p})}{\partial p_1^2}\right)^2 = 0.$$

1.1.2 ELSV, Hodge Integrals and Witten's Conjecture

Suppose $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford compactification of the moduli space of genus g curves with n marked points. Let $\mathcal{C}$ be a genus g curve and $x_1, \ldots, x_n$ be distinct points on $\mathcal{C}$ so that $X = (\mathcal{C}, x_1, \ldots, x_n)$ is a point in $\overline{\mathcal{M}}_{g,n}$. For each i, we can define a line bundle $\mathcal{L}_i$ on $\overline{\mathcal{M}}_{g,n}$ whose fiber at the point X is the holomorphic cotangent line to $\mathcal{C}$ at x_i. Let ψ_i be the first Chern class of $\mathcal{L}_i$. In addition, each $\overline{\mathcal{M}}_{g,n}$ admits a natural rank g vector bundle $\mathbb{E}$, the Hodge bundle, whose fiber at X corresponds to the space of global holomorphic differentials on $\mathcal{C}$. We let λ_k be the kth Chern class of $\mathbb{E}$ and we let $\lambda_0 = 1$. We denote the intersection numbers (also called *Hodge integrals*) by

$$\Phi(\lambda_k \tau_{m_1} \cdots \tau_{m_n}) = \int_{\overline{\mathcal{M}}_{g,n}} \lambda_k \psi_1^{m_1} \cdots \psi_n^{m_n}.$$

Note that since the intersection numbers are independent of the order of the ψ_i classes, we may write the intersection numbers using exponential notation so that

$\tau_i^{m_i}$ denotes m_i copies of τ_i. Witten [110] conjectured that the generating series

$$F(t_0, t_1, \ldots) = \Phi\left(\exp\left(\sum_{i\geq 0} t_i \tau_i\right)\right),$$

satisfies the KdV equation, and more generally the KdV hierarchy, a fact which is known as *Witten's conjecture*. That is, it can be written as a solution to the KP hierarchy which does not depend on any of the even indexed parameters. There are multiple proofs of Witten's conjecture in the literature each with its own strengths. In what follows we will outline a combinatorial proof of the fact that $F(t_0, t_1, \ldots)$ satisfies the KdV hierarchy given by Kazarian [71] which is related to an earlier method used by Kazarian and Lando [72] to show that $F(t_0, t_1, \ldots)$ satisfies the KdV equation. The proof proceeds by using the fact that the generating series for simple Hurwitz numbers satisfies the KP hierarchy.

The first step in the proof of Witten's conjecture is a result of T. Ekadahl, S. Lando, M. Shapiro and A. Vainshtein [28] called the ELSV formula which relates Hodge integrals and simple Hurwitz numbers. That is,

$$\frac{h_k(\alpha)}{k!} = \prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots + (-1)^g \lambda_g}{\prod_{i=1}^{\ell(\alpha)} (1 - \alpha_i \psi_i)},$$

where the genus g can be determined by the Riemann-Hurwitz formula,

$$k = 2g - 2 + \ell(\alpha) + |\alpha|.$$

Following Kazarian [71], let

$$\widetilde{G}(u; T_0, T_1, \ldots) = \sum_{j, k_0, k_1, \ldots} (-1)^j \Phi(\lambda_j \tau_0^{k_0} \tau_1^{k_1} \cdots) u^{2j} \frac{T_0^{k_0}}{k_0!} \frac{T_1^{k_1}}{k_1!} \cdots,$$

and let $G(u; q_1, q_2, \ldots)$ be the series obtained from $\widetilde{G}$ by the linear substitution of variables define by the recursion

$$T_{k+1} = \sum_{m\geq 1} m(u^2 q_m + 2u q_{m+1} + q_{m+2}) \frac{\partial}{\partial q_m} T_k,$$

for $k \geq 0$ and $T_0 = q_1$. Using the ELSV formula, we can show that the Hurwitz series H and the generating series G are related by a change of variable.

Let x and y be related by

$$x = \frac{y}{1 + zy} e^{-\frac{zy}{1+zy}},$$
$$y = \sum_{b\geq 1} \frac{b^b}{b!} z^{b-1} x^b,$$

where the indeterminate z is the same one that appears in the generating function H. Note that x and y are inverse to one another with respect to the Lagrange inversion formula [53, Theorem 1.2.4]. Now, suppose that the family of variables p_b and the family of variables q_k are related by,

$$p_b = \sum_{k \geq b} c_k^b z^{k-b} q_k,$$

where the rational coefficients c_k^b are determined by the expansion

$$x^b = \sum_{k \geq b} c_k^b z^{k-b} y^k.$$

Also, let

$$\begin{aligned} H_1 &= \sum_{b \geq 1} \frac{b^{b-2}}{b!} p_b z^{b-1}, \\ H_2 &= \frac{1}{2} \sum_{b_1, b_2 \geq 1} \frac{b_1^{b_1} b_2^{b_2}}{(b_1 + b_2) b_1! b_2!} p_{b_1} p_{b_2} z^{b_1 + b_2}. \end{aligned}$$

Then Kazarian [71] proved that

$$(H(z, \mathbf{p}) - H_1 - H_2) = G(z^{1/3}; z^{4/3} q_1, z^{8/3} q_2, z^{12/3} q_3, \ldots).$$

Note that H_1 and H_2 are subtracted from $H(z, \mathbf{p})$ to make up for the fact that $\overline{\mathcal{M}_{0,1}}$ and $\overline{\mathcal{M}_{0,2}}$ do not exist. In addition to showing that the Hurwitz series H and the Hodge integral series G are related by a change of variables, Kazarian also showed that the change of variables is an automorphism (up to scaling) of the KP hierarchy. Explicitly, $G(u; q_1, q_2, \ldots)$ satisfies the KP hierarchy in the variables $\frac{q_i}{i}$.

Lastly, to see that $F(t_0, t_2, \ldots)$ satisfies the KdV hierarchy we need to show that F can be written as a solution to the KP hierarchy which does not depend on the even indexed variables. Suppose we set $u = 0$ in the generating series $\widetilde{G}$. Then since

$$T_{k+1} = \sum_{m \geq 1} m(u^2 q_m + 2u q_{m+1} + q_{m+2}) \frac{\partial}{\partial q_m} T_k,$$

we see that

$$T_{k+1}|_{u=0} = (2k-1)!! q_{2k+1},$$

where we have used the double factorial to mean $(2k-1)!! = (2k-1)(2k-3)\cdots(1)$. Thus, since

$$G(0; q_1, q_2, \ldots) = \sum_{k_0, k_2, \ldots} \Phi(\tau_0^{k_0} \tau_1^{k_1} \cdots) \frac{(T_0|_{u=0})^{k_0}}{k_0!} \frac{(T_1|_{u=0})^{k_1}}{k_1!} \cdots,$$

we have

$$F(t_0, t_1, \ldots) = G(0; t_0, 0, t_1, 0, \frac{t_2}{3!!}, 0, \frac{t_3}{5!!}, 0, \ldots).$$

In particular, this tells us that $F(t_0, t_1, \ldots)$ is a solution to the KdV hierarchy, proving Witten's conjecture.

Though there have subsequently been many other enumerative applications of the KP hierarchy, Witten's conjecture provided much of the initial motivation to pursue those connections. In Chapter 5 we consider an enumeration problem which is related to the Hurwitz problem with the additional constraint that the transposition factors be ordered, in a way made clear in Chapter 5. Another major result arising from the connection between combinatorics and the KP hierarchy comes from the topic of map enumeration, which we discuss now.

1.1.3 Map Asymptotics and Painlevé I

A *map* is a connected graph embedded in a compact connected surface in such a way that the regions delimited by the graph, called the *faces*, are homeomorphic to discs. We allow for loops and multiple edges and we say that a map is orientable or non-orientable if the underlying surface is. We may also refer to a rooted map, one in which a vertex, edge and face are distinguished, all of which are incident with one another. The distinguished vertex is called the *root vertex*, the distinguished edge is called the *root edge* and the distinguished face is called the *root face*.

Given a map $\mathfrak{m}$, the Euler characteristic may be computed via

$$\chi = V(\mathfrak{m}) - E(\mathfrak{m}) + F(\mathfrak{m}),$$

where V, E and F are the number of vertices, edges and faces respectively. Given the Euler characteristic χ we may compute the genus of the underlying surface, and thus the genus of the map $\mathfrak{m}$ by $\chi = 2 - 2g$.

The classical approach to enumerating maps is the one initiated by Tutte [108] and later used by Bender and Canfield [6] among other authers. The idea is to begin with the root vertex / root edge pair and then remove the root edge, taking care that there is some canonical rooting of the resulting object. One then analyzes the resulting object and constructs a recurrence for the generating series of interest. The downside to using this method is that the resulting recursive identities involve generating series with an arbitrarily large number of distinguished faces, causing the actual computation of the generating series in question to become very impractical. Despite these difficulties, Bender and Canfield were able to show that the generating series for maps (both the orientable series and the non-orientable series) could be written as rational functions in an auxiliary algebraic series. Later, using a similar approach, Gao [37, 39] was able to show that the generating series for a variety of classes of maps on all surfaces had a similar structure. One such class of maps is triangulations. For more details we refer to Bender and Canfield's paper [6] as well as Gao's papers [37, 39] and Section 4.4 in Chapter 4.

As a result of being able to determine the structure of various map generating series, Bender and Canfield were able to determine the asymptotic behaviour of many families of maps. It was determined that many families had asymptotic behaviour whose genus dependence was uniform. That is, if C denotes some class of rooted maps (say 2-connected, triangulations, etc.) then we may let $M_g(C,n)$ be the number of maps in class C with genus g and which have n edges. Bender and Canfield showed that for many classes of maps,

$$M_g(C,n) \sim \alpha t_g (\beta n)^{5(g-1)/2} \gamma^n,$$

if the maps are orientable and

$$M_g(C,n) \sim \alpha p_g (\beta n)^{5(g-1)/2} \gamma^n,$$

if the maps are non-orientable. In the formulas above, α, β and γ depend on the class of map under consideration and can be determined in a mostly straightforward way, however, the constants t_g and p_g (called the *map asymptotics constants*) do not depend on the map class and are rather difficult to compute.

Similar to our description above of the solution to Witten's conjecture by means of the Hurwitz problem, the question of enumerating orientable maps can also be re-written as a permutation factorization problem. Briefly, we may start with the class of bipartite rooted orientable maps with edges labelled from 1 to n. Then, by encoding the order in which the edge labels are encountered around the white and black vertices we may construct two permutations which encode the white and black vertices seperately. Then, by analyzing the combinatorial significance of multiplying the two vertex permutations, we see that the product of the white and black vertex permutations must encode the incidence of the edge labels as each face is traversed. For more details we refer to the proof of Proposition 3.2.1 in Chapter 3. As a result, it can be shown that the generating series for bipartite rooted orientable maps is a content-type series and thus satisfies the KP hierarchy.

Now, consider a rooted orientable bipartite map $\mathfrak{m}$. If we restrict ourselves to the case that each black face must have degree two, then we may contract each black face to an edge, resulting in a rooted orientable map which is not necessarily bipartite. In fact, this gives a bijection between rooted orientable bipartite maps in which every black face has degree two and the class of rooted orientable maps. As a result, this shows that the generating series for rooted orientable maps also satisfies the KP hierarchy. Another proof of this fact can be found in Goulden and Jackson [50] where the authors used the KP hierarchy to derive a quadratic recurrence for the number of rooted orientable triangulations.

After Goulden and Jackson showed that the number of rooted orientable triangulations satisfied a quadratic recurrence, Bender, Richmond and Gao [9] used this fact to analyze the map asymptotics constants t_g. In particular, since the asymptotic behaviour of the number of rooted orientable triangulations can be determined up to t_g, the quadratic recurrence determined by Goulden and Jackson results in a quadratic recursion for the map asymptotics constants t_g. Specifically, Bender, Richmond and Gao were able to show that if $u_g = -2^{g-2}\Gamma\left(\frac{5g-1}{2}\right) t_g$ is a rescaling of the map asymptotics constant and if

$$u(z) = z^{1/2} \sum_{g \geq 0} u_g z^{-5g/2},$$

is the generating series encoding the u_g then the series $u(z)$ satisfies the nonlinear differential equation given by

$$u^2 - \frac{1}{6}u'' = z,$$

which is called the Painléve I equation. This quadratic relationship allows for a much more efficient method of computing the t_g and also allowed Bender, Richmond and Gao to analyze the asymptotic behaviour of t_g as g becomes large.

In Chapter 3 we consider the problem of enumerating rooted bipartite quadrangulations, or equivalently, all rooted orientable maps. In particular, we are able to derive a quadratic recursion for the number of rooted orientable bipartite quadrangulations as well as provide a number of refinements.

Further, in Chapter 4 we apply similar methodology to the problem of enumerating rooted triangulations on all surfaces (not necessarily orientable). This requires the use of the BKP hierarchy rather than the KP hierarchy and the algebraic manipulations become much more tedious. As a result, we are able to prove a conjecture of Garoufalidis and Mariño concerning the map asymptotics constants p_g, providing a much more efficient method of computing p_g. In particular, we are able to show that if

$$v_g = 2^{\frac{g-3}{2}}\Gamma\left(\frac{5g-1}{4}\right)p_{\frac{g+1}{2}},$$

is a rescaling of the map asymptotics constants p_g then the generating series

$$v(z) = z^{1/4}\sum_{g\geq 0} v_g z^{-5g/4},$$

satisfies the differential equation

$$2v' - v^2 + 3u = 0,$$

where $u(z)$ is the series described earlier which encodes the map asymptotics constants t_g.

In the remainder of this chapter we record the requisite results which will be used in the chapters to come.

1.2 Partitions

In this section and the next two we briefly review some facts about partitions, the ring of symmetric functions and asymptotics. For the most part we follow Macdonald [78] for notation and results concerning partitions and symmetric functions. Full details and proofs can be found there or in Stanley [105].

A *partition* is any (finite or infinite) sequence

$$\lambda = (\lambda_1, \lambda_2, \cdots)$$

of non-negative integers in non-increasing order

$$\lambda_1 \geq \lambda_2 \geq \cdots$$

where at most finitely many λ_i are non-zero. The λ_i are called the *parts* of the partition λ and we use ϵ to denote the empty partition that has all parts equal to zero.

Given a partition λ, we say that the *length* of the partition, $\ell(\lambda)$, is equal to the number of non-zero parts and the *size* of the partition, $|\lambda|$, is the sum of the parts

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots.$$

If λ is a partition of size n then we write $\lambda \vdash n$. We will also use similar notation for sets. That is, if S is a finite set then we will write $|S|$ for the cardinality of the set.

Sometimes it is convenient to use the following notation for a partition λ:

$$\lambda = (1^{m_1} 2^{m_2} \cdots),$$

where $m_i = m_i(\lambda)$ is the number of times that i occurs as a part of λ and is called the *multiplicity* of i in λ. We may also write a partition in this form with the size of the parts in decreasing rather than increasing order, as in Example 1.2.1 below.

We sometimes represent a partition graphically as a connected collection of unit squares on the integer lattice. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ we say that its (Young) *diagram* is a collection of unit squares in the integer lattice, left aligned such that the first row consists of λ_1 squares, the second row consists of λ_2 squares, etc. We will often use λ to denote both the partition and its diagram with the usage clear from the context.

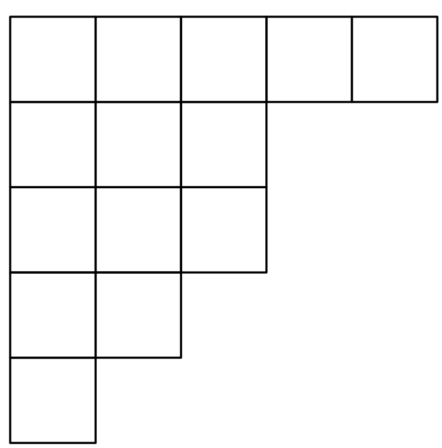


Figure 1.1: Young Diagram Example

Example 1.2.1. *Consider the partition* $\lambda = (5,3,3,2,1) = 123^25 = 53^221$. *This partition has the Young diagram in Figure 1.1 and is such that* $|\lambda| = 14$, *which is the number of squares in Figure 1.1, and* $\ell(\lambda) = 5$, *which is the number of rows in Figure 1.1.*

The unit squares which comprise the diagram of a partition are called the *cells* of the partition. We may also refer to a cell in a partition λ as a pair (i, j) where i and j are the row and column indices of the cell. Given a cell (i, j) in a partition λ we define the *content* of the cell to be the value $j - i$, as shown in Example 1.2.2.

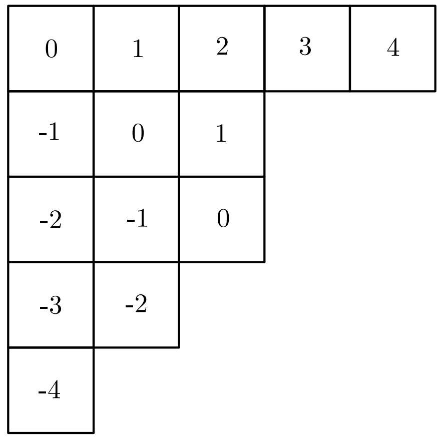


Figure 1.2: Young Diagram Content Example

Example 1.2.2. *In Figure 1.2 the diagram of the partition* $\lambda = (5, 3, 3, 2, 1)$ *is depicted with the integer in each cell corresponding to the content of that cell.*

For a partition λ we form its *conjugate*, denoted by λ', by taking the diagram of λ and reflecting in the main diagonal. The partition corresponding to this new diagram is the conjugate. From this description it is easy to see that the parts of λ' are the lengths of the columns of λ, or in other words

$$\lambda'_i = |\{j : \lambda_j \geq i\}|.$$

For any two partitions λ, μ we say that $\mu \subseteq \lambda$ if the diagram of μ fits inside the diagram of λ, or in other words $\lambda_i \geq \mu_i$ for all i.

Let λ, μ be partitions such that $\mu \subseteq \lambda$. If we draw the diagram for λ and then remove the cells corresponding to the diagram of μ, what we are left with is called the *skew diagram* λ/μ. If $\theta = \lambda/\mu$ is a skew diagram then $\theta_i = \lambda_i - \mu_i$ is the number of cells in the ith row of the skew diagram. We let $\theta' = \lambda'/\mu'$ be the conjugate of the skew diagram θ and as such $\theta'_i = \lambda'_i - \mu'_i$ is the number of cells in the ith row of the conjugate skew diagram θ' and hence the number of cells in the ith column of the skew diagram θ.

A skew diagram $\theta = \lambda/\mu$ is called a *horizontal m-strip* if $|\theta| = m$ and $\theta'_i \leq 1$ for each i. In other words, θ is a horizontal m-strip if it contains m cells and no two cells are vertically adjacent. Similarly we say that θ is a *vertical m-strip* if $|\theta| = m$ and $\theta_i \leq 1$ for each i.

Example 1.2.3. *Let* $\lambda = 53^2 21$ *and let* $\mu = 321$. *Then* $\mu \subseteq \lambda$ *and we can form the skew diagram* $\theta = \lambda/\mu$. *This can be seen in Figure 1.3 where the partition formed by*

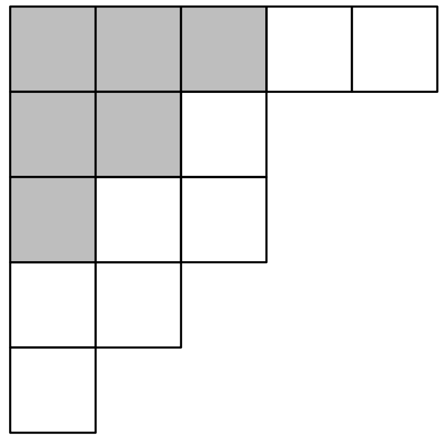


Figure 1.3: Skew Diagram Example

the shaded and unshaded cells is λ*, the partition formed by just the shaded squares is* μ *and the skew diagram* θ *is the set of cells that are unshaded.*

Given a partition λ and a partition μ we will write $\lambda \cup \mu$ for the partition whose parts comprise the parts of both λ and μ with multiplicity. For example, if $\lambda = (5,5,3,1)$ and $\mu = (6,3,3,2)$ then $\lambda \cup \mu = (6,5,5,3,3,3,2,1)$. In other words, for each $i \geq 1$, $m_i(\lambda \cup \mu) = m_i(\lambda) + m_i(\mu)$.

Throughout this thesis we will use alternate interpretations of the definition for a partition, sometimes viewing partitions as consisting of only the non-zero parts and other times viewing them as countable sequences with finitely many non-zero entries. Both descriptions are equivalent and which version we are using will be clear from the context. We will also let $\mathcal{P}$ denote the set of all partitions, including the empty partition ϵ.

1.3 Symmetric Functions

Let $\mathcal{S}_n$ be the group of permutations on n elements. The group $\mathcal{S}_n$ is also called the *symmetric group* on n elements. Thus, $\mathcal{S}_n$ is the set of bijections from $\{1, \cdots, n\}$ to itself with composition as the group operation. Let $\mathbb{Q}[x_1, \cdots, x_n]$ be the ring of polynomials in the n algebraically independent variables $x_1, \cdots, x_n$ with rational coefficients. There is a natural action of $\mathcal{S}_n$ on $\mathbb{Q}[x_1, \cdots, x_n]$ where $\sigma \in \mathcal{S}_n$ takes $p(x_1, \cdots, x_n) \in \mathbb{Q}[x_1, \cdots, x_n]$ to $\sigma p = p(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$. We say that a polynomial $p \in \mathbb{Q}[x_1, \cdots, x_n]$ is a *symmetric polynomial* if $\sigma p = p$ for all $\sigma \in \mathcal{S}_n$. We let Λ_n be the ring of all symmetric polynomials in n variables.

The ring Λ_n is a graded ring,

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k,$$

where Λ_n^k contains polynomials $p \in \Lambda_n$ which are homogeneous of total degree k, along with the zero polynomial.

For integers m, n such that $m \geq n$, the homomorphism

$$\mathbb{Q}[x_1, \cdots, x_m] \to \mathbb{Q}[x_1, \cdots, x_n]$$

formed by taking $x_{n+1}, \cdots, x_m$ to zero and all other x_i to themselves restricts to a homomorphism

$$\Lambda_m^k \to \Lambda_n^k$$

and so we may form the inverse limit

$$\Lambda^k = \varprojlim_n \Lambda_n^k.$$

We call the ring

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k$$

the ring of *symmetric functions*.

We will now describe four families of symmetric functions, each of which is a basis for Λ and will be important later on.

Every symmetric function is a formal infinite sum of monomials in a countable family of indeterminates. For ease of notation we will suppress these variables when possible. Also, we will occasionally use a bold variable name to refer to the vector consisting of all the variables. For example, we will write $\mathbf{x}$ to mean $\mathbf{x} = (x_1, x_2, \cdots)$ so that we can write $f(\mathbf{x}) \in \Lambda$ when we wish to make clear which family of indeterminates we are considering.

For each $r \geq 0$ the rth *elementary symmetric function* e_r is defined by the generating function

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t) \tag{1.2}$$

where t and the x_i are algebraically independent. The elementary symmetric functions have the form

$$e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}. \tag{1.3}$$

For each partition λ we define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots.$$

Also, the e_r are algebraically independent and they generate Λ.

For each $r \geq 0$ the rth *complete symmetric function* h_r is defined by the generating function

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} \frac{1}{1 - x_i t}, \tag{1.4}$$

so that in general,

$$h_n = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}. \tag{1.5}$$

Similar to the elementary symmetric functions, we define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots,$$

From the generating functions (1.2) and (1.4) we have

$$H(t)E(-t) = 1,$$

or

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0, \qquad \forall n \ge 1. \tag{1.6}$$

Since the e_r are algebraically independent we may define a homomorphism of graded rings

$$\omega : \Lambda \to \Lambda$$

by

$$\omega(e_r) = h_r.$$

From (1.6) we see that ω is an involution and hence an automorphism of Λ and so the h_r are also an algebraically independent set of generators for Λ. This involution is often called the *fundamental involution*.

For each $r \ge 1$ the rth *power sum symmetric* function is defined by

$$p_r = \sum_i x_i^r,$$

and $p_0 = 1$. We can construct the corresponding generating function

$$\begin{aligned} P(t) &= \sum_{r \ge 1} p_r t^{r-1} = \sum_{i \ge 1} \sum_{r \ge 1} x_i^r t^{r-1}, \\ &= \sum_{i \ge 1} \frac{x_i}{1 - x_i t} = \sum_{i \ge 1} \frac{\partial}{\partial t} \log \frac{1}{1 - x_i t}. \end{aligned}$$

From this we get

$$\begin{aligned} P(t) &= \frac{\partial}{\partial t} \log \prod_{i \ge 1} \frac{1}{1 - x_i t}, \\ &= \frac{\partial}{\partial t} \log H(t) = \frac{H'(t)}{H(t)}, \end{aligned} \tag{1.7}$$

and likewise

$$P(-t) = \frac{\partial}{\partial t} \log E(t) = \frac{E'(t)}{E(t)}. \tag{1.8}$$

From (1.7) and (1.8) we get

$$nh_n = \sum_{r=1}^{n} p_r h_{n-r}$$
$$ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r}$$

and so we see that the p_r are algebraically independent and generate Λ. Furthermore, for any partition λ we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

and the set of all such p_λ forms a linear basis for Λ. Note that since the p_r are algebraically independent we may view them as being indeterminates themselves. This will be of use throughout this thesis and when we think of the power sums as being algebraically independent indeterminates we will write them as $\mathbf{p} = (p_1, p_2, \ldots)$.

Since ω changes e_r to h_r it follows from (1.7) and (1.8) that

$$\omega(p_n) = (-1)^{n-1} p_n,$$

and hence for any partition λ,

$$\omega(p_\lambda) = \epsilon_\lambda p_\lambda,$$

where

$$\epsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)}.$$

In order to describe the elementary and complete symmetric functions in terms of the power sum symmetric functions we let

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!,$$

where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i. Then

$$H(t) = \exp\left(\sum_{r \geq 1} \frac{p_r t^r}{r} \right), \text{ and}$$
$$E(t) = \exp\left(-\sum_{r \geq 1} \frac{(-1)^r p_r t^r}{r} \right) = \frac{1}{H(-t)},$$

or equivalently,

$$h_n = \sum_{|\lambda| = n} z_\lambda^{-1} p_\lambda, \text{ and} \tag{1.9}$$
$$e_n = \sum_{|\lambda| = n} \epsilon_\lambda z_\lambda^{-1} p_\lambda.$$

The last family of symmetric functions that we will make use of is the Schur functions. For any partition λ the *Schur function* s_λ will be defined as

$$s_\lambda = \det_{1\le i,j\le \ell(\lambda)} \left(h_{\lambda_i - i + j}\right), \qquad (1.10)$$
$$= \det_{1\le i,j\le \ell(\lambda')} \left(e_{\lambda'_i - i + j}\right),$$

where the second equality is a relatively routine calculation which can be found in Macdonald [78]. Here we use the convention that $h_i = 0$ (or $e_i = 0$) if $i < 0$. In addition to defining Schur functions indexed by a partition λ, we may also define the *skew Schur functions* which are indexed by a skew partition λ/μ. Given partitions $\lambda, \mu \in \mathcal{P}$ with $\mu \subseteq \lambda$ we define

$$s_{\lambda/\mu} = \det_{1\le i,j\le \ell(\lambda)} \left(h_{\lambda_i - \mu_j - i + j}\right).$$

In the case that the underlying variables $\mathbf{x} = (x_1, x_2, \ldots)$ form a finite set (i.e., all but a finite number of them have been set to zero) we can write $s_\lambda(\mathbf{x})$ as a ratio of determinants in the variables $\mathbf{x}$. Suppose that $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Given a partition λ and a partition μ we write $\lambda + \mu$ to mean the partition whose parts are given by $(\lambda + \mu)_i = \lambda_i + \mu_i$. If we let $\delta = (n-1, n-2, \ldots, 0)$ and define for $\lambda \in \mathcal{P}$,

$$a_\lambda(\mathbf{x}) = \det_{1\le i,j\le n} (x_i^{\lambda_j}),$$

then

$$s_\lambda(\mathbf{x}) = \frac{a_{\lambda+\delta}(\mathbf{x})}{a_\delta(\mathbf{x})},$$

where $a_\delta(\mathbf{x})$ is the Vandermonde determinant.

From (1.10) we see that

$$\omega(s_\lambda) = s_{\lambda'}$$

and also that

$$s_{(n)} = h_n, \qquad s_{(1^n)} = e_n.$$

It is also the case that the Schur functions form a linear basis for Λ. Note that since the fundamental involution acts on Schur functions by taking s_λ to $s_{\lambda'}$ we sometimes use ω to mean the involution on the set of partitions of n that takes λ to λ'.

Also, since (1.9) allows us to write each h_n as a polynomial in the family $\mathbf{p} = (p_1, p_2, \ldots)$, (1.10) tells us that the Schur functions can also be viewed as polynomials in $\mathbf{p}$. Throughout this thesis we will need to view Schur functions as both polynomials in the 'indeterminates' $\mathbf{p}$, and as functions in the underlying indeterminates $\mathbf{x}$. To make this distinction clear we will refer to $s_\lambda(\mathbf{p})$ as a Schur polynomial and $s_\lambda(\mathbf{x})$ as a Schur function, even when the underlying set $\mathbf{x}$ may be finite.

We may define a bilinear form $\langle \cdot, \cdot \rangle$ on Λ (called the *Hall inner product*) by

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}, \tag{1.11}$$

so that the power sum symmetric functions form an orthogonal basis for Λ. With respect to this inner product, we have

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu},$$

so that the Schur functions are orthonormal. This inner product is symmetric and positive definite and since $\omega(p_\lambda) = \pm p_\lambda$ we have

$$\langle \omega(p_\lambda), \omega(p_\mu) \rangle = \langle p_\lambda, p_\mu \rangle,$$

so that ω is an isometry.

For any symmetric function $f \in \Lambda$ we let $f^\perp$ be the adjoint of multiplication by f as a linear operator on Λ (i.e., so that $\langle f^\perp g, h \rangle = \langle g, fh \rangle$ for $f, g, h \in \Lambda$). In the case of the power sum symmetric function p_n we see that

$$\begin{aligned} \langle p_n^\perp p_\lambda, p_\mu \rangle &= \langle p_\lambda, p_n p_\mu \rangle \\ &= \begin{cases} z_\lambda & \text{if } \lambda = \mu \cup \{n\} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence $p_n^\perp p_\lambda = z_\lambda z_\mu^{-1} p_\mu$ if n is a part of λ and μ is obtained from λ by removing a part of size n. Since $z_\lambda z_\mu^{-1} = n m_n(\lambda)$ where $m_n(\lambda)$ is the number of parts of size n in λ we see that

$$p_n^\perp = n \frac{\partial}{\partial p_n}.$$

Since each $f \in \Lambda$ can be written as a polynomial in terms of power sums, $f = \phi(p_1, p_2, \cdots)$, we have

$$f^\perp = \phi\left(\frac{\partial}{\partial p_1}, \frac{2\partial}{\partial p_2}, \frac{3\partial}{\partial p_3}, \cdots \right).$$

We now turn to the problem of writing the Schur and power sum symmetric functions in terms of one another. In order to do this we will require a few basic facts about the symmetric group. Complete details can be found in [78] or [105]. A permutation $\sigma \in \mathcal{S}_n$ has *cycle type* λ, where λ is a partition of n, if σ is comprised of distinct cycles of length $\lambda_1, \lambda_2, \cdots$. We denote this by cyc($\sigma$)= λ. The *conjugacy classes* in $\mathcal{S}_n$ are indexed by partitions of n so that the conjugacy class $\mathcal{C}_\lambda$, where λ is a partition of n, corresponds to the set of permutations $\sigma \in \mathcal{S}_n$ with $\mathrm{cyc}(\sigma) = \lambda$.

It is well known that the *irreducible characters* of $\mathcal{S}_n$ are indexed by partitions of n. If λ and ρ are partitions of n, then we use χ_ρ^λ to denote the evaluation of the irreducible character indexed by λ at a permutation with cycle type ρ. It is also well known that

$$\chi_\rho^\lambda = \langle s_\lambda, p_\rho \rangle.$$

This allows us to write the Schur functions and the power sum symmetric functions in terms of one another:

$$s_\lambda = \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho,$$
$$p_\rho = \sum_\lambda \chi_\rho^\lambda s_\lambda.$$

Example 1.3.1. *We list a number of examples of Schur functions written in terms of power sum symmetric functions which will be useful later.*

$$s_\epsilon = 1, \qquad s_{(1)} = p_1, \qquad s_{(1^2)} = \frac{1}{2}p_1^2 - \frac{1}{2}p_2, \qquad s_{(2)} = \frac{1}{2}p_1^2 + \frac{1}{2}p_2,$$

$$s_{(1^3)} = \frac{1}{6}p_1^3 - \frac{1}{2}p_1p_2 + \frac{1}{3}p_3, \qquad s_{(2,1)} = \frac{1}{3}p_1^3 - \frac{1}{3}p_3, \qquad s_{(3)} = \frac{1}{6}p_1^3 + \frac{1}{2}p_1p_2 + \frac{1}{3}p_3,$$
$$s_{(1^4)} = \frac{1}{24}p_1^4 - \frac{1}{4}p_1^2p_2 + \frac{1}{3}p_1p_3 + \frac{1}{8}p_2^2 - \frac{1}{4}p_4, \qquad s_{(21^2)} = \frac{1}{8}p_1^4 - \frac{1}{4}p_1^2p_2 - \frac{1}{8}p_2^2 + \frac{1}{4}p_4,$$
$$s_{(2^2)} = \frac{1}{12}p_1^4 + \frac{1}{4}p_2^2 - \frac{1}{3}p_1p_3, \qquad s_{(31)} = \frac{1}{8}p_1^4 + \frac{1}{4}p_1^2p_2 - \frac{1}{8}p_2^2 - \frac{1}{4}p_4,$$
$$s_{(4)} = \frac{1}{24}p_1^4 + \frac{1}{4}p_1^2p_2 + \frac{1}{3}p_1p_3 + \frac{1}{8}p_2^2 + \frac{1}{4}p_4.$$

The final fact that we will need concerning symmetric functions is *Pieri's formula*, which says that

$$h_r s_\mu = \sum_\lambda s_\lambda, \tag{1.12}$$

where the sum is over all partitions λ such that λ/μ is a horizontal r-strip. Using ω we get

$$e_r s_\mu = \sum_\lambda s_\lambda \tag{1.13}$$

where the sum is over all partitions λ such that λ/μ is a vertical r-strip. By taking adjoints we see that

$$h_r^\perp s_\mu = \sum_\lambda s_\lambda \tag{1.14}$$

where the sum is over partitions λ such that μ/λ is a horizontal r-strip and

$$e_r^\perp s_\mu = \sum_\lambda s_\lambda \tag{1.15}$$

where the sum is over partitions λ such that μ/λ is a vertical r-strip.

In addition to the symmetric function theory outlined above, we will have need of the following notation concerning the various rings which will appear in this thesis and some combinatorial operations on them. Throughout this thesis we will use R to denote a generic commutative ring with unity. Typically we will think of this ring as being the field of complex numbers or rational numbers but sometimes it can be as general as a ring of formal power series. Given a ring R we will write

$R[x,y,z,\ldots]$ to denote the ring of polynomials in the indeterminates $x,y,z,\ldots$ with coefficients in R and we will write $R[[x,y,z,\ldots]]$ to denote the ring of formal power series in $x,y,z,\ldots$ with coefficients in R. We will denote by $[\cdot]$ the coefficient extraction operator. That is, if $G(x) \in R[[x]]$ has the form

$$G(x) = \sum_{i \geq 0} a_i x^i,$$

then for any $n \geq 0$,

$$[x^n]G(x) = a_n.$$

Lastly, given a family of indeterminates $\mathbf{p} = (p_1, p_2, \ldots)$ and a family of indeterminates $\mathbf{q} = (q_1, q_2, \ldots)$ we write $\mathbf{p} + \mathbf{q}$ to mean the family of indeterminates $(p_1 + q_1, p_2 + q_2, \ldots)$. For example, if $G(\mathbf{p}) \in R[[\mathbf{p}]]$ then $G(\mathbf{p}+\mathbf{q}) = G(p_1+q_1, p_2+q_2, \ldots)$.

1.4 Asymptotics

Following Bender [4] and Bender and Canfield [6] we will use the following asymptotic notation. Suppose $f(n)$ and $g(n)$ are functions of a non-negative integer n. We write

$$f(n) = O(g(n)),$$

if $f(n)/g(n)$ is bounded as $n \to \infty$,

$$f(n) = o(g(n)),$$

if $f(n)/g(n) \to 0$ as $n \to \infty$, and

$$f(n) \sim g(n),$$

if $f(n)/g(n) \to 1$ as $n \to \infty$. Thus, $f(n) \sim g(n)$ is the same as saying that $f(n) = g(n)(1 + o(1))$.

Further, suppose that $h_1(x)$ and $h_2(x)$ are functions of a variable x. We write

$$h_1(x) \approx h_2(x) \qquad \text{as } x \to x_0$$

to mean that

1. h_1 and h_2 are each analytic in $|x| < |x_0|$, and $x = x_0$ is the only singularity of each on the circle $|x| = |x_0|$;

2. h_1 and h_2 may each be written in the form

$$\sum_{i=1}^{d} (1 - x/x_0)^{e_i} g_i(x) + p(x),$$

where $g_1, \ldots, g_d$ are all analytic near x_0, $e_1 < e_2 < \ldots < e_d$ are rational numbers, e_1 is not in the set $\{0, 1, 2, \ldots,\}$, $g_1(x_0) \neq 0$, and $p(x)$ is a polynomial;

3. h_1 and h_2 have the same e_1 in the above representation, and also the same $g_1(x_0)$.

In addition, we will make use of the following variant of *Darboux's theorem* which can be found in Bender's paper [4]. Suppose f is a function with a singularity at α. This singularity is algebraic if $f(z)$ can be written as a function analytic near α plus a finite sum of terms of the form

$$(1 - z/\alpha)^{-\omega} g(z), \tag{1.16}$$

where g is a function which is analytic and non-zero near α and ω is a complex number not equal to $0, -1, -2, \ldots$. The weight of (1.16) is the real part of ω.

Theorem 1.4.1. *Suppose $A(z) = \sum_{n\geq 0} a_n z^n$ and is analytic near 0 and has only algebraic singularities on its circle of convergence. Let w be the maximum of the weights at these singularities on its circle of convergence. Denote by α_k, ω_k and g_k the values of α, ω and g for those terms of the form* (1.16) *of weight w. Then*

$$a_n - \frac{1}{n} \sum_k \frac{g_k(\alpha_k) n^{\omega_k}}{\Gamma(\omega_k)\alpha_k^n} = o(r^{-n} n^{w-1}),$$

where $r = |\alpha_k|$, the radius of convergence of $A(z)$, and $\Gamma(s)$ is the gamma function.

As a simple corollary, given two functions $h_1(x)$ and $h_2(x)$ of a variable x, if $h_1(x) \approx h_2(x)$ as $x \to x_0$ then

$$[x^n] h_1(x) \sim [x^n] h_2(x) \sim \frac{g_1(x_0) x_0^{-n}}{n^{1+e_1}\Gamma(-e_1)},$$

as $n \to \infty$.

1.5 Thesis Outline

The remainder of this thesis is organized as follows. In Chapter 2 we discuss the KP and BKP hierarchies. We begin by describing each of the hierarchies as an identity involving the Bernstein operators. Using the action of the Bernstein operators on Schur polynomials we are able to describe each of the KP and BKP hierarchies in a number of equivalent forms including as a collection of identities satisfied by the Schur coefficients and as a family of partial differential equations. We also give an alternate form of the hierarchies in which the partial differential equations can more easily be extracted. We continue by giving an alternate construction of each of the KP and BKP hierarchies by taking as a starting point series which have Schur polynomial coefficients of a particular form. In the case of the KP hierarchy this corresponds to coefficients which are determinant minors and in the BKP case they correspond to Pfaffian minors. In each case the coefficients descriptions allow us to give examples of classes of solutions which arise in enumerative problems.

In the remaining chapters we consider various applications of the preceding results. We begin in Chapter 3 by considering the problem of enumerating rooted bipartite quadrangulations on orientable surfaces (or equivalently all maps). Using the KP equation along with some combinatorial manipulation we are able to show that the generating series satisfies a quadratic differential equation, allowing us to give an efficient method for computing the number of such objects.

In Chapter 4 we consider the problem of enumerating rooted triangulations on all surfaces. This is the first example of an enumerative problem whose generating series is a solution to the BKP hierarchy. Using the BKP equation and some combinatorial manipulation, we are able to show that the resulting series satisfies a cubic differential equation. Using this equation we are able to make a connection to the non-orientable map asymptotics constants and, in particular, we are able to prove a conjecture of Garoufalidis and Mariño.

Lastly, in Chapter 5 we consider the problem of enumerating monotone Hurwitz numbers. More specifically, we consider the case of enumerating simple monotone Hurwitz numbers, or monotone factorizations of the identity. We are able to show that the generating series for such factorizations satisfies a quadratic differential equation and the resulting asymptotic behaviour is determined by the map asymptotics constant.

Chapter 2

Generating Series Solutions to Integrable Hierarchies

The primary tools we will be developing and using in this thesis are, most generally, integrable hierarchies. In particular we will examine the KP and BKP hierarchies. There are two ways to develop these hierarchies which we will describe here. The first is based on the Bernstein operators, a family of differential operators which act on Schur polynomials in a nice way. The second is in terms of generating series whose translations have coefficients which satisfy certain algebraic identities.

We begin by describing a simple property of generating series which will be used multiple times in the remainder of this chapter.

Lemma 2.0.1. *Let $G(\mathbf{p})$ be a generating series in $R[[\mathbf{p}]]$ and suppose that*

$$G(\mathbf{p}+\mathbf{q}) = \sum_{\lambda\in\mathcal{P}} a_\lambda s_\lambda(\mathbf{p}),$$

where a_λ does not depend on $\mathbf{p}$. Then

$$a_\lambda = s_\lambda^\perp(\mathbf{q})G(\mathbf{q}).$$

Proof. We compute,

$$\begin{aligned} a_\lambda &= s_\lambda^\perp(\mathbf{p})G(\mathbf{p}+\mathbf{q})\big|_{\mathbf{p}=0}, \\ &= s_\lambda^\perp(\mathbf{q})G(\mathbf{p}+\mathbf{q})\big|_{\mathbf{p}=0}, \\ &= s_\lambda^\perp(\mathbf{q})G(\mathbf{q}). \end{aligned}$$

□

Lemma 2.0.1 allows us to write algebraic relations among the Schur coefficients of the translation of a generating series (the a_λ's) as differential equations satisfied by the original series. We shall see in Section 2.4 that if the algebraic identities are the Plücker relations satisfied by determinantal minors, the corresponding differential equations form the KP hierarchy. Analogously, in Section 2.5 we shall see that if the algebraic identities are the Pfaffian minor analogs to the Plücker relations then the corresponding differential equations form the BKP hierarchy.

2.1 Operations on Partitions

In order to describe the action of the Bernstein operators on Schur polynomials we must first describe some operations on partitions and the properties they enjoy.

Given a partition λ we may view its diagram as continuing on to the right indefinitely and downward indefinitely. By doing so we may describe a partition as a bi-infinite sequence of up (**U**) and right (**R**) steps which is infinitely **U** to the left and infinitely **R** to the right. For example, the partition $\lambda = (5,3,2,2,2)$ and its code are displayed in Figure 2.1. We call this sequence the *code* of λ.

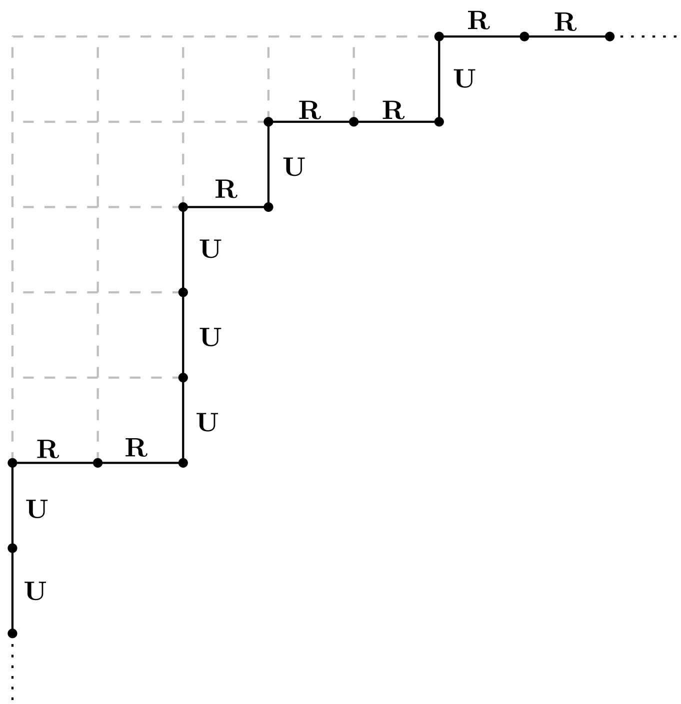


Figure 2.1: The Code of a Partition

The partition λ can be determined by its code by noting that each **U** beyond the first **R** corresponds to a part in λ and that the size of a part is equal to the number of **R**s preceeding the corresponding **U**.

Given the code of λ, we define the partition $\lambda \uparrow i$ for $i \geq 1$ to be the partition whose code is given by the code of λ with the ith **R** (from the left) changed to a **U**. For example, see Figure 2.2.

Since in the code of a partition λ every **U** beyond the first **R** corresponds to a part of λ, the operation $\lambda \uparrow i$ results in adding a part to λ. In addition, since this new **U** has $(i-1)$ **R**s preceeding it, the new part must have size $i-1$. Similarly, for each **U** beyond the new one, the number of **R**s to the left has decreased by one. Thus,

$$\lambda \uparrow i = (\lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_j - 1, i-1, \lambda_{j+1}, \cdots), \tag{2.1}$$

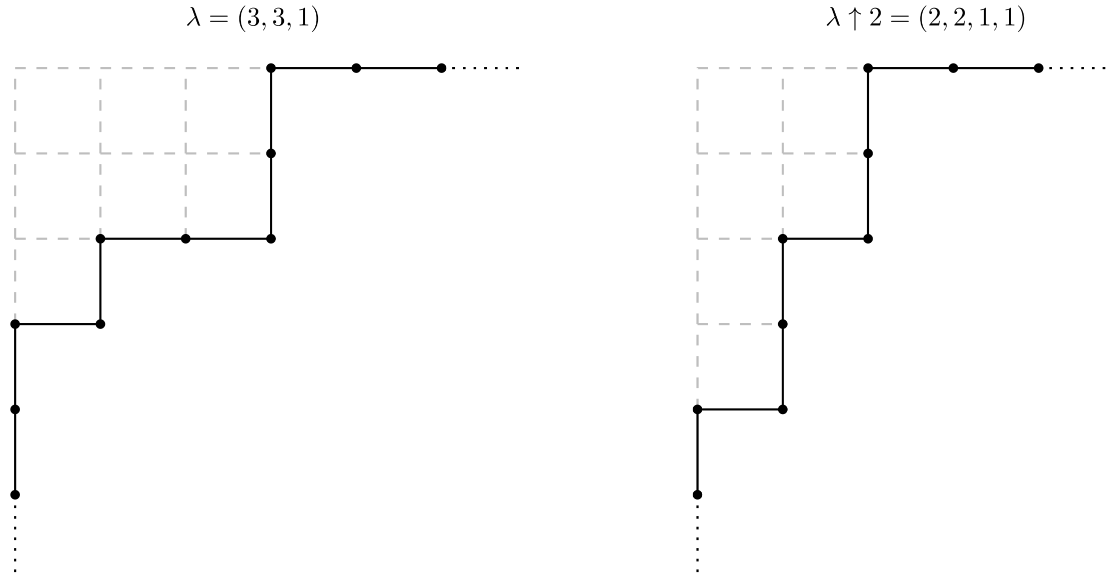


Figure 2.2: The $\uparrow$ Operator

where j is uniquely chosen such that $\lambda_j \geq i > \lambda_{j+1}$ (with the convention that $\lambda_0 = \infty$). Now, if we define $u_i(\lambda)$ to be the number of **U**s that follow the ith **R** from the left in the code of λ then $u_i(\lambda) = j$.

It is straightforward to compute

$$|\lambda \uparrow i| = |\lambda| - j + i - 1,$$

from (2.1) and so since $u_i(\lambda) = j$ we have that

$$u_i(\lambda) = |\lambda| - |\lambda \uparrow i| + i - 1.$$

Note that, by definition, $u_i(\lambda)$ weakly decreases as i increases, so we have

$$|\lambda| - \ell(\lambda) = |\lambda \uparrow 1| < |\lambda \uparrow 2| < \cdots,$$

and,

$$|\lambda \uparrow i| = |\lambda| + i - 1, \qquad i \geq \lambda_1.$$

Analogous to the description of the $\uparrow$ operation on partitions, we may define $\lambda \downarrow j$ for $j \geq 1$ as the partition which results from switching the jth **U** (from the right) in the code of λ to an **R**. For example, see Figure 2.3.

Similar to the previous case, it is straightforward to see that

$$\lambda \downarrow j = (\lambda_1 + 1, \lambda_2 + 1, \cdots, \lambda_{j-1} + 1, \lambda_{j+1}, \cdots). \tag{2.2}$$

From (2.2) we may compute

$$|\lambda \downarrow j| = |\lambda| + j - \lambda_j - 1,$$

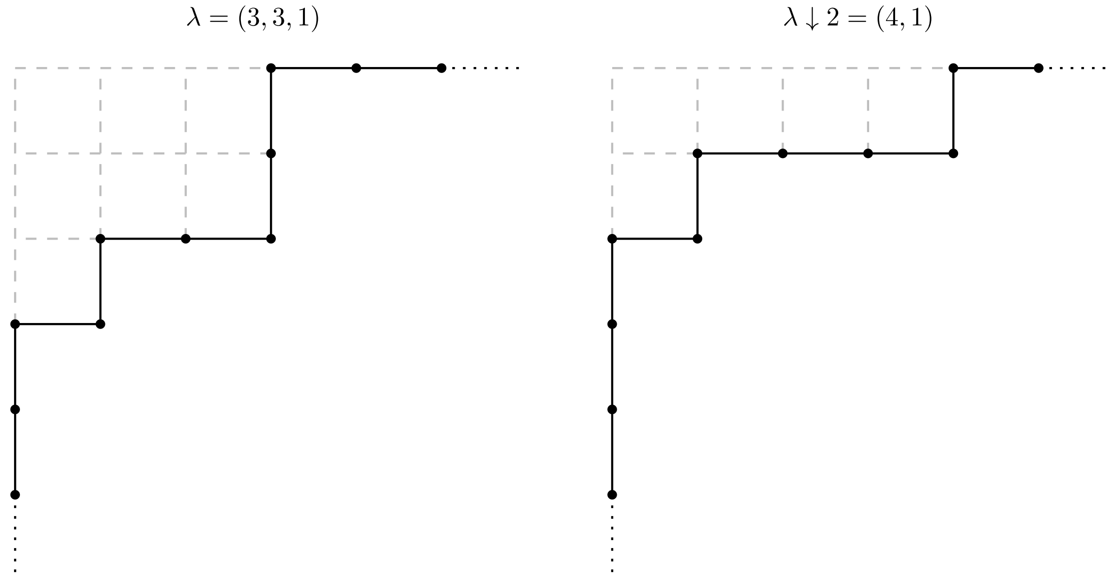


Figure 2.3: The $\downarrow$ Operator

as well as

$$|\lambda| - \lambda_1 = |\lambda \downarrow 1| < |\lambda \downarrow 2| < \cdots.$$

From the description of $\uparrow$ and $\downarrow$ acting on partitions, it follows that

$$(\lambda \uparrow i) \downarrow (u_i(\lambda) + 1) = \lambda,$$

and that

$$(\lambda \downarrow j) \uparrow (\lambda_j + 1) = \lambda.$$

In some sense this implies that $\uparrow$ and $\downarrow$ are inverse operations. More than that, since taking the transpose of a partition corresponds to reversing the order of the code and then exchanging **U**s for **R**s and **R**s for **U**s, we see that $\uparrow$ and $\downarrow$ are in fact dual operations, i.e.,

$$(\lambda^t) \uparrow i = (\lambda \downarrow i)^t.$$

2.2 Bernstein Operators

We define the *Bernstein operator* as

$$B(\mathbf{p}; t) = \exp\left(\sum_{k\geq 1} \frac{t^k}{k} p_k\right) \exp\left(-\sum_{k\geq 1} \frac{t^{-k}}{k} p_k^\perp\right),$$

and the *adjoint Bernstein operator* as

$$B^\perp(\mathbf{p}; t) = \exp\left(-\sum_{k\geq 1} \frac{t^k}{k} p_k\right) \exp\left(\sum_{k\geq 1} \frac{t^{-k}}{k} p_k^\perp\right).$$

These operators were introduced by Bernstein in the context of modular representation theory [111, p.69]. See also [78, p.95] for a combinatorial account. In [15] we showed that B and $B^{\perp}$ act nicely on Schur polynomials. In particular, though the Bernstein operators are bi-infinite series involving arbitrarily large positive and negative powers of t, the action of the Bernstein operators on a Schur function always results in a Laurent series.

Theorem 2.2.1. *Suppose that for all $\lambda \in \mathcal{P}$, $a_\lambda \in R$. Then*

$$B(\mathbf{p};t)\sum_{\lambda\in\mathcal{P}} a_\lambda s_\lambda(\mathbf{p}) = \sum_{\beta\in\mathcal{P}} s_\beta(\mathbf{p})\sum_{k\geq 1}(-1)^{k-1}t^{|\beta|-|\beta\downarrow k|}a_{\beta\downarrow k},$$

and

$$B^{\perp}(\mathbf{p};t)\sum_{\lambda\in\mathcal{P}} a_\lambda s_\lambda(\mathbf{p}) = \sum_{\alpha\in\mathcal{P}} s_\alpha(\mathbf{p})\sum_{m\geq 1}(-1)^{|\alpha|-|\alpha\uparrow m|+m-1}t^{|\alpha|-|\alpha\uparrow m|}a_{\alpha\uparrow m}.$$

□

The Bernstein operators are of interest to us because many integrable hierarchies can be defined in terms of them. In particular, both the KP and BKP hierarchies can be described in terms of $B(\mathbf{p};t)$ and $B^{\perp}(\mathbf{p};t)$. For other applications of the Bernstein operators see [55, 56, 64].

In addition to showing that the Bernstein operators act nicely on Schur polynomials, we also showed that the commutator of $B(\mathbf{p};t)$ and a certain translation operator was simple. Specifically, define

$$\Theta(\mathbf{p},\mathbf{q}) = \exp\left(\sum_{k\geq 1} q_k \frac{\partial}{\partial p_k}\right),$$

and

$$\Gamma(\mathbf{q};t) = \exp\left(\sum_{k\geq 1}\frac{t^i}{i}q_i\right).$$

Using multivariate Taylor series, we see that if $f(\mathbf{p})$ is a formal power series then

$$\Theta(\mathbf{p},\mathbf{q})f(\mathbf{p}) = f(\mathbf{p}+\mathbf{q}).$$

It is not difficult to show that the following relations hold between $B(\mathbf{p};t)$ and $\Theta(\mathbf{p},\mathbf{q})$. Note that this result essentially arises in the proof of Theorem 5.3 in [15].

Proposition 2.2.2. *We have*

$$B(\mathbf{p};t)\Theta(\mathbf{p},\mathbf{q}) = \Gamma(\mathbf{q};t)^{-1}\Theta(\mathbf{p},\mathbf{q})B(\mathbf{p};t),$$

and

$$B^{\perp}(\mathbf{p};t)\Theta(\mathbf{p},\mathbf{q}) = \Gamma(\mathbf{q};t)\Theta(\mathbf{p},\mathbf{q})B^{\perp}(\mathbf{p};t).$$

2.2.1 KP Hierarchy

To begin we will describe the KP hierarchy. The results in this section can be found in [14]. Many of these results were known to physicists before this, though finding a clear exposition has proven difficult.

Definition 2.2.3. *A generating series $G(\mathbf{p}) \in R[[\mathbf{p}]]$ is said to be a solution to the KP hierarchy if and only if*

$$[t^{-1}]\,(B(\mathbf{p};t)G(\mathbf{p}))\,(B^{\perp}(\mathbf{q};t)G(\mathbf{q})) = 0.$$

The differential equations which $G(\mathbf{p})$ satisfies as a result of this identity are collectively called the KP Hierarchy.

Using Theorem 2.2.1 (and Proposition 2.2.2) we may write a series of equivalent descriptions of the KP hierarchy.

Theorem 2.2.4. *Suppose that for all $\lambda \in \mathcal{P}$, $a_\lambda \in R$ and that*

$$G(\mathbf{p}) = \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda(\mathbf{p}).$$

The following are equivalent:

(i) *The formal power series $G(\mathbf{p})$ is a solution to the KP hierarchy.*

(ii) *The formal power series $G(\mathbf{p}+\mathbf{q})$ is a solution to the KP hierarchy in the variables $\mathbf{p}$.*

(iii) *For all $\alpha, \beta \in \mathcal{P}$,*

$$\sum_{i,j} (-1)^{|\alpha|-|\alpha\uparrow i|+i+j} a_{\alpha\uparrow i} a_{\beta\downarrow j} = 0,$$

where the sum is over all integers $i, j \geq 1$ such that $|\alpha \uparrow i| + |\beta \downarrow j| = |\alpha| + |\beta| + 1$.

(iv) *For all $\alpha, \beta \in \mathcal{P}$,*

$$\sum_{i,j} (-1)^{|\alpha|-|\alpha\uparrow i|+i+j} (s^{\perp}_{\alpha\uparrow i}(\mathbf{p})G(\mathbf{p}))(s^{\perp}_{\beta\downarrow j}(\mathbf{p})G(\mathbf{p})) = 0,$$

where the sum is over all integers $i, j \geq 1$ such that $|\alpha \uparrow i| + |\beta \downarrow j| = |\alpha| + |\beta| + 1$.

Proof. The fact that (ii) implies (i) follows immediately by setting $q_i = 0$ for all $i \geq 1$. The fact that (i) implies (ii) follows from the definition of the KP hierarchy and Proposition 2.2.2. That (i) and (iii) are equivalent follows by taking coefficients and using Theorem 2.2.1. Lastly, that (ii) and (iv) are equivalent follows by coefficient extraction, Theorem 2.2.1 and Lemma 2.0.1. $\square$

Example 2.2.5. *Suppose* $\alpha = (1)$. *Then* $\alpha \uparrow 1 = \epsilon, \alpha \uparrow 2 = (1,1), \alpha \uparrow 3 = (2,1)$ *and* $|\alpha \uparrow i| > 3$ *for* $i > 3$. *If* $\beta = (1)$ *then* $\beta \downarrow 1 = \epsilon, \beta \downarrow 2 = (2), \beta \downarrow 3 = (2,1)$ *and* $|\beta \downarrow i| > 3$ *for* $i > 3$. *As a result, the quadratic equation appearing in part (iii) of Theorem 2.2.4 is given by*

$$\sum_{\substack{i,j\geq 1\\ |\alpha\uparrow i|+|\beta\downarrow j|=3}} (-1)^{|\alpha|-|\alpha\uparrow i|+i+j} a_{\alpha\uparrow i} a_{\beta\downarrow j} = -a_\epsilon a_{(2,1)} + a_{(2,1)} a_\epsilon = 0,$$

which gives a redundant identity. However, if we choose $\alpha = \epsilon$ *then* $\alpha \uparrow 1 = \epsilon, \alpha \uparrow 2 = (1), \alpha \uparrow 3 = (2), \alpha \uparrow 4 = (3)$ *and* $|\alpha \uparrow i| > 3$ *for* $i > 4$. *Also, if* $\beta = (1,1,1)$ *we get* $\beta \downarrow 1 = (1,1), \beta \downarrow 2 = (2,1), \beta \downarrow 3 = (2,2)$ *and* $|\beta \downarrow i| > 4$ *for* $i > 3$. *Thus, the algebraic identity in part (iii) of Theorem 2.2.4 becomes*

$$a_\epsilon a_{(2,2)} - a_{(1)} a_{(2,1)} + a_{(2)} a_{(1,1)} = 0.$$

Additionally, if we take $\alpha = (2)$ *we get* $\alpha \uparrow 1 = (1), \alpha \uparrow 2 = (1,1), \alpha \uparrow 3 = (2,2)$ *and* $|\alpha \uparrow i| > 4$ *for* $i > 3$. *If* $\beta = (1)$ *we get* $\beta \downarrow 1 = \epsilon, \beta \downarrow 2 = (2), \beta \downarrow 3 = (2,1)$ *and* $|\beta \downarrow i| > 3$ *for* $i > 3$. *The corresponding algebraic equation then becomes*

$$-a_{(1)} a_{(2,1)} + a_{(1,1)} a_{(2)} + a_{(2,2)} a_\epsilon = 0. \tag{2.3}$$

These examples show that not only does the family of identities given in part (iii) of Theorem 2.2.4 contain redundant equations but also repetition.

Continuing on with the example of $\alpha = (2)$ *and* $\beta = (1)$, *recall that*

$$s_\epsilon(\mathbf{p}) = 1, \qquad s_{(1)}(\mathbf{p}) = p_1, \qquad s_{(2)}(\mathbf{p}) = \frac{1}{2}(p_1^2 + p_2), \qquad s_{(1,1)}(\mathbf{p}) = \frac{1}{2}(p_1^2 - p_2),$$

$$s_{(2,1)}(\mathbf{p}) = \frac{1}{3}(p_1^3 - p_3), \qquad s_{(2,2)}(\mathbf{p}) = \frac{1}{12}(p_1^4 - 4p_1p_3 + 3p_2^2).$$

In this case, the differential equation described in part (iv) of Theorem 2.2.4 becomes (after rearranging),

$$\frac{1}{12}G(\mathbf{p})\left(\frac{\partial^4 G(\mathbf{p})}{\partial p_1^4} - 12\frac{\partial^2 G(\mathbf{p})}{\partial p_1 \partial p_3} + 12\frac{\partial^2 G(\mathbf{p})}{\partial^2 p_2}\right) - \frac{1}{3}\frac{\partial G(\mathbf{p})}{\partial p_1}\left(\frac{\partial^3 G(\mathbf{p})}{\partial p_1^3} - 3\frac{\partial G(\mathbf{p})}{\partial p_3}\right)$$
$$+ \frac{1}{4}\left(\frac{\partial^2 G(\mathbf{p})}{\partial p_1^2} + 2\frac{\partial G(\mathbf{p})}{\partial p_2}\right)\left(\frac{\partial^2 G(\mathbf{p})}{\partial p_1^2} - 2\frac{\partial G(\mathbf{p})}{\partial p_2}\right) = 0.$$

Making the substitution $g(\mathbf{p}) = \log G(\mathbf{p})$ *and dividing the result by* $\exp(2g(\mathbf{p}))$ *this differential equation becomes (after simplification),*

$$\frac{1}{12}\frac{\partial^4 g(\mathbf{p})}{\partial p_1^4} - \frac{\partial^2 g(\mathbf{p})}{\partial p_1 \partial p_3} + \frac{\partial^2 g(\mathbf{p})}{\partial p_2^2} + \frac{1}{2}\left(\frac{\partial^2 g(\mathbf{p})}{\partial p_1^2}\right)^2 = 0,$$

which is known as the KP equation.

2.2.2 BKP Hierarchy

The BKP hierarchy has a description similar to the KP hierarchy in terms of the Bernstein operators. The definition below can be found in [109] and [67]. To our knowledge, the equivalent forms found in Theorem 2.2.7 (particularly the description in terms of coefficients) are new although related work has been carried out in [95].

Definition 2.2.6. *A sequence of generating series $\{G_n(\mathbf{p})\}_{n\in\mathbb{Z}}$ with each $G_n(\mathbf{p}) \in R[[\mathbf{p}]]$ is said to be a solution to the BKP hierarchy if and only if*

$$\begin{aligned}[t^{m-n+1}]\left(B(\mathbf{p};t)G_{n-1}(\mathbf{p})\right)&\left(B^{\perp}(\mathbf{q};t)G_{m+1}(\mathbf{q})\right)\\ &+(-1)^{n+m}[t^{n-m+1}]\left(B^{\perp}(\mathbf{p};t)G_{n+1}(\mathbf{p})\right)\left(B(\mathbf{q};t)G_{m-1}(\mathbf{q})\right)\\ &\qquad=\frac{1}{2}(1-(-1)^{n+m})G_n(\mathbf{p})G_m(\mathbf{q}),\end{aligned}$$

for all $n, m \in \mathbb{Z}$. The differential equations satisfied by $\{G_n(\mathbf{p})\}_{n\in\mathbb{Z}}$ as a result of this identity are collectively called the BKP hierarchy.

Again, using Theorem 2.2.1 (and Proposition 2.2.2) we can write a series of conditions equivalent to the BKP hierarchy.

Theorem 2.2.7. *Suppose that for all $\lambda \in \mathcal{P}$ and $n \in \mathbb{Z}$, $b_\lambda(n) \in R$ and that for each non-negative integer n,*

$$G_n(\mathbf{p}) = \sum_{\lambda\in\mathcal{P}} b_\lambda(n)s_\lambda(\mathbf{p}).$$

The following are equivalent.

(i) The sequence of formal power series $\{G_n(\mathbf{p})\}_{n\in\mathbb{Z}}$ is a solution to the BKP hierarchy.

(ii) The sequence of formal power series $\{G_n(\mathbf{p}+\mathbf{q})\}_{n\in\mathbb{Z}}$ is a solution to the BKP hierarchy in the variables $\mathbf{p}$.

(iii) For all $n, m \in \mathbb{Z}$ and $\alpha, \beta \in \mathcal{P}$,

$$\begin{aligned}\sum_{i,j}&(-1)^{|\alpha|-|\alpha\uparrow i|+i+j}b_{\alpha\uparrow i}(m+1)b_{\beta\downarrow j}(n-1)\\ &+(-1)^{n+m}\sum_{i,j}(-1)^{|\beta|-|\beta\uparrow i|+i+j}b_{\beta\uparrow i}(n+1)b_{\alpha\downarrow j}(m-1)\\ &\qquad=\frac{1}{2}(1-(-1)^{n+m})b_\alpha(n)b_\beta(m),\end{aligned}$$

where the first sum is over all integers $i, j \geq 1$ such that $|\alpha\uparrow i| + |\beta\downarrow j| = |\alpha| + |\beta| + n - m - 1$ and the second sum is over integers $i, j \geq 1$ such that $|\alpha\downarrow j| + |\beta\uparrow i| = |\alpha| + |\beta| + m - n - 1$.

(iv) *For all $n, m \in \mathbb{Z}$ and $\alpha, \beta \in \mathcal{P}$,*

$$\begin{aligned}\sum_{i,j} &(-1)^{|\alpha|-|\alpha\uparrow i|+i+j} (s^{\perp}_{\alpha\uparrow i}(\mathbf{p}) G_{m+1}(\mathbf{p}))(s^{\perp}_{\beta\downarrow j}(\mathbf{p}) G_{n-1}(\mathbf{p})) \\ &+ (-1)^{n+m} \sum_{i,j} (-1)^{|\beta|-|\beta\uparrow i|+i+j} (s^{\perp}_{\beta\uparrow i}(\mathbf{p}) G_{n+1}(\mathbf{p}))(s^{\perp}_{\alpha\downarrow j} G_{m-1}(\mathbf{p})) \\ &= \frac{1}{2}(1-(-1)^{n+m})(s^{\perp}_{\alpha}(\mathbf{p}) G_n(\mathbf{p}))(s^{\perp}_{\beta}(\mathbf{p}) G_m(\mathbf{p})),\end{aligned}$$

where the first sum is over all integers $i, j \geq 1$ such that $|\alpha \uparrow i| + |\beta \downarrow j| = |\alpha| + |\beta| + n - m - 1$ and the second sum is over integers $i, j \geq 1$ such that $|\alpha \downarrow j| + |\beta \uparrow i| = |\alpha| + |\beta| + m - n - 1$.

Proof. The proof proceeds in exactly the same way as the proof of Theorem 2.2.4. □

Example 2.2.8. *Suppose that $m = N-1, n = N+1, \alpha = (2)$ and $\beta = (1)$. Then the identity described in part (iii) of Theorem 2.2.7 becomes*

$$\sum_{\substack{i,j\geq 1 \\ |\alpha\uparrow i|+|\beta\downarrow j|=4}} (-1)^{|\alpha|-|\alpha\uparrow i|+i+j} b_{\alpha\uparrow i}(N) b_{\beta\downarrow j}(N) + \sum_{\substack{i,j\geq 1 \\ |\beta\uparrow i|+|\alpha\downarrow j|=0}} (-1)^{|\beta|-|\beta\uparrow i|+i+j} b_{\beta\uparrow i}(N+2) b_{\alpha\downarrow j}(N-2).$$

Recall from Example 2.2.5 that $\alpha\uparrow 1 = (1), \alpha\uparrow 2 = (1,1), \alpha\uparrow 3 = (2,2)$ and $|\alpha\uparrow i| > 4$ for $i > 3$. Also, $\beta\downarrow 1 = \epsilon, \beta\downarrow 2 = (2), \beta\downarrow 3 = (2,1)$ and $|\beta\downarrow i| > 3$ for $i > 3$ so that the first sum becomes

$$-b_{(1)}(N) b_{(2,1)}(N) + b_{(1,1)}(N) b_{(2)}(N) + b_{(2,2)}(N) b_{\epsilon}(N).$$

We may compute $\alpha\downarrow 1 = \epsilon, \alpha\downarrow 2 = 3$ and $|\alpha\downarrow i| > 3$ for $i > 2$. Also, $\beta\uparrow 1 = \epsilon, \beta\uparrow 2 = (1,1)$ and $|\beta\uparrow i| > 2$ for $i > 2$ so that the second sum becomes

$$-b_{\epsilon}(N+2) b_{\epsilon}(N-2).$$

Putting these together gives the identity

$$b_{\epsilon}(N+2) b_{\epsilon}(N-2) = b_{\epsilon}(N) b_{(2,2)}(N) - b_{(1)}(N) b_{(2,1)}(N) + b_{(2)}(N) b_{(1,1)}(N). \tag{2.4}$$

Notice that the right hand side of Equation (2.4) is the same algebraic expression appearing in Equation (2.3) in Example 2.2.5 which gave rise to the KP equation. In particular, this tells us that for $m = N-1, n = N+1, \alpha = (2)$ and $\beta = (1)$ part (iv) of Theorem 2.2.7 gives the differential equation

$$\begin{aligned}G_{N+2}(\mathbf{p}) G_{N-2}(\mathbf{p}) = &\frac{1}{12} G_N(\mathbf{p}) \left(\frac{\partial^4 G_N(\mathbf{p})}{\partial p_1^4} - 12 \frac{\partial^2 G_N(\mathbf{p})}{\partial p_1 \partial p_3} + 12 \frac{\partial^2 G_N(\mathbf{p})}{\partial^2 p_2} \right) \\ &- \frac{1}{3} \frac{\partial G_N(\mathbf{p})}{\partial p_1} \left(\frac{\partial^3 G_N(\mathbf{p})}{\partial p_1^3} - 3 \frac{\partial G_N(\mathbf{p})}{\partial p_3} \right) \\ &+ \frac{1}{4} \left(\frac{\partial^2 G_N(\mathbf{p})}{\partial p_1^2} + 2 \frac{\partial G_N(\mathbf{p})}{\partial p_2} \right) \left(\frac{\partial^2 G_N(\mathbf{p})}{\partial p_1^2} - 2 \frac{\partial G_N(\mathbf{p})}{\partial p_2} \right).\end{aligned}$$

Letting $g(\mathbf{p}) = \log G_N(\mathbf{p})$ and dividing by $G_N(\mathbf{p})$, this equation becomes

$$\frac{G_{N+2}(\mathbf{p})G_{N-2}(\mathbf{p})}{G_N^2(\mathbf{p})} = \frac{1}{12}\frac{\partial^4 g(\mathbf{p})}{\partial p_1^4} - \frac{\partial^2 g(\mathbf{p})}{\partial p_1 \partial p_3} + \frac{\partial^2 g(\mathbf{p})}{\partial p_2^2} + \frac{1}{2}\left(\frac{\partial^2 g(\mathbf{p})}{\partial p_1^2}\right)^2,$$

which is known as the BKP equation.

2.2.3 2-Toda Hierarchy

Though we will not make use of it here, for completeness we will also mention the 2-Toda hierarchy. It is of interest to see the description of the 2-Toda hierarchy as its presentation shares many similarities with both the KP and BKP hierarchies. For more about the 2-Toda hierarchy and its application to combinatorics see [14, 92]. The primary difference between the 2-Toda hierarchy and the KP and BKP hierarchies is that in the KP and BKP cases there is only a single family of indeterminates. In the 2-Toda case there is a second family of indeterminates along with a discrete parameter (similar to the one which appears in the BKP hierarchy).

Definition 2.2.9. *A sequence of formal power series $\{G_n(\mathbf{p},\mathbf{q})\}_{n\in\mathbb{Z}}$ where each $G_n(\mathbf{p},\mathbf{q}) \in R[[\mathbf{p},\mathbf{q}]]$ is a solution to the 2-Toda hierarchy if and only if for all $k, m \in \mathbb{Z}$,*

$$\begin{aligned}[t^{k-m}]\left(B(\mathbf{p};t)\ G_m(\mathbf{p},\mathbf{q})\right)\left(B^{\perp}(\mathbf{w};t)G_{k+1}(\mathbf{w},\mathbf{z})\right) = \\ [t^{m-k}]\left(B^{\perp}(\mathbf{q};t)G_{m+1}(\mathbf{p},\mathbf{q})\right)\left(B(\mathbf{z};t)G_k(\mathbf{w};\mathbf{z})\right).\end{aligned}$$

The differential equations satisfied by $\{G_n(\mathbf{p},\mathbf{q})\}_{n\in\mathbb{Z}}$ as a result of this identity are collectively called the 2-Toda hierarchy.

Similar to each of the other hierarchies, we may write a sequence of equivalent statements.

Theorem 2.2.10. *Suppose that for all $\lambda, \mu \in \mathcal{P}$ and $n \in \mathbb{Z}$, $c^{\lambda}_{\mu}(n) \in R$ and that for all $n \in \mathbb{Z}$,*

$$G_n(\mathbf{p},\mathbf{q}) = \sum_{\lambda,\mu\in\mathcal{P}} c^{\lambda}_{\mu}(n) s_{\lambda}(\mathbf{p}) s_{\mu}(\mathbf{q}).$$

The following are equivalent.

(i) The sequence of formal power series $\{G_n(\mathbf{p},\mathbf{q})\}_{n\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy.

(ii) The sequence of formal power series $\{G_n(\mathbf{p}+\mathbf{w},\mathbf{q}+\mathbf{z})\}_{n\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy in the variables $\mathbf{p}$ and $\mathbf{q}$.

(iii) *For all $m, k \in \mathbb{Z}$ and $\alpha, \beta, \lambda, \mu \in \mathcal{P}$,*

$$\sum_{i,j} (-1)^{|\alpha|-|\alpha\uparrow j|+i+j} c_{\mu}^{\lambda\downarrow i}(m) c_{\beta}^{\alpha\uparrow j}(k+1) = \sum_{s,t} (-1)^{|\mu|+|\mu\uparrow s|+s+t} c_{\mu\uparrow s}^{\lambda}(m+1) c_{\beta\downarrow t}^{\alpha}(k),$$

where the first sum is over integers $i, j \geq 1$ such that $|\lambda \downarrow i| + |\alpha \uparrow j| = |\lambda| + |\alpha| + m - k$ and the second sum is over integers $s, t \geq 1$ such that $|\mu \uparrow s| + |\beta \downarrow t| = |\mu| + |\beta| + k - m$.

(iv) *for all $m, k \in \mathbb{Z}$ and $\alpha, \beta, \lambda, \mu \in \mathcal{P}$,*

$$\begin{aligned}&\sum_{i,j} (-1)^{|\alpha|-|\alpha\uparrow j|+i+j} \left(s_{\lambda\downarrow i}^{\perp}(\mathbf{p}) s_{\mu}^{\perp}(\mathbf{q}) G_m(\mathbf{p},\mathbf{q})\right) \left(s_{\alpha\uparrow j}^{\perp}(\mathbf{p}) s_{\beta}^{\perp}(\mathbf{q}) G_{k+1}(\mathbf{p},\mathbf{q})\right) \\ &= \sum_{r,s} (-1)^{|\mu|-|\mu\uparrow r|+r+s} \left(s_{\lambda}^{\perp}(\mathbf{p}) s_{\mu\uparrow r}^{\perp}(\mathbf{q}) G_{m+1}(\mathbf{p},\mathbf{q})\right) \left(s_{\alpha}^{\perp}(\mathbf{p}) s_{\beta\downarrow s}^{\perp}(\mathbf{q}) G_k(\mathbf{p},\mathbf{q})\right),\end{aligned}$$

where the first sum is over integers $i, j \geq 1$ such that $|\lambda \downarrow i| + |\alpha \uparrow j| = |\lambda| + |\alpha| + m - k$ and the second sum is over integers $r, s \geq 1$ such that $|\mu \uparrow r| + |\beta \downarrow s| = |\mu| + |\beta| + k - m$.

Proof. The proof is essentially the same as Theorem 2.2.4 and Theorem 2.2.7. □

Example 2.2.11. *In the case of the 2-Toda hierarchy, choose $k = m-1, \alpha = \lambda = \beta = \epsilon$ and $\mu = 1$. We compute $\epsilon \uparrow 1 = \epsilon$, $\epsilon \uparrow 2 = 1$, $\epsilon \downarrow 1 = \epsilon$, $\epsilon \downarrow 2 = 1$, $1 \uparrow 1 = \epsilon$ and $1 \uparrow 2 = (1,1)$. The only solutions (i,j) to $|\lambda \downarrow i| + |\alpha \uparrow j| = |\lambda| + |\alpha| + 1$ are $(1,2)$ and $(2,1)$. Similarly, the only solution (r,s) to $|\mu \uparrow r| + |\beta \downarrow s| = |\mu| + |\beta| - 1$ is $(1,1)$. Theorem 2.2.10(iii) then gives*

$$c_1^{\epsilon}(m) c_{\epsilon}^{1}(m) - c_1^{1}(m) c_{\epsilon}^{\epsilon}(m) = -c_{\epsilon}^{\epsilon}(m+1) c_{\epsilon}^{\epsilon}(m-1),$$

as one of the coefficient constraints. Similarly, Theorem 2.2.10(iv) gives

$$\begin{aligned}&\left(s_{\epsilon}^{\perp}(\mathbf{p}) s_{\epsilon}^{\perp}(\mathbf{q}) G_{m+1}(\mathbf{p},\mathbf{q})\right) \left(s_{\epsilon}^{\perp}(\mathbf{p}) s_{\epsilon}^{\perp}(\mathbf{q}) G_{m-1}(\mathbf{p},\mathbf{q})\right) \\ &\quad + \left(s_{1}^{\perp}(\mathbf{p}) s_{\epsilon}^{\perp}(\mathbf{q}) G_{m}(\mathbf{p},\mathbf{q})\right) \left(s_{\epsilon}^{\perp}(\mathbf{p}) s_{1}^{\perp}(\mathbf{q}) G_{m}(\mathbf{p},\mathbf{q})\right) \\ &\quad = \left(s_{1}^{\perp}(\mathbf{p}) s_{1}^{\perp}(\mathbf{q}) G_{m}(\mathbf{p},\mathbf{q})\right) \left(s_{\epsilon}^{\perp}(\mathbf{p}) s_{\epsilon}^{\perp}(\mathbf{q}) G_{m}(\mathbf{p},\mathbf{q})\right),\end{aligned}$$

as one of the partial differential equations in the 2-Toda hierarchy. Using the fact that $s_{\epsilon}^{\perp}(\mathbf{p}) = 1$ and $s_1^{\perp}(\mathbf{p}) = \frac{\partial}{\partial p_1}$, this gives

$$G_{m+1}(\mathbf{p},\mathbf{q}) G_{m-1}(\mathbf{p},\mathbf{q}) + \left(\frac{\partial}{\partial p_1} G_m(\mathbf{p},\mathbf{q})\right) \left(\frac{\partial}{\partial q_1} G_m(\mathbf{p},\mathbf{q})\right) = G_m(\mathbf{p},\mathbf{q}) \left(\frac{\partial^2}{\partial p_1 \partial q_1} G_m(\mathbf{p},\mathbf{q})\right),$$

which can be further simplified to

$$\frac{\partial^2}{\partial p_1 \partial q_1} \log G_m(\mathbf{p},\mathbf{q}) = \frac{G_{m+1}(\mathbf{p},\mathbf{q}) G_{m-1}(\mathbf{p},\mathbf{q})}{G_m^2(\mathbf{p},\mathbf{q})}.$$

This last equation is called the 2-Toda equation.

2.3 Hirota Equations

The primary advantage to having a description of integrable hierarchies in terms of the Bernstein operators is that it means we can describe the equations appearing in the hierarchies more easily. We have already seen in the Example 2.2.5 and Example 2.2.8 above that we can write down the differential equations in each hierarchy. However, doing so becomes increasingly labourious. Fortunately we may use Hirota's [58] method of rewriting the KP and BKP hierarchies in such a way that the differential equations can more easily be constructed.

Definition 2.3.1. *Suppose that f and g are generating series in the countable family of indeterminates $\mathbf{p} = \{p_1, p_2, \cdots\}$ and that ϕ is a power series with polynomial (in $\mathbf{p}$) coefficients. We define the Hirota differential operator by*

$$\mathfrak{H}(\phi; f, g) = \phi(\mathbf{q}^\perp) f(\mathbf{p}-\mathbf{q}) g(\mathbf{p}+\mathbf{q})\big|_{\mathbf{q}=0}.$$

Example 2.3.2. *For any integer $k > 0$ and generating series $f, g \in R[[\mathbf{p}]]$ we have*

$$\mathfrak{H}(p_k; f, g) = q_k^\perp f(\mathbf{p}-\mathbf{q}) g(\mathbf{p}+\mathbf{q})\big|_{\mathbf{q}=0} = f(p_k^\perp g) - (p_k^\perp f) g.$$

More generally, for any integer $n > 0$,

$$\mathfrak{H}(p_k^n; f, g) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (p_k^{\perp i} f)(p_k^{\perp (n-i)} g).$$

Similarly, for any partition λ,

$$\mathfrak{H}(p_\lambda; f, g) = \sum_{\substack{\mu,\nu \\ \mu\cup\nu=\lambda}} (-1)^{|\mu|} \binom{\lambda}{\mu} (p_\mu^\perp f)(p_\nu^\perp g),$$

where $\binom{\lambda}{\mu} = \prod_{i\geq 1} \binom{m_i(\lambda)}{m_i(\mu)}$.

Note that in the last example above, given a partition λ, if $\ell(\lambda)$ is odd then the map $(\mu,\nu) \to (\nu,\mu)$ for partitions μ,ν with $\mu\cup\nu = \lambda$ is a sign reversing involution and so

$$\mathfrak{H}(p_\lambda; f, f) = 0.$$

Both the KP and BKP hierarchies discussed above make use of the expression

$$(B(\mathbf{p};t) f(\mathbf{p}))(B^\perp(\mathbf{q};t) g(\mathbf{q})), \tag{2.5}$$

where $f, g \in R[[\mathbf{p}]]$. Using the definition of the Bernstein operators, this can be written as

$$\left\{\exp\left(\sum_{k\geq 1} \frac{t^k}{k} p_k\right) \exp\left(-\sum_{k\geq 1} \frac{t^{-k}}{k} p_k^\perp\right) f(\mathbf{p})\right\} \left\{\exp\left(-\sum_{k\geq 1} \frac{t^k}{k} q_k\right) \exp\left(\sum_{k\geq 1} \frac{t^{-k}}{k} q_k^\perp\right) g(\mathbf{q})\right\}.$$

Writing this so that multiplication appears to the left of differentiation and letting $p_k = p'_k, q_k = q'_k$, we arrive at

$$[t^{-1}]\exp\left(\sum_{k\geq 1}\frac{t^k}{k}(p'_k - q'_k)\right)\exp\left(-\sum_{k\geq 1}\frac{t^{-k}}{k}(p'^{\perp}_k - q'^{\perp}_k)\right)f(\mathbf{p}')g(\mathbf{q}').$$

Applying the linear substitution $p'_k = q_k - u_k$, $q'_k = q_k + u_k$ we see that $p'^{\perp}_k - q'^{\perp}_k = -u^{\perp}_k$ so that the expression becomes

$$[t^{-1}]\exp\left(\sum_{k\geq 1}\frac{t^k}{k}(-2u_k)\right)\exp\left(\sum_{k\geq 1}\frac{t^{-k}}{k}u^{\perp}_k\right)f(\mathbf{q}-\mathbf{u})g(\mathbf{q}+\mathbf{u}).$$

The Hirota differential operators appear since given any generating series $f, g \in R[[\mathbf{p}]]$ and any partition λ,

$$\begin{aligned}
u^{\perp}_\lambda f(\mathbf{q}-\mathbf{u})g(\mathbf{q}+\mathbf{u}) &= u^{\perp}_\lambda f(\mathbf{q}-\mathbf{u}-\mathbf{p})g(\mathbf{q}+\mathbf{u}+\mathbf{p})\big|_{\mathbf{p}=0}, \\
&= p^{\perp}_\lambda f(\mathbf{q}-\mathbf{u}-\mathbf{p})g(\mathbf{q}+\mathbf{u}+\mathbf{p})\big|_{\mathbf{p}=0}, \\
&= p^{\perp}_\lambda \exp\left(\sum_{k\geq 1}\frac{u_k p^{\perp}_k}{k}\right)f(\mathbf{q}-\mathbf{p})g(\mathbf{q}+\mathbf{p})\Bigg|_{\mathbf{p}=0}, \\
&= \mathfrak{H}\left(p_\lambda \exp\left(\sum_{k\geq 1}\frac{u_k p_k}{k}\right); f, g\right).
\end{aligned}$$

This allows us to write expression (2.5) as

$$\mathfrak{H}\left(\exp\left(\sum_{k\geq 1}\frac{t^k}{k}(-2u_k)\right)\exp\left(\sum_{k\geq 1}\frac{t^{-k}}{k}p_k\right)\exp\left(\sum_{k\geq 1}\frac{u_k p_k}{k}\right); f, g\right).$$

Simplifying, this becomes

$$\mathfrak{H}\left(\exp\left(\sum_{k\geq 1}\frac{u_k}{k}(p_k - 2t^k)\right)H(t^{-1}); f, g\right), \tag{2.6}$$

where $H(t)$ is the generating series for complete symmetric functions (in the power sums $\mathbf{p}$). Since expression (2.6) will appear in multiple places below it is convenient to have some notation to describe the coefficients of u_λ for each partition λ.

Definition 2.3.3. *For any partition λ and any integer i,*

$$\begin{aligned}
\phi^{(i)}_\lambda &= \left[t^{i-1}\frac{u_\lambda}{z_\lambda}\right]\exp\left(\sum_{k\geq 1}(p_k - 2t^k)\right)H(t^{-1}), \\
&= \left\{t^{1-i}\prod_{j=1}^{\ell(\lambda)}(p_{\lambda_j} - 2t^{\lambda_j})\right\}\circ H(t),
\end{aligned}$$

where $\circ$ denotes umbral composition (with respect to t). That is, $t^m \circ H(t) = h_m$ extended linearly.

Example 2.3.4. *We can compute a few of the* $\phi_\lambda^{(i)}$ *as follows,*

$$\phi_k^{(i)} = [t^{1-i}(p_k - 2t^k)] \circ H = p_k h_{1-i} - 2h_{k+1-i},$$

and,

$$\phi_{k,1}^{(i)} = [t^{1-i}(p_k - 2t^k)(p_1 - 2t)] \circ H(t) = p_k p_1 h_{1-i} - 2p_1 h_{k+1-i} - 2p_k h_{2-i} + 4h_{k+2-i}.$$

Using Definition 2.3.3 and (2.6) we have the following useful Lemma.

Lemma 2.3.5. *For any generating series* $f, g \in R[[\mathbf{p}]]$*,*

$$(B(\mathbf{p};t)f(\mathbf{p}))(B^\perp(\mathbf{q};t)g(\mathbf{q})) = \mathfrak{H}\left(\exp\left(\sum_{k\geq 1}\frac{u_k}{k}(p_k - 2t^k)\right)H(t^{-1}); f, g\right),$$

under the linear change of variables $p_k \mapsto q_k - u_k$*,* $q_k \mapsto q_k + u_k$*. In particular,*

$$(B(\mathbf{p};t)f(\mathbf{p}))(B^\perp(\mathbf{q};t)g(\mathbf{q})) = \sum_{\substack{\lambda\in\mathcal{P}\\ i\in\mathbb{Z}}} t^{i-1}\frac{u_\lambda}{z_\lambda}\mathfrak{H}(\phi_\lambda^{(i)}; f, g).$$

2.3.1 KP Hierarchy

Recall that by definition a generating series $G(\mathbf{p}) \in R[[\mathbf{p}]]$ is a solution to the KP hierarchy if and only if

$$[t^{-1}]\,(B(\mathbf{p};t)G(\mathbf{p}))\,(B^\perp(\mathbf{q};t)G(\mathbf{q})) = 0.$$

Applying Lemma 2.3.5 then gives us an alternate description of the differential equations appearing in the KP hierarchy.

Proposition 2.3.6. *A generating series* $G(\mathbf{p}) \in R[[\mathbf{p}]]$ *satisfies the KP hierarchy if and only if for each partition* λ*,*

$$\mathfrak{H}(\phi_\lambda^{(0)}; G, G) = 0.$$

Proof. Since the definition of the KP hierarchy is that $G(\mathbf{p})$ satisfies it if and only if

$$[t^{-1}](B(\mathbf{p};t)G(\mathbf{p}))(B(\mathbf{q};t)G(\mathbf{q})) = 0,$$

we may apply Lemma 2.3.5 and extract coefficients. ☐

Example 2.3.7. *Using Proposition 2.3.6 we know that*

$$\mathfrak{H}(\phi_k^{(0)}; G, G) = 0 = \mathfrak{H}(\phi_{k,1}^{(0)}; G, G),$$

for each $k > 0$*. We must also then have*

$$\mathfrak{H}(\phi_{k+1}^{(0)} + \frac{1}{2}\phi_{k,1}^{(0)}; G, G) = 0.$$

But, using the results in Example 2.3.4 we know that

$$\phi_{k+1}^{(0)} + \frac{1}{2}\phi_{k,1}^{(0)} = p_{k+1}p_1 + \frac{1}{2}p_k p_1^2 - p_1 h_{k+1} - p_k h_2.$$

Thus,

$$\begin{aligned}\mathfrak{H}(\phi_{k+1}^{(0)} + \frac{1}{2}\phi_{k,1}^{(0)}; G, G) &= \mathfrak{H}(p_{k+1}p_1 + \frac{1}{2}p_k p_1^2 - p_1 h_{k+1} - p_k h_2; G, G),\\ &= \mathfrak{H}(p_{k+1}p_1 - \frac{1}{2}p_k p_2 - p_1 h_{k+1}; G, G) = 0,\end{aligned}$$

where we have used the fact that $h_2 = \frac{1}{2}p_2 + \frac{1}{2}p_1^2$ *and that if* $\ell(\lambda)$ *is odd then* $\mathfrak{H}(p_\lambda; \tau, \tau) = 0$.

If we make the change of variables $G(\mathbf{p}) = \exp(g(\mathbf{p}))$ *and divide by* $\exp(2g(\mathbf{p}))$ *then (up to scaling) the first few equations become:*

$$g_{31} = g_{2^2} + \frac{1}{12}g_{1^4} + (g_{1^2})^2,$$

$$g_{41} = \frac{1}{6}g_{21^3} + (g_{1^2})(g_{21}) + g_{32},$$

$$g_{51} = \frac{1}{16}g_{1^4}g_{1^2} + \frac{1}{4}g_{1^2}g_{2^2} + \frac{3}{4}g_{1^2}g_{31} + \frac{1}{480}g_{1^6} + \frac{1}{8}g_{1^2}^3 + \frac{1}{8}g_{1^2 2^2} + \frac{1}{2}g_{21}^2 + \frac{1}{8}g_{31^3} + g_{42},$$

where we have used the notation

$$g_{a_1 a_2 \cdots a_n} = \frac{\partial^n g}{\partial_{p_{a_1}}\partial_{p_{a_2}}\cdots\partial_{p_{a_n}}}.$$

2.3.2 BKP Hierarchy

Similar to the KP case, recall that a sequence of generating series $\{G_n(\mathbf{p})\}_{n\in\mathbb{Z}}$ where each $G_n(\mathbf{p}) \in R[[\mathbf{p}]]$ is a solution to the BKP hierarchy if and only if

$$\begin{aligned}&[t^{m-n+1}]\left(B(\mathbf{p};t)G_{n-1}(\mathbf{p})\right)\left(B^\perp(\mathbf{q};t)G_{m+1}(\mathbf{q})\right)\\ &\quad + (-1)^{n+m}[t^{n-m+1}]\left(B^\perp(\mathbf{p};t)G_{n+1}(\mathbf{p})\right)\left(B(\mathbf{q};t)G_{m-1}(\mathbf{q})\right)\\ &\qquad = \frac{1}{2}(1-(-1)^{n+m})G_n(\mathbf{p})G_m(\mathbf{q}),\end{aligned}$$

for all $n, m \in \mathbb{Z}$. Applying Lemma 2.3.5 then gives the following description.

Proposition 2.3.8. *A sequence of generating series* $\{G_n(\mathbf{p})\}_{n\in\mathbb{Z}}$ *where each* $G_n(\mathbf{p})$ *is in* $R[[\mathbf{p}]]$ *satisfies the BKP hierarchy if and only if for each partition* λ *and each* $n, m \in \mathbb{Z}$,

$$\begin{aligned}&\mathfrak{H}(\phi_\lambda^{(m-n+2)}; G_{n-1}, G_{m+1})\\ &\quad + (-1)^{n+m}\mathfrak{H}(\phi_\lambda^{(n-m+2)}; G_{n+1}, G_{m-1})\\ &\qquad = \frac{1}{2}(1-(-1)^{n+m})\mathfrak{H}(1; G_n, G_m).\end{aligned}$$

Proof. As in the KP case this follows by applying Lemma 2.3.5 to the definition of the BKP hierarchy and then extracting coefficients. □

Example 2.3.9. *Let $n = N + 1$ and $m = N - 1$ so that the corresponding equation becomes*

$$\mathfrak{H}(\phi_\lambda^{(0)}; G_N, G_N) + \mathfrak{H}(\phi_\lambda^{(4)}; G_{N+2}, G_{N-2}) = 0.$$

If we consider the case $\lambda = (3)$ we see that (using Example 2.3.4)

$$\phi_3^{(0)} = p_3 p_1 - 2h_4,$$

and that

$$\phi_3^{(4)} = p_3 h_{1-4} - 2h_0 = -2.$$

Thus the differential equation becomes

$$\mathfrak{H}(p_3 p_1 - 2h_4; G_N, G_N) + \mathfrak{H}(-2; G_{N+2}, G_{N-2}) = 0,$$

or,

$$2G_{N+2}(\mathbf{p}) G_{N-2}(\mathbf{p}) = \mathfrak{H}(p_3 p_1 - 2h_4; G_N, G_N).$$

Lastly, using the fact that $h_4 = \frac{1}{24}p_1^4 + \frac{1}{4}p_1^2 p_2 + \frac{1}{3}p_1 p_3 + \frac{1}{8}p_2^2 + \frac{1}{4}p_4$ and that $\mathfrak{H}(p_\lambda; f, f) = 0$ if $\ell(\lambda)$ is odd, we arrive at the differential equation

$$2G_{N+2}(\mathbf{p}) G_{N-2}(\mathbf{p}) = \mathfrak{H}\left(\frac{1}{3}p_3 p_1 - \frac{1}{12}p_1^4 - \frac{1}{4}p_2^2; G_N, G_N\right).$$

2.4 Determinantal Series and the KP Hierarchy

In addition to constructing the KP and BKP hierarchies using the Bernstein operators, we can take a more mechanical approach and instead consider generating series whose Schur polynomial coefficients have some specified structure. In particular, for the KP hierarchy we will assume that the coefficients are certain determinantal minors and for the BKP hierarchy (covered in Section 2.5) we will assume that they are certain Pfaffian minors.

First we recall the Cauchy-Binet formula for the determinant of a product of matrices. Given an $m \times n$ matrix A and subsets $R \subseteq \{1, 2, \cdots, m\}$ and $C \subseteq \{1, 2, \cdots, n\}$, we will denote by A_C^R the submatrix of A whose rows are indexed by R and whose columns are indexed by C. Further, we will write $[n] = \{1, 2, \cdots, n\}$.

Theorem 2.4.1 (Cauchy-Binet)**.** *Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Then*

$$\det(AB) = \sum_{\substack{S \subseteq [n] \\ |S| = m}} \det(A_S^{[m]}) \det(B_{[m]}^S).$$

Using the *Cauchy-Binet* theorem, we can now show that a generating series whose Schur coefficients are given by determinant minors has the same property after translation. Though Theorem 2.4.2 appears to be a foundational result for symmetric functions we have been unable to find a literature reference.

Theorem 2.4.2. *Let $G(\mathbf{p})$ be a generating series in $R[[\mathbf{p}]]$ and suppose for some positive integer n the generating series $G(\mathbf{p})$ can be written as*

$$G(\mathbf{p}) = \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le n}} a_\lambda(n) s_\lambda(\mathbf{p}),$$

where

$$a_\lambda(n) = \det_{1\le i,j\le n}(a_{i,\lambda_j+n-j+1}),$$

for some $a_{i,j}\in R$. Then

$$G(\mathbf{p}+\mathbf{q}) = \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le n}} \widetilde{a_\lambda}(n) s_\lambda(\mathbf{p}),$$

where

$$\widetilde{a_\lambda}(n) = \det_{1\le i,j\le n}(\widetilde{a}_{i,\lambda_j+n-j+1})$$

and

$$\widetilde{a}_{i,j} = \sum_{k\ge 0} a_{i,k} h_{k-j}(\mathbf{q}).$$

Proof. First, recall that for any $\lambda\in\mathcal{P}$

$$s_\lambda(\mathbf{p}+\mathbf{q}) = \sum_{\mu\subseteq\lambda} s_{\lambda/\mu}(\mathbf{q}) s_\mu(\mathbf{p}).$$

Then

$$\begin{aligned}
G(\mathbf{p}+\mathbf{q}) &= \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le n}} a_\lambda(n) s_\lambda(\mathbf{p}+\mathbf{q}),\\
&= \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le n}} \sum_{\mu\subseteq\lambda} a_\lambda(n) s_{\lambda/\mu}(\mathbf{q}) s_\mu(\mathbf{p}),\\
&= \sum_{\substack{\mu\in\mathcal{P}\\ \ell(\lambda)\le n}} s_\mu(\mathbf{p}) \sum_{\substack{\lambda\supseteq\mu\\ \ell(\lambda)\le n}} a_\lambda(n) s_{\lambda/\mu}(\mathbf{q}).
\end{aligned}$$

Now, suppose that A is the $n\times\infty$ matrix with $A_{i,j} = a_{i,j}$ and further suppose that S_μ is the $\infty\times n$ matrix with $(S_\mu)_{i,j} = h_{i-(\mu_j+n-j+1)}(\mathbf{q})$. Then if $C = \{\lambda_1+n, \lambda_2+n-1, \cdots, \lambda_n+1\}$,

$$\det(A_C^{[n]}) = \det_{1\le i,j\le n}(a_{i,\lambda_j+n-j+1}) = a_\lambda(n),$$

and

$$\begin{aligned}
\det((S_\mu)_{[n]}^S) &= \det_{1\le i,j\le n}(h_{\lambda_i+n-i+1-\mu_j+j-n-1}),\\
&= \det_{1\le i,j\le n}(h_{\lambda_i-\mu_j-i+j}),\\
&= s_{\lambda/\mu}(\mathbf{q}).
\end{aligned}$$

Using the Cauchy-Binet theorem we then get

$$\det(AS_\mu) = \sum_{\substack{S\subseteq\{1,2,\ldots\}\\|S|=n}} \det(A_S^{[n]})\det((S_\mu)_{[n]}^S),$$
$$= \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le n}} a_\lambda(n)s_{\lambda/\mu}(\mathbf{q}),$$
$$= \sum_{\substack{\lambda\supseteq\mu\\ \ell(\lambda)\le n}} a_\lambda(n)s_{\lambda/\mu}(\mathbf{q}),$$

where the last line follows from the observation that $\det_{1\le i,j\le n}(h_{\lambda_i-\mu_j-i+j}) = 0$ if $\mu \notin \lambda$. Now, we have

$$(AS_\mu)_{i,j} = \sum_{k\ge0} a_{i,k}h_{k-(\mu_j+n-j+1)}(\mathbf{q}),$$

so that if we let $\widetilde{a}_{i,j} = \sum_{k\ge0} a_{i,k}h_{k-j}(\mathbf{q})$ then

$$\det(AS_\mu) = \det_{1\le i,j\le n}(\widetilde{a}_{i,\mu_j+n-j+1}) = \widetilde{a}_\mu(n).$$

□

The coefficients $a_\lambda(n)$ appearing in Theorem 2.4.2 can be seen as determinant minors of the $\infty\times\infty$ matrix A where $A_{i,j} = a_{i,j}$. In particular, if for any $\lambda\in\mathcal{P}, \ell(\lambda)\le n$ we let $C_\lambda = \{\lambda_1+n, \lambda_2+n-1, \cdots, \lambda_n+1\}$ then $a_\lambda(n) = \det(A_{C_\lambda}^{[n]})$. We will see that the collection of determinant minors of this form satisfies a family of identities known as the Plücker relations. In the definition of the Plücker relations that follows we will make use of the notation found in [50].

Definition 2.4.3. *We consider a family $\{a_\lambda\in R : \lambda\in\mathcal{P}\}$. It will be convenient to adopt two conventions which allow us to write $a_{(\alpha_1,\alpha_2,\cdots,\alpha_n)}$ where $(\alpha_1,\alpha_2,\cdots,\alpha_n)$ is a sequence which may not be a partition. First, we will take*

$$a_{(\alpha_1,\alpha_2,\cdots,\alpha_n)} = 0 \quad \text{if } \alpha_n < 0; \qquad a_{(\alpha_1,\alpha_2,\cdots,\alpha_n)} = a_{(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})} \quad \text{if } \alpha_n = 0.$$

Also, given a sequence $(\alpha_1,\alpha_2,\cdots,\alpha_n)$ we define the operator Δ_j for each $j = 1,\cdots,n-1$ to be

$$\Delta_j(\alpha_1,\cdots,\alpha_n) = (\alpha_1,\cdots,\alpha_{j-1},\alpha_{j+1}-1,\alpha_j+1,\alpha_{j+2},\cdots,\alpha_n).$$

The second convention is then that

$$a_{(\alpha_1,\alpha_2,\cdots,\alpha_n)} = -a_{\Delta_j(\alpha_1,\alpha_2,\cdots,\alpha_n)}.$$

Suppose α,β are partitions where α and β have i and j parts respectively for some $i,j\ge0$, $(i,j)\ne(0,0)$ and suppose $\alpha = (\alpha_1,\cdots,\alpha_i)$ and $\beta = (\beta_1,\cdots,\beta_j)$. Let $m = \max\{i+1, j-1, 2\}$ and set $\alpha_{i+1} = \cdots = \alpha_{m-1} = \beta_{j+1} = \cdots = \beta_{m+1} = 0$. we say that *a family $\{a_\lambda \in R : \lambda \in \mathcal{P}\}$ which follows the conventions above satisfies the Plücker relations if and only if it satisfies the equation*

$$\sum_{k=0}^{m}(-1)^k a_{(\alpha_1-1,\cdots,\alpha_{m-1}-1,\beta_{k+1}+m-k)} \cdot a_{(\beta_1+1,\cdots,\beta_k+1,\beta_{k+2},\cdots,\beta_{m+1})} = 0,$$

for each pair of partitions α and β.

Example 2.4.4. *Consider the Plücker relation arising from the partitions $\alpha = \epsilon$ and $\beta = (1,1,1)$. The resulting relation is then*

$$a_{(-1,3)} \cdot a_{(1,1)} - a_{(-1,2)} \cdot a_{(2,1)} + a_{(-1,1)} \cdot a_{(2,2)} = 0.$$

Using the conventions in Definition 2.4.3 we have $a_{(-1,3)} = -a_{(2,0)} = -a_{(2)}$, $a_{(-1,2)} = -a_{(1,0)} = -a_{(1)}$ and $a_{(-1,1)} = -a_{(0,0)} = -a_\epsilon$. Thus the Plücker relation arising from the partitions $\alpha = \epsilon$ and $\beta = (1,1,1)$ can be written as,

$$a_{(2)} \cdot a_{(1,1)} - a_{(1)} \cdot a_{(2,1)} + a_\epsilon a_{(2,2)} = 0.$$

Proposition 2.4.5. *Fix some positive integer n. For each partition $\lambda \in \mathcal{P}$ if $\ell(\lambda) \leq n$ let*

$$a_\lambda(n) = \det_{1\leq i,j\leq n}(a_{i,\lambda_j+n-j+1}),$$

with $a_{i,j} \in R$. If $\ell(\lambda) > n$ let $a_\lambda(n) = 0$. Then the set $\{a_\lambda(n) : \lambda \in \mathcal{P}\}$ satisfies the Plücker relations.

Proof. First note that the family $\{a_\lambda(n) : \lambda \in \mathcal{P}\}$ satisfies the conventions described in Definition 2.4.3 by definition. Let α, β be partitions and $m = \max\{\ell(\alpha)+1, \ell(\beta)-1, 2\}$ as in Definition 2.4.3. If $m > n$ then the Plücker relations are trivially satisfied so we may assume that $m \leq n$. Also, if $m \leq n$ then using the conventions described in Definition 2.4.3 we may proceed as though $m = n$. Recall the Laplace expansion of a determinant. Suppose A is an $n \times n$ matrix and $R \subseteq \{1,2,\cdots,n\}$. Then letting $\sigma(R) = \sum_{r\in R} r$,

$$\det(A) = (-1)^{\sigma(R)} \sum_{\substack{C\subseteq[n] \\ |C|=|R|}} (-1)^{\sigma(C)} \det(A_C^R)\det(A_{[n]\setminus C}^{[n]\setminus R}).$$

Let A be an $n\times(n-1)$ matrix with entries $A_{i,j} = a_{i,\alpha_j+n-j}$ and let B be an $n\times(n+1)$ matrix with entries $B_{i,j} = a_{i,\beta_j+n-j+2}$. Consider the $2n \times 2n$ matrix given by

$$M = \begin{pmatrix} A & B \\ 0 & B \end{pmatrix},$$

where 0 is an $n \times (n-1)$ matrix with all zero entries. Certainly $\operatorname{rank}(M) < 2n$, so $\det(M) = 0$. Performing the Laplace expansion on $\det(M)$ with $R = \{1,2,\cdots,n\}$ gives

$$(-1)^{\binom{n+1}{2}} \sum_{\substack{C\subseteq[2n] \\ |C|=n}} (-1)^{\sigma(C)} \det(M_C^{[n]})\det(M_{[2n]\setminus C}^{[2n]\setminus[n]}) = 0.$$

However, notice that $\det(M^{[2n]\backslash[n]}_{[2n]\backslash C}) = 0$ if $[n-1]\cap C \neq [n-1]$. Thus, C is uniquely determined by the single column index which is not in $[n-1]$. Now, suppose that $C = \{1,2,\cdots,n-1,k+n\}$ with $0 \le k \le n$. Then $\det(M^{[n]}_C)$ is the determinant of A with the column indexed by $k+1$ in B adjoined. In other words, $(M^{[n]}_C)_{i,j} = a_{i,\alpha_j+n-j}$ for $1 \le j \le n$ and $(M^{[n]}_C)_{i,n} = a_{i,\beta_{k+1}+n-k+1} = a_{i,(\beta_{k+1}+n-k)+1}$, or,

$$\det(M^{[n]}_C) = a_{(\alpha_1-1,\cdots,\alpha_{n-1}-1,\beta_{k+1}+n-k)}(n).$$

Similarly, $M^{[2n]\backslash[n]}_{[2n]\backslash C}$ is the matrix B with the column indexed by $k+1$ removed, so,

$$M^{[2n]\backslash[n]}_{[2n]\backslash C} = a_{(\beta_1+1,\cdots,\beta_k+1,\beta_{k+2},\cdots,\beta_{n+1})}.$$

Lastly, notice that

$$\sigma(C) = \sigma(R) + k,$$

so that

$$0 = \det(M) = \sum_{k=0}^{n}(-1)^k a_{(\alpha_1-1,\cdots,\alpha_{n-1}-1,\beta_{k+1}+n-k)} \cdot a_{(\beta_1+1,\cdots,\beta_k+1,\beta_{k+2},\cdots,\beta_{n+1})} = 0,$$

as required. $\square$

Now that we know that the determinantal minors described in Theorem 2.4.2 satisfy the Plücker relations we can use Lemma 2.0.1 to show that generating series whose Schur polynomial coefficients are matrix minors satisfy a family of differential equations. The main components of Theorem 2.4.6 below appear in [50].

Theorem 2.4.6. *Suppose that for some fixed positive integer n,*

$$G(\mathbf{p}) = \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le n}} a_\lambda(n) s_\lambda(\mathbf{p}),$$

where

$$a_\lambda(n) = \det_{1\le i,j\le n}(a_{i,\lambda_j+n-j+1}),$$

and each $a_{i,j} \in R$. Then for each pair of partitions α and β, $G(\mathbf{p})$ satisfies the differential equation,

$$\sum_{k=0}^{m}(-1)^k s^\perp_{(\alpha_1-1,\cdots,\alpha_{m-1}-1,\beta_{k+1}+m-k)}(\mathbf{p})G(\mathbf{p}) \cdot s^\perp_{(\beta_1+1,\cdots,\beta_k+1,\beta_{k+2},\cdots,\beta_{m+1})}(\mathbf{q})G(\mathbf{q}) = 0,$$

where $m = \max\{\ell(\alpha)+1, \ell(\beta)-1, 2\}$. In particular, $G(\mathbf{p})$ satisfies the KP hierarchy.

Proof. Using Theorem 2.4.2 we know that

$$G(\mathbf{p}+\mathbf{q}) = \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le n}} \widetilde{a}_\lambda(n) s_\lambda(\mathbf{p}),$$

where

$$\widetilde{a}_\lambda(n) = \det_{1\le i,j\le n}(\widetilde{a}_{i,\lambda_j+n-j+1}),$$

and

$$\widetilde{a}_{i,j} = \sum_{k\ge 0} a_{i,k} h_{k-j}(\mathbf{q}).$$

Using Proposition 2.4.5 we know that the coefficients $\widetilde{a}_\lambda(n)$ satisfy the Plücker relations and so, using Lemma 2.0.1, we know that for each pair of partitions α and β, $G(\mathbf{p})$ satisfies

$$\sum_{k=0}^{m}(-1)^k s^{\perp}_{(\alpha_1-1,\cdots,\alpha_{m-1}-1,\beta_{k+1}+m-k)}(\mathbf{p})G(\mathbf{p})\cdot s^{\perp}_{(\beta_1+1,\cdots,\beta_k+1,\beta_{k+2},\cdots,\beta_{m+1})}(\mathbf{q})G(\mathbf{q}) = 0.$$

To see that this is equivalent to the KP hierarchy first note that by re-arranging the entries in the sequence $(\alpha_1-1,\cdots,\alpha_{m-1}-1,\beta_{k+1}+m-k)$ there must exist indices i and j so that

$$(\alpha_1-1,\cdots,\alpha_{m-1}-1,\beta_{k+1}+m-k) = \Delta_{m-1}\cdots\Delta_{j+1}\Delta_j(\alpha\uparrow i).$$

Also, $(\beta_1+1,\cdots,\beta_k+1,\beta_{k+2},\cdots,\beta_{m+1}) = \beta\downarrow(k+1)$ by definition. Since $|\Delta_j\lambda| = |\lambda|$ for any partition λ we can compute,

$$|(\alpha_1-1,\cdots,\alpha_{m-1}-1,\beta_{k+1}+m-k)| = |\alpha\uparrow i| = |\alpha| - (m-1) + \beta_{k+1} + m - k.$$

Similarly,

$$|(\beta_1+1,\cdots,\beta_k+1,\beta_{k+2},\cdots,\beta_{m+1})| = |\beta\downarrow(k+1)| = |\beta| - \beta_{k+1} + k.$$

Thus,

$$\begin{aligned}|(\alpha_1-1,\cdots,\alpha_{m-1}-1,\beta_{k+1}+m-k)| + |(\beta_1+1,\cdots,\beta_k+1,\beta_{k+2},\cdots,\beta_{m+1})|\\ = |\alpha\uparrow i| + |\beta\downarrow(k+1)| = |\alpha|+|\beta|+1.\end{aligned}$$

This allows us to rewrite the differential equations which $G(\mathbf{p})$ satisfies as

$$\sum_{i,k}(-1)^{m-j+k}(s^{\perp}_{\alpha\uparrow i}(\mathbf{p})G(\mathbf{p}))(s^{\perp}_{\beta\downarrow(k+1)}(\mathbf{p})G(\mathbf{p})) = 0,$$

where the sum is over all integers $i,k\ge 1$ such that $|\alpha\uparrow i| + |\beta\downarrow(k+1)| = |\alpha|+|\beta|+1$. Lastly, by multiplying the equation above by $(-1)^m$ and re-writing $(-1)^{k-j}$ as $(-1)^{k+j}$ we may use the fact that $|\alpha\uparrow i| = |\alpha| - j + i - 1$ and get (shifting the index k by one),

$$\sum_{i,k}(-1)^{|\alpha|-|\alpha\uparrow i|+i+k}(s^{\perp}_{\alpha\uparrow i}(\mathbf{p})G(\mathbf{p}))(s^{\perp}_{\beta\downarrow k}(\mathbf{p})G(\mathbf{p})) = 0,$$

which by Theorem 2.2.4 part (iv) means that $G(\mathbf{p})$ satisfies the KP hierarchy. $\square$

Example 2.4.7. *Recall from Example 2.4.4 that the Plücker relation arising from the partitions $\alpha = \epsilon$ and $\beta = (1,1,1)$ is given by*

$$a_{(2)} \cdot a_{(1,1)} - a_{(1)} \cdot a_{(2,1)} + a_\epsilon a_{(2,2)} = 0.$$

Using the fact that

$$s_\epsilon = 1, \qquad s_{(1)} = p_1, \qquad s_{(2)} = \frac{1}{2}(p_1^2 + p_2), \qquad s_{(1,1)} = \frac{1}{2}(p_1^2 - p_2),$$

$$s_{(2,1)} = \frac{1}{3}(p_1^3 - p_3), \qquad s_{(2,2)} = \frac{1}{12}(p_1^4 - 4p_1p_3 + 3p_2^2),$$

Theorem 2.4.6 implies that if $G(\mathbf{p})$ has the form described in the statement of the theorem then it satisfies the differential equation,

$$\frac{1}{12}G(\mathbf{p})\left(\frac{\partial^4 G(\mathbf{p})}{\partial p_1^4} - 12\frac{\partial^2 G(\mathbf{p})}{\partial p_1 \partial p_3} + 12\frac{\partial^2 G(\mathbf{p})}{\partial^2 p_2}\right) - \frac{1}{3}\frac{\partial G(\mathbf{p})}{\partial p_1}\left(\frac{\partial^3 G(\mathbf{p})}{\partial p_1^3} - 3\frac{\partial G(\mathbf{p})}{\partial p_3}\right)$$
$$+ \frac{1}{4}\left(\frac{\partial^2 G(\mathbf{p})}{\partial p_1^2} + 2\frac{\partial G(\mathbf{p})}{\partial p_2}\right)\left(\frac{\partial^2 G(\mathbf{p})}{\partial p_1^2} - 2\frac{\partial G(\mathbf{p})}{\partial p_2}\right) = 0,$$

which is the same equation as the one found in Example 2.2.5 which led to the KP equation.

2.4.1 Matrix Integrals

One of the ways in which series of the type described in Theorem 2.4.6 arise is via the study of matrix integrals. In particular, for the KP hierarchy, we consider the iterated integral operator given by

$$\langle f(\mathbf{x})\rangle_n = \frac{1}{n!}\int_{\mathbb{R}^n} f(\mathbf{x})|a_\delta(\mathbf{x})|^2 \prod_{j=1}^{n} w(x_j)dx_1 \cdots dx_n,$$

where n is a positive integer and $w(x)$ is a weight function suitably normalized so that $\langle 1\rangle_n = 1$. Also, recall that

$$a_\lambda(\mathbf{x}) = \det_{1\le i,j,\le n}(x_i^{\lambda_j}),$$

and that $\delta = (n-1, n-2, \cdots, 0)$ so that $a_\delta(\mathbf{x})$ is the Vandermonde determinant.

We will also make use of the following well known integration theorem which is essentially the continuous analog of the Cauchy-Binet theorem.

Theorem 2.4.8. *Suppose n is a positive integer, ϕ_i and ψ_i are functions for $1 \le i \le n$ and $w(x)$ is a weight function. Then*

$$\frac{1}{n!}\int_{\mathbb{R}^n} \det_{1\le i,j\le n}(\phi_i(x_j)) \det_{1\le i,j\le n}(\psi_i(x_j)) \prod_{i=1}^{n} w(x_i)dx_1\cdots dx_n = \det_{1\le i,j\le n}(a_{i,j}),$$

where,

$$a_{i,j} = \int_{\mathbb{R}} \phi_i(x)\psi_j(x)w(x)dx.$$

Proof. We see that

$$\begin{aligned}
&\frac{1}{n!}\int_{\mathbb{R}^n}\det_{1\le i,j\le n}(\phi_i(x_j))\det_{1\le i,j\le n}(\psi_i(x_j))\prod_{i=1}^n w(x_i)dx_1\cdots dx_n \\
&\quad = \frac{1}{n!}\sum_{\sigma,\rho\in S_n}\int_{\mathbb{R}^n}\prod_{i=1}^n(\phi_{\sigma(i)}(x_i)\psi_{\rho(i)}(x_i)w(x_i))dx_1\cdots dx_n, \\
&\quad = \frac{1}{n!}\sum_{\sigma,\rho\in S_n}\prod_{i=1}^n a_{\sigma(i),\rho(i)}, \\
&\quad = \det_{1\le i,j\le n}(a_{i,j}).
\end{aligned}$$

□

Using Theorem 2.4.8 we can now describe the action of the operator $\langle\cdot\rangle_n$ on Schur functions.

Proposition 2.4.9. *For any positive integer n let $\mathbf{x} = \{x_1,\dots,x_n\}$ and suppose partition $\lambda \in \mathcal{P}$ is such that $\ell(\lambda) \le n$. Then*

$$\langle s_\lambda(\mathbf{x})\rangle_n = \det_{1\le i,j\le n}(a_{i,\lambda_j+n-j+1}),$$

where

$$a_{i,j} = \int_{\mathbb{R}} x^{n+j-i-1}w(x)dx.$$

In particular, if

$$G(\mathbf{x},\mathbf{y}) = \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le n}} s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}),$$

then

$$\langle G(\mathbf{x},\mathbf{y})\rangle_n = \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le n}} a_\lambda(n)s_\lambda(\mathbf{y}),$$

where

$$a_\lambda(n) = \det_{1\le i,j\le n}(a_{i,\lambda_j+n-j+1}).$$

Proof. We have,

$$\begin{aligned}
\langle s_\lambda(\mathbf{x})\rangle_n &= \frac{1}{n!}\int_{\mathbb{R}^n} s_\lambda(\mathbf{x})a_\delta(\mathbf{x})^2\prod_{j=1}^n w(x_j)dx_1\cdots dx_n, \\
&= \frac{1}{n!}\int_{\mathbb{R}^n}\frac{a_{\lambda+\delta}(\mathbf{x})}{a_\delta(\mathbf{x})}a_\delta(\mathbf{x})^2\prod_{j=1}^n w(x_j)dx_1\cdots dx_n, \\
&= \frac{1}{n!}\int_{\mathbb{R}^n} a_{\lambda+\delta}(\mathbf{x})a_\delta(\mathbf{x})\prod_{j=1}^n w(x_j)dx_1\cdots dx_n.
\end{aligned}$$

Applying Theorem 2.4.8 with $\phi_i(x) = x^{\lambda_i+n-i}$ and $\psi_i(x) = x^{n-i}$ then gives the result.

□

Proposition 2.4.9 gives a class of examples of generating series which are solutions to the KP hierarchy. We will discuss an example of this type of series in Chapter 3 when we consider rooted maps on orientable surfaces.

2.4.2 Content-Type Series

In addition to the generating series of the type described above arising from matrix integrals, another class of generating series can be described which is more combinatorial in appearance called content-type series. First, recall that given a cell (i,j) in a partition λ, the *content* of the cell (i,j) is $j-i$. Suppose y_k for $k \in \mathbb{Z}$ is a family of indeterminates, then series of the form

$$\sum_{\lambda \in \mathcal{P}} \prod_{(i,j)\in\lambda} y_{j-i} s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q}),$$

are called *content-type series*. For more examples of content-type series as they arise in enumeration, see [14] and [50].

Theorem 2.4.10. *Suppose n is a positive integer and λ is a partition with $\ell(\lambda) \le n$, then*

$$\prod_{(i,j)\in\lambda} y_{j-i} s_\lambda(\mathbf{q}) = c_n \det_{1\le i,j\le n}(a_{i,\lambda_j+n-j+1}),$$

where

$$c_n = \prod_{j=1}^{n}\prod_{i=1}^{j-1} y_{j-i},$$

and

$$a_{i,j} = \begin{cases} y_{j-n-1}\cdots y_0 h_{i+j-n-1}(\mathbf{q}), & \text{if } j-n-1 \ge 0,\\ \frac{1}{y_{j-n}\cdots y_1} h_{i+j-n-1}(\mathbf{q}), & \text{if } j-n-1 < 0.\end{cases}$$

Proof. Let

$$\phi_j = \begin{cases} y_j \cdots y_0, & \text{if } j \ge 0,\\ \frac{1}{y_{j+1}\cdots y_1}, & \text{if } j < 0.\end{cases}$$

Then

$$\begin{aligned}\det_{1\le i,j\le n}(a_{i,\lambda_j+n-j+1}) &= \prod_{j=1}^{n} \phi_{\lambda_j-j} \det(h_{\lambda_j+i-j}(\mathbf{q})),\\ &= \prod_{j=1}^{n} \phi_{\lambda_j-j} s_\lambda(\mathbf{q}),\end{aligned}$$

using the Jacobi-Trudi identity. Now, for some $j \in \{1,\cdots,n\}$ suppose $\lambda_j - j \ge 0$. Then

$$\begin{aligned}\prod_{i=1}^{j-1} y_{-i}\phi_{\lambda_j-j} &= y_{-j+1}y_{-j+2}\cdots y_{-1}y_0y_1\cdots y_{\lambda_j-j},\\ &= \prod_{i=0}^{\lambda_j-1} y_{\lambda_j-j-i},\end{aligned}$$

which is the product of y_{j-i} over the cells in the jth row of λ. If $\lambda_j - j < 0$ then

$$\begin{aligned}\prod_{i=1}^{j-1} y_{-i}\phi_{\lambda_j - j} &= \frac{\prod_{i=1}^{j-1} y_{-i}}{y_{\lambda_j - j+1}\cdots y_1},\\ &= y_{-j+1}y_{-j+2}\cdots y_{\lambda_j - j},\\ &= \prod_{i=0}^{\lambda_j - 1} y_{\lambda_j - j - i},\end{aligned}$$

which again is the product of the y_{j-i} over the cells in the jth row of λ. Thus,

$$c_n \prod_{j=1}^{n} \phi_{\lambda_j - j} = \prod_{(i,j)\in\lambda} y_{j-i},$$

and the result follows. □

Theorem 2.4.10 above tells us that content-type series satisfy the KP hierarchy with respect to the indeterminates $\mathbf{p}$ which we will record now as a Corollary.

Corollary 2.4.11. *Content-type series satisfy the KP hierarchy in the indeterminates* $\mathbf{p}$.

Proof. This follows from Theorem 2.4.6 and Theorem 2.4.10 above along with the fact that the KP equations are homogeneous and so if $G(\mathbf{p})$ is a solution, so is $cG(\mathbf{p})$ for any constant c. □

We will make use of Corollary 2.4.11 in Chapter 5 when we examine the monotone Hurwitz problem.

2.5 Pfaffian Series and the BKP Hierarchy

In addition to series whose Schur coefficients are determinantal minors we may also consider series whose Schur coefficients are Pfaffian minors. In practice, this change of perspective allows us to analyze generating series corresponding to locally orientable structures rather than the orientable ones which arise in the determinantal case. An example of this will be considered in Chapter 4 where we look at rooted cubic maps on locally orientable surfaces.

Given a skew-symmetric matrix, its determinant is the square of a polynomial in its entries. This polynomial is called the Pfaffian of the matrix. For our purposes it will be most useful to take the following recursive definition of the Pfaffian of a matrix. Given a skew symmetric pairing $(i,j) \in R$ for all $i,j \geq 1$ (that is, $(i,j) = -(j,i)$) we will define the *Pfaffian* of an even length sequence of positive integers $(a_1, a_2, \ldots, a_{2n})$ as $pf(a_1,a_2) = (a_1,a_2)$ with $a_1 < a_2$ and for $n > 1$,

$$\mathrm{pf}(a_1,a_2,\ldots,a_{2n}) = \sum_{i=1}^{2n} (-1)^i \mathrm{pf}(a_1,a_i)\mathrm{pf}(a_2,a_3,\ldots,\widehat{a_i},\ldots,a_{2n}),$$

where $\widehat{a_i}$ means that the index a_i has been omitted. A fundamental property of Pfaffians is that exchanging two adjacent indices results in changing the sign of the Pfaffian. That is,

$$\mathrm{pf}(a_1, \dots, a_i, a_{i+1}, \dots, a_{(2n)}) = -\mathrm{pf}(a_1, \dots, a_{i+1}, a_i, \dots, a_{(2n)}).$$

This property also implies that if there is a repeated index then the corresponding Pfaffian is zero. We may also use this property to write the recursive definition of a Pfaffian in the following, equivalent, way. For any $1 \le k \le 2n$,

$$\mathrm{pf}(a_1, a_2, \dots, a_{2n}) = \sum_{i=1}^{2n} (-1)^{i+k-1} \mathrm{pf}(a_k, a_i) pf(a_1, \dots, \widehat{a_i}, \dots, \widehat{a_k} \dots, a_{2n}).$$

Using this notation the underlying matrix is left implicit, however, sometimes we will want to make the matrix explicit. Suppose Q is a $(2n) \times (2n)$ skew symmetric matrix. We may write

$$\mathrm{pf}(Q) = \mathrm{pf}(1, 2, \dots, 2n),$$

where the Pfaffian above is with respect to the matrix Q. Here the skew symmetric matrix Q is made explicit, however, the indices with which we are taking the Pfaffian are now implicit.

Often in this Thesis we will describe a skew-symmetric submatrix of a skew-symmetric matrix by giving the set of row (and thus corresponding column) indices. Since the order that the rows and columns appear in the submatrix matter when computing the Pfaffian we will always take the indices to be in increasing order.

We now recall the *minor summation formula* for Pfaffians which was proven by Ishikawa and Wakayama in [61]. This will be of use when we prove Theorem 2.4.2 in the Pfaffian case.

Theorem 2.5.1. *Suppose n and N are even integers with $n \le N$, B is an $N \times N$ skew symmetric matrix and T is an $n \times N$ matrix. Then*

$$\sum_{\substack{I \subseteq [N] \\ |I| = n}} \mathrm{pf}(B_I^I) \det(T_I^{[n]}) = \mathrm{pf}(Q),$$

where Q is the $n \times n$ skew symmetric matrix given by $Q = TBT^t$, i.e.,

$$Q_{i,j} = \sum_{1 \le k < \ell \le N} b_{k,\ell} \det(T_{k,\ell}^{i,j}).$$

Using the minor summation formula we can now show that, as in the determinantal case considered in Theorem 2.4.2, a series whose Schur coefficients are Pfaffian minors maintains this property under translation.

Theorem 2.5.2. *Let B be an $\infty \times \infty$ skew-symmetric matrix with entries in R and with a special row and column indexed by 0 (that is, B has row and column indices* *given by $\{0,1,2,\ldots\}$). Let $G(\mathbf{p})$ be a generating series in $R[[\mathbf{p}]]$ and suppose for some positive integer n the generating series $G(\mathbf{p})$ can be written as*

$$G(\mathbf{p}) = \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\le n}} b_\lambda(n) s_\lambda(\mathbf{p}),$$

where

$$b_\lambda(n) = \mathrm{pf}(B_\lambda),$$

and B_λ is the submatrix of the skew-symmetric matrix B with row and column indices given by the set $\{\lambda_1+n, \lambda_2+n-1, \ldots, \lambda_n+1\}$ if n is even and the set $\{\lambda_1+n, \lambda_2+n-1, \ldots, \lambda_n+1, 0\}$ if n is odd. Then

$$G(\mathbf{p}+\mathbf{q}) = \sum_\lambda \widetilde{b}_\lambda(n) s_\lambda(\mathbf{p}),$$

where

$$\widetilde{b}_\lambda(n) = \mathrm{pf}(\widetilde{B}_\lambda),$$

and $\widetilde{B}_\lambda$ is the submatrix of the skew-symmetric matrix $\widetilde{B}$ with row and column indices given by the set $\{\lambda_1+n, \lambda+2+n-1, \ldots, \lambda_n+1\}$ if n is even and the set $\{\lambda_1+n, \lambda_2+n-1, \ldots, \lambda_n+1, 0\}$ if n is odd. Furthermore, the matrix $\widetilde{B}$ has entries given by

$$\left(\widetilde{B}_\lambda\right)_{i,j} = \begin{cases} \sum_{1\le k,\ell\le\infty} B_{k,\ell}(h_{k-i}(\mathbf{q})h_{\ell-j}(\mathbf{q}) - h_{\ell-i}(\mathbf{q})h_{k-j}(\mathbf{q})), & \text{if } i,j\in\{1,2,\ldots\},\\ \sum_{1\le\ell\le\infty} B_{0,\ell} h_{\ell-(\mu_j+n-j+1)}(\mathbf{q}), & \text{if } i=0 \text{ and } j\in\{1,2,\ldots\},\\ -\sum_{1\le\ell\le\infty} B_{0,\ell} h_{\ell-(\mu_i+n-i+1)}(\mathbf{q}), & \text{if } j=0 \text{ and } i\in\{1,2,\ldots\},\\ 0, & \text{if } i=j=0. \end{cases}$$

Proof. As in the proof of Theorem 2.4.2, first recall that

$$s_\lambda(\mathbf{p}+\mathbf{q}) = \sum_\mu s_{\lambda/\mu}(\mathbf{p}) s_\mu(\mathbf{q}),$$

so that after some manipulation,

$$G(\mathbf{p}+\mathbf{q}) = \sum_\mu s_\mu(\mathbf{q}) \sum_\lambda b_\lambda(n) s_{\lambda/\mu}(\mathbf{p}).$$

Now, suppose that S_μ is the $\infty\times n$ matrix with

$$(S_\mu)_{i,j} = \begin{cases} h_{i-(\mu_j+n-j+1)}(\mathbf{p}) & \text{if } i,j\in\{1,2,\ldots\},\\ 0 & \text{if } i \text{ or } j \text{ are equal to } 0,\\ 1 & \text{if } i=j=0. \end{cases}$$

Given a partition λ we may construct the set $R = \{\lambda_1+n, \lambda_2+n-1, \ldots, \lambda_n+1\}$ if n is even or $R = \{\lambda_1+n, \lambda_2+n-1, \ldots, \lambda_n+1, 0\}$ if n is odd so that $b_\lambda(n)s_{\lambda/\mu}(\mathbf{p}) = \mathrm{pf}(B_R^R)\det((S_\mu)_{[n]}^R)$. Thus,

$$\begin{aligned}\sum_\lambda b_\lambda(n) s_{\lambda/\mu}(\mathbf{p}) &= \sum_{\substack{R\subseteq\{1,2,\cdots\}\\ |R|=n}} \mathrm{pf}(B_R^R)\det((S_\mu)_{[n]}^R),\\ &= \mathrm{pf}(S_\mu^t B S_\mu),\end{aligned}$$

using the minor summation formula. Now, if $i, j \in \{1, 2, \dots\}$ then

$$\begin{aligned}
(S_\mu^t B S_\mu)_{i,j} &= \sum_{1\le \ell\le\infty} B_{0,\ell}\det((S_\mu)_{i,j}^{0,\ell}) + \sum_{1\le k<\ell\le\infty} B_{k,\ell}\det((S_\mu)_{i,j}^{k,\ell}),\\
&= \sum_{1\le k<\ell\le\infty} B_{k,\ell}\det((S_\mu)_{i,j}^{k,\ell}),\\
&= \sum_{1\le k<\ell\le\infty} B_{k,\ell}\big(h_{k-(\mu_i+n-i+1)}(\mathbf{p})h_{\ell-(\mu_j+n-j+1)}(\mathbf{p})\\
&\qquad\qquad -h_{\ell-(\mu_i+n-i+1)}(\mathbf{p})h_{k-(\mu_j+n-j+1)}(\mathbf{p})\big),\\
&= \widetilde{B}_{\mu_i+n-i+1,\mu_j+n-j+1}(n).
\end{aligned}$$

If $j \in \{1, 2, \dots\}$ then

$$\begin{aligned}
(S_\mu^t B S_\mu)_{0,j} &= \sum_{1\le \ell\le\infty} B_{0,\ell}\det((S_\mu)_{0,j}^{0,\ell}) + \sum_{1\le k<\ell\le\infty} B_{k,\ell}\det((S_\mu)_{0,j}^{k,\ell}),\\
&= \sum_{1\le \ell\le\infty} B_{0,\ell}\det((S_\mu)_{0,j}^{0,\ell}),\\
&= \sum_{1\le \ell\le\infty} B_{0,\ell}h_{\ell-(\mu_j+n-j+1)},\\
&= \widetilde{B}_{0,\mu_j+n-j+1}(n).
\end{aligned}$$

Since B is skew-symmetric, so is $S_\mu^t B S_\mu$ and so if $i \in \{1, 2, \dots\}$ then $(S_\mu^t B S_\mu)_{i,0} = -(S_\mu^t B S_\mu)_{0,i} = -\widetilde{B}_{0,\mu_i+n-i+1}$. Also since $S_\mu^t B S_\mu$ is skew-symmetric, $(S_\mu^t B S_\mu)_{0,0} = 0$ and so the result follows. □

Similar to the determinantal case, the coefficients $b_\lambda(n)$ appearing in Theorem 2.5.2 are Pfaffian minors of an $\infty \times \infty$ matrix B. Also similar to the determinantal case is the fact that these minors satisfy a family of algebraic identities which are similar to the Plücker relations.

Definition 2.5.3. *We consider a family $\{b_\lambda(n) \in R : n \in \{0, 1, 2, \dots\}, \lambda \in \mathcal{P}, \ell(\lambda) \le n\}$. As in the definition of the Plücker relations we will take the convention that given any sequence of non-negative integers $\alpha_1, \dots, \alpha_k$ with $k \le n$,*

$$b_{(\alpha_1,\alpha_2,\dots,\alpha_k)}(n) = 0 \ \ \text{if } \alpha_n < 0; \qquad b_{(\alpha_1,\alpha_2,\dots,\alpha_k)}(n) = b_{(\alpha_1,\alpha_2,\dots,\alpha_{k-1})}(n) \ \ \text{if } \alpha_k = 0.$$

Also, we take the convention that

$$b_{(\alpha_1,\alpha_2,\dots,\alpha_k)}(n) = -b_{\Delta_j(\alpha_1,\alpha_2,\dots,\alpha_k)}.$$

These two conventions allow us to write $b_{(\alpha_1,\alpha_2,\dots,\alpha_k)}(n)$ for any sequence.

Suppose n and m are non-negative integers and α and β are partitions where α and β have i and j parts respectively for some $i, j \ge 0$, $(i, j) \ne (0, 0)$ and $i \le n$, $j \le m$. Also, set $\alpha_{i+1} = \dots = \alpha_n = 0 = \beta_{j+1} = \dots = \beta_m$. We say that a family

$\{b_\lambda(n) \in R : n \in \{0,1,2,\ldots\}, \lambda \in \mathcal{P}, \ell(\lambda) \leq n\}$ *satisfies the Pfaffian minor identities if and only if it satisfies the equation*

$$\begin{aligned}
&\sum_{k=1}^{m}(-1)^k b_{(\alpha_1-1,\ldots,\alpha_n-1,\beta_k+m-k)}(n+1) b_{(\beta_1+1,\ldots,\beta_{k-1}-1,\beta_{k+1},\ldots,\beta_m)}(m-1) \\
&\quad + (-1)^{n+m}\sum_{k=1}^{n}(-1)^k b_{(\alpha_1+1,\ldots,\alpha_{k-1}+1,\alpha_{k+1},\ldots,\alpha_n)}(n-1) b_{(\beta_1-1,\ldots,\beta_m-1,\alpha_k+n-k)}(m+1) \\
&\qquad = \frac{1}{2}(1-(-1)^{n+m}) b_{(\alpha_1,\ldots,\alpha_n)}(n) b_{(\beta_1,\ldots,\beta_m)}(m).
\end{aligned}$$

for all non-negative integers n, m and each pair of partitions α and β with $\ell(\alpha) \leq n$ and $\ell(\beta) \leq m$.

Example 2.5.4. *Suppose $n = N-1, m = N+1, \alpha = \epsilon$ and $\beta = (1,1,1)$. Then the corresponding algebraic identity is given by*

$$-b_{(-1,3)}(N)b_{(1,1)}(N)+b_{(-1,2)}(N)b_{(2,1)}(N)-b_{(-1,1)}(N)b_{(2,2)}(N)-b_\epsilon(N-2)b_\epsilon(N+2) = 0.$$

Using the conventions in Definition 2.5.3 we see that, as in Example 2.4.4, $b_{(-1,3)}(N) = -b_{(2)}(N), b_{(-1,2)}(N) = -b_{(1)}(N)$ and $b_{(-1,1)}(N) = -b_\epsilon(N)$ so that the corresponding algebraic identity is

$$b_\epsilon(N-2)b_\epsilon(N+2) = b_{(2)}(N)b_{(1,1)}(N) - b_{(1)}(N)b_{(2,1)}(N) + b_\epsilon(N)b_{(2,2)}(N).$$

As in the determinantal case, we can now show that the family of Pfaffian minors satisfies the Pfaffian minor identities. The components of the proof of Proposition 2.5.5 appear in the paper [107] by Hirota and Tsujimoto. In [107] the authors are concerned with finding a family of solutions which satisfy the BKP equation, rather than the entire hierarchy.

Proposition 2.5.5. *Suppose B is a skew-symmetric matrix with row and column indices given by $\{0,1,2,\ldots\}$. Given a partition λ and a non-negative integer n with $\ell(\lambda) \leq n$ let B_λ be the submatrix of B with row and column indices given by $\{\lambda_1+n,\ldots,\lambda_n+1\}$ if n is even and $\{\lambda_1+n,\ldots,\lambda_n+1,0\}$ if n is odd. Then the family $\{b_\lambda(n) \in R : n \in \{0,1,2,\ldots\}, \lambda \in \mathcal{P}, \ell(\lambda) \leq n\}$ with*

$$b_\lambda(n) = \mathrm{pf}(B_\lambda),$$

satisfies the Pfaffian minor identities.

Proof. That the family $\{b_\lambda(n) \in R : n \in \{0,1,2,\ldots\}, \lambda \in \mathcal{P}, \ell(\lambda) \leq n\}$ follows the conventions described in the definition of the Pfaffian minor identities follows from the definition of $b_\lambda(n)$.

Consider the pairing $(i,j) = B_{i,j}$ for $i,j \in \{0,1,2,\ldots\}$. Recall the definition of a Pfaffian: given a sequence of indices $a_1, a_2, \ldots, a_{2n}$ with $n > 1$ and some $j \in \{1,2,\ldots\}$,

$$\mathrm{pf}(a_1,a_2,\ldots,a_{2n}) = \sum_{i=1}^{2n}(-1)^{i+j-1}\mathrm{pf}(a_i,a_j)\mathrm{pf}(a_1,\ldots,\widehat{a_i},\ldots,\widehat{a_j},\ldots,a_{2n}).$$

To begin, we will prove a general identity which is satisfied by Pfaffian minors. Suppose L and K are positive odd integers and that $a_1,\ldots,a_K,b_1,\ldots,b_L \in \{0,1,2,\ldots\}$. Then

$$\begin{aligned}
&\sum_{\ell=1}^{L}(-1)^{\ell}\mathrm{pf}(a_1,\ldots,a_K,b_\ell)\mathrm{pf}(b_1,\ldots,\widehat{b_\ell},\ldots,b_L)\\
&\quad=\sum_{\ell=1}^{L}(-1)^{\ell}\sum_{k=1}^{K}(-1)^{k-1}\mathrm{pf}(a_k,b_\ell)\mathrm{pf}(a_1,\ldots,\widehat{a_k},\ldots,a_K)\mathrm{pf}(b_1,\ldots,\widehat{b_\ell},\ldots,b_L),\\
&\quad=\sum_{k=1}^{K}(-1)^{k}\sum_{\ell=1}^{L}(-1)^{\ell-1}(-\mathrm{pf}(b_\ell,a_k))\mathrm{pf}(a_1,\ldots,\widehat{a_k},\ldots,a_K)\mathrm{pf}(b_1,\ldots,\widehat{b_\ell},\ldots,b_L),\\
&\quad=-\sum_{k=1}^{K}(-1)^{k}\mathrm{pf}(a_1,\ldots,\widehat{a_k},\ldots,a_K)\mathrm{pf}(b_1,\ldots,b_L,a_k).
\end{aligned} \tag{2.7}$$

Let n and m be non-negative integers and let α,β be partitions with $\ell(\alpha)\le n$ and $\ell(\beta)\le m$. We wish to show that

$$\begin{aligned}
&\sum_{k=1}^{m}(-1)^{k}b_{(\alpha_1-1,\ldots,\alpha_n-1,\beta_k+m-k)}(n+1)b_{(\beta_1+1,\ldots,\beta_{k-1}-1,\beta_{k+1},\ldots,\beta_m)}(m-1)\\
&\quad+(-1)^{n+m}\sum_{k=1}^{n}(-1)^{k}b_{(\alpha_1+1,\ldots,\alpha_{k-1}+1,\alpha_{k+1},\ldots,\alpha_n)}(n-1)b_{(\beta_1-1,\ldots,\beta_m-1,\alpha_k+n-k)}(m+1)\\
&\qquad=\frac{1}{2}(1-(-1)^{n+m})b_{(\alpha_1,\ldots,\alpha_n)}(n)b_{(\beta_1,\ldots,\beta_m)}(m),
\end{aligned} \tag{2.8}$$

which will prove that the family $\{b_\lambda(n)\in R : n\in\{0,1,2,\ldots\},\lambda\in\mathcal{P},\ell(\lambda)\le n\}$ satisfies the Pfaffian minor identities.

If n and m are odd, consider equation (2.7) with $K=n$, $L=m$, $a_i=\alpha_i+n-i+1$ and $b_i=\beta_i+m-i+1$. Then

$$\begin{aligned}
&\mathrm{pf}(a_1,\ldots,a_K,b_\ell)\\
&\qquad=\mathrm{pf}(\alpha_1+n,\alpha_2+n-1,\ldots,\alpha_n+1,\beta_\ell+m-\ell+1),\\
&\qquad=\mathrm{pf}((\alpha_1-1)+(n+1),(\alpha_2-1)+(n+1)-1,\ldots,(\alpha_n-1)+2,\beta_\ell+m-\ell+1),\\
&\qquad=b_{(\alpha_1-1,\ldots,\alpha_n-1,\beta_\ell+m-\ell)}(n+1),\\
&\mathrm{pf}(b_1,\ldots,\widehat{b_\ell},\ldots,b_L)\\
&\qquad=\mathrm{pf}(\beta_1+m,\ldots,\beta_{\ell-1}+m-(\ell-1)+1,\beta_{\ell+1}+m-(\ell+1)+1,\ldots,\beta_m+1),\\
&\qquad=b_{(\beta_1+1,\ldots,\beta_{\ell-1}+1,\beta_{\ell+1},\ldots,\beta_m)}(m-1),\\
&\mathrm{pf}(a_1,\ldots,\widehat{a_k},\ldots,a_K)\\
&\qquad=b_{(\alpha_1+1,\ldots,\alpha_{k-1}+1,\alpha_{k+1},\ldots,\alpha_n)}(n-1),\\
&\mathrm{pf}(b_1,\ldots,b_L,a_k)\\
&\qquad=b_{(\beta_1-1,\ldots,\beta_m-1,\alpha_k+n-k)}(m+1).
\end{aligned}$$

Substituting these into equation (2.7) then gives the identity (2.8) in the case that n and m are odd.

If n and m are even, consider equation (2.7) with $K = n+1$, $L = m+1$, $a_i = \alpha_i + n - i + 1$ for $i \le n$, $a_{n+1} = 0$, $b_i = \beta_i + m - i + 1$ for $i \le m$ and $b_{m+1} = 0$. Then,

$$\mathrm{pf}(a_1, \ldots, a_K, b_\ell) = \begin{cases} b_{(\alpha_1 - 1, \ldots, \alpha_n - 1, \beta_\ell + m - \ell)}(n+1), & \text{if } \ell \neq m+1, \\ 0 & \text{if } \ell = m+1, \end{cases}$$

where the 0 above comes from the fact that $a_K = b_{m+1} = 0$. Since $\mathrm{pf}(a_1, \ldots, a_K, b_{m+1}) = 0$ we need only compute $\mathrm{pf}(b_1, \ldots, \widehat{b_\ell}, \ldots, b_L)$ in the case that $\ell < m+1$. In this case

$$\mathrm{pf}(b_1, \ldots, \widehat{b_\ell}, \ldots, b_L) = b_{\beta_1+1, \ldots, \beta_{\ell-1}+1, \beta_{\ell+1}, \ldots, \beta_m)}(m-1).$$

Similarly,

$$\mathrm{pf}(b_1, \ldots, b_L, a_k) = \begin{cases} b_{(\beta_1 - 1, \ldots, \beta_m - 1, \alpha_k + n - k)}(m+1), & \text{if } k \neq n+1, \\ 0 & \text{if } k = n+1, \end{cases}$$

and in the case that $k \neq n+1$,

$$\mathrm{pf}(a_1, \ldots, \widehat{a_k}, \ldots, a_K) = b_{(\alpha_1+1, \ldots, \alpha_{k-1}+1, \alpha_{k+1}, \ldots, \alpha_n)}(n-1).$$

Again, substituting these into equation (2.7) gives identity (2.8) when n and m are even.

Lastly suppose that, without loss of generality, n is even and m is odd. Consider equation (2.7) with $K = n+1$, $L = m$, $a_i = \alpha_i + n - i + 1$ for $i \le n$, $a_{n+1} = 0$ and $b_i = \beta_i + m - i + 1$. Then,

$$\begin{aligned} \mathrm{pf}(a_1, \ldots, a_K, b_\ell) &= \mathrm{pf}(\alpha_1 + n, \ldots, \alpha_n + 1, 0, \beta_\ell + m - \ell + 1), \\ &= -\mathrm{pf}(\alpha_1 + n, \ldots, \alpha_n + 1, \beta_\ell + m - \ell + 1, 0), \\ &= -b_{(\alpha_1 - 1, \ldots, \alpha_n - 1, \beta_\ell + m - \ell)}(n+1), \end{aligned}$$

$$\mathrm{pf}(b_1, \ldots, \widehat{b_\ell}, \ldots, b_L) = b_{(\beta_1+1, \ldots, \beta_{\ell-1}+1, \beta_{\ell+1}, \ldots, \beta_m)}(m-1),$$

$$\mathrm{pf}(a_1, \ldots, \widehat{a_k}, \ldots, a_K) = \begin{cases} b_{(\alpha_1+1, \ldots, \alpha_{k-1}+1, \alpha_{k+1}, \ldots, \alpha_n)}(n-1), & \text{if } k \neq n+1, \\ b_{(\alpha_1, \ldots, \alpha_n)}(n), & \text{if } k = n+1, \end{cases}$$

$$\mathrm{pf}(b_1, \ldots, b_L, a_k) = \begin{cases} b_{(\beta_1 - 1, \ldots, \beta_m - 1, \alpha_k + n - k)}(m+1), & \text{if } k \neq n+1, \\ b_{(\beta_1, \ldots, \beta_m)}(m), & \text{if } k = n+1. \end{cases}$$

As in the previous two cases, substituting these into equation (2.7) gives identity (2.8) in the case that n is even and m is odd, giving the desired result. □

Now, as in the determinant case, we can use the fact that Pfaffian minors satisfy the Pfaffian minor identities to show that a family of generating series whose coefficients are Pfaffian minors satisfy a family of differential equations. In particular, that they satisfy the BKP hierarchy.

Theorem 2.5.6. *Suppose B is a skew-symmetric matrix with row and column indices given by $\{0,1,2,\ldots\}$. Given a partition λ and a non-negative integer n with $\ell(\lambda) \leq n$ let B_λ be the submatrix of B with row and column indices given by $\{\lambda_1+n,\ldots,\lambda_n+1\}$ if n is even and $\{\lambda_1+n,\ldots,\lambda_n+1,0\}$ if n is odd. If $\{G_n(\mathbf{p}) : n \in \{0,1,2,\ldots\}\}$ is a family of generating series such that for each non-negative integer n,*

$$G_n(\mathbf{p}) = \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\leq n}} b_\lambda(n) s_\lambda(\mathbf{p}),$$

with $b_\lambda(n) = \mathrm{pf}(B_\lambda)$ then the family of generating series satisfies the differential equations given by

$$\begin{aligned}
&\sum_{k=1}^{m}(-1)^k s^\perp_{(\alpha_1-1,\ldots,\alpha_n-1,\beta_k+m-k)}(\mathbf{p})G_{n+1}(\mathbf{p})s^\perp_{(\beta_1+1,\ldots,\beta_{k-1}-1,\beta_{k+1},\ldots,\beta_m)}(\mathbf{q})G_{m-1}(\mathbf{q})\\
&\quad+(-1)^{n+m}\sum_{k=1}^{n}(-1)^k s^\perp_{(\alpha_1+1,\ldots,\alpha_{k-1}+1,\alpha_{k+1},\ldots,\alpha_n)}(\mathbf{p})G_{n-1}(\mathbf{p})s^\perp_{(\beta_1-1,\ldots,\beta_m-1,\alpha_k+n-k)}(\mathbf{q})G_{m+1}(\mathbf{q})\\
&\qquad= \frac{1}{2}(1-(-1)^{n+m})s^\perp_{(\alpha_1,\ldots,\alpha_n)}(\mathbf{p})G_n(\mathbf{p})s^\perp_{(\beta_1,\ldots,\beta_m)}(\mathbf{q})G_m(\mathbf{q}),
\end{aligned}$$

for all non-negative integers n,m and each pair of partitions α and β with $\ell(\alpha)\leq n$ and $\ell(\beta)\leq m$. In particular they satisfy the BKP hierarchy.

Proof. This proof proceeds in almost exactly the same way as the proof of Theorem 2.4.6. That is, we use Theorem 2.5.2 to show that $G_n(\mathbf{p}+\mathbf{q})$ has coefficients which satisfy the Pfaffian minor identitites and then we apply Lemma 2.0.1. That these differential equations are equivalent to the BKP hierarchy also follows in much the same way as in the proof of Theorem 2.4.6 by writing the modifications done to the indices in terms of the ↑ and ↓ operations on partitions. □

Example 2.5.7. *Recall from Example 2.5.4 that the Pfaffian minor identity arising from $n = N-1, m = N+1, \alpha = \epsilon$ and $\beta = (1,1,1)$ is given by*

$$b_\epsilon(N-2)b_\epsilon(N+2) = b_{(2)}(N)b_{(1,1)}(N) - b_{(1)}(N)b_{(2,1)}(N) + b_\epsilon(N)b_{(2,2)}(N).$$

Using the fact that

$$s_\epsilon = 1,\qquad s_{(1)} = p_1,\qquad s_{(2)} = \frac{1}{2}(p_1^2+p_2),\qquad s_{(1,1)} = \frac{1}{2}(p_1^2-p_2),$$

$$s_{(2,1)} = \frac{1}{3}(p_1^3-p_3),\qquad s_{(2,2)} = \frac{1}{12}(p_1^4-4p_1p_3+3p_2^2),$$

Theorem 2.5.6 implies that if $G_n(\mathbf{p})$ is of the form specified in the statement of the theorem then it satisfies the differential equation,

$$\begin{aligned} G_{N+2}(\mathbf{p})G_{N-2}(\mathbf{p}) = \frac{1}{12}G_N(\mathbf{p})\left(\frac{\partial^4 G_N(\mathbf{p})}{\partial p_1^4} - 12\frac{\partial^2 G_N(\mathbf{p})}{\partial p_1 \partial p_3} + 12\frac{\partial^2 G_N(\mathbf{p})}{\partial^2 p_2}\right) \\ - \frac{1}{3}\frac{\partial G_N(\mathbf{p})}{\partial p_1}\left(\frac{\partial^3 G_N(\mathbf{p})}{\partial p_1^3} - 3\frac{\partial G_N(\mathbf{p})}{\partial p_3}\right) \\ + \frac{1}{4}\left(\frac{\partial^2 G_N(\mathbf{p})}{\partial p_1^2} + 2\frac{\partial G_N(\mathbf{p})}{\partial p_2}\right)\left(\frac{\partial^2 G_N(\mathbf{p})}{\partial p_1^2} - 2\frac{\partial G_N(\mathbf{p})}{\partial p_2}\right), \end{aligned}$$

which is the same equation as the one found in Example 2.2.8 which led to the BKP equation.

Remark. It is important to note that Theorem 2.5.6 is distinctly different from Theorem 2.4.6 in that in the determinantal case we only need a single generating series. In the Pfaffian case, however, we need a family of generating series indexed by a non-negative integer where the coefficients of each of the generating series are Pfaffian minors of a single matrix.

2.5.1 Matrix Integrals

As in the determinantal case, one of the ways in which series of the type described in Theorem 2.5.6 arise is via matrix integrals. In fact, for the purposes of this thesis, matrix integrals will be the sole source for solutions to the Pfaffian minor identitites. Consider the matrix integral operator given by

$$\langle\!\langle f(\mathbf{x}) \rangle\!\rangle_n = \frac{1}{n!}\int_{\mathbb{R}^n} f(\mathbf{x})|a_\delta(\mathbf{x})|\prod_{j=1}^{n} w(x_j)dx_1 \cdots dx_n,$$

where n is a positive integer and $w(x)$ is a weight function. The variant of Theorem 2.4.8 which applies to iterated integrals of the form defined above (where the absolute value of the Vandermonde determinant appears rather than its square) is called de Bruijn's [26] thoerem and is as follows.

Theorem 2.5.8. *Let A be the $n \times n$ matrix with entries*

$$A_{i,j} = \int_{\mathbb{R}^2} \phi_i(x)\phi_j(y)\mathrm{sgn}(y-x)w(x)w(y)dxdy,$$

where $w(x)$ is a weight function and $\phi_i(x), i = 1 \cdots n$ are functions. If $n = 2m+1$ is odd, add an additional row and column to A with index 0 so that

$$A_{0,i} = -A_{i,0} = \int_{\mathbb{R}} \phi_i(x)w(x)dx, \qquad i = 1 \cdots 2m+1,$$

and $A_{0,0} = 0$. Then

$$\frac{1}{n!}\int_{\mathbb{R}^n} |\det_{1\le i,j\le n}(\phi_i(x_j))|\prod_{j=1}^{n} w(x_j)dx_1 \cdots dx_n = \mathrm{pf}(A).$$

Using Theorem 2.5.8 we can describe the action of the operator $\langle\!\langle \cdot \rangle\!\rangle_n$ on Schur functions.

Proposition 2.5.9. *Let B be an $\infty \times \infty$ matrix with row and column indices given by $\{0, 1, 2, \ldots\}$. For $i, j \in \{1, 2, \ldots\}$ let*

$$\begin{aligned} B_{i,j} &= \int_{\mathbb{R}^2} x^{i-1} y^{j-1} \mathrm{sgn}(y - x) w(x) w(y) dx dy, \\ &= \int_{\infty < x < y < \infty} (x^{i-1} y^{j-1} - x^{j-1} y^{i-1}) w(x) w(y) dx dy. \end{aligned}$$

Let $B_{0,0} = 0$ and for $i \in \{1, 2, \ldots\}$ let

$$B_{0,i} = -B_{i,0} = \int_{\mathbb{R}} x^{i-1} w(x) dx.$$

Suppose for some positive integer n, $\mathbf{x} = \{x_1, \ldots, x_n\}$ and that $\lambda \in \mathcal{P}$ is such that $\ell(\lambda) \leq n$. Then

$$\langle\!\langle s_\lambda(\mathbf{x}) \rangle\!\rangle_n = \mathrm{pf}(B_\lambda),$$

where B_λ is the submatrix of B with rows and columns given by $\{\lambda_1 + n, \ldots, \lambda_n + 1\}$ if n is even and $\{\lambda_1 + n, \ldots, \lambda_n + 1, 0\}$ if n is odd. In particular, if for each non-negative integer n,

$$G_n(\mathbf{x}, \mathbf{p}) = \sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) \leq n}} s_\lambda(x_1, \ldots, x_n) s_\lambda(\mathbf{p}),$$

then the family of generating series $\{\langle\!\langle G_n(\mathbf{x}, \mathbf{p}) \rangle\!\rangle_n : n = 0, 1, 2, \ldots\}$ satisfies the BKP hierarchy in the indeterminates $\mathbf{p}$.

Proof. We have,

$$\begin{aligned} \langle\!\langle s_\lambda(\mathbf{x}) \rangle\!\rangle_n &= \frac{1}{n!} \int_{\mathbb{R}^n} s_\lambda(\mathbf{x}) |a_\delta(\mathbf{x})| \prod_{i=1}^n w(x_i) dx_1 \cdots dx_n, \\ &= \frac{1}{n!} \int_{\mathbb{R}^n} \frac{|a_{\lambda+\delta}(\mathbf{x})|}{|a_\delta(\mathbf{x})|} |a_\delta(\mathbf{x})| \prod_{i=1}^n w(x_i) dx_1 \cdots dx_n, \\ &= \frac{1}{n!} \int_{\mathbb{R}^n} |a_{\lambda+\delta}(\mathbf{x})| \prod_{i=1}^n w(x_i) dx_1 \cdots dx_n. \end{aligned}$$

Recalling that $a_\mu(\mathbf{x}) = \det_{1 \leq i,j \leq n}(x_i^{\mu_j})$ and that $\delta = (n-1, n-2, \ldots, 0)$ we see that

$$\langle\!\langle s_\lambda(\mathbf{x}) \rangle\!\rangle_n = \frac{1}{n!} \int_{\mathbb{R}^n} \left| \det_{1 \leq i,j \leq n} \left(x_i^{\lambda_j + n - j} \right) \right| \prod_{i=1}^n w(x_i) dx_1 \cdots dx_n.$$

Applying Thereom 2.5.8 with $\phi_i(x) = x^{i-1}$ then gives the result. That the family of generating series $\{\langle\!\langle G_n(\mathbf{x}, \mathbf{p}) \rangle\!\rangle_n : n = 0, 1, 2, \ldots\}$ satisfies the BKP hierarchy follows from Theorem 2.5.6. □

Proposition 2.5.9 gives us a class of examples of series which satisfy the differential equations described in Theorem 2.5.6 which includes the BKP equation. We shall use this in Chapter 4 when we discuss the problem of enumerating rooted cubic maps on surfaces which are not necessarily orientable.

Chapter 3

Orientable Bipartite Quadrangulations

3.1 Main Results

The contents of this chapter are joint work with Guillaume Chapuy and appear in the paper [16] as well as an extended abstract which appears in the FPSAC 2014 proceedings.

Recall from Chapter 1 that an orientable *map* is a connected graph embedded in a compact connected orientable surface in such a way that the regions delimited by the graph, called *faces*, are homeomorphic to open discs. Loops and multiple edges are allowed. For the remainder of this chapter we will use map to refer to an orientable map. A rooted map is one in which a vertex, edge and face are distinguished, all of which are incident with one another. The distinguished vertex is called the *root vertex*, the distinguished edge is called the *root edge* and the distinguished face is called the *root face*.

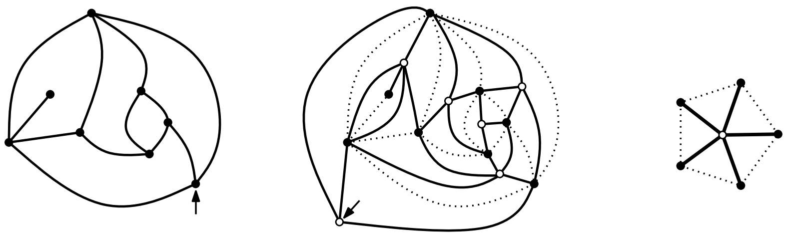
Also recall from Chapter 1 that given a map $\mathfrak{m}$, the Euler characteristic may be computed via

$$\chi = V(\mathfrak{m}) - E(\mathfrak{m}) + F(\mathfrak{m}),$$

where V, E and F are the number of vertices, edges and faces respectively. Given the Euler characteristic χ we may compute the genus of the underlying surface, and thus the genus of the map $\mathfrak{m}$ by $\chi = 2 - 2g$.

A map is *bipartite* if its vertices can be coloured with two colors, say black and white, in such a way that each edge links a white and a black vertex. Unless otherwise mentioned, bipartite maps will be endowed with their *canonical bicolouration* in which the root vertex is coloured white. The *degree* of a face in a map is equal to the number of edge sides along its boundary, counted with multiplicity. Note that in a bipartite map every face has even degree, since colours alternate along its boundary.

A *quadrangulation* is a map in which every face has degree 4. There is a classical bijection that goes back to Tutte [108], between bipartite quadrangulations with n faces and genus g, and rooted maps with n edges and genus g. The bijection proceeds as follows. Given a (not necessarily bipartite) map $\mathfrak{m}$ of genus g with



(a) A map $\mathfrak{m}$ (b) its associated bipartite quadrangulation $\mathfrak{q}$ (thick edges) (c) the local rules of the construction around a face of $\mathfrak{m}$

Figure 3.1: Tutte's Bijection

n edges, add a new (white) vertex inside each face of $\mathfrak{m}$, and link it by a new edge to each of the corners incident to the face. The bipartite quadrangulation $\mathfrak{q}$ is obtained by erasing all the original edges of $\mathfrak{m}$, *i.e.* by keeping only the new (white) vertices, the old (black) vertices, and the newly created edges. The root edge of $\mathfrak{q}$ is the one created from the root corner of $\mathfrak{m}$ (which is enough to root $\mathfrak{q}$ if we demand that its root vertex is white). In Figure 3.1, (a) and (b) display an example of the construction for a map of genus 0 (embedded on the sphere). Root corners are indicated by arrows. This bijection transports the number of faces of the map to the number of white vertices of the quadrangulation (in the canonical bicolouration).

For $g, n \geq 0$, we let Q_g^n be the number of rooted bipartite quadrangulations of genus g with n faces. Equivalently, by Tutte's construction, Q_g^n is the number of rooted maps of genus g with n edges. By convention we admit a single map with no edges and which has genus zero, one face, and one vertex. Our first result is the following recurrence formula:

Theorem 3.1.1. *The number Q_g^n of rooted maps of genus g with n edges (which is also the number of rooted bipartite quadrangulations of genus g with n faces) satisfies the following recurrence relation:*

$$\frac{n+1}{6}Q_g^n = \frac{4n-2}{3}Q_g^{n-1} + \frac{(2n-3)(2n-2)(2n-1)}{12}Q_{g-1}^{n-2} + \frac{1}{2}\sum_{\substack{k+\ell=n \\ i,j\geq 0}} (2k-1)(2\ell-1)Q_i^{k-1}Q_j^{\ell-1},$$

for $n \geq 1$, with the initial conditions $Q_g^0 = \delta_{g,0}$, and $Q_g^n = 0$ if $g < 0$ or $n < 0$.

We actually prove a more general result, where in addition to edges and genus, we also control the number of faces of the map. Let x be a formal variable, and let

$Q_g^n(x)$ be the generating polynomial of maps of genus g with n edges, where the exponent of x records the number of faces of the map:

$$Q_g^n(x) := \sum_{\mathfrak{m}} x^{\#(faces\,of\,\mathfrak{m})}, \tag{3.1}$$

where the sum is taken over rooted maps of genus g with n edges. We then have the following generalization of Theorem 3.1.1:

Theorem 3.1.2. *The generating polynomial $Q_g^n(x)$ of rooted maps of genus g with n edges and a weight x per face (which is also the generating polynomial of rooted bipartite quadrangulations of genus g with n faces with a weight x per white vertex) satisfies the following recurrence relation:*

$$\frac{n+1}{6}Q_g^n(x) = \frac{(1+x)(2n-1)}{3}Q_g^{n-1}(x) + \frac{(2n-3)(2n-2)(2n-1)}{12}Q_{g-1}^{n-2}(x) \\ + \frac{1}{2}\sum_{\substack{k+\ell=n\\k,\ell\geq 1}}\sum_{\substack{i+j=g\\i,j\geq 0}}(2k-1)(2\ell-1)Q_i^{k-1}(x)Q_j^{\ell-1}(x),$$

for $n \geq 1$, with the initial conditions $Q_g^0(x) = x\delta_{g,0}$, and $Q_g^n = 0$ if $g < 0$ or $n < 0$.

Of course, Theorem 3.1.1 is a straightforward corollary of Theorem 3.1.2 (it just corresponds to the case $x = 1$). By extracting the coefficient of x^f in Theorem 3.1.2, for $f \geq 1$, we obtain yet another corollary that enables one to count maps by edges, vertices, and genus:

Corollary 3.1.3. *The number $Q_g^{n,f}$ of rooted maps of genus g with n edges and f faces (which is also the number of rooted bipartite quadrangulations of genus g with n faces and f white vertices) satisfies the following recurrence relation:*

$$\frac{n+1}{6}Q_g^{n,f} = \frac{(2n-1)}{3}Q_g^{n-1,f} + \frac{(2n-1)}{3}Q_g^{n-1,f-1} + \frac{(2n-3)(2n-2)(2n-1)}{12}Q_{g-1}^{n-2,f} \\ + \frac{1}{2}\sum_{\substack{k+\ell=n\\k,\ell\geq 1}}\sum_{\substack{u+v=f,\\u,v\geq 1}}\sum_{\substack{i+j=g\\i,j\geq 0}}(2k-1)(2\ell-1)Q_i^{k-1,u}Q_j^{\ell-1,v},$$

for $n, f \geq 1$, with the initial conditions $Q_g^{0,f} = \delta_{g,0}\delta_{f,1}$ and $Q_g^{n,f} = 0$ whenever $f, g,$ or n is negative.

Corollary 3.1.3 has interesting specializations when the number of faces f is small. In particular, when $f = 1$, the equation becomes linear, and one recovers the celebrated Harer-Zagier formula ([54], see [19] for a bijective proof):

Corollary 3.1.4 (Harer-Zagier recurrence formula, [54])**.** *The number $\epsilon_g(n) = Q_g^{n,1}$ of rooted maps of genus g with n edges and one face satisfies the following recurrence relation:*

$$\frac{n+1}{6}\epsilon_g(n) = \frac{(2n-1)}{3}\epsilon_g(n-1) + \frac{(2n-3)(2n-2)(2n-1)}{12}\epsilon_{g-1}(n-2),$$

with the initial conditions $\epsilon_g(0) = \delta_{g,0}$ and $\epsilon_g(n) = 0$ if $n < 0$ or $g < 0$.

We conclude this list of corollaries with yet another formulation of Corollary 3.1.3 that takes a nice symmetric form and emphasizes the duality between vertices and faces inherent to maps. Let $M_g^{i,j}$ be the number of rooted maps of genus g with i vertices and j faces. Euler's relation ensures that such a map has n edges where:

$$i + j = n + 2 - 2g,$$

which shows that $M_g^{i,j} = Q_g^{i+j+2g-2,j}$. Corollary 3.1.3 thus takes the following form:

Theorem 3.1.5. *The number $M_g^{i,j}$ of rooted maps of genus g with i vertices and j faces (which is also the number of rooted bipartite quadrangulations of genus g with i black vertices and j white vertices) satisfies the following recurrence relation:*

$$\frac{n+1}{6} M_g^{i,j} = \frac{(2n-1)}{3}\left(M_g^{i-1,j} + M_g^{i,j-1} + \frac{(2n-3)(2n-2)}{4} M_{g-1}^{i,j}\right) + \frac{1}{2} \sum_{\substack{i_1+i_2=i \\ i_1,i_2\geq 1}} \sum_{\substack{j_1+j_2=j, \\ j_1,j_2\geq 1}} \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 0}} (2n_1-1)(2n_2-1) M_{g_1}^{i_1,j_1} M_{g_2}^{i_2,j_2},$$

for $i, j \geq 1$, with the initial conditions that $M_g^{i,j} = 0$ if $i+j+2g < 2$, that if $i+j+2g = 2$ then $M_g^{i,j} = \delta_{i,1}\delta_{j,1}$, and where we use the notation $n = i+j+2g-2$, $n_1 = i_1+j_1+2g_1-1$, and $n_2 = i_2 + j_2 + 2g_2 - 1$.

The remainder of this chapter is organized as follows. In Section 3.2, we prove Theorem 3.1.2 (and therefore all the theorems and corollaries stated above). This result relies on both classical facts about the KP equation for bipartite maps, and an elementary Lemma obtained by combinatorial means (Lemma 3.2.2). In Section 3.3, we give corollaries of our results in terms of generating functions. In particular, we obtain a very efficient recurrence formula that can be used to compute the generating function of maps of fixed genus inductively (Theorem 3.3.1).

3.2 Proof of the Main Formula

3.2.1 Bipartite Maps and the KP Equation

The first element of our proof is the fact that the generating function for bipartite maps is a solution to the KP equation (Proposition 3.2.1 below). In the rest of this chapter, the *weight* of a map is one over its number of edges, and a generating function of some family of maps is *weighted* if each map is counted with its weight in this generating function. The keystone of this chapter is the following result

Proposition 3.2.1 ([50], see also [96])**.** *For $n, v, k \geq 1$, and $\alpha \vdash n$ a partition of n, let $H_\alpha(n, v, k)$ be the number of rooted bipartite maps with n edges and v vertices, k of which are white, where the half face degrees are given by the parts of α. Let $H = H(z, w, x; \mathbf{p})$ be the weighted generating function of bipartite maps, with* *z marking edges, w marking vertices, x marking white vertices and the p_i marking the number of faces of degree $2i$ for $i \geq 1$:*

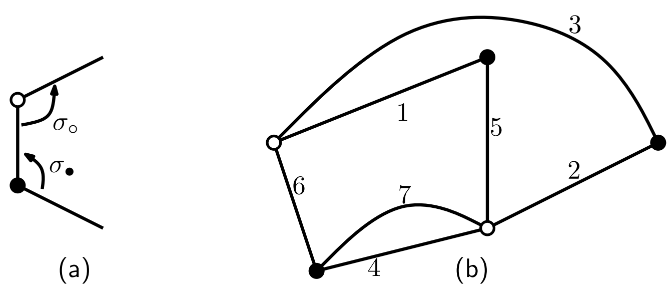
$$H(z,w,x;\mathbf{p}) := 1 + \sum_{\substack{n\geq 1\\ v\geq 1\\ k\geq 1}} \frac{w^v z^n x^k}{n} \sum_{\alpha\vdash n} H_\alpha(n,v,k) p_\alpha.$$

Then $\exp(H)$ *is a solution to the KP hierarchy. In particular, H is a solution to the KP equation:*

$$-H_{3,1} + H_{2,2} + \frac{1}{12}H_{1^4} + \frac{1}{2}(H_{1,1})^2 = 0, \tag{3.2}$$

where indices indicate partial derivatives with respect to the variables p_i, for example $H_{3,1} := \frac{\partial^2}{\partial p_3 \partial p_1} H$.

Proof. First recall that a bipartite orientable map $\mathfrak{m}$ with n edges labelled from 1 to n can be encoded by a triple of permutations $(\sigma_\circ, \sigma_\bullet, \phi) \in (S_n)^3$ such that $\sigma_\circ \sigma_\bullet = \phi$. In this correspondence, the cycles of the permutation $\sigma_\circ$ (resp. $\sigma_\bullet$) encode the counterclockwise ordering of the edges around the white (resp. black) vertices of $\mathfrak{m}$, while the cycles of ϕ encode the clockwise ordering of the white to black edge-sides around the faces of $\mathfrak{m}$. This encoding gives a 1 to $(n-1)!$ correspondence between rooted bipartite maps with n edges and triples of permutations as above that are *transitive*, i.e. that generate a transitive subgroup of S_n. We refer to [24], or Figure 3.2 for more about this encoding (see also [74, 62]).



$$\sigma_\circ = (1,3,6)(2,5,7,4)$$
$$\sigma_\bullet = (1,5)(2,3)(4,7,6)$$
$$\sigma_\circ\sigma_\bullet = (1,7)(2,6)(3,5)(4)$$

(c)

Figure 3.2: (a) The rules defining the permutations $\sigma_\circ$ and $\sigma_\bullet$. (b) A bipartite map with 7 edges arbitrarily labelled from 1 to 7. (c) The corresponding permutations $\sigma_\circ$ and $\sigma_\bullet$.

Let $b_{\alpha,\beta}^{(a_1,a_2,\cdots)}$ be the number of tuples of permutations $(\sigma,\gamma,\pi_1,\pi_2,\cdots)$ on $\{1,\cdots,n\}$ such that

1. σ has cycle type α, γ has cycle type β and π_i has $n-a_i$ cycles for each $i \geq 1$;
2. $\sigma\gamma\pi_1\pi_2\cdots = 1$ in S_n where 1 is the identity;
3. the subgroup generated by $\sigma, \gamma, \pi_1, \pi_2, \cdots$ acts transitively on $\{1,\cdots,n\}$.

Suppose $q_1, q_2, \dots$ and $u_1, u_2, \dots$ are two infinite sets of auxiliary indeterminates, using the notation $q_\beta = \prod_i q_{\beta_i}$. Then it is well known (see the proof of Theorem 3.1 in [50] for example) that the series

$$B = \sum_{\substack{|\alpha|=|\beta|=n\geq 1,\\ a_1,a_2,\cdots\geq 0}} \frac{1}{n!} b_{\alpha,\beta}^{(a_1,a_2,\cdots)} p_\alpha q_\beta u_1^{a_1} u_2^{a_2} \cdots$$

can be written as $B = \log(\widetilde{B})$ where

$$\widetilde{B} = \sum_{\lambda\in\mathcal{P}} \prod_{i\geq 1} \prod_{(j,k)\in\lambda} (1 + u_i(k-j)) s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q}). \tag{3.3}$$

In (3.3) the second product is over the cells (j,k) in λ and $(k-j)$ is the content of the cell (j,k). Also, (3.3) shows that $\widetilde{B}$ is a content-type series like those discussed in Section 2.4.2 where $y_j = \prod_{i\geq 1}(1 + ju_i)$. Thus, $\widetilde{B}$ satisfies the KP hierarchy in the indeterminates $\mathbf{p}$ by Corollary 2.4.11. Using Example 2.2.5 we then see that B satisfies the differential equation

$$-B_{3,1} + B_{2,2} + \frac{1}{12} B_{1^4} + \frac{1}{2}(B_{1,1})^2 = 0. \tag{3.4}$$

Now, using the encoding of maps as triples of permutations described above, we see that $(n-1)! H_\alpha(n,v,k) = b_{\alpha,1^n}^{(n-k,n+k-v,0,\cdots)}$, since the coefficient on the right hand side is the number of solutions to the equation $\sigma\gamma\pi_1\pi_2 = 1$ where the total number of cycles in π_1 and π_2 is v, π_1 has k cycles, σ has cycle type α and where γ is the identity. Multiplying by σ^{-1} then gives $\pi_1\pi_2 = \sigma^{-1}$ which matches the encoding of bipartite maps given above. Thus, by setting $q_1 = w^2 zx$, $q_i = 0$ for $i \geq 2$, $u_1 = w^{-1}x^{-1}, u_2 = w^{-1}$ and $u_i = 0$ for $i \geq 3$ in B, we get the series H as required.

Note that this proof is essentially the same as the one presented in [50]. The result was referred to before this reference in the mathematical physics literature, however, it is hard to find references in which the result is properly stated or proved. We refer to Chapter 5 of the book [74] as an entry point for the interested reader.

□

3.2.2 Bipartite quadrangulations

Our goal is to use Proposition 3.2.1 to get information on the generating function of bipartite quadrangulations. To this end, we let θ denote the operator that substitutes the variable p_2 to 1 and all the variables p_i to 0 for $i \neq 2$. When we apply θ to (3.2) we get four terms:

$$-\theta H_{3,1} + \theta H_{2,2} + \frac{1}{12}\theta H_{1^4} + \frac{1}{2}(\theta H_{1,1})^2 = 0. \tag{3.5}$$

Note that since all the derivatives appearing in (3.2) are with respect to p_1, p_2 or p_3, any monomial in H that contains a variable p_i for some $i \neq \{1,2,3\}$ gives a zero contribution to (3.5). Therefore each of the four terms appearing in (3.5) can be interpreted as the generating function of *some* family of bipartite maps having only faces of degree 2,4, or 6 (subject to further restrictions). However, thanks to local operations on maps, we will be able to relate each term to maps having only faces of degree 4, as shown by the next lemma.

If $A(z,w)$ is a formal power series in z and w with coefficients in $\mathbb{C}[x]$ we denote by $[z^p w^q]A(z,w)$ the coefficient of the monomial $z^p w^q$ in $A(z,w)$. It is a polynomial in x.

Lemma 3.2.2. *Let $n, g \geq 1$. Then we have:*

$$[z^{2n} w^{n+2-2g}]\theta H_{2,2} = \frac{n-1}{2} Q_g^n(x), \tag{3.6}$$

$$[z^{2n} w^{n+1-2g}]\theta H_{1,1} = (2n-1) Q_g^{n-1}(x), \tag{3.7}$$

$$[z^{2n} w^{n+2-2g}]\theta H_{1^4} = (2n-1)(2n-2)(2n-3) Q_{g-1}^{n-2}(x), \tag{3.8}$$

$$[z^{2n} w^{n+2-2g}]\theta H_{3,1} = \frac{2n-1}{3}\Big(Q_g^n(x) - (1+x) Q_g^{n-1}(x)\Big). \tag{3.9}$$

We now prove the lemma. By definition, if $v \geq 1$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition of some integer, then $[z^{2n} w^v]\theta H_\lambda$ is $\frac{1}{2n}$ times the generating polynomial (the variable x marking white vertices) of rooted bipartite maps with $2n$ edges, v vertices, ℓ marked (numbered) faces of degrees $2\lambda_1, 2\lambda_2, \ldots, 2\lambda_\ell$, and all other (unmarked) faces of degree 4. If r is the number of unmarked faces, such a map has $r+\ell$ faces, and by Euler's formula, the genus g of this map satisfies: $v - 2n + (r+\ell) = 2 - 2g$. Moreover the number of edges is equal to the sum of the half face degrees so $2n = 2r + |\lambda|$, therefore we obtain the relation:

$$2g = n + 2 - v + \frac{|\lambda|}{2} - \ell, \tag{3.10}$$

which we shall use repeatedly. We now proceed with the proof of Lemma 3.2.2.

Proof of (3.6). As discussed above, $H_{2,2}$ is the weighted generating function of rooted bipartite maps with two marked faces of degree 4, so $\theta H_{2,2}$ is the weighted generating function of rooted quadrangulations with two marked faces. Moreover, by (3.10), the maps that contribute to the coefficient $[z^{2n} w^{n+2-2g}]$ in $\theta H_{2,2}$ have genus g. Now, there are $n(n-1)$ ways of marking two faces in a quadrangulation with n faces, and the weight of such a map is $\frac{1}{2n}$ since it has $2n$ edges. Therefore: $[z^{2n} w^{n+2-2g}]\theta H_{2,2} = \frac{1}{2n} n \cdot (n-1) Q_g^n(x)$. ◻

Proof of (3.7) *and* (3.8). As discussed above, for $k \geq 1$, $\theta H_{1^{2k}}$ is the weighted generating function of bipartite maps carrying $2k$ marked (numbered) faces of degree 2, having all other faces of degree 4. Moreover, by (3.10), the genus of maps that contribute to the coefficient $[z^{2n} w^{n+k-2g}]$ in this series is equal to $g+1-k$. Therefore:

$$[z^{2n} w^{n+k-2g}]\theta H_{1^{2k}} = \frac{1}{2n} P_{g+1-k}^{2n,2k}(x) \tag{3.11}$$

where $P_h^{m,\ell}(x)$ denotes the generating polynomial (the variable x marking white vertices) of rooted bipartite maps of genus h with ℓ numbered marked faces of degree 2, all other faces of degree 4, and m edges in total. Now, we claim that for all h and all m, ℓ with $m+\ell$ even one has:

$$P_h^{m,\ell}(x) = m(m-1)\dots(m-\ell+1)Q_h^{\frac{m-\ell}{2}}(x). \tag{3.12}$$

This is obvious for $\ell = 0$ since a quadrangulation with m edges has $m/2$ faces. For $\ell \geq 1$, consider a bipartite map with all faces of degree 4, except ℓ marked faces of degree 2, and m edges in total. By contracting the first marked face into an edge, one obtains a map with one less marked face, and a marked edge. This marked edge can be considered as the root edge of that map (keeping the canonical bicolouration of vertices). Conversely, starting with a map having $\ell-1$ marked faces, and $m-1$ edges, and expanding the root-edge into a face of degree 2, there are m ways of choosing a root corner in the resulting map in a way that preserves the canonical bicolouration of vertices. Since the contraction operation does not change the number of white vertices, we deduce that $P_h^{m,\ell}(x) = m \cdot P_h^{m-1,\ell-1}(x)$ and (3.12) follows by induction. (3.7) and (3.8) then follow from (3.11) for $k=1$ and $k=2$, respectively. □

Proof of (3.9). This case starts in the same way as the three others, but we will have to use an additional tool (a simple Tutte equation) in order to express everything in terms of quadrangulation numbers only. First, $\theta H_{3,1}$ is the weighted generating function of rooted bipartite maps with one face of degree 6, one face of degree 2, and all other faces of degree 4. Moreover, by (3.10), maps that contribute to the coefficient of $[z^{2n}w^{n+2-2g}]$ in this series all have genus g. We first get rid of the face of degree 2 by contracting it into an edge, and declare this edge as the root of the new map, keeping the canonical bicolouration. If the original map has $2n$ edges, we obtain a map with $2n-1$ edges in total. Conversely, if we start with a map with $2n-1$ edges and we expand the root edge into a face of degree 2, we have $2n$ ways of choosing a new root corner in the newly created map, keeping the canonical bicolouration. Therefore if we let $X_g^n(x)$ be the generating polynomial (the variable x marking white vertices) of rooted bipartite maps having a face of degree 6, all other faces of degree 4, and $2n-1$ edges in total, we have:

$$[z^{2n}w^{n+2-2g}]\theta H_{3,1} = \frac{1}{2n} \cdot 2nX_g^n(x) = X_g^n(x),$$

where the first factor is the weight coming from the definition of H. Thus to prove (3.9) it is enough to establish the following equation:

$$Q_g^n(x) = \frac{3}{2n-1}X_g^n(x) + (1+x)Q_g^{n-1}(x). \tag{3.13}$$

The reader well acquainted with map enumeration may have recognized in (3.13) a (very simple case of a) Tutte/loop equation. It is proved as follows. Let $\mathfrak{q}$ be a rooted bipartite quadrangulation of genus g with n faces, and let e be the root edge of $\mathfrak{q}$. There are two cases: 1. the edge e is bordered by two distinct faces, and 2. the edge e is bordered twice by the same face.

In case 1, removing the edge e gives rise to a map of genus g with a marked face of degree 6. By marking one of the $2n-1$ white corners of this map as the root, we obtain a rooted map counted by $X_g^n(x)$, and since there are 3 ways of placing a diagonal in a face of degree 6 to create two quadrangles, the generating polynomial $N_1(x)$ corresponding to case 1. satisfies $(2n-1)N_1(x) = 3X_g^n(x)$.

In case 2, the removal of the edge e creates two faces (*a priori*, either in the same or in two different connected components) of degrees k_1, k_2 with $k_1+k_2+2=4$. Now since $\mathfrak{q}$ is bipartite, k_1 and k_2 are even which shows that one of the k_i is zero and the other is equal to 2. Therefore, in $\mathfrak{q}$, e is a single edge hanging in a face of degree 2. By removing e and contracting the degree 2 face, we obtain a quadrangulation with $n-1$ faces (and a marked edge that serves as a root, keeping the canonical bicolouration). Conversely, there are two ways to attach a hanging edge in a face of degree 2, which respectively keep the number of white vertices equal or increase it by one. Therefore the generating polynomial corresponding to case 2. is $N_2(x) = (1+x)Q_g^{n-1}(x)$.

Writing that $Q_g^n(x) = N_1(x) + N_2(x)$, we obtain (3.13) and complete the proof. □

Proof of Theorem 3.1.2. Just extract the coefficient of $[z^n w^{n+2-2g}]$ in Equation (3.5) using Lemma 3.2.2, and group together the two terms containing $Q_g^n(x)$, namely $\frac{n-1}{2}Q_g^n(x) - \frac{2n-1}{3}Q_g^n(x) = -\frac{n+1}{6}Q_g^n(x)$. □

3.3 Genus generating functions

3.3.1 The univariate generating functions $Q_g(t)$

We start by studying the generating functions of maps of fixed genus by the number of edges. In particular through the whole Section 3.3.1 we set $x=1$ and we use the notation $Q_g^n \equiv Q_g^n(1)$ as in Theorem 3.1.1. We let $Q_g(t) := \sum_{n\geq 0} Q_g^n t^n$ be the generating function of rooted maps of genus g by the number of edges. It was shown in [7] that $Q_g(t)$ is a rational function of $\rho := \sqrt{1-12t}$. In genus 0, the result goes back to Tutte [108] and one has the explicit expression:

$$Q_0(t) = T - tT^3, \tag{3.14}$$

where $T = \frac{1-\sqrt{1-12t}}{6t}$ is the unique formal power series solution of the equation

$$T = 1 + 3tT^2. \tag{3.15}$$

In the following we will give a very simple recursive formula to compute the series $Q_g(t)$ as a rational function of T, and we will study some of its properties [1].

Theorem 3.3.1. *For $g \geq 0$, we have $Q_g(t) = R_g(T)$ where T is given by* (3.15) *and R_g is a rational function that can be computed iteratively via:*

$$\frac{d}{dT}\left(\frac{(T-1)(T+2)}{3T}R_g(T)\right) = \frac{(T-1)^2}{18T^4}(2D+1)(2D+2)(2D+3)R_{g-1}(t) \qquad (3.16)$$
$$+ \frac{(T-1)^2}{3T^4}\sum_{\substack{i+j=g \\ i,j\geq 1}}((2D+1)R_i(T))((2D+1)R_j(T)),$$

where $D = \dfrac{T(1-T)}{T-2}\dfrac{d}{dT}$.

Proof. First, one easily checks that Theorem 3.1.1 is equivalent to the following differential equation:

$$(D+1)Q_g = 4t(2D+1)Q_g + \frac{1}{2}t^2(2D+1)(2D+2)(2D+3)Q_{g-1} \qquad (3.17)$$
$$+ 3t^2\sum_{\substack{i+j=g \\ i,j\geq 1}}((2D+1)Q_i)((2D+1)Q_j),$$

where D is the operator $D := t\cdot\frac{d}{dt}$. Using (3.15) one checks that $tT'(t) = \frac{T(1-T)}{T-2}$, so that $D = \left(\frac{dT(t)}{dt}\right)\frac{d}{dT} = \frac{T(1-T)}{T-2}\frac{d}{dT}$ and the definition of D coincides with the one given in the statement of the theorem.

Now, for $h \geq 0$ let R_h be the unique formal power series such that $Q_h(t) = R_h(T)$. Grouping all the genus g generating functions on the left hand side, we can put (3.17) in the form:

$$AR_g(t) + B\frac{d}{dT}R_g(t) = R.H.S. \qquad (3.18)$$

where $A = 1-4t-6t^2(2D+1)Q_0$, $B = t(1-8t-12t^2(2D+1)Q_0))$, and the R.H.S. is the same as in (3.16). Using the explicit expression (3.14) of Q_0 in terms of T, we can then rewrite the L.H.S. of (3.18) as

$$\frac{(T-1)(T+2)}{3T}R_g'(T) + \frac{T^2+2}{3T^2}R_g(T) = \frac{d}{dT}\left(\frac{(T-1)(T+2)}{3T}R_g(T)\right),$$

and we obtain (3.16). Note that we have not proved that $R_g(T)$ is a rational function: we admit this fact from [7]. □

[1]Note that being a rational function of T or ρ is equivalent, but we prefer to work with T, since as a power series T has a clear combinatorial meaning. Indeed, T is the generating function of labelled/blossomed trees, which are the fundamental building blocks that underly the bijective decomposition of maps [102, 21, 20]. It is thus tempting to believe that those rationality results have a combinatorial interpretation in terms of these trees, even if it is still an open problem to find one. Indeed, so far the best rationality statement that is understood combinatorially is that the series of rooted bipartite quadrangulations of genus g with a distinguished vertex is a rational function in the variable U such that $1 = tT^2(1+U+U^{-1})$, which is weaker than the rationality in T. See [20] for this result.

Observe that we have $R_g(1) = Q_g(0) < \infty$ so the quantity $\frac{(T-1)(T+2)}{3T} R_g(T)$ vanishes at $T = 1$, and we have:

$$\frac{(T-1)(T+2)}{3T} R_g(T) = \int_1^T R.H.S.,$$

with the R.H.S. given by (3.16), which shows that (3.16) indeed enables one to compute the R_g's recursively. Note that it is not obvious *a priori* that no logarithm appears during this integration, although this is true since it is known that R_g is rational[2] [7]. Moreover, since all generating functions considered are finite at $T = 1$ (which corresponds to the point $t = 0$) we obtain via an easy induction that R_g has only poles at $T = 2$ or $T = -2$ for $g \geq 1$. More precisely, by an easy induction, we obtain a bound on the degrees of the poles:

Corollary 3.3.2. *For $g \geq 1$ we have $Q_g(t) = R_g(T)$ where R_g can be written as:*

$$R_g = c_0^{(g)} + \sum_{i=1}^{5g-3} \frac{\alpha_i^{(g)}}{(2-T)^i} + \sum_{i=1}^{3g-2} \frac{\beta_i^{(g)}}{(T+2)^i}, \tag{3.19}$$

for rational numbers $c_0^{(g)}$ and $\alpha_i^{(g)}, \beta_i^{(g)}$.

Note that by plugging the ansatz (3.19) into the recursion (3.16), we obtain a very efficient way of computing the R_g 's inductively.

We conclude this section with (known) considerations on asymptotics. From (3.19), it is easy to see that the dominant singularity of $Q_g(t)$ is unique, and is reached at $t = \frac{1}{12}$, *i.e.* when $T = 2$. In particular the dominant term in (3.19) is $\frac{\alpha_{5g-3}^{(g)}}{(2-T)^{5g-3}}$. Using the fact that $2 - T = 2\sqrt{1-12t} + O(1-12t)$ when t tends to $\frac{1}{12}$, and using a standard transfer theorem for algebraic functions [33], we obtain that for fixed g, n tending to infinity:

$$Q_g^n \sim t_g n^{\frac{5(g-1)}{2}} 12^n, \tag{3.20}$$

with $t_g = \frac{1}{2^{5g-3}\Gamma(\frac{5g-3}{2})} \alpha_{5g-3}^{(g)}$. Moreover, by extracting the leading order coefficient in (3.16) when $T \sim 2$, we see with a short computation that the sequence $\tau_g = (5g-3)\alpha_{5g-3}^{(g)} = 2^{5g-2}\Gamma(\frac{5g-1}{2}) t_g$ satisfies the following Painleve-I type recursion

$$\tau_g = \frac{1}{3}(5g-4)(5g-6)\tau_{g-1} + \frac{1}{2}\sum_{h=1}^{g-1} \tau_h \tau_{g-h}, \tag{3.21}$$

which enables one to compute the t_g's easily by induction starting from $t_1 = \frac{1}{24}$ (*i.e.* $\tau_1 = \frac{1}{3}$). These results are well known (for (3.20) see [6], or [20, 18] for bijective interpretations; for (3.21) see [74, p.201] for historical references, or [9] for a proof along the same lines as ours but starting from the Goulden and Jackson recurrence [50]). So far, as far as we know, all the known proofs of (3.21) rely on the use of integrable hierarchies.

[2] We unfortunately haven't been able to reprove this fact from our approach

3.3.2 Genus generating function at fixed n.

We now indicate a straightforward consequence of Theorem 3.1.2 in terms of "genus generating functions", *i.e.* generating functions of maps where the number of edges is fixed and genus varies. Such generating functions have been considered in the combinatorial literature before (with a slightly different scaling), especially in the case of one-face maps, where they admit elegant combinatorial interpretations (see [10]). They also appear naturally in formal expansions of matrix integrals (see, e.g., [74] or [54]). Let s be a variable, and for $n \geq 1$ let $H_n(x,s)$ be the generating polynomial of maps with n edges by the number of faces (variable x), and the genus (variable s), *i.e.*:

$$H_n(x,s) = \sum_{g\geq 0} Q_g^n(x) s^g,$$

in the notation of Theorem 3.1.2. Then Theorem 3.1.2 has the following equivalent formulation:

Corollary 3.3.3. *The genus generating function $H_n \equiv H_n(x,s)$ is solution of the following recurrence equation:*

$$\frac{n+1}{6} H_n = \frac{(1+x)(2n-1)}{3} H_{n-1} + \frac{s(2n-3)(2n-2)(2n-1)}{12} H_{n-2} + \frac{1}{2} \sum_{\substack{k+\ell=n \\ k,\ell\geq 1}} (2k-1)(2\ell-1) H_{k-1} H_{\ell-1},$$

for $n \geq 1$, with the initial condition $H_0(x,s) = x$.

3.3.3 The bivariate generating functions $M_g(x;y)$

We let $M_g^{i,j}$ be defined as in Theorem 3.1.5 and we consider the bivariate generating function of rooted maps of genus g by vertices (variable x) and faces (variable y):

$$M_g(x,y) = \sum_{i,j\geq 1} M_g^{i,j} x^i y^j.$$

Bender, Canfield, and Richmond [8] showed that $M_g(x,y)$ is a rational function in the two parameters p and q such that:

$$x = p(1-p-2q), \quad y = q(1-2p-q). \tag{3.22}$$

Arquès and Giorgetti [3] later refined this result and showed that

$$M_g(x,y) = \frac{pq(1-p-q)P_g(p,q)}{\left((1-2p-2q)^2-4pq\right)^{5g-3}} \tag{3.23}$$

where P_g is a polynomial of total degree at most $6g-6$. However, similarly as in the case of univariate functions discussed in the previous section, the recursions given in [8, 3] to compute the polynomials P_g involve additional variables and are complicated to use except for the very first values of g.

It would be natural to try to compute these polynomials by reformulating Theorem 3.1.5 as a recursive partial differential equation for the series $M_g(x,y)$ in the variables (x,y) or (u,v), in the same way as we did for the univariate series in Section 3.3.1. However due to the bivariate nature of the problem, this approach does not seem to lead easily to an efficient way of computing the polynomials $P_g(u,v)$.

Instead, we prefer to remark that since $P_g(u,v)$ has total degree at most $(6g-6)$ one can use the method of undetermined coefficients. In order to determine $P_g(u,v)$ we need to determine $\binom{6g-5}{2}$ coefficients, which can be done by computing the same number of terms of the sequence $M_g^{i,j}$. More precisely it is easy to see that computing $M_g^{i,j}$ for all i,j such that $2 \leq i+j \leq 6g-4$ gives enough data to determine the polynomial P_g, whose coefficients can then be obtained by solving a linear system. Since Theorem 3.1.5 gives a very efficient way of computing the numbers $M_g^{i,j}$, this technique seems more efficient (and much simpler) than trying to solve recursively the functional equations of the papers [8, 3]. We have implemented it on Maple and checked that we recovered the expressions of [3] for $g = 1, 2, 3$, and computed the next terms with no difficulty (for example computing $P_{10}(p,q)$ took less than a minute on a standard computer).

Chapter 4

Non-Orientable Triangulations

4.1 Introduction

In this chapter we will use map to refer to a map which is either orientable or non-orientable. Sometimes such maps are called locally orientable in order to distinguish from the common usage of the term map referring to orientable maps.

As in Chapter 1 and Chapter 3, given a map $\mathfrak{m}$ the Euler characteristic may be computed via Euler's formula,

$$\chi = V(\mathfrak{m}) - E(\mathfrak{m}) + F(\mathfrak{m}),$$

where V, E and F are the number of vertices, edges and faces respectively. Given the Euler characteristic χ we may compute the genus of the underlying surface, and thus the genus of the map $\mathfrak{m}$ by $\chi = 2 - 2g$. At times it will be most convenient to use Euler genus which is twice the genus.

In [6], Bender and Canfield studied the number $T_g(n)$ of n-edged rooted maps on an orientable surface of genus g and the number $P_g(n)$ of n-edged rooted maps on a non-orientable surface of genus g. They showed that asymptotically the numbers behaved as

$$\begin{aligned} T_g(n) &\sim t_g n^{5(g-1)/2} 12^n, \\ P_g(n) &\sim p_g n^{5(g-1)/2} 12^n, \text{ when } g > 0, \end{aligned}$$

where t_g and p_g are constants which depend only on g. Unfortunately, determining the constants t_g and p_g proved to be very difficult. Later, Gao[39] showed that if C denotes a class of rooted maps (for example, 2-connected, triangulations and $2d$-regular) and $M_g(C, n)$ is the number of maps in class C which are of genus g and which have n edges then for many such classes,

$$M_g(C, n) \sim \alpha t_g (\beta n)^{5(g-1)/2} \gamma^n,$$

if the maps are orientable and

$$M_g(C,n) \sim \alpha p_g(\beta n)^{5(g-1)/2}\gamma^n,$$

if the maps are non-orientable. Here α, β and γ depend on the class of maps considered.

Goulden and Jackson[50] showed that the number of rooted orientable triangulations satisfies a quadratic recurrence equation. Equivalently, this implies that the generating series for rooted orientable triangulations satisfies a quadratic differential equation. Goulden and Jackson proved this using the fact that the generating series for rooted maps with respect to vertex degrees satisfies a family of differential equations known as the KP hierarchy, essentially coming from the fact that the generating series in question has a determinantal structure (the Schur expansion has coefficients which satisfy the Plücker relations). Using this quadratic differential equation and the fact that the class of triangulations has the asymptotic behaviour described by Gao, it was shown by Bender, Richmond and Gao[9] that the map asymptotics constants t_g satisfied a quadratic recurrence. It was then shown by Garoufalidis, Lê and Mariño[40] that the scaled generating series for the map asymptotics constant t_g satisfied a quadratic differential equation equivalent to the Painlevé I equation. In particular they showed that if $u_g = -2^{g-2}\Gamma\left(\frac{5g-1}{2}\right)t_g$ then the series

$$u(z) = z^{1/2}\sum_{g=0}^{\infty} u_g z^{-5g/2},$$

satisfies

$$u^2 - \frac{1}{6}u'' = z.$$

By analogy to the orientable case and motivated by some results in mathematical physics concerning integrals over real symmetric matrices, Garoufalidis and Mariño[41] conjecture that similar to the series $u(z)$ above, the scaled generating series for the non-orientable map asymptotics constants also satisfies a simple differential equation. The main result of this chapter is the following, which appears as Conjecture 1 in Garoufalidis and Mariño[41].

Theorem 4.1.1. *Let*

$$v_g = 2^{\frac{g-3}{2}}\Gamma\left(\frac{5g-1}{4}\right)p_{\frac{g+1}{2}}.$$

The series

$$v(z) = z^{1/4}\sum_{g=0}^{\infty} v_g z^{-5g/4}$$

satisfies the differential equation

$$2v' - v^2 + 3u = 0,$$

where the series u is the same as the series u above.

One of the main advantages to the differential equations satisfied by the generating series for the t_g and p_g constants is that they can be used to determine the corresponding asymptotic behaviour. The differential equation for $u(z)$ was used by Garoufalidis, Lê and Mariño[40] to determine the asymptotic behaviour of t_g to all orders. In particular, the following Theorem is proven in Appendix A of [40].

Theorem 4.1.2. *Suppose the series*

$$u(z) = z^{1/2} \sum_{g=0}^{\infty} u_g z^{-5g/2}$$

satisfies

$$u^2 - \frac{1}{6} u'' = z.$$

Then as $g \to \infty$,

$$u_g \sim A^{-2g+\frac{1}{2}} \Gamma\left(2g - \frac{1}{2}\right) \frac{S}{2\pi i} \left\{1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell A^\ell}{\prod_{m=1}^{\ell} (2g - 1/2 - m)}\right\},$$

where $A = \frac{8\sqrt{3}}{5}, S = -i\frac{3^{1/4}}{\sqrt{\pi}}$ and the μ_ℓ are defined by the recursion relation

$$\mu_\ell = \frac{5}{16\sqrt{3}\ell} \left\{\frac{192}{25} \sum_{k=0}^{\ell-1} \mu_k u_{(\ell-k+1)/2} - \left(\ell - \frac{9}{10}\right)\left(\ell - \frac{1}{10}\right) \mu_{\ell-1}\right\}, \qquad \mu_0 = 1,$$

with the convention that $u_{n/2} = 0$ if n is odd.

Similarly, the differential equation in Theorem 4.1.1 was used by Garoufalidis and Mariño[41] to determine the complete asymptotic behaviour of v_g and hence, by Theorem 4.1.1, p_g up to an unknown constant S'. The complete details can be found in Garoufalidis and Mariño[41] and the following appears as Theorem 1 in [41].

Theorem 4.1.3. *Suppose the series*

$$v(z) = z^{1/4} \sum_{g=0}^{\infty} v_g z^{-5g/4}$$

satisfies the differential equation

$$2v' - v^2 + 3u = 0,$$

where $u(z)$ is the series described in Theorem 4.1.2. Then the sequence v_g has an asymptotic expansion of the form

$$v_g \sim (A/2)^{-g} \Gamma(g) \frac{S'}{2\pi i} \left\{1 + \sum_{\ell=1}^{\infty} \frac{\nu_\ell (A/2)^\ell}{\prod_{m=1}^{\ell} (g - m)}\right\},$$

where A is given in Theorem 4.1.2, $S' \neq 0$ is some non-zero constant (called the Stokes constant) and the sequence ν_ℓ is defined by the recursion relation

$$\nu_\ell = -\frac{4}{5\ell} \sum_{k=0}^{\ell-1} v_{\ell+1-k} \nu_k, \qquad \nu_0 = 1.$$

The rest of this chapter is organized as follows. In Section 4.2 we discuss how the generating series for maps (counting both orientable and non-orientable maps) has a Pfaffian structure. This is in analogy with the generating series for orientable maps which has a determinantal structure. As a result we show that the map series satisfies the BKP equation. In addition, we discuss some linear differential equations which are satisfied by the map series which follow by considering the removal of vertices with degree one or two. In Section 4.3 we use the differential equations described in Section 4.2 to derive a cubic differential equation for the specialization of the map series to triangulations. In Section 4.4 we give a structural result for $L_g^{(3)}(z)$, the generating series for triangulations. In particular, we show that $L_g^{(3)}(z)$ can be written as a rational series in terms of an auxilliary algebraic series. This type of structure seems to arise frequently in map enumeration and in permutation factorization problems as in, for example, Goulden, Guay-Paquet and Novak's work on monotone Hurwtiz numbers[47], however the reason for this is not clear. Lastly, in Section 4.5, we combine the structure theorem and the cubic differential equation to derive some results about the asymptotic behaviour of triangulations on all surfaces. As an application, we are able to prove Garoufalidis and Mariño's conjecture about the non-orientable map asymptotics constants.

4.2 Symmetric Matrix Integrals and Maps

Recall the matrix integral operator given by

$$\langle\!\langle f(\mathbf{x})\rangle\!\rangle_N = \frac{1}{N!}\int_{\mathbb{R}^N} f(\mathbf{x})|a_\delta(\mathbf{x})|\prod_{j=1}^{N} w(x_j)dx_1\cdots dx_N,$$

where N is a positive integer and $w(x)$ is a weight function. In this chapter we will fix the weight $w(x)$ to be $w(x) = \exp\left(\frac{-x^2}{4}\right)$. This averaging operator is related to a similar operator over the vector space W_N of all $N\times N$ real symmetric matrices $M \in W_N$ with measure $e^{-\mathrm{Tr}\ M^2/4}$. In fact, up to a multiplicative constant,

$$\langle\!\langle f(\lambda)\rangle\!\rangle_N = \int_{W_N} f(M)e^{-\mathrm{Tr}\ M^2/4}dM.$$

This relationship follows from the fact that the integrand is conjugation invariant and so we may use the polar decomposition for real symmetric matrices to reduce the integral to one over $\mathbb{R}^N$.

Theorem 4.2.1 (Kakei[69], Van de Leur[109], Adler and Moerbeke[1])**.** *Suppose $t_1, t_2, \ldots$ is an infinte set of auxiliary indeterminates and that*

$$Z_N = \left\langle\!\!\left\langle \exp\left(\sum_{k\geq 1}\frac{p_k(x_1,\ldots,x_N)}{2k}t_k\right)\right\rangle\!\!\right\rangle_N.$$

Then the family of generating series $\{Z_N : N = 0, 1, 2, \ldots\}$ satisfies the BKP hierarchy in the indeterminates $\{\frac{t_1}{2}, \frac{t_2}{2}, \frac{t_3}{2}, \ldots\}$. In particular,

$$(\partial_1^4 + 3\partial_2^2 + 3\partial_3\partial_1) \log Z_N + 6(\partial_1^2 \log Z_N)^2 = \frac{3}{4} \frac{Z_{N+2} Z_{N-2}}{Z_N^2},$$

where we have used the notation $\partial_i = \frac{\partial}{\partial t_i}$.

Proof. Adler and Moerbeke[1] showed that for even N the Z_N satisfies each of the differential equations in the BKP hierarchy. Van de Leur[109] then generalized this result to any N. In both cases the authors used the Fock space approach to integrable hierarchies. Our proof here is essentially the general version of the direct proof of the fact that Z_N satisfies the BKP equation as shown by Kakei[69] and which can also be found in Hirota[58].

First, recall that

$$\sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) \le N}} s_\lambda(x_1, x_2, \ldots, x_N) s_\lambda(\mathbf{p}) = \exp\left(\frac{p_k(x_1, \ldots, x_N) p_k}{k}\right).$$

Letting $p_k = \frac{t_k}{2}$, Proposition 2.5.9 then says that the family of generating series $\{Z_N : N = 0, 1, 2, \ldots\}$ satisfies the BKP hierarchy in the indeterminates $\{\frac{t_1}{2}, \frac{t_2}{2}, \frac{t_3}{2}, \ldots\}$. In particular, Example 2.2.8 says that the family of generating series $\{Z_N : N = 0, 1, 2, \ldots\}$ satisfies the BKP equation given by

$$\frac{Z_{N+2} Z_{N-2}}{Z_N^2} = \frac{1}{12} \frac{\partial^4 \log Z_N}{\partial p_1^4} - \frac{\partial^2 \log Z_N}{\partial p_1 \partial p_3} + \frac{\partial^2 \log Z_N}{\partial p_2^2} + \frac{1}{2}\left(\frac{\partial^2 \log Z_N}{\partial p_1^2}\right)^2.$$

Making the substitution $p_k = \frac{t_k}{2}$ for each k then gives the desired result. □

Theorem 4.2.2 (Mehta[80], Van de Leur[109], Adler and Moerbeke[1])**.** *Suppose $t_1, t_2, \ldots$ is an infinite set of auxiliary indeterminates and that*

$$Z_N = \left\langle\!\!\left\langle \exp\left(\sum_{k \ge 1} \frac{p_k(x_1, \ldots, x_N)}{2k} t_k\right)\right\rangle\!\!\right\rangle_N.$$

Then

$$\partial_1 \log Z_N = \sum_{i \ge 1} i t_{i+1} \partial_i \log Z_N + \frac{N t_1}{2},$$

and

$$\partial_2 \log Z_N = \sum_{i \ge 1} \frac{i}{2} t_i \partial_i \log Z_N + \frac{N(N+1)}{4},$$

where $\partial_i = \frac{\partial}{\partial t_i}$.

Proof. This can be shown directly using the fact that the integration measure is translation invariant and the details of this method can be found in Mehta[80]. Alternate proofs using Fock space methods can be found in Van de Leur[109] and Adler and Moerbeke[1].

For our purposes, via Theorem 4.2.3, the series of interest is a generating series for rooted maps. In this case, the two equations correspond to adding / removing a vertex of degree one and adding / removing a vertex of degree two. □

Let $\ell_{k,\alpha}$ be the number of maps with k faces and vertex partition given by α. Then the generating series for maps in all surfaces is defined to be

$$L(t;y) = \sum_{k\geq 1}\sum_{\alpha} \ell_{k,\alpha} y^k t_\alpha.$$

Remark. Let

$$\ell_\alpha(y) = [t_\alpha]L(t;y) = \sum_{k\geq 1} \ell_{k,\alpha} y^k.$$

Then, since for any fixed map the sum of the vertex degrees is equal to the sum of the face degrees, we must have $k \leq |\alpha|$. Thus, each $\ell_\alpha(y)$ is a polynomial in y.

Theorem 4.2.3 (Goulden and Jackson[48])**.** *For any positive integer N,*

$$L(t;N) = 2E \log \left\langle\!\!\left\langle \exp\left(\sum_{k\geq 1} \frac{p_k(x_1,\ldots,x_N)}{2k} t_k\right)\right\rangle\!\!\right\rangle_N,$$

where

$$E = \sum_{k\geq 1} k t_k \partial_k.$$

In particular, Theorem 4.2.3 above implies that the differential equations in Theorem 4.2.1 and Theorem 4.2.2 are also satisfied by the generating series for all maps. This collection of differential equations is the primary tool used in the remainder of this chapter.

4.3 A Cubic Differential Equation for Triangulations

Let $L^{(3)}(x,y) = L(t;y)\big|_{t_i = x\delta_{i,3}}$. That is, let $L^{(3)}(x,y)$ be the generating series for cubic maps on all surfaces. Similarly, let $L^{(3)}(x;w)$ be the generating series for cubic maps with vertices marked by x and Euler genus (twice the genus) marked by w. By Euler's formula,

$$L^{(3)}(x;w) = w^2\, L^{(3)}(x,y)\big|_{x\mapsto xw^{1/2},\, y\mapsto w^{-1}}.$$

We begin by using Theorem 4.2.1 and Theorem 4.2.2 to derive a cubic differential equation for $L^{(3)}(x;w)$.

Theorem 4.3.1. *Let*

$$\begin{aligned}
T &= L^{(3)}(x;w) + w + 1 - \frac{1}{2x^2},\\
V &= w^{-2}\left((1+2w)^2 L^{(3)}\left(x(1+2w)^{1/2}, \frac{w}{1+2w}\right)\right.\\
&\qquad \left. + (1-2w)^2 L^{(3)}\left(x(1-2w)^{1/2}, \frac{w}{1-2w}\right) - 2L^{(3)}(x;w)\right).
\end{aligned}$$

Then

$$\begin{aligned}
&4x^4w^2(D+12)^2(D+8)(D+4)T - (D+6)DT + 12x^4(D+12)((D+4)T)^2\\
&\quad = V\left(2x^4w^2(D+12)(D+8)(D+4)T - \frac{1}{2}(D+6)T + 6x^4((D+4)T)^2\right),
\end{aligned}$$

where $D = 3x\partial_x$.

Proof. Recall from Theorem 4.2.2 that if

$$Z_N = \left\langle\!\!\left\langle \exp\left(\sum_{k\geq 1} \frac{p_k(\lambda)}{2k} t_k\right)\right\rangle\!\!\right\rangle_N,$$

then

$$\begin{aligned}
\partial_1 \log Z_N &= \sum_{i\geq 1} i t_{i+1}\partial_i \log Z_N + \frac{Nt_1}{2},\\
\partial_2 \log Z_N &= \sum_{i\geq 1} \frac{i}{2} t_i \partial_i \log Z_N + \frac{N(N+1)}{4},
\end{aligned}$$

and (from Theorem 4.2.1) that

$$(\partial_1^4 + 3\partial_2^2 - 3\partial_3\partial_1)\log Z_N + 6(\partial_1^2 \log Z_N)^2 = \frac{3}{4}\frac{Z_{N+2}Z_{N-2}}{Z_N^2}.$$

Let $Y_N = \log Z_N|_{t_i=0, i>3}$. Then the equations above imply that

$$\partial_1 Y_N = t_2\partial_1 Y_N + 2t_3\partial_2 Y_N + \frac{Nt_1}{2}, \tag{4.1}$$

$$\partial_2 Y_N = \frac{1}{2}t_1\partial_1 Y_N + t_2\partial_2 Y_N + \frac{3}{2}t_3\partial_3 Y_N + \frac{N(N+1)}{4}, \tag{4.2}$$

$$\partial_1^4 Y_N + 3\partial_2^2 Y_N - 3\partial_3\partial_1 Y_N + 6(\partial_1^2 Y_N)^2 = \frac{3}{4}\exp(Y_{N+2} + Y_{N-2} - 2Y_N). \tag{4.3}$$

The $\partial_2 Y_N$ term can be eliminated from (4.1) using (4.2) and similarly the $\partial_1 Y_N$ term can be eliminated from (4.2) using (4.1). Letting $D = 3t_3\partial_3$ this gives

$$\partial_1 Y_N = \frac{t_3}{A(1-t_2)} DY_N + C, \tag{4.4}$$

$$\partial_2 Y_N = \frac{1}{2A} DY_N + \frac{B}{A}, \tag{4.5}$$

where

$$A = 1 - \frac{t_1 t_3}{1 - t_2} - t_2,$$
$$B = \frac{N t_1^2}{4(1 - t_2)} + \frac{N(N+1)}{4},$$

and

$$C = \frac{2 t_3 B}{A(1 - t_2)} + \frac{N t_1}{2(1 - t_2)}.$$

Letting $T = 2DY_n|_{t_i = x\delta_{i,3}} + N(N+1) - \frac{N}{2x^2} = L^{(3)}(x, N) + N(N+1) - \frac{N}{2x^2}$ (the second equality follows from Theorem 4.2.3), (4.4) and (4.5) can be used to determine

$$\partial_1^2 Y_N\big|_{t_i = x\delta_{i,3}} = \frac{x^2}{2}(D+4)T, \tag{4.6}$$
$$\partial_1^4 Y_N\big|_{t_i = x\delta_{i,3}} = \frac{x^4}{2}(D+12)(D+8)(D+4)T, \tag{4.7}$$
$$\partial_2^2 Y_N\big|_{t_i = x\delta_{i,3}} = \frac{1}{8}(D+2)T - \frac{N}{4x^2}, \tag{4.8}$$
$$\partial_1 \partial_3 Y_N\big|_{t_i = x\delta_{i,3}} = \frac{1}{6}(D+3)T - \frac{N}{4x^2}, \tag{4.9}$$

where now $D = 3x\partial_x$. Using the equations above, (4.3) becomes

$$\begin{aligned} &\frac{x^4}{2}(D+12)(D+8)(D+4)T + \frac{3}{8}(D+2)T - \frac{3}{4}\frac{N}{x^2} - \frac{1}{2}(D+3)T + \frac{3}{4}\frac{N}{x^2} \\ &\quad + 6\left(\frac{x^2}{2}(D+4)T\right)^2 = \frac{3}{4}\left(Y_{N+2}\big|_{t_i = x\delta_{i,3}} + Y_{N-2}\big|_{t_i = x\delta_{i,3}} - 2Y_N\big|_{t_i = x\delta_{i,3}}\right). \end{aligned}$$

Simplifying, this becomes

$$\begin{aligned} &2x^4(D+12)(D+8)(D+4)T - \frac{1}{2}(D+6)T + 6x^2((D+4)T)^2 \\ &\quad = \frac{3}{4}\left(Y_{N+2}\big|_{t_i = x\delta_{i,3}} + Y_{N-2}\big|_{t_i = x\delta_{i,3}} - 2Y_N\big|_{t_i = x\delta_{i,3}}\right). \end{aligned}$$

Applying the operator $2D$ to both sides of this equation gives

$$\begin{aligned} &4x^4(D+12)^2(D+8)(D+4)T - (D+6)DT + 12x^4(D+12)((D+4)T)^2 \\ &\quad = V(2x^4(D+12)(D+8)(D+4)T - \frac{1}{2}(D+6)T + 6x^4((D+4)T)^2), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} V &= 2D\, Y_{N+2}\big|_{t_i = x\delta_{i,3}} + 2D\, Y_{N-2}\big|_{t_i = x\delta_{i,3}} - 4D\, Y_N\big|_{t_i = x\delta_{i,3}} \\ &= L^{(3)}(x, N+2) + L^{(3)}(x, N-2) - 2L^{(3)}(x, N). \end{aligned}$$

Now, using the fact that the coefficient of x^n in $L^{(3)}(x,y)$ is a polynomial in y (see Remark 4.2), we may extract coefficients in (4.10) to get a countable number of polynomial identities which are satisfied and thus equation (4.10) holds with

$$\begin{aligned}T &= L^{(3)}(x,y) + y(y+1) - \frac{y}{2x^2},\\ V &= L^{(3)}(x,y+2) + L^{(3)}(x,y-2) - 2L^{(3)}(x,y).\end{aligned}$$

Recall that

$$L^{(3)}(xw^{1/2}, w^{-1}) = w^{-2}L^{(3)}(x;w).$$

We have

$$\begin{aligned}L^{(3)}(xw^{1/2}, w^{-1}+2) &= L^{(3)}\left(x(1+2w)^{1/2}\left(\frac{w}{1+2w}\right)^{1/2}, \left(\frac{w}{1+2w}\right)^{-1}\right)\\ &= w^{-2}(1+2w)^2L^{(3)}\left(x(1+2w)^{1/2}, \frac{w}{1+2w}\right).\end{aligned}$$

Similarly,

$$L^{(3)}(xw^{1/2}, w^{-1}-2) = w^{-2}(1-2w)^2L^{(3)}\left(x(1-2w)^{1/2}, \frac{w}{1-2w}\right).$$

Making the change of variables $x \mapsto w^{1/2}x$, setting $y = w^{-1}$ and then simplifying gives the desired result. □

Since any cubic map must have an even number of vertices, we let $x = z^{1/2}$. Note that in this case D becomes $D = 6z\partial_z$. Also, let

$$L_g^{(3)}(z) = [w^g]L^{(3)}(z^{1/2};w),$$

and let

$$T_g(z) = [w^g]\left(L^{(3)}(z^{1/2};w) + w + 1 - \frac{1}{2z}\right) = L_g^{(3)}(z) + \delta_{g,1} + \left(1 - \frac{1}{2z}\right)\delta_{g,0}.$$

Corollary 4.3.2. *For $g \geq 0$,*

$$\begin{aligned}&4z^2(D+12)^2(D+8)(D+4)T_{g-2} - (D+6)DT_g\\ &\qquad + 12z^2(D+12)\left(\sum_{i=0}^{g}\{(D+4)T_i\}\{(D+4)T_{g-i}\}\right)\\ &= \sum_{k=0}^{g} V_k\Bigg\{2z^2(D+12)(D+8)(D+4)T_{g-k-2}\\ &\qquad -\frac{1}{2}(D+6)T_{g-k} + 6z^2\left(\sum_{i=0}^{g-k}\{(D+4)T_i\}\{(D+4)T_{g-k-i}\}\right)\Bigg\},\end{aligned}$$

where

$$V_k = \sum_{t=0}^{k} 2^{k-t+2}(1+(-1)^{k-t})\sum_{i=0}^{k-t+2}\binom{2-t}{k-t-i+2}\frac{z^i}{i!}\partial_z^i L_t^{(3)}(z).$$

Proof. This follows after making the substitution $x = z^{1/2}$ in Theorem 4.3.1 and extracting the coefficient of w^g. □

4.4 A Structure Theorem for $L_g^{(3)}(z)$

In [37] Gao studied the generating series for rooted triangulations on arbitrary surfaces. Using a Tutte type equation for the generating series he was able to find compact representations of the generating series in the projective plane, sphere and torus cases. In each case the generating series is rational in the auxilliary, algebraic series $s = s(z)$ which is the unique power series solution to $z = \frac{1}{2}s(1-s)(1-2s)$ with $s(0) = 0$. In particular, Gao proves Theorem 4.4.1 below. Using this same method we show in Theorem 4.4.2 that the generating series for a surface of any genus is always rational when expressed in terms of the series s. Note that the method used below is essentially the same as the method used in [37] and also in [6]. The difference is that here we consider cubic maps enumerated with respect to the number of vertices and in [37] Gao considers triangulations with respect to the number of vertices.

To simplify some of the expressions below we let $\eta = 1 - 6s + 6s^2$.

Theorem 4.4.1 (Gao[37])**.** *The generating series $L_0^{(3)}(z)$ and $L_1^{(3)}(z)$ are given in terms of s by*

$$L_0^{(3)}(z) = \frac{2s(1-4s+2s^2)}{(1-s)(1-2s)^2},$$
$$L_1^{(3)}(z) = \frac{(1-2s)(1-s+s^2) - (1-6s+6s^2)^{1/2}}{s(1-s)(1-2s)}.$$

Theorem 4.4.2. *For $g \geq 0$ the series $L_g^{(3)}$ is a polynomial in $\eta^{1/2}$ with coefficients which are rational series in s.*

Proof. Let $t_g(n, k, u_1, u_2, \cdots)$ be the number of maps with genus g which have n vertices, whose root face has degree k, which have a finite number of distinguished faces the ith of which has degree u_i and where every other face has degree three. Let $I = \{i_1 < i_2 < \cdots\}$ be a finite set and let

$$T_g(x, y, I) = \sum t_g(n, k, u_1, u_2, \cdots) x^n y^k z_{i_1}^{u_1} \cdots.$$

The series of near-triangular maps (T_g) was studied by Gao[37] using a Tutte type recursion. Here we repeat part of the argument in order to prove the Theorem but we refer to the paper for more details. Recall the Tutte type recursion, letting I be a finite set and $w \notin I$ an integer, then

$$\begin{aligned} T_g(x,y,I) = y^2 \sum_{j=0/2}^{g} \sum_{S \subseteq I} & T_j(x,y,S) T_{g-j}(x,y,I-S) \\ & + 2y^3 \left. \frac{\partial}{\partial z_w} T_{g-1}(x,y,I+\{w\}) \right|_{z_w=y} \\ & + y^2 \frac{\partial}{\partial y} \Big(y T_{g-\frac{1}{2}}(x,y,I) \Big) \\ & + y^{-1} \left[T_g(x,y,I) - \delta_{0,g} \delta_{\varnothing,I} x - y T_g^1(x,I) \right] \\ & + \sum_{i \in I} \frac{y z_i}{z_i - y} \left[z_i T_g(x,z_i,I-\{i\}) - y T_g(x,y,I-\{i\}) \right] \\ & + \delta_{0,g} \delta_{\varnothing,I} x, \end{aligned} \tag{4.11}$$

where here the summation index beginning at 0/2 means to sum over half integers from 0 to g and where $T_g^1(x,I) = [y] T_g(x,y,I)$. Note also that if we let $T_g^3(x,I) = [y^3] T_g(x,y,I)$ then

$$T_0^3(x,\varnothing) = T_0^1(x,\varnothing) - x^2, \qquad T_{\frac{1}{2}}^3(x,\varnothing) = T_{\frac{1}{2}}^1(x,\varnothing) - x,$$

and

$$T_g^3(x,\varnothing) = T_g^1(x,\varnothing) \text{ for } g \geq 1.$$

Let

$$\begin{aligned} A(x,y) &= 2y^3 T_0(x,y,\varnothing) + 1 - y, \\ B(x,y) &= (1-y)^2 + 4y^3 (x - xy + y T_0^1(x,\varnothing)). \end{aligned}$$

Then (4.11) when $(g,I) = (0,\varnothing)$ is equivalent to

$$A(x,y)^2 = B(x,y),$$

and for $(g,I) \neq (0,\varnothing)$ (4.11) is equivalent to

$$\begin{aligned} -A(x,y) T_g(x,y,I) = y^3 & \sum_{\substack{j=0/2 \\ (j,S) \neq (0,\varnothing),(g,I)}}^{g} \sum_{S \subseteq I} T_j(x,y,S) T_{g-j}(x,y,I-S) \\ & + 2y^4 \left. \frac{\partial}{\partial z_w} T_{g-1}(x,y,I+\{w\}) \right|_{z_w=y} \\ & + y^3 \frac{\partial}{\partial y} \Big(y T_{g-\frac{1}{2}}(x,y,I) \Big) \\ & + y^2 \sum_{i \in I} \frac{z_i}{z_i - y} \left[z_i T_g(x,z_i,I-\{i\}) - y T_g(x,y,I-\{i\}) \right] \\ & - y T_g^1(x,I), \end{aligned}$$

as in [37]. Let $f(x)$ be a series such that $A(x,f(x)) = 0$. Then, if we define $F^{(k)} = \left(\frac{\partial^k F}{\partial y^k}\right)\Big|_{y=f}$ we see that

$$B^{(0)} = (1-f)^2 + 4f^3(x - xf + fT_0^1(x,\varnothing)) = 0,$$
$$B^{(1)} = -2(1-f) + 4f^2(3x - 4xf + 4fT_0^1(x,\varnothing)) = 0.$$

Solving this and setting $f = \frac{1}{1-s}$ we get

$$x = \frac{1}{2}s(1-s)(1-2s),$$

and

$$T_0^1(x,\varnothing) = \frac{1}{4}s^2(1-s)(1-3s).$$

This implies that

$$T_0^3(x,\varnothing) = \frac{1}{2}s^3(1-s)(1-4s+2s^2).$$

By a Theorem of Brown[13], since $B(x,y)$ is a square it can be uniquely written as

$$B(x,y) = Q(x,y)^2 R(x,y),$$

where $Q(x,y)$ and $R(x,y)$ are polynomials in y and where

$$R(x,0) = 1.$$

We may suppose that $B(x,y) = (1+yQ_1)^2(1+R_1y+R_2y^2)$ where Q_1, R_1, R_2 are rational series in s. Solving this gives

$$Q_1 = s-1, \qquad R_1 = -2s, \qquad \text{and } R_2 = -2s+3s^2.$$

It follows by the same argument as in [37] that $T_g(x,y,I)$ is a polynomial in $R^{\frac{1}{2}}(x,y)$, $R^{\frac{1}{2}}(x,z_i), i \in I$ and $R^{\frac{1}{2}}\left(x,\frac{1}{1-s}\right)$ with coefficients which are rational series in $x, y, z_i, i \in I, R_1, R_2$ and $\frac{1}{1-s}$. More specifically, if $T_g(x) = [y^3]T_g(x,y,\varnothing)$ then $T_g(x)$ is a polynomial in $R^{\frac{1}{2}}\left(x,\frac{1}{1-s}\right)$ with coefficients which are rational series in s. Since $R^{\frac{1}{2}}\left(x,\frac{1}{1-s}\right) = \frac{\eta^{\frac{1}{2}}}{1-s}$ it follows that $T_g(x)$ is a polynomial in $\eta^{\frac{1}{2}}$ with coefficients which are rational series in s. The result then follows from the fact that $L_g^{(3)}(x) = x^{2g-2}T_g(x)$. $\square$

We will now make use of Theorem 4.4.2 and Theorem 4.3.1 to prove a stronger structure theorem for $L_g^{(3)}(z)$. In addition, we will show that the recurrence implied by Theorem 4.3.1 can be used to determine $L_g^{(3)}(z)$ inductively. It will be convenient for what follows to define a basis with which to work. For $i \geq 0$, define

$$\psi_i = \begin{cases} \frac{(1-2s)}{\eta^{1+i/2}}, & \text{if } i \text{ is odd,} \\ \frac{1}{\eta^{1+i/2}}, & \text{if } i \text{ is even.} \end{cases}$$

Proposition 4.4.3. *The basis ψ_i defined above satisfies the multiplicative identity,*

$$\psi_i \psi_j = \chi(i,j)\psi_{i+j+2} + (1 - \chi(i,j))\psi_{i+j},$$

where

$$\chi(i,j) = \begin{cases} 1, & \text{if either } i \text{ or } j \text{ is even,} \\ \frac{1}{3}, & \text{if both } i \text{ and } j \text{ are odd.} \end{cases}$$

Also, the action of the operator D on the ψ_i basis is given by

$$D\psi_i = \begin{cases} (i+2)\psi_{i+4} + (i+2)\psi_{i+2} - 2(i+2)\psi_i, & \text{if } i \text{ is even,} \\ (i+2)\psi_{i+4} + i\psi_{i+2} - 2(i+1)\psi_i, & \text{if } i \text{ is odd.} \end{cases}$$

Proof. Each of these identites follow from the definition of ψ_i, η, s and D. $\square$

First, we state a technical lemma which will allow us to prove Theorem 4.4.5. The proof of this lemma will appear in the Appendix since though the proof is relatively straightforward, it requires a very lengthy computation.

Lemma 4.4.4. *Suppose for $g \geq 3$,*

$$R_g(s) = \left\{ \frac{4}{3}D^2 \; -4(g\psi_2 + \psi_0 - 4)D - \frac{4}{3}(2\psi_4 + 9g(3-g)\psi_2 + 12\psi_0 - 32) \right\} L_g^{(3)},$$

and that

$$L_k^{(3)} = \sum_{i=0}^{5k-8} \mu_k(i)\psi_i,$$

for $2 \leq k < g$. Then

$$R_g(s) = \sum_{i=0}^{5g} r_g(i)\psi_i,$$

for some constants $r_g(i)$. Furthermore,

$$\begin{aligned} r_g(5g) = {} & \frac{1}{162}(5g-8)(5g-12)(5g-6)w_{g-2} \\ & - \frac{1}{3^6 5!}(5g-6)(5g-10)(5g-14)(5g-18)(5g-22)w_{g-4} \\ & + \frac{(5g-6)}{54}\sum_{i=1}^{g-1}\chi(i,g-i)w_i w_{g-i} \\ & - \frac{(5g-6)}{2^3 3^6}\sum_{i=2}^{g-1}\chi(i,g-i)(5i-8)(5i-12)w_{i-2}w_{g-i} \\ & - \frac{1}{9 \cdot 216}\sum_{i=1}^{g-2}\sum_{j=1}^{g-i-1}(5i-2)\chi(i,g-i)\chi(j,g-i-j)w_i w_j w_{g-i-j}, \end{aligned}$$

where $w_k = (5k-6)\mu_k(5k-8)$.

Proof. See the Appendix. $\square$

Theorem 4.4.5. *For $g \geq 2$ the generating series for cubic maps on all surfaces is of the form*

$$L_g^{(3)} = \sum_{i=0}^{5g-8} \mu_g(i)\psi_i,$$

where the $\mu_g(i)$ do not depend on z. Furthermore, $L_g^{(3)}$ for $g \geq 2$ can be determined recursively using the equation given in Corollary 4.3.2.

Proof. Using Theorem 4.4.1, Corollary 4.3.2 can be rearranged so that when written in the ψ_i basis we have

$$\left\{\frac{4}{3}D^2 \ -4(g\psi_2 + \psi_0 - 4)D - \frac{4}{3}(2\psi_4 + 9g(3-g)\psi_2 + 12\psi_0 - 32)\right\} L_g^{(3)} = R_g(s),$$

where $R_g(s)$ depends only on $L_i^{(3)}$, $i = 0 \cdots g-1$. That $L_g^{(3)} = \sum_{i=0}^{5g-8} \mu_g(i)\psi_i$ then follows by induction by directly computing the base case

$$L_2^{(3)} = \frac{23}{12}\psi_0 - 3\psi_1 + \frac{13}{12}\psi_2,$$

and then using Lemma 4.4.4, in particular, that

$$R_g(s) = \sum_{i=0}^{5g} r_g(i)\psi_i.$$

The result then follows using Theorem 4.4.2 and comparing coefficients. □

4.5 Asymptotic Behaviour

In this final section we will consider some of the implications of the results above. In particular, we examine the leading coefficient in the ψ basis expansion of $L_g^{(3)}(z)$ for each g. Using Theorem 4.3.2 we can determine a recursion which the leading coefficients satisfy. Since the asymptotic behaviour of each element in the ψ basis can be determined, this allows us to determine the asymptotic behaviour of the series $L_g^{(3)}(z)$ itself. In what follows we will let

$$\alpha_g = \mu_g(5g-8),$$

i.e., α_g is the leading coefficient of $L_g^{(3)}$ in the ψ_i basis. Let

$$\beta_g = \begin{cases} \frac{(5g-6)}{\sqrt{3}}\alpha_g, & \text{if } g \text{ is odd,} \\ (5g-6)\alpha_g, & \text{if } g \text{ is even.} \end{cases}$$

Theorem 4.5.1. *For all $g > 1$,*

$$\begin{aligned}\frac{8}{3}(g-1)\beta_g = \frac{1}{162}&(5g-8)(5g-12)(5g-6)\beta_{g-2}\\ &- \frac{1}{3^6 5!}(5g-6)(5g-10)(5g-14)(5g-18)(5g-22)\beta_{g-4}\\ &+ \frac{8(5g-6)}{432}\sum_{i=1}^{g-1}\beta_i\beta_{g-i} - \frac{(5g-6)}{2^3 3^6}\sum_{i=2}^{g-1}(5i-8)(5i-12)\beta_{i-2}\beta_{g-i}\\ &- \frac{1}{2^3 3^5}\sum_{i=1}^{g-1}\sum_{j=1}^{g-i-1}(5i-2)\beta_i\beta_j\beta_{g-i-j}.\end{aligned}$$

Proof. Using the notation in the proof of Theorem 4.4.5, we have

$$p_g(D)L_g^{(3)} = \sum_{i=0}^{5g} r_g(i)\psi_i,$$

where

$$p_g(D) = \frac{4}{3}D^2 - 4(g\psi_2 + \psi_0 - 4)D - \frac{4}{3}(2\psi_4 + 9g(3-g)\psi_2 + 12\psi_0 - 32).$$

Now,

$$\begin{aligned}[\psi_{5g}]p_g(D)L_g^{(3)} &= \frac{4}{3}(5g-2)(5g-6)\alpha_g - 4g(5g-6)\alpha_g\\ &= \frac{8}{3}(5g-6)(g-1)\alpha_g.\end{aligned}$$

So,

$$\frac{8}{3}(5g-6)(g-1)\alpha_g = r_g(5g).$$

Letting $w_k = (5k-6)\alpha_k$ and applying Lemma 4.4.4 we have,

$$\begin{aligned}\frac{8}{3}(g-1)w_g &= r_g(5g),\\ &= \frac{1}{162}(5g-8)(5g-12)(5g-6)w_{g-2}\\ &\quad - \frac{1}{3^6 5!}(5g-6)(5g-10)(5g-14)(5g-18)(5g-22)w_{g-4}\\ &\quad + \frac{(5g-6)}{54}\sum_{i=1}^{g-1}\chi(i,g-i)w_i w_{g-i}\\ &\quad - \frac{(5g-6)}{2^3 3^6}\sum_{i=2}^{g-1}\chi(i,g-i)(5i-8)(5i-12)w_{i-2}w_{g-i}\\ &\quad - \frac{1}{9\cdot 216}\sum_{i=1}^{g-2}\sum_{j=1}^{g-i-1}(5i-2)\chi(i,g-i)\chi(j,g-i-j)w_i w_j w_{g-i-j}.\end{aligned}$$

If we make the substitution

$$w_k = \begin{cases} \sqrt{3}\beta_k, & \text{if } k \text{ is odd,} \\ \beta_k, & \text{if } k \text{ is even,} \end{cases}$$

then it is straightforward to check that if g is even, $\chi(i,g-i)w_iw_{g-i} = \beta_i\beta_{g-i}$ and $\chi(i,g-i)\chi(j,g-i-j)w_iw_jw_{g-i-j} = \beta_i\beta_j\beta_{g-i-j}$ for any i and j and that if g is odd then $\chi(i,g-i)w_iw_{g-i} = \sqrt{3}\beta_i\beta_{g-i}$ and $\chi(i,g-i)\chi(j,g-i-j)w_iw_jw_{g-i-j} = \beta_i\beta_j\beta_{g-i-j}$. So, taking the recursion for w_g above and making the substitution, we get (after dividing by $\sqrt{3}$ if g is odd) the desired result. □

Let

$$\beta(z) = \sum_{n\geq 0} \beta_n \left(\frac{3}{2}\right)^n z^{-(5n-2)/4},$$

with $\beta_0 = -36$ and $\beta_1 = \frac{18}{\sqrt{3}}$.

Corollary 4.5.2. *The generating series $\beta(z)$ satisfies the differential equation*

$$-\frac{8}{15}z\beta'(z)-\frac{16}{15}\beta(z)+\frac{8}{135}\beta'''''(z)+\frac{2}{81}\beta''(z)\beta'(z)+\frac{2}{81}\beta'''(z)\beta(z)+\frac{1}{486}\beta(z)^2\beta'(z) = 0.$$

Proof. That the differential equation above is equivalent to the recursion for the coefficients β_k is easily checked by extracting coefficients. □

From the structure theorem for $L_g^{(3)}$, we know that

$$L_g^{(3)} \approx \begin{cases} \frac{\alpha_g}{\eta^{(5g-6)/2}}, & \text{if } g \text{ is even,} \\ \frac{\alpha_g(1-2s)}{\eta^{(5g-6)/2}}, & \text{if } g \text{ is odd.} \end{cases}$$

Further, it is a result of Gao[37] that

$$\eta \approx \left(\frac{\sqrt{6}}{2}\right)^{(5g-6)/2} \left(1-12\sqrt{3}z\right)^{-(5g-6)/4}.$$

Darboux's theorem then implies that

$$\ell_g(n) = [z^n]L_g^{(3)} \sim \frac{\beta_g}{5g-6}\left(\frac{3}{2}\right)^{(5g-6)/4} \frac{n^{(5g-2)/4}}{\Gamma\left(\frac{5g-6}{4}\right)}\left(12\sqrt{3}\right)^n.$$

In terms of the orientable map asymptotics constant t_g and the nonorientable map asymptotics constant p_g, it is straightforward to show that

$$\beta_k = \left(\frac{2}{3}\right)^k 9\left((5k-6)v_{k-1} - 4u_{k/2}\right),$$

where

$$u_n = 2^{n-2}\Gamma\left(\frac{5n-1}{4}\right)t_n,$$
$$v_n = 2^{(n-3)/2}\Gamma\left(\frac{5n-1}{4}\right)p_{(n+1)/2},$$

and we adopt the convention that $u_n = 0$ if n is not an integer. In particular, $u_0 = 1$ and $v_0 = -\sqrt{3}$

Let

$$u(z) = z^{1/2}\sum_{n\geq 0} u_n z^{-5n/2},$$
$$v(z) = z^{1/4}\sum_{n\geq 0} v_n z^{-5n/2}.$$

Bender, Richmond and Gao[9] showed that $u(z)$ satisfies the ordinary differential equation

$$u^2 - \frac{1}{6}u'' = z.$$

Using the differential equation for u and Corollary 4.5.2 we can now prove Theorem 4.1.1 (Conjecture 1 of Garoufalidis and Mariño[41]).

Proof of Theorem 4.1.1. Let v_n be the unique sequence of numbers such that the generating series

$$v(z) = z^{1/4}\sum_{n\geq 0} v_n z^{-5n/4}$$

satisfies the differential equation $2v' - v^2 + 3u = 0$ and with $v_0 = -\sqrt{3}$. Let

$$\beta(z) = -36(u(z) + v'(z)).$$

Substituting this β into the differential equation in Corollary 4.5.2 and reducing using the relations $u^2 - \frac{1}{6}u'' = z$ and $2v' - v^2 + 3u = 0$ we find that β is a solution.

Also, checking β_0 and β_1 we see that this is the unique solution which determines the map asymptotics constants. □

Chapter 5

Monotone Hurwitz Numbers

5.1 Introduction

Suppose f is a symmetric function of bounded degree and for any partition λ let A_λ be the multiset of contents of the cells in λ (recall that the content of a cell is the difference $j - i$ where j is the column index and i is the row index of the cell). We define the *Plancherel average* of f to be

$$\Phi_n(f) = \sum_{\lambda \vdash n} \frac{f(A_\lambda)(\dim \lambda)^2}{n!},$$

where $\dim \lambda$ is the dimension of the irreducible representation of S_n indexed by λ. Alternatively, $\dim \lambda$ is the number of standard Young tableaux of shape λ. As a simple example, since $\sum_{\lambda \vdash n} (\dim \lambda)^2 = n!$ we see that $\Phi_n(1) = 1$. In other words, $\Phi_n(f)$ may be thought of as the expected value of f evaluated over the partitions of size n where the probability of a partition λ is given by $\frac{(\dim \lambda)^2}{n!}$. This measure is called the Plancherel measure.

The problem of computing Plancherel averages of various symmetric functions was considered by Stanley [106] where it was shown that $\Phi_n(f)$ is always a polynomial in n. In addition, Stanley gave a closed form for the Plancherel average of one and two part elementary symmetric functions. That is, suppose e_k are the elementary symmetric functions. Then Theorem 2.2 in [106] and the result immediately following it are that,

$$\sum_{n \geq 0} \sum_{k=0}^{n} \frac{\Phi_n(e_k(x_1^2, x_2^2, \cdots, x_n^2))t^{n-k}x^n}{n!} = (1 - x)^{-t},$$

and

$$\sum_{n \geq 0} \sum_{k,m=0}^{n} \frac{\Phi_n(e_k e_m)t^{n-k}v^{n-m}x^n}{n!} = (1 - x)^{-tv}.$$

Also, Fujii et al [35] showed that if $F_r = \sum_{i \geq 0} \prod_{j=0}^{r-1} (x_i^2 - j^2)$ then

$$\Phi_n(F_r) = \frac{(2r)!}{(r+1)!^2}(n)_{r+1},$$

where $(n)_{r+1} = n(n-1)\cdots(n-r)$. This formula implies the following identity for the Plancherel average of power sum symmetric functions:

$$\Phi_n(p_{2k}) = \sum_{j=1}^{k} T(k,j)\frac{(2j)!}{(j+1)!^2}(n)_{j+1},$$

where $T(k,j) = 2\sum_{i=1}^{j} \frac{(-1)^{j-i} i^{2k}}{(j-i)!(j+i)!}$.

The topic of this chapter is the study of the Plancherel average of complete symmetric functions. In particular, we consider the exponential generating series

$$\mathcal{H} = \sum_{n\geq 0}\sum_{k\geq 0}\frac{t^k x^n}{n!}\Phi_n(h_k),$$

and show that the related series, $\gamma = \frac{x\partial}{\partial x}\log\mathcal{H}$ satisfies the non-linear differential equation

$$t^2x^2\gamma''' + t^2x\gamma'' - \gamma' - 4t^2\gamma\gamma' + 6t^2x(\gamma')^2 + 1 = 0,$$

where $'$ denotes differentiation with respect to x. Using this equation we are able to derive closed form expressions for some of the coefficients in the generating series γ as well as for Plancherel averages $\Phi_n(h_{2k})$ for small values of k. In addition, we determine the asymptotics of coefficients in γ. Specifically, we show that

$$[t^{2(n+g-1)}x^n]\gamma \sim a_g \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^{5g-3}\frac{n^{\frac{5g-7}{2}}}{\Gamma\left(\frac{5g-3}{2}\right)}\left(\frac{27}{2}\right)^n,$$

where a_g is a constant depending only on g. We also show that a_g is a rescaling of the map asymptotics constant studied by Richmond, Bender and Gao [9] and also appearing in the asymptotics of Hurwitz numbers [99]. As applications of these results, we give a combinatorial interpretation of the generating series γ, relating its coefficients to a restricted permutation factorization problem. Also, we discuss some applications to random matrix theory, specifically the identity correlators in $U(N)$ and the HCIZ integral.

The remainder of this chapter is organized as follows. In Section 5.2 we give some background results which we will use throughout. In particular, we describe the more general class expansion problem associated to complete symmetric functions, as studied by Lassalle [76] and Feray [32] and the monotone Hurwitz problem studied by Goulden, Guay-Paquet and Novak [44, 45].

In Section 5.3 we prove the differential equation for γ, making use of the results of Lassale and Feray. We then apply this differential equation to the structural results of Goulden, Guay-Paquet and Novak to obtain a recursive method for constructing the graded components, γ_g of γ in Section 5.4. In Section 5.5 we then derive the asymptotics of the coefficients in γ and describe some applications in Section 5.6.

5.2 Background

We begin by defining some special elements in the group algebra $\mathbb{C}[S_n]$. For $k = 2, \cdots, n$, let

$$J_k = \sum_{i=1}^{k-1} (i\ k),$$

where $(i\ k)$ is the transposition that swaps i and k and leaves everything else fixed.

The problem of evaluating symmetric functions at the elements J_k was studied independently by Jucys [65] and Murphy [83, 84] in relation to representation theory of the symmetric group and as such the elements J_k are referred to as *Jucys-Murphy elements*. Jucys and Murphy showed that

$$e_k(J_2, \cdots, J_n) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = n-k}} C_\lambda,$$

where C_λ is the sum of all permutations (in S_n) with cycle type λ. In particular, if σ appears in the sum C_λ then $\ell(\lambda)$ is the number of cycles in σ.

Farahat and Higman [31] showed that the set of class sums, denoted C_λ, generate a subalgebra of the center of $\mathbb{C}[S_n]$ and, in particular, generate the center of $\mathbb{C}[S_n]$. Thus, the result of Jucys and Murphy implies that any symmetric function evaluated at $J_2, \cdots, J_n$, lies in the center of $\mathbb{C}[S_n]$.

Thus, we may consider the class of problems of the form: given a symmetric function F and a partition $\lambda \vdash n$, what is the coefficient of C_λ in $F(J_2, \cdots, J_n)$? Such a problem is known as a class expansion problem and arises in many areas such as algebraic geometry[89], physics[35], random matrix theory[44, 22, 79] and various other enumerative contexts.

The particular problem we consider here is a specialization of the class expansion problem corresponding to complete symmetric functions. That is, we wish to determine the coefficient of C_{1^n} in $h_k(J_2, \cdots, J_n)$ (the equivalence between this problem and the one stated in the introduction will be made clear in the following section). More generally, we define the generating series for all of the class expansion coefficients

$$\overrightarrow{\mathcal{H}} = \sum_{n \geq 0} \sum_{k \geq 0} t^k \sum_{\lambda \vdash n} \frac{|C_\lambda|}{n!} [C_\lambda] h_k(J_2, \cdots, J_n) p_\lambda z^n$$

where $[C_\lambda] h_k(J_2, \cdots, J_n)$ denotes the coefficient of C_λ in $h_k(J_2, \cdots, J_n)$. We also define the related series

$$\Gamma = \log \overrightarrow{\mathcal{H}}.$$

The problem of computing the coefficients of $\overrightarrow{\mathcal{H}}$ was considered by Lassale [76] (using the algebra of shifted symmetric functions) and Feray [32] (using combinatorics of Jucys-Murphy elements) where it was shown that the generating series Γ satisfied the following partial differential equations.

Proposition 5.2.1. *The generating series Γ satisfies the following partial differential equations,*

$$\frac{\partial \Gamma}{\partial p_1} = 1 + t\sum_{i\geq 1}(i+1)p_i\frac{\partial \Gamma}{\partial p_{i+1}}, \tag{5.1}$$

and

$$\sum_{i\geq 1}(i+1)p_i\frac{\partial \Gamma}{\partial p_{i+1}} = t\sum_{i,j\geq 1} ijp_{i+j-1}\frac{\partial^2\Gamma}{\partial p_i \partial p_j} + t\sum_{i,j\geq 1} ijp_{i+j-1}\frac{\partial \Gamma}{\partial p_i}\frac{\partial \Gamma}{\partial p_j} + t\sum_{i,j\geq 1}(i+j+1)p_ip_j\frac{\partial \Gamma}{\partial p_{i+j+1}}. \tag{5.2}$$

The problem of determining the coefficients of Γ was also studied by Goulden, Guay-Paquet and Novak [45] where it was called the monotone Hurwitz problem. One of the main results in [45] was that there is an alternate grading of Γ in which the components of Γ exhibit uniform structural properties. In particular, making the substitutions $p_i \mapsto p_i w^{\frac{-i-1}{2}}$, $t \mapsto w^{1/2}$ allows us to write

$$w\Gamma = \sum_{g\geq 0} w^g \Gamma_g.$$

Thus

$$[p_\lambda z^n]\Gamma_g = [p_\lambda t^k z^n]\Gamma$$

where $2g = k + 2 - n - \ell(\lambda)$. The parameter g is called the genus.

Theorem 5.2.2. *(Theorem 0.5 in [45]) Let s be the unique formal power series solution of the functional equation*

$$s = z(1-\omega)^{-2}$$

in the ring $\mathbb{Q}[[z, p_1, p_2, \cdots]]$, where $\omega = \sum_{k\geq 1}\binom{2k}{k}p_k s^k$. Also, define $\eta = \sum_{k\geq 1}(2k+1)\binom{2k}{k}p_k s^k$. Then the genus one monotone Hurwitz series is

$$\Gamma_1 = \frac{1}{24}\log\frac{1}{1-\eta} - \frac{1}{8}\log\frac{1}{1-\omega},$$

and for $g \geq 2$ we have

$$\Gamma_g = -c_{g,(0)} + \frac{1}{(1-\eta)^{2g-2}}\sum_{d=0}^{3g-3}\sum_{\alpha\vdash d}\frac{c_{g,\alpha}\eta_\alpha}{(1-\eta)^{\ell(\alpha)}}, \tag{5.3}$$

where

$$\eta_j = \sum_{k\geq 1} k^j(2k+1)\binom{2k}{k}p_k s^k, \; j \geq 1,$$

$\eta_\alpha = \prod_i \eta_{\alpha_i}$ and the $c_{g,\alpha}$ are rational constants.

5.3 The Differential Equation for γ

So far we have been discussing the generating series

$$\overrightarrow{\mathcal{H}} = \sum_{n\geq 0}\sum_{k\geq 0} t^k \sum_{\lambda\vdash n} \frac{|C_\lambda|}{n!}[C_\lambda] h_k(J_2, \cdots, J_n) p_\lambda z^n$$

for the class expansion coefficients of the complete symmetric functions. For convenience in what follows we stop writing the indeterminate z which appears in $\overrightarrow{\mathcal{H}}$. This indeterminate may be recovered by making the substitution $p_k \mapsto p_k z^k$. To relate the series $\overrightarrow{\mathcal{H}}$ and the generating series

$$\mathcal{H} = \sum_{n\geq 0}\sum_{k\geq 0} \frac{t^k x^n}{n!}\Phi_n(h_k),$$

we use the fact that the Jucys-Murphy elements act diagonally (as operators with respect to multiplication) on the orthogonal idempotent basis in the center of the symmetric group. In particular, the eigenvalue of any symmetric function F in Jucys-Murphy elements corresponding to the orthogonal idempotent labelled by the partition λ is given by $F(A_\lambda)$.

Using this it is straightforward to show that

$$\overrightarrow{\mathcal{H}} = \sum_{\lambda\in\mathcal{P}}\sum_{k\geq 0} h_k(A_\lambda) t^k s_\lambda \frac{\dim\lambda}{|\lambda|!} = \sum_{\lambda\in\mathcal{P}}\prod_{(i,j)\in\lambda}\frac{1}{1-(j-i)t} s_\lambda \frac{\dim\lambda}{|\lambda|!}. \tag{5.4}$$

For more details see [79] or the last section in [44]. Equation (5.4) tells us that in addition to the linear differential equations satisfied by Γ given in Theorem 5.2.1, Γ also satisfies the nonlinear differential equations arising from the KP hierarchy.

Proposition 5.3.1. *The series*

$$\overrightarrow{\mathcal{H}} = \sum_{\lambda\in\mathcal{P}}\prod_{(i,j)\in\lambda}\frac{1}{1-(j-i)t} s_\lambda \frac{\dim\lambda}{|\lambda|!},$$

satisfies the KP hierarchy. In particular, if $\Gamma = \log\overrightarrow{\mathcal{H}}$ *then the series* Γ *satisfies the partial differential equation*

$$\frac{1}{12}\frac{\partial^4\Gamma}{\partial p_1^4} - \frac{\partial^2\Gamma}{\partial p_1\partial p_3} + \frac{\partial^2\Gamma}{\partial p_2^2} + \frac{1}{2}\left(\frac{\partial^2\Gamma}{\partial p_1^2}\right)^2 = 0. \tag{5.5}$$

Proof. Suppose

$$\Omega = \sum_{\lambda\in\mathcal{P}}\prod_{(i,j)\in\lambda}\frac{1}{1-(j-i)t} s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q}).$$

Then Ω is a content-type series and so Corollary 2.4.11 tells us that Ω satisfies the KP hierarchy. Recall that

$$[p_1^{|\lambda|}] s_\lambda(\mathbf{p}) = \frac{\dim\lambda}{|\lambda|!},$$

so that

$$\overrightarrow{\mathcal{H}} = \Omega\big|_{q_i=\delta_{i,1}}.$$

Thus, $\overrightarrow{\mathcal{H}}$ also satisfies the KP hierarchy. Lastly, that Γ satisfies the KP equation (Equation (5.5)) follows from Example 2.2.5. □

To make the connection from $\overrightarrow{\mathcal{H}}$ to $\mathcal{H}$ we again use the fact that $[p_1^{|\lambda|}]s_\lambda = \frac{\dim\lambda}{|\lambda|!}$ so that

$$\begin{aligned}\overrightarrow{\mathcal{H}}\Big|_{\substack{p_i=x\delta_{i,1}\\ z=1}} &= \sum_{\lambda}\sum_{k\geq 0} h_k(A_\lambda)t^k x^n \frac{(\dim\lambda)^2}{|\lambda|!^2}\\ &= \sum_{n\geq 0}\sum_{k\geq 0}\frac{t^k x^n}{n!}\sum_{\lambda\vdash n} h_k(A_\lambda)\frac{(\dim\lambda)^2}{n!} = H.\end{aligned}$$

Using this we can prove one of the main results in this chapter.

Theorem 5.3.2. *If*

$$\gamma = x\frac{d}{dx}\log\mathcal{H} = x\frac{d}{dx}\log\overrightarrow{\mathcal{H}}\Big|_{\substack{p_i=x\delta_{i,1}\\ z=1}} = x\frac{d}{dx}\,\Gamma\big|_{\substack{p_i=x\delta_{i,1}\\ z=1}}$$

then

$$t^2x^2\gamma''' + t^2x\gamma'' - \gamma' - 4t^2\gamma\gamma' + 6t^2x\gamma'^2 + 1 = 0, \tag{5.6}$$

where $'$ denotes differentiation with respect to x.

Proof. First, recall Equation (5.1) in Proposition 5.2.1,

$$\frac{\partial\Gamma}{\partial p_1} = 1 + t\sum_{i\geq 1}(i+1)p_i\frac{\partial\Gamma}{\partial p_{i+1}}. \tag{5.7}$$

Setting $z = 1$, $p_1 = x$ and $p_i = 0$ for $i > 1$ (abbreviated $p_i = x\delta_{i,1}$) in Equation (5.7) gives

$$\frac{d\Gamma}{dx}\Big|_{\substack{p_i=x\delta_{i,1}\\ z=1}} = 1 + 2tx\,\frac{\partial\Gamma}{\partial p_2}\Big|_{\substack{p_i=x\delta_{i,1}\\ z=1}}. \tag{5.8}$$

Using the fact that $\gamma = x\,\frac{d\Gamma}{dx}\big|_{\substack{p_i=x\delta_{i,1}\\ z=1}}$ then gives

$$\frac{\gamma}{x} = 1 + 2tx\,\frac{\partial\Gamma}{\partial p_2}\Big|_{\substack{p_i=x\delta_{i,1}\\ z=1}},$$

or,

$$\frac{\partial\Gamma}{\partial p_2}\Big|_{\substack{p_i=x\delta_{i,1}\\ z=1}} = \frac{\gamma - x}{2tx^2}. \tag{5.9}$$

Differentiating Equation (5.1) in Proposition 5.2.1 (Equation (5.7) above) gives

$$\frac{\partial^2\Gamma}{\partial p_1\partial p_2} = 3t\frac{\partial\Gamma}{\partial p_3} + t\sum_{i\geq 1}(i+1)p_i\frac{\partial^2\Gamma}{\partial p_2\partial p_{i+1}}. \tag{5.10}$$

Setting $z = 1$ and $p_i = x\delta_{i,1}$ in equation (5.10) gives

$$\frac{d}{dx}\left(\frac{\partial \Gamma}{\partial p_2}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}} \right) = 3t\, \frac{\partial \Gamma}{\partial p_3}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}} + 2tx\, \frac{\partial^2 \Gamma}{\partial p_2^2}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}}. \tag{5.11}$$

Using Equation (5.9) in Equation (5.11) gives

$$\left(\frac{\gamma - x}{2tx^2} \right)' = 2t\, \frac{\partial \Gamma}{\partial p_3}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}} + 2tx\, \frac{\partial^2 \Gamma}{\partial p_2^2}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}}. \tag{5.12}$$

Similarly, recall Equation (5.2) in Proposition 5.2.1,

$$\begin{aligned} \sum_{i\geq 1}(i+1)p_i \frac{\partial \Gamma}{\partial p_{i+1}} = t \sum_{i,j\geq 1} ijp_{i+j-1}\frac{\partial^2 \Gamma}{\partial p_i \partial p_j} &+ t\sum_{i,j\geq 1} ijp_{i+j-1}\frac{\partial \Gamma}{\partial p_i}\frac{\partial \Gamma}{\partial p_j} \\ &+ t\sum_{i,j\geq 1}(i+j+1)p_i p_j \frac{\partial \Gamma}{\partial p_{i+j+1}}. \end{aligned} \tag{5.13}$$

Setting $z = 1$ and $p_i = x\delta_{i,1}$ in Equation (5.13) gives

$$2x\, \frac{\partial \Gamma}{\partial p_2}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}} = tx\, \frac{d^2 \Gamma}{dx^2}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}} + tx\left(\frac{d\Gamma}{dx}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}} \right)^2 + 3tx^2\, \frac{\partial \Gamma}{\partial p_3}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}},$$

or,

$$2x\, \frac{\partial \Gamma}{\partial p_2}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}} = tx\left(\frac{\gamma}{x}\right)' + tx\left(\frac{\gamma}{x}\right)^2 + 3tx^2\, \frac{\partial \Gamma}{\partial p_3}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}}. \tag{5.14}$$

Using Equation (5.9) in Equation (5.14) gives

$$\frac{\gamma - x}{tx} = tx\left(\frac{\gamma}{x}\right)' + tx\left(\frac{\gamma}{x}\right)^2 + 3tx^2\, \frac{\partial \Gamma}{\partial p_3}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}},$$

or,

$$\frac{\partial \Gamma}{\partial p_3}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}} = \frac{1}{3tx^2}\left(\frac{\gamma - x}{tx} - tx\left(\frac{\gamma}{x}\right)' - tx\left(\frac{\gamma}{x}\right)^2 \right). \tag{5.15}$$

Using Equation (5.15) in Equation (5.12) gives

$$\left(\frac{\gamma - x}{2tx^2} \right)' = \frac{1}{x^2}\left(\frac{\gamma - x}{tx} - tx\left(\frac{\gamma}{x}\right)' - tx\left(\frac{\gamma}{x}\right)^2 \right) + 2tx\, \frac{\partial^2 \Gamma}{\partial p_2^2}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}},$$

or,

$$\frac{\partial^2 \Gamma}{\partial p_2^2}\bigg|_{\substack{p_i = x\delta_{i,1} \\ z=1}} = \frac{1}{2tx}\left(\frac{\gamma - x}{2tx^2} \right)' - \frac{1}{2tx^3}\left(\frac{\gamma - x}{tx} - tx\left(\frac{\gamma}{x}\right)' - tx\left(\frac{\gamma}{x}\right)^2 \right). \tag{5.16}$$

Lastly, recall Equation (5.5) in Proposition 5.3.1

$$\frac{1}{12}\frac{\partial^4 \Gamma}{\partial p_1^4} - \frac{\partial^2 \Gamma}{\partial p_1 \partial p_3} + \frac{\partial^2 \Gamma}{\partial p_2^2} + \frac{1}{2}\left(\frac{\partial^2 \Gamma}{\partial p_1^2} \right)^2 = 0. \tag{5.17}$$

Setting $z = 1$ and $p_i = x\delta_{i,1}$ in Equation (5.17) as before gives

$$\frac{1}{12}\left(\frac{\gamma}{x}\right)''' - \left(\frac{1}{3tx^2}\left(\frac{\gamma - x}{tx} - tx\left(\frac{\gamma}{x}\right)' - tx\left(\frac{\gamma}{x}\right)^2\right)\right)' \\ + \frac{1}{2tx}\left(\frac{\gamma - x}{2tx^2}\right)' - \frac{1}{2tx^3}\left(\frac{\gamma - x}{tx} - tx\left(\frac{\gamma}{x}\right)' - tx\left(\frac{\gamma}{x}\right)^2\right) + \frac{1}{2}\left(\left(\frac{\gamma}{x}\right)'\right)^2 = 0. \tag{5.18}$$

After simplification, Equation (5.18) becomes

$$t^2x^2\gamma''' + t^2x\gamma'' - \gamma' - 4t^2\gamma\gamma' + 6t^2x\gamma'^2 + 1 = 0,$$

as required. $\square$

If we make the substitutions $x \mapsto xw^{-1}$, $t \mapsto w^{1/2}$ and $\gamma \mapsto w^{-1}\gamma$ then, as mentioned in Section 5.2, we may write

$$\gamma = \sum_{g\geq 0} w^g \gamma_g.$$

In particular, this allows us to rewrite the partial differential equation (5.6) in Theorem 5.3.2 as a partial differential equation involving w instead of t. Using this we can extract coefficients which will allow us, in the following section, to compute the γ_g recursively.

Corollary 5.3.3. *After making the variable substitutions $x \mapsto xw^{-1}$, $t \mapsto w^{1/2}$ and $\gamma \mapsto w^{-1}\gamma$, γ satisfies the following differential equation,*

$$wx^2\gamma''' + wx\gamma'' - \gamma' - 4\gamma\gamma' + 6x\gamma'^2 + 1 = 0. \tag{5.19}$$

Additionally, if $\gamma_g = [w^g]\gamma$ and $s = s(x)$ is the unique formal power series solution of the functional equation $s = x(1-2s)^{-2}$ with $s(0) = 0$ then $\gamma_0 = s(1-3s)$ and for $g \geq 1$,

$$-(1-2s)(1-6s)\gamma_g' - \frac{4}{1-2s}\gamma_g + x^2\gamma_{g-1}''' + x\gamma_{g-1}'' + \sum_{k=1}^{g-1}\{6x\gamma_k' - 4\gamma_k\}\gamma_{g-k}' = 0. \tag{5.20}$$

Proof. Equation (5.19) follows immediately after making the variable substitutions in Equation (5.6).

Extracting the constant term in Equation (5.19) gives

$$6x\gamma_0'^2 - 4\gamma_0\gamma_0' - \gamma_0' + 1 = 0. \tag{5.21}$$

This equation has a unique power series solution with $\gamma_0(0) = 0$ (the coefficients are determined recursively) and it is straightforward to check that $\gamma_0 = s(1-3s)$ solves equation (5.21).

Lastly, taking the coefficient of w^g in Equation (5.19) for $g \geq 1$ gives

$$\gamma_g'(12x\gamma_0' - 4\gamma_0 - 1) - 4\gamma_0'\gamma_g + x^2\gamma_{g-1}''' + x\gamma_{g-1}'' + \sum_{k=1}^{g-1}\{6x\gamma_k' - 4\gamma_k\}\gamma_{g-k}' = 0. \tag{5.22}$$

After making the substitution $\gamma_0 = s(1-3s)$ and simplifying, Equation (5.22) becomes

$$-(1-2s)(1-6s)\gamma_g' - \frac{4}{1-2s}\gamma_g + x^2\gamma_{g-1}''' + x\gamma_{g-1}'' + \sum_{k=1}^{g-1}\{6x\gamma_k' - 4\gamma_k\}\gamma_{g-k}' = 0,$$

as required. □

5.4 A Recursion for γ_g

We begin by giving a recursive method of computing the series γ_g as rational functions in s. This method essentially uses the fact that the differential equation relating the various γ_g (Equation (5.20) in Corollary 5.3.3) is a linear differential equation in γ_g. The method given in Theorem 5.4.1 below is efficient for computing γ_g for small g but does not seem very useful for determining the asymptotic behaviour of the coefficients.

Theorem 5.4.1. *For $g \geq 1$,*

$$\gamma_g = (1-2s)^2 \int \frac{\kappa_g(s)}{(1-2s)^2} ds,$$

where

$$\kappa_g(s) = \left\{ x^2\gamma_{g-1}''' + x\gamma_{g-1}'' - 2\sum_{i=1}^{k-1}(2\gamma_i - 3x\gamma_i')\gamma_{k-i}' \right\}\Bigg|_{x=s(1-2s)^2}.$$

Proof. Recall from Corollary 5.3.3 that if s is given by $s = x(1-2s)^{-2}$ and $s(0) = 0$ then

$$-(1-2s)(1-6s)\frac{d\gamma_g}{dx} - \frac{4}{1-2s}\gamma_g + \kappa_g(s) = 0. \tag{5.23}$$

Making the variable substitution $x = s(1-2s)^2$ we see that

$$\frac{d\gamma_g}{dx} = \frac{ds}{dx}\frac{d\gamma_g}{ds} = \frac{1}{(1-2s)(1-6s)}\frac{d\gamma_g}{ds},$$

so that equation (5.23) becomes

$$\frac{d\gamma_g}{ds} + \frac{4}{1-2s}\gamma_g = -\kappa_g(s),$$

a first order, linear differential equation in s whose solution is given by

$$\gamma_g = (1-2s)^2\left\{\int \frac{\kappa_g(s)}{(1-2s)^2}ds + C\right\}$$

for some constant C. Since $\gamma_g(0) = 0$, $s(0) = 0$ and $\kappa_g(0) = 0$ we see that $C = 0$. □

Example 5.4.2. *Using Theorem 5.4.1 we may compute (using Maple or otherwise) the series γ_g for small values of g explicitly as rational functions in the series s as in Table 5.1 below.*

Table 5.1: Series γ_g for small g.

γ_0	$s(1-3s)$
γ_1	$\frac{s^2}{(1-6s)^2}$
γ_2	$\frac{s^2(1-2s)(1+36s)}{(1-6s)^7}$
γ_3	$\frac{s^2(1-2s)(1+262s+6788s^2+14904s^3-38880s^4)}{(1-6s)^{12}}$
γ_4	$\frac{s^2(1-2s)(1+1256s+146224s^2+3569872s^3+17017680s^4-29253312s^5-78335424s^6+97977600s^7)}{(1-6s)^{17}}$

Theorem 5.2.2 gives structural information about the genus g series Γ_g for $g \geq 1$. Using the fact that $\gamma_g = \frac{xd}{dx}\Gamma_g$ we can thus derive structural information about the series γ_g. In particular, we get the following corollary.

Lemma 5.4.3. *Suppose we have a series $\mathcal{M}$ of the form*

$$\mathcal{M} = \sum_{\ell=0}^{3g-3} \frac{p_\ell(s)(6s)^{a_\ell} b_\ell}{(1-6s)^{c_\ell}},$$

where $p_\ell(s)$ is a polynomial in s and a_ℓ, b_ℓ and c_ℓ are positive integers. Then

$$\frac{\partial \mathcal{M}}{\partial s} = 6 \sum_{\ell=0}^{3g-3} \frac{\widetilde{p_\ell}(s)(6s)^{a_\ell - 1} b_\ell}{(1-6s)^{c_\ell + 1}},$$

where

$$\widetilde{p_\ell}(s) = s(1-6s)p'_\ell(s) + (a_\ell(1-6s) + 6c_\ell s)p_\ell(s).$$

Proof. Computing, we have

$$\begin{aligned}\frac{\partial \mathcal{M}}{\partial s} &= \sum_{\ell=0}^{3g-3} \left(\frac{p'_\ell(s)(6s)^{a_\ell} b_\ell}{(1-6s)^{c_\ell}} + \frac{6p_\ell(s)(6s)^{a_\ell - 1} a_\ell b_\ell}{(1-6s)^{c_\ell}} + \frac{6p_\ell(s)(6s)^{a_\ell} c_\ell b_\ell}{(1-6s)^{c_\ell + 1}} \right), \\ &= 6 \sum_{\ell=0}^{3g-3} \frac{\widetilde{p_\ell}(s)(6s)^{a_\ell - 1} b_\ell}{(1-6s)^{c_\ell + 1}}.\end{aligned}$$

□

Corollary 5.4.4. *(To Theorem 5.2.2) Let s be the unique formal power series solution of the functional equation*

$$s = x(1-2s)^{-2}$$

with $s(0) = 0$. Then $\gamma_0 = s(1-3s)$, $\gamma_1 = \frac{s^2}{(1-6s)^2}$ and for $g \geq 2$,

$$\Gamma_g\big|_{\substack{p_i = x\delta_{i,1} \\ z=1}} = -b_{g,0} + \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^\ell}{(1-6s)^{2g+\ell}}, \tag{5.24}$$

and

$$\gamma_g = \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^\ell}{(1-6s)^{2g+\ell}}(1-2s)(12s(g-1)+\ell), \tag{5.25}$$

where the $b_{g,\ell}$ are rational constants.

Proof. First, note that γ_0 was determined in Corollary 5.3.3 and γ_1 was determined in Example 5.4.2.

For Equation (5.24), recall Equation (5.3) from Theorem 5.2.2 that for $g \geq 2$,

$$\Gamma_g = -c_{g,(0)} + \frac{1}{(1-\eta_0)^{2g-2}} \sum_{d=0}^{3g-3} \sum_{\alpha \vdash d} \frac{c_{g,\alpha}\eta_\alpha}{(1-\eta_0)^{\ell(\alpha)}},$$

where

$$\eta_j = \sum_{k\geq 1} k^j(2k+1)\binom{2k}{k} p_k \overline{s}^k, \ j \geq 0,$$

and $\overline{s}$ is the unique solution to the functional equation $\overline{s} = z(1-\omega)^{-2}$ with $\overline{s}(0) = 0$ and

$$\omega = \sum_{k\geq 1} \binom{2k}{k} p_k \overline{s}^k.$$

We see that $\omega\big|_{\substack{z=x \\ p_i=\delta_{i,1}}} = 2\,\overline{s}\big|_{\substack{z=x \\ p_i=\delta_{i,1}}}$ so that $\overline{s}\big|_{\substack{z=x \\ p_i=\delta_{i,1}}} = x(1-2\,\overline{s}\big|_{\substack{z=x \\ p_i=\delta_{i,1}}})^{-2}$, or in other words, $\overline{s}\big|_{\substack{z=x \\ p_i=\delta_{i,1}}} = s$. Also, $\eta_j = 6s, \ j \geq 0$. Thus,

$$\Gamma_g\big|_{\substack{p_i=x\delta_{i,1} \\ z=1}} = \Gamma_g\big|_{\substack{z=x \\ p_i=\delta_{i,1}}} = -c_{g,(0)} + \sum_{d=0}^{3g-3} \sum_{\alpha \vdash d} \frac{c_{g,\alpha}(6s)^{\ell}(\alpha)}{(1-6s)^{2g+\ell(\alpha)-2}} = -b_{g,\ell} + \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^\ell}{(1-6s)^{2g+\ell-2}},$$

where

$$b_{g,\ell} = \sum_{d=0}^{3g-3} \sum_{\substack{\alpha \vdash d \\ \ell(\alpha)=\ell}} c_{g,\alpha},$$

proving Equation (5.24).

To see why Equation (5.25) holds, making the substitution $x = s(1-2s)^2$ we see that

$$\gamma_g = \frac{xd}{dx}\,\Gamma_g\big|_{\substack{p_i=x\delta_{i,1} \\ z=1}} = \frac{s(1-2s)}{(1-6s)}\frac{d}{ds}\,\Gamma_g\big|_{\substack{p_i=x\delta_{i,1} \\ z=1}}.$$

Thus, applying Lemma 5.4.3 with $a_\ell = \ell, b_\ell = b_{g,\ell}, c_\ell = 2g+\ell-2$ and $p_\ell(s) = 1$ gives the result. $\square$

Remark. The constants $b_{g,\ell}$ are sums of the constants $c_{g,\alpha}$ appearing in Theorem 5.2.2. That is,

$$b_{g,\ell} = \sum_{d=0}^{3g-3} \sum_{\substack{\alpha \vdash d \\ \ell(\alpha)=\ell}} c_{g,\alpha}.$$

In particular,

$$b_{g,3g-3} = \sum_{d=0}^{3g-3} \sum_{\substack{\alpha\vdash d \\ \ell(\alpha)=\ell}} c_{g,\alpha} = c_{g,1^{3g-3}},$$

and

$$b_{g,0} = c_{g,(0)} = -\frac{B_{2g}}{4g(g-1)},$$

where B_{2g} is a Bernoulli number and where the last equality follows from [45, Theorem 0.5].

As a straightforward consequence we may write down formulas for the coefficients in the series $\Gamma_g|_{\substack{p_i=x\delta_{i,1} \\ z=1}}$.

Lemma 5.4.5. *Let $s = s(x)$ be the series determined by the functional equation $s = x(1-2s)^{-2}$ and $s(0) = 0$. Then for any polynomial $p(s)$ and positive integer k,*

$$[x^n]\frac{p(s)}{(1-6s)^k} = \frac{2^{n-1}}{n}\sum_{i=0}^{d}\frac{c_i}{2^i}\sum_{j\geq 0}3^j\binom{k+j}{j}\binom{3n-i-j-2}{2n-1},$$

where the constants c_i are the coefficients in the following polynomial

$$\sum_{i=0}^{d} c_i s^i = (1-6s)\frac{dp(s)}{ds} + 6kp(s).$$

Proof. This follows from an application of Lagrange's implicit function theorem [53]. □

Theorem 5.4.6. *For $g = 0, 1$, the coefficients of x^n, $n > 0$ in the exponential series $\Gamma_g|_{\substack{p_i=x\delta_{i,1} \\ z=1}}$ are given by*

$$\left[\frac{x^n}{n!}\right]\Gamma_0|_{\substack{p_i=x\delta_{i,1} \\ z=1}} = 2^n(3n-3)\cdots(2n+1), \tag{5.26}$$

and

$$\left[\frac{x^n}{n!}\right]\Gamma_1|_{\substack{p_i=x\delta_{i,1} \\ z=1}} = \frac{(n-1)!}{n}2^{n-1}\sum_{j\geq 0}3^j\binom{j+2}{j}\binom{3n-j-3}{2n-1}. \tag{5.27}$$

For $g \geq 2$ the coefficients are given by

$$\left[\frac{x^n}{n!}\right]\Gamma_g|_{\substack{p_i=x\delta_{i,1} \\ z=1}} = 2^n(n-1)!\sum_{\ell=0}^{3g-3}\sum_{j\geq 0}3^{\ell+j}\binom{2g+\ell+j-2}{j}\binom{3n-\ell-j-3}{2n-1} \tag{5.28}$$

$$\left(6(g-1)+\ell+\frac{6(g-1)(2n-1)}{n-\ell-j-1}\right)b_{g,\ell}. \tag{5.29}$$

Proof. Using Lagrange's implicit function theorem [53], we have

$$\begin{aligned}
\left[\frac{x^n}{n!}\right]\Gamma_0\big|_{\substack{p_i=x\delta_{i,1}\\ z=1}} &= \frac{1}{n}\left[\frac{x^n}{n!}\right]\gamma_0 \\
&= (n-1)![x^n]s(1-3s) \\
&= \frac{(n-1)!}{n}[\lambda^{n-1}](1-6\lambda)(1-2\lambda)^{-2n} \\
&= \frac{(n-1)!}{n}\left\{2^{n-1}\binom{3n-2}{n-1} - 3\cdot 2^{n-1}\binom{3n-3}{n-2}\right\} \\
&= 2^n(3n-3)\cdots(2n+1).
\end{aligned}$$

Next, we have

$$\left[\frac{x^n}{n!}\right]\Gamma_1\big|_{\substack{p_i=x\delta_{i,1}\\ z=1}} = (n-1)![x^n]\gamma_1 = (n-1)![x^n]\frac{s^2}{(1-6s)^2}.$$

Letting $p(s) = s^2$ and $k = 2$ we have

$$(1-6s)\frac{dp(s)}{ds} + 6kp(s) = 2s$$

and so, applying Lemma 5.4.5,

$$\left[\frac{x^n}{n!}\right]\Gamma_1\big|_{\substack{p_i=x\delta_{i,1}\\ z=1}} = \frac{(n-1)!}{n}2^{n-1}\sum_{j\geq 0}3^j\binom{j+2}{j}\binom{3n-j-3}{2n-1}.$$

From Corollary 5.4.4 for $g \geq 2$ we have

$$\left[\frac{x^n}{n!}\right]\Gamma_g\big|_{\substack{p_i=x\delta_{i,1}\\ z=1}} = n!\sum_{\ell=0}^{3g-3}[x^n]\frac{b_{g,\ell}(6s)^\ell}{(1-6s)^{2g+\ell}}.$$

For fixed g, ℓ we let $p(s) = b_{g,\ell}(6s)^\ell$ and $k = 2g+\ell$ so that

$$(1-6s)\frac{dp(s)}{ds} + 6kp(s) = 6^\ell b_{g,\ell}(12(g-1)s^\ell + \ell s^{\ell-1}).$$

Applying Lemma 5.4.5 then gives

$$\begin{aligned}
[x^n]\frac{b_{g,\ell}(6s)^\ell}{(1-6s)^{2g+\ell}} &= \frac{2^{n-1}}{n}\sum_{j\geq 0}3^j\binom{3g+\ell-2+j}{j}6^\ell b_{g,\ell} \\
&\qquad\times\left\{\frac{12(g-1)}{2^\ell}\binom{3n-\ell-j-2}{2n-1} + \frac{\ell}{2^{\ell-1}}\binom{3n-\ell-j-3}{2n-1}\right\} \\
&= \frac{2^n}{n}\sum_{j\geq 0}3^{\ell+j}\binom{2g+\ell+j-2}{j}b_{g,\ell} \\
&\qquad\times\left\{12(g-1)\binom{3n-\ell-j-2}{2n-1} + 2\ell\binom{3n-\ell-j-3}{2n-1}\right\} \\
&= \frac{2^n}{n}\sum_{j\geq 0}3^{\ell+j}\binom{2g+\ell+j-2}{j}\binom{3n-\ell-j-3}{2n-1} \\
&\qquad\times\left(6(g-1) + \ell + \frac{6(g-1)(2n-1)}{n-\ell-j-1}\right)b_{g,\ell}
\end{aligned}$$

from which the result follows. $\square$

Corollary 5.4.4 tells us that the series γ_g are completely determined by the constants $b_{g,\ell}$. Using the form of the series γ_g and the differential equation in Corollary 5.3.3 relating the various γ_g we can derive a polynomial identity which depends on the $b_{g,\ell}$. We can then extract coefficients to find a system of equations which determine the $b_{g,\ell}$ (See the following Corollary 5.4.8). We shall see in the following section how this polynomial identity can also be used to determine the asymptotics of the coefficients in the series γ_g.

Theorem 5.4.7. *For $g \geq 3$ the following polynomial identity in s holds:*

$$\sum_{\ell=0}^{3g-3}(6s)^{\ell}(1-6s)^{3g-3-\ell}T_{1,g,\ell}(s)b_{g,\ell} + \sum_{\ell=0}^{3g-6}(6s)^{\ell}(1-6s)^{3g-6-\ell}T_{2,g,\ell}(s)b_{g-1,\ell}$$
$$+\sum_{k=2}^{g-2}\sum_{\ell=0}^{3k-3}\sum_{m=0}^{3(g-k)-3}(6s)^{\ell+m}(1-6s)^{3g-6-\ell-m}T_{3,g,k,\ell,m}(s)b_{k,\ell}b_{g-k,m} = 0$$

where the polynomials $T_{1,g,\ell}(s), T_{2,g,\ell}(s)$ and $T_{3,g,k,\ell,m}(s)$ are given below:

$$\begin{aligned}T_{1,g,\ell}(s) = (1-2s)(&216gs^2 - 72s^2 + 10s\ell - 12gs + 12s - 288gs^3 - 12s^2\ell \\ &- 144g^2s^2 - 24gs\ell + 288g^2s^3 + 48gs^2\ell - \ell^2 + 2s\ell^2),\end{aligned}$$

$$\begin{aligned}T_{2,g,\ell}(s) = &-48s + 3648s^2 + 84sg\ell - 6120s^2g\ell + 66336s^3g\ell - 271296s^4g\ell + 2160s^2g^2\ell \\
&- 38016s^3g^2\ell + 108sg\ell^2 - 2304s^2g\ell^2 + 183168s^4g^2\ell + 11808s^3g\ell^2 \\
&+ 898560s^5g^3 - 40s\ell^3 + 232s^2\ell^3 - 2380032s^5g^2 - 786720s^4g - 36408s^3\ell \\
&+ 12096s^3g^3 + 1672s^2\ell^2 - 7184s^3\ell^2 + 2725056s^5g + 132048s^4\ell \\
&- 193536s^4g^3 + 607104s^4g^2 + 24sg - 4704s^2g - 148s\ell + 102000s^3g \\
&+ 1440s^2g^2 - 61344s^3g^2 - 138s\ell^2 - 55392s^3 + 361536s^4 - 1127808s^5 \\
&- 1658880s^6g^3 - 480s^3\ell^3 + 4064256s^6g^2 - 4288896s^6g - 211680s^5\ell \\
&+ 12960s^4\ell^2 + 1078272s^7g^3 + 336s^4\ell^3 - 2529792s^7g^2 + 2488320s^7g \\
&+ 248832s^6g^4 + 20736s^4g^4 - 124416s^5g^4 - 6s\ell^4 + 12s^2\ell^4 + 1610496s^6 \\
&+ 475200s^5g\ell - 345600s^5g^2\ell - 23040s^4g\ell^2 - 300672s^6g\ell + 228096s^6g^2\ell \\
&+ 6912s^3g^3\ell - 41472s^4g^3\ell + 864s^2g^2\ell^2 - 5184s^3g^2\ell^2 + 82944s^5g^3\ell \\
&+ 15552s^5g\ell^2 + 10368s^4g^2\ell^2 - 288s^2g\ell^3 + 576s^3g\ell^3 + 48sg\ell^3 \\
&- 8s^3\ell^4 - 829440s^7 - 55296s^6g^3\ell - 6912s^5g^2\ell^2 - 384s^4g\ell^3 \\
&- 8352s^5\ell^2 - 165888s^7g^4 + 4280s^2\ell + 122688s^6\ell + \ell^3 + \ell^4,\end{aligned}$$

$$\begin{aligned}T_{3,g,k,\ell,m}(s) = 2(1-2s)(&12ks - 576gs^3k + 24ksm + 288gs^2k + 48gs^2m - 24gsm \\
&- 48ks^2m - 120s^2 - 576gs^3 - 144g^2s^2 + 288g^2s^3 - 12gs \\
&+ 288s^3 + 12s - 264ks^2 + 14sm + 576ks^3 - 36s^2m \\
&- 144k^2s^2 + 288k^2s^3 - m^2 + 2sm^2 + 264gs^2) \\
&(504ks^2 - 72s^2 + 18s\ell - 12ks + 12s - 864ks^3 - 36s^2\ell - 432k^2s^2 \\
&- 72ks\ell + 864k^2s^3 + 144ks^2\ell - 3\ell^2 + 6s\ell^2 + 2\ell).\end{aligned}$$

Proof. First, recall that Corollary 5.4.4 says that for $g \geq 2$

$$\gamma_g = \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^\ell}{(1-6s)^{2g+\ell}} p_{0,g,\ell}(s)$$

where $p_{0,g,\ell}(s) = (1-2s)(12s(g-1)+\ell)$. Applying Lemma 5.4.3 we have

$$\gamma_g' = \frac{1}{(1-2s)(1-6s)} \frac{d\gamma_g}{ds} = \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^{\ell-1}}{(1-6s)^{2g+\ell+2}} p_{1,g,\ell}(s)$$

with $p_{1,g,\ell}(s) = \frac{6}{1-2s}(s(1-6s)p_{0,g,\ell}'(s) + (12gs+\ell)p_{0,g,\ell})$,

$$\gamma_g'' = \frac{1}{(1-2s)(1-6s)} \frac{d\gamma_g'}{ds} = \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^{\ell-2}}{(1-6s)^{2g+\ell+4}} p_{2,g,\ell}(s)$$

with $p_{2,g,\ell}(s) = \frac{6}{1-2s}(s(1-6s)p_{1,g,\ell}'(s) + (12(g+1)s+\ell)p_{1,g,\ell})$ and

$$\gamma_g''' = \frac{1}{(1-2s)(1-6s)} \frac{d\gamma_g''}{ds} = \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^{\ell-3}}{(1-6s)^{2g+\ell+6}} p_{3,g,\ell}(s)$$

with $p_{3,g,\ell}(s) = \frac{6}{1-2s}(s(1-6s)p_{2,g,\ell}'(s) + (12(g+2)s+\ell)p_{2,g,\ell})$.

Computing the polynomials $p_{1,g,\ell}(s), p_{2,g,\ell}(s)$ and $p_{3,g,\ell}(s)$ explicitly we see that

$$\begin{aligned}
\gamma_g &= (1-2s) \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^\ell}{(1-6s)^{2g+\ell}} q_{0,g,\ell}(s), \\
\gamma_g' &= \frac{6}{(1-2s)} \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^{\ell-1}}{(1-6s)^{2g+\ell+2}} q_{1,g,\ell}(s), \\
\gamma_g'' &= \frac{6}{(1-2s)^3} \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^{\ell-2}}{(1-6s)^{2g+\ell+4}} q_{2,g,\ell}(s), \\
\gamma_g''' &= \frac{6}{(1-2s)^5} \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^{\ell-3}}{(1-6s)^{2g+\ell+6}} q_{3,g,\ell}(s),
\end{aligned}$$

where

$$
\begin{aligned}
q_{0,g,\ell}(s) &= 12s(g-1)+\ell,\\
q_{1,g,\ell}(s) &= -264gs^2+120s^2-14s\ell+12gs-12s+576gs^3-288s^3+36s^2\ell\\
&\quad+144g^2s^2+24gs\ell-288g^2s^3-48gs^2\ell+\ell^2-2s\ell^2,\\
q_{2,g,\ell}(s) &= -8928gs^4-264s^3\ell+14400g^2s^4+6912gs^5+144s^4\ell-13824g^2s^5\\
&\quad+1728g^3s^3-6912g^3s^4+16s^2\ell^2-4s\ell^3+6912g^3s^5-24s^3\ell^2+4s^2\ell^3\\
&\quad+1392gs^3\ell-1440gs^4\ell+432g^2s^2\ell-1728g^2s^3\ell+36gs\ell^2\\
&\quad+1728g^2s^4\ell+144gs^3\ell^2+1440s^4+12gs-600gs^2+168s^2-26s\ell\\
&\quad+3696gs^3+132s^2\ell+432g^2s^2-4608g^2s^3-2s\ell^2-816s^3-12s\\
&\quad-144gs^2\ell^2+\ell^3+36gs\ell-408gs^2\ell,\\
q_{3,g,\ell}(s) &= 8640gs^4+1776s^3\ell+62784g^2s^4-88128gs^5-5856s^4\ell-72576g^2s^5\\
&\quad-82944g^3s^4+172s^2\ell^2+24s\ell^3+248832g^3s^5-232s^3\ell^2-144s^2\ell^3\\
&\quad+2976gs^3\ell+4416gs^4\ell+1728g^2s^2\ell-10368g^2s^3\ell+72gs\ell^2\\
&\quad+20736g^2s^4\ell-1728gs^3\ell^2-31104gs^5\ell-13824g^2s^5\ell+4608gs^4\ell^2\\
&\quad+34560gs^6\ell-288gs^2\ell^3+576gs^3\ell^3+6912g^3s^3\ell-41472g^3s^4\ell\\
&\quad+864g^2s^2\ell^2-5184g^2s^3\ell^2+82944g^3s^5\ell+10368g^2s^4\ell^2+48gs\ell^3\\
&\quad-3456gs^5\ell^2-384gs^4\ell^3-55296g^3s^6\ell-6912g^2s^5\ell^2-9216s^4\\
&\quad+36288s^5+864s^5\ell^2-165888g^4s^7-8s^3\ell^4+217728gs^6+9504s^5\ell\\
&\quad-82944g^2s^6-331776g^3s^6+288s^3\ell^3-432s^4\ell^2-165888gs^7\\
&\quad+165888g^2s^7+165888g^3s^7-192s^4\ell^3+20736g^4s^4-124416g^4s^5\\
&\quad-6s\ell^4+12s^2\ell^4-51840s^6+\ell^4+12gs-1080gs^2+72s^2-38s\ell\\
&\quad-116s^2\ell+1008g^2s^2-15264g^2s^3-28s\ell^2+768s^3-12s+48gs\ell\\
&\quad+10368g^3s^3-5184s^6\ell+248832g^4s^6+4128gs^3-960gs^2\ell.
\end{aligned}
$$

We also compute (using Example 5.4.2)

$$\gamma_1' = \frac{2s}{(1-2s)(1-6s)^4},$$

and

$$6x\gamma_1' - 4\gamma_1 = \frac{8s^2(1+6s)(1-3s)}{(1-6s)^4}.$$

For $g \geq 3$, Equation (5.20) from Corollary 5.3.3 becomes

$$
\begin{aligned}
-(1-2s)(1-6s)\gamma_g' - \frac{4}{1-2s}\gamma_g + x^2\gamma_{g-1}''' + x\gamma_{g-1}'' + \{6x\gamma_1' - 4\gamma_1\}\gamma_{g-1}'&\\
+ \gamma_1'\{6x\gamma_{g-1}' - 4\gamma_{g-1}\} + \sum_{k=2}^{g-2}\{6x\gamma_k' - 4\gamma_k\}\gamma_{g-k}' &= 0.
\end{aligned}
$$

Substituting the expressions above then gives

$$
\begin{aligned}
0 = -\sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^\ell}{(1-6s)^{2g+\ell+1}} q_{1,g,\ell}(s) - 4s \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^\ell}{(1-6s)^{2g+\ell}} q_{0,g,\ell}(s) \\
+ \frac{1}{1-2s} \sum_{\ell=0}^{3g-6} \frac{b_{g-1,\ell}(6s)^\ell}{(1-6s)^{2g+\ell+4}} q_{3,g-1,\ell}(s) + \frac{1}{1-2s} \sum_{\ell=0}^{3g-6} \frac{b_{g-1,\ell}(6s)^\ell}{(1-6s)^{2g+\ell+2}} q_{2,g-1,\ell}(s) \\
+ \frac{8s^2(1+6s)(1-3s)}{1-2s} \sum_{\ell=0}^{3g-6} \frac{b_{g-1,\ell}(6s)^\ell}{(1-6s)^{2g+\ell+4}} q_{1,g-1,\ell}(s) \\
+ 2s^2 \left(6 \sum_{\ell=0}^{3g-6} \frac{b_{g-1,\ell}(6s)^\ell}{(1-6s)^{2g+\ell+4}} q_{1,g-1,\ell}(s) - 4 \sum_{\ell=0}^{3g-6} \frac{b_{g-1,\ell}(6s)^\ell}{(1-6s)^{2g+\ell+2}} q_{0,g-1,\ell}(s) \right) \\
+ \sum_{k=2}^{g-2} \left(6 \sum_{\ell=0}^{3k-3} \frac{b_{k,\ell}(6s)^\ell}{(1-6s)^{2k+\ell+2}} q_{1,k,\ell}(s) - 4 \sum_{\ell=0}^{3k-3} \frac{b_{k,\ell}(6s)^\ell}{(1-6s)^{2k+\ell}} q_{0,k,\ell}(s) \right) \\
\times \left(\sum_{\ell=0}^{3(g-k)-3} \frac{b_{g-k,\ell}(6s)^\ell}{(1-6s)^{2(g-k)+\ell+2}} q_{1,g-k,\ell}(s) \right).
\end{aligned}
$$

Multiplying by $(1-2s)(1-6s)^{5g-2}$ and simplify gives

$$
\begin{aligned}
\sum_{\ell=0}^{3g-3} (6s)^\ell (1-6s)^{3g-3-\ell} T_{1,g,\ell}(s) b_{g,\ell} + \sum_{\ell=0}^{3g-6} (6s)^\ell (1-6s)^{3g-6-\ell} T_{2,g,\ell}(s) b_{g-1,\ell} \\
+ \sum_{k=2}^{g-2} \sum_{\ell=0}^{3k-3} \sum_{m=0}^{3(g-k)-3} (6s)^{\ell+m} (1-6s)^{3g-6-\ell-m} T_{3,g,k,\ell,m}(s) b_{k,\ell} b_{g-k,m} = 0
\end{aligned}
$$

where

$$
\begin{aligned}
T_{1,g,\ell}(s) &= (1-2s)(-q_{1,g,\ell}(s) - 4s(1-6s)q_{0,g,\ell}(s)), \\
T_{2,g,\ell}(s) &= q_{3,g-1,\ell}(s) + (1-6s)^2 q_{2,g-1,\ell}(s) \\
&\quad + 8s^2(1+6s)(1-3s)q_{1,g-1,\ell}(s) + 12s^2(1-2s)q_{1,g-1,\ell}(s) \\
&\quad - 8s^2(1-2s)(1-6s)^2 q_{0,g-1,\ell}(s), \\
T_{3,g,k,\ell,m}(s) &= 2(1-2s)q_{1,g-k,m}(s)(3q_{1,k,\ell}(s) - 2(1-6s)^2 q_{0,k,\ell}(s)).
\end{aligned}
$$

Explicitly, the polynomials $T_{1,g,\ell}(s), T_{2,g,\ell}(s)$ and $T_{3,g,k,\ell,m}(s)$ are those given in the statement of the theorem. □

Corollary 5.4.8. *For $g \geq 3$ and $n \geq 1$, the constants $b_{g,\ell}$ satisfy the following recurrence:*

$$
\begin{aligned}
\sum_{\ell=0}^{n} \sum_{i=0}^{4} (-1)^{n-\ell-i} 6^{n-i} \alpha_{g,\ell}^i \binom{3g-\ell-3}{n-\ell-i} b_{g,\ell} + \sum_{\ell=0}^{n} \sum_{i=0}^{7} (-1)^{n-\ell-i} 6^{n-i} \beta_{g,\ell}^i \binom{3g-\ell-6}{n-\ell-i} b_{g-1,\ell} \\
+ \sum_{k=2}^{g-2} \sum_{\ell=0}^{3k-3} \sum_{m=0}^{3(g-k)-3} \sum_{i=0}^{7} (-1)^{n-\ell-m-i} 6^{n-i} \omega_{g,k,\ell,m}^i \binom{3g-\ell-m-6}{n-\ell-m-i} b_{k,\ell} b_{g-k,m} = 0,
\end{aligned}
$$

where $\alpha_{g,\ell}^i = [s^i]T_{1,g,\ell}(s), \beta_{g,\ell}^i = [s^i]T_{2,g,\ell}(s)$ and $\omega_{g,k,\ell,m}^i = [s^i]T_{3,g,k,\ell,m}(s)$.

Proof. This follows by taking the coefficient of s^n in the polynomial identity in Theorem 5.4.7. $\square$

Remark. The recursion given in Corollary 5.4.8 is not sufficient to completely determine the constants $b_{g,\ell}$. In particular, the initial conditions need to be determined and in addition, so does $b_{g,0}$ (this is because $\alpha^0_{g,\ell} = -\ell$ and so $\alpha^0_{g,0} = 0$). Using Table 5.1 we see that

$$b_{2,0} = \frac{1}{240}, \; b_{2,1} = -\frac{1}{120}, \; b_{2,2} = \frac{19}{720}, \quad \text{and} \quad b_{2,3} = \frac{7}{180}.$$

Also, as mentioned in the remark following Corollary 5.4.4,

$$b_{g,0} = -\frac{B_{2g}}{4g(g-1)},$$

where B_{2g} is a Bernoulli number.

5.5 Asymptotics

We now turn to the asymptotic behaviour of the coefficients in γ_g, $g \geq 1$. From the structure of the γ_g (Corollary 5.4.4) we expect the asymptotic behaviour of the γ_g to be related to the asymptotics of the algebraic series $s(x)$. Fortunately, $s(x)$ is rather straightforward to analyze and, in fact, arose in the enumeration of 2-connected maps on the projective plane [36]. In paticular, by considering the Newton-Puiseux expansion of the algebraic series $s(x)$, the asymptotic behaviour can be determined.

Lemma 5.5.1. *(Lemma 4 in [36]) Let $s = s(x)$ be the solution to $s = x(1-2s)^{-2}$ with $s(0) = 0$. Then $s(x)$ is analytic at 0, and has a unique singularity at $x = 2/27$. Moreover, as $x \to 2/27$,*

$$1 - 6s \sim \frac{2}{\sqrt{3}}\left(1 - \frac{27}{2}x\right)^{1/2}.$$

We now combine the asymptotic behaviour of the series $s(x)$ with the structural results of Corollary 5.4.4 to determine the asymptotics of the series $\gamma_g(x)$ as $x \to 2/27$. It is then a simple matter to apply *Darboux's theorem* [5, Theorem 4] to find the asymptotic behaviour of $[x^n]\gamma_g$ as $n \to \infty$.

Theorem 5.5.2. *For $g \geq 1$, as $n \to \infty$*

$$[x^n]\gamma_g \sim a_g \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^{5g-3} \frac{n^{\frac{5g-5}{2}}}{\Gamma\left(\frac{5g-3}{2}\right)}\left(\frac{27}{2}\right)^n,$$

where $a_1 = \frac{1}{24}$ and $a_g = (5g-5)b_{g,3g-3}$ for $g \geq 2$.

Proof. From Example 5.4.2 we have

$$\gamma_1 = \frac{s^2}{(1-6s)^2}.$$

Using Lemma 5.5.1 we see that γ_1 has a unique singularity at $x = 2/27$ and, since $s(2/27) = 1/6$ we see that, as $x \to 2/27$,

$$\gamma_1 \sim \frac{1}{36}\left(\frac{\sqrt{3}}{2}\right)^2 \left(1 - \frac{27}{2}x\right)^{-1} \sim a_1 \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^2 \left(1 - \frac{27}{2}x\right)^{-1},$$

with $a_1 = 1/24$. Similarly, for $g \geq 2$ from Corollary 5.4.4 we see that

$$\gamma_g = \sum_{\ell=0}^{3g-3} \frac{b_{g,\ell}(6s)^\ell}{(1-6s)^{2g+\ell}}(1-2s)(12s(g-1)+\ell)$$

so that, as $x \to 2/27$,

$$\gamma_g \sim (5g-5)b_{g,3g-3}\frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^{5g-3}\left(1-\frac{27}{2}x\right)^{-\frac{5g-3}{2}} \sim a_g \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^{5g-3}\left(1-\frac{27}{2}x\right)^{-\frac{5g-3}{2}}.$$

Thus, for $g \geq 1$ as $x \to 2/27$,

$$\gamma_g \sim a_g \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^{5g-3}\left(1-\frac{27}{2}x\right)^{-\frac{5g-3}{2}}.$$

By Darboux's theorem [5, Theorem 4], this gives the asymptotic expansion of the coefficients in γ_g given in the statement of the theorem. □

Remark. Provided the Newton-Puiseux expansion for $s(x)$ could be determined to higher order, Theorem 5.5.2 can be extended to determine lower order asymptotic behaviour of the coefficients in γ_g. Doing so would likely involve the constants $b_{g,\ell}$ for $\ell < 3g-3$.

Theorem 5.5.2 tells us that the asymptotic behaviour of $[x^n]\gamma_g$ as $n \to \infty$ is determined by the constants $a_g = (5g-5)b_{g,3g-3}$ (and $a_1 = 1/24$). We now use the polynomial identity in Theorem 5.4.7 to determine a recursive method for computing the constants a_g.

Theorem 5.5.3. *The constants a_g appearing in Theorem 5.5.2 satisfy the following quadratic recursion for $g \geq 2$:*

$$(5g-3)a_g = \frac{2}{3}(5g-8)(5g-6)(5g-4)a_{g-1} + 4\sum_{k=1}^{g-1}(5k-3)(5(g-k)-3)a_k a_{g-k}.$$

Proof. From Example 5.4.2 or by comparing our results with those from [45, Equation (A.1)] we see that $b_{2,3} = 7/180$ so that $a_2 = 7/36$. It is then easily checked that the recurrence above gives the same value for a_2 with $a_1 = 1/24$.

For $g \geq 3$ recall that Theorem 5.4.7 states the following polynomial identity in s is satisfied:

$$\sum_{\ell=0}^{3g-3}(6s)^{\ell}(1-6s)^{3g-3-\ell}T_{1,g,\ell}(s)b_{g,\ell}+\sum_{\ell=0}^{3g-6}(6s)^{\ell}(1-6s)^{3g-6-\ell}T_{2,g,\ell}(s)b_{g-1,\ell}$$
$$+\sum_{k=2}^{g-2}\sum_{\ell=0}^{3k-3}\sum_{m=0}^{3(g-k)-3}(6s)^{\ell+m}(1-6s)^{3g-6-\ell-m}T_{3,g,k,\ell,m}(s)b_{k,\ell}b_{g-k,m}=0, \quad (5.30)$$

where the polynomials $T_{1,g,\ell}(s), T_{2,g,\ell}(s)$ and $T_{3,g,k,\ell,m}(s)$ are as in the statement of Theorem 5.4.7. In particular, setting $s = 1/6$ in Equation (5.30) gives

$$T_{1,g,3g-3}(1/6)b_{g,3g-3}+T_{2,g,3g-6}(1/6)b_{g-1,3g-6}$$
$$+\sum_{k=2}^{g-2}T_{3,g,k,3k-3,3(g-k)-3}(1/6)b_{k,3k-3}b_{g-k,3(g-k)-3}=0. \quad (5.31)$$

Evaluating $T_{1,g,3g-3}(s), T_{2,g,3g-6}(s)$ and $T_{3,g,k,3k-3,3(g-k)-3}(s)$ at $s = 1/6$ gives

$$\begin{aligned}T_{1,g,3g-3}(1/6) &= -\frac{4}{9}(5g-3)(5g-5),\\ T_{2,g,3g-6}(1/6) &= \frac{8}{27}(5g-5)(5g-6)(5g-8)(5g-10)+\frac{8}{27}(5g-8)(5g-10),\\ T_{3,g,k,3k-3,3(g-k)-3}(1/6) &= \frac{16}{9}(5(g-k)-3)(5(g-k)-5)(5k-3)(5k-5).\end{aligned}$$

Thus, equation (5.31) becomes

$$\begin{aligned}&\frac{4}{9}(5g-3)(5g-5)b_{g,3g-3}\\ &=\frac{8}{27}(5g-4)(5g-6)(5g-8)(5g-10)b_{g-1,3g-6}+\frac{8}{27}(5g-8)(5g-10)b_{g-1,3g-6}\\ &\quad+\frac{16}{9}\sum_{k=2}^{g-2}(5(g-k)-3)(5(g-k)-5)(5k-3)(5k-5)b_{k,3k-3}b_{g-k,3(g-k)-3}.\end{aligned}$$

Or, after dividing by 4/9 and setting $a_g = (5g-5)b_{g,3g-3}$,

$$\begin{aligned}(5g-3)a_g &= \frac{2}{3}(5g-4)(5g-6)(5g-8)a_{g-1}+\frac{2}{3}(5g-8)a_{g-1}\\ &\quad+4\sum_{k=2}^{g-2}(5k-3)(5(g-k)-3)a_k a_{g-k}\\ &=\frac{2}{3}(5g-8)(5g-6)(5g-4)a_{g-1}+4\sum_{k=1}^{g-1}(5k-3)(5(g-k)-3)a_k a_{g-k},\end{aligned}$$

with $a_1 = 1/24$. □

Remark. Let $f_g = \frac{5g-3}{3\cdot 2^5}\left(\frac{\sqrt{6}}{2}\right)^{g-1}a_g$. Then the recursion in Theorem 5.5.3 becomes

$$f_g=\frac{\sqrt{6}}{96}(5g-4)(5g-6)f_{g-1}+6\sqrt{6}\sum_{k=1}^{g-1}f_k f_{g-k}$$

and $f_1 = \frac{1}{2^7 3^2}$. Comparing the recursion for the f_g and f_1 to the f_g's appearing in the paper of Bender, Gao and Richmond on the map asymptotics constant [9], we see that they are in fact the same so that $f_g = 24^{-3/2} 6^{g/2} \Gamma\left(\frac{5g-1}{2}\right) t_g$ where t_g is the map asymptotics constant. In other words, the constants a_g are a rescaling of the map asymptotics constants.

5.6 Applications

5.6.1 Simple Monotone Hurwitz Numbers

As an initial application, we describe a combinatorial interpretation of the preceding results.

For $g, n \geq 0$ let $\overrightarrow{h}_g(n)$ be the number of r-tuples of transpositions, $\tau_1, \cdots, \tau_r \in S_n$ such that

- $\tau_1 \cdots \tau_r = 1$ (the identity in S_n),
- $r = 2(n + g - 1)$,
- if $\tau_i = (a_i\ b_i)$ with $a_i < b_i$ then $b_1 \leq \cdots \leq b_r$, and
- the transpositions τ_i generate a transitive subgroup of S_n.

We call $\overrightarrow{h}_g(n)$ the *simple monotone Hurwitz numbers* (these are the monotone Hurwitz number [44, 45] analog of the simple Hurwitz numbers [97, 104, 51]). It follows from Section 5.2 (see also [45]) that for g fixed,

$$\sum_{n \geq 0} \overrightarrow{h}_g(n) \frac{x^n}{n!} = \Gamma_g \big|_{\substack{p_i = x\delta_{i,1} \\ z=1}}.$$

In particular, this implies the following interpretation of the coefficients of γ_g:

$$\left[\frac{x^n}{n!}\right] \gamma_g = n \overrightarrow{h}_g(n).$$

Thus, each of the results in Sections 5.3 and 5.4 can be interpreted as statements about the number of simple monotone Hurwitz numbers. Of particular interest is Theorem 5.5.2 which implies that the asymptotic behaviour of the fixed genus simple monotone Hurwitz numbers are governed by the map asymptotics constant studied by Bender, Gao and Richmond [9]. For additional results concerning the asymptotic behaviour of Hurwitz numbers and their relation to the map asymptotics constant, see [99].

5.6.2 Plancherel Averaged Complete Symmetric Functions

Recall from Section 5.2, the exponential generating series for the Plancherel averaged complete symmetric functions is given by

$$\mathcal{H} = \mathcal{H}(t,x) = \sum_{n\geq 0}\sum_{k\geq 0} \frac{t^k x^n}{n!}\Phi_n(h_k),$$

and is related to the series Γ by

$$\Gamma\big|_{\substack{p_i = x\delta_{i,1}\\ z=1}} = \log \mathcal{H}.$$

After making the variable substitutions $x \mapsto xw^{-1}$ and $t \mapsto w^{1/2}$ this gives

$$\mathcal{H}(w^{1/2}, xw^{-1}) = \exp\left(\sum_{g\geq 0} w^{g-1}\, \Gamma_g\big|_{\substack{p_i = x\delta_{i,1}\\ z=1}}(x)\right).$$

For the remainder of this subsection we will use Γ_g to mean $\Gamma_g|_{\substack{p_i = x\delta_{i,1}\\ z=1}}$, in an effort to reduce notation. Thus,

$$\mathcal{H}(w^{1/2}, xw^{-1}) = \exp\left(\sum_{g\geq 0} w^{g-1}\Gamma_g(x)\right). \tag{5.32}$$

Note also that $\gamma_g = x\frac{d\Gamma_g}{dx}$.

Since $(\dim\lambda)^2 = (\dim\lambda^t)^2$ where λ^t is the conjugate partition to λ,

$$\begin{aligned}
\mathcal{H}(-t,x) &= \sum_{n,k\geq 0} \frac{(-t)^k x^n}{n!} \sum_{\lambda\vdash n} \frac{h_k(A_\lambda)(\dim\lambda)^2}{n!},\\
&= \sum_{n,k\geq 0} \frac{t^k x^n}{n!} \sum_{\lambda\vdash n} \frac{h_k(-A_\lambda)(\dim\lambda)^2}{n!},\\
&= \sum_{n,k\geq 0} \frac{t^k x^n}{n!} \sum_{\lambda\vdash n} \frac{h_k(A_{\lambda^t})(\dim\lambda^t)^2}{n!} = \mathcal{H}(t,x),
\end{aligned}$$

where we have used the fact that h_k is homogeneous of degree k and $A_{\lambda^t} = -A_\lambda$ (here $-A_\lambda$ is the multiset given by $\{-c : c \in A_\lambda\}$). This implies that for all $n,k\geq 0$,

$$\Phi_n(h_{2k+1}) = \left[\frac{t^{2k+1}x^n}{n!}\right]\mathcal{H}(t,x) = 0.$$

Thus $\mathcal{H}(t^{1/2},x)$ is a power series in t. If we now make the substitutions $w\mapsto t$ and $x \mapsto tx$ in equation (5.32) we see that

$$\mathcal{H}(t^{1/2},x) = \exp\left(\sum_{g\geq 0} t^{g-1}\Gamma_g(tx)\right),$$

and in particular, for $n, k \geq 0$,

$$\Phi_n(h_{2k}) = \left[t^k \frac{x^n}{n!}\right] \exp\left(\sum_{g\geq 0} t^{g-1}\Gamma_g(tx)\right).$$

Now, since $\Gamma_g(0) = 0$ for all $g \geq 0$ and $[x]\Gamma_g(x) = 0$ for $g > 0$ (these facts and that $[x]\Gamma_0(x) = 1$ follow from the combinatorial description) we see that

$$t^{g-1}\Gamma_g(tx) = x\delta_{g,0} + \sum_{i\geq 0} \frac{\overrightarrow{h}_g(i+2)}{(i+2)!} x^{i+2} t^{g+i+1}.$$

Thus,

$$\sum_{g\geq 0} t^{g-1}\Gamma_g(tx) = x + \sum_{i\geq 1} \frac{\xi_i(x)}{i} t^i,$$

where

$$\begin{aligned}
\xi_i(x) &= i[t^i] \sum_{g\geq 0} t^{g-1}\Gamma_g(tx), \\
&= i[t^i] \sum_{g\geq 0}\sum_{j\geq 0} \frac{\overrightarrow{h}_g(j+2)}{(j+2)!} x^{j+2} t^{g+j+1}, \\
&= i \sum_{g=0}^{i-1} \frac{\overrightarrow{h}_g(i-j+1)}{(i-j+1)!} x^{i-g+1}.
\end{aligned}$$

Thus, we arrive at the following Theorem.

Theorem 5.6.1. *Let $\varphi : \Lambda \to \mathbb{C}[x]$ be the map from the ring of symmetric functions to the ring of polynomials in x given by*

$$\varphi(p_k) = \xi_k(x),$$

extended as an algebra homomorphism. Then for $n, k \geq 0$,

$$\Phi_n(h_{2k}) = \left[\frac{x^n}{n!}\right] \varphi(h_k) e^x.$$

Proof. We have

$$\begin{aligned}
\Phi_n(h_{2k}) &= \left[t^k \frac{x^n}{n!}\right] \mathcal{H}(t^{1/2}, x) \\
&= \left[t^k \frac{x^n}{n!}\right] \exp\left(\sum_{g\geq 0} t^{g-1}\Gamma_g(tx)\right) \\
&= \left[t^k \frac{x^n}{n!}\right] \exp\left(x + \sum_{i\geq 1} \frac{\xi_i(x)}{i} t^i\right) \\
&= \left[t^k \frac{x^n}{n!}\right] \varphi\left(\exp\left(\sum_{i\geq 1} \frac{p_i}{i} t^i\right)\right) e^x \\
&= \left[t^k \frac{x^n}{n!}\right] \varphi\left(\sum_{i\geq 0} h_i t^i\right) e^x \\
&= \left[\frac{x^n}{n!}\right] \varphi(h_k) e^x.
\end{aligned}$$

$\square$

Example 5.6.2. *It may be determined using the results in Sections 5.3 and 5.4 that*

$$\vec{h}_0(2) = 1,\ \vec{h}_0(3) = 8,\ \vec{h}_0(4) = 144,\ \vec{h}_0(5) = 4224,$$
$$\vec{h}_1(2) = 1,\ \vec{h}_1(3) = 40,\ \vec{h}_1(4) = 1944,$$
$$\vec{h}_2(2) = 1,\ \vec{h}_2(3) = 168,\ \vec{h}_3(2) = 1.$$

Thus,

$$\begin{aligned}
\xi_1(x) &= \frac{1}{2}x^2,\\
\xi_2(x) &= \frac{8}{3}x^3 + x^2,\\
\xi_3(x) &= 18x^4 + 20x^3 + \frac{3}{2}x^2,\\
\xi_4(x) &= \frac{704}{5}x^5 + 324x^4 + 112x^3 + 2x^2,
\end{aligned}$$

so that

$$\begin{aligned}
\varphi(h_1) &= \frac{1}{2}x^2,\\
\varphi(h_2) &= \frac{1}{8}x^4 + \frac{4}{3}x^3 + \frac{1}{2}x^2,\\
\varphi(h_3) &= \frac{1}{48}x^6 + \frac{2}{3}x^5 + \frac{25}{4}x^4 + \frac{20}{3}x^3 + \frac{1}{2}x^2,\\
\varphi(h_4) &= \frac{1}{384}x^8 + \frac{1}{6}x^7 + \frac{697}{144}x^6 + \frac{598}{15}x^5 + \frac{163}{2}x^4 + 28x^3 + \frac{1}{2}x^2.
\end{aligned}$$

This allows us to give closed form expressions for $\Phi_n(h_{2k})$ for small k. In particular, the Plancherel averages $\Phi_n(h_{2k})$ for small k have the following closed form expressions:

$$\begin{aligned}
\Phi_n(h_2) &= \binom{n}{2},\\
\Phi_n(h_4) &= 3\binom{n}{4} + 8\binom{n}{3} + \binom{n}{2},\\
\Phi_n(h_6) &= 15\binom{n}{6} + 80\binom{n}{5} + 150\binom{n}{4} + 40\binom{n}{3} + \binom{n}{2},\\
\Phi_n(h_8) &= 105\binom{n}{8} + 840\binom{n}{7} + 3485\binom{n}{6} + 4784\binom{n}{5} + 1956\binom{n}{4} + 168\binom{n}{3} + \binom{n}{2}.
\end{aligned}$$

5.6.3 Identity Correlators

Let $U(N)$ denote the group of $N \times N$ unitary matrices, and let $d\mu$ be the normalized Haar measure on $U(N)$. The problem of computing group integrals of the form

$$\int_{U(N)} f(U)g(U^*)d\mu,$$

where f and g are polynomial functions on $U(N)$ arises often in physics and probability theory ([22, 23, 82] and the references therein). In [22, 23] it was shown that the group integrals above could be reduced to those of the form

$$I_N(\pi) = \int_{U(N)} u_{11}\cdots u_{nn}\overline{u_{1\pi(1)}}\cdots\overline{u_{n\pi(n)}}d\mu,$$

called the *permutation correlators*, where $\pi \in S_n$ and u_{ij} is the i,j entry in the unitary matrices over which the integration is performed. Furthermore, the permutation correlators, $I_N(\pi)$ were expressed as certain character sums of S_n. In [88] it was observed that the character sum expression for $I_N(\pi)$ was equivalent to the expression

$$I_N(\pi) = \frac{1}{N^n}\sum_{k\geq 0}(-1)^k\frac{[C_\mu]h_k(J_2,\cdots,J_n)}{N^k},$$

where $N \geq n$ and π has cycle type μ. In particular, using the results of Section 5.2, we see that if id_n is the identity permutation in S_n then

$$\begin{aligned} I_N(id_n) &= \frac{1}{N^n}\sum_{k\geq 0}(-1)^k\frac{\Phi_n(h_k)}{N^k}, \\ &= \left[\frac{x^n}{n!}\right]H\left(\frac{1}{N},\frac{x}{N}\right). \end{aligned}$$

Example 5.6.2 thus allows us to give higher order approximations to $I_N(id_n)$. For example,

$$I_N(id_n) = \frac{1}{N^n} + \frac{\binom{n}{2}}{N^{n+2}} + \frac{3\binom{n}{4}+8\binom{n}{3}+\binom{n}{2}}{N^{n+4}} + O\left(\frac{1}{N^{n+6}}\right).$$

5.6.4 HCIZ Integral

The *Harish-Chandra Itzykson Zuber (HCIZ) integral* is a Unitary group integral given by

$$I_{HCIZ}(N) = \int_{U(N)} e^{zN\mathrm{tr}(AUBU^*)}d\mu,$$

where A and B are matrices and arises in a number of physics models (see [82] and the references therein). In [44], the coefficients appearing in the asymptotic expansion of the free energy of $I_{HCIZ}(N)$, $N \to \infty$ was given a combinatorial interpretation. In particular, the formal expansion of the free energy of the HCIZ integral is given by

$$F_N(z) = \frac{1}{N^2}\log I_{HCIZ}(N) = \sum_{g\geq 0}\sum_{d\geq 1}\sum_{\alpha,\beta\vdash d}\frac{\vec{H}_g(\alpha,\beta)p_\alpha q_\beta z^d}{N^{2g}d!},$$

where $p_k = \frac{1}{N}\mathrm{tr}(A^k)$, $p_\alpha = \prod_{i\geq 0}p_{\alpha_i}$ and similarly for q_β and B. Note that this expansion is formal and we refer to [44] for results pertaining to the analytic expansion.

The formal expansion of $F_N(z)$ and its relation to the 2-Toda hierarchy was considered in [112] (for a symmetric function approach see [14]). In particular, in [112] the degeneration of $F_N(z)$ occuring when $p_k = q_k = \delta_{k,1}$ was considered. In this case, using the preceding results, we have

$$\begin{aligned} F_N(z)\big|_{p_k=q_k=\delta_{k,1}} &= \sum_{g\geq 0}\sum_{d\geq 1} \frac{\overrightarrow{h}_g(d) z^d}{N^{2g} d!} \\ &= \sum_{g\geq 0} \frac{1}{N^{2g}} \Gamma_g(z). \end{aligned}$$

In particular, this gives

$$\gamma\big|_{w=\frac{1}{N},x=z} = \frac{zd}{dz}\, F_N(z)\big|_{p_k=q_k=\delta_{k,1}}.$$

Thus, Corollary 5.3.3 gives a nonlinear differential equation satisfied by the free energy of the HCIZ integral in the case that $p_k = q_k = \delta_{k,1}$.

Chapter 6

Future Work

In Chapter 2 we gave a development of the KP and BKP hieraries in a way which was most suitable for enumerative applications. As a result we were able to apply these hierarchies to the problems of enumerating orientable bipartite quadrangulations in Chapter 3, non-orientable triangulations in Chapter 4 and monotone Hurwitz numbers / Plancherel averaged complete symmetric functions in Chapter 5. This collection of applications is, of course, not exhaustive and many other possible applications have yet to be worked out. As possible avenues of future research we mention a few examples of other possible enumerative applications of integrable hierarchies.

6.1 Mixed Hurwitz Numbers

In Section 1.1.1 in the introduction we discussed the Hurwitz enumeration problem. That is, counting the number of factorizations of a permutation into transpositions. Further, in Chapter 5, we discussed the monotone Hurwitz problem. That is, the problem of counting factorizations into transpositions where an additional monotonicity constraint is present. In [46] Goulden, Guay-Paquet and Novak studied a more general problem that interpolated between the normal Hurwitz problem and the monotone Hurwitz problem.

Suppose that $k, l \geq 0$ are integers and that $\alpha, \beta \in \mathcal{P}$ with $|\alpha| = |\beta| = d$. Let $W^{k,l}(\alpha, \beta)$ be the number of $(k+l+2)$−tuples $(\sigma, \rho, \tau_1, \dots, \tau_k, \tau_{k+1}, \dots, \tau_{k+l})$ of permutations in S_d for which

1. σ has cycle type α and ρ has cycle type β,
2. each τ_i is a transposition of the form $(a_i\, b_i)$,
3. $b_1 \leq \ldots \leq b_k$,
4. the identity $\sigma = \rho\tau_1 \ldots \tau_{k+l}$ holds in S_d.

The numbers $W^{k,l}(\alpha,\beta)$ are called the mixed double Hurwitz numbers.

In [46] the authors consider the generating series

$$\mathbf{W}(z,t,u,\mathbf{p},\mathbf{q}) = 1 + \sum_{d\geq 1}\frac{z^d}{d!}\sum_{k,l\geq 0} t^k\frac{u^l}{l!}\sum_{\alpha,\beta\vdash d} W^{k,l}(\alpha,\beta)p_\alpha q_\beta,$$

which encodes the mixed double Hurwitz numbers. Similar to the methods used in Section 5.2 for the monotone Hurwitz problem, Goulden, Guay-Paquet and Novak used Jucys-Murphy elements and the center of the group algebra of S_d to show that $\mathbf{W}$ can be written as

$$\mathbf{W}(z,t,u,\mathbf{p},\mathbf{q}) = \sum_{\lambda\in\mathcal{P}}\prod_{(i,j)\in\lambda}\frac{ze^{(j-i)u}}{1-(j-i)t}s_\lambda(\mathbf{p})s_\lambda(\mathbf{q}).$$

In particular, this shows that the generating series $\mathbf{W}$ is a content-type series like those considered in Section 2.4.2. Thus, using Theorem 2.4.11 we know that the series $\mathbf{W}$ satisfies the KP hierarchy in the variables $\mathbf{p}$. The second component that we need in order to analyze the series $\mathbf{W}$ in a manner similar to the monotone Hurwitz problem is some sort of linear differential equation like the ones in Proposition 5.2.1. A combinatorial derivation of this sort of relationship seems difficult, however, as we shall see in Section 6.4, it should be possible to construct a family of linear differential equations algebraically.

6.2 Truncated Random Matrices

A number of enumerative problems to which integrable hierarchies are applicable arise in mathematical physics. In particular, many problems which arise in the study of random matrices have combinatorial interpretations. One such problem is the calculation of moments for truncated random matrices.

Let $U(d)$ be the unitary group of $d\times d$ matrices and let $d\mu$ be the normalized Haar measure on $U(d)$. We may think of the space $U(d)$ as being a probability space. Given a matrix $U\in U(d)$ and k an integer such that $1\leq k\leq d$, let U_k denote the upper left $k\times k$ corner of U. We call U_k a truncated unitary matrix. Let

$$G_d(x;k) = \sum_{n\geq 0}\int_{U(d)} |\mathrm{Tr}\ (U_k)|^{2n} d\mu\frac{x^{2n}}{(n!)^2},$$

be the generating series for the moments of the trace of truncated unitary matrices. In [86] Novak showed that the generating series $G_d(x;k)$ could be written as

$$G_d(x;k) = \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\leq k}}\prod_{(i,j)\in\lambda}\frac{k+(j-i)}{d+(j-i)}\frac{(f^\lambda)^2}{(|\lambda|!)^2}x^{2|\lambda|}.$$

In particular, this means that $G_d(x;k)$ can be written as an evaluation of a generating series which is a solution to the KP hierarchy. The way of doing this is very similar to the method used in Chapter 5 where we encoded the generating series for simple monotone Hurwitz numbers as an evaluation of the more general generating series for all monotone Hurwitz numbers. In both cases we construct a generating series which depends on the indeterminates $\mathbf{p}$ and then we evaluate at $\mathbf{p} = (1, 0, 0, \ldots)$ to get the specialized series.

In [87] Novak gave a combinatorial description of the generating series $G_d(x; k)$ as the generating series for a vicious walker model on the integer lattice. In doing so he also showed that the special case of $G_d(x; d)$ is related to the generating series for permutations of length n with maximal increasing subsequence bounded by d.

Similar to the mixed Hurwitz problem, there doesn't seem to be any combinatorially derived linear differential equations available for the truncated matrices problem. In the special case of $G_d(x; d)$ the matrix integral interpretation can be used to find linear differential equations called the Virasoro equations and this was used by Adler and van Moerbeke in [2] to derive a differential equation for $G_d(x; d)$ which could be used to analyze its coefficients. The tools discussed in Section 6.4 could be used to perform a similar analysis on the more general series $G_d(x; k)$.

6.3 m-Hypermap Numbers

Let $b_{\alpha,\beta}^{(a_1,a_2,\ldots)}$ be the number of tuples of permutations $(\sigma, \gamma, \pi_1, \pi_2, \ldots)$ in S_n such that

1. σ has cycle type α, γ has cycle type β,
2. each π_i has $n - a_i$ cycles,
3. $\sigma\gamma\pi_1\pi_2\ldots = 1$ in S_n where 1 is the identity,
4. the subgroup generated by $\sigma, \gamma, \pi_1, \pi_2, \ldots$ acts transitively on $\{1, \ldots, n\}$.

The numbers $b_{\alpha,\beta}^{(a_1,a_2,\ldots)}$ were considered in the proof of Proposition 3.2.1 where it was shown that the series

$$B = \sum_{\substack{|\alpha|=|\beta|=n\geq 1 \\ a_1,a_2,\ldots,\geq 0}} \frac{1}{n!} b_{\alpha,\beta}^{(a_1,a_2,\ldots)} p_\alpha q_\beta u_1^{a_1} u_2^{a_2} \ldots,$$

could be written as $B = \log(\widetilde{B})$ where

$$\widetilde{B} = \sum_{\lambda\in\mathcal{P}} \prod_{i\geq 1} \prod_{(j,k)\in\lambda} (1 + u_i(k-j)) s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q}).$$

In particular, $\widetilde{B}$ is a content-type series and so by Theorem 2.4.11 is a solution to the KP hierarchy.

That $\widetilde{B}$ satisfies the KP hierarchy was used in Chapter 3 because the generating series for bipartite maps is a specialization of the generating series $\widetilde{B}$.

More generally, we may consider the numbers $c_{\alpha,\beta}^{g,m}$ defined by

$$c_{\alpha,\beta}^{(g,m)} = \sum b_{\alpha,\beta}^{(a_1,a_2,\ldots)},$$

where the sum is over all $(a_1, a_2, \ldots)$ with $a_i = 0$ for $i > m$ and

$$a_1 + a_2 + \ldots + a_m = \ell(\alpha) + \ell(\beta) + 2g - 2.$$

If we let $N_{\alpha,\beta}^{(g,m)}$ be a rescaling of the $c_{\alpha,\beta}^{(g,m)}$ so that

$$N_{\alpha,\beta}^{(g,m)} = \frac{1}{|\alpha|!}|\text{Aut }\alpha||\text{Aut }\beta|c_{\alpha,\beta}^{(g,m)},$$

then we call the $N_{\alpha,\beta}^{(g,m)}$ the m-hypermap numbers. The reason for this is that in the case that $m = 1$ we get rooted hypermaps, or rooted bipartite maps.

If we now consider the generating series for m-hypermap numbers,

$$N^{(m)} = \sum_{\substack{|\alpha|=|\beta|=d\geq 1,\\ g\geq 0}} \frac{N_{\alpha,\beta}^{(g,m)}}{|\text{Aut }\alpha||\text{Aut }\beta|} p_\alpha q_\beta t^{\ell(\alpha)+\ell(\beta)+2g-g},$$

then it is not too difficult to see that

$$N^{(m)} = B\big|_{u_1=\ldots=u_m=t, u_j=0, j>m}.$$

In particular, this tells us that $\exp(N^{(m)})$ satisfies the KP hierarchy. This fact was initially proven in [50].

The m-hypermap numbers were studied by Bousquet-Mélou and Schaeffer in [12] where they showed that

$$N_{\alpha,(1^d)}^{(0,m)} = d!m\frac{((m-1)d-1)!}{((m-1)d-\ell(\alpha)+2)!}\prod_{i=1}^{\ell(\alpha)}\binom{m\alpha_i - 1}{\alpha_i}.$$

Further combinatorial analysis of the m-hypermap numbers seems very difficult, however, an algebraic approach may be possible using the fact that the series $N^{(m)}$ satisfies the KP hierarchy coupled with the algebraically derived linear differential equations mentioned in Section 6.4. Currently the m-hypermap numbers have resisted algebraic attemps at their determination and the only approach which has shown results has been the combinatorial one.

6.4 Linear Differential Equations

In the three applications of integrable hierarchies studied in this thesis some sort of linear differential equation or equations was necessary in addition to an integrable hierarchy. In each of the cases presented the linear differential equations could be derived using combinatorial means, however, the additional problems mentioned in this chapter seem resistant to the type of combinatorial analysis necessary to derive similar linear differential equations. Fortunately there appears to be a way to derive the auxiliary result algebraically, something which we will briefly present here. A precise and complete description of this method along with any possible applications seems to be a very fruitful area of future research.

Consider the differential operator given by

$$\Delta = \frac{1}{2}\sum_{i,j\geq 1} p_i p_j p_{i+j}^{\perp} + \frac{1}{2}\sum_{i,j\geq 1} p_{i+j} p_i^{\perp} p_j^{\perp}.$$

This is called the join-cut operator and is the differential operator alluded to in Section 6.1 above in relation to the Hurwitz enumeration problem. Combinatorially the join-cut operator corresponds to the operation of multiplying a permutation by a transposition. If the two non-fixed points of the transposition are in different cycles to begin with, then after multiplying those two cycles are joined together, a 'join'. If the two non-fixed points of the transposition are in the same cycle to begin with, then after multiplying those two cycles will appear in different cycles in the resulting permutation, a 'cut'.

Using the combinatorial description of the join-cut operator (or similarly using Jucys-Murphy elements and the center of the S_n group algebra) Goulden [43] showed that Δ acts diagonally on Schur polynomials as follows,

$$\Delta s_\lambda = \left(\sum_{(i,j)\in\lambda}(j-i)\right) s_\lambda.$$

Similar to the approach used by Lassalle [76], we may construct the related operators given by,

$$\mathcal{U}_0 = p_1, \qquad \mathcal{U}_k = [\Delta, \mathcal{U}_{k-1}] \text{ for } k > 0,$$

and

$$\mathcal{D}_0 = p_1^{\perp}, \qquad \mathcal{D}_k = [\mathcal{D}_{k-1}, \Delta] \text{ for } k > 0,$$

where here $[A,B] = AB - BA$ is the commutator of the operators A and B. As an example, $\mathcal{U}_1 = [\Delta, p_1] = \sum_{k\geq 1} p_{k+1}p_k^{\perp}$ and $\mathcal{D}_1 = \sum_{k\geq 1} p_k p_{k+1}^{\perp}$.

Now, recall the Murnaghan-Nakayama rule (see [78] for example) that

$$p_1 s_\lambda = \sum_{\mu > \lambda} s_\mu,$$

and dually,

$$p_1^{\perp} s_\lambda = \sum_{\mu < \lambda} s_\mu,$$

where here $\mu > \lambda$ means that μ/λ is a single cell and $\mu < \lambda$ means that λ/μ is a single cell. The Murnaghan-Nakayama rule then allows us to describe the action of the $\mathcal{U}_k$ and $\mathcal{D}_k$ operators on Schur polynomials. First, a bit of notation, if $\lambda > \mu$ are partitions so that λ/μ is a single cell then we write $c(\lambda/\mu)$ for the content of the single cell in λ which is not in μ. In other words,

$$c(\lambda/\mu) = \frac{\prod_{(i,j)\in\lambda}(j-i)}{\prod_{(i,j)\in\mu}(j-i)}.$$

Proposition 6.4.1. *The operators $\mathcal{U}_k$ and $\mathcal{D}_k$ for $k \geq 0$ act on the Schur polynomials as,*

$$\mathcal{U}_k s_\lambda = \sum_{\mu>\lambda} c(\mu/\lambda)^k s_\mu,$$

and

$$\mathcal{D}_k s_\lambda = \sum_{\mu<\lambda} c(\lambda/\mu)^k s_\mu.$$

Proof. We will begin with the action of $\mathcal{U}_k$. The Murnaghan-Nakayama rule shows that this is true when $k = 0$. Suppose it is true up to $k-1$, then

$$\begin{aligned}
\mathcal{U}_k s_\lambda &= [\Delta, \mathcal{U}_{k-1}]s_\lambda = \Delta\mathcal{U}_{k-1}s_\lambda - \mathcal{U}_{k-1}\Delta s_\lambda, \\
&= \Delta\sum_{\mu>\lambda} c(\mu/\lambda)^{k-1}s_\mu - \sum_{(i,j)\in\lambda}(j-i)\mathcal{U}_{k-1}s_\lambda, \\
&= \sum_{\mu>\lambda} c(\mu/\lambda)^{k-1}\sum_{(i,j)\in\mu}(j-i)s_\mu - \sum_{\mu>\lambda} c(\mu/\lambda)^{k-1}\sum_{(i,j)\in\lambda}(j-i)s_\mu, \\
&= \sum_{\mu>\lambda} c(\mu/\lambda)^{k-1}\left(\sum_{(i,j)\in\mu}(j-i) - \sum_{(i,j)\in\lambda}(j-i)\right)s_\mu = \sum_{\mu>\lambda} c(\mu/\lambda)^k s_\mu.
\end{aligned}$$

The proof for $\mathcal{D}_k$ is essentially the same. □

We will now consider a family of generating series which are a specialization of the content-type series appearing in Section 2.4.2. Suppose that $p(x)$ and $q(x)$ are polynomials. We say that the series $\Phi^{p,q}$ of the form

$$\Phi^{p,q} = \sum_{\lambda\in\mathcal{P}}\prod_{(i,j)\in\lambda}\frac{p(j-i)}{q(j-i)}\frac{f^\lambda}{|\lambda|!}s_\lambda(\mathbf{p}),$$

are called rational content-type series. It is easily seen that the series $\Phi^{p,q}$ are a specialization of the content-type series studied in Section 2.4.2 and, as a result, are solutions to the KP hierarchy in the variables $\mathbf{p}$.

Using the results of Proposition 6.4.1 we may compute the action of $\mathcal{U}_k$ and $\mathcal{D}_k$ on rational content-type series. To make the formulas more compact we introduce the umbral composition operator $\circ$ which we define in the following way. If $p(x) = \sum_{k\geq0} a_k x^k$ and $q(x) = \sum_{k\geq0} b_k x^k$ are polynomials then,

$$(p\circ\mathcal{U}) = \sum_{k\geq0} a_k\mathcal{U}_k, \qquad (q\circ\mathcal{D}) = \sum_{k\geq0} b_k\mathcal{D}_k,$$

and we extend this definition similarly for any family of operators $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1, \ldots)$. Then, using Proposition 6.4.1 we see that

$$(p \circ \mathcal{U})\Phi^{p,q} = \sum_{\lambda \in \mathcal{P}} \left\{ \prod_{(i,j)\in\lambda} \frac{p(j-i)}{q(j-i)} \right\} \left(\sum_{\mu \prec \lambda} q(\lambda/\mu) \frac{|\lambda| f^{\mu}}{f^{\lambda}} \right) \frac{f^{\lambda} s_{\lambda}}{|\lambda|!},$$

and that a similar result holds for $(q \circ \mathcal{D})\Phi^{p,q}$.

Now, suppose that there exist operators $\mathcal{C}_k$ for $k \geq 0$ which act diagonally on Schur polynomials as,

$$\mathcal{C}_k s_{\lambda} = \left\{ \sum_{\mu \prec \lambda} c(\lambda/\mu)^k \frac{|\lambda| f^{\mu}}{f^{\lambda}} \right\} s_{\lambda},$$

and that similarly for $k \geq 0$ there exist operators $\mathcal{T}_k$ which act as

$$\mathcal{T}_k s_{\lambda} = \left\{ \sum_{\mu \succ \lambda} c(\mu/\lambda)^k \frac{f^{\mu}}{|\mu| f^{\lambda}} \right\} s_{\lambda}.$$

Then we may arrive at the following theorem.

Theorem 6.4.2. *Given operators $\mathcal{C}_k$ and $\mathcal{T}_k$ for $k \geq 0$ as described above and an integer $i \geq 0$,*

$$(x^i p \circ \mathcal{U})\Phi^{p,q} = (x^i q \circ \mathcal{C})\Phi^{p,q},$$

and

$$(x^i q \circ \mathcal{D})\Phi^{p,q} = (x^i p \circ \mathcal{T})\Phi^{p,q}.$$

In particular, Theorem 6.4.2 gives a method of computing a countable number of linear differential equations for rational content-type series, provided the operators $\mathcal{C}_k$ and $\mathcal{T}_k$ exist. Fortunately, it is essentially a result of Okounkov [90, 91, 94] that the operators $\mathcal{C}_k$ and $\mathcal{T}_k$ exist and are differential operators. In fact, Okounkov showed that these operators are part of a commuting algebra of differential operators which act diagonally on Schur functions and are called the Sekiguchi-Debiard operators (see also [75, 76]). The first few operators are given by

$$\mathcal{T}_0 = 1, \qquad \mathcal{T}_1 = 0, \qquad \mathcal{T}_2 = E, \qquad \mathcal{T}_3 = 2\Delta, \qquad \mathcal{C}_0 = E, \qquad \mathcal{C}_1 = 2\Delta,$$

where here

$$E = \sum_{i \geq 1} p_i p_i^{\perp},$$

is the Eulerian operator which acts on Schur polynomials as $E s_{\lambda} = |\lambda| s_{\lambda}$.

Using Theorem 6.4.2 we can now construct families of linear differential equations which are satisfied by rational content-type series. For example, the linear differential equations appearing in Proposition 5.2.1 can be derived in this way as well as the monotone join-cut equation studied by Goulden, Guay-Paquet and Novak in [44, 45, 47]. This method is also applicable to the mixed Hurwitz problem and the m-hypermap problem described above. Note that the mixed Hurwitz generating series is not a rational content-type series, however, the proof of Theorem 6.4.2 and the generalized version in Theorem 6.4.3 does not depend on $p(x)$ and $q(x)$ being polynomials. Unfortunately, the linear differential equations that are generated with this method grow in complexity with $p(x)$ and $q(x)$. This difficulty also appears in the m-hypermap case as the degree of the polynomials $p(x)$ and $q(x)$ can be arbitrarily large.

In order to apply this method to the truncated random matrix problem we will need to extend Theorem 6.4.2 to include bounded rational content-type series. That is, rational content-type series where the sum is over all partitions where the length of the partitions are bounded (or equivalently the size of the first part is bounded). Fortunately we can prove a generalization of Theorem 6.4.2. For $A = \{(k, i_k) : k, i_k \geq 1\}$ define

$$\mathcal{P}(A) = \{\lambda \in \mathcal{P} : \lambda_k \leq i_k \ \forall (k, i_k) \in A\}.$$

In particular, if we let $A = (n+1, 0)$ then $\mathcal{P}(A) = \{\lambda \in \mathcal{P} : \ell(\lambda) \leq n\}$ and similarly if $A = (1, n)$ then $\mathcal{P}(A) = \{\lambda \in \mathcal{P} : \lambda_1 \leq n\}$. We may define the bounded rational content-type series to be series of the form

$$\Phi_A^{p,q} = \sum_{\lambda \in \mathcal{P}(A)} \left\{ \prod_{(i,j) \in \lambda} \frac{p(j-i)}{q(j-i)} \right\} \frac{f^\lambda}{|\lambda|!} s_\lambda.$$

Then it is straightforward to prove the following generalization of Theorem 6.4.2.

Theorem 6.4.3. *Suppose $r(x)$ is a formal power series such that $r(i_k - k + 1)q(i_k - k + 1) = 0$ for all $(k, i_k) \in A$. Then the series $\Phi_A^{p,q}$ satisfies the linear differential equation,*

$$(rp \circ \mathcal{U})\Phi_A^{p,q} = (rq \circ \mathcal{C})\Phi_A^{p,q}.$$

Similarly, if r is chosen so that $r(i_k - k + 1)p(i_k - k + 1) = 0$ for all $(k, i_k) \in A$, then

$$(rq \circ \mathcal{D})\Phi_A^{p,q} = (rp \circ \mathcal{T})\Phi_A^{p,q}.$$

Proof. For a pair (k, i_k) with $k, i_k \geq 1$ let

$$\mathcal{P}^\bullet(k, i_k) = \{\lambda \in \mathcal{P} : \lambda_k = i_k + 1\}.$$

Also, for any $k \geq 1$ we let e_k be the vector with 1 in the kth position and zero elsewhere. We compute,

$$\begin{aligned}
(rp \circ \mathcal{U})\Phi_A^{p,q} &= \sum_{\lambda \in \mathcal{P}(A)} \left\{ \prod_{s \in \lambda} \frac{p(s)}{q(s)} \right\} \frac{f^\lambda}{|\lambda|!} \left(\sum_{\mu > \lambda} r(\mu/\lambda) p(\mu/\lambda) s_\mu \right) \\
&= \sum_{\lambda \in \mathcal{P}(A)} \sum_{\mu > \lambda} \left\{ \prod_{s \in \mu} \frac{p(s)}{q(s)} \right\} \frac{f^\mu}{|\mu|!} s_\mu \left(\frac{|\mu| f^\lambda}{f^\mu} r(\mu/\lambda) q(\mu/\lambda) \right) \\
&= \sum_{\mu \in \mathcal{P}(A)} \left\{ \prod_{s \in \mu} \frac{p(s)}{q(s)} \right\} \frac{f^\mu}{|\mu|!} s_\mu \left(\sum_{\lambda < \mu} \frac{|\mu| f^\lambda}{f^\mu} r(\mu/\lambda) q(\mu/\lambda) \right) \\
&\quad + \sum_{(k, i_k) \in A} \sum_{\substack{\mu \in \mathcal{P}^\bullet(k, i_k) \\ \mu - e_k \in \mathcal{P}(A)}} \left\{ \prod_{s \in \mu} \frac{p(s)}{q(s)} \right\} \frac{f^\mu}{|\mu|!} s_\mu \left(\frac{|\mu| f^{\mu - e_k}}{f^\mu} r(\mu/\mu - e_k) q(\mu/\mu - e_k) \right).
\end{aligned}$$

Now, since $c(\mu/\mu - e_k) = \mu_k - k$ and $\mu \in \mathcal{P}^\bullet(k, i_k)$, $c(\mu/\mu - e_k) = i_k - k + 1$. Thus, $r(\mu/\mu - e_k)q(\mu/\mu - e_k) = 0$ and so

$$\begin{aligned}(rp \circ \mathcal{U})\Phi_A^{p,q} &= \sum_{\mu \in \mathcal{P}(A)} \left\{\prod_{s \in \mu} \frac{p(s)}{q(s)}\right\} \frac{f^\mu}{|\mu|!} s_\mu \left(\sum_{\lambda \prec \mu} \frac{|\mu| f^\lambda}{f^\mu} r(\mu/\lambda) q(\mu/\lambda)\right) \\ &= \sum_{\mu \in \mathcal{P}(A)} \left\{\prod_{s \in \mu} \frac{p(s)}{q(s)}\right\} \frac{f^\mu}{|\mu|!} (rq \circ c(\lambda)) s_\mu = (rq \circ \mathcal{C})\Phi_A^{p,q}.\end{aligned}$$

The proof for $(rq \circ \mathcal{D})\Phi_A^{p,q} = (rp \circ \mathcal{T})\Phi_A^{p,q}$ proceeds similarly. □

If we consider the special case of Theorem 6.4.3 when $A = \varnothing$ then the truncated rational content-type series $\Phi_A^{p,q}$ is just $\Phi^{p,q}$ and if we take $r(x) = x^i$ then Theorem 6.4.2 follows.

6.4.1 Plancherel Averaged Hall-Littlewood Polynomials

Recall from Chapter 5 that given a symmetric function f the Plancherel average of f is

$$\Phi_n(f) = \sum_{\lambda \vdash n} \frac{f(A_\lambda)(\dim \lambda)^2}{n!},$$

where $\dim \lambda$ is the dimension of the irreducible representation of S_n indexed by the partition λ and $A_\lambda = \{j - i : (i,j) \in \lambda\}$ is the multiset of contents of the cells in λ. In Chapter 5 we consider the problem of computing the Plancherel average of complete symmetric functions. However, given Theorem 6.4.3 we may consider the more general problem of determining properties of the Plancherel average for a more general class of symmetric function, the Hall-Littlewood polynomials. This problem was also considered by Lassalle in [76].

The Hall-Littlewood symmetric functions indexed by a single integer (the one-part Hall-Littlewood polynomials) can be written as $P_k(t) = (1-t)g_k(t)$ for $k > 0$ and $g_0(t) = 1$ where the polynomials $g_k(t)$ have generating series

$$\sum_{k \ge 0} g_k(t) u^k = \prod_{i \ge 1} \frac{1 - x_i t u}{1 - x_i u}.$$

In particular, this means that the problem of computing the Plancherel average of Hall-Littlewood polynomials is related to computing the coefficients of the series $\Phi_{HL} = \Phi_\varnothing^{1-tux, 1-ux}$.

Applying Theorem 6.4.3 to Φ_{HL} gives the following differential equations:

$$p_1^\perp \Phi_{HL} - u \sum_{i \ge 1} p_i p_{i+1}^\perp \Phi_{HL} = \Phi_{HL},$$

$$\sum_{i \ge 1} p_i p_{i+1}^\perp \Phi_{HL} - u \sum_{i,j \ge 1} p_{i+j-1} p_i^\perp p_j^\perp \Phi_{HL} - u \sum_{i,j \ge 1} p_i p_j p_{i+j+1}^\perp \Phi_{HL} = -tuE\Phi_{HL},$$

$$p_1\Phi_{HL} - tu\sum_{i\geq 1} p_{i+1}p_i^{\perp}\Phi_{HL} = E\Phi_{HL} - u\sum_{i,j\geq 1} p_ip_jp_{i+j}^{\perp}\Phi_{HL} - u\sum_{i,j\geq 1} p_{i+j}p_i^{\perp}p_j^{\perp}\Phi_{HL}.$$

The first two equations are equivalent to those found by Lassalle analytically while the last equation appears to be new and may play the role of the monotone join cut equation in [45] for the series Φ_H.

6.4.2 Truncated Random Matrices

If we modify the series Φ_{HL} slightly and add a restriction on the partitions appearing as summation indices we arrive at a series which contains the problem discussed earlier in Section 6.2. Namely, consider the series $\Phi_T^{(d)}(k) = \Phi_{\{(k+1,0)\}}^{k+x,d+x}$. As mentioned earlier, the condition $\lambda_{k+1} \leq 0$ is equivalent to $\ell(\lambda) \leq k$. Using the hook-content formula,

$$s_{\lambda,k}(1) = \prod_{(i,j)\in\lambda} \frac{k+(j-i)}{h_\lambda(i,j)},$$

where $s_{\lambda,k}(1)$ is the result of evaluating s_λ at $x_1 = \cdots = x_k = 1$, $x_{k+1} = \cdots = 0$ and $h_\lambda(i,j)$ is the hook length of the cell (i,j). That is, $h_\lambda(i,j)$ is the number of cells to the right of (i,j) in λ plus the number of cells below (i,j) in λ plus one (to account for the cell (i,j) itself). Thus, we have

$$\Phi_T^{(d)}(k) = \sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)\leq k}} \frac{s_{\lambda,k}(1)}{s_{\lambda,d}(1)}\frac{f^\lambda}{|\lambda|!}s_\lambda.$$

In particular, if we set $p_i = x^2\delta_{i,1}$ then the series $\Phi_T^{(d)}(k)$ is equal to $G_d(x;k)$.

Theorem 6.4.3 can now be applied to $\Phi_T^{(d)}(k)$ and combined with the fact that $\Phi_T^{(d)}(k)$ satisfies the KP hierarchy (since it is a content-type series). One hopes that this will lead to some interesting results about the series $G_d(x;k)$.

Appendix

In Chapter 4 the structure theorem for the generating series for triangulations on all surfaces (orientable and non-orientable) as well as the asymptotic behaviour of the number of such triangulations relied on a technical lemma whose proof involves a lengthy computation. It is the goal of this chapter to provide the details of this computation. First, we recall some of the notation used in Chapter 4. For our purposes, z will be an indeterminate and $s = s(z)$ is the unique power series solution to $z = \frac{1}{2}s(1-s)(1-2s)$ with $s(0) = 0$. We will also write $\eta = 1 - 6s + 6s^2$ and define, for $i \geq 0$,

$$\psi_i = \begin{cases} \frac{(1-2s)}{\eta^{\frac{i}{2}+1}}, & \text{if } i \text{ is even,} \\ \frac{1}{\eta^{\frac{i}{2}+1}}, & \text{if } i \text{ is odd.} \end{cases}$$

Furthermore, if $L_g^{(3)}(z)$ is the generating series for triangulations on a surface with Euler characteristic g then we define

$$T_g(z) = L_g^{(3)}(z) + \delta_{g,1} + \left(1 - \frac{1}{2z}\right)\delta_{g,0}.$$

For convenience we will also write $D = 6z\partial_z$. We wish to prove the following lemma.

Lemma 4.4.4. *Suppose for $g \geq 3$,*

$$R_g(s) = \left\{\frac{4}{3}D^2 \; -4(g\psi_2 + \psi_0 - 4)D - \frac{4}{3}(2\psi_4 + 9g(3-g)\psi_2 + 12\psi_0 - 32)\right\} L_g^{(3)},$$

and that

$$L_k^{(3)} = \sum_{i=0}^{5k-8} \mu_k(i)\psi_i,$$

for $2 \leq k < g$. Then

$$R_g(s) = \sum_{i=0}^{5g} r_g(i)\psi_i,$$

for some constants $r_g(i)$. *Furthermore,*

$$\begin{aligned}
r_g(5g) = &\frac{1}{162}(5g-8)(5g-12)(5g-6)w_{g-2}\\
&-\frac{1}{3^6 5!}(5g-6)(5g-10)(5g-14)(5g-18)(5g-22)w_{g-4}\\
&+\frac{(5g-6)}{54}\sum_{i=1}^{g-1}\chi(i,g-i)w_i w_{g-i}\\
&-\frac{(5g-6)}{2^3 3^6}\sum_{i=2}^{g-1}\chi(i,g-i)(5i-8)(5i-12)w_{i-2}w_{g-i}\\
&-\frac{1}{9\cdot 216}\sum_{i=1}^{g-2}\sum_{j=1}^{g-i-1}(5i-2)\chi(i,g-i)\chi(j,g-i-j)w_i w_j w_{g-i-j},
\end{aligned}$$

where $w_k = (5k-6)\mu_k(5k-8)$.

To prove Lemma 4.4.4 we will require Corollary 4.3.2 and Theorem 4.4.1. In an effort to make this proof more readable we will reproduce the statements of these results here.

Corollary 4.3.2. *For* $g \geq 0$,

$$\begin{aligned}
&4z^2(D+12)^2(D+8)(D+4)T_{g-2} - (D+6)DT_g\\
&\qquad + 12z^2(D+12)\left(\sum_{i=0}^{g}\{(D+4)T_i\}\{(D+4)T_{g-i}\}\right)\\
&= \sum_{k=0}^{g} V_k \Big\{2z^2(D+12)(D+8)(D+4)T_{g-k-2}\\
&\qquad -\frac{1}{2}(D+6)T_{g-k} + 6z^2\left(\sum_{i=0}^{g-k}\{(D+4)T_i\}\{(D+4)T_{g-k-i}\}\right)\Big\},
\end{aligned}$$

where

$$V_k = \sum_{t=0}^{k} 2^{k-t+2}(1+(-1)^{k-t})\sum_{i=0}^{k-t+2}\binom{2-t}{k-t-i+2}\frac{z^i}{i!}\partial_z^i L_t^{(3)}(z).$$

Theorem 4.4.1. *The generating series* $L_0^{(3)}(z)$ *and* $L_1^{(3)}(z)$ *are given in terms of* s *by*

$$\begin{aligned}
L_0^{(3)}(z) &= \frac{2s(1-4s+2s^2)}{(1-s)(1-2s)^2},\\
L_1^{(3)}(z) &= \frac{(1-2s)(1-s+s^2)-(1-6s+6s^2)^{\frac{1}{2}}}{s(1-s)(1-2s)}.
\end{aligned}$$

Our first step will be to rewrite Corollary 4.3.2 so that $R_g(s)$ appears. In order to break up the calculations which follow we will begin by defining the following series. Let

$$
\begin{aligned}
A(k) &= z^2(D+12)(D+8)(D+4)T_k,\\
B(k) &= z^2\sum_{i=1}^{k-1}\{(D+4)T_i\}\{(D+4)T_{k-i}\},\\
\overline{B(k)} &= z^2\sum_{i=0}^{k}\{(D+4)T_i\}\{(D+4)T_{k-i}\},\\
\widetilde{B(k)} &= \frac{1}{3}(\psi_2+2\psi_0)\overline{B(k)},\\
C(k) &= (D+6)T_k,\\
\alpha_k(t) &= \sum_{j=0}^{k-t+2}\binom{2-t}{k-t-j+2}\frac{z^j}{j!}\partial_z^j L_t^{(3)}(z),\\
\overline{V_k} &= \sum_{t=0}^{k-2}(1+(-1)^{k-t})2^{k-t+2}\alpha_k(t), \text{ and}\\
V_k &= \sum_{t=0}^{k}(1+(-1)^{k-t})2^{k-t+2}\alpha_k(t).
\end{aligned}
$$

Proposition 6.4.4. *For $g \geq 2$ let*

$$
R_g(s) = \left\{\frac{4}{3}D^2 - 4(g\psi_2+\psi_0-4)D - \frac{4}{3}(2\psi_4+9g(3-g)\psi_2+12\psi_0-32)\right\}L_g^{(3)}.
$$

Then

$$
\begin{aligned}
R_g(s) = {}& \frac{4}{3}(\psi_2+2\psi_0)DA(g-2)+4(\psi_2+2\psi_0)DB(g)\\
&-\frac{8}{3}(\psi_2+2\psi_0)(\psi_0-1)(2A(g-2)+6B(g))-3\psi_2\overline{V_g} \qquad (6.1)\\
&-\sum_{i=1}^{g-1}V_i\left(\frac{2}{3}(\psi_2+2\psi_0)A(g-i-2)-\frac{1}{6}(\psi_2+2\psi_0)C(g-i)+6\widetilde{B(g-i)}\right).
\end{aligned}
$$

Proof. From Corollary 4.3.2 we have

$$
\begin{aligned}
&4z^2(D+12)^2(D+8)(D+4)T_{g-2}-(D+6)DT_g\\
&\qquad +12z^2(D+12)\left(\sum_{i=0}^{g}\{(D+4)T_i\}\{(D+4)T_{g-i}\}\right)\\
&= \sum_{k=0}^{g}V_k\left\{2z^2(D+12)(D+8)(D+4)T_{g-k-2}-\frac{1}{2}(D+6)T_{g-k}\right. \qquad (6.2)\\
&\qquad \left. +6z^2\left(\sum_{i=0}^{g-k}\{(D+4)T_i\}\{(D+4)T_{g-k-i}\}\right)\right\},
\end{aligned}
$$

with

$$
\begin{aligned}
V_k &= \sum_{t=0}^{k}2^{k-t+2}(1+(-1)^{k-t})\sum_{i=0}^{k-t+2}\binom{2-t}{k-t-i+2}\frac{z^i}{i!}\partial_z^i L_t^{(3)}(z),\\
&= q_k(D)L_k^{(3)}(z)+\overline{V_k},
\end{aligned}
$$

and

$$q_k(D) = 4(k-1)(k-2) + \frac{2(3-2k)}{3}D + \frac{D^2}{9}.$$

If we let

$$r_g(D) = 12z^2\{(D+4)T\}(D+4) - \frac{1}{2}(D+6),$$

and $U_k = (D+4)T_k$ then (6.2) becomes (note that $T_g = L_g^{(3)}$ for $g \geq 2$),

$$\begin{aligned} p_g(D)L_g^{(3)} &= 4z^2(D+12)^2(D+8)(D+4)U_{g-2} && (6.3)\\ &\quad + 12z^2(D+12)\sum_{i=1}^{g-1} U_iU_{g-i} - V_0(2z^2(D+12)(D+8)U_{g-2}\\ &\quad + 6z^2\sum_{j=1}^{g-1} U_jU_{g-j}) - \overline{V_g}(6z^2U_0^2 - \frac{1}{2}(D+6)T_0) && (6.4)\\ &\quad - \sum_{i=1}^{g-1} V_i(2z^2(D+12)(D+8)U_{g-i-2} - \frac{1}{2}(D+6)T_{g-i} + 6z^2\sum_{j=0}^{g-i} U_jU_{g-i-j}), \end{aligned}$$

where

$$p_g(D) = (6z^2U_0^2 - \frac{1}{2}(D+6)T_0)q_g(D) - V_0r_g(D) - 2Dr_g(D).$$

Using Gao's formula for $L_0^{(3)}$ in terms of s (Theorem 4.4.1) we may compute

$$6z^2U_0^2 - \frac{1}{2}(D+6)T_0 = \frac{3}{(1-2s)^2},$$

and

$$V_0 = \left(8 + 2D + \frac{D^2}{9}\right)L_0^{(3)} = 8\left(\frac{1}{\eta} - 1\right) = 8(\psi_0 - 1).$$

Similarly,

$$r_g(D) = \left(\frac{1}{2} - \eta\right)D + (1-4\eta),$$

and

$$Dr_g(D) = \left(\frac{1}{2} - \eta\right)D^2 + \left(\frac{2}{\eta} + 3 - 8\eta\right)D + \left(\frac{8}{\eta} + 8 - 16\eta\right).$$

So,

$$\begin{aligned} p_g(D) = \frac{3}{(1-2s)^2}&\left(\frac{D^2}{9} + \frac{2}{3}(3-2g)D + 4(g-1)(g-2)\right)\\ &+ 8\left(\frac{1}{\eta} - 1\right)\left(\left(\frac{1}{2} - \eta\right)D + (1-4\eta)\right)\\ &- 2\left(\left(\frac{1}{2} - \eta\right)D^2 + \left(\frac{2}{\eta} + 3 - 8\eta\right)D + \left(\frac{8}{\eta} + 8 - 16\eta\right)\right). \end{aligned}$$

In particular, we have

$$\frac{(1-2s)^2}{\eta^2}p_g(D) = \frac{4}{3}D^2 - 4(g\psi_2 + \psi_0 - 4)D - \frac{4}{3}(2\psi_4 + 9g(3-g)\psi_2 + 12\psi_0 - 32).$$

Multiplying (6.4) by $\frac{(1-2s)^2}{\eta^2}$ on both sides, using the formula for $\frac{(1-2s)^2}{\eta^2}p_g(D)$ above and the fact that

$$\frac{(1-2s)^2}{\eta^2} = \frac{1}{3}\left(\frac{1}{\eta^2} + \frac{2}{\eta}\right) = \frac{1}{3}(\psi_2 + 2\psi_0),$$

then gives the desired result. □

The proof of Lemma 4.4.4 will proceed in two parts. The first will be to show that $R_g(s) = \sum_{i=0}^{5g+2} r_g(i)\psi_i$. Then, we will show that $r_g(5g+2) = r_g(5g+1) = 0$ and that $r_g(5g)$ has the description given in Lemma 4.4.4. In order to do this we will need to show that each of the series appearing in the right hand side of (6.1) in Proposition 6.4.4 can be expanded in terms of the ψ_i basis and work out the first few terms in this expansion. Before we continue, first recall some fundamental properties of the ψ_i basis. Proposition 4.4.3 tells us that the ψ_i basis satisfies the multiplicative identity

$$\psi_i\psi_j = \chi(i,j)\psi_{i+j+2} + (1-\chi(i,j))\psi_{i+j},$$

where

$$\chi(i,j) = \begin{cases} 1, & \text{if either } i \text{ or } j \text{ is even,} \\ \frac{1}{3}, & \text{if both } i \text{ and } j \text{ are odd.} \end{cases}$$

Suppose $F = \sum_{i\geq 0} f_i\psi_i$ and $G = \sum_{i\geq 0} g_i\psi_i$ are series which can be written in the ψ_i basis. Using the multiplicative property of the ψ_i basis we may define

$$\pi(F,G;r) = \sum_{i+j=r-2} \chi(i,j) f_i g_j + \sum_{i+j=r} (1-\chi(i,j)) f_i g_j,$$

so that

$$FG = \sum_{r\geq 0} \pi(F,G;r)\psi_r.$$

In particular, if $F = \sum_{i=0}^{M} f_i\psi_i$ and $G = \sum_{i=0}^{N} g_i\psi_i$, then

$$FG = \sum_{i=0}^{M+N+2} \pi(F,G,i)\psi_i,$$

where the first few terms can be determined explicitly as,

$$\begin{aligned} \pi(F,G;M+N+2) &= \chi(M,N) f_M g_N, \\ \pi(F,G;M+N+1) &= \chi(M,N-1) F_M g_{N-1} + \chi(M-1,N) f_{M-1} g_N, \text{ and} \\ \pi(F,G;M+N) &= \sum_{i+j=M+N-2} \chi(i,j) f_i g_j + (1-\chi(M,N)) f_M g_N. \end{aligned}$$

When working out the top terms of the expansion of (6.1) it will often be the case that we have two series, say F and $\overline{F}$, which agree at the top few terms but which may differ afterward. It will be useful to know that the products of F and $\overline{F}$ with an additional series G also agree at the top few terms. In particular, if we suppose that $\overline{F} = \sum_{i=0}^{M} \overline{f_i}\psi_i$ with $f_{M-j} = \overline{f_{M-j}}$, $j = 0,1,2$ then $F - \overline{F} = \sum_{i=0}^{M-3}(f_i - \overline{f_i})\psi_i$ so that

$$(F - \overline{F})G = \sum_{i=0}^{M+N-1} \pi(F - \overline{F}, G; i)\psi_i.$$

Thus, $\pi(F, G; M + N + 2 - i) = \pi(\overline{F}, G, M + N + 2 - i)$ for $i = 0, 1, 2$.

In addition to making use of the multiplicative properties of the ψ_i basis we will also make use of the operator D on the ψ_i. Again, recall from Proposition 4.4.3 that

$$D\psi_i = \begin{cases} (i+2)\psi_{i+4} + (i+2)\psi_{i+2} - 2(i+2)\psi_i, & \text{if } i \text{ is even,} \\ (i+2)\psi_{i+4} + i\psi_{i+2} - 2(i+1)\psi_i, & \text{if } i \text{ is odd.} \end{cases}$$

So, if $F = \sum_{i\geq 0} f_i\psi_i$ is a series in the ψ_i then we may define $\delta(F; i)$ by

$$\delta(F; i) = \begin{cases} (i-2)f_{i-4} + if_{i-2} - 2(i+2)f_i, & \text{if } i \text{ is even,} \\ (i-2)f_{i-4} + (i-2)f_{i-2} - 2(i+1)f_i, & \text{if } i \text{ is odd,} \end{cases}$$

so that

$$DF = \sum_{i\geq 0} \delta(F; i)\psi_i.$$

In particular, if $F = \sum_{i=0}^{M} f_i\psi_i$, then

$$DF = \sum_{i=0}^{M+4} \delta(F; i)\psi_i,$$

where the top few terms may be computed explicitly as,

$$\begin{aligned} \delta(F; M+4) &= (M+2)f_M, \\ \delta(F; M+3) &= (M+1)f_{M-1}, \text{ and} \\ \delta(F; M+2) &= \begin{cases} Mf_{M-2} + (M+2)F_M, & \text{if } M \text{ is even,} \\ Mf_{M-2} + MF_M, & \text{if } M \text{ is odd.} \end{cases} \end{aligned}$$

As in the multiplication case, it is easy to check (using the above formulas for example) that if $F = \sum_{i=0}^{M} f_i\psi_i$ and $\overline{F} = \sum_{i=0}^{M} \overline{f_i}\psi_i$ are two series in the ψ_i which agree on the top few terms so that $f_{M-j} = \overline{f_{M-j}}$ for $j = 0, 1, 2$ then so do DF and $D\overline{F}$ i.e., $\delta(F, M + 4 - j) = \delta(\overline{F}, M + 4 - j)$ for $j = 0, 1, 2$.

As a notational convenience we will write $\delta^k(F; i)$ to mean $[\psi_i]D^kF$, which is the same as

$$\delta^k(F; i) = \delta^{k-1}(DF; i) = \delta(D^{k-1}F; i).$$

In particular, $\delta^k(F; i)$ can be computed recursively. Also, for $k \geq 2$ we will write,

$$U_k = (D+4)L_k^{(3)} = \sum_{i=0}^{5k-4} u_k(i)\psi_i,$$

where $u_k(i) = \delta(L_k^{(3)}; i) + 4\mu_k(i)$. The top few terms of U_k may be worked out explicitly as,

$$u_k(5k-4) = \delta(L_k^{(3)}; 5k-4) = (5k-6)\mu_k(5k-8) \text{ and}$$
$$u_k(5k-5) = \delta(L_k^{(3)}; 5k-5) = (5k-7)\mu_k(5k-9).$$

In addition, the bottom few terms may be worked out. In this case,

$$u_k(0) = \delta(L_k^{(3)}; 0) + 4\mu_k(0) = 0 \text{ and}$$
$$u_k(1) = \delta(L_k^{(3)}; 1) + 4\mu_k(1) = 0.$$

so that

$$U_k = \sum_{i=2}^{5k-4} u_k(i)\psi_i.$$

More generally, we may work out the bottom few terms of $\delta(F; i)$ for a series F in the ψ_i. Since we will have need of this in the lemmas that follow we will record this here.

Lemma 6.4.5. *If $F = \sum_{i\geq M} f_i\psi_i$ is a series in the ψ_i then $\delta(F; i) = 0$ for $i < M$,*

$$\delta(F; M) = \begin{cases} -2(M+2)F_M, & \text{if } M \text{ is even,} \\ -2(M+1)F_M, & \text{if } M \text{ is odd,} \end{cases}$$

and

$$\delta(F; M+1) = \begin{cases} -2(M+3)F_{M+1}, & \text{if } M \text{ is even,} \\ -2(M+2)F_{M+1}, & \text{if } M \text{ is odd.} \end{cases}$$

Proof. This follows via straightforward computation. □

Along with the definition of U_k for $k \geq 2$ above we will also let $U_1 = 18\psi_1 - 12\psi_0$ and $U_0 = 12\psi_0$. Note that this U_1 and U_0 are not the same as $(D+4)T_1$ and $(D+4)T_0$ as was used in the proof of Corollary 6.4.4.

We now proceed by determining the structure of each of the summands on the right hand side of (6.1) in Proposition 6.4.4.

Lemma 6.4.6. *Let $k \geq 0$ be a non-negative integer and if $k > 1$ further suppose that*

$$L_k^{(3)} = \sum_{i=0}^{5k-8} \mu_k(i)\psi_i.$$

Then

$$A(k) = \sum_{i=0}^{5k+4} a_k(i)\psi_i,$$

where

$$a_k(5k+4) = \frac{\delta^2(U_k, 5k+4)}{432},$$
$$a_k(5k+3) = \frac{\delta^2(U_k, 5k+3)}{432}, \ \textit{and}$$
$$a_k(5k+2) = \frac{\delta^2(U_k, 5k+2)}{432}.$$

Proof. Recall that
$$A(k) = z^2(D+12)(D+8)(D+4)T_k.$$
For $k > 1$, $T_k = L_k^{(3)} = \sum_{i=0}^{5k-8} \mu_k(i)\psi_i$ by assumption. Using Lemma 6.4.5 we may determine that
$$(D+12)(D+8)(D+4)L_k^{(3)} = \sum_{i=6}^{5k+4} \gamma_k(i)\psi_i$$
for some constants $\gamma_k(i)$. Also recall that $z = \frac{1}{2}s(1-s)(1-2s)$ so that
$$\frac{z^2}{\eta^3} = \left(\frac{1}{216} - \frac{1}{144}\psi_0 + \frac{1}{432}\psi_4\right).$$

Thus, for $k > 1$,
$$\begin{aligned} A(k) &= z^2(D+12)(D+8)(D+4)T_k, \\ &= z^2 \sum_{i=6}^{5k+4} \gamma_k(i)\psi_i, \\ &= \frac{z^2}{\eta^3} \sum_{i=0}^{5k-2} \gamma_k(i+6)\psi_i, \\ &= \left(\frac{1}{216} - \frac{1}{144}\psi_0 + \frac{1}{432}\psi_4\right) \sum_{i=0}^{5k-2} \gamma_k(i+6)\psi_i. \end{aligned}$$

Extracting the first few coefficients then gives the desired results. For $k = 0, 1$ we compute
$$A(0) = z^2(D+12)(D+8)(D+4)T_0 = -\frac{2}{3}\psi_0 + \frac{1}{3}\psi_2 + \frac{1}{3}\psi_4,$$
and
$$\begin{aligned} A(1) &= z^2(D+12)(D+8)(D+4)T_1, \\ &= -\frac{4}{9}\psi_0 + \frac{25}{24}\psi_1 + \frac{2}{3}\psi_2 - \frac{1}{2}\psi_3 + \frac{2}{3}\psi_4 - \frac{9}{4}\psi_5 - \frac{5}{9}\psi_6 + \frac{5}{6}\psi_7 - \frac{1}{3}\psi_8 + \frac{7}{9}\psi_9. \end{aligned}$$

In both cases $A(k)$ has the same form as when $k \geq 2$ and after checking the top three terms of $A(1)$ and $A(0)$ we see that the result follows. $\square$

Lemma 6.4.7. *Let $k \geq 2$ and suppose that for all $2 \leq i < k$,*

$$L_i^{(3)} = \sum_{j=0}^{5i-8} \mu_i(j)\psi_j.$$

Then

$$B(k) = \sum_{i=0}^{5k-6} b_k(i)\psi_i,$$

where

$$b_k(5k-6) = \frac{1}{432}\sum_{i=1}^{k-1} \pi(U_i, U_{k-i}; 5k-6),$$
$$b_k(5k-7) = \frac{1}{432}\sum_{i=1}^{k-1} \pi(U_i, U_{k-i}; 5k-7), \text{ and}$$
$$b_k(5k-8) = \frac{1}{432}\sum_{i=1}^{k-1} \pi(U_i, U_{k-i}; 5k-8).$$

Proof. Recall that

$$B(k) = z^2 \sum_{i=1}^{k-1} \{(D+4)T_i\}\{(D+4)T_{k-i}\}.$$

For $k > 2$ we have

$$B(k) = z^2 \sum_{i=2}^{k-2} U_i U_{k-i} + 2z^2\{(D+4)T_1\}U_{k-1}.$$

Now,

$$z^2\{(D+4)T_1\} = \eta^2\left(\frac{1}{18}\psi_0 - \frac{1}{24}\psi_1 - \frac{1}{36}\psi_2 - \frac{1}{36}\psi_4 + \frac{1}{24}\psi_5\right),$$

so that

$$\begin{aligned} z^2\{(D+4)T_1\}U_{k-1} &= \eta^2\left(\frac{1}{18}\psi_0 - \frac{1}{24}\psi_1 - \frac{1}{36}\psi_2 - \frac{1}{36}\psi_4 + \frac{1}{24}\psi_5\right)\sum_{i=2}^{5k-9} u_{k-1}(i)\psi_i, \\ &= \eta^2 \sum_{i=4}^{5k-2} \gamma_k(i)\psi_i, \\ &= \sum_{i=0}^{5k-6} \gamma_k(i+4)\psi_i, \end{aligned}$$

where

$$\gamma_k(i) = \pi(\frac{1}{18}\psi_0 - \frac{1}{24}\psi_1 - \frac{1}{36}\psi_2 - \frac{1}{36}\psi_4 + \frac{1}{24}\psi_5, U_{k-1}; i).$$

In particular, it is straightforward to check that

$$\gamma_k(5k-2) = \frac{1}{432}\pi(U_1, U_{k-1}; 5k-6),$$
$$\gamma_k(5k-3) = \frac{1}{432}\pi(U_1, U_{k-1}; 5k-7),$$
$$\gamma_k(5k-4) = \frac{1}{432}\pi(U_1, U_{k-1}; 5k-8),$$

where $U_1 = 18\psi_1 - 12\psi_0$. We also see that

$$
\begin{aligned}
z^2 \sum_{i=2}^{k-2} U_i U_{k-i} &= z^2 \sum_{i=2}^{k-2} \sum_{j=6}^{5k-6} \pi(U_i, U_{k-i}; j)\psi_j, \\
&= \frac{z^2}{\eta^3} \sum_{i=2}^{k-2} \sum_{j=0}^{5k-12} \pi(U_i, U_{k-i}; j+6)\psi_j, \\
&= \left(\frac{1}{216} - \frac{1}{144}\psi_0 + \frac{1}{432}\psi_4\right) \sum_{i=2}^{k-2} \sum_{j=0}^{5k-12} \pi(U_i, U_{k-i}; j+6)\psi_j.
\end{aligned}
$$

Putting this together we have for $k > 2$,

$$B(k) = \sum_{i=0}^{5k-6} b_k(i)\psi_i,$$

where

$$b_k(i) = 2\gamma_k(i+4) + \pi\left(\frac{1}{216} - \frac{1}{144}\psi_0 + \frac{1}{432}\psi_4, \eta^3 \sum_{j=2}^{k-2} U_j U_{k-j}; i\right).$$

In particular, the top three terms are given by the expressions in the statement of the lemma.

For $k = 2$,

$$B(2) = z^2\{(D+4)T_1\}^2 = \frac{11}{12}\psi_0 - \psi_1 + \frac{5}{6}\psi_2 - \psi_3 + \frac{1}{4}\psi_4,$$

and it can be checked that the top terms coincide with the formulas given in the statement of the lemma. □

Lemma 6.4.8. *Let $k \geq 2$ and suppose that for all $2 \leq i < k$,*

$$L_i^{(3)} = \sum_{j=0}^{5i-8} \mu_i(j)\psi_j.$$

Then

$$\overline{B(k)} = \sum_{i=0}^{5k-4} \overline{b_k(i)}\psi_i,$$

where

$$
\begin{aligned}
\overline{b_k(5k-4)} &= \frac{1}{6}u_k(5k-4), \\
\overline{b_k(5k-5)} &= \frac{1}{6}u_k(5k-5), \quad \textit{and} \\
\overline{b_k(5k-6)} &= \frac{1}{6}u_k(5k-6) - \frac{1}{6}u_k(5k-4) + b_k(5k-6).
\end{aligned}
$$

Proof. Recall that for $k \geq 2$,

$$\overline{B(k)} = z^2 \sum_{i=0}^{k} \{(D+4)T_i\}\{(D+4)T_{k-i}\},$$
$$= 2z^2\{(D+4)T_0\}U_k + B(k).$$

Now,

$$z^2\{(D+4)T_0\} = \frac{\eta}{12}(\psi_0 - 1),$$

so

$$\overline{B(k)} = \frac{\eta}{6}(\psi_0 - 1)\sum_{i=2}^{5k-4} u_k(i)\psi_i + \sum_{i=0}^{5k-6} b_k(i)\psi_i,$$

from which the result follows. $\square$

Lemma 6.4.9. *Let $k \geq 1$ and if suppose that for all $2 \leq i < k$,*

$$L_i^{(3)} = \sum_{j=0}^{5i-8} \mu_i(j)\psi_j.$$

Then

$$\widetilde{B(k)} = \sum_{i=0}^{5k} \widetilde{b_k(i)}\psi_i,$$

where

$$\widetilde{b_k(5k)} = \frac{1}{18}u_k(5k-4),$$
$$b_k\widetilde{(5k-1)} = \frac{1}{18}u_k(5k-5), \text{ and}$$
$$b_k\widetilde{(5k-2)} = \frac{1}{18}u_k(5k-6) + \frac{1}{18}u_k(5k-4) + \frac{1}{3}b_k(5k-6).$$

Proof. Recall that

$$\widetilde{B(k)} = \frac{1}{3}(\psi_2 + 2\psi_0)\overline{B(k)}$$

so that for $k \geq 2$ the result follows immediately from Lemma 6.4.8.

For $k = 1$, using the fact that

$$\frac{1}{3}(\psi_2 + 2\psi_0) = \frac{(1-2s)^2}{\eta^2},$$

we have

$$\widetilde{B(1)} = \frac{2(1-2s)^2 z^2}{\eta^2}\{(D+4)T_0\}\{(D+4)T_1\},$$
$$= -\frac{4}{3}\psi_2 + \psi_3 - \frac{2}{3}\psi_4 + \psi_5.$$

After checking we see that the top three terms of $\widetilde{B(1)}$ agree with the formulas given in the statement of the lemma provided that $U_1 = 18\psi_1 - 12\psi_0$. $\square$

Lemma 6.4.10. *Let $k \geq 1$ and suppose that for all $2 \leq i < k$,*

$$L_i^{(3)} = \sum_{j=0}^{5i-8} \mu_i(j)\psi_j.$$

Then

$$C(k) = \sum_{i=0}^{5k-4} c_k(i)\psi_i,$$

where

$$\begin{aligned} c_k(5k-4) &= u_k(5k-4), \\ c_k(5k-5) &= u_k(5k-5), \text{ and} \\ c_k(5k-6) &= u_k(5k-6). \end{aligned}$$

Proof. Recall that $C(k) = (D+6)T_k$. So, for $k > 1$

$$C(k) = U_k + 2L_k^{(3)},$$

and the result follows. For $k = 1$ we compute

$$C(1) = (D+6)T_1 = 18\psi_1 - 12\psi_0.$$

$\square$

Lemma 6.4.11. *Let $k \geq 1$ and $t \geq 0$. If $t \geq 2$ suppose that*

$$L_t^{(3)} = \sum_{i=0}^{5t-8} \mu_t(i)\psi_i.$$

Then

$$\alpha_k(t) = \sum_{i=0}^{4k+t} \alpha_{k,t}(i)\psi_i,$$

where

$$\begin{aligned} \alpha_{k,t}(4k+t) &= \frac{\delta^{k-t+1}(U_t, 4k+t)}{6^{k-t+2}(k-t+2)!}, \\ \alpha_{k,t}(4k+t-1) &= \frac{\delta^{k-t+1}(U_t, 4k+t-1)}{6^{k-t+2}(k-t+2)!}, \text{ and} \\ \alpha_{k,t}(4k+t-2) &= \frac{\delta^{k-t+1}(U_t, 4k+t-2)}{6^{k-t+2}(k-t+2)!}. \end{aligned}$$

Proof. Recall that

$$\alpha_k(t) = \sum_{j=0}^{k-t+2} \binom{2-t}{k-t-j+2} \frac{t^j}{j!} \partial_z^j L_t^{(3)}.$$

Let $R_k = z^k \partial_z^k$ and let $\overline{D} = \frac{1}{6}D = z\partial_z$. Then

$$\overline{D}R_k = kR_k + R_{k+1},$$

so that $R_{k+1} = (\overline{D} - k)R_k$. Thus,

$$R_k = (\overline{D} - k + 1)(\overline{D} - k + 2)\cdots\overline{D} = \left(\frac{1}{6}D - k + 1\right)\left(\frac{1}{6}D - k + 2\right)\cdots\left(\frac{1}{6}D\right).$$

So, for $t \geq 2$,

$$\begin{aligned}
\alpha_k(t) &= \sum_{j=0}^{k-t+2} \binom{2-t}{k-t-j+2} \frac{R_j}{j!} L_t^{(3)}, \\
&= \sum_{j=0}^{k-t+2} \binom{2-t}{k-t-j+2} \frac{(\overline{D} - j + 1)(\overline{D} - j + 2)\cdots\overline{D}}{j!} L_t^{(3)}, \\
&= \sum_{i=0}^{4k+t} \alpha_{k,t}(i)\psi_i,
\end{aligned}$$

since the highest power of D appearing is $k - t + 2$ and so the highest index of ψ_i appearing is $5t - 8 + 4(k - t + 2) = 4k + t$. The result then follows for $t \geq 2$ since for $0 \leq i \leq 3$,

$$\delta^{k-t+2}(L_t^{(3)}; 4k + t - i) = \delta^{k-t+1}(DL_t^{(3)}; 4k + t - i) = \delta^{k-t+1}(U_t; 4k + t - i).$$

For $t = 1$,

$$\alpha_k(1) = \left(\frac{R_k}{k!} + \frac{R_{k+1}}{(k+1)!}\right) L_1^{(3)} = \frac{R_k}{(k+1)!}\left(\frac{D}{6} + 1\right) L_1^{(3)}.$$

Also note that $\alpha_0(1) = \left(\frac{D}{6} + 1\right) L_1^{(3)}$ so

$$\alpha_k(1) = \frac{R_k}{(k+1)!}\alpha_0(1).$$

Now, $\alpha_0(1) = \frac{1}{6}(D + 6)L_1^{(3)} = \frac{1}{6}(18\psi_1 - 12\psi_0) = \frac{1}{6}U_1$. Since R_k is a polynomial in D with leading term $\frac{D^k}{6^k}$ we see that

$$\alpha_k(1) = \sum_{i=0}^{4k+1} \alpha_{k,1}(i)\psi_i,$$

where the top three terms are given by the statement of the lemma.

For $t = 0$,

$$\begin{aligned}
\alpha_k(0) &= \left(\frac{R_k}{k!} + \frac{2R_{k+1}}{(k+1)!} + \frac{R_{k+2}}{(k+2)!}\right) L_0^{(3)}, \\
&= \frac{R_k}{(k+2)!}\left(\frac{D^2}{6^2} + 3\frac{D}{6} + 2\right) L_0^{(3)}, \\
&= \frac{2R_k}{(k+1)!}\alpha_0(0).
\end{aligned}$$

We can compute,
$$\alpha_0(0) = \frac{1}{2}\left(\frac{D^6}{36} + \frac{D}{2} + 2\right) L_0^{(3)} = \psi_0 - 1.$$
So, as before, for $k \geq 1$
$$\alpha_k(0) = \frac{2R_k}{(k+2)!}(\psi_0 - 1) = \frac{2R_k}{(k+2)!}\psi_0 = \sum_{i=0}^{4k} \alpha_{k,0}(i)\psi_i,$$
where the top three terms are given in the statement of the lemma. □

Lemma 6.4.12. *Let $k \geq 2$ and suppose that*
$$L_i^{(3)} = \sum_{j=0}^{5i-8} \mu_i(j)\psi_j,$$
for $2 \leq i \leq k-2$. Then
$$\overline{V(k)} = \sum_{i=0}^{5k-2} \overline{v_k(i)}\psi_i,$$
where
$$\overline{v_k(5k-2)} = \frac{2^5}{6^4 4!}\delta^3(U_{k-2}, 5k-2),$$
$$\overline{v_k(5k-3)} = \frac{2^5}{6^4 4!}\delta^3(U_{k-2}, 5k-3), \text{ and}$$
$$\overline{v_k(5k-4)} = \frac{2^5}{6^4 4!}\delta^3(U_{k-2}, 5k-4) + \frac{2^7}{6^6 6!}\delta^5(U_{k-4}, 5k-4).$$

Proof. Recall that
$$\overline{V_k} = \sum_{t=0}^{k-2}(1 + (-1)^{k-t})2^{k-t+2}\alpha_k(t),$$
so that
$$\overline{V_k} = \sum_{i=0}^{5k-2} \overline{v_k(i)}\psi_i,$$
where
$$\overline{v_k(i)} = \sum_{t=0}^{k-2}(1 + (-1)^{k-t})2^{k-t+2}\alpha_{k,t}(i).$$
The result then follows from Lemma 6.4.11. □

Lemma 6.4.13. *Let $k \geq 1$ and suppose that*
$$L_i^{(3)} = \sum_{j=0}^{5i-8} \mu_i(j)\psi_j,$$
for $2 \leq i \leq k$. Then
$$V(k) = \sum_{i=0}^{5k} v_k(i)\psi_i,$$

where

$$v_k(5k) = \frac{2^3}{6^2 2!}\delta(U_k, 5k),$$
$$v_k(5k-1) = \frac{2^3}{6^2 2!}\delta(U_k, 5k-1), \text{ and}$$
$$v_k(5k-2) = \frac{2^3}{6^2 2!}\delta(U_k, 5k-2) + \frac{2^5}{6^4 4!}\delta^3(U_{k-2}, 5k-2).$$

Proof. This follows in the same way as Lemma 6.4.12 by applying Lemma 6.4.11. □

Now that we have worked out the structure of each of the summands appearing on the right hand side of (6.1) in Proposition 6.4.4, we may complete the proof of Lemma 4.4.4.

Proof of Lemma 4.4.4. Recall from Proposition 6.4.4 that

$$\begin{aligned} R_g(s) = &\frac{4}{3}(\psi_2 + 2\psi_0)DA(g-2) + 4(\psi_2 + 2\psi_0)DB(g) \\ &- \frac{8}{3}(\psi_2 + 2\psi_0)(\psi_0 - 1)(2A(g-2) + 6B(g)) - 3\psi_2\overline{V_g} \\ &- \sum_{i=1}^{g-1} V_i\left(\frac{2}{3}(\psi_2 + 2\psi_0)A(g-i-2) - \frac{1}{6}(\psi_2 + 2\psi_0)C(g-i) + 6\widetilde{B(g-i)}\right). \end{aligned} \tag{6.5}$$

Using the lemmas above, this implies that $R_g(s)$ has an expansion in the ψ_i basis of the form

$$R_g(s) = \sum_{i=0}^{5g+2} r_g(i)\psi_i.$$

Again, using the lemmas above we see that the top term can be written out explicitly as

$$\begin{aligned} r_g(5g+2) = &\frac{4}{3}\delta(A(g-2); 5g-2) + 4\delta(B(g); 5g-2) - 3\overline{v_g(5g-2)} \\ &- \sum_{i=1}^{g-1}\pi\left(V_i, \frac{2}{3}(\psi_2 + 2\psi_0)A(g-i-2) - \frac{1}{6}(\psi_2 + 2\psi_0)C(g-i) + 6\widetilde{B(g-i)}; 5g+2\right). \end{aligned}$$

Using Lemma 6.4.6 we see that

$$\frac{4}{3}\delta(A(g-2); 5g-2) = \frac{1}{324}\delta^3(U_{g-2}; 5g-2),$$

using Lemma 6.4.7 we have

$$4\delta(B(g); 5g-2) = \frac{1}{108}(5g-4)\sum_{i=1}^{g-1}\pi(U_i, U_{g-i}; 5g-6),$$

and using Lemma 6.4.12 we have

$$3\overline{v_g(5g-2)} = \frac{1}{324}\delta^3(U_{g-2}; 5g-2).$$

If we let

$$\frac{2}{3}(\psi_2 + 2\psi_0)A(g-i-2) - \frac{1}{6}(\psi_2 + 2\psi_0)C(g-i) + 6\widetilde{B(g-i)} = \sum_{j=0}^{5(g-i)} t_{g-i}(j)\psi_j,$$

then (note that we can do this using the lemmas above),

$$\begin{aligned}
t_{g-i}(5(g-i)) &= -\frac{1}{6}c_{g-i}(5(g-i)-4) + 6b_{g-i}(\widetilde{5(g-i)}),\\
&= -\frac{1}{6}u_{g-i}(5(g-i)-4) + \frac{1}{3}u_{g-i}(5(g-i)-4),\\
&= \frac{1}{6}u_{g-i}(5(g-i)-4).
\end{aligned}$$

Putting this together we have

$$\begin{aligned}
r_g(5g+2) &= \frac{1}{324}\delta^3(U_{g-2}; 5g-2) + \frac{1}{108}(5g-4)\sum_{i=1}^{g-1}\pi(U_i, u_{g-i}; 5g-6)\\
&\quad - \frac{1}{324}\delta^3(U_{g-2}; 5g-2) - \sum_{i=1}^{g-1}\pi\left(V_i, \sum_{j=0}^{5(g-i)} t_{g-i}(j)\psi_j; 5g+2\right),\\
&= \frac{(5g-4)}{108}\sum_{i=1}^{g-1}\chi(i,g-i)u_i(5i-4)u_{g-i}(5(g-i)-4)\\
&\quad - \sum_{i=1}^{g-1}\chi(i,g-i)v_i(5i)u_{g-i}(5(g-i)-4),\\
&= \frac{(5g-4)}{108}\sum_{i=1}^{g-1}\chi(i,g-i)u_i(5i-4)u_{g-i}(5(g-i)-4)\\
&\quad - \sum_{i=1}^{g-1}\chi(i,g-i)\frac{(5i-4)}{54}u_i(5i-4)u_{g-i}(5(g-i)-4),\\
&= 0.
\end{aligned}$$

Using a very similar argument we can show that $r_g(5g+1) = 0$ as well. Thus, we have shown that

$$R_g(s) = \sum_{i=0}^{5g} r_g(i)\psi_i.$$

What is left is to show that $r_g(5g)$ has the form specified in the statement of the lemma. Using (6.5) and the lemmas above we see that $r_g(5g)$ can be written as

$$r_g(5g) = \Phi_1 + \Phi_2,$$

where

$$\Phi_1 = \frac{4}{3}\delta(A(g-2); 5g-4) - \frac{16}{3}a_{g-2}(5g-6) + \frac{8}{3}\delta(A(g-2); 5g-2) - 3\overline{v_g(5g-4)},$$

and

$$\Phi_2 = 4\delta(B(g); 5g-4) - 16b_g(5g-6) + 8\delta(B(g); 5g-2) - \sum_{i=1}^{g-1} \pi\left(V_i, \sum_{j=0}^{5(g-i)} t_{g-i}(j)\psi_j; 5g\right).$$

Since Lemma 6.4.6 shows that $A(g-2)$ and $\frac{D^2U_{g-2}}{432}$ agree on the top three terms, we have

$$\frac{4}{3}\delta(A(g-2); 5g-4) = \frac{1}{324}\delta^3(U_{g-2}; 5g-4).$$

Also,

$$\begin{aligned}
\frac{16}{3}a_{g-2}(5g-6) &= \frac{\delta^2(U_{g-2}; 5g-6)}{81}, \\
\frac{8}{3}\delta(A(g-2); 5g-2) &= \frac{1}{162}\delta^3(U_{g-2}; 5g-2), \text{ and} \\
3\overline{v_g(5g-4)} &= \frac{2}{3^3 4!}\delta^3(U_{g-2}; 5g-4) + \frac{2}{3^5 6!}\delta^5(U_{g-4}; 5g-4).
\end{aligned}$$

So,

$$\begin{aligned}
\Phi_1 &= \frac{1}{324}\delta^3(U_{g-2}; 5g-4) - \frac{1}{81}\delta^2(U_{g-2}; 5g-6) + \frac{1}{162}\delta^3(U_{g-2}; 5g-2) \\
&\quad - \frac{2}{3^3 4!}\delta^3(U_{g-2}; 5g-4) - \frac{2}{3^5}6!\delta^5(U_{g-4}; 5g-4), \\
&= \frac{1}{162}\left(\delta^3(U_{g_2}; 5g-2) + 2\delta^3(U_{g-2}; 5g-6)\right) - \frac{1}{3^6 5!}\delta^5(U_{g-4}; 5g-4).
\end{aligned}$$

We showed above that

$$t_{g-i}(5(g-i)) = \frac{1}{6}u_{g-i}(5(g-i)-4).$$

It can be similarly shown that

$$t_{g-i}(5(g-i)-1) = \frac{1}{6}u_{g-i}(5(g-i)-5).$$

We may also compute

$$\begin{aligned}
t_{g-i}(5(g-i)-2) &= \frac{2}{3}a_{g-i-2}(5(g-i)-6) - \frac{1}{6}(c_{g-i}(5(g-i)-6) \\
&\quad + 2c_{g-i}(5(g-i)-4)) + 6b_{g-i}(5\widetilde{(g-i)}-2), \\
&= \frac{1}{6}u_{g-i}(5(g-i)-6) + \frac{\delta^2(U_{g-i-2}; 5(g-i)-6)}{648} + 2b_{g-i}(5(g-i)-6).
\end{aligned}$$

This shows that $\sum_{j=0}^{5(g-i)} t_{g-i}(j)\psi_j$ and $\frac{1}{6}\psi_2 U_{g-i} + \frac{1}{648}D^2U_{g-i-2} + 2B(g-i)$ agree for the first three terms. Also, Lemma 6.4.13 shows that V_i and $\frac{1}{9}DU_i + \frac{2}{3^4 4!}D^3U_{i-2}$ agree on the first three terms. So,

$$\sum_{i=1}^{g-1}\pi\left(V_i, \sum_{j=0}^{5(g-i)} t_{g-i}(j)\psi_j; 5g\right)$$

$$= \sum_{i=1}^{g-1}\pi\left(\frac{1}{9}DU_i + \frac{2}{3^4 4!}D^3U_{i-2}, \frac{1}{6}\psi_2 U_{g-i} + \frac{1}{648}D^2U_{g-i-2} + 3B(g-i); 5g\right),$$

$$= \frac{1}{23^3}\sum_{i=1}^{g-1}\pi(DU_i, U_{g-i}; 5g-4)$$

$$+ \frac{1}{9}\sum_{i=1}^{g-1}\chi(5i, 5(g-i)-2)\delta(U_i; 5i)\left(\frac{\delta^2(U_{g-i-2}; 5(g-i)-6)}{648} + 2b_{g-i}(5(g-i)-6)\right)$$

$$+ \frac{1}{3^5 4!}\sum_{i=1}^{g-1}\chi(5i-2, 5(g-i))\delta^3(U_{i-2}; 5i-2)u_{g-i}(5(g-i)-4).$$

Also, since Lemma 6.4.7 agrees with $\frac{1}{432}\sum_{i=1}^{g-1}U_iU_{g-i}$ for the top three terms,

$$4\delta(B(g); 5g-4) = \frac{1}{108}\delta(\sum_{i=1}^{g-1}U_iU_{g-i}; 5g-4) = \frac{1}{54}\sum_{i=1}^{g-1}\pi(DU_i, U_{g-i}; 5g-4).$$

So,

$$\Phi_2 = 4\delta(B(g), 5g-4) - 16b_g(5g-6) + 8\delta(B(g); 5g-2) - \sum_{i=1}^{g-1}\pi(V_i, \sum_{j=0}^{5(g-i)} t_{g-i}(j)\psi_j; 5g),$$

$$= \frac{1}{54}\sum_{i=1}^{g-1}\pi(DU_i, U_{g-i}; 5g-4) - 16b_g(5g-6) + 8(5g-4)b_g(5g-6)$$

$$-\left(\frac{1}{54}\sum_{i=1}^{g-1}\pi(DU_i, U_{g-i}; 5g-4) + \frac{1}{9}\sum_{i=1}^{g-1}\chi(i, g-i)\delta(U_i; 5i)\frac{\delta^2(U_{g-i-2}; 5(g-i)-6)}{648}\right.$$

$$+ \frac{2}{9}\sum_{i=1}^{g-1}\chi(i, g-i)\delta(U_i, 5i)b_{g-i}(5(g-i)-6)$$

$$\left.+\frac{1}{3^5 4!}\sum_{i=1}^{g-1}\chi(i, g-i)\delta^3(U_{i-2}; 5i-2)U_{g-i}(5(g-i)-4)\right),$$

$$= 8(5g-6)b_g(5g-6) - \frac{(5g-6)}{2^4 3^6}\sum_{i=1}^{g-1}\chi(i, g-i)u_i(5i-4)\delta^2(U_{g-i-2}; 5(g-i)-6)$$

$$- \frac{2}{9}\sum_{i=1}^{g-1}\chi(i, g-i)\delta(U_i; 5i)b_{g-i}(5(g-i)-6).$$

Putting this together we get

$$
\begin{aligned}
r_g(5g) = \frac{1}{162}\big(\delta^3(U_g - 2; 5g-2) - 2\delta^2(U_{g-2}; 5g-6)\big) \\
- \frac{1}{3^6 5!}\delta^5(U_{g-4}; 5g-4) + 8(5g-6)b_g(5g-6) \\
- \frac{(5g-6)}{2^3 3^6}\sum_{i=1}^{g-2}\chi(i, g-i)u_i(5i-4)\delta^2(U_{g-i-2}; 5(g-i)-6) \\
- \frac{2}{9}\sum_{i=1}^{g-2}\chi(i, g-i)\delta(U_i, 5i)b_{g-i}(5(g-i)-6),
\end{aligned}
$$

as required. □

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