Pairwise Balanced Designs and Related Codes

by

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erasure code. We generalize the existing constructions to obtain some new infinite classes of anti-Pasch Steiner triple systems. In addition, we study some related problems concerning Steiner triple systems avoiding certain configurations.

Abstract

This thesis deals with pairwise balanced designs, group divisible designs and related codes. We study pairwise balanced designs with three consecutive block sizes. In particular, we investigate the spectrum of pairwise balanced designs with block sizes five, six and seven; six, seven and eight; seven, eight and nine; and eight, nine and ten. We have standardized the known techniques for constructing pairwise balanced designs with consecutive block sizes. New constructions employing certain line configurations in finite projective planes are also developed. The direct and recursive constructions both require the existence of finite projective planes, particularly ovals in desarguesian projective planes. Combining known and new techniques, we have essentially determined the spectra for these pairwise balanced designs.

We also study uniform group divisible designs with block size five. We prove that uniform group divisible designs with block size five exist for all parameters satisfying the basic necessary conditions with a finite number of possible exceptions. Many of direct constructions are required to obtain this strong existence result. In particular, we have constructed many group divisible designs with block five admitting a large automorphism group. Several new recursive constructions are presented and used to settle this problem. One recursive construction requires a relatively new type of combinatorial design, the modified group divisible design, which is also studied in this thesis.

Finally, we study some coding theoretic problems arising from computer science which have design theoretic connections. We have related a well known problem in combinatorial design theory and finite geometry to coding theory. The existence of anti-Pasch Steiner triple systems corresponds to the existence of a certain type of

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Chapter 1

Introduction

Pairwise balanced designs are of fundamental importance in combinatorial theory. They are of interest in their own right, and have many applications in the construction of other types of designs. Standard texts (e.g., [9], [24]) treat the role of pairwise balanced designs well.

1.1 Definitions

In this section we define the common terms that are used in the thesis. Terms specific to a proof or construction are defined as the need arises.

Throughout the thesis, we use the notation $\mathbb{N}$ to denote the set of positive integers; $[a, b]$ to denote the set of integer v such that $a \leq v \leq b$; $\mathbb{Z}_n$ to be the ring (or group) of residues modulo n; and $\mathbf{F}_q$ to be the Galois field on q elements.

A *pairwise balanced design* (or PBD) with index λ is a pair $(V, \mathcal{B})$ where

1. V is a finite set of points,

2. $\mathcal{B}$ is a collection of subsets of V called blocks,

3. every pair of distinct points of V occur in exactly λ blocks.

We use the notation ($\text{PBD}_\lambda(v, K)$) when $|V| = v$ and $|B| \in K$. When $\lambda = 1$, we simply denote it by PBD(v,K). We denote B(K)=$\{v$: there exists a PBD(v,K)$\}$. A set K, a subset of the positive integers, is said to be PBD-closed if B(K)=K.

The notation PBD$(v, K \cup k^*)$ denotes a PBD containing a block of size k. If $k \notin K$, this indicates that there is exactly one block of size k in the PBD. On the other hand, if $k \in K$, then there is at least one block of size k in the PBD.

In a sequence of three papers [117, 118, 119], Wilson developed a theory concerning the structure of PBD-closed sets. We give a brief summary of his results. Let K be a subset of the set of all positive integers; we define two integers: $\alpha(K)$ $=$ gcd$\{v - 1 : v \in K\}$ and $\beta(K)$ =gcd$\{v(v-1) : v \in K\}$. Wilson's main result is that there exists a constant N (depending on K) such that, for all $v > N$, $v \in$B(K) if and only if $v - 1 \equiv 0 \pmod{\alpha(K)}$ and $v(v-1) \equiv 0 \pmod{\beta(K)}$. Although the proofs are constructive in a sense, this theory does not yield any reasonable upper bounds on N.

A *balanced incomplete block design* (BIBD) with index λ is a ($\text{PBD}_\lambda(v, K)$) with K=$\{k\}$. We use the notation BIBD(v, k, λ) where $|B| = k$ for all $B \in \mathcal{B}$.

If the blocks of the design, BIBD(v, k, λ), can be partitioned into classes C_1, C_2, $\ldots, C_m$ such that every point in V appears in exactly one block in each C_i for $i = 1, 2, \ldots, m$, the design is called *resolvable* and is denoted by RBIBD$(v, k, 1)$.

Often we denote B(K) $=$ $\{v$: there exists a PBD$(v, K)\}$, in the case when $K = \{k\}$, we simply write B(K) as B(k). Also, we use the notation RB(k) $=$ $\{v$: there exists a RBIBD$(v, k, \lambda)\}$.

A *group divisible design* (GDD) of index λ is a triple (V,$\mathcal{G}$,$\mathcal{B}$) where

1. V is a finite set of points,

2. $\mathcal{G}$ is a set of subsets of V, called *groups*, which partition V,

3. $\mathcal{B}$ is a collection of subsets of V called *blocks*,

4. every pair of distinct points of V not found together in a group, occur in exactly λ blocks,

5. $|G \cap B| \leq 1$ for all $G \in \mathcal{G}$ and $B \in \mathcal{B}$.

We do not require all groups to have the same size. When it is important to consider the sizes of the groups explicitly, we refer to the group-type of the GDD. The *group-type*, or more simply, *type*, of a $(\mathrm{GDD}_\lambda(V, \mathcal{G}, \mathcal{B}))$ is the multiset $H=(|G|:G \in \mathcal{G})$. For convenience, we use the notation $H = (g_1^{n_1}, ..., g_r^{n_r})$ where $g_i^{n_i}$ means n_i groups of size g_i. We use the notation $GDD_\lambda(v, H, K)$ for a $GDD_\lambda(V, \mathcal{G}, \mathcal{B})$ where $|G| \in H$ is defined as above and $K=\{|B| \in \mathcal{B}\}$. A K-GDD is a group divisible design with block sizes from the set K.

A *transversal design* $\mathrm{TD}_\lambda(k, n)$ is a GDD with kn points, k groups of size n, and index λ. Every group and every block of a transversal design intersect in a point. In the case of $\lambda = 1$, we simply denote it by $\mathrm{TD}(k, n)$. It is well-known that a $\mathrm{TD}(k, n)$ is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order n. For a list of lower bounds on the number of MOLS of all orders up to 10000, we refer the reader to [3].

A *parallel class* in a design is a set of blocks that partition the point set. If the blocks of a design can be partitioned into parallel classes, then it is said to be *resolvable*. In the sequel we write RTD and RGDD with the appropriate parameters to denote a resolvable TD and GDD respectively. The existence of a resolvable $\mathrm{TD}(k, n)$ is equivalent to the existence of a $\mathrm{TD}(k + 1, n)$.

A *K-modified group divisible design* (K-MGDD) of type a^b with index λ is a set of ab points, equipped with a parallel class of blocks of size a, a parallel class of blocks of size b, and all other blocks having size in the set K, so that every unordered pair of points occurs together in exactly λ block. Any two points appearing in a block of either parallel class appears in no other block. As with GDD, when $K = \{k\}$, we simply denote the K-MGDD by k-MGDD.

Next, we need definitions on incomplete objects.

An *incomplete group divisible design* (IGDD) with block-sizes from K and index λ is a quadruple $(V, \mathcal{G}, \mathcal{H}, \mathcal{B})$ where V is a finite set of cardinality v, $\mathcal{G} = (G_1, G_2, \dots, G_s)$ is a partition of V, $\mathcal{H} = \{(H_{11}, H_{12}, \dots, H_{1t}), (H_{21}, H_{22}, \dots, H_{2t}), \dots, (H_{s1}, H_{s2}, \dots, H_{st})$ is a collection of subsets of V with the property that $H_{ij} \subseteq G_i$ for all $j = 1, 2, \dots, t$ and $H_{aj} \cap H_{bj} = \emptyset$ for all $a \neq b$, (the G_i are *groups* and H_{ij} are *holes*, and $\mathcal{B}$ is a family of subsets of V called *blocks* which satisfies the properties:

(1) Any pair of distinct elements of $\mathcal{V}$ which occurs in a group does not occur in any block.

(2) If a pair of distinct elements from V comes from distinct groups and each element occurs are not both in H_{aj} and H_{bj} for some a, b, j, then that pair occurs in exactly in λ blocks. If there exists a, b and j so that $a \neq b$, then that pair appears in no block.

An IGDD is of type

$$(g_1; h_{11}, h_{12}, \dots, h_{1t})^{a_1} (g_2; h_{21}, h_{22}, \dots, h_{2t})^{a_2} \dots (g_r; h_{r1}, h_{r2}, \dots, h_{rt})^{a_r}$$

if there are a_i groups of size g_i with hole sizes $h_{i1}, h_{i2}, \dots, h_{it}$.

Related to incomplete group divisible design is a *holey* group divisible design (HGDD). A K-HGDD of type $(\{u_i : 1 \leq i \leq r\}, h)$ is a structure $(X, \{Y_i\}_{1 \leq i \leq r}, \mathcal{G}, \mathcal{B})$

with index λ where X is a hm-set (of points), $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ is a partition of X into h *groups* of m points each, $\{Y_1, Y_2, \ldots, Y_r\}$ is a partition of X into r *holes*, each hole $Y_i (1 \leq i \leq r)$ is a set of hu_i points such that $|Y_i \cap G_j| = u_i$ for $1 \leq j \leq k$, and $\mathcal{B}$ is a collection of subsets with sizes in K of X (called *blocks*), such that no block contains two distinct points of any group or any hole, but any other pairs of points of X is contained in exactly λ block of $\mathcal{B}$.

If we remove one or more sub-designs from a $\mathrm{TD}_\lambda(k, v)$, we obtain a transversal design with holes. In the case of one hole, it is called an *incomplete transversal design* (ITD). More formally, an ITD, denoted by $\mathrm{TD}_\lambda(k, m) - \mathrm{TD}_\lambda(k, n)$, is a quadruple $(X, Y, \mathcal{G}, \mathcal{B})$, where X is a set of km points, $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ is a partition of X into k *groups* of m points each, $Y \subseteq X$ is a set of kn points such that $|Y \cap G_j| = n$ for $1 \leq j \leq k$, and $\mathcal{B}$ is a set of subsets (called *blocks*) of X, each of which intersects each group in exactly one point such that every pair of points $\{x, y\}$ from distinct groups is either in Y or occurs in a λ block but not both.

The set Y is referred to as a hole. If $Y = \emptyset$, then the ITD is a TD.

Related to incomplete transversal is a *holey* TD (HTD). A k-HTD of type $\{u_i : 1 \leq i \leq r\}$ is a structure $(X, \{Y_i\}_{1 \leq i \leq r}, \mathcal{G}, \mathcal{B})$ with index λ where X is a km-set (of points), $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ is a partition of X into k *groups* of m points each, $\{Y_1, Y_2, \ldots, Y_r\}$ is a partition of X into r *holes*, each hole $Y_i (1 \leq i \leq r)$ is a set of ku_i points such that $|Y_i \cap G_j| = u_i$ for $1 \leq j \leq k$, and $\mathcal{B}$ is a collection of subsets of X (called *blocks*), each meeting each group in exactly one point, such that no block contains two distinct points of any group or any hole, but any other pairs of points of X is contained in exactly λ block of $\mathcal{B}$.

When λ is not mentioned, we assume that $\lambda = 1$.

Let k be a positive integer and let v and α be positive integers. Let $\mathcal{V}$ be a set of size v. Let any subset of size k of V be a *block*. Then a (k,α)-*partial* resolution class is a collection C of blocks such that every element of V occurs in either exactly α or exactly zero blocks of C. The set of elements of V not occurring in the partial resolution class is the *complement* of the class.

Let k, α and λ be positive integer. A $(\lambda,\alpha;k)$-*frame* is a triple $(V,\mathcal{G},\mathcal{B})$ where V is a set of size v, $\mathcal{G}$ is a partition of V into parts (*groups*), and $\mathcal{B}$ is a collection of (k,α)-partial resolution class of V which satisfies the conditions:

1. The complement of each (k,α)-partial resolution class B of $\mathcal{B}$ is a group $G \in \mathcal{G}$;
2. Each unordered pair $\{x,y\}$ of V which does not lie in some group G of $\mathcal{G}$ lies in precisly λ blocks of $\mathcal{B}$;
3. No unordered pair $\{x,y\}$ of elements of V which lies in some group G of $\mathcal{G}$ also lies in a block of $\mathcal{B}$.

The *type* of the $(\lambda,\alpha;k)$-frame is the multiset $T = [|G| : G \in \mathcal{G}]$. If $\mathcal{G}$ contains a_i groups of size g_i for $i = 1,2,\ldots,r$, then the exponential notation $g_1^{a_1}g_2^{a_2}\ldots g_r^{a_r}$ is also used. By convention, factors of the type 0^a can be included in the expoential form of the type to accommodate null groups when necessary.

A *k-frame* of type T is a $(1,1;k)$-frame of type T.

A *finite projective plane of order* n, $n \geq 2$, is a collection of $n+1$ subsets (called *lines*) of a n^2+n+1-set V points, such that every two points of V occur together in exactly one of the lines. An *oval* of a projective plane of order n is a set of $n+1$ points such that no three are collinear. An *hyperoval* of a projective plane of order n is a set of $n+2$ points such that no three are collinear.

Let A be a set of non-negative integers, and let $\mathcal{D}$ be a PBD(v, K). Then an *A-arc with w points* in $\mathcal{D}$ is a set of w points $\mathcal{S}$ of $\mathcal{D}$ such that if B is a block, then $|B \cap S| \in \mathcal{A}$. The *order* of an arc is the number of points in the arc. Suppose that a projective plane of order n contains a A-arc of order w. Then it also contains a *complementary* $(n + 1 - A)$-arc of order $n^2 + n + 1 - w$, where $n+1-A = \{n+1-a : a \in A\}$.

Let x be a nonnegative integer, and let $\mathcal{I}$=$\{i_1, i_2, ..., i_s\}$ with $0 \leq i_1 < i_2 < ... < i_s \leq x$. Further suppose that $0 \leq s_1 \leq s_2 \leq \ldots s_x \leq n$. Let $(X, \mathcal{G}, \mathcal{B})$ be a $TD(k+x, n)$ with $\mathcal{G}$= $\{G_1, G_2, ..., G_k, H_1, H_2, ..., H_x\}$. Then an $(x, \mathcal{I}, s_1, s_2, \ldots, s_x)$-*thwart* is a set $S = \cup_{j=1}^{x} S_j$, where $S_j \subseteq H_j$ with $|S_j|= s_j$ for each $1 \leq j \leq x$, such that for every $B \in \mathcal{B}$, $|B \cap S| \in \mathcal{I}$.

A (v, k, λ) packing design (briefly packing) is a pair $(\mathcal{X}, \mathcal{B})$ where $\mathcal{X}$ is a v-set, $\mathcal{B}$ is a collection of some k-subsets (called blocks) of $\mathcal{X}$ such that every pair $\{x, y\} \subset \mathcal{X}$ is contained in at most λ blocks of $\mathcal{B}$. The packing number $D(v, k, \lambda)$ is defined to be the maximum number of blocks in a (v, k, λ) packing. A (v, k, λ) packing with $D(v, k, \lambda)$ blocks will be called a maximum packing and we called $D(v, k, \lambda)$ the packing number for v points, block sizes k and index λ.

Next, we introduce the concept of *difference families*. Let G be an additive abelian group. Let $B = \{b_1, b_2, \ldots, b_k\}$ be a subset of G. Define

$$B + g = \{b_1 + g, b_2 + g, \ldots, b_k + g\}$$

for $g \in G$ and define the *development* of B as

$$\text{dev } B = \{B + g : g \in G\}$$

The development of B is just of orbit of the set B under the action of the group G. Since we are allowing repeating blocks in our designs, we wish to point out that the definition of the development of a block B excludes repeated blocks from occurring in dev B. If $|\text{dev } B| < |G|$ then B is said to be in a *short orbit* under G.

Let $\mathcal{F} = \{B_1, B_2, \ldots, B_t\}$ be a family of subsets of G and define the *development* of $\mathcal{F}$ as

$$\text{dev } \mathcal{F} = \textstyle\sum_{i=1}^{t} \text{dev} B_i.$$

If dev $\mathcal{F}$ is a BIBD(v, k, λ), we say that $\mathcal{F}$ is a (v, k, λ) difference family, and denote it by $DF_\lambda(v, k)$. The sets $B_1, B_2, \ldots, B_t$ are called *base blocks*. The group G will certainly be contained in the automorphism group of dev $\mathcal{F}$. If G is cyclic group $\mathbb{Z}_n$, the design is also cyclic.

In some cases, we can determine whether or not an arbitrary collection of blocks $\mathcal{F}$ will be a difference family simply by examining $\mathcal{F}$. We do this as follows. Let the elements of block B_i be $\{b_{i1}, b_{i2}, \ldots, b_{ik}\}$. If the list of differences D,

$$D = (b_{ij} - b_{il} : i = 1, 2, \ldots, t; l = 1, 2, \ldots, k; j \neq l)$$

contains every nonzero element of G exactly λ time then dev $\mathcal{F}$ is a BIBD(v, k, λ). It may be the case that the list of differences does not contain every element exactly λ times but dev $\mathcal{F}$ is still a BIBD(v, k, λ). This occur only when short orbits are present.

We remark that the difference method is very useful in constructing PBDs and GDDs. In both cases, it is not necessary that all base blocks have the same size. In the case of GDDs, often we construct the groups by taking certain short orbits.

1.2 Constructions

Theorem 1 *[24] If q is a power of a prime number, there exists a projective plane* *of order q.*

The proof of the above theorem depends on the existence of a finite field of size q. We denote the projective plane that arises from this finite field by $\mathrm{PG}(2,q)$.

Theorem 2 *[24] Let π be a $PG(2,q)$ where q is odd. Then π contains an oval.*

Theorem 3 *[24] Let π be a $PG(2,q)$ where q is even. Then π contains a hyperoval.*

The existence of ovals (and hyperovals) in $\mathrm{PG}(2,q)$ is very useful in constructing pairwise balanced designs. Here is a well known theorem about constructing pairwise balanced designs using ovals (and hyperovals).

Theorem 4 *[87] Let q be a prime power. Then for $0 \leq t \leq q+1$, $q^2 - t \in B(\{q-2, q-1, q\})$ and $q^2+q+1-t \in B(\{q-1, q, q+1\})$.*

Theorem 5 *[87] If q is a prime power and n is a positive integer where $1 \leq n \leq q-1$ and t is a non-negative integer such that $0 \leq q+1-n$ then $nq+t \in B(\{n, n+1, n+2, q\})$.*

Theorem 6 *(Greig, see [87]) Let q be an odd prime power. Then,*

(a) there exists a $\{(q-1)/2, (q+1)/2\}$-GDD of type $((q-1)/2)^q$ and

(b) there exists a $\{(q+1)/2, (q+3)/2\}$-GDD of type $((q+1)/2)^q$.

A slight generalization of above is presented in [87].

Theorem 7 *If q is an odd prime power, then for any integer t such that $0 \leq t \leq q+1$, we have $\frac{q(q-1)}{2} + t \in B(\{\frac{q-1}{2}, \frac{q+1}{2}, \frac{q+3}{2}\})$.*

Theorem 8 *[97] (Denniston) There exists a $\{0,2^n\}$-arc of order $w = 2^{2n+m} - 2^{n+m} + 2^n$ in $PG(2,2^{m+n})$.*

Theorem 9 *[87] If there is a $TD(k+x,n)$ containing an $(x,\mathcal{I},s_1,s_2,\dots,s_x)$-thwart, and if $\{s_1,\dots,s_x\} \subseteq B((k+\mathcal{I})\cup\{n\})$, then $nk+\sum_{i=1}^{x} s_i \in B((k+\mathcal{I})\cup\{n\})$ where $k+\mathcal{I} = \{k+i : i \in \mathcal{I}\}$.*

Theorem 10 *[43] For p a prime, $0 \leq k \leq p-2$, there is a $TD(k+3,p)$ containing a $(3,\{0,1,2\},a,b,c)$-thwart and its complementary $(3,\{1,2,3\},p-a,p-b,p-c)$-thwart, whenever $a+b+c \leq p+1$.*

Let A and B be subsets of $\mathbb{Z}_n$. Then define $A -_n B = \{a - b \bmod n : a \in A, b \in B\}$. Now define $m(n,a,b) = min\{|A -_n B| : A, B \subseteq \mathbb{Z}_n, |A| = a, |B| = b\}$.

In [42], the following is proved.

Theorem 11 *For q a prime or prime power, there exists a $TD(q+1,q)$ containing the thwart $(l+2,\{0,1,2,(l+\alpha+\beta)^*\},a+\alpha,b+\beta,1,1,\dots,1)$ for all $0 \leq l \leq q-1-m(q-1,a,b)$, and $\alpha,\beta \in \{0,1\}$.*

Theorem 12 *[97](Truncation of groups in a transversal design) Let k be an integer, $k \geq 2$. Let $K = \{k,k+1,...,k+s\}$. Suppose that there exists a $TD(k+s,m)$. Let $g_1,g_2,...g_s$ be integers satisfying $0 \leq g_i \leq m, i = 1,2,...,s$. Then there exists a K-GDD of type $(m,m,...,m(k\ times),g_1,g_2,...,g_s)$.*

Theorem 13 *[97](Spike-type construction) Let k, s, and n be integers with $k \geq 2$, $s \geq 0$, and $n \geq 1$. Let $K = \{k,k+1,...,k+s+1\}$. Suppose there exists a $TD(k+s+n,m)$. Let $g_1,g_2,...,g_s$ be integers satisfying $1 \leq g_i \leq m, i = 1,2,...,s$. Then there exists a $K \cup \{k+s+n\}$-GDD of type $(m,m,...,m(k\ times),g_1,g_2,...,g_s,1,1,...,1(n\ times))$.*

Theorem 12 is similar to Theorem 13, except that certain points on a particular block are retained. Some of these points remain as groups of size one, hence the name 'spike'. Since we are interested in PBDs with 3 consecutive block sizes, Theorem 12 is often used when $s = 1$.

Theorem 14 *[97](Line flipped spike construction) Let k be an integer, $k \geq 3$, and let $K = \{k-1, k, ..., k+s+1\}$. Suppose that there exists a $TD(k+s+n, m)$, where n is a positive integer. Let $g_1, g_2, ..., g_s$ be integers satisfying $0 \leq g_i \leq m-1, i = 1, 2, ..., s$. Then there exists a K-GDD of type $((m-1), (m-1), ..., (m-1)(k$ times$), g_1, g_2, ..., g_s, n)$.*

Theorem 15 *[97](Singular Indirect Product) Let K be a set of positive integers and $k \in K$; Suppose there exists a $TD(k, m+n) - TD(k, m)$. If $n + m + h \in B(K \cup (m+h)^\star)$ and $km + h \in B(K)$ then $nk + km + h \in B(K)$.*

Theorem 16 *[97](Singular Direct Product) Let K be a set of positive integers; Suppose there exists a $TD(k, n)$. If $n + h \in B(K \cup h^\star)$ for all $i = 1, 2, \ldots, k$ and $h \in B(K)$ then $nk + h \in B(K)$.*

Theorem 17 *[97](Filling in Holes) If there exists a K-GDD of type $g_1 g_2 \ldots g_n$, and $g_i + h \in B(K \cup h^\star)$ for $1 \leq i \leq n-1$ and $g_n + h \in B(K)$ then $\sum_{i=1} n g_i + h \in B(K)$.*

Theorem 18 *[97](Wilson's Fundamental Construction) Let $(V, \mathcal{G}, \mathcal{B})$ be a GDD (the* master *GDD) with groups $G_1, G_2, \ldots, G_t$. Suppose there exists a function $w : V \to \mathbb{Z}^+ \cup \{0\}$ (a weight function) which has the property that for each block $B = \{x_1, x_2, \ldots, x_k\} \in \mathcal{B}$ there exists a K-GDD of type $(w(x_1), w(x_2), \ldots, w(x_k))$ (such a GDD is an* ingredient *GDD). Then there exists a K-GDD of type*

$$(\sum_{x \in G_1} w(x), \sum_{x \in G_2} w(x), \ldots, \sum_{x \in G_t} w(x)).$$

The existence of PBDs, especially BIBDs, has attracted considerable attention. We recall some known results concerning PBDs.

Theorem 19 *1. [63] For all positive integers $v \equiv 1$ or $5 \pmod{20}$, there exists a $B(5,1;v)$.*

2. [62] There exists a $(v, \{5, 9^*\})$-PBD for any integer $v \equiv 9$ or $17 \pmod{20}$ and $v \geq 37$ with the possible exception of $v = 49$.

3. [62] There exists a $(v, \{5, 13^*\})$-PBD for any integer $v \equiv 13 \pmod{20}$ and $v \geq 53$.

4. [7, 21] There exists a $(v, \{5, 17^*\})$-PBD for any integer $v \equiv 9, 17 \pmod{20}$ and $v \geq 69$, with the possible exception of $v \in \{77, 89, 137, 209, 249, 269, 289\}$

5. [7, 21] There exists a $(v, \{5, 21^*\})$-PBD for all $v \equiv 1$ or $5 \pmod{20}$, $v \geq 85$.

6. [21] There exists a $(v, \{5, 25^*\})$-PBD for all $v \equiv 1$ or $5 \pmod{20}$, $v \geq 101$, with the possible exception of $v = 141$.

In this thesis, we employ mainly TD$(5, m)$ and TD$(6, m)$ (see [3] and references therein):

Lemma 1 *There exists a TD$(5, n)$ for every integer $n \geq 4$ and $n \neq 6$ or 10.*

Lemma 2 *There exists a TD$(6, n)$ for every integer $n \geq 5$ and $n \neq 6, 10, 14, 18, 22$.*

The following is well known.

Lemma 3 *For every prime power q, there exists a RTD(q, q).*

To create HTDs, we employ the following technique.

Lemma 4 *[31] Suppose that a TD*$(k+1,t)$*, a TD*(k,m) *and a TD*$(k,m+1)$ *all exist. Then there is a k-HTD of type* $m^t u^1$*, where* $0 \leq u \leq t-1$.

Chapter 2

Pairwise Balanced Designs

In this chapter, we study pairwise balanced designs. In particular, the emphasis is on pairwise balanced designs with consecutive block sizes. Also, some other existence and non-existence results for pairwise balanced designs are also discussed.

2.1 Deleting Lines in Projective Planes

In this section, we present a construction of pairwise balanced designs with various block sizes from projective planes.

In particular, we show that certain line configurations can be removed from the projective plane to obtain some interesting PBDs. For example, we establish

$50, 51, 52, 53, 54 \in B(\{5,6,7\}),$

$72 \in B(\{6,7,8\}),$

$68, 69 \in B(\{5,6,7,8\}),$

$82, 83, 84, 85, 86, 87, 88, 89 \in B(\{7, 8, 9\})$, and

$93, 94, 95, 110, 114 \in B(\{8, 9, 10\})$.

Numerous applications of PBDs with three but not four consecutive block sizes are given in next section. In determining existence of PBDs on v points with block sizes $\{k, k+1, k+2\}$, often the most difficult cases seem to arise when v is greater than $(k+2)^2$, but not much greater. For example, when $k = 7$, deletions of points in arcs of a projective plane of order 8, and of an affine plane of order 9, establish that if $63 \leq x \leq 81$, then $x \in B(\{7, 8, 9\})$. However, the range following this is not amenable to quite as simple a method (indeed, the next known member of B($\{7, 8, 9\}$) was 91, from the (91,7,1) design). It is in this range that we find deletions of various configurations from finite projective planes to be most useful. While we have not been able to settle all open cases in B($\{k, k+1, k+2\}$) for $k = 5, 6, 7, 8$ using the techniques described here, the extension of the initial sequence of values for which such PBDs are available both simplifies the determination of closure for these sets, and provide simple direct constructions for PBDs. For more complete results on closures of sets with three consecutive block sizes, see [87] and [46]. Naturally, the idea of employing configurations in finite planes to produce PBDs is far from new; see [58] and [97] for related results. The results here are general; while we illustrate them primarily with their consequences for B($\{k, k+1, k+2\}$) when k is small, the goal is really to develop general observations about simple configurations in planes.

One particular importance of the line deletion techniques explored here is in the construction of incomplete transversal designs. Letting $N^*(k)$ be the number of idempotent mutually orthogonal latin squares of side k, it follows from $v \in B(\{k, k+1, k+2, a^*\})$ that a $TD(\ell, v) - TD(\ell, a)$ exists with $\ell = \min(N^*(k), N^*(k+1), N^*(k+$

2)) + 2. Taking $k = 7$, we obtain $TD(7, v) - TD(7, a)$ whenever $v \in B(\{7, 8, 9, a^*\})$.

In providing motivation, we have concentrated on applications to the construction of various designs. It is perhaps important to remark that deleting any set of points at all in a projective plane yields a PBD of some kind. The only surprise, then, is that fairly simple considerations can be used to limit the block sizes to a small set. This goal of restricting the block sizes leads in some cases to interesting new geometric questions; we shall see that our goal of few block sizes leads to a notion of a scattering dual k-arc.

2.1.1 The Mia Configuration

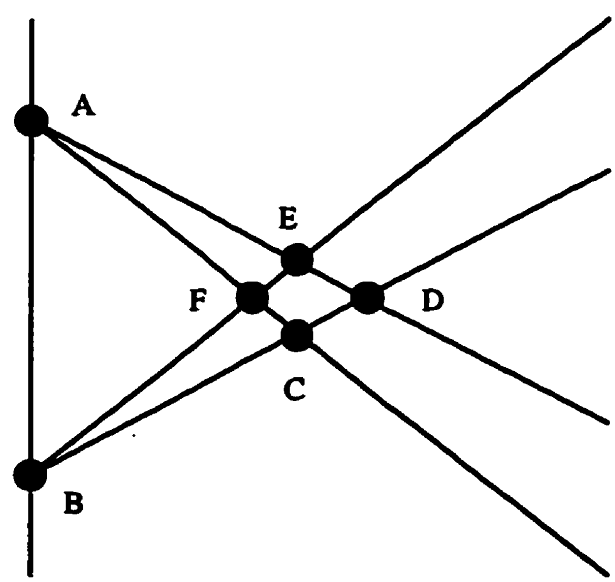


Figure 2.1: The Mia Configuration

A *Mia configuration* is a set of five lines l_1, l_2, l_3, l_4 and l_5 so that $l_2 \cap l_3$ and $l_4 \cap l_5$ are two distinct points on l_1. Figure 2.1 shows the Mia configuration.

Lemma 1 *The Mia configuration exists in any finite projective plane.*

Proof: Take a line l_1 in the plane, and identify two distinct points A, B on the line. For each of the two points, identity two distinct lines intersecting l_1 at that point. The intersections of the four lines define four more points in the finite projective plane. The five chosen lines form the Mia configuration. □

Now, we examine how each line intersects the Mia configuration.

Lemma 2 *Every line intersects the Mia configuration in either $q+1$,3,4 or 5 points where q is the order of the projective plane.*

Proof: The proof uses the labels in Figure 2.1. Trivially, any one of the five lines intersects the Mia configuration at $q+1$ points. If a line intersects the Mia configuration at point A, then it intersects line BE and line CD. Hence, the line intersects the Mia configuration at 3 points. The situation is similar if the line intersects point B. If the line intersects point C, then there are two possible cases. Either it intersects point E so the line intersects the Mia configuration at 3 points or it does not intersect E so the line intersects the configuration at 4 points. The situation is exactly the same by symmetry for points D, E and F. If a line does not hit any of the five lines except in the configuration, then it intersects the Mia configuration at five points; hence the result follows. □

Lemma 3 *If $q \geq 4$ is a prime power, then $q^2 - 4q + 4 \in B(\{q-4, q-3, q-2\})$.*

Proof: The Mia configuration has $5q-3$ points. The result follows by removing the Mia configuration from a projective plane of order q and Lemma 2. □

We can also add back some points from the Mia configuration to obtain some interesting PBDs.

Lemma 4 *If* $0 \le a \le q-3$ *and* $q \ge 4$ *is a prime power, then* $q^2 - 4q + 4 + a \in B(\{q-4, q-3, q-2, a^*\})$.

Proof: From the proof of Lemma 2, we can add any a points on the line AB as long as we do not include the point of intersection of lines AB and CE or the point of intersection of lines AB and DF. □

As a consequence, we have the following corollary.

Corollary 1 $50, 54 \in B(\{5,6,7\})$.

Proof: Apply Lemma 4 with $q = 9$ and $a = 1, 5$. □

Corollary 2 $82, 88, 89 \in B(\{7,8,9\})$.

Proof: Apply Lemma 4 with $q = 11$ and $a = 1, 7, 8$. □

2.1.2 The Dual k-Arc

A *dual k-arc* is a set of k lines in a finite projective plane with the property that no three points of intersection of any two lines are concurrent. We begin with the existence of the dual k-arc in the finite projective plane. The *dual plane* of a projective plane π is the projective plane obtained by interchanging the role of lines and points in π.

Lemma 5 *For* q *a prime power, and any* $1 \le k \le q+1$, *there exists a projective plane of order* q *containing a dual k-arc.*

Proof: Every desarguesian projective plane contains k points such that no three of them are collinear. The result follows by taking the lines corresponding to the k points in the dual plane. □

Figure 2.2 shows a dual 6-arc.

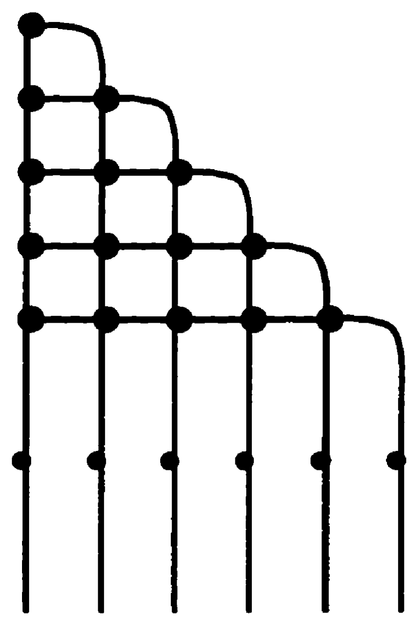


Figure 2.2: The Dual Arc Configuration

We call P a *corner* point if P is on two of the k lines and Q, a *ray* point if Q is on exactly one of the k lines. Let $\mathcal{A}$ be any dual k-arc, and ℓ be any line of the plane not in $\mathcal{A}$. If a points on ℓ are ray points and b points on ℓ are corner points, we must have $a + 2b = k$. Using this observation, we have the following theorem.

Theorem 20 *If q is a prime power and $1 \leq k \leq q+1$, then $q^2+q+1-k(q+1)+\frac{k(k-1)}{2} \in B(\{q+1-k, q+1-(k-1), \ldots, q+1-\lceil\frac{k}{2}\rceil\})$.*

Proof: Take a desarguesian projective plane of order q. By Lemma 5, there exists a dual k-arc. There are $k(q+1)-\frac{k(k-1)}{2}$ points in the dual k-arc. If a points on

l are ray points and b points on l are corner points, since $a + 2b = k$, one has $a+b \in \{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil + 1, \ldots, k\}$. The result follows by removing the points in the dual k-arc. $\square$

As in Lemma 4, it is possible to identify some points in the dual k-arc whose retention does not increase the block sizes.

Theorem 21 *Let $k \geq 4$. If $q+1-k > \frac{(k-1)(k-2)(k-3)(k-4)}{8}$ and q is a prime power, then $q^2+q+1-k(q+1)+\frac{k(k-1)}{2}+1 \in B(\{q+1-k, q+1-(k-1), \ldots, q+1-\lceil \frac{k}{2} \rceil\})$; in addition, if $q+1-k > \frac{(k-1)(k-2)(k-3)(k-4)}{8} + \frac{(k-2)(k-3)}{2}$ then $q^2+q+1-k(q+1)+\frac{k(k-1)}{2}+2 \in B(\{\{q+1-k, q+1-(k-1), \ldots, q+1-\lceil \frac{k}{2} \rceil\})$.*

Proof: Choose a line ℓ_1 of the dual arc. There are $\frac{(k-1)(k-2)}{2}$ corner points not on ℓ_1, and $\frac{(k-1)(k-2)(k-3)(k-4)}{8}$ pairs of corner points defined by disjoint pairs of lines of the dual arc other than ℓ_1. Each such pair defines a line; the line so defined meets ℓ_1, and we call the intersection point *bad*. Under the stated requirement on q and k, one of the ray points, say p_1, is not bad. Adding p_1 therefore does not increase the size of any line whose size was already at least $q+1-k+2$.

Having chosen to add p_1, we next choose a line $\ell_2 \neq \ell_1$ from the dual arc. As before, pairs of corners make up to $\frac{(k-1)(k-2)(k-3)(k-4)}{8}$ of the ray points on ℓ_2 bad. In addition, in this case, a point is bad if it lies on a line defined by p_1 and one of the corners of the dual arc. Having fixed p_1 and ℓ_2, there are $\frac{(k-2)(k-3)}{2}$ ways to choose the corner, and hence at most this number of points are, in addition, classified as bad. Hence, under the stated condition, there remains a ray point p_2 that is not bad. Adding p_1 and p_2 does not increase the size of any line to more than $q+1-k+2$. $\square$

Corollary 3 $51, 52 \in B(\{5,6,7\})$.

Proof: Apply Theorems 20 and 21 with $q = 9$ and $k = 5$. □

Corollary 4 $83, 84, 85 \in B(\{7, 8, 9\})$.

Proof: Apply Theorems 20 and 21 with $q = 11$ and $k = 5$. □

Corollary 5 $83 + a, 84 + a \in B(\{7, 8, 9, a^*\})$ *for* $0 \leq a \leq 6$.

Proof: Retain a points on one of the rays. □

Corollary 6 $93, 94, 95 \in B(\{8, 9, 10\})$.

Proof: Apply Theorems 20 and 21 with $q = 11$ and $k = 4$. □

If we only remove the ray points instead of all points in the dual k-arc, then we can also obtain some interesting PBDs.

Theorem 22 *If q is a prime power, then* $q^2 + q + 1 - k(q - k + 2) \in B(\{k - 1, q + 1 - k, q + 1 - k + 2, \ldots, q + 1 - \alpha\})$ *where* $\alpha \in \{0, 1\}$ *and* α *and* k *have the same parity.*

Proof: If a line intersects i corner points, then it intersects exactly $k - 2i$ ray points. So, by removing all the ray points, the result follows. □

Corollary 7 $48 \in B(\{4, 6, 8, 4^*\})$.

Proof: The corollary follows by taking $q = 8$ and $k = 5$. □

In Theorem 21 we have given a counting argument to ensure the presence of certain PBDs. However, it is possible that the bad points overlap to result in an overestimate of the number of bad points. We consider the cases when $q = 9$ and $q = 11$ to get a better result than Theorem 21.

Lemma 6 . $53 \in B(\{5,6,7\})$.

Proof: A difference set for a projective plane of order 9 is

$$D = \{0, 1, 3, 9, 27, 49, 56, 61, 77, 81\}.$$

Let five lines be $D+0$, $D+1$, $D+3$, $D+5$ and $D+9$. One can check that the five lines form a dual 5-arc. Removing all points on the five lines except for 49 and 65, all lines have sizes 5,6 or 7. Hence, we obtain $53 \in B(\{5,6,7\})$. □

Lemma 7 $86, 87 \in B(\{7,8,9\})$.

Proof: A difference set for projective plane of order 11 is

$$D = \{1, 11, 16, 40, 41, 43, 52, 60, 74, 78, 121, 128\}.$$

Let five lines in the plane be $D+0$, $D+13$, $D+104$, $D+5$ and $D+39$. By removing all points in the five lines except $52, 53, 120$ and 6, all lines have sizes 7, 8 or 9. This gives $87 \in B(\{7,8,9\})$. In addition, if we also remove the point 6, we also obtain $86 \in B(\{7,8,9\})$. □

So far, we have no restriction on the intersection pattern of the corners. However, if we restrict that no three corners in a dual k-arc are collinear, we can obtain some more PBDs with consecutive block sizes. We call a dual k-arc *scattering* if no three of the corner points obtained from six different lines are collinear. ¿From Lemma 9.1.1 in [68], one obtains a necessary condition on scattering dual k-arcs.

Lemma 8 *A scattering dual k-arc in a projective plane of order q must satisfy* $k(k-1)(k-2)(k-3) + 8k \leq 8(q^2 + q + 1)$.

However, the necessary condition is not sufficient. A complete search was attempted for scattering dual 7-arcs in desarguesian projective planes of order 11 and 13. However, there is no scattering dual 7-arc in these projective planes. Also, there is no scattering dual 6-arc in the desarguesian projective plane of order 9. However, scattering dual 6-arcs exist in the desarguesian projective planes of order 11 and 13.

Lemma 9 *There exists a scattering dual 6-arc in a projective plane of order 11.*

Proof: A difference set for projective plane of order 11 is

$$D = \{1, 11, 16, 40, 41, 43, 52, 60, 74, 78, 121, 128\}.$$

Let the six lines be $D + 0$, $D + 13$, $D + 104$, $D + 39$, $D + 1$ and $D + 2$. It is a straightforward matter to check that these 6 lines form a scattering dual 6-arc. □

Lemma 10 *There exists a scattering dual 6-arc in a projective plane of order 13.*

Proof: A difference set for projective plane of order 13 is

$$D = \{0, 2, 3, 10, 26, 39, 43, 61, 109, 121, 130, 136, 155, 141\}.$$

Let the six lines be $D + 0$, $D + 1$, $D + 4$, $D + 5$, $D + 6$ and $D + 9$. One can check that these six lines from a scattering dual 6-arc. □

So far, we have only considered the existence of scattering dual k-arcs. Now, we show how to use them to obtain PBDs.

Theorem 23 *If there exists a scattering dual k-arc in a projective plane of order q then* $q^2 + q + 1 - k(q+1) + \frac{k(k-1)}{2} \in B(\{q+1-k, q+1-(k-1), q+1-(k-2)\})$.

Proof: The proof of this theorem is parallel to Theorem 20 and thus omitted. □

Theorem 22 can also be generalized for the scattering dual k-arc.

Corollary 8 $68, 69 \in B(\{5, 6, 7, 8\})$.

Proof: Apply Theorem 23 with the scattering dual 6-arc in Lemma 9 to obtain a PBD$(76, \{6, 7, 8\} \cup 8^*)$. The result follows by removing seven or eight points in a block of size eight. □

Corollary 9 $114 \in B(\{8, 9, 10\})$.

Proof: Apply Theorem 23 with the scattering dual 6-arc in Lemma 10 to obtain $114 \in B(\{8, 9, 10\})$. □

One general question is to decide when scattering dual k-arcs exist, as they appear to be very useful in constructing PBDs.

2.1.3 The Anti-Fano Configuration

Let π be a projective plane. Let A, B, C and D be 4 points such that no three are collinear. Let $G = AC \cap BD$, $E = AD \cap BC$ and $F = AB \cap CD$. The six lines AB, AC, AD, BC, BD and CD form an *anti-Fano configuration* if the three points E, F and G are non-collinear.

Lemma 11 *If q is an odd prime power, then there exists a projective plane of order q containing an anti-Fano configuration.*

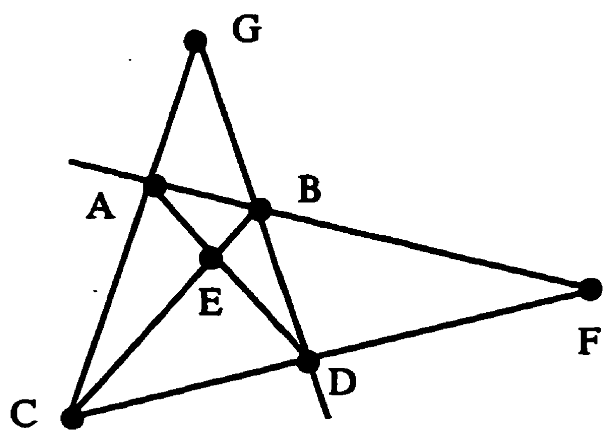


Figure 2.3: The Anti-Fano Configuration

Proof: It is known that the desarguesian projective plane of order q, q odd, does not contain a projective sub-plane of order 2 [24]. The result follows since if points E, F and G are collinear, then the seven points form a projective sub-plane of order 2 (a Fano configuration). □

Theorem 24 *If there exists a projective plane of order q containing an anti-Fano configuration, then* $q^2 - 5q + 6 \in B(\{q-5, q-4, q-3\})$.

Proof: In the proof, we often refer to Figure 2.3. Let l be any line. If l does not intersect any of the seven vertices, then l intersects the configuration at precisely six points. If l intersects the configuration at any one of the A, B, C and D, then l does not hit any other vertices in the configuration. Hence, l intersects the configuration at precisely four points. If l intersects one or two of E, F and G, then again by counting, it intersects precisely four or five times. Also, the number of points in the configuration is $6(q+1) - 11$. We obtain the result by removing the anti-Fano configuration from the plane. □

Corollary 10 $72 \in B(\{6,7,8\})$.

Proof: Apply Theorem 24 with $q = 11$. □

Corollary 11 $110 \in B(\{8,9,10\})$.

Proof: Apply Theorem 24 with $q = 13$. □

2.2 An Update

In this section, we update some results on the closure of B({4, 5, 6}), B({5, 6, 7}) and B({6, 7, 8}).

First of all, we begin with pairwise balanced designs with block sizes four, five and six.

The following is proved by Lenz [77].

Theorem 25 *For any integer $v \geq 4$, $v \in B(\{4,5,6\})$ except when*

$$v \in \{7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}$$

and possibly when $v \in \{43, 47\}$.

We deal with the last two possible exceptions.

Lemma 12 $43, 47 \in B(\{4,5,6\})$.

Proof: By Theorem 6 with $q = 9$, this gives a PBD$(45, \{5, 6, 6^*\})$; deleting two points from a block of size six gives a PBD$(43, \{4, 5, 6\})$. For $v = 47$, we start with the $(66, 6, 1)$ design in Example I.2.34 of [6]. Delete the points x_3 for $x \in \{0, 1, 3, 4, 7, 8, 9, 10, 11, 12\}$, and the points y_4 for $y \in \{0, 1, 2, 3, 5, 6, 7, 8, 9, 11\}$. This leaves a PBD$(46, \{4, 5, 6^*\})$. There is a parallel class of blocks of sizes 4 and 5 consisting of $\{0_0, 7_0, 2_2, 3_2\}$, $\{1_0, 11_0, 0_1, 5_1, 2_3\}$, $\{2_0, 5_0, 4_1, 9_1, 6_3\}$, $\{3_0, 4_0, 8_0, 1_2, 10_4\}$, $\{6_0, 3_1, 5_2, 10_2, 4_4\}$, $\{9_0, 6_1, 0_2, 8_2\}$, $\{10_0, 2_1, 8_1, 11_2\}$, $\{12_0, 12_1, 12_2, 12_4, \infty\}$, $\{1_1, 10_1, 7_2, 9_2\}$ and $\{7_1, 11_1, 4_2, 6_2, 5_3\}$. Add an infinite point to these blocks to get a PBD$(47, \{4, 5, 6\})$. □

Next, we study the closure containing block sizes five, six and seven. The following is proved in [82]. Let $Q_1 = [8, 20] \cup [22, 24] \cup [27, 29] \cup [32, 34]$ and $Q_2 = \{39\} \cup [50, 54] \cup [68, 69] \cup [92, 94] \cup [98, 99] \cup \{104\} \cup [108, 109] \cup \{114\} \cup [123, 124]$.

We are able to obtain a slight improvement of the result.

Lemma 13 $39, 50, 51, 52, 53, 54, 92, 123 \in B(\{5, 6, 7\})$.

Proof: For 39 (due to Greig), consider three non-concurrent lines in PG$(2, 7)$. Delete the three points of intersection, and five other points per line; a block of size eight only remains if the three single points retained from the three lines are collinear. $39 \in B(\{5, 6, 7\})$ is also proved in [94]. For $[50, 54]$, see Corollaries 1, 3 and Lemma 6. For $v = 92$, the BIBD$(96, 6, 1)$ in [24] is a 6-GDD of type 6^{16}; add one infinite point to the groups to get 97 points. Now delete five points from a 6-block to get 92 points. For $v = 123$, let $V = \mathbb{Z}_{123}$ and $B_1 = \{0, 3, 9, 21, 36, 19, 80\}$, $B_2 = \{0, 24, 75, 25, 109\}$, $B_3 = \{0, 30, 7, 88, 83\}$, $B_4 = \{0, 45, 13, 67, 41\}$ and $B_5 = \{0, 57, 49, 112, 20\}$. Develop these five blocks over $\mathbb{Z}_{123}$ to obtain a PBD$(123, \{5, 7\})$. □

Let $E_{567} = [8, 20] \cup [22, 24] \cup [27, 29] \cup [32, 34]$ and $X_{567} = [68, 69] \cup [93, 94] \cup [98, 99] \cup \{104\} \cup [108, 109] \cup \{114, 124\}$.

Theorem 26 *For any integer $v \geq 5$, $v \in B(\{5, 6, 7\})$ with the possible exceptions in X_{567} and the definite exceptions in E_{567}.*

We now turn our attention to PBDs with block sizes six, seven and eight. Again, the following result is proved in [82].

Let $M_1 = [9,30] \cup [32,36] \cup [38,41]$ and $M_2 = \{37\} \cup [44,47] \cup \{65\} \cup [68,75] \cup \{77\} \cup [93,95] \cup [97,103] \cup \{108\} \cup [122,125] \cup [128,131] \cup \{135\} \cup [137,150] \cup [152,155] \cup [159,161] \cup [165,167] \cup [170,180] \cup [184,185] \cup [233,240] \cup [242,246] \cup [250,251] \cup \{255\}$.

Theorem 27 *For any integer $v \geq 9$, $v \in PBD(\{6,7,8\})$ with the possible exceptions in M_2 and definite exceptions in M_1.*

We have made the following improvement.

Lemma 14 $37, 44, 45, 47 \notin B(\{6,7,8\})$.

Proof: The result follows from a theorem of Batten [16]. □

Lemma 15 $72, 97, 102, 103, 108, 171, 234, 246, 250, 251, 255 \in B(\{6,7,8\})$.

Proof: For $v = 72$, see Corollary 10. For $v = 97$, take a 6-GDD of type 6^{16} [6] and add a point at infinity to each group to obtain a $97 \in B(\{6,7\})$. $[102,120] \in B(\{6,7,8\})$ can be seen as follow: the BIBD$(120,8,1)$ appearing as a Denniston arc in PG$(2,16)$ contains a hyperoval of the plane (in fact, Denniston arcs are nested). Deleting i of the hyperoval points for $0 \leq i \leq 18$ gives $120 - i \in B(\{6,7,8\})$. Also, $171 \in B(6)$ [2]. For $v = 234$, take a TD$(6,38)$ and fill in the group with six infinite points and apply Singular Direct Product. The required PBD$(44, \{6,7,8,6^*\})$ is constructed by removing six points in two groups in a TD$(8,7)$. For the remaining value, we take a resolvable (288,8,1) design [2] on $(\mathbb{Z}_7 \times \mathbb{Z}_{41}) \cup \{\infty\}$, whose starter blocks are $\{(0,9t),(0,32t),(1,3t),(1,38t),(2,t),(2,40t),(4,14t),(4,27t)\}$ for

$t = 1, 37, 16, 18, 10$, and $\{\infty, (0,0), (1,0), (2,0), (3,0), (4,0), (5,0), (6,0)\}$. Every block meets the set $(\{0\} \times \mathbb{Z}_{41}) \cup \{\infty\}$ in 0 or in 2 points, and hence deleting any subset of i of these 42 points yields $288 - i \in B(\{6,7,8\})$. □

This (288,8,1) design is remarkable, since deleting points $(\{0\} \times Z_{41}) \cup \{\infty\}$ gives $246 \in B(\{6,8\})$, and deleting points $\{0,1\} \times \mathbb{Z}_{41}$ gives $206 \in B(\{4,6,8\})$.

Also, much research has been done on designs having a maximal arc [120, 121]. Designs having structure similar to the above BIBD are of interest.

Let $E_{678} = [9,30] \cup [32,41] \cup \{45,47\}$ and $X_{678} = \{46,65\} \cup [68,71] \cup [73,75] \cup \{77\} \cup [93,95] \cup [98,101] \cup [122,125] \cup [128,131] \cup \{135\} \cup [137,150] \cup [152,155] \cup [159,161] \cup [165,167] \cup \{170\} \cup [172,180] \cup [184,185] \cup \{233\} \cup [235,240] \cup [242,245]$.

Theorem 28 *For any integer $v \geq 9$, $v \in B(\{6,7,8\})$ with the possible exceptions in X_{678} and definite exceptions in E_{678}.*

2.3 PBDs with Block Sizes Seven, Eight and Nine

In this section, we study pairwise balanced designs with block sizes seven, eight and nine. We do not comment on the non-existence result as it is a special case of a theorem in [16].

Lemma 16 $49, 50, 56, 57, 58 \in B(\{7,8,9\})$.

Proof: Remove six or seven points in a group in TD$(8,7)$ to obtain $49, 50 \in B(\{7,8,9\})$. Remove seven or eight points in two different groups in TD$(9,8)$ to obtain $56, 57, 58 \in B(\{7,8,9\})$. □

Lemma 17 $90 \in B(\{7,8,9\})$.

Proof: A difference set for projective plane of order 11 is

$$D = \{0, 15, 39, 59, 10, 42, 40, 127, 120, 73, 51, 77\}.$$

Let three lines in the plane be $D+12$, $D+15$, $D+80$ and an oval be $-D$. By removing all points in the three lines and the oval, this gives $90 \in B(\{7,8,9\})$. □

Lemma 18 $[63, 92] \subseteq B(\{7,8,9\})$.

Proof: Apply Theorem 4 with $q = 8$ and $q = 9$ to handle [63,81]. For $[82, 90] \subseteq B(\{7,8,9\})$, see Section 2.1. Apply Theorem 6 with $q = 13$ to obtain a $\{7,8\}$-GDD of type 7^{13}. This gives $91 \in B(\{7,8\})$. Adding an infinite point to the groups gives $92 \in B(\{7,8\})$. □

Lemma 19 *If there exists a RB*$(v, 8)$ *and* $v \geq 120$, *then* $[v-8, v-6] \cup [v-1, v+9] \subseteq B(\{7,8,9\})$. *Hence,* $[112, 114] \cup [119, 129] \cup [224, 226] \cup [231, 241] \cup [280, 282] \cup [287, 297] \cup [336, 338] \cup [343, 353] \cup [392, 394] \cup [399, 409] \subseteq B(\{7,8,9\})$.

Proof: Remove a block or seven points in a block in RB$(v, 8)$ to obtain $v-8, v-7 \in B(\{7,8,9\})$. Remove seven points in a block and add a point at infinity to a parallel class not containing that block to obtain $v-6 \in B(\{7,8,9\})$. Remove zero or one points from RB$(v, 8)$ to obtain $v-1, v \in B(\{7,8,9\})$. Since $v \geq 120$, there are at least 15 parallel classes. Remove seven points in a block and add nine points forming a block at infinity to nine parallel classes not containing that block to obtain $v+2 \in B(\{7,8,9\})$. Take any two blocks in a parallel class and remove two points from two of the blocks. There are at most five parallel classes containing at least one block of size six. Then adjoining seven, eight or nine infinite points in a block at infinity to the parallel classes including all those with a block of size six to get $v+3, v+4, v+5 \in B(\{7,8,9\})$. Remove a point from RB$(v,8)$ and add seven infinite points to obtain $v+6 \in B(\{7,8,9\})$. Finally, if we add $7,8$ or 9 infinite points to RB$(v,8)$ in a block at infinity, then we get $v+7, v+8, v+9 \in B(\{7,8,9\})$. Employ $120, 232, 288, 344, 400 \in RB(8)$ [56]. □

Lemma 20 $115, 227 \in B(\{7,8,9\})$.

Proof: From Theorem 8, there is a $\{0,8\}$-arc A of order 120 in $PG(2,16)$. Add two points x and y not in the arc to A. Then $A \cup \{x,y\}$ contains a unique block of size ten. Remove seven points in a line of size eight intersecting the block of size ten including the point of intersection to get $115 \in B(\{7,8,9\})$. That $227 \in B(\{7,8,9\})$ follows similarly from the fact that there is a $\{0,8\}$-arc of order 232 in $PG(2,32)$. □

Lemma 21 $130, 131, 133, 134, 135, 136, 137 \in B(\{7,8,9\})$.

Proof: Theorem 6 with $q = 17$ gives a $\{8,9\}$-GDD of type 8^{17}. There are three points in the design such that any two points induce a block of size nine. Hence, removing 0, 1, 2, or 3 of the three points yields $[133,136] \in B(\{7,8,9\})$. Add a point at infinity to the groups of the GDD to get $137 \in B(\{7,8,9\})$. Add a point at infinity and remove seven points from a block of size eight to obtain $130 \in B(\{7,8,9\})$. Finally, the GDD is embedded in a projective plane of order 17 and is obtained on the set of exterior points. Adding two oval points gives a unique block of size ten. Remove 7 points including the point of intersection of a block of size eight with the block of size ten to obtain $131 \in B(\{7,8,9\})$. □

Lemma 22 $169, 217, 218, 301, 302 \in B(\{7,8,9\})$.

Proof: Observe that $169 \in B(7)$, and that there exist 7-GDDs of type 7^{31} and 7^{43} [2]. Add zero or one infinite point. □

Lemma 23 $[308, 310] \in B(\{7,8,9\})$.

Proof: Take a TD(9, 43) and truncate two groups to zero or one points each; fill the groups using $50 \in B(\{7,8,7^*\})$ and seven infinite points. □

Lemma 24 $[386, 401] \in B(\{7,8,9\})$.

Proof: Take a projective plane of order 43 with 44 oval points. Remove an oval point to get a TD(44, 43) with one oval point in 43 of the 44 groups. Remove all but nine groups which all have an oval point. Remove any t oval points where $0 \leq t \leq 9$. This gives a $\{7,8,9\}$-GDD of type $42^t 43^{9-t}$. Add seven infinite points to obtain $[385, 394] \subseteq B(\{7,8,9\})$.

Now start with a TD(9, 49) with nine oval points. Choose four oval points and a group not containing any of the four oval points. These four oval points define six distinct lines which intersect the group in at most six distinct points. Truncate the group to size $6, 7$ or 8 leaving those intersecting points. Now, remove any t of those four oval points to get a $\{7,8,9\}$-GDD of type $49^{8-t} 48^t a^1$ for $0 \leq t \leq 4$ and $a = 6, 7, 8$. With these GDDs, apply Theorem 17 with $h = 1$ to obtain $[395, 401] \subseteq B(\{7,8,9\})$. □

Lemma 25 $410, 411, 412 \in B(\{7,8,9\})$.

Proof: Take a TD(9, 56) and truncate to nine points in two different groups. This gives $410 \in B(\{7,8,9\})$. A $(433, \{9\})$-PBD exists, obtained by developing the starter blocks $5^i \cdot \{0, 1, 3, 30, 52, 61, 84, 280, 394\}$, $i = 0, 36, 72, 108, 144, 180$, over the cyclic group of order 433 [2]. Choose a set P of four points, no three collinear. These define six blocks, each containing seven points not in P; let C be the union of these six blocks. Then suppose that there are two blocks B_1 and B_2, so that B_1, B_2 and C are pairwise disjoint. Removing the points of all three sets would then establish the statement for 411, and removing all but one of the points would settle 412. It remains to exhibit the set P and the blocks B_1 and B_2. Take $P = \{0, 1, 2, 7\}$, and $B_1 = \{80, 108, 127, 151, 271, 338, 344, 412, 426\}$ $B_2 = \{81, 109, 128, 152, 272, 339, 345, 413, 427\}$. □

Lemma 26 $[413, 417] \subseteq B(\{7,8,9\})$.

Proof: Take a TD(9, 57) and truncate to x and y points in two different groups where $7 \leq x, y \leq 9$. □

Lemma 27 $[418, 433] \subseteq B(\{7,8,9\})$.

Proof: A (433,{9})-PBD $\mathcal{D}$ exists [2]. Simple counting ensures that $\mathcal{D}$ contains seven points, no three collinear. By deleting any subset of these, we obtain $[426, 433] \in B(\{7,8,9\})$. In $\mathcal{D}$, remove eight or nine points in a block of size nine to obtain $424, 425 \in B(\{7,8,9\})$. Remove a block and 1 or 2 points in pair of disjoint blocks from $\mathcal{D}$ to obtain $422, 423 \in B(\{7,8,9\})$. If we have removed a block and two points from a pair of disjoint blocks, this induces 19 blocks of size seven. There are at most 133 points lying in a block of size seven. Hence, it is possible to pick one more point to remove, so $421 \in B(\{7,8,9\})$. Continue in this way to show $419, 420 \in B(\{7,8,9\})$. Finally, take two intersecting blocks from $\mathcal{D}$ and remove eight points from each block including the point of intersection. This gives $418 \in B(\{7,8,9\})$. □

Lemma 28 $[430, 436] \subseteq B(\{7,8,9\})$.

Proof: In a $PG(2,8)$, there is a sub-plane $PG(2,2)$. Choose three collinear points (P_1, P_2, P_3 on line l) in the sub-plane; the remaining four points in the sub-plane form a $\{0,1,2\}$-arc Q. The six points of $l \setminus \{P_1, P_2, P_3\}$ together form a $\{0,1,2,6^*\}$-arc on 10 points. Remove three points from Q and remove any h points from $l \setminus \{P_1, P_2, P_3\}$ to obtain $61+h \in B(\{7,8,9,h^*\})$ for $3 \leq h \leq 9$. Add h infinite points to the groups of a TD$(7,61)$, for $3 \leq h \leq 9$. This gives $[430, 436] \subseteq B(\{7,8,9\})$. □

Lemma 29 $[437, 440] \subseteq B(\{7,8,9\})$.

Proof: By [2], $400 \in RB(8)$. Take any two blocks in a parallel class. Remove one block and two points from the other block. This gives 17 blocks of size six and at most 102 points in a block of size six. Hence, it is possible to pick a point so that its removal does not shorten any block by more than two. Now, there are 25 blocks of size six and at most 150 points in a block of size six. So a further point can be removed so that every block has size at least six. This gives at most 37 blocks of size six. Adding 49 infinite points to 49 parallel classes including all parallel classes containing a block of size six. This gives $437 \in B(\{7,8,9\})$. It is easy to establish that $438, 439, 440 \in B(\{7,8,9\})$ in a similar way. □

Lemma 30 $[441, 449] \subseteq B(\{7,8,9\})$.

Proof: Take a projective plane of order 49 with 50 oval points. Remove one oval point. This gives a TD(50, 49). Remove 41 groups including the group that does not contain an oval point. This gives a TD(9, 49) with one oval point in each group. Now, remove any t of the oval points for $0 \leq t \leq 9$. This gives a $\{7,8,9\}$-GDD of type 48^t49^{9-t}. Add eight points at infinity and use the fact that $56, 57 \in B(\{7,8,9,8^*\})$. □

Lemma 31 *If q is a prime power and $q \in B(\{7,8,9\})$, then $[7q, 8q-6] \in B(\{7,8,9\})$. Hence* $[343, 386] \cup [448, 642] \cup [791, 1090] \cup [1183, 1346] \cup [1589, 1906] \cup [1967, 2338] \cup [2359, 2818] \in B(\{7,8,9\})$.

Proof: Apply Theorem 5 with $n = 7$. Then employ $q = 49, 64, 71, 79, 81, 113, 121, 128, 137, 169, 227, 233, 239, 281, 293, 337, 341, 353$. □

Lemma 32 *If there exists a $\{9,10\}$-GDD of type m^{10} and for each $0 \leq t_i \leq m$ $0 \leq i \leq 10$, then there exists a $\{7,8,9\}$-GDD of type $(7m+t_1)(7m+t_2)\ldots(7m+t_{10})$.*

Proof: Assign weight 8 to t_i elements in group i for $0 \leq i \leq 10$ and assign the remaining elements weight 7. All we need to show is the existence of the ingredient GDDs: $\{7,8,9\}$-GDD of type 7^t8^{9-t} for $0 \leq t \leq 9$ can be obtained by removing any $t+1$ oval points in a projective plane of order 8. Remove a line disjoint from all oval points and an oval point from a projective plane of order 9. This gives a 9-GDD of type 8^{10} with 9 oval points. For each $0 \leq t \leq 9$, delete any t oval points to produce $\{7,8,9\}$-GDDs of type 7^t8^{10-t}; for $t = 10$, delete two blocks, and one point in the same group as their point of intersection, from a TD(10, 9). □

Lemma 33 $[643, 790] \in B(\{7,8,9\})$.

Proof: Remove a line containing only one oval point from the projective plane of order 9. This gives a TD$(9,9)$ with one oval point in each group. Remove any subset of the oval points to obtain a $\{7,8,9\}$-GDD of type 9^t8^{9-t} where $0 \le t \le 9$. Take a TD$(9,m)$, $m \in \{8,9\}$ and assign weight 9 to t_i points in group i, $0 \le i \le 9$ and give weight 8 to all remaining points. By Theorem 18, this results in a $\{7,8,9\}$-GDD of type $(8m+t_1)(8m+t_2)\dots(8m+t_9)$. This gives $[576,729] \subseteq B(\{7,8,9\})$. Apply Lemma 32 with a $\{9,10\}$-GDD of type 10^{10} to obtain $[700,800] \subseteq B(\{7,8,9\})$. The $\{9,10\}$-GDD of type 10^{10} can be obtained by removing a block from TD$(10,11)$. □

Lemma 34 $[1091,1182] \cup [1295,1457] \cup [1799,1961] \cup [2191,2353] \subseteq B(\{7,8,9\})$.

Proof: Any integer $v \in [112,274]$ can be written as a sum of two integers a and b such that $a,b \in B(\{7,8,9\})$ and $a,b \le 137$. Take a TD$(9,m)$ and truncate $137-a$ and $137-b$ points from two different groups for $m \in \{137,169,241,297\}$. □

Lemma 35 *There exists* $\{7,8,9\}$*-GDD of type* 7^7a*,* 7^8a *and* 7^9a *for* $0 \le a \le 7$.

Proof: Truncate one group of a TD$(8,7)$ to obtain a $\{7,8,9\}$-GDD of type 7^7a. Take a TD$(8,8)$ and remove one point to produce an 8-GDD of type 7^9. Truncate points in one group to obtain $\{7,8,9\}$-GDDs of type 7^8a. Take a TD$(8,9)$ and remove one point to give a $\{8,9\}$-GDD of type 7^98^1. Truncate the long group to obtain $\{7,8,9\}$-GDDs of type 7^9a. □

Lemma 36 $[1458,1588] \subseteq B(\{7,8,9\})$.

Proof: Take a TD$(10,25)$ and remove one block. This gives a $\{9,10\}$-GDD of type 24^{10}. Assign weight 7 to eight groups, weight 0 or 7 to the ninth group and assign arbitrarily weights from 0 to 7 to the last group. This gives a $\{7,8,9\}$-GDD of type $168^8(7u)^1v$ where $0 \le u \le 24$ and $0 \le v \le 168$. Adding a point at infinity to obtain a $168 \cdot 8 + 7u + v + 1 \in B(\{7,8,9\})$ if $7u+1, v+1 \in B(\{7,8,9\})$. Choose $7u \in \{49,63,77,91,112,126\}$ and $v \in [62,80]$ to obtain $[1456,1551] \setminus \{1517,1518\} \in B(\{7,8,9\})$. Choose $7u \in \{49,63,77,91,112,126\}$ and $v \in [118,130] \cup [132,136]$ to obtain $[1512,1607] \setminus \{1525,1539,1553,1567,1573,1574,1588,1602\} \in B(\{7,8,9\})$. Choose $7u \in \{133,168\}$ and $v \in [62,80]$ to obtain $[1540,1558] \cup [1575,1593] \in B(\{7,8,9\})$. Take a TD(9, 216) [3] and truncate two groups to 6 and 48 and add a point at infinity. This gives $1567 \in B(\{7,8,9\})$. Take a TD(17, 224) [3] and take a 'stair' of length of size 5 or 6 (see [41]) together with 7 of the groups. This gives $1573, 1574 \in B(\{7,8,9\})$. □

Lemma 37 $[1959,1977] \cup [2341,2358] \subseteq B(\{7,8,9\})$.

Proof: Take a TD$(9,m)$ and truncate one group to a points where $v \in [63,81]$, $m \in \{237,293\}$. □

Combining the above lemmas, we have $[343,2470] \subseteq B(\{7,8,9\})$.

Lemma 38 *If* $[343 \cdot 7^a, 343 \cdot 7^{a+1} + 69] \subseteq B(\{7,8,9\})$ *where* a *is any non-negative integer then* $[343 \cdot 7^{a+1}, 343 \cdot 7^{a+2} + 69] \subseteq B(\{7,8,9\})$.

Proof: For any $v \in [343 \cdot 7^{a+1}, 343 \cdot 7^{a+2} + 69]$, v can be written as $7u + v$ where $u \in [343 \cdot 7^a, 343 \cdot 7^{a+1} + 69]$ and $v \in \{63,64,65,66,67,68,69\}$. Since $u \ge 76$, a TD$(8,u)$ exists [3]; truncate a group to size v. □

Corollary 12 *For any* $v \ge 343$, $v \in B(\{7,8,9\})$.

Let $E_{789} = [10,48] \cup [51,55] \cup [59,62]$ and $X_{789} = [93,111] \cup [116,118] \cup \{132\} \cup [138,168] \cup [170,216] \cup [219,223] \cup [228,230] \cup [242,279] \cup [283,286] \cup [298,300] \cup [303,307] \cup [311,335] \cup [339,342]$.

Theorem 29 *For any integer $v \geq 10$, $v \in B(\{7,8,9\})$ with the possible exceptions in X_{789} and definite exceptions in E_{789}.*

2.4 PBDs with Block Sizes Eight, Nine and Ten

In [82], it is shown that if $v \geq 1864$, then $v \in B(\{8,9,10\})$. We can make considerable progress on this. Due to the complication of stating the result in [82], we reproduce part of the proof here. We do not comment on the non-existence here, as it is included in [16].

Define $f(n) = n + n \cdot 2^m(8 \cdot 2^m + 1 - n) + (8 \cdot 2^m - 7 - n) \cdot \binom{n}{2}$ for positive integers m, n. The following theorem in [82] is useful.

Theorem 30 *Let m be a positive integer. If there exists a positive integer k such that $f(1), f(2), \ldots, f(k)$ are all less than $2^{6+2m} + 2^{4+m} - 2^{6+m} - 7$, then $2^{6+m} - 2^{3+m} + 8 + t \in B(\{8,9,10\})$ for all integers t such that $0 \leq t \leq k+1$.*

The number of blocks in a $B(v, 8)$ with $v = 2^{m+6} - 2^{m+3} + 8$ is $56(2^{2m}) + 15(2^m) + 1$ because the number of blocks is equal to $\frac{v(v-1)}{56}$. So, in $PG(2, 2^{3+m})$, there are $2^{6+2m} + 2^{3+m} + 1 - 56(2^{2m}) - 15(2^m) - 1 = 2^m(2^{m+3} - 7)$ lines missing the set v points of the $\{0,8\}$-arc.

We need the following result from [82].

Corollary 13 *Suppose there exists a positive integer k such that $f(1), f(2), \ldots, f(k-1) < 2^{6+2m} + 2^{4+m} - 2^{6+m} - 7$ and $k(2^m) < 2^m(2^{m+3} - 7)$. Let a be a positive integer such that $a + k(2^m) + \frac{k(k-1)}{2} \leq 2^{m+3} + 1$, then $2^{m+6} - 2^{m+3} + k + a + 8 \in B(\{8,9,10,a^*\})$.*

Lemma 39 $57, 64, 65, 72, 73, 74, 80 \in B(\{8, 9, 10\})$.

Proof: By [2], we have $57, 64 \in B(8)$. Add a point to each group in $\mathrm{TD}(8,8)$ to obtain $65 \in B(\{8,9,10\})$. Now $72 \in B(\{8,9,10\})$ because $\mathrm{TD}(9,8)$ exists; add one point to each group of $\mathrm{TD}(9,8)$ to obtain $73 \in B(\{8,9,10\})$. Take a $\mathrm{TD}(10,9)$ and remove eight points in two groups to obtain $74 \in B(\{8,9,10\})$. Finally, remove one point from $\mathrm{TD}(9,9)$ to obtain $80 \in B(\{8,9,10\})$. □

Lemma 40 $[81, 91] \subseteq B(\{8, 9, 10\})$.

Proof: The existence of the projective plane of order 9 establishes that $91 \in B(\{8,9,10\})$. Remove an oval point in the plane to produce a $\mathrm{TD}(10,9)$ with 9 groups containing an oval point. Deleting any t of the oval points gives $90 - t \in B(\{8,9,10\})$ for $0 \leq t \leq 9$. □

Lemma 41 $92, 93 \in B(\{8, 9, 10\})$.

Proof: Take a $\mathrm{TD}(10,11)$ and remove nine points in two blocks so that no group of size 11 remains. □

Lemma 42 $96, 97, 98, 99, 100 \in B(\{8, 9, 10\})$.

Proof: First of all, if $n = 10$ and $A = \{0,5\} = B$ then $A -_n B = \{0,5\}$. Hence $m(10,2,2) \leq 2$. If we take $q = 11$ and apply Theorem 11 with $l = 8$, $\alpha = \beta = 1$ to produce a $(10, \{0,1,2,10^*\}, 3, 3, 1, 1, \ldots, 1)$-thwart in $\mathrm{TD}(10,11)$. By complementing the thwart, we obtain a $(10, \{8,9,10,0^*\}, 8, 8, 10, 10, \ldots, 10)$-thwart in $\mathrm{TD}(10,11)$. Hence we obtain $[96, 100] \subseteq B(\{8,9,10\})$. □

Lemma 43 $94, 95, 110, 114 \in B(\{8, 9, 10\})$.

Proof: See Section 2.1. □

Theorem 31 *[2]* $120, 232, 288, 344, 400, 456 \in RB(8)$.

Lemma 44 $120, 121, 122, 123, 124, 125, 128, 129, 130, 131, 132 \in B(\{8, 9, 10\})$.

Proof: In Theorem 30, taking $m = 1$, we have $f(1) = 33$, $f(2) = 69$, $f(3) = 105$ and $f(4) = 138$, therefore we obtain $[120, 125] \subseteq B(\{8, 9, 10\})$. Apply Corollary 13 with $a = 8, 9, 10$ when $k = 2$ to get $130, 131, 132 \in B(\{8, 9, 10\})$. Finally, by Lemma 31, it is possible to add 8 or 9 infinite points to the resolvable design RB(120, 8) and a block at infinity to obtain $128, 129 \in B(\{8, 9, 10\})$. □

Lemma 45 $[136, 154] \subseteq B(\{8, 9, 10\})$.

Proof: Taking $q = 17$, apply Lemma 7. □

Lemma 46 $161, 162, 163, 164, 165, 168, 169, 170, 171, 172 \in B(\{8, 9, 10\})$.

Proof: Apply Lemma 7 with $q = 19$. The set of exterior points induces a design with block sizes 9, 10 (See [82]). All block sizes of 10 are induced by the exterior lines; it is possible to choose three exterior lines so that their pairwise intersection are distinct and the points of intersection are exterior points. By removing the three points we obtain $168 \in B(\{8, 9, 10\})$. Remove 0, 1 or 2 points from the $\{9, 10\}$-GDD of type 9^{19} to obtain $169, 170, 171 \in B(\{8, 9, 10\})$. By adding a point at infinity to each group, we get $172 \in B(\{8, 9, 10\})$. Remove 9 or 10 points from a block of size 10 in the GDD to obtain $161, 162 \in B(\{8, 9, 10\})$. Remove eight points from a group to obtain $163 \in B(\{8,9,10\})$. Adding two oval points from the $\{9,10\}$-GDD of type 9^{19} arising from the finite projective plane of order 19. This gives a PBD$(173, \{9,10\} \cup 11^*)$. Remove eight points or nine points from a nine-block intersecting the unique block of size 11 to obtain $164, 165 \in B(\{8,9,10\})$. □

Lemma 47 $232, 233, 234, 235, 236, 237, 238, 240, 241, 242, 243, 244, 245, 246 \in B(\{8,9,10\})$..

Proof: Apply Theorem 30 with $m = 2$; we have $f(1) = 129$, $f(2) = 273$, $f(3) = 429$, $f(4) = 594$ and $f(5) = 765$. They are all less than 812 which is number of exterior points. Hence $[232, 238] \subseteq B(\{8,9,10\})$. Apply Corollary 13 with $a = 10$ and $k = 1,2,3,4$ respectively to obtain $[243, 246] \subseteq B(\{8,9,10\})$. Finally, $232 \in RB(8)$ has more than 10 resolution classes, and hence $[240, 242] \subseteq B(\{8,9,10\})$. □

Lemma 48 $288, 289, 296, 297, 298 \in B(\{8,9,10\})$.

Proof: Add $0, 1, 8, 9, 10$ infinite points to a RB$(288, 8)$ design. □

Lemma 49 $344, 345, 352, 353, 354 \in B(\{8,9,10\})$.

Proof: Add $0, 1, 8, 9, 10$ infinite points to a RB$(344, 8)$ design. □

Lemma 50 $400, 401, 408, 409, 410, 457 \in B(\{8,9,10\})$.

Proof: Add $0, 1, 8, 9, 10, 57$ infinite points to a RB$(400, 8)$ design. □

Lemma 51 $402, 449, 450, 451, 498 \in B(\{8,9,10\})$.

Proof: Truncate a group of TD(k, 49) for $k \in \{9, 10\}$ to 0, 1, 2, or 49 points. Fill the groups using eight infinite points and $57 \in B(8)$. □

Lemma 52 $424, 425, 432, 433 \in B(\{8, 9\})$.

Proof: Remove $0, 1, 8$ and 9 points from the B(433, 9) design [2]. □

Lemma 53 $434, 440, 441, 442 \in B(\{8, 9\})$.

Proof: Greig [59] gives a 9-GDD of type 9^{49}, so we obtain 441. Delete any point to obtain 440. Add an infinite point to the groups to obtain 442, and delete eight points from a 9-block in this PBD to obtain 434. □

Lemma 54 $[456, 471] \subseteq B(\{8, 9, 10\})$.

Proof: Apply Theorem 30, taking $m = 3$; we have $f(1) = 513$, $f(2) = 1065$, $f(3) = 1653$, $f(4) = 2274$, $f(5) = 2925$ and $f(6) = 3603$. They are all less than 3705 which is number of exterior points. Hence $[456, 463] \subseteq B(\{8, 9, 10\})$. Apply Corollary 13 with $a = 10$ and $k = 1, 2, 3, 4, 5$ respectively to obtain $[467, 471] \subseteq B(\{8, 9, 10\})$. $456 \in RB(8)$ has more than 10 resolution classes, and hence $[464, 466] \subseteq B(\{8, 9, 10\})$. □

Lemma 55 $[504, 506] \subseteq B(\{8, 9, 10\})$.

Proof: Since $513 \in RB(9)$, deleting eight or nine points from a block gives 505 or 504. Instead adding an infinite point to one parallel class and then deleting eight points from a 9-block gives 506. □

Lemma 56 $569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579 \in B(\{8, 9, 10\})$.

Proof: By a simple counting argument, there is a TD$(13, 64)$ containing a $(5, \{0, 1, 2\}, 57, 1, 1, 1, 1)$-thwart. This gives $[569, 573] \subseteq B(\{8, 9, 10\})$. Take a TD$(10, 64)$ and truncate two groups to $56, a$ where $a \in \{7, 8, 9\}$ and add a point at infinity to obtain $[576, 578] \subseteq B(\{8, 9, 10\})$. Take a TD$(10, 64)$ and truncate two groups to 57 and 10 to obtain $579 \in B(\{8, 9, 10\})$. Finally, take a RB$(513, 9)$ design. Remove two points in a block and zero points or one point in another block in the same resolution class. This gives at most three blocks of size seven. $574, 575 \in B(\{8, 9, 10\})$ can be obtained by adding 64 infinite points to the resulting design. □

Lemma 57 $583 \in B(\{8, 9, 10\})$.

Proof: Truncate two groups of TD$(10, 71)$ to seven points each, and fill the groups using one infinite point. □

Lemma 58 *If q is a prime power and $q \in B(\{8, 9, 10\})$ then* $[8q, 9q-7] \subseteq B(\{8, 9, 10\})$.

Proof: Apply Theorem 5 with $n = 8$. □

Lemma 59 $[512, 569] \cup [584, 866] \cup [968, 1514] \cup [1864, 2162] \cup [2312, 2594] \cup [2824, 3170] \subseteq B(\{8, 9, 10\})$

Proof: Apply Lemma 58 with $q = 64, 73, 81, 89, 97, 121, 128, 137, 151, 169, 233, 241, 289, 353$. □

Lemma 60 $[867, 967] \in B(\{8, 9, 10\})$.

Proof: Take a TD$(10, a)$ and truncate two groups to sizes x and y. For [867,910], take $a \in \{97, 100\}$, $x \in \{0, 10\}$ and $y \in [80, a]$ For 912, take $a = 97$, $x = 64$ and $y = 72$. For [913,945] take $a = 97$, $x \in \{57, 72\}$ and $y \in [80, 97]$. For [946,967] take $a = 97$ and $x, y \in [80, 97]$. Finally, for 911 apply Theorem 30 to add 7 points to a Denniston arc. □

Lemma 61 $[1512, 1664] \subseteq B(\{8, 9, 10\})$.

Proof: Any integer in [160,.312] can be written as sum of two integers a, b where $a, b \in B(\{8, 9, 10\})$. So take a TD$(10, 169)$ and truncate two groups to a, b points. □

Lemma 62 *There exists* $\{8, 9, 10\}$*-GDD of type* 9^8a^1, 9^9a^1 *and* $9^{10}a^1$ *where* $0 \leq a \leq 9$.

Proof: Take a TD$(9, 9)$ and TD$(10, 9)$. Truncate one group to obtain $\{8, 9, 10\}$-GDDs of types 9^8a^1 and 9^9a^1. Take a TD$(10, 11)$, remove a block and truncate one group. Use one deleted point to define groups to obtain $\{8, 9, 10\}$-GDDs of type $9^{10}a^1$. □

Lemma 63 $[1648, 1864] \in B(\{8, 9, 10\})$.

Proof: Take a TD$(11, 19)$ and apply weight nine to first eight groups, assign weight zero or nine in two groups and assign arbitrary weights from $\{0, 1, \ldots, 9\}$ to the last group. All required ingredients exist by Lemma 62. Hence, we obtain a $\{8, 9, 10\}$-GDD of type $(171)^8(9a)^1(9b)^1c^1$ where $0 \leq a, b \leq 19$ and $0 \leq c \leq 171$. Choose

$a, b \in \{0,1,8,9,10,11,16,17,18,19\}$ and $c \in [136,154]$. Then $9a, 9b \in B(\{8,9,10\})$ and

$$a+b \in \{16,17,18,19,20,21,22,24,25,26,27,28,29,30,32,33,34,35,36,37,38\}.$$

Then $[1368+280, 1368+496] \subseteq B(\{8,9,10\})$. □

Let $E_{8,9,10} = [11,56] \cup [58,63] \cup [66,71] \cup [75,79]$ and $X_{8,9,10} = [101,109] \cup [111,113] \cup [115,119] \cup [126,127] \cup [133,135] \cup [155,160] \cup [166,167] \cup [173,231] \cup \{239\} \cup [247,287] \cup [290,295] \cup [299,343] \cup [346,351] \cup [355,399] \cup [403,407] \cup [411,423] \cup [426,431] \cup [435,439] \cup [443,448] \cup [452,455] \cup [472,497] \cup [499,503] \cup [507,511] \cup [580,582]$.

Combining the above lemmas, we obtain the following theorem.

Theorem 32 *For any integer $v \geq 11$, $v \in B(\{8,9,10\})$ with the possible exceptions in $X_{8,9,10}$ and definite exceptions in $E_{8,9,10}$.*

2.5 Some Non-Existence Results

In this section, we prove some non-existence results to supplement the results in [98] and [17].

An *incidence structure* is a triple $D = (V, \mathcal{B}, I)$ where V and $\mathcal{B}$ are any two disjoint sets and I is a binary relation between V and $\mathcal{B}$. The elements of V are called *points*, those of $\mathcal{B}$ *blocks* and those of I *flags*.

Give an incidence structure, we define the dual as follows: Let $V' = \mathcal{B}$ and $\mathcal{B}' = V$, we define I' as $(p', B') \in I'$ if and only if $(B, p) \in I$. We call $D' = (V', \mathcal{B}', I')$ a *dual incidence structure of D*.

By taking the dual of any $(v, k, 1)$-packing design, we can obtain another incidence structure which is often a lot easier to analyze.

We denote $OQ_{\geq 5}$ be the set of odd prime power greater than or equal to five and $Q_{\geq 5}$ be the set of prime power greater than or equal to five.

The following theorem is very useful.

Theorem 33 *[51] Let K be a set of positive integers and let m denote the smallest integer in K. Suppose that there exists a PBD(v, K) which contains blocks B_h and B_k of sizes h and k, respectively. Then*

1. *$v \geq (m-1)k + h - m + 1$; hence*
2. *$v \geq (m-1)k + 1$, with equality if and only if there exists a resolvable BIBD$(k(m-2)+1, m-1, 1)$;*
3. *if B_h and B_k do not intersect, then $v \geq (m-1)k + h$.*

Lemma 64 $48 \notin B(\{4, 9\})$.

Proof: Suppose to the contrary that a PBD$(48, \{4, 9\})$ design exists. Let x be a point of the design and r_i be the number of blocks of size i that point x is on. Evidently, $47 = 3r_4 + 8r_9$ by considering the neighbours of a point. This gives $r_9 \equiv 1 \pmod 3$. Hence, every point in on at least one block of size nine. Let b be the number of blocks of size nine. Since every point is on at least one block of size nine, we must have $b \geq 6$. Let a_i be the number of points in the design so that it is on i blocks of size nine. We have shown that $a_{3k} = a_{3k+2} = 0$ for all k positive integer. Note that $a_i = 0$ for all $i \geq 7$ since otherwise, there are more than 48 points in the design. So, we have the relation $48 = a_1 + a_4$. Also, we know that

$9b = a_1 + 4a_4$. Solving yields $a_4 = 3b - 16$. Now, we consider only blocks of size nine. In the dual, it forms a packing design with b points and $3b - 16$ blocks of size 4 with replication number at most 9. The packing number for v points is at most $\lfloor \frac{v}{4} \lfloor \frac{v-1}{3} \rfloor \rfloor$. We know that $3b - 16 \leq 48$. A simple check reveals that for such range of b, it is always impossible to have a packing of the given size. □

Lemma 65 $39 \notin B(\{5, 7\})$.

Proof: Suppose to the contrary that there exists a PBD$(39, \{5, 7\})$. Let x be a point in the design. By considering the neighbours of x, x is on an odd number of blocks of size seven. Let b_i be the number of blocks of size i in the design. By counting pairs, we obtain $741 = 10b_5 + 21b_7$. This means $b_7 \equiv 1 \pmod{10}$. The possibility of $b_7 = 1$ is ruled out immediately because every point is on at least one block of size seven. Every point is either on one, three or five blocks of size seven. If a point is on at least seven blocks of size seven, then the design must have at least 43 points. Let a_i be the number of points on i blocks of size seven. If b_7=11, then $77 = a_1 + 3a_3 + 5a_5$ and $a_1 + a_3 + a_5 = 39$ imply that $19 = a_3 + 2a_5$. Next, we consider the dual incidence structure; it is a packing with block sizes three or five. A block of size five can be replaced by two blocks of size three. If we ignore the condition on replication, then we must be able to pack 19 triples on 11 points and this is impossible since the packing number is 17 [95]. If $b_7 = 21$, then $147 = a_1 + 3a_3 + 5a_5$ and this means $54 = a_3 + 2a_5$. Since, $a_3 + a_5 \leq 39$; this means $a_5 \geq 15$ and $a_3 = 54 - 2a_5$. If the design exists, we must be able to pack a_5 blocks of size five and $54 - 2a_5$ blocks of size three in 21 points. However, there are exactly 210 unordered pairs on 21 points. On the other hand, a_5 blocks of size five give $10a_5$ pairs and $54 - 2a_5$ triples give $3(54 - 2a_5)$ pairs. As $a_5 \geq 15$, the number of pairs is always greater than 210. Hence, $b_7 = 21$ is impossible. If $b_7 \geq 31$, the relations $a_3+a_5 \leq 39$ and $2a_3+4a_5 = 7b_7-39$ do not have any solution in positive integers. $\square$

Lemma 66 $49 \notin B(\{5,8\})$.

Proof: Suppose to the contrary that a PBD$(49, \{5,8\})$ exists. Let b_i be the number of blocks of size i in the design. By counting pairs, we must have $1176 = 10b_5+28b_8$. This gives $b_8 \equiv 2 \pmod 5$. Also, every point must on either zero blocks or four blocks of size eight by considering the neighbour of a point. Let a_i be the number of points on i blocks of size eight. We have $a_0 + a_4 = 49$. Also, $8b_8 = 4a_4$ giving $a_4 = 2b_8$. This forces $b_8 = 2, 7, 12, 17$ or 22. Since there is at lesat one point on four blocks of size eight, then there must be at least 29 points on at least one block of size eight. By considering another point of the 29 points, we see that there must be at least 41 points on four blocks of size eight. Hence $a_4 \geq 41$ and this eliminates $b_8 = 2, 7, 12, 17$. Suppose $b_8 = 22$, by considering the dual strucure forms by the blocks of size eight, we must have 44 blocks of size four packed in 22 points. This violates the packing bound [95]. $\square$

Lemma 67 $52 \notin B(\{5,8\})$.

Proof: Suppose to the contrary that a PBD$(52, \{5,8\})$ exists. Let b_i be the number of blocks of size i in the design. By counting pairs, we must have $1326 = 10b_5+28b_8$. Hence, $b_8 \equiv 2 \pmod 5$. Also, every point is on one or five blocks of size eight. Let a_i be the number of points on i block of size eight. We have $a_1 + a_5 = 52$. Also, $8b_8 = a_1 + 5a_5$. This gives $a_5 = 2b_8 - 13$. It implies $b_8 = 2, 7, 12, 17, 22, 27$ or 32. A upper bound for the packing number is $\lfloor \frac{v}{5} \lfloor \frac{v-1}{4} \rfloor \rfloor$ [95]. If $b_8 = 7$, then this means there are seven blocks of size eight and five of them intersect in one point. However, the remaining 16 points are partitioned by two blocks of size eight. By considering the point on five blocks of size eight, it is impossible to have any block of size five passing through that point becuase the remaining 16 points are partitioned in two blocks of eight. If $b_8 = 12$, $a_8 = 11$ and packing bound is four. If $b_8 = 17$, then $a_8 = 21$ and packing bound is 13. If $b_8 = 22$, then $a_8 = 31$ and packing bound is 22. If $b_8 = 27$, then $a_8 = 41$ and packing bound is 26. If $b_8 = 32$, then $a_8 = 51$ and packing bound is 31. Hence, no such design exists. □

Lemma 68 $39 \notin B(OQ_5)$ *and in particular* $39 \notin B(\{5,7,9\})$.

Proof: We have shown that $39 \notin B(\{5,7\})$ (Lemma 65). If $39 \in B(\{5,7,9\})$, then it must contain a block of size nine. Using Theorem 33, there cannot be another block of size nine. Also, every block must intersect the block of size nine. By removing the block of size nine, we obtain a PBD$(30, \{4,6\})$ with nine parallel classes. Let the parallel types are: $A : 6^5$, $B : 6^3 4^3$ and $C : 6^1 4^6$. Let a, b, c be the number of parallel classes of type A, B, C respectively. We must have $a + b + c = 8$ and $435 = 75a + 63b + 51c$. However, this set of equations has no solution in positive integer. Hence, $39 \notin B(\{5,7,9\})$. Now, if $39 \in B(OQ_5)$, then it must contain a block of size $h \geq 11$. We obtain a contradiction by using Theorem 33. □

Lemma 69 $44 \notin B(\{5,8,9\})$.

Proof: Suppose to the contrary that such a design exists. Consider the point type of a point x: $43 = 4r_5 + 7r_8 + 8r_9$ where r_i denotes number of blocks of size i through point x. This means that every point is on 1 (mod 4) of block of size eight. Since 44 is not a multiple of eight, so there must be a point on five blocks of size eight since if there exists a point on at least nine blocks of size eight, then the design has at least 50 points. By removing the point on five blocks of size eight, it is a $\{5, 8, 9\}$-GDD of type $7^5 4^2$. However, every point on a block of size four must be on a block of size eight. This is impossible because there are only seven groups. $\Box$

Lemma 70 $38 \notin B(Q_{\geq 5})$ *and* $38 \notin B(\{5, 7, 8, 9\})$

Proof: Suppose to the contrary that such a design exists. By Theorem 33, it cannot have a block of size at least eleven. If it has a block of size nine, then by Theorem 33, we can conclude that all other blocks must have size five. But it is known that $38 \notin B(\{5, 9\})$ [19]. Hence, this shows that it can not have a block of size nine. We now show that $38 \notin B(\{5, 7, 8\})$. Every point is on odd number of blocks of size eight by considering the degree of any point in the design. Let b_8 be the number of blocks of size eight. It is evident that $b_8 \geq 5$. If $b_8 = 5$, it must be the case that exactly one point lies on three blocks of size eight. But a pair containing the point of intersection and a point in other two blocks of size eight cannot occur in a block. If $b_8 \neq 5$, no point can be on five or more blocks of size eight, or otherwise, the design must have more than 38 points. Hence, every point must be on either one or three blocks of size eight. A simple counting reveals that the number of points on three blocks of size eight is $4b_8 - 19$. By considering the dual incidence structure of the block of size eight, we must be able to pack $4b_8 - 19$ triples on b_8 points. It is impossible for $b_8 \leq 14$. However, if $b_8 \geq 15$, then $4b_8 - 19 \geq 39$. $\Box$

Lemma 71 $37 \notin B(\{5, 7, 8\})$.

Proof: It has been shown in [58] that $37 \notin B(\{5, 7\})$. If $37 \in B(\{5, 7, 8\})$, then it must has a block of size eight. Since every point must be on even number of blocks of size eight, it must have at least nine blocks of size eight. (In fact, if it exists, it must have exactly nine blocks of size eight, otherwise the design would have more than 37 points.) The structure of the blocks of size eight must be the 'dual-8-arc' as in Section 2.1. However, any point on two blocks of size eight must also be on a block of size seven. Now one can not have a tranverse block of size seven. □

2.6 Pairwise Balanced Designs with Holes

In this section, we consider a problem of Hartman and Heinrich on pairwise balanced designs with holes.

Let $\mathbb{Z}_{\geq 3}$ be the set of all integers that are at least three. The problem of constructing designs PBD$(v, \mathbb{Z}_{\geq 3} \cup \{k^*\})$ was considered by Hartman and Heinrich in [64], where the following result is established.

Theorem 34 *A $PBD(v, \mathbb{Z}_{\geq 3} \cup \{k^*\})$ exists if and only if $v \geq 2k+1$ except when*

(i) $v = 2k+1$ *and* $k \equiv 0 \pmod 2$*;*

(ii) $v = 2k+2$ *and* $k \not\equiv 4 \pmod 6$, $k > 1$*;*

(iii) $v = 2k+3$ *and* $k \equiv 0 \pmod 2$, $k > 6$*;*

(iv) $(v, k) \in \{(7,2), (8,2), (9,2), (10,2), (11,4), (12,2), (13,2)\}$, *and possibly when* $(v, k) \in \mathcal{P} = \{(17,6), (21,8), (26,9), (28,11), (29,10), (29,12), (30,11), (33,14), (35,12), (37,14), (38,13), (39,14), (42,17), (47,18), (49,20), (55,20)\}$.

The possible exception $(v, k) = (17, 6)$ in Theorem 34 was subsequently removed by Heathcote [66] who showed that there cannot exist a PBD$(17, \mathbb{Z}_{\geq 3} \cup \{6^*\})$. Since then, there remain fifteen pairs $(v, k) \in \mathcal{P}$ for which the existence of a PBD$(v, \mathbb{Z}_{\geq 3} \cup \{k^*\})$ is undetermined. We construct PBDs settling the problem for all of the pairs in $\mathcal{P}$.

The strategy we used in constructing a PBD$(v, \mathbb{Z}_{\geq 3} \cup \{k^*\})$ $(\mathcal{X}, \mathcal{B})$ is to completely specify the set of blocks $\mathcal{A} \subseteq \mathcal{B}$ with sizes greater than three, that is, $\mathcal{A} = \{B \in \mathcal{B} \mid |B| \geq 4\}$. Following [61], we call the partial design $(\mathcal{X}, \mathcal{A})$ the *prestructure* of the PBD. The remaining blocks of size three (*triples*) are then filled in by a variant of Stinson's hillclimbing algorithm [110] similar to the one described in [61].

The most difficult task in the construction of PBD$(v, \mathbb{Z}_{\geq 3} \cup \{k^*\})$ is the determination of suitable prestructures. The prestructures $(\mathcal{X}, \mathcal{A})$ used in this paper are constructed manually, taking into account the following elementary conditions that must be satisfied:

(a) $\sum_{A \in \mathcal{A}} \binom{|A|}{2} \equiv \binom{v}{2} \pmod 3$;

(b) for every $x \in \mathcal{X}$, $\sum_{A \in \mathcal{A} | x \in A} (|A| - 1) \equiv v - 1 \pmod 2$.

In Table 2.1 and 2.2, we give prestructures of designs PBD$(v, \mathbb{Z}_{\geq 3} \cup \{k^*\})$ for which the hillclimbing algorithm succeeds in completing them to PBDs. In each case, the prestructure consists of only one block of size k, and the remaining blocks have sizes four and five. The point-set of a PBD of order v is taken to be the set consisting of the first v elements of $P = \{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}, \mathtt{A}, \mathtt{B}, \ldots, \mathtt{Z}, \mathtt{1}, \mathtt{2}, \mathtt{3}\}$. The block of size k in each prestructure is the set consisting of the first k elements of P, and we omit it from the listing in Table 2.1 and 2.2.

Given these prestructures, it is easy to complete them with triples to PBDs using hillclimbing. Our program, running on a DEC 2000 4/200 Alpha system, took less

(v,k)	$(21,8)$	$(26,9)$	$(28,11)$	$(29,10)$	$(29,12)$	$(30,11)$	$(33,14)$	$(35,12)$
	aijkl	amouz	amxyz	alszC	erstv	anuvw	auvwx	aopqr
	bimno	ajkl	ilvAB	akpu	amqx	klmzA	bAEFG	amsy
	ampq	bjmn	alot	bksv	bmry	alot	aotA	bmtA
	anrs	cjop	blps	clpw	cmsz	blpu	bouC	cnuC
	aotu	dkqs	clqu	dlqu	dmtA	clqv	covE	dnvz
	bjpr	emqt	dmpv	emrv	enqy	dmrw	dowG	eowB
	bkqt	fquv	emqs	fmsw	fnrx	empx	epxB	foxD
Blocks in	blsu	gkrw	fmrw	gnqx	gosB	fmqy	fpyD	gpsA
prestructure	cnpu	hmrx	gnpx	hnty	hotC	gnrz	gpzF	hptC
	doqr	iryz	hnqy	iory	ipvz	hnpA	hptE	iquz
	emst		inrz	jotx	jpwA	inqB	iquG	jqvB
	fkps		jorA		kuvB	jorC	jqvB	krwD
	gjqu		kosB		luwC	kosD	krwD	lrxy
	hlrt						lrxF	
							msyA	
							nszC	

Table 2.1: Prestructures for PBD$(v, \mathbb{Z}_{\geq 3} \cup \{k^*\})$ (i)

than two seconds on the largest design. For the actual blocks of the design, see [34].

2.7 Direct Constructions for Pairwise Balanced Designs

In this section, we present some direct constructions of pairwise balanced designs. These constructions supplement the result in [17].

Lemma 72 *There exists a {5,6}-GDD of type* 5^{28}.

Proof: Let $V = \mathbb{Z}_{135}$ and develop the following blocks over $\mathbb{Z}_{135}$:

$$\{0,10,25,70,66,102\},\ \{0,5,55,6,128\},$$
$$\{0,30,21,61,112\},\ \{0,20,37,72,34\},$$
$$\{0,11,127,68,39\},\ \{0,16,42,18,64\},$$
$$\{0,27,54,81,108\}.$$

The last three blocks generate 11 parallel classes of block size five. Add five infinite points to obtain a $\{5,6\}$-GDD of type 5^{27}. □

Lemma 73 *There exists a PBD*$(103, \{5,7\})$.

Proof: Let $V = \mathbb{Z}_{103}$. Develop the following blocks over $\mathbb{Z}_{103}$:

$$\{0,1,46,56,6,70,27\},\ \{0,2,18,67,80\},$$
$$\{0,92,4,95,75\},\ \{0,9,81,44,51\}.$$

□

Lemma 74 *There exists a PBD*$(123, \{5,7\})$.

Proof: Let $V = \mathbb{Z}_{123}$. Develop the following blocks over $\mathbb{Z}_{123}$:

$$\{0,3,9,21,36,19,80\},\ \{0,24,75,25,109\},$$
$$\{0,30,7,88,83\},\ \{0,45,13,67,41\},$$
$$\{0,57,49,112,20\}.$$

□

Lemma 75 *There exists a PBD*$(163, \{5, 7\})$.

Proof: Let $V = \mathbb{Z}_{163}$. Develop the following blocks over $\mathbb{Z}_{163}$:

$$\{0,1,58,104,5,127,31\},\ \{0,2,8,18,21\},$$
$$\{0,116,138,66,77\},\ \{0,45,17,79,65\},$$
$$\{0,7,49,81,119\},\ \{0,80,71,134,56\},$$
$$\{0,76,43,111,151\}.$$

□

Lemma 76 *There exists a PBD*$(223, \{5, 7\})$.

Proof: Let $V = \mathbb{Z}_{223}$. Develop the following blocks over $\mathbb{Z}_{223}$:

$$\{0,1,183,39,3,103,117\},\ \{0,4,9,16,24\},$$
$$\{0,63,86,29,155\},\ \{0,156,128,178,44\},$$
$$\{0,6,37,93,119\},\ \{0,206,81,71,146\},$$
$$\{0,11,105,59,181\},\ \{0,13,32,62,83\},$$
$$\{0,149,58,196,25\},\ \{0,61,133,188,115\}.$$

□

Lemma 77 *There exists a PBD*$(197, \{5, 8\})$.

Proof: Let $V = \mathbb{Z}_{197}$. Develop the following blocks over $\mathbb{Z}_{197}$:

$$\{0,5,126,102,167,32,176,180\},\ \{0,1,3,46,88\}.$$

Multiply the second block by 104^i for $i = 1,2,3,4,5,6$ to obtain six more blocks. □

Lemma 78 *There exists a PBD*$(133, \{5,9\})$.

Proof: Let $V = \mathbb{Z}_{133}$. Develop the following blocks over $\mathbb{Z}_{133}$:

$$\{1,11,121,5,55,73,44,85,4\},\ \{0,2,28,37,75\},$$
$$\{0,22,42,8,27\},\ \{0,109,63,88,31\}.$$

□

Lemma 79 *There exists a PBD*$(193, \{5,9\})$.

Proof: Let $V = \mathbb{Z}_{193}$. Develop the following blocks over $\mathbb{Z}_{193}$:

$$\{1,108,84,5,154,34,12,138,43\},\ \{0,2,10,28,55\},$$
$$\{0,23,115,129,150\},\ \{0,168,68,36,181\},$$
$$\{0,1,6,77,136\},\ \{0,108,69,17,20\},$$
$$\{0,84,118,99,37\}.$$

□

Lemma 80 *There exists a PBD*$(129, \{5,17^{\star}\})$.

Proof: Let $V = \mathbb{Z}_{112}$. Develop the following blocks over $\mathbb{Z}_{112}$:

$$\{0,1,3,10\},\ \{0,5,11,34\},$$
$$\{0,14,49,67\},\ \{0,19,41,74\},$$
$$\{0,28,56,84\},\ \{0,4,24,36,66\},$$
$$\{0,8,48,73,99\},\ \{0,16,60,97,43\}.$$

The blocks of size four generate 17 parallel classes. Attach 17 infinite points to obtain a PBD$(129, \{5, 17^*\})$. □

Lemma 81 *There exists a PBD*$(125, \{5, 21^*\})$ *and a PBD*$(115, \{4, 5, 11^*\})$.

Proof: Let $V = \mathbb{Z}_{104}$. Develop the following blocks over $\mathbb{Z}_{104}$:

$$\{0,1,3,10\},\ \{0,5,11,38\},$$
$$\{0,13,30,55\},\ \{0,15,46,65\},$$
$$\{0,18,41,75\},\ \{0,26,52,78\},$$
$$\{0,4,28,40,60\},\ \{0,8,53,69,90\}.$$

The blocks of size four generate 21 parallel classes. Attach either 11 or 21 infinite points to obtain the result. □

Lemma 82 *There exists a PBD*$(149, \{5, 17^*\})$.

Proof: Let $V = \mathbb{Z}_{132}$. Develop the following blocks over $\mathbb{Z}_{132}$:

$$\{0,4,1,9,22\},\ \{0,12,29,61,102\},$$
$$\{0,16,52,76,50\},\ \{0,20,48,88,125\},$$
$$\{0,6,25,87\},\ \{0,10,53,67\},$$
$$\{0,11,46,85\},\ \{0,15,38,69\},$$
$$\{0,33,66,99\}.$$

The blocks of size four generate 17 parallel classes. Add 17 infinite points to obtain the result. □

Lemma 83 *There exists a PBD*$(169, \{5, 17^*\})$.

Proof: Let $V = \mathbb{Z}_{152}$. Develop the following blocks over $\mathbb{Z}_{152}$:

$$\{0,4,1,9,22\},\ \{0,12,29,45,2\},$$
$$\{0,20,64,100,70\},\ \{0,24,56,84,103\},$$
$$\{0,7,62,85\},\ \{0,15,61,86\},$$
$$\{0,26,57,115\},\ \{0,34,69,111\},$$
$$\{0,40,93,141,54\},\ \{0,38,76,114\}.$$

The blocks of size four generate 17 parallel classes. Add 17 infinite points to obtain the result. □

(v, k)	$(37,14)$	$(38,13)$	$(39,14)$	$(42,17)$	$(47,18)$	$(49,20)$	$(55,20)$
Blocks in prestructure	auvwx	zABCD	auvwx	rstuv	KLMNO	DEFGH	DEFGH
	bAEFG	anrz	bAEFG	LMNOP	asBK	STUVW	STUVW
	aotA	bnsA	aotA	arwF	bsCM	auDN	auDN
	bouC	cntB	bouC	brxC	ctDO	buEP	buEP
	covE	dorC	covE	cryB	dtEQ	cuFR	cuFR
	dowG	eosD	dowG	dszC	euFS	duGT	duGT
	epxB	fotE	epxB	esxB	fuGL	evHV	evHV
	fpyD	gpuF	fpyD	fsyG	gvHN	fvIO	fvIO
	gpzF	hpsG	gpzF	gtBH	hvIP	gwJQ	gwJQ
	hptE	iptH	hptE	htxI	iwJR	hwKS	hwKS
	iquG	jqvI	iquG	ityC	jwBM	ixLU	ixLU
	jqvB	kqwJ	jqvB	juzA	kxCO	jxMW	jxMW
	krwD	lqxK	krwD	kuDJ	lxDQ	kyDP	kyDP
	lrxF	mryL	lrxF	luEK	myES	lyER	lyER
	msyA		msyA	mvzL	nyFL	mzFT	mzFT
	nszC		nszC	nvAM	ozGN	nzGV	nzGV
				ovDN	pzHP	oAHO	oAHO
				pwDO	qAIR	pAIQ	pAIQ
				qwAP	rAJK	qBJS	qBJS
						rBKU	rBKU
						sCLW	sCLW
						tCMN	tCMN

Table 2.2: Prestructures for PBD$(v, \mathbb{Z}_{\geq 3} \cup \{k^*\})$ (ii)

Chapter 3

Group Divisible Designs

In this chapter, we study the existence of group divisible designs and modified group divisible designs.

Group divisible designs have been instrumental in the construction of other types of designs. Many researchers have been involved in investigating the existence of group divisible designs. Our interest here is in the existence of uniform GDDs with block size k. Simple counting arguments show that if there is a uniform k-GDD of type g^u with $u > 1$, then

$$\begin{aligned} u &\geq k \\ (u-1)g &\equiv 0 \bmod (k-1) \\ u(u-1)g^2 &\equiv 0 \bmod k(k-1). \end{aligned} \tag{3.1}$$

The necessary conditions for the existence of a uniform k-GDD of type g^u have been proved to be sufficient for $k = 3$ and 4 [63, 30], with the definite exception of 4-GDDs of type 2^4 and 6^4. However, little was known about the case $k = 5$ other than the following result due to Hanani [63].

Theorem 35 *If $q \equiv 1 \pmod 4$ is a prime power, then there exists a 5-GDD of type 5^q.*

In first two section of this chapter, we construct 5-GDDs of type g^u. From (3.1), the necessary conditions for the existence of such a GDD with $u > 1$ are tabulated here:

$$\begin{array}{cc} g \pmod{20} & \text{Condition on } u \\ 0 & u \geq 5 \\ 1,3,7,9,11,13,17,19 & u \equiv 1,5 \pmod{20} \\ 2,6,14,18 & u \equiv 1,5 \pmod{10} \\ 4,8,12,16 & u \equiv 0,1 \pmod 5 \\ 5,15 & u \equiv 1 \pmod 4 \\ 10 & u \equiv 1 \pmod 2,\ u \geq 5 \end{array} \tag{3.2}$$

We establish a strong existence result:

Theorem 36 *The necessary condition (3.2) is also sufficient, except when $g^u \in \{2^5, 2^{11}, 3^5, 6^5\}$, and possibly where*

1. $g^u = 3^{45}$, 3^{65};
2. $g \equiv 2,6,14,18 \pmod{20}$ *and*
   - *(a)* $g = 2$ *and* $u \in \{15,35,71,75,85,95,111,115,135,195,215,335\}$;
   - *(b)* $g = 6$ *and* $u \in \{15,35,45,75,95,115,135\}$;
   - *(c)* $g = 18$ *and* $u \in \{11,15,35,71,111,115,135,195\}$;
   - *(d)* $g = 2\alpha$ *for* $\alpha > 1$ *and* $(\alpha, 30) = 1$, *and* $u \in \{11,15,35,71,75,111,115,135, 195\}$;

(e) $g = 6\gamma$, $\gamma \not\equiv 0 \pmod 5$, $\gamma \neq 3$ *odd, and* $u = 15$;

3. $g \equiv 10 \pmod{20}$ *and*

(a) $g = 10$ *and* $u \in \{5, 7, 15, 23, 27, 33, 35, 39, 47, 63\}$.

(b) $g = 30$ *and* $u \in \{9, 15\}$;

(c) $g = 90$ *and* $u \in \{7, 23, 27, 39, 47\}$;

(d) $g = 10\alpha$, $\alpha \equiv 1, 5 \pmod 6$, *and* $u \in \{7, 15, 23, 27, 35, 39, 47\}$.

(e) $g = 30\gamma$, $\gamma \geq 5$ *odd*, $\gamma \not\equiv 0 \pmod 3$ *or* $\gamma = 9$, $u = 15$.

Using this theorem as a starting point and employing quite different techniques, we show in Section 3 that, for α sufficiently large, 5-GDDs of type $(2\alpha)^u$ exist whenever the basic necessary conditions are met. This leaves a finite (but large) number of possible exceptions for uniform 5-GDDs.

In section 4, we study optimal packing with block size five when $v \equiv 2 \pmod 4$.

In section 4, we study a problem of Assaf concerning modified group divisible designs with block size four. We are able to solve all but a very small list of possible exceptions.

3.1 Direct Constructions

In this section, we present direct constructions for a large number of uniform group divisible designs with block size five.

Lemma 5 *There exist 5-GDDs of type* g^u *for*

1. $g = 2$ *and* $u = 41$;

2. $g = 8$ *and* $u \in \{10, 11, 15, 16, 20\}$;

3. $g = 10$ *and* $u \in \{9, 13, 17\}$;

4. $g = 12$ *and* $u \in \{10, 11, 15, 16, 20\}$; *and*

5. $g = 15$ *and* $u = 9$.

Proof: Let $v = g \cdot u$. For $g = 8, 12$, $u \in \{10, 15, 20\}$ we take $X = \mathbb{Z}_{v-g}$ plus g infinite points; there is one group on the infinite points and other groups consist of points which are equal modulo $(v-g)/g$. In these cases, all base blocks of size four have elements which are distinct modulo 4, and hence generate four parallel classes each on the non-infinite points. For the others, $X = \mathbb{Z}_v$ and groups consist of points which are equal modulo v/g.

2^{41}: {0,1,3,7,34}, {0,5,16,30,70}, {0,8,23,43,61}, {0,9,19,45,69}

8^{10}: {0,1,3,13,35}, {0,4,20,28,43}, {0,5,19,26}, {0,6,17,47}

8^{11}: {0,1,3,7,59}, {0,5,23,51,68}, {0,8,21,35,47},{0,9,19,57,73}

8^{15}: {0,4,9,10,12}, {0,20,37,52,88}, {0,29,48,55,78}, {0,22,40,65,81}, {0,13,46,67}, {0,11,38,73}

8^{16}: {0,1,3,7,12}, {0,8,18,45,79}, {0,13,38,53,100}, {0,14,35,74,98}, {0,17,43,76,99}, {0,19,50,70,92}

8^{20}: {0,16,39,48,88}, {0,4,12,46,83}, {0,13,20,67,137}, {0,22,24,65,90}, {0,33,60,135,149}, {0,18,29,74,126}, {0,1,6,59}, {0,21,31,122}

10^{9}: {0,1,3,8,58}, {0,4,21,51,70}, {0,6,16,29,44}, {0,11,25,37,59}

10^{13}: {0,1,3,7,12}, {0,8,18,43,80}, {0,14,44,63,90}, {0,15,38,60,94}, {0,16,33,75,99}, {0,20,41,73,102}

10^{17}: {0,1,3,7,12}, {0,13,39,91,156}, {0,8,18,54,115}, {0,22,64,104,135}, {0,15,45,77,101}, {0,25,75,123,151}, {0,16,37,96,137}, {0,38,58,81,141}

12^{10}: {0,4,6,16,73}, {0,8,52,76,101}, {0,20,37,58,80}, {0,3,29,34}, {0,1,14,47}, {0,11,30,53}

12^{11}: {0,1,3,7,15}, {0,5,18,39,68}, {0,9,36,61,92}, {0,10,42,72,95}, {0,16,51,75,94}, {0,17,45,65,91}

12^{15}: {0,4,20,67,85}, {0,8,9,68,104}, {0,5,12,27,29}, {0,13,32,58,106}, {0,69,80,113,119}, {0,46,86,117,138}, {0,25,35,78}, {0,3,37,114}, {0,41,79,102}

12^{16}: {0,2,24,37,86}, {0,4,25,65,140}, {0,6,15,124,125}, {0,8,66,107,145}, {0,12,39,90,162}, {0,18,44,63,139}, {0,20,23,54,111}, {0,28,87,98,187}, {0,29,36,46,178}

12^{20}: $\{0,4,59,92,131\}$, $\{0,5,12,86,184\}$, $\{0,18,34,204,217\}$, $\{0,54,85,219\}$ (multiply by 1, 49, and 121 $\pmod{228}$)

15^{9}: {0,1,8,20,30}, {0,2,28,60,93}, {0,3,40,55,79}, {0,4,38,51,86}, {0,5,21,71,94}, {0,6,17,31,74}

□

Lemma 6 *[8] There exists a 5-GDD of type* 2^{21}.

Proof: Take as point set $\{0,1\} \times \mathbb{Z}_{21}$, and as groups $\{0,1\} \times \{y\}$ for $y \in \mathbb{Z}_{21}$. Develop the following blocks mod $(-, 21)$:

$$\{(0,0),(0,2),(0,5),(0,11),(1,4)\},\ \{(0,0),(1,1),(1,3),(1,7),(1,12)\},$$
$$\{(0,0),(0,1),(0,8),(1,16),(1,19)\},\ \{(0,0),(0,4),(1,9),(1,10),(1,17)\}$$

$\square$

Lemma 7 *If $q = 25$, 45, or 65, then there exists a 5-GDD of type 2^q.*

Proof: In each case, the point set is $X = (\{0,1\} \times \mathbb{Z}_{q-1}) \cup \{\infty_1, \infty_2\}$, and the groups are $\{x\} \times \{y, y + (q-1)/2\}$, for $x \in \{0,1\}$, $0 \le y \le (q-1)/2 - 1$, plus $\{\infty_1, \infty_2\}$. Develop the following blocks mod $(-, q-1)$:

$q = 25$:

$$\{(0,0),(0,1),(0,3),(0,7),(1,1)\},\ \{(0,0),(1,3),(1,9),(1,10),(1,23)\},$$
$$\{(0,0),(0,8),(1,14),(1,16),(1,19)\},\ \{(0,0),(0,9),(0,14),(1,5),(1,21)\},$$
$$\{(0,0),(0,11),(1,4),(1,13)\}$$

$q = 45$:

$$\{(0,0),(0,2),(0,3),(0,10),(1,3)\},\ \{(0,0),(1,4),(1,5),(1,18),(1,42)\},$$
$$\{(0,0),(0,23),(1,7),(1,32),(1,40)\},\ \{(0,0),(0,11),(0,16),(1,27),(1,30)\},$$
$$\{(0,0),(0,9),(0,13),(0,38),(1,33)\},\ \{(0,0),(1,8),(1,26),(1,31),(1,35)\},$$
$$\{(0,0),(0,20),(1,10),(1,12),(1,22)\},\ \{(0,0),(0,18),(0,30),(1,15),(1,43)\},$$
$$\{(0,0),(0,17),(1,23),(1,38)\}$$

$q = 65$:

$$\{(0,0)(0,2)(0,22)(0,45)(1,8)\},\ \{(0,0)(0,15)(0,26)(0,54)(1,9)\},$$
$$\{(0,0)(0,51)(0,58)(0,63)(1,51)\},\ \{(0,0)(0,16)(0,24)(1,13)(1,14)\},$$

$$\{(0,0)(0,18)(0,47)(1,43)(1,63)\},\ \{(0,0)(0,30)(0,33)(1,2)(1,40)\},$$
$$\{(0,0)(0,9)(1,31)(1,39)(1,55)\},\ \{(0,0)(0,14)(1,17)(1,35)(1,56)\},$$
$$\{(0,0)(0,60)(1,1)(1,24)(1,37)\},\ \{(0,0)(1,4)(1,11)(1,26)(1,38)\},$$
$$\{(0,0)(1,15)(1,34)(1,44)(1,48)\},\ \{(0,0)(1,18)(1,20)(1,23)(1,29)\},$$
$$\{(0,0)(0,37)(1,32)(1,49)\}$$

In each case the last block generates 2 parallel classes on the non-infinite points; add each infinite point to one parallel class. □

Lemma 8 *(Mills; see [102]) There is a* 5*-GDD of type* 2^{31}.

Lemma 9 *There exist 5-GDDs of types* 2^{51} *and* 2^{91}.

Proof: Solutions are given over $\mathbb{Z}_u \times \{0,1\}$ for $u \in \{51, 91\}$.

$u = 51$: The groups are $\{(i,0),(i+32,1)\}$ for $i \in \mathbb{Z}_{51}$. Base blocks are:

$$\{(0,0),(3,0),(7,0),(12,0),(18,0)\},\ \{(0,0),(1,0),(14,0),(22,0),(0,1)\},$$
$$\{(0,1),(3,1),(7,1),(12,1),(2,0)\},\ \{(0,0),(2,0),(19,0),(6,1),(27,1)\},$$
$$\{(0,0),(10,0),(26,1),(28,1),(41,1)\},\ \{(0,0),(16,0),(13,1),(33,1),(39,1)\},$$
$$\{(0,0),(20,0),(3,1),(14,1),(22,1)\},\ \{(0,0),(23,0),(12,1),(30,1),(47,1)\},$$
$$\{(0,0),(24,0),(15,1),(43,1),(44,1)\},\ \{(0,0),(25,0),(9,1),(36,1),(46,1)\}$$

$u = 91$: The groups are $\{(i,0),(i,1)\}$ for $i \in \mathbb{Z}_{91}$. Base blocks are:

$$\{(0,0),(3,0),(7,0),(12,0),(20,0)\},\ \{(0,0),(48,0),(21,0),(10,0),(47,0)\},$$
$$\{(0,0),(40,0),(63,0),(69,0),(24,0)\},\ \{(0,0),(2,0),(1,1),(4,1),(5,1)\},$$
$$\{(0,0),(32,0),(16,1),(64,1),(80,1)\},\ \{(0,0),(57,0),(74,1),(23,1),(6,1)\},$$
$$\{(0,0),(14,0),(7,1),(9,1),(59,1)\},\ \{(0,0),(42,0),(21,1),(53,1),(34,1)\},$$

$\{(0,0),(35,0),(63,1),(29,1),(89,1)\}$, $\{(0,0),(15,0),(33,1),(39,1),(77,1)\}$,
$\{(0,0),(58,0),(73,1),(78,1),(49,1)\}$, $\{(0,0),(18,0),(76,1),(65,1),(56,1)\}$,
$\{(0,0),(19,0),(27,1),(60,1),(88,1)\}$, $\{(0,0),(31,0),(68,1),(50,1),(43,1)\}$,
$\{(0,0),(41,0),(87,1),(72,1),(51,1)\}$, $\{(0,0),(25,0),(13,1),(55,1),(67,1)\}$,
$\{(0,0),(36,0),(26,1),(61,1),(71,1)\}$, $\{(0,0),(30,0),(52,1),(66,1),(44,1)\}$

□

Lemma 10 *A 5-GDD of type 2^{55} exists.*

Proof: Take the point set as $\{0,1\} \times \mathbb{Z}_5 \times \mathbb{Z}_{11}$; let $t_1(x,y,z) = (x,y,z+1)$, and $t_2(x,y,z) = (x,y+1,4z)$. Apply the group of order 55 generated by t_1 and t_2 to the 10 blocks

$$\{(0,0,0),(1,0,0),(1,0,9),(1,2,1),(1,4,10)\}$$
$$\{(0,0,0),(1,0,8),(1,1,9),(1,2,7),(1,3,0)\}$$
$$\{(0,0,0),(0,1,5),(1,2,2),(1,2,4),(1,4,3)\}$$
$$\{(0,0,0),(0,2,9),(1,2,3),(1,3,4),(1,4,6)\}$$
$$\{(0,0,0),(0,1,9),(0,2,8),(1,0,4),(1,0,7)\}$$
$$\{(0,0,0),(0,0,3),(0,1,2),(1,0,5),(1,0,6)\}$$
$$\{(0,0,0),(0,0,5),(0,3,4),(1,2,10),(1,3,6)\}$$
$$\{(0,0,0),(0,0,1),(0,2,7),(1,1,5),(1,1,7)\}$$
$$\{(0,0,0),(0,0,2),(0,4,1),(1,1,3),(1,2,0)\}$$
$$\{(0,0,0),(0,0,4),(0,2,3),(0,4,8),(1,3,3)\}$$

Apply the group of order 11 generated by t_1 to the blocks

$$\{(0,0,z),(0,1,4z),(0,2,5z),(0,3,9z),(0,4,3z)\} \text{ for } z = 0,1, \text{ and}$$
$$\{(1,0,z),(1,1,4z),(1,2,5z),(1,3,9z),(1,4,3z)\} \text{ for } z = 0,2.$$

Groups are of the form $\{(0, y, z), (1, y + 3, z + 9 \cdot (4^y))\}$. □

Lemma 11 *There is a 5-GDD of type 2^{61}, and a 5-GDD of type 2^{81}.*

Proof: Over $\mathbb{Z}_2 \times \mathbb{Z}_{61}$, form base blocks

$$\{(0,0),(0,1),(0,4),(0,25),(1,11)\},\{(0,0),(0,8),(0,23),(1,25),(1,27)\}$$

Multiply by $(1, t)$ for $t = 1, 13, 47$ to obtain three blocks from each.

Over $\mathbb{Z}_2 \times \mathbb{F}_{81}$, form base blocks

$$\{(0,0),(0,1),(0,x),(0,x^3+1),(1,x^3+x^2+2x+1)\},$$
$$\{(0,0),(0,x+1),(0,2x^2+1),(1,2x^3+x+1),(1,2x^3+x^2+2)\}$$

where x is a primitive element satisfying $x^4 = x^3 + 1$. Multiply by $(1, x^{10t})$ for $0 \le t \le 3$ to obtain four blocks from each. □

Lemma 12 *There exists a 5-GDD of type 2^{131}.*

Proof: On $\mathbb{Z}_{131} \times \{0, 1\}$, $\{(i, 0), (i, 1)\}$ for $i \in \mathbb{Z}_{131}$ form groups. Base blocks are obtained as follows. Take

$$\{(0,0),(2,0),(6,0),(14,0),(16,1)\},\{(0,1),(4,1),(9,1),(16,1),(1,0)\},$$
$$\{(0,0),(1,0),(22,0),(19,1),(77,1)\},\{(0,0),(11,0),(91,0),(35,1),(120,1)\},$$
$$\{(0,1),(6,1),(66,1),(30,0),(73,0)\},\{(8,1),(31,1),(71,1),(95,1),(57,1)\}.$$

Multiply these each by the fifth root of unity 53; the first five yield five blocks each, and the last is invariant under multiplication by 53. This gives 26 base blocks, which can be developed over $\mathbb{Z}_{131}$ to obtain the 5-GDD. □

Lemma 13 *There is a 5-GDD of type* 2^{191}.

Proof: On $\mathbb{Z}_{191} \times \{0,1\}$, take $\{(i,0),(i,1)\}$ as groups for $i \in \mathbb{Z}_{191}$. Consider the base blocks

$$\{(0,0),(3,0),(9,0),(20,0),(1,1)\},\ \{(0,1),(2,1),(5,1),(11,1),(3,0)\},$$
$$\{(0,0),(2,0),(15,0),(6,1),(18,1)\},\ \{(0,0),(143,0),(9,1),(28,1),(66,1)\},$$
$$\{(0,0),(71,0),(84,1),(148,1),(173,1)\},\ \{(0,0),(37,0),(17,1),(123,1),(141,1)\},$$
$$\{(0,0),(130,0),(85,1),(92,1),(168,1)\},$$

and the base blocks

$$\{(41,0),(71,0),(95,0),(76,0),(99,0)\},\ \{(5,0),(4,0),(156,0),(163,0),(54,0)\},$$
$$\{(32,0),(102,0),(158,0),(50,0),(40,0)\}$$

Multiply each by the fifth root of unity 39; those in the first set produce five blocks each, while those in the last produce one. The 38 blocks that result are developed under $\mathbb{Z}_{191}$ to produce the GDD. □

Lemma 14 *There is a 5-GDD of type* 2^{211}.

Proof: On $\mathbb{Z}_{211}\times\{0,1\}$, form base blocks from $\{(1,0),(2^{42},0),(2^{84},0),(2^{126},0),(2^{168},0)\}$ and $\{(0,0),(1,0),(2,1),(9,1),(57,1)\}$ by multiplying by 2^6. The first generates seven blocks and the second generates 35. Develop the 42 blocks so obtained under $\mathbb{Z}_{211}$ to obtain the GDD. □

Lemma 15 *If* $q = 21, 41, 61$ *or* 81, *then there exists a 5-GDD of type* 3^q *over* $\mathbb{Z}_3 \times X_q$, *where* $X_q = \mathbb{Z}_q$ *for* $q \in \{21,41,61\}$, *and* $\mathbb{F}_{81}$ *otherwise.*

Proof: $q = 21$:

$$\{(0,0),(0,1),(0,3),(1,2),(1,9)\},\ \{(0,0),(0,4),(0,13),(1,7),(1,17)\},$$
$$\{(0,0),(0,5),(1,10),(1,16),(2,7)\}$$

$q = 41$:

$$B_1 = \{(0,1),(0,10),(0,16),(0,18),(0,37)\},$$
$$B_2 = \{(0,0),(1,20),(1,21),(2,17),(2,24)\}$$

Multiply B_2 by $(1,y)$ for $y = 1,10,16,18,37$.

$q = 61$:

$$\{(0,0),(0,1),(0,28),(1,13),(1,45)\},\ \{(0,0),(0,3),(0,57),(1,33),(1,53)\},$$
$$\{(0,0),(0,6),(1,10),(1,20),(2,39)\}$$

Multiply each of these blocks by $(1,13)$ and $(1,47)$ to produce 6 further base blocks.

$q = 81$: Let x be a primitive element of $\mathbb{F}_{81}$ satisfying $x^4 = x^3 + 1$.

$$B_1 = \{(0,0),(0,x^{16}),(0,x^{32}),(0,x^{48}),(0,x^{64})\},$$
$$B_2 = \{(0,0),(1,1),(1,-1),(2,x^{13}),(2,-x^{13})\},$$
$$B_3 = \{(0,0),(1,x^{6}),(1,-x^{6}),(2,x^{7}),(2,-x^{7})\},$$

Multiply B_1 by $(1,x^4),(1,x^5)$, and B_2, B_3 by $(1,x^{8t})$ for $0 \le t \le 4$. □

Lemma 16 *There exists a 5-GDD of type 3^q for $q = 25$.*

Proof: Take $X = \mathbb{Z}_3 \times \mathbb{F}_{25}$ where $x^2 = x + 3$, and multiply the following 2 blocks by $(1,x^{8t})$ for $t = 0,1,2$. Then develop mod $(3,5^2)$. (B_1 and its multiples each generate 15 blocks; B_2 and its multiples each generate 75 blocks).

$$B_1 = \{(0,0),(0,x),(0,2x),(0,3x),(0,4x)\},$$
$$B_2 = \{(0,0),(1,1),(1,4),(2,x+3),(2,4x+2)\}$$

□

Lemma 17 *If $q = 11, 31$ or 71 then there exists a 5-GDD of type 6^q.*

Proof: Take the point set as $X = (\mathbb{Z}_5 \cup \infty) \times \mathbb{F}_q$ and the groups as $(\mathbb{Z}_5 \cup \infty) \times \{y\}, y \in \mathbb{F}_q$. In $\mathbb{F}_q$, let z be a primitive root of unity and w a fifth root (for $q = 11, 31, 71$ respectively, we take $z = 2, 3, 11$ and $w = 4, 4, 5$). Define automorphisms T_1, T_2, T_3 by $T_1(x,y) = (x, y+1)$, $T_2(x,y) = (x+1, w \cdot y)$, and $T_3(x,y) = (x, z^{10} \cdot y)$. In each case, apply the group of order $q(q-1)/10$ generated by T_1, T_3 to the first base block (and its multiples) and the group generated by T_1, T_2, T_3 to the other base blocks given.

$q = 11$: Base blocks:

$$\{(0,t),(1,4t),(2,5t),(3,9t),(4,3t)\} \text{ for } t = 2,4,10.$$
$$\{(0,0),(0,1),(4,3),(\infty,2),(\infty,5)\},$$
$$\{(0,0),(0,4),(1,9),(3,6),(\infty,10)\},$$
$$\{(0,0),(0,2),(0,8),(1,4),(2,9)\}$$

$q = 31$: Base blocks:

$$\{(0,t),(1,4t),(2,16t),(3,2t),(4,8t)\} \text{ for } t = 1,9,20.$$
$$\{(0,0),(0,1),(4,24),(\infty,3),(\infty,6)\},$$
$$\{(0,0),(0,12),(1,20),(3,3),(\infty,4)\},$$
$$\{(0,0),(0,3),(0,11),(1,2),(3,21)\}$$

$q = 71$: Base blocks:

$$\{(0,t),(1,5t),(2,25t),(3,54t),(4,57t)\} \text{ for } t = 14,40,61.$$
$$\{(0,0),(0,1),(4,57),(\infty,2),(\infty,8)\},$$
$$\{(0,0),(0,13),(1,28),(3,8),(\infty,35)\},$$
$$\{(0,0),(0,3),(0,21),(1,22),(3,54)\}$$

□

Lemma 18 *There is a 5-GDD of type* 10^{11}.

Proof: Take the point set as $\{0,1\} \times \mathbb{Z}_5 \times \mathbb{Z}_{11}$; then apply the automorphism group of order 55 generated by t_1, t_2 to the blocks below, where $t_1(x,y,z) = (x,y,z+1)$, and $t_2(x,y,z) = (x,y+1,4z)$. The base blocks are:

$$\{(0,0,0),(1,0,6),(1,0,7),(1,1,8),(1,2,3)\},\ \{(0,0,0),(1,1,7),(1,1,10),(1,2,6),(1,3,4)\},$$
$$\{(0,0,0),(1,1,1),(1,2,2),(1,2,4),(1,4,8)\},\ \{(0,0,0),(0,1,6),(1,0,9),(1,2,7),(1,3,10)\},$$
$$\{(0,0,0),(0,1,10),(0,2,8),(1,0,1),(1,0,4)\},\ \{(0,0,0),(0,0,3),(0,1,7),(1,0,2),(1,0,8)\},$$
$$\{(0,0,0),(0,0,2),(0,2,6),(1,2,10),(1,3,7)\},\ \{(0,0,0),(0,0,1),(0,2,10),(1,1,5),(1,4,2)\},$$
$$\{(0,0,0),(0,0,5),(0,1,3),(0,2,7),(1,3,6)\},\ \{(0,0,0),(0,0,7),(0,2,1),(0,4,5),(1,3,9)\}.$$

□

Lemma 19 *There is a 5-GDD of type* 10^q *for* $q \in \{19,43,67,79\}$.

Proof: On $\mathbb{Z}_{5q} \times \{0,1\}$, take as groups the translates of $\{0,q,2q,3q,4q\} \times \{0,1\}$. For blocks, start with a set of six blocks determined given by the rows of the matrices to follow. The second coordinates of the elements are specified by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The first coordinates of the elements are given by:

$$q = 19: \begin{pmatrix} 0 & 1 & 3 & 7 & 28 \\ 0 & 5 & 1 & 2 & 4 \\ 0 & 13 & 3 & 21 & 53 \\ 0 & 14 & 12 & 63 & 68 \\ 0 & 20 & 14 & 30 & 67 \\ 0 & 31 & 66 & 81 & 87 \end{pmatrix} \qquad q = 43: \begin{pmatrix} 0 & 1 & 3 & 7 & 53 \\ 0 & 5 & 1 & 2 & 4 \\ 0 & 8 & 3 & 26 & 13 \\ 0 & 9 & 7 & 38 & 108 \\ 0 & 35 & 124 & 100 & 197 \\ 0 & 36 & 66 & 181 & 127 \end{pmatrix}$$

$$q = 67: \begin{pmatrix} 0 & 1 & 3 & 7 & 28 \\ 0 & 10 & 1 & 2 & 4 \\ 0 & 38 & 3 & 21 & 33 \\ 0 & 9 & 7 & 63 & 68 \\ 0 & 20 & 94 & 65 & 207 \\ 0 & 51 & 101 & 281 & 222 \end{pmatrix} \qquad q = 79: \begin{pmatrix} 0 & 1 & 8 & 32 & 93 \\ 0 & 10 & 1 & 2 & 9 \\ 0 & 13 & 3 & 31 & 53 \\ 0 & 4 & 37 & 128 & 163 \\ 0 & 20 & 59 & 75 & 242 \\ 0 & 66 & 76 & 261 & 372 \end{pmatrix}$$

Multiply each by the element 11, 41, 131, or 176 of order $(q-1)/6$ for $q = 19$, 43, 67, or 79, respectively, to obtain $(q-1)/6$ blocks from each. The $q-1$ base blocks resulting are developed over $\mathbb{Z}_{5q}$ to obtain the GDD. $\square$

Lemma 20 *There is a 5-GDD of type* 10^q *for* $q \in \{29, 37, 53\}$.

Proof: On $\mathbb{Z}_{5q} \times \{0,1\}$, groups are formed as the translates of $\{0,q,2q,3q,4q\} \times \{0,1\}$. For $q = 29$, start with blocks

$$\{(0,0),(1,0),(3,0),(25,0),(2,1)\},\ \{(0,0),(7,0),(21,0),(10,1),(35,1)\},$$
$$\{(0,0),(10,0),(7,1),(40,1),(56,1)\},\ \{(0,0),(6,1),(23,1),(54,1),(73,1)\}$$

For $q = 37$, start with blocks

$$\{(0,0),(6,0),(13,0),(40,0),(7,1)\},\ \{(0,0),(17,0),(71,0),(5,1),(25,1)\},$$
$$\{(0,0),(5,0),(92,1),(15,1),(36,1)\},\ \{(0,0),(6,1),(38,1),(84,1),(103,1)\}$$

For $q = 53$, start with blocks

$$\{(0,0),(6,0),(18,0),(5,0),(7,1)\},\ \{(0,0),(42,0),(76,0),(5,1),(10,1)\},$$
$$\{(0,0),(10,0),(77,1),(35,1),(51,1)\},\ \{(0,0),(6,1),(38,1),(209,1),(198,1)\}$$

In each case, we multiply by an element of order $(q-1)/4$; for $q = 29$, 37, and 53, the multiplier elements are 36, 16, and 16, respectively. The resulting $q-1$ blocks can be developed over $\mathbb{Z}_{5q}$ to obtain the 5-GDD. □

Lemma 21 *There is a 5-GDD of type* 15^q *for* $q \in \{13, 29\}$.

Proof: Points are taken to be $\mathbb{Z}_{15q}$, and groups are formed by the translates of the multiples of q. In both cases, three base blocks are given; these are multiplied by the element 16 for $q = 13$, or 181 for $q = 29$, which is an element of order $(q-1)/4$. The $3(q-1)/4$ resulting blocks are developed under $\mathbb{Z}_{15q}$ to form the 5-GDD. For $q = 13$, the blocks are $\{0,1,3,7,18\}$, $\{0,5,38,74,105\}$, and $\{0,10,44,94,152\}$. For $q = 29$, the blocks are $\{0,1,3,7,21\}$, $\{0,5,13,46,241\}$, and $\{0,9,62,159,244\}$. □

Lemma 22 *There is a 5-GDD of type* 30^q *for* $q \in \{7, 19, 23\}$.

Proof: On $\mathbb{Z}_{15q} \times \{0,1\}$, take as groups the translates of $X \times \{0,1\}$ where $X = \{q \cdot i : 0 \leq i < 15\}$. The same technique as in Lemma 19 is used, choosing a multiplier of order $(q-1)/2$ in this case. Again six blocks are chosen with second coordinates as in Lemma 19, and first coordinates as follows:

$$q = 7: \quad \begin{pmatrix} 0 & 1 & 3 & 11 & 69 \\ 0 & 4 & 1 & 2 & 31 \\ 0 & 5 & 4 & 8 & 90 \\ 0 & 12 & 17 & 30 & 67 \\ 0 & 15 & 54 & 81 & 101 \\ 0 & 29 & 58 & 94 & 97 \end{pmatrix} \qquad q = 19: \quad \begin{pmatrix} 0 & 1 & 3 & 11 & 159 \\ 0 & 4 & 1 & 2 & 136 \\ 0 & 50 & 4 & 8 & 15 \\ 0 & 12 & 17 & 180 & 97 \\ 0 & 15 & 9 & 36 & 101 \\ 0 & 44 & 103 & 94 & 7 \end{pmatrix}$$

$$q = 23: \quad \begin{pmatrix} 0 & 1 & 3 & 11 & 144 \\ 0 & 19 & 1 & 2 & 106 \\ 0 & 5 & 4 & 8 & 30 \\ 0 & 12 & 17 & 75 & 22 \\ 0 & 15 & 9 & 6 & 56 \\ 0 & 29 & 88 & 199 & 82 \end{pmatrix}$$

Multiply each by the element 16 ($q = 7$), 61 ($q = 19$), or 301 ($q = 23$) to obtain $(q-1)/2$ blocks from each. The $3(q-1)$ base blocks resulting are developed over $\mathbb{Z}_{15q}$ to obtain the GDD. □

Lemma 23 *There is a 5-GDD of type* 30^{27}.

Proof: On $\mathbb{F}_{27} \times \mathbb{Z}_{15} \times \{0,1\}$, take as groups $\mathbb{F}_{27} \times \{i\} \times \{j\}$ for $i \in \mathbb{Z}_{15}$ and $j \in \{0,1\}$. Let α be a primitive element of $\mathbb{F}_{27}$ satisfying $\alpha^3 = \alpha^2 + 2\alpha + 2$. For blocks, start with

$$\{(\alpha^0,0,0),(\alpha^1,1,0),(\alpha^2,3,0),(\alpha^3,11,0),(\alpha^4,9,0)\},$$
$$\{(\alpha^0,0,0),(\alpha^1,4,0),(\alpha^2,1,1),(\alpha^3,2,1),(\alpha^8,1,1)\},$$
$$\{(\alpha^0,0,0),(\alpha^1,5,0),(\alpha^2,4,1),(\alpha^3,8,1),(\alpha^5,0,1)\},$$
$$\{(\alpha^0,0,0),(\alpha^2,12,0),(\alpha^4,2,1),(\alpha^8,0,1),(\alpha^{12},7,1)\},$$
$$\{(\alpha^0,0,0),(\alpha^1,0,0),(\alpha^5,9,1),(\alpha^6,6,1),(\alpha^{25},11,1)\},$$
$$\{(\alpha^0,0,0),(\alpha^1,14,0),(\alpha^5,13,1),(\alpha^3,4,1),(\alpha^{25},7,1)\}$$

and multiply each in the first component by the element α^2 of order 13 to obtain 13 blocks. The 78 base blocks obtained are developed over $\mathbb{F}_{27} \times \mathbb{Z}_{15}$ to produce the GDD. □

Lemma 24 *There exist 5-GDDs of type g^7 for $g \in \{40, 60\}$.*

Proof: Let $X = \mathbb{Z}_{g/4} \times GF(2^2, x^2 = x+1) \times \mathbb{Z}_7$. Groups consist of points that are equal mod 7.

For 40^7, base blocks are

$$\{(0,0,0),(0,0,6),(8,0,2),(1,1,1),(6,x+1,3)\},$$
$$\{(0,0,0),(6,0,4),(7,0,5),(2,1,2),(2,x,1)\},$$
$$\{(0,0,0),(5,0,1),(4,0,2),(2,1,3),(3,x+1,4)\},$$
$$\{(0,0,0),(3,0,3),(3,1,4),(0,x,6),(4,x+1,2)\}.$$

For 60^7, base blocks are

$$\{(0,0,0),(0,0,3),(1,1,1),(3,x,6),(5,x+1,5)\},$$
$$\{(0,0,0),(2,0,5),(6,0,3),(0,1,1),(7,x+1,6)\},$$
$$\{(0,0,0),(5,0,5),(12,0,2),(0,1,4),(4,x,1)\},$$
$$\{(0,0,0),(7,0,6),(6,0,1),(0,1,2),(2,x,3)\},$$

$$\{(0,0,0),(1,0,2),(4,0,4),(10,1,1),(12,x+1,3)\},$$
$$\{(0,0,0),(10,0,5),(1,1,3),(3,1,4),(4,x,6)\}.$$

Multiply the base blocks by $(1,x^i,2^i)$ for $0 \le i \le 2$ and develop modulo $(g/4, 2^2, 7)$.
□

3.2 Recursive Constructions

To obtain the required designs we employ several new constructions listed below.

The first one provides a new way to obtain GDDs by using HTDs.

Construction 1 *Suppose that a k-HTD of type $h^{r-1}s^1$ and a k-GDD of type $h^{r-1}(s+w)^1$ both exist. Then a k-GDD of type $(kh)^{r-1}(ks+w)^1$ exist.*

Proof: Let $(X, \{Y_i\}_{1\le i\le r}, \mathcal{G}, \mathcal{B})$ be a k-HTD of type $h^{r-1}s^1$ with $|Y_r \cap G| = s$, for each $G \in \mathcal{G}$. Add a set F of w extra points to all groups of the GDD. For each $G \in \mathcal{G}$, we then construct a k-GDD of type $h^{r-1}(s+w)^1(G \cup F, \{Y_i \cap G : 1 \le i \le r-1\} \cup \{(Y_r \cap G) \cup F\}, \mathcal{B}_G)$. Thus the required GDD is obtained by taking point set $X \cup F$, block set $\mathcal{B} \cup (\cup_{G\in\mathcal{G}}\mathcal{B}_G)$ and group set $\{Y_1, Y_2, \ldots, Y_{r-1}, Y_r \cup F\}$. □

The following construction is simple but useful.

Construction 2 *Suppose that there exists a k-GDD of type $\{s_i : 1 \le i \le r\}$. Let $a \ge 0$ be an integer. If, for each i satisfying $1 \le i \le r$, there exists a k-GDD of type $\{s_{ij} : 1 \le j \le k(i)\} \cup \{a\}$ where $s_i = \sum_{1\le j\le k(i)} s_{ij}$, then there is a k-GDD of type $\{s_{ij} : 1 \le j \le k(i), 1 \le i \le r\} \cup \{a\}$.*

Construction 3 *[111] Suppose that there exists a TD$(k+1, kn) -$ TD$(k+1, n)$. Then there exists a k-GDD of type $((k-1)n)^{k+1}$.*

For convenience, we now restrict ourselves to the case for block size 5.

Construction 4 *Suppose that there exists a $(v, \{5, w^*\})$-PBD. Then there is a 5-GDD of type $4^{(v-w)/4}(w-1)^1$.*

Proof: This follows from deleting one point from the distinguished block of the PBD. □

Construction 5 *Let d be a prime power and w a nonnegative integer. Suppose that a 5-GDD of type $4^d w^1$ exists. Then*

- *a 5-GDD of type $40^d(w + 4a + 12b)^1$ if $d \geq 10$, and*
- *a 5-GDD of type $60^d(w+4a+12b)^1$ if $d \geq 15$, where $0 \leq a, b$, and $a+b \leq d-1$.*
- *a 5-GDD of type $80^d(w + 4a + 8b + 12c + 20f + 24e)^1$ if $d \geq 20$, where $0 \leq a, b, c, f, e$ and $a+b+c+f+e \leq d-1$.*

Proof: By Lemma 3, an RT$(10, d)$ exists. Take as groups the blocks of one of the parallel classes from an RT$(10, d)$ to obtain a $\{10, d\}$-RGDD of type 10^d, in which all groups of the RTD form a distinguished parallel class. Adjoin $a + b + 1$ infinite points to the RGDD, where one infinite point is adjoined to each of $a+b+1$ parallel classes including the distinguished one. In the resulting design, give weight w to one infinite point which is adjoined to the distinguished parallel class, and weight 12 to b infinite points and give the remaining points weight 4. Then apply Theorem 18 to obtain a 5-GDD of type $40^d(w + 4a + 12b)^1$. The input designs used are 5-GDDs of types $4^{10}, 4^{11}, 4^{10}12^1, 4^d w^1$. The first three designs are obtained by using Construction 4 with appropriate PBDs in Theorem 19(1) and (3). Similarly, we can construct a 5-GDD of type $60^d(w + 4a + 12b)^1$ beginning with an RTD$(15, d)$; the last case is also similar using an RTD$(20,d)$. □

3.2.1 Existence Results: $g \equiv 0, 1, 3 \pmod 4$

In this subsection, we apply previous constructions to establish our existence results on 5-GDDs. First, we treat cases with five and six groups.

Lemma 25 *Let g and $u \geq 5$ be positive integers satisfying $u \equiv 1$ or $5 \pmod{20}$ and $g \neq 2, 3, 6, 10$. Then there exists a 5-GDD of type g^u.*

Proof: For each value of u, a $B(5, 1; u)$ exists by Theorem 19(1). Regard the BIBD as a 5-GDD of type 1^u and give every point weight g. Applying Theorem 18 gives the result. □

Lemma 26 *Suppose that $g \equiv 0 \pmod 4$ and $g \geq 4$. Then there exists a 5-GDD of type g^6.*

Proof: For $g = 4$, the result follows from Theorem 19(1) by deleting a point of the BIBD. Now take a 5-GDD of type 4^6 and apply Theorem 18 with weight n where a TD$(5, n)$ exists. By Lemma 1, this takes care of all values of g except for $g \in \{8, 12, 24, 40\}$. Deleting one block from a TD$(6, 7)$ yields a $\{5, 6\}$-GDD of type 6^6. We then give weight 4 to every point of the GDD and apply Theorem 18 to get a 5-GDD of type 24^6. For $g = 8$ and 12, the result follows from using Construction 3 with $k = 5$, and $n = 2, 3$, since both a TD$(6, 10)-$TD$(6, 2)$ and a TD$(6, 15)-$TD$(6, 3)$ exist (see Brouwer [27] and Colbourn [37]). Finally, we take a 5-GDD of type 8^6 and inflate every point by 5 using Theorem 18. This covers the case for $g = 40$ and the proof is complete. □

The use of these two lemmas requires designs with block sizes five and six. Bennett, Colbourn and Mullin [17] prove two results on such designs:

Theorem 37 *There is a* $\{5,6\}$*-GDD of type* 5^n *for all* $n \geq 5$ *except possibly when* $n \in Q = \{7,8,10,16\}$.

Theorem 38 *A* $(v, \{5,6\})$*-PBD exists if and only if* $v \equiv 0,1 \pmod 5$ *except when* $v \in \{10,11,15,16,20,35\}$ *and possibly when* $v \in \{40,50,51,80\}$.

Now we can treat the cases when $g \equiv 0 \pmod{20}$:

Lemma 27 *Let* g *and* u *be integers satisfying* $u \geq 5$, $u \notin Q$ *and* $g \equiv 0 \pmod{20}$. *Then there exists a 5-GDD of type* g^u.

Proof: Apply Theorem 18 to those $\{5,6\}$-GDDs in Theorem 37 with the necessary input designs from Theorem 19(1) and Lemma 26. □

We also require some 5-GDDs obtained by deleting points in incomplete PBDs:

Lemma 28 *Let* a, b, c *and* d *be integers satisfying* $a \geq 1, b \geq 2, c \geq 3, d \geq 3$ *and* $d \neq 5$. *Then there exist 5-GDDs with following types:* 4^{5a}, 4^{5a+1}, $4^{5a+2}8^1$, $4^{5b}12^1$, $4^{5c}8^1$, *and* $4^{5d+4}24^1$.

Proof: Applying Construction 4 with those PBDs in Theorem 19 produces the desired result. □

Lemma 29 *Let* g *and* u *satisfy* $u \in Q$, $g \equiv 0 \pmod{20}$ *and* $g \notin \{40,60,120,200\}$. *Then there exists a 5-GDD of type* g^u.

Proof: Write $g = 20n$; then $n \notin \{2,3,6,10\}$. So, we can use Theorem 18 with weight n to get a 5-GDD of type $(20n)^u$ from a 5-GDD of type 20^u. Thus we need only consider $g = 20$. The construction of a 5-GDD of type 20^u is as follows. The case $u = 7$ is handled by the construction of the (141,5,1) BIBD in [63].

For $u \in \{10, 16\}$ we have a 5-GDD of 4^u from Theorem 19(1). The result then comes from using Theorem 18 with weight 5.

For $u = 8$, we first use Construction 1 with $k = 5$ and $h = 4$ to yield a 5-GDD of type $20^{r-1}(5s + w)^1$, with $r = 8$, $s = 3$, $w = 5$, and thus $5s + w = 20$. The ingredients in this construction are a 5-GDD of type $4^{r-1}(s + w)^1$ and a 5-HTD of type $4^{r-1}s^1$. The first one comes from Lemma 28 and the second can be easily constructed by Lemmas 4 and 2. We then break up the group of size $5s + w$ by an appropriate 5-GDD with group size 20 already obtained and apply Construction 2 to get the desired result. □

Lemma 30 *If $u \in Q$, then there exists a 5-GDD of type 40^u and of type 200^u.*

Proof: For $u = 16$, the construction is as follows. We first use Construction 5 to yield a 5-GDD of type $40^d(w + 4a + 12b)^1$ with the parameters $u = d = 16$, $a = b = w = 0$, Apply Construction 2 to get the desired results.

For $u = 10$, we take a TD(9,9) and delete one point from one group. The resulting design is a 9-GDD of type 8^{10}. We then give weight 5 to the GDD and apply Theorem 18 to get the desired result. The input design is a 5-GDD of type 5^9 which comes from Theorem 35.

For $u = 8$, form a $\{7, 8\}$-GDD of type $8^7 6^1$ by deleting two points of an affine plane of order 8, and use this to produce a 5-HTD of type $8^7 6^1$. A 5-GDD of type $8^7 16^1$ exists via a construction of Greig (see [17]). Apply Construction 1 with $k = 5$, $h = 8$, $s = 6$, and $w = 10$ to get a 5-GDD of type 40^8. For $u = 7$, see Lemma 24.

Give weight 5 to 5-GDDs of type 40^u to get 5-GDDs of type 200^u. □

Lemma 31 *If $u \in Q\backslash\{8\}$, then there exists a 5-GDD of type* 60^u.

Proof: For $u \in \{10, 16\}$, we have a 5-GDD of type 4^u. Give weight 15 to the GDD and apply Theorem 18. For $u = 7$, see Lemma 24. □

Lemma 32 *For $1 \leq a \leq 19$, there is a 5-GDD of type $60^7(4a)^1$, and hence a 5-GDD of type* 60^8.

Proof: First we form a 6-HTD of type 3^7 on $\mathbb{Z}_{21} \times \{1, 2, 3, 4, 5, 6\}$ as follows. Consider the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 5 & 9 & 10 \\ 2 & 1 & 6 & 15 & 20 & 19 \\ 3 & 19 & 9 & 20 & 1 & 11 \\ 5 & 11 & 15 & 16 & 19 & 6 \\ 6 & 20 & 11 & 18 & 10 & 16 \end{pmatrix}$$

Multiply each column by 16^i for $0 \leq i \leq 2$, arithmetic modulo 21, to produce 18 columns. Then develop the columns over $\mathbb{Z}_{21}$ to produce $21 \cdot 18$ columns. Each column $(a, b, c, d, e, f)^T$ then forms a block $\{(a, 1), (b, 2), (c, 3), (d, 4), (e, 5), (f, 6)\}$ of the 6-HTD. The manner of construction ensures that the blocks of the HTD can be partitioned into 18 parallel classes of 21 blocks each (the action of $\mathbb{Z}_{21}$ turns a base block into a parallel class). Hence there is a *resolvable* k-HTD of type 3^7 for each $k \leq 6$. We require only the one with $k = 5$.

A second main ingredient is a 5-GDD of type $12^7 4^1$ produced as in Lemma 5. Over $\mathbb{Z}_{84}$, consider the starter blocks $\{0, 1, 10, 27\}$, $\{0, 2, 5, 20, 50\}$, $\{0, 4, 12, 23, 55\}$, $\{0, 6, 22, 46, 59\}$; developing these over $\mathbb{Z}_{84}$ gives blocks of size 5, and 4 parallel classes of blocks of size 4, in a GDD of type 12^7. Extending the four parallel classes gives the 5-GDD of type $12^7 4^1$.

Now we proceed as follows. Extend $a-1$ parallel classes in the resolvable 5-HTD of type 3^7. Give the resulting "design" weight 4, using 5-GDDs of type 4^5, 4^6, and $12^7 4^1$. The result is a 5-GDD of type $60^7(4a)^1$. □

Lemma 33 *If $u \in Q$, then there exists a 5-GDD of type* 120^u.

Proof: First, we observe that when an RTD$(6,m)$ exists, we have a $\{6,m\}$-GDD of type 6^m by taking as groups the blocks of one of the parallel class from the RTD. Furthermore, we also have a $\{5,6,m-1\}$-GDD of type 6^{m-1} by deleting one group from the GDD. It is known [3] that either an RTD$(6,u)$ or an RTD$(6,u+1)$ exists for all stated values of u, and hence a $\{5,6,u\}$-GDD of type 6^u exists. The result then is obtained by applying Theorem 18 with the necessary input designs from Lemmas 27 and 29. □

Summarizing the results of Lemmas 30–33, we have proved

Theorem 39 *Let g and u be integers satisfying $u \geq 5$ and $g \equiv 0 \pmod{20}$. Then there exists a 5-GDD of type g^u.*

Theorem 39 may be applied to establish the following two existence results.

Theorem 40 *Let $g \equiv 0 \pmod 4$ and $g \not\equiv 0 \pmod{20}$. Let $u \geq 5$ and $u \equiv 0$ or $1 \pmod 5$. Then there exists a 5-GDD of type g^u.*

Proof: For each value of $u \equiv 0$ or $1 \pmod 5$, a 5-GDD of type 4^u exists by Lemma 28. So, a 5-GDD of type g^u can be constructed by applying Theorem 18. But this construction does not work for $g \in \{8,12,24\}$. To deal with them, we first apply Theorem 18 to the $\{5,6\}$-GDDs of type 1^u from Theorem 38. For all values of $g \equiv 0 \pmod 4$, this handles all values of u except for $u \in \{10,11,15,16,20,35,40,50,51,80\}$.

Next, it has been proved in Lemmas 1 and 26 that 5-GDDs of type g^5 and g^6 exist when $g \in \{8, 12, 24\}$. Take a 5-GDD of type $(5g)^m$ from Theorem 39 and break up each group by a TD$(5, g)$ to obtain a 5-GDD of type g^{5m}. Furthermore, we add g infinite points to a 5-GDD of type $(5g)^m$ and break up each group by a 5-GDD of type g^6 in such a way that the g infinite points become a common group. This gives a $\{5\}$-GDD of type g^{5m+1}. In this way all cases are handled except when $g \in \{8, 12, 24\}$ and $u \in \{10, 11, 15, 16, 20\}$.

When $g = 24$, since either an RTD$(6, u)$ or an RTD$(6, u + 1)$ exists for all remaining values of u (see [3]), we have a $\{5, 6, u\}$-GDD of type 6^u as noted in the proof of Lemma 33. Therefore, the result can be obtained by Theorem 18 and Lemma 28.

For $g \in \{8, 12\}$, Lemma 5 gives direct constructions for the remaining cases. □

Theorem 41 *Let* $g \equiv 5 \pmod{10}$, $u \equiv 1 \pmod 4$ *and* $u \geq 5$. *Then there exists a 5-GDD of type* g^u.

Proof: Because of Theorem 18 and Lemma 1, it suffices to give the proof for $g = 5$ and 15. For $g = 5$, we apply Construction 2 as follows. Take a 5-GDD of type 20^m from Theorem 39 and adjoin 5 infinite points to each group. We then break up each group by a TD$(5, 5)$ to obtain a 5-GDD of type g^{4m+1}. This takes care of the case where $u \geq 21$. When $u < 21$, the result follows from Theorem 35.

Similarly, Construction 2 with Theorem 39 can be used to get the result for $g = 15$ and all stated values of u except $u \in \{9, 13, 17\}$. Lemma 5 handles $u = 9$. Lemma 21 handles $u = 13$. For $u = 17$, employ a 4-RGDD of type 3^{16} [102], and extend all parallel classes to obtain a 5-GDD of type $3^{16}15^1$. Next form a 5-HTD of type 3^{16} [18]. Now apply Construction 1 with $k = 5$, $h = 3$, $r = 17$, $s = 0$, and $w = 15$ to get a 5-GDD of type 15^{17}. □

In the cases when $g \not\equiv 2 \pmod 4$, it remains only to treat the case when $g = 3$, in view of Theorem 25. First we treat the easier half:

Theorem 42 *Let* $u \equiv 1 \pmod{20}$. *Then a 5-GDD of type* 3^u *exists.*

Proof: Direct constructions when $u \in \{21, 41, 61, 81\}$ are given in Lemma 15. In the remaining cases, write $x = (u-1)/20$ and use Lemma 31 to form a 5-GDD of type 60^x. Add three infinite points and fill the holes using a 5-GDD of type 3^{21} to get a 5-GDD of type 3^u. □

Theorem 43 *Let* $u \equiv 5 \pmod{20}$ *and* $u \notin \{5, 45, 65\}$. *Then a 5-GDD of type* 3^u *exists.*

Proof: In Lemma 16, a 5-GDD of type 3^{25} is given. Since 5-GDDs of type 72^z exist whenever $z \equiv 0, 1 \pmod 5$, adding three infinite points and filling with the 5-GDD of type 3^{25} gives a 5-GDD of type 3^{24z+1} whenever $z \equiv 0, 1 \pmod 5$. Hence we obtain the existence of a 5-GDD of type 3^u whenever $u \equiv 25 \pmod{120}$. A second infinite class is obtained by using a $\mathrm{TD}(5, 3u)$ when $u \equiv 1 \pmod{20}$, and filling its groups using a 5-GDD of type 3^u from Theorem 42 to get a 5-GDD of type 3^{5u}. This yields all $u \equiv 5 \pmod{100}$ except for $u = 5$. A third class is obtained by taking a 5-GDD of type 75^{4x+1} for all positive x and filling its groups with 5-GDDs of type 3^{25} to get 5-GDDs of type $3^{100x+25}$ for all $x \geq 1$.

Now Construction 5 with $d \geq 16$ a prime power, $w \in \{0, 4, 8, 12\}$ can be used to produce a 5-GDD of type $60^d 72^1$, and hence a 5-GDD of type 3^{20d+25}. This handles cases when $u \in \{345, 365, 565, 645, 665, 765, 845, 965\}$.

Next we take a $\mathrm{RTD}(6, m)$, and truncate a group to x points to obtain a $\{5, 6, x, m\}$-GDD of type $6^x 5^{m-x}$, and give weight 12 using 5-GDDs of type 12^5,

12^6, 12^x, and 12^m. Then adding three infinite points and filling using 5-GDDs of type 3^{21} and 3^{25}, we obtain a 5-GDD of type $3^{20m+4x+1}$. Applications of this follow:

$20m+4x+1$	m	x	Block Sizes
445	21	6	{5,6,21}
465	21	11	{5,6,11,21}
545	25	11	{5,6,11,25}

In a similar manner, extending six parallel classes of a (65,5,1)-RBIBD to get a {5,6}-GDD of type $5^{13}6^1$ handles $u = 285$; and truncating a group of a TD(6,11) to six points gives a {5,6,11}-GDD of type $5^{11}6^1$, which handles $u = 245$.

For $u \in \{85, 165\}$, first form a 5-GDD of type $3^{16}15^1$ from a resolvable 4-GDD of type 3^{16} [102]. Then, whenever a 5-GDD of type 48^n exists (i.e., $n \equiv 0,1 \pmod 5$), fill $n-1$ of its groups using 15 infinite points and the 5-GDD of type $3^{16}15^1$; fill the last group using the 5-GDD of type 3^{21}. When $n \in \{5, 10\}$, this handles $u \in \{85, 165\}$. For $u = 185$, start with a 4-RGDD of type 3^8 [102]; inflate using RTD(4,4) to obtain a 4-RGDD of type 12^8 and extend all parallel classes to get a 5-GDD of type $12^8 28^1$. Using Wilson's transversal design construction with the master design TD(9,8), and applying weight 12 using $\text{TD}(5, 12+x) - \text{TD}(5, x)$ for $x = 0, 1, 2, 3$, produces 5-HTDs of type $12^8 y^1$ for $0 \leq y \leq 21$. In particular, a 5-HTD of type $12^8 11^1$ exists. Now apply Construction 1 with $k = 5$, $h = 12$, $s = 11$, and $w = 17$ to get a 5-GDD of type $60^8 72^1$, and fill in groups using three infinite points to obtain a 5-GDD of type 3^{185}.

In particular, these classes give 5-GDDs of type 3^u for $u = 145$, 165, 85. Now under the stated conditions on u, if the desired value of u is not in one of the classes already handled, let $\alpha = \frac{u-1}{4}$. Note that $\alpha \equiv 1 \pmod 5$. Choose β so that $\alpha \equiv 5\beta + 1 \pmod{25}$ and $\beta \in \{7, 8, 4\}$ (all cases with $\beta \equiv 0, 1 \pmod 5$ have been completed above). Then write $n = \frac{\alpha-5\beta-1}{25}$. It is easily checked that $n > \beta$.

Form a TD$(6, 5n)$ (since $n > 2$ in each case, such a TD exists [3]). Truncate one group to leave $5\beta + 1$ points in it, to produce a $\{5, 6\}$-GDD of type $(5n)^5(5\beta + 1)^1$. Using 5-GDDs of type 12^5 and 12^6, give weight 5 to obtain a 5-GDD of type $(60n)^5(60\beta + 12)^1$. Add three infinite points and fill the holes using 5-GDDs of type 3^{20n+1} and $3^{20\beta+4+1}$ to get a 5-GDD of type $3^{100n+20\beta+5}$. □

3.2.2 Existence Results: $g \equiv 10 \pmod{20}$

In this subsection, we examine classes when $g \equiv 10 \pmod{20}$. Write $g = 10\alpha$ throughout. First we introduce some general observations.

Lemma 34 *A 5-GDD of type* $(10\alpha)^u$ *exists whenever* $\alpha \geq 5$, α *is odd, and* $u \equiv 1 \pmod{4}$.

Proof: In Lemma 5, 5-GDDs of type 10^u are constructed for $u \in \{9, 13, 17\}$; giving weight α yields 5-GDDs of type $(10\alpha)^u$ in these cases. A TD$(5, 10\alpha)$ exists except possibly when $\alpha = 1$, and hence the case when $u = 5$ is settled. Now write $v = (u-1)/4$, so that $v \geq 5$, and form a 5-GDD of type $(40\alpha)^v$. Add 10α infinite points, and fill the groups using the 5-GDD of type $(10\alpha)^5$ to get the 5-GDD of type $(10\alpha)^u$. □

Lemma 35 *A 5-GDD of type* $(10\alpha)^u$ *exists when* $\alpha \geq 1$, $\alpha \neq 3$ *is odd,* $u \notin \{7, 15, 23, 27, 35, 39, 47\}$, *except possibly when* $u \in \{5, 33, 63\}$ *and* $\alpha = 1$.

Proof: In Lemma 5, 5-GDDs of types 10^9, 10^{13} and 10^{17} are given; also given are 5-GDDs of types 2^u for $u \in \{21, 25, 45\}$ to which weight 5 can be given to get

5-GDDs of types 10^u for the same values. Lemma 18 gives a 5-GDD of type 10^{11}. Lemma 20 gives 5-GDDs of types 10^{29}, 10^{37}, and 10^{53}. There is a 5-GDD of type 2^u for $u = 31$ (Lemma 8), and for $u = 51$ (Lemma 9); give weight 5α to get 5-GDDs for $(10\alpha)^u$. There is a 5-GDD of type 10^q by Lemma 19 for $q \in \{19, 43, 67, 79\}$.

When there is a PBD with block sizes from {5,6,7,9} of order v, deleting a point gives a {5,6,7,9}-GDD with group sizes 4, 5, 6, and 8. Giving weight 20α and filling the groups with 10α infinite points yields a 5-GDD of type $(10\alpha)^{2v-1}$. Using the result in [19], this establishes the existence of 5-GDDs of type $(10\alpha)^u$ for $u \in \{41, 49, 51, 59, 61\}$ and all $u \geq 69$ with the exception of $u \in \{135, 185, 195, 197, 207, 215, 247\}$.

Forming an idempotent TD(6,m) and truncating a group to $x < m$ points gives a $\{5, 6, m, x\}$-GDD of type $5^{m-x}6^x$; when $x \notin \{2, 3, 4\}$, weight 20α can be given and groups filled to produce a 5-GDD of type $(10\alpha)^{10m+2x+1}$. Apply with $(m, x) =$ (12,7), (17,7), (17,12), (17,13), (19,8), (19,12), (23,8) to handle $u =$ 135, 185, 195, 197, 207, 215, 247, respectively. Filling groups of 5-GDDs of type 80^u using the 5-GDD of type 10^9 and 10 infinite points gives 5-GDDs of types 10^u for $u \in \{57, 65\}$. Fill groups of a 5-GDD of type 110^5 using 5-GDDs of type 10^{11} to handle $u = 55$.

When $\alpha > 1$, fill a 5-GDD of type $(90\alpha)^7$ (obtained later in Lemma 38) using a 5-GDD of type $(10\alpha)^9$ to handle $u = 63$. Lemma 34 handles $u \in \{5, 33\}$ when $\alpha > 1$. □

It remains to treat cases when $\alpha \equiv 0 \pmod 3$.

A remarkably simple construction of 5-MGDDs follows:

Lemma 36 *If a TD*$(6, 5n+1)$*−TD*$(6, n)$ *exists, then there exists a 5-MGDD of type* 6^{4n+1}.

Proof: Delete all points in the hole. □

Lemma 37 *A 5-GDD of type* 30^u *exists whenever* $u \equiv 1 \pmod 4$, $u \neq 9$.

Proof: A TD(5,30) is a 5-GDD of type 30^5; since 5-GDDs of type 120^n exist for all $n \geq 5$, filling groups using a 5-GDD of type 30^5 and 30 infinite points settles all cases when $u \geq 21$. For $n \in \{3,4\}$, a $\mathrm{TD}(6,5n+1)-\mathrm{TD}(6,n)$ exists [39], so Lemma 36 gives 5-MGDDs of types 6^{13} and 6^{17}. Give weight 5 using 5-GDDs of type 5^5 and 5^{13} or 5^{17} to get 5-GDDs of types 30^{13} and 30^{17}. □

Again, it remains to treat the more difficult class:

Lemma 38 *Let* $\gamma \geq 1$ *be odd. A 5-GDD of type* $(30\gamma)^u$ *exists whenever* $u \equiv 3 \pmod 4$, $u \notin \{3, 15\}$, *except possibly when* $\gamma = 3$ *and* $u \in \{7,23,27,35,39,47,59,63,67\}$.

Proof: Give weight 5γ to the 5-GDDs of types 6^{11} and 6^{31} from Lemma 17 to obtain 5-GDDs of types $(30\gamma)^{11}$ and $(30\gamma)^{31}$. Lemma 19 gives a 5-GDD of type 10^q and hence of type $(30\gamma)^q$ for all $\gamma \geq 3$, and $q \in \{19,43\}$. Lemma 22 gives a 5-GDD of type 30^q and hence of type $(30\gamma)^q$ for all $\gamma \neq 3$, and $q \in \{7,19,23\}$. Lemma 23 gives a 5-GDD of type 30^{27} and hence of type $(30\gamma)^{27}$ for all $\gamma \neq 3$. Filling groups of 5-GDDs of type $(300\gamma)^n$ for $n \geq 5$ using the 5-GDD of type $(30\gamma)^{11}$ handles all cases when $u \equiv 1 \pmod{10}$, $u \geq 51$.

In general, we form a GDD on $v = (u-1)/2$ points with block sizes at least five, and group sizes 5 or 15, and even sizes other than four. Then giving weight 60γ and filling holes using 30γ infinite points yields the required GDD. When v can be written in the form $v = 5n + h$ with $n \geq 5$, n odd, and $0 \leq h < n$, $h \neq 4$ if $\gamma = 3$, h even, we proceed as follows. Form a TD(6,n), and truncate one group to h points. Use one of the deleted points to define groups, to obtain a $\{5,6,n\}$-GDD of type $5^n h^1$, which can be given weight 60γ. This general method treats all values for u when $u \geq 91$ except for $u \in \{99, 119, 139, 159\}$. Employing an *idempotent* TD(6,n), we can instead permit h to be any value other than 2, 3, or 4, and form a $\{5, 6, n, h\}$-GDD of type $5^{n-h}6^h$. Use this construction taking $69 = 5 \cdot 12 + 9$ to handle $u = 139$.

Filling groups of 5-GDDs of type $(330\gamma)^n$ using the 5-GDD of type $(30\gamma)^{11}$ produces 5-GDDs of type $(30\gamma)^{11n}$ when $n \equiv 1 \pmod 4$. This treats the cases when $u \in \{55, 99\}$. The remaining cases when $u \in \{75, 79, 83, 87, 119, 159\}$ are treated as follows. Write $d = \frac{u-11}{4}$. Form a TD(7,d) or TD(7,$d+1$), and truncate a group to 15 points; if group size $d+1$ is chosen, then delete a block of size 6. In either case, a {5,6,7}-GDD of type $6^d 15^1$ is obtained. Give weight 20γ to get a 5-GDD of type $(120\gamma)^d(300\gamma)^1$. Then add 30γ infinite points and fill the groups using 5-GDDs of type $(30\gamma)^{11}$ and $(30\gamma)^5$.

Now start with a TD(10,9) and truncate five groups to 0, 6 or 9 points each; give weight 20γ when $\gamma \neq 3$ to obtain a 5-GDD of type $(180\gamma)^a(120\gamma)^b$ for $a \geq 5$ and $a + b \leq 10$. Filling using 30γ infinite points and 5-GDDs of type $(30\gamma)^5$ and $(30\gamma)^7$ handles $u \in \{35, 39, 43, 47, 59\}$. Instead filling groups of a 5-GDD of type $(210\gamma)^9$ or $(180\gamma)^{11}$ using a 5-GDD of type $(30\gamma)^7$ handles $u \in \{63, 67\}$. □

Lemma 38 is made more complicated by the fact that no 5-GDD of type 30^9 is known; however, when $\gamma \geq 3$ is odd, a 5-GDD of type $(30\gamma)^9$ is known. This permits us to treat some of the omissions when $\gamma = 3$:

Lemma 39 *A 5-GDD of type* 90^u *exists whenever* $u \equiv 3 \pmod 4$, $u \notin$ *{3, 7, 23, 27, 39, 47}.*

Proof: We treat the required cases left by Lemma 38, first using the same main technique as does its proof. Write $29 = 5 \cdot 5 + 4$ to handle $u = 59$. Baker [15] found a 7-GDD of type 3^{15}; since there is a 5-GDD of type 30^7, give weight 30 to the points of the 7-GDD, resulting in a 5-GDD of type 90^{15}. For $u = 35$, fill the groups of a 5-GDD of type 540^7 using a 5-GDD of type 90^5. For $u = 63$, fill the groups of a 5-GDD of type 810^7 using a 5-GDD of type 90^9. For $u = 67$, fill the groups of a 5-GDD of type 900^6 using a 5-GDD of type 90^{11}. □

Giving weight γ to the 5-GDD of type 90^{15} produces a 5-GDD of type $(90\gamma)^{15}$ for all $\gamma \neq 3$ odd.

3.2.3 Existence Results: $g \equiv 2, 6, 14, 18 \pmod{20}$

The problem when $g \equiv 2, 6, 14, 18 \pmod{20}$ falls naturally into two cases, an easier one when $u \equiv 1, 5 \pmod{20}$ and a harder one when $u \equiv 11, 15 \pmod{20}$. We treat the easier case first.

Lemma 40 *Let $g \equiv 2, 6, 14, 18 \pmod{20}$ and $u \equiv 1, 5 \pmod{20}$. Then a 5-GDD of type g^u exists except when $g \in \{2, 6\}$ and $u = 5$, and possibly when $g^u \in \{2^{85}, 6^{45}\}$.*

Proof: If $g \neq 2, 6$, form a 5-GDD of type 1^u and give weight g to each point. When $g = 2$, we proceed as follows. In Lemmas 5, 6, 7, and 11, solutions are given for $u \in \{21, 25, 41, 45, 61, 65, 81\}$. Using the 5-GDD of type 2^{21} to fill the groups of a 5-GDD of type 40^n yields a 5-GDD of type 2^{20n+1} for all $n \geq 5$. Next we treat $u \equiv 5 \pmod{20}$. When possible, write $u = 4(5m + x) + 1$ where $m \equiv 0, 1 \pmod 5$, $m \notin \{5, 6, 10, 15, 26, 30\}$, $x \equiv 0, 1 \pmod 5$, $0 \leq x \leq m$. Form an idempotent TD(6,m) and truncate one group to x points; taking the parallel class of blocks that results from idempotence as groups gives a $\{5, 6, x, m\}$-GDD of type $5^{m-x}6^x$. Give weight 8, and fill in the groups using 5-GDDs of type 2^{21} and 2^{25}. This handles all $u \equiv 5 \pmod{20}$, $u \geq 225$ except for $u = 285, 305$. Extending six parallel classes of a resolvable (65,5,1) design gives a $\{5,6\}$-GDD of type $5^{13}6^1$; similar inflation and filling handles $u = 285$. There is a (45,5,1) BIBD having two parallel classes which share precisely one block (See appendix AA??). Extend one of the parallel classes and use the other to define groups; this gives a $\{5,6\}$-GDD of type $5^8 6^1$, which can be inflated to settle $u = 185$. For $u = 20n+1$, $n \in \{1,2,3\}$, form a 5-GDD of type $(40n+2)^5$ and fill its groups with the 5-GDD of type 2^{20n+1} to get 5-GDDs for $u \in \{105, 205, 305\}$. Filling groups of a 5-GDD of type 50^5 with a 5-GDD of type 2^{25} handles $u = 125$. Fill the groups of a 5-GDD of type 48^6 using the 5-GDD of type 2^{25} to get a 5-GDD of type 2^{145}. Finally, there is a 5-GDD of type $8^7 16^1$ [17] and a 5-HTD of type 8^8; apply Construction 1 with $k = 5$, $h = s = w = r = 8$ to get a 5-GDD of type $40^7 48^1$, and fill its holes using 5-GDDs of types 2^{21} and 2^{25} to get a 5-GDD of type 2^{165}.

When $g = 6$, we proceed as follows. Apply Lemma 36 with $n = 5, 6, 10, 15$, obtaining the first two incomplete TDs from [38] and [96], and the latter two from $V(4,t)$ vectors [39]. In each case a 5-MGDD of type 6^{4n+1} results; filling the blocks of size $4n+1$ with a 5-GDD of type 1^{4n+1} gives a 5-GDD of type 6^{4n+1}, settling the cases when $u = 21$, 25, 41, and 61. In a similar way, we apply Lemma 36 with $3 \leq n \leq 9$ where, in addition to those above, one finds solutions for $n = 3, 4, 7$, and 9 in [39] and for $n = 8$ in [4]. Form a 5-HTD of type $(4n+1)^6$ (see [18]), and fill its groups using the 5-MGDD of type 6^{4n+1} (aligning the blocks of size $4n+1$ on the holes of the HTD) to produce a 5-MGDD of type 6^{20n+5}. Fill the blocks of size $20n+5$ using 5-GDDs of type 1^{20n+5} to obtain 5-GDDs of type 6^{20n+5}, hence settling $u \in \{65, 85, 105, 125, 145, 165, 185\}$. Now using a 5-GDD of type 6^{21} to fill holes in a 5-GDD of type 120^n for $n \geq 5$ yields 5-GDDs of type 6^{20n+1} for all $n \geq 5$. When $u \geq 205$ and $u \equiv 5 \pmod{20}$, the proof parallels the case when $g = 2$ closely.

Filling groups of a 5-GDD of type 60^8 using six infinite points and a 5-GDD of type 6^{11} (Lemma 17) handles $u = 81$. □

We recall a known result:

Theorem 44 *[5] A resolvable* $(v, 5, 1)$ *exists for all* $v \equiv 5 \pmod{20}$ *except possibly when* $v \in$ *{45, 105, 145, 185, 225, 345, 465, 585, 645, 665, 705, 785, 885, 925, 945, 1045, 1065, 1145, 1165, 1185, 1305, 1385, 1485, 1545, 1665, 1905, 2265, 2385, 2505, 2745, 2865, 2985, 3105, 3225, 3345, 3585, 3785, 3945, 4065, 4185, 4425, 4665, 4905}.*

Lemma 41 *If* $u \equiv 11, 15 \pmod{20}$ *and* $g = 6\gamma$ *for* $\gamma \not\equiv 0 \pmod 5$, $\gamma \neq 3$ *odd, then a 5-GDD of type* g^u *exists except possibly when* $u = 15$, *and in addition when* $u \in$ *{35, 75, 95, 115, 135} and* $\gamma = 1$.

Proof: Lemma 17 gives 5-GDDs of types 6^{11} and 6^{31}, to which weight γ can be given, settling $u \in \{11, 31\}$. Then filling the groups of a 5-GDD of type $(60\gamma)^n$ with 5-GDDs of type $(6\gamma)^{11}$ yields 5-GDDs of type $(6\gamma)^{10n+1}$ for all $n \geq 5$. Hence all cases with $u \equiv 11 \pmod{20}$ are treated.

Forming a 5-GDD of type $((20n + 11) \cdot 6\gamma)^5$ for $n \geq 0$, and using the 5-GDDs of type $(6\gamma)^{20n+11}$ to fill its holes yields 5-GDDs of type $(6\gamma)^{100n+55}$ for all $n \geq 0$. It remains to treat $u \equiv 15, 35, 75, 95 \pmod{100}$. Most cases are settled as follows. Form a GDD on $(u-1)/2$ points with block sizes congruent to 0 or 1 modulo 5, and group sizes chosen from 5, 27 and integers congruent to 2 modulo 10 other than 2 and 22. Giving each point weight 12γ, one obtains a 5-GDD whose groups can be filled using 6γ infinite points and 5-GDDs of types $(6\gamma)^{11}$ and $(6\gamma)^{20n+5}$ for $n = 2$ and $n \geq 4$ to get a 5-GDD of type 6^u. When possible, write $(u-1)/2 = 5m + x$ with $m \equiv 1 \pmod{10}$, $2 \leq x \leq m - 9$, $x \equiv 2 \pmod{10}$ and $x \notin \{2, 22\}$. Form a

TD(6,m), and delete $m-x$ points from one group; use one deleted point to define groups of type $5^m x^1$ in a $\{5,6,m\}$-GDD. It is easily checked that suitable choices exist whenever $u \geq 635$, and for the following values of u: 235, 335, 435, 475, 535, 575, 595. Instead truncating a group of a TD(6,15) to 12 points gives a {5,6,15}-GDD of type $5^{15}12^1$, settling $u = 175$. Apply Construction 5 with $d = 19$ to obtain a 5-GDD of type $60^{19}144^1$; fill its holes using 5-GDDs of type 6^{11} and 6^{25} to handle $u = 215$. Truncating a group of a TD(6,35) to 11 or 31 points, then appending an infinite points to its groups, and finally using one of the deleted points to define groups gives $\{5,6,36\}$-GDDs of types $5^{35}12^1$ and $5^{35}32^1$, settling $u \in \{375, 415\}$. For $u = 295$, truncate a group of TD(6,81) to 36 points and give weight 4 to get a 5-GDD of type 324^5144^1; fill using 6 infinite points and 5-GDDs of types 6^{55} and 6^{25}.

A number of the remaining cases can be settled by extending x parallel classes of a resolvable $(20n+5,5,1)$ design from Theorem 44, to form a $\{5,6\}$-GDD of type $5^{4n+1}x^1$, when $x \equiv 2 \pmod{10}$, $2 \leq x \leq 5n-3$ and $x \notin \{2,22\}$. Then inflation and filling is as before. We simply give pairs (u,n) where $u = 2(20n+5+x)+1$ satisfies the requirements of this construction: (195,4), (275,6), (395,8), (495,10), (515,12), (615,13). This completes all of the cases when $\gamma = 1$.

When $\gamma > 1$, the construction is made more flexible by permitting $x = 2, 22$ (since 5-GDDs of types $(6\gamma)^5$ and $(6\gamma)^{45}$ exist when $\gamma > 1$ is odd. The resolvable BIBD construction settles in addition (135,3). Truncating a group of a TD(6,11) to 2 points gives a {5,6,11}-GDD of type $5^{11}2^1$, settling $u = 115$. Now, filling groups of a 5-GDD of type $(30\gamma)^7$ or $(30\gamma)^{19}$ using a 5-GDD of type $(6\gamma)^5$ handles $u \in \{35, 95\}$. Finally, form a 5-HTD of type 4^7 (for example, by giving weight 4 to a 5-HTD of type 1^7). There also exists a 5-GDD of type $4^7 8^1$, so applying Construction 1 with $h = 4$, $s = 0$, $w = 8$, $k = 5$, $r = 8$ and the specified ingredients gives a 5-GDD of type $20^7 8^1$; give it weight 3γ to obtain a 5-GDD of type $(60\gamma)^7(24\gamma)^1$, and fill using 6γ infinite points and 5-GDDs of types $(6\gamma)^{11}$ and $(6\gamma)^5$ to settle $u = 75$. □

Next we treat cases when g is not a multiple of 3. We start with the case when $g = 2$.

Lemma 42 *If $u \equiv 11, 15 \pmod{20}$, a 5-GDD of type 2^u exists except when $u = 11$ and possibly when $u \in \{15, 35, 71, 75, 95, 111, 115, 135, 195, 215, 335\}$.*

Proof: A putative 5-GDD of type 2^{11} would have 22 blocks, and thus would form a symmetric GDD; however, the necessary condition in Theorem 5.1 of [75] fails. By Lemma 8, a 5-GDD of type 2^{31} exists. By Lemma 10, a 5-GDD of type 2^{55} exists. First we complete a closure using this GDD, and then treat "small" cases. For $d \geq 17$ an odd prime power, use Construction 5 to produce a 5-GDD of type $(60)^d h^1$, where $h \equiv 0, 8 \pmod{40}$, $h \notin \{128, 168\}$, and $40 \leq h \leq 12d - 12$. Fill its groups using 2 infinite points to get a 5-GDD of type $2^{30d+1+\frac{h}{2}}$. Some quite tedious calculations show that choosing d to be an odd prime power at most 67, this succeeds for all u in the range $535 \leq u \leq 2395$ except for the values: 575, 635, 655, 755, 1115, 1175, 1195. To obtain closure, let $67 \leq d < 6571$ be an odd prime power, and let $\widehat{d}$ be the next odd prime power. One can verify that $5(\widehat{d} - d) \leq d - 33$ in this range, and hence one can always choose an odd prime power q for which $30q + 161 \leq u \leq 36q - 35$ when $2171 \leq u \leq 236435$. Forming a 5-GDD of type $60^q(2(u - 1 - 30q))^1$ using Construction 5 then settles these cases. To complete the closure, observe that it suffices to have 14 MOLS of order d to apply Construction 5 (i.e., d need not be a prime power). Since 14 MOLS exist when $d \geq 7875$ [3], when $u > 236435$, write $d = \frac{u-x}{30}$ where $x \in \{21, 25, 41, 45, 61, 125\}$; then an RTD(15,$d$) exists (since $d \geq 7877$), and Construction 5 completes the closure.

Now we turn to smaller cases. Lemma 9 settles $u \in \{51, 91\}$. Lemmas 12, 13, and 14 settle $u \in \{131, 191, 211\}$. Forming a 5-GDD of type 60^n for $n \geq 5$ odd, and filling using the 5-GDD of type 2^{31} handles $u \in \{151, 271, 331, 391, 451, 511\}$. Using Lemma 32 to form a 5-GDD of type $60^7 48^1$, and filling its groups using two infinite points and 5-GDDs of type 2^{25} and 2^{31} handles $u = 235$. When $d \geq 11$ is a prime power, Construction 5 can be applied to form a 5-GDD of type $40^d 60^1$ and hence a 5-GDD of type $2^{20d+30+1}$; applications handle $u \in \{251, 291, 351, 371, 411, 491, 531\}$. Construction 5 can be also applied to form a 5-GDD of type $40^d 108^1$ and hence a 5-GDD of type $2^{20d+54+1}$; applications handle $u \in \{315, 375, 395, 435, 515\}$. Forming a {5,6}-PBD on $v+1$ points, with $v = \frac{u-1}{6}$, and deleting a point to form a GDD on v points with block sizes {5,6} and group sizes {4,5}, inflating by weight 12, and filling groups using 2^{25} and 2^{31}, handles $u \in \{175, 355, 415, 655, 1195\}$. Similarly, extend 15 parallel classes of a resolvable (65,5,1) design to form a 6-GDD of type $5^{13}15^1$; delete a point not in the long group to form a {5,6,15}-GDD of type $5^{15}4^1$; give weight 12 as above and fill to obtain a 5-GDD of type 2^{475}. Truncating a group of a TD(6,15) to 10 points gives a {5,6}-GDD of type $15^5 10^1$, to which weight 4 can be given; then filling groups settles $u = 171$. In a similar way, truncate a group of TD(6,20) to 15 points to settle $u = 231$. Truncating a group of a TD(6,25) to 12 or 22 points handles $u \in \{275, 295\}$. Truncating a group of a TD(6,45) to 10 or 22 points handles $u \in \{471, 495\}$. Truncating a group of a TD(6,52) to 27 points handles $u = 575$.

Filling groups of a 5-GDD of type 62^5 using the 5-GDD of type 2^{31} handles $u = 155$; similarly, $u \in \{255, 455, 755\}$ are handled from $u \in \{51, 91, 151\}$. Construction 5 can be used to make 5-GDDs of types $40^{13}100^1$ and $40^{47}348^1$ to obtain $u = 311$, and 1115. Construction 5 can also be used to make a 5-GDD of type $80^{25}348^1$ to handle 1175.

There is a 5-GDD of type $2^{80}42^1$ obtained by filling four groups of a 5-GDD of type 40^5 using two infinite points and a 5-GDD of type 2^{21}. Form a {5,6}-GDD of type $8^5 1^1$ and give weight 20 to get a 5-GDD of type $160^5 20^1$; add 42 infinite points and fill its groups using the 5-GDD of type $2^{80}42^1$ to get a 5-GDD of type $2^{400}62^1$; then fill the group of size 62 using a 5-GDD of type 2^{31} to settle $u = 431$. □

Lemma 43 *A 5-GDD of type* $(2\alpha)^u$ *with* $\alpha \geq 5$, α *odd,* $\alpha \not\equiv 0 \pmod 3$ *or* $\alpha = 9$, $\alpha \not\equiv 0 \pmod 5$ *exists whenever* $u \equiv 11, 15 \pmod{20}$ *except for* $u \in \{11, 15, 35, 71, 75, 111, 115, 135, 195\}$.

Proof: Under the stated conditions on α, a 5-GDD of type 2^u can be inflated to form a 5-GDD of type $(2\alpha)^u$. Thus one needs only consider the exceptions in Lemma 42. Filling a 5-GDD of type $(10\alpha)^q$ using a 5-GDD of type $(2\alpha)^5$ for $q \in \{19, 43, 67\}$ handles $u \in \{95, 215, 335\}$. □

Lemma 44 *A 5-GDD of type* 18^u *exists whenever* $u \equiv 11, 15 \pmod{20}$ *and* $u \notin \{11, 15, 35, 71, 111, 115, 135, 195\}$.

Proof: Starting with the list from Lemma 43, treat the case when $u = 75$ by forming a 5-GDD of type $90^{u/5}$, and filling its groups using 5-GDDs of type 18^5. □

3.3 More Constructions

In this section, we prove that there exists a 5-GDD of type g^u for all but a finite number of pairs of (g, u). We use the notation $[a, b]_{c(d)}$ to denote the set of integer v such that $a \leq v \leq b$ and $v \equiv c \pmod d$. We have the following construction for group divisible designs.

Construction 6 *If there exists a M-GDD of type $g_1g_2\dots g_n$, so that for each $k \in M$, there is a K-MGDD of type u^k, then there exists a K-IGDD of type $(g_1+g_2+\dots+g_n; g_1, g_2, \dots, g_n)^u$.*

Proof: Let V be the set of points of the M-GDD of type $g_1g_2\dots g_n$. We construct the K-IGDD of type $(g_1+g_2+\dots+g_n; g_1, g_2, \dots, g_n)^u$ with point set $V \times \{1,2,\dots,u\}$. For every block $B = \{x_1, x_2, \dots, x_m\}$ of size m in the GDD, we put the MGDD of type u^m on the set of mu points corresponding to the $\{x_1, x_2, \dots, x_m\} \times \{1,2,\dots,u\}$ so that the two parallel classes of blocks align on $\{x_1, x_2, \dots, x_m\} \times \{i\}$ for $i = 1,2,\dots,u$ and $x_i \times \{1,2,\dots,u\}$. □

We need one more construction for group divisible designs.

Construction 7 *Let $(X, \mathcal{G}, \mathcal{B})$ be a TD(k+l,t) where*

$$\mathcal{G} = \{G_1, G_2, \dots, G_k, H_1, H_2, \dots, H_l\}.$$

For $1 \le i \le l$, let $H_i = H_{i1} \cup H_{i2} \cup \dots \cup H_{ip_i}$ be a partition of H_i. Let nonnegative numbers m, m_{ij} be given such that for any block $A \in \mathcal{A}$ intersecting $H_{ij(i)}$ $(1 \le i \le l)$ there exists a K-IGDD of type $(m + \sum_{i=1}^{l} m_{ij(i)}; m_{1j(1)}, m_{2j(2)}, \dots, m_{lj(l)})^k$ Then a K-IGDD of type

$$(mt + \sum_{i=1}^{l}\sum_{j=1}^{p} m_{ij}h_{ij}; \sum_{j=1}^{p_1} m_{1j}h_{1j}, \sum_{j=1}^{p_2} m_{2j}h_{2j}, \dots, \sum_{j=1}^{p_l} m_{lj}h_{lj})^k$$

exists.

Proof (Sketch): The proof is a modification of Brouwer-Van Rees theorem for transversal designs [31], replacing each sub-TD by a sub-GDD. □

Construction 8 *Suppose we have a K-GDD of type $g_1 g_2 \dots g_l$ and there is a $TD(5, g_1 + g_2 + \dots, g_{l-1} + a) - TD(5, a)$ where $0 \leq a \leq g_l$. Then there exists a $\{5\} \cup K$-GDD of type $g_1^5 g_2^5 \dots g_{l-1}^5 (g_l + 4a)^1$.*

Proof: This is just a simple variant of Singular Indirect Product (See [97]). □

Construction 9 *[14] The set of $\{r$: there exists a k-MGDD of type $g^r\}$ is PBD-closed.*

First of all, we establish that there exists a sequence of integers a_i such that a_i is odd, there exists a TD$(32, a_i)$, and $151a_i \geq 120a_{i+1} + 30$. Such a sequence can always be chosen with $a_1 = 31$ when each $a_i \leq 10000$ ([3]). Choose $a_n = 31(317)$, $a_{n+1} = 31(389)$, $a_{n+2} = 31(479)$, $a_{n+3} = 31(593)$, $a_{n+4} = 31(739)$, $a_{n+5} = 31(919)$, $a_{n+6} = 31(1129)$, $a_{n+7} = 31(1409)$, $a_{n+8} = 31(1753)$. Note that $a_{n+8} = 54343$ and it is known [3] that if $k \geq 54343$, then there exists a TD(32,k).

Lemma 84 *If $g \geq 3750$ and $g \equiv 2 \pmod 4$ then there exists a $\{5, 31\}$-GDD with block sizes a multiple of four or a multiple of 30 but not equal to 90.*

Proof: Take a TD(32,m) where m is odd and give weight 4 to each point in first 30 groups. Give weights 0 or 4 to each point in the 31^{st} group and weights 0 or 30 to each point in the last group. All 5-GDDs of type 4^{30}, 4^{31} and $\{5, 31\}$-GDD of type $4^{30}30^1$ and $4^{31}30^1$ exists. The last two designs are obtained from a TD(5,30): add a point to each group, and remove another point in the design to obtain $4^{30}30^1$; for $4^{31}30^1$, remove a point from TD(5,31). This gives a $\{5, 31\}$-GDD of type $(4m)^{30}(4x)^1(30y)^1$. Since $m \geq 31$, if $g \in [120m + 30, 154m]_{2(4)}$, then there exists a $\{5, 31\}$-GDD of type $g_1 g_2 \dots g_k$ where $g_i \equiv 0 \pmod 4$ or $g_i \equiv 0 \pmod{30}$

and $g_i \neq 90$. By the above remark, whenever $g \geq 3750$ and $g \equiv 2 \pmod 4$, there exists a $\{5,31\}$-GDD with block sizes a multiple of four or a multiple of 30 but not equal to 90. □

Next we show that if $g \equiv 10 \pmod{20}$, we can obtain a similar result.

Lemma 85 *If $g \equiv 10 \pmod{30}$ and $g \geq 3250$ and if $g \equiv 50 \pmod{60}$ and $g \geq 4490$, then there exists a $\{5,31\}$-GDD with group sizes $20x$ or $30k$ where $k \neq 3$.*

Proof: By Lemma 84, there exists a $\{5,31\}$-GDD on v points with group sizes $4m$ and $30k$ where $m \geq 1$ and $k \neq 3$. Give it weight five, we obtain a $\{5,31\}$-GDD on $5v$ points with block sizes $20m$ and $150k$. This proves the claim for $g \geq 18600$. To deal with remaining cases, we start with a TD(21,31) and remove a point to obtain a $\{5,31\}$-GDD of type $20^{31}30^1$. Now take a TD(32,31) and truncate 27 groups to $g_1, g_2, \ldots, g_{27}$ so that $g_i \in \{0,6,9,15,18,24,27,30\}$ to obtain a GDD of type $31^5 g_1 g_2 \ldots g_{27}$ with block sizes at least five. Inflate the GDD by giving weight 20 and add 30 infinite points; replace each group by either a $\{5,31\}$-GDD of type $20^{31}30^1$ or 5-GDD of type 30^r for $r = 5,7,11,13,17,19$ to show that if $g = 31(20)5 + 30 + 20(3k)$ and $g \equiv 10 \pmod{20}$ where $2 \leq k \leq 270$ then there exists a $\{5,31\}$-GDD on g points with group sizes $20x$ and 30. This proves that if $g \equiv 10 \pmod{60}$ and $g \in [3250, 19330]_{10(60)}$, then there exists a $\{5,31\}$-GDD with group sizes $20x$ and $30k$. To deal with the case when $g \equiv 50 \pmod{60}$, we essentially use the same technique except we retain seven groups of size 31 and truncate the remaining 25 groups. This proves that if $g \in [4490, 19370]_{50(60)}$, then there exists a $\{5,31\}$-GDD on g points with group sizes $20x$ and $30k$. □

If there exists a 5-GDD of type g^7, then $g \equiv 0 \pmod{10}$. Also, there exists a 5-GDD of type $(20g)^7$ for all $g \geq 1$ and a 5-GDD of type 30^7 by Theorem 36.

Before we proceed, we need a modified group divisible design.

Lemma 86 *There exists a 5-MGDD of type* 31^7.

Proof: Let $V = \mathbb{Z}_{217}$ and the two parallel classes are the development of $\{31i : i \in \{0, 1, \dots, 6\}\}$ and $\{7i : i \in \{0, 1, \dots, 30\}\}$. The blocks are $\{0, 1, 3, 11, 48\}$, $\{0, 4, 23, 59, 11\}$, $\{0, 6, 72, 88, 101\}$ and multiply each by 191^i where 191 is a cube root of unity over $\mathbb{Z}_{217}$ to obtain six further blocks. These nine blocks together with their translates generate the MGDD. □

Lemma 87 *If* $g \equiv 10 \pmod{30}$ *and* $g \geq 3250$ *and if* $g \equiv 50 \pmod{60}$ *and* $g \geq 4490$, *then there exists a 5-GDD of type* g^7.

Proof: From Lemma 85, if $g \equiv 10 \pmod{30}$ and $g \geq 3250$ and if $g \equiv 50 \pmod{60}$ and $g \geq 4490$, then there exists a $\{5, 31\}$-GDD with group sizes $20x$ or $30y$ where $y \neq 3$. Use this GDD together with Construction 6 to obtain a 5-GDD of type g^7. Lemma 86 constructs a 5-MGDD of type 31^7 and both 5-GDD of type $20x^7$ and $30y^7$ ($y \neq 3$) exist by Theorem 36. □

Now we deal with the existence of 5-GDD of type g^{11}. Theorem 36 establishes that the necessary condition is $g \equiv 0 \pmod 2$. Also, the necessary condition is also sufficient for $g \equiv 0 \pmod 4$ and $g \equiv 0 \pmod 6$ where $g \neq 18$. First of all, we have a new direct construction.

Lemma 88 *There exists a 5-GDD of type* 22^{11}.

Proof: Let $V = \mathbb{Z}_{121} \times \{0, 1\}$. The groups are $\{(11i, 0), (11i, 1) : i \in \{0, 1, 2, \dots, 10\}\}$ and its translates. The blocks are

$$\{(1,0),(3,0),(9,0),(27,0),(81,0)\},\ \{(16,1),(48,1),(23,1),(69,1),(86,1)\},$$
$$\{(0,0),(1,0),(5,0),(21,0),(2,1)\},\ \{(0,0),(10,0),(4,1),(5,1),(53,1)\},$$
$$\{(0,0),(19,0),(75,1),(87,1),(105,1)\},\ \{(0,0),(17,0),(30,1),(49,1),(91,1)\}.$$

Multiply the last four blocks by 3^i in the first coordinates where 3 is a fifth root of unity over $\mathbb{Z}_{121}$ to obtain 16 more blocks. □

Lemma 89 *There is a 5-MGDD of type* 11^9.

Proof: Let $V = \mathbb{Z}_{99}$; the two parallel classes of groups are $\{11i : i = 0, 1, \dots, 8\}$ and $\{9i : i = 0, 1, \dots, 10\}$. The blocks are $\{0, 1, 3, 8, 43\}$, $\{0, 4, 24, 34, 53\}$, $\{0, 6, 21, 68, 82\}$, $\{0, 12, 25, 51, 83\}$ and their translates. □

Lemma 90 *There is a 5-MGDD of type* 11^q *for* $q = 11, 13, 17, 19, 23, 31, 43, 67, 79, 103, 127, 139$.

Proof: Let $V = \mathbb{Z}_{11} \times \mathbb{Z}_q$ and the two parallel classes of blocks are $\{11i : i = 0, 1, \dots, q-1\}$ and $\{qi : i = 0, 1, \dots, 10\}$. For each q, we have a base block B and multiply by a multiplier of order $(q-1)/2$ to obtain $(q-1)/2 - 1$ further blocks.

q	B	*multiplier*
11	$\{(1,0), (3,1), (4,2), (5,8), (9,6)\}$	$(1,4)$
13	$\{(1,0), (3,1), (4,2), (5,7), (9,3)\}$	$(1,4)$
17	$\{(1,0), (3,1), (4,2), (5,7), (9,5)\}$	$(1,9)$
19	$\{(1,0), (3,1), (4,2), (5,4), (9,14)\}$	$(1,6)$
23	$\{(1,0), (3,1), (4,2), (5,12), (9,3)\}$	$(1,2)$
29	$\{(1,0), (3,1), (4,2), (5,4), (9,22)\}$	$(1,4)$
31	$\{(1,0), (3,1), (4,2), (5,13), (9,10)\}$	$(1,7)$
43	$\{(1,0), (3,1), (4,2), (5,4), (9,12)\}$	$(1,9)$
67	$\{(1,0), (3,1), (4,2), (5,4), (9,11)\}$	$(1,56)$
79	$\{(1,0), (3,1), (4,2), (5,8), (9,11)\}$	$(1,40)$
103	$\{(1,0), (3,1), (4,2), (5,7), (9,13)\}$	$(1,63)$
127	$\{(1,0), (3,1), (4,2), (5,7), (9,8)\}$	$(1,98)$
139	$\{(1,0), (3,1), (4,2), (5,4), (9,19)\}$	$(1,35)$

□

Lemma 91 *If* $g \geq 3750$ *and* $g \equiv 2 \pmod 4$ *then there exists a* 5*-GDD of type* g^{11}.

Proof: By Lemma 84, there exists a $\{5, 31\}$−GDD on g points with group sizes $4x$ and $30y$ where $y \neq 3$. Apply Construction 6 to the GDD to obtain the result. A 5-MGDD of type 31^{11} is constructed in Lemma 90 and 5-GDD of type $(4x)^{11}$ and $(30y)^{11}$ $(y \neq 3)$ are constructed in Theorem 36. □

Lemma 92 *If* $g \equiv 10 \pmod{12}$ *and* $g \in [2314, 2746]$, *then there exists a* 5*-GDD of type* g^{11}.

Proof: Take a TD(5,22(5)) and a TD(5,22(4)(5)+a)-TD(5,a) where $0 \leq a \leq 22(5)$. Apply Construction 8 to obtain a 5-GDD of type $(22(5))^{20}(22(5) + 4a)^1$. If we set $a \equiv 1 \pmod 3$, we have $22(5) + 4a \equiv 0 \pmod 6$. By inflating from a 5-GDD of type 6^{11} and 22^{11}, we know that both 5-GDD of type $(22(5) + 4a)^{11}$ and $(22(5))^{11}$ exist. Apply Construction 6 to obtain a 5-GDD of type $(2310 + 4a)^{11}$. □

Lemma 93 *If* $g \in [470, 542]_{2(12)}$, *then there exists a* 5*-GDD of type* g^{11}.

Proof: Take a TD(5,88+a)-TD(5,a) where $0 \leq a \leq 22$ and apply Singular Indirect Product with $22 - a$ infinite points to obtain a 5-GDD of type $22^{20}(22+4a)$. Choose a so that $22 + 4a$ is a multiple of 6. Apply Construction 6 to obtain the result. □

Lemma 94 *There is a* 5*-GDD of type* 206^{11}.

Proof: Remove a point from a TD(9,23) to obtain a $\{9, 23\}$-GDD of type $8^{23}22^1$. □

Lemma 95 *If* $g \in [1874, 2306]_{2(12)}$, *then there is a* 5*-GDD of type* g^{11}.

Proof: Take a TD(5,88) with 22 infinite points to obtain a 5-GDD of type $22^{16}110^1$. Apply Singular Indirect Product to obtain a 5-GDD of type $22^{80}(110+4a)^1$. Choose $a \equiv 1 \pmod 3$ and apply Construction 6. □

Lemma 96 *If $g \in [2750, 2882]_{2(12)} \cup [2990, 3062]_{2(12)}$, then there is a 5-GDD of type g^{11}.*

Proof: Take a TD$(5, 4(12t+8)+a)$-TD(5,a) with $12t+8-a$ infinite points to obtain a 5-GDD of type $(12t+8)^{20}(12t+8+4a)^1$. Choose a so that $12t+8+4a \in [470, 542]_{2(12)}$ when $t = 8, 9$. □

Lemma 97 *If $g \in [3242, 3842]_{2(12)}$, then there is a 5-GDD of type g^{11}.*

Proof: Inflating a 5-GDD of type 22^{11} yields a 5-GDD of type $(22(7))^{11}$. Take a TD(5,88(7)+a)-TD(5,a) where $0 \leq a \leq 22(7)$ and apply singular indirect product to obtain a 5-GDD of type $22(7)^{20}(22(7)+4a)^1$. Choose a so that $22(7)+4a$ is a multiple of six. Apply Construction 6 to obtain a 5-GDD of type $(22(7)(21)+4a)^{11}$. □

Lemma 98 *There exists a 5-MGDD of type 11^{12t+19} for $0 \leq t \leq 11$.*

Proof: From Lemma 90, we only need to deal with the cases when $t = 3, 7, 9$. In [14]. The set of r such that there exists a 5-MGDD of type g^r is PBD-closed (Construction 9). When $t = 3$, take a TD(5,11) to obtain $55 \in B(\{5, 11\})$. When $t = 7$, take a TD(5,18) and add a point at infinity to obtain $91 \in B(\{5, 19\})$. When $t = 9$, take a TD(5,23) to obtain $115 \in B(\{5, 23\})$. □

Lemma 99 *There exists a 5-GDD of type $(114, 4)^{11}$.*

Proof: Take a TD(5,23) and remove a point to obtain a $\{5,23\}$-GDD of type $4^{23}(22)^1$. Apply Construction 6 to obtain the result. We have both 5-GDD of type 4^{11} and 22^{11} and 5-MGDD of type 11^{23}. □

In the next lemma, we obtain a bound on the existence of 5-GDD of type g^{11}.

Let $A = [1210,1254]_{2(4)} \cup [1430,1482]_{2(4)} \cup [1870,1938]_{2(4)} \cup [2090,2166]_{2(4)} \cup [2530,2622]_{2(4)} \cup [2750,2850]_{2(4)} \cup [2970,3078]_{2(4)} \cup [3190,3306]_{2(4)} \cup [3410,3534]_{2(4)}$.

Lemma 100 *If $g \in A$, then there exists a 5-GDD of type g^{11}.*

Proof: Let t be an odd integer such that TD(12,t) exists. Give weight 0 or 4 to each point in one group and apply Construction 7 with $m = 110$. We have a 5-GDD of type 110^{11} by simply inflating a 5-GDD of type 22^{11} by weight five. Also, we have constructed a 5-IGDD of type $(114,4)^{11}$. We obtain a 5-GDD of type $(110m+4y)^{11}$ where $0 \le y \le m$. We display the values in the following table.

m	Interval
11	$[1210,1254]_{2(4)}$
13	$[1430,1482]_{2(4)}$
17	$[1870,1938]_{2(4)}$
19	$[2090,2166]_{2(4)}$
23	$[2530,2622]_{2(4)}$
25	$[2750,2850]_{2(4)}$
27	$[2970,3078]_{2(4)}$
29	$[3190,3306]_{2(4)}$
31	$[3410,3534]_{2(4)}$

□

We only use a simple construction to obtain an asymptotic bound on the existence of 5-GDD of type g^{15}.

Lemma 101 *There is a 5-MGDD of type* 31^{15}.

Proof: Let $V = \mathbb{Z}_{465}$ and the two groups are $\{31i : i = 0, 1, \dots, 14\}$ and $\{15i : i = 0, 1, \dots, 30\}$. The base blocks are $\{0, 1, 17, 5, 51\}$, $\{0, 61, 32, 6, 53\}$, $\{0, 76, 347, 141, 204\}$, $\{0, 2, 409, 110, 233\}$, $\{0, 77, 363, 36, 158\}$, $\{0, 107, 321, 187, 25\}$, $\{0, 18, 290, 218, 101\}$; multiply each by 346 and 346^2 to obtain 14 more blocks. □

Lemma 102 *There is a* 5-*GDD of type* g^{15} *for all* $g \geq 6090$ *and* $g \equiv 2 \pmod 4$.

Proof: As in Lemma 84, there exists a $\{5, 31\}$-GDD of type 4^{30}, 4^{31}, $4^{30}30^1$ and $4^{31}30^1$. Take a TD(31,m) where m is odd. Give weight 4 to each point in first 30 groups, give weight 0 or 4 to each point in the 31^{st} groups and give weight 0 or 30 to each point in the last group. Apply Wilson's Fundamental Construction to obtain a $\{5, 31\}$-GDD of type $(4m)^{30}(4x)^1(30y)^1$ where $0 \leq x, y \leq m$. If we insist on $y \equiv 3 \pmod 6$ and $y \neq 9$, then $30y \equiv 0 \pmod{90}$ and $30y \neq 270$. Hence, we can apply Construction 6 to obtain a 5-GDD of type $(120m + 4x + 30y)^{15}$. A simple calculation yields that that if $g \geq 6090$ and $g \equiv 2 \pmod 4$, then there exists a 5-GDD of type g^{15}. □

Many values less than 6090 can also be constructed.

Lemma 103 *There is a 5-MGDD of type* 31^{23}.

Proof: Let $V = \mathbb{Z}_{31} \times \mathbb{Z}_{23}$. Let the two parallel classes are $\{(0, i) : i = 0, 1, \dots, 22\}$ and $\{(i, 0) : i = 0, 1, \dots, 30\}$. The base blocks are $\{(0,0), (1,1), (3,2), (7,3), (15,4)\}$,

$\{(0,0),(3,5),(9,1),(14,7),(21,6)\}$ and $\{(0,0),(4,5),(5,10),(13,1),(15,22)\}$. Multiply the three blocks by $(1,25)^i$ for $i=1,2,\dots,10$ to obtain 30 further blocks. $\square$

Lemma 104 *There is a 5-MGDD of type 31^{27}.*

Proof: Let $V=\mathbb{Z}_{31}\times\mathbb{F}_{27}$ where α be a primitive element of $\mathbb{F}_{27}$ satisfying $\alpha^3=\alpha^2+2\alpha+2$. For blocks, start with $\{(0,\alpha^0),(1,\alpha^1),(3,\alpha^2),(7,\alpha^3),(15,\alpha^4)\}$, $\{(0,\alpha^0),(3,\alpha^1),(9,\alpha^2),(14,\alpha^3),(21,\alpha^{11})\}$, $\{(0,\alpha^0),(4,\alpha^2),(5,\alpha^4),(13,\alpha^1),(15,\alpha^8)\}$ and multiply each in the second component by α^2 of order 13 to obtain 13 blocks. These base blocks obtained are developed over $\mathbb{Z}_{31}\times\mathbb{F}_{27}$ to produce the GDD. $\square$

Lemma 105 *If $g\equiv 10 \pmod{60}$ and $g\geq 3250$ and if $g\equiv 50 \pmod{60}$ and $g\geq 4490$, then there exists a 5-GDD of type g^{23} and g^{27}.*

Proof: In Lemma 85, we prove that if $g\equiv 10 \pmod{30}$ and $g\geq 3250$ and if $g\equiv 50 \pmod{60}$ and $g\geq 4490$, then there exists a $\{5,31\}$-GDD on g points with block sizes $20x$ and/or 30. Use this GDD and apply Construction 6. Both 5-MGDDs of type 31^{23} and 31^{27} exist by Lemmas 103, 104. $\square$

Now, we show that if g is large enough, the basic necessary condition of the existence of 5-GDD is also sufficient. First we deal with the case when $g\equiv 10 \pmod{20}$.

Lemma 106 *If $g\equiv 10 \pmod{60}$ and $g\geq 3250$ and if $g\equiv 50 \pmod{60}$ and $g\geq 4490$, then there exists a 5-GDD of type g^r for all r except possibly for $r=15$.*

Proof: Both 5-GDD of type g^5 and g^7 exists from [3] and Lemma 87. When the number of groups is not in the set $\{7,11,15,23,27,35,39,47,55,59,63,71,75,83,87\}$, the result has been established in Theorem 36. The case when the number of groups is $7, 11, 23, 27$ has been established in Lemma 87 and 105. Take a TD(9,6k+3) (which exists for all $6k+3 \geq 335$ [3]) and truncate four groups to $\{0, 4(2k+1), 6(2k+1)\}$ points and give weight 20 to obtain a 5-GDD with group sizes $120k+60$, $80k+40$. Add $20k+10$ infinite points and fill in each group by a 5-GDD of type $(20k+10)^5$ or $(20k+10)^7$ to obtain a 5-GDD of type $(20k+10)^r$ for $r = 35, 39, 43, 47, 59$. When the number of the groups is 63 or 67, we fill in the groups of a 5-GDD of type $(7g)^9$ and $6g^{11}$. Take a TD(9,10k+5) (which exists for all $10k+5 \geq 335$ [3]) and truncate four groups to $\{0, 4(2k+1), 6(2k+1), 8(2k+1), 10(2k+1)\}$ and give weight 20 to each point and fill in the groups with $20k+10$ infinite points. This solves the remaining cases. □

Next, we deal with the case when $g \not\equiv 10 \pmod{20}$.

Lemma 107 *If $g \equiv 2 \pmod{12}$ and $g \geq 650$ or if $g \equiv 10 \pmod{12}$ and $g \geq 898$, then there exists a 5-GDD of type g^{35}, g^{115}, g^{135}, g^{195}.*

Proof: Break up the groups of a 5-GDD of type $(5g)^r$ where $r = 7, 23, 27, 39, 87$; such GDDs exist by Lemma 106. □

Lemma 108 *If $g \geq 3750$ and $g \equiv 2 \pmod{4}$, then there exists a 5-GDD of type g^{71} and g^{111}.*

Proof: We take a 5-GDD of type $10g^7$ and a 5-GDD of type $10g^{11}$ with g infinite points and fill each group by a 5-GDD of type g^{11} to obtain a 5-GDD of type g^{71} and g^{111}. □

Finally, we note the following. Let $K = \{5, 9, 11, 13, 17, 29, 31\}$.

Lemma 109 *If there exists a PBD(v,K), then there exists a 5-GDD of type* $(v-1)^{11}$.

Proof: Take a PBD(v, K) and remove one point. Apply Construction 6 to obtain the result. All required GDDs and MGDDs exist by Lemma 90 and Theorem 36.

3.4 Optimal Packings with Block Size Five

In this section, we study optimal packings with block size five on v points with $v \equiv 2 \pmod 4$ and $\lambda = 1$.

The function $D(v, k, 1)$ is of importance in coding theory since the block incidence vectors of a $(v, k, 1)$ packing from the codewords of a binary code of length v with minimum distance $2(k-1)$ and constant weight k. Thus $D(v, k, 1)$ is the maximum number of codewords in such a code.

Schönheim [105] has shown that

$$D(v,k,\lambda) \leq \lfloor \frac{v}{k} \lfloor \frac{\lambda(v-1)}{k-1} \rfloor \rfloor = B(v,k,\lambda)$$

Other upper bounds on the function $D(v, k, 1)$ have been given by Johnson [71] and Best et al. [23]. Lower bounds on the function $D(v, k, \lambda)$ are generally given by construction of (v, k, λ) packings.

The values of $D(v, 3, \lambda)$ for all v and λ have been determined by Schönheim [105], and Hanani [63]. The values of $D(v, 4, 1)$ have been determined for all v by Brouwer [28].

3.4.1 $v \equiv 2, 6, 10 \pmod{20}$

In this subsection, we consider the case when $v \equiv 2, 6, 10 \pmod{20}$.

Lemma 110 *If there exists a 5-GDD of type 2^n, then $D(2n, 5, 1) = B(2n, 5, 1)$.*

Proof: Simple counting yields the number of blocks in a 5-GDD of type 2^n meets the Schönheim bound. □

As a corollary, we obtain the following result.

Corollary 14 *If $v \equiv 2, 10 \pmod{20}$, $v \neq 10, 22, 30, 70, 142, 150, 170, 190, 222, 230, 270, 390, 430, 670$, then $D(v, 5, 1) = B(v, 5, 1)$.*

In the remaining of the section, we focus on the case when $v \equiv 6 \pmod{20}$.

Lemma 111 *If there exists a 5-GDD of type $2^n 6^1$, then $n \equiv 0 \pmod{10}$.*

Proof: The result follows immediately by counting the pairs and the neighbours of a point. □

We need some direct constructions.

Lemma 112 *There exists a 5-GDD of type $2^{40} 6^1$.*

Proof; Let $V = \mathbb{Z}_{40} \times \{0, 1\}$. The groups are $\{(i, j), (20+i, j)\}$ for $i = 0, 1, 2, \ldots, 19$ and $j = 0, 1$. The blocks are

$$\{(0,0),(2,0),(3,0),(2,1),(7,1)\},\ \{(0,1),(2,1),(3,1),(6,0),(27,0)\},$$
$$\{(0,0),(4,0),(12,0),(30,0),(23,1)\},\ \{(0,0),(16,0),(23,0),(29,0),(14,1)\},$$
$$\{(0,1),(4,1),(10,1),(28,1),(18,0)\},\ \{(0,1),(8,1),(15,1),(29,1),(20,0)\},$$
$$\{(0,0),(5,0),(6,1),(29,1)\},\ \{(0,0),(9,0),(30,1),(17,1)\},$$
$$\{(0,0),(15,0),(18,1),(27,1)\}.$$

The last three blocks of size four generate six parallel classes on V. Add an infinite point to each parallel class and a group of infinite points gives a 5-GDD of type $2^{40}6^1$. □

Lemma 113 *[1] There exists a 5-GDD of type* $2^{60}6^1$.

Proof: Let $V = \mathbb{Z}_{60} \times \{0,1\}$; and the groups are $\{(i,i),(i+30,j)\}$ for $i = 0,1,\ldots,29$ and $j = 0,1$. The blocks are

$$\{(0,0),(0,1),(0,20),(0,27),(1,13)\},\ \{(0,0),(0,8),(0,39),(0,43),(1,23)\},$$
$$\{(0,0),(0,13),(0,22),(0,50),(1,29)\},\ \{(0,0),(0,12),(0,18),(1,1),(1,36)\},$$
$$\{(0,0),(0,44),(0,55),(1,5),(1,17)\},\ \{(0,0),(0,3),(1,3),(1,9),(1,45)\},$$
$$\{(0,0),(0,24),(1,2),(1,55),(1,59)\},\ \{(0,0),(1,25),(1,27),(1,41),(1,58)\},$$
$$\{(0,0),(1,37),(1,48),(1,56),(1,57)\},\ \{(0,0),(1,4),(1,14),(1,19),(1,51)\},$$
$$\{(0,0),(0,15),(1,26),(1,47)\},\ \{(0,0),(0,2),(1,30),(1,52)\},$$
$$\{(0,0),(0,14),(1,8),(1,34)\}.$$

Each of the last three blocks generates two parallel classes on V. Add an infinite point to each parallel class and a group of infinite points to get a 5-GDD of type $2^{60}6^1$. □

Lemma 114 *If there exists a 5-GDD of type* 2^n, *then there exists a 5-GDD of type* $2^{5(n-1)}6^1$.

Proof: This is a variant of singular indirect product with one infinite point. □

Corollary 15 *There exists a 5-GDD of type* $2^{10n}6^1$ *for* $n = 4,6,10,12,15$.

Proof: The case when $n = 4, 6$ are constructed in Lemmas 112 and 113. When $n = 10, 12, 15$, they are constructed using Lemma 114 by taking a 5-GDD of type 2^n for $n = 21, 25, 31$. $\square$

Lemma 115 *If there exists a 5-GDD of type $(20g_1)(20g_2)\dots(20g_k)$ and for each $i = 1, 2, \dots, k$, there exists a 5-GDD of type $2^{10g_i}6^1$, then there exists a 5-GDD of type $2^{10(g_1+g_2+\dots+g_k)}6^1$.*

Proof: This is a variant of singular direct product by taking six infinite points. $\square$

Lemma 116 *If n is even, then there exists a 5-GDD of type $2^{10n}6^1$ for all $n \geq 78$.*

Proof: There exists a 5-GDD of type 80^n for all $n \geq 5$ by Theorem 36. Adding six infinite points and fill in each group by a 5-GDD of type $2^{40}6^1$ yields a 5-GDD of type $2^{40n}6^1$. Take a TD$(13, 13)$ and remove a point to obtain a 13-GDD of type 12^{14}. Truncate eight groups to sizes zero, four,six, ten or twelve. Each block has size at least five. Give weight twenty and apply Wilson's Fundamental Construction to obtain a 5-GDD of type $240^6g_1g_2\dots g_8$ where $g_i \in \{0, 80, 200, 240\}$ for $i = 1, 2, \dots, 8$. Apply Lemma 115 to obtain a 5-GDD of type $2^{10n}6^1$ for all $78 \leq n \leq 168$ and $n \equiv 0 \pmod 2$. Take a TD$(24, 23)$ and truncate three blocks which intersect in one point. This gives a GDD with group type $20^{23}22^1$ with block sizes at least 21. Keep eight groups of size twenty and truncate other groups to sizes $\{0, 4, 10, 12, 20\}$. Give weight 20 and apply Wilson's Fundamental Construction to obtain a 5-GDD. Apply Lemma 115 to obtain a 5-GDD of type $2^{10n}6^1$ for all $170 \leq n \leq 480$ and $n \equiv 0 \pmod 2$. Similarly, take a TD$(25, 25)$ and remove a point to obtain a 25-GDD of type 24^{26} and truncate points in 20 groups and give weight 20 to obtain a 5-GDD of type $2^{10n}6^1$ for $154 \leq n \leq 624$. Finally, take a TD$(6, 2n)$

for $n \geq 54$ and truncate in a group and give weight 20 to obtain a 5-GDD of type $(40n)^5(20g)^1$ for $94 \leq g \leq 120$ to obtain a 5-GDD of type $2^{10(10n+g)}6^1$. Hence, there exists a 5-GDD of type $2^{10n}6^1$ for all $n \geq 94$ and $n \equiv 0 \pmod 2$. □

Next, we deal with the case when n is odd.

Lemma 117 *If n is odd, then there exists a 5-GDD of type $2^{10n}6^1$ for all $n \geq 105$.*

Proof: Take a TD(16, 16) and remove a point to obtain a 16-GDD of type 15^{17}. Truncate ten groups to sizes $\{0, 4, 6, 10, 12, 15\}$ to obtain a GDD with block sizes at least five. Give weight 20 and apply Lemma 115 to obtain a 5-GDD of type $2^{10n}6^1$ for all $235 \geq n \geq 105$ and $n \equiv 1 \pmod 2$. In general, take a TD$(4n+1, 4n+1)$ for $n \geq 6$ and $4n+1$ a prime power and remove a point and truncate to all but six groups to sizes $\{0, 4, 10, 12, 15\}$. Give weight 20 and apply Lemma 115 to obtain a 5-GDD of type $2^{10k}6^1$ for $(4n)(6) + 15 + 12(4n-4) \geq k \geq (4n)(6) + 25$ and $k \equiv 1 \pmod 2$. Take $n = 6, 7, 9, 10, 12, 13, 15$ to obtain a 5-GDD of type $2^{10k}6^1$ for all $169 \leq k \leq 1047$ and $k \equiv 1 \pmod 2$. Take a TD$(6, 4n+1)$ for $n \geq 150$ and truncate a group to at least 109 points and give weight 20. Apply Lemma 115 and induction simply yield that if $k \geq 109$ and $k \equiv 1 \pmod 2$, then there exists a 5-GDD of type $2^{10k}6^1$. □

Lemma 118 *There exists a 5-GDD of type $2^{180}6^1$.*

Proof: We first construct a 5-GDD of type 10^9 with a parallel class. Let $V = \mathbb{Z}_{90}$ and the groups are the translates of $\{0, 9, 18, \ldots, 81\}$. The blocks are

$$\{0,6,7,18,34\},\ \{0,2,5,15,44\},$$
$$\{0,4,23,37,68\},\ \{0,8,17,38,58\}.$$

The five points in the first starter block are distinct (mod 5). Hence, the first starter block generates five parallel classes. By adding one infinite point, we obtain a $\{5,6\}$-GDD of type $10^9 1^1$. Give weight four to obtain a 5-GDD of type $40^9 4^1$. Add two infinite points and fill in with a 5-GDD of type 2^{21} to obtain a 5-GDD of type $2^{180} 6^1$. □

Lemma 119 *If n is even and $n \notin \{2,8,14,16,26,38,54,58\}$, then there exists a 5-GDD of type $2^{10n} 6^1$.*

Proof: Take a 5-GDD of type 120^n $(n = 5,7)$ and add six infinite points to obtain a 5-GDD of type $2^{60n} 6^1$. Take a $TD(6,30)$ and truncate a group to 20 points to obtain a $\{5,6\}$-GDD of type $30^5 20^1$. Give weight four to obtain a 5-GDD of type $120^5 80^1$. Add six points to obtain a 5-GDD of type $2^{340} 6^1$. Take a $TD(7,7)$ and remove a point to obtain a 7-GDD of type 6^8. Truncate a group to four points and give weight 20 to obtain a 5-GDD of type $120^7 80^1$. Add six infinite points to obtain a 5-GDD of type $2^{460} 6^1$. Take a $TD(11,11)$ and remove a point to obtain a 11-GDD of type 10^{12}. Keep six groups of size 10 and truncate one group to six points, one group to zero or four points and the remaining groups to zero points. Give weight 20 and add six infinite points to obtain a 5-GDD of type $2^{10n} 6^1$ for $n = 66, 74$. □

Lemma 120 *If there exists a 5-GDD of type $2^{10n} 6^1$, then $D(20n+6,5,1) = B(20n+6,5,1)$.*

Proof: In the group of size six, we put a further block of size five. The result follows by a simple counting argument. □

Corollary 16 *$D(20n+6,5,1) = B(20n+6,5,1)$ for all n but possibly $n \in \{2,8,14,16,18,26,54,58\}$ if n is even and $n \leq 101$ if n is odd.*

3.4.2 $v \equiv 14, 18 \pmod{20}$

In this subsection, we discuss the asymptotic behavior of $D(v, 5, 1)$ when $v \equiv 14, 18 \pmod{20}$.

Before we proceed, we need a result on 5-GDD of type $2^{10n}14^1$ and $2^{10n}18^1$.

Lemma 121 *If there exists a 5-GDD of type 2^n, a TD$(6, \frac{n-1}{2})$, then there exists a 5-GDD of type $2^{5(n-1)}a^1$ for $a = 14$ and $a = 18$.*

Proof: Take a TD$(6, \frac{n-1}{2})$ and truncate a group to three or four points. Give weight four and apply Wilson's Fundamental Construction to obtain a 5-GDD of type $(2(n-1))^5 12^1$ and $(2(n-1))^5 16^1$. Add two infinite points and fill in the group by a 5-GDD of type 2^n. □

Lemma 122 *There exists a 5-GDD of type $2^{10n}a^1$ for $a = 14, 18$ and $n \geq 181$ or $n = 137$.*

Proof: Take a TD$(26, 25)$ and truncate 21 groups to sizes $\{0, 12, 15, 20, 22, 25\}$. Give weight 20 and fill in each group to obtain a 5-GDD of type $2^x a^1$ where $x \in \{120, 150, 200, 220, 250\}$ and $a \in \{14, 18\}$. This gives a 5-GDD of type $2^{10n}a^1$ for $a = 14, 18$ and $181 \leq n \leq 500$. Similar argument can prove that there exists a 5-GDD of type $2^{10n}a^1$ for all $a = 14, 18$ and $n \geq 181$ by using a larger TDs to obtain a larger interval, then apply induction. To handle the case $n = 137$, use a TD$(6, 25)$ and truncate a group to 12; then give weight 20. □

Lemma 123 $D(2574, 5, 1) = B(2574, 5, 1)$ *and* $D(2078, 5, 1) = B(2078, 5, 1)$.

Proof: In [107], a 4-RGDD of type 3^8 is given. By completing all resolution classes, we obtain a 5-GDD of type $3^8 7^1$. Give weight 67 to obtain a 5-GDD of type $201^8 469^1$. Add a point at infinity and fill in the group by a 5-GDD of type 2^{101} or a 5-GDD of type 2^{235}. Simple counting show that it is indeed an optimal packing on 2078 points. Instead, we can give weight 83, to obtain a 5-GDD of type $249^8 581^1$. Add one point and fill in the groups by a 5-GDD of type 2^{125} or a 5-GDD of type 5-GDD of type 2^{291}. This gives an optimal packing on 2574 points. □

Theorem 45 *$D(20n+2574, 5, 1) = B(20n+2574, 5, 1)$ and $D(20n+2078, 5, 1) = B(20n+2078, 5, 1)$ for all $n \geq 751$.*

Proof: Take a TD(138, 137) and truncate one group to size 128 or 103, and 132 groups to sizes $\{0, 12, 15, 20, 22, 25, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130\}$. Give weight 20 and fill in with 14 or 18 infinite points corresponds to the case when the one group has size 128 or 103. Fill in all other groups by a 5-GDD of type $2^n a^1$ for some n and $a = 14, 18$. This gives a 5-GDD of type $2^{10n} b^1$ for $751 \leq n \leq 10000$ and $b \in \{2078, 2574\}$. A simple induction proves that there exists a 5-GDD of type $2^{10n} b^1$ for all $n \geq 751$. Filling in the group of size 2078 or 2574 by an optimal packing on the same number of points; the result follows easily by simple counting. □

In the case when $v \equiv 2 \pmod 4$, we have proved that if v is large enough, then $D(v, 5, 1) = B(v, 5, 1)$.

3.5 MGDDs with Block Size Four

In this section, we investigate the existence of modified group divisible designs with block size four.

The existence of the modified group divisible designs has been studied by Assaf [11] and Assaf and Wei [14]. They have applications in constructing various type of combinatorial objects; see [10] and Section 3.3. The existence of modified group divisible with block size three has been completely settled in [11]. In [14], the following result is proved. Let E = {(10,8), (10,15), (10,18), (10,23), (19,11), (19,12), (19,14), (19,15), (19,18), (19,23)}.

Theorem 46 *If $m, n \neq 6$, then a 4-MGDD of type m^n exists if and only if $(m-1)(n-1) \equiv 0 \pmod 3$ with the possible exception of $(m, n) \in E$.*

The case when one of the m or n takes on the value six is completely open, mainly due to the nonexistence of a 4-MGDD of type 6^4. We address the case of the existence of 4-MGDD of type 6^n. We develop some new constructions for MGDDs to settle this problem with few possible exceptions. We then settle the existence of 4-MGDDs with index greater than one completely.

3.5.1 Some Direct Constructions

Before we proceed, we need some direct constructions.

Lemma 124 *[14] There is a 4-MGDD of type 6^7.*

Proof: Let $V = \mathbb{Z}_{21} \times \{0,1\}$. A parallel class is $G_1 = \{(3i, j) : i = 0, 1, \ldots, 6\}$ for $j = 0, 1$ and their translates. The second parallel class is $\{(7i, j) : i = 0, 1, 2; j = 0, 1\}$. The base blocks are:

$$\{(0,0),(1,0),(5,0),(2,1)\},\ \{(0,0),(6,1),(17,1),(19,1)\},\ \{(0,0),(2,0),(10,1),(15,1)\},$$
$$\{(0,0),(8,0),(11,1),(12,1)\},\ \{(0,0),(10,0),(5,1),(9,1)\}.$$

Develop these under $\mathbb{Z}_{21}$ to obtain the blocks of the 4-MGDD. □

Lemma 125 *[14] There is a 4-MGDD of type* 6^{10}.

Proof: Let $V = \mathbb{Z}_5 \times \mathbb{Z}_{10} \cup H_{10}$, where $H_{10} = \{h_0, h_1, \ldots, h_9\}$. The first parallel class is $\{(0,a) : a \in \mathbb{Z}_{10}\}$ and its translates together with H_{10}. The second parallel class is $\{(a,0) : a \in \mathbb{Z}_5\} \cup h_0$ and its translates. The base blocks are:

$$\{(3,0),(4,1),(6,2),(7,3)\},\ \{(4,0),(5,1),(7,3),(8,2)\},\ \{(5,0),(6,1),(8,2),(9,3)\},$$
$$\{(0,0),(6,1),(7,3),(9,2)\},\ \{(0,0),(1,1),(7,2),(8,3)\},\ \{(1,0),(2,1),(8,4),(9,2)\},$$
$$\{(0,0),(2,1),(3,2),(9,4)\},\ \{(1,0),(4,2),(6,4),(9,3)\},\ \{(0,0),(3,1),(5,3),(8,2)\},$$
$$\{(2,0),(4,2),(7,1),(9,4)\},\ \{(1,0),(3,3),(6,2),(8,1)\},\ \{(0,0),(2,2),(5,1),(7,4)\},$$
$$\{(0,0),(1,3),(3,4),(4,1)\},\ \{(2,0),(3,3),(5,2),(6,1)\},\ \{(1,0),(2,3),(4,4),(5,1)\},$$
$$\{(0,4),(3,6),(1,8),h_7\},\ \{(0,5),(4,7),(1,9),h_8\},\ \{(0,0),(4,6),(1,8),h_9\},$$
$$\{(0,1),(3,7),(4,9),h_0\},\ \{(0,0),(3,2),(4,8),h_1\},\ \{(0,1),(2,3),(1,9),h_2\},$$
$$\{(0,3),(4,8),(1,9),h_5\},\ \{(0,2),(4,7),(2,8),h_4\},\ \{(0,0),(4,4),(3,9),h_6\},$$
$$\{(0,3),(3,4),(2,8),h_0\},\ \{(0,4),(4,5),(3,9),h_1\},\ \{(0,5),(3,9),(2,5),h_3\},$$
$$\{(0,0),(3,6),(1,9),h_4\},\ \{(0,1),(4,2),(3,8),h_6\},\ \{(0,2),(4,3),(2,9),h_7\},$$
$$\{(0,4),(3,7),(1,8),h_2\},\ \{(0,3),(1,5),(2,7),h_6\},\ \{(0,1),(3,6),(2,7),h_3\},$$
$$\{(0,2),(2,3),(3,7),h_9\},\ \{(0,0),(2,1),(1,7),h_5\},\ \{(0,3),(1,6),(4,7),h_1\},$$
$$\{(0,0),(4,2),(3,4),h_3\},\ \{(0,1),(4,3),(2,5),h_4\},\ \{(0,0),(4,5),(2,6),h_2\},$$
$$\{(0,2),(3,4),(2,6),h_5\},\ \{(0,0),(4,1),(2,5),h_7\},\ \{(0,1),(2,2),(1,6),h_8\},$$
$$\{(0,0),(3,3),(2,4),h_8\},\ \{(0,1),(1,4),(4,5),h_9\},\ \{(0,2),(1,5),(3,6),h_0\}.$$

These base blocks under the group $\alpha : (x,y) \mapsto (x+1,y)$ and $\alpha : h_i \mapsto h_{i+1}$ generate the design. □

Lemma 126 *There is a 4-MGDD of type* 6^{13}.

Proof: Let $V = \mathbb{Z}_{78}$. A parallel class is $\{6i : i = 0, 1, \ldots, 12\}$ and its translates. The second parallel class is $\{13i : i = 0, 1, \ldots, 5\}$ and its translates. The base blocks are {0,1,3,10}, {0,4,27,38}, {0,5,25,33}, {0,14,29,61}, {0,16,35,57}. Develop these blocks over $\mathbb{Z}_{78}$. □

Lemma 127 *There is a* 4*-MGDD of type* 6^{19}*.*

Proof: Let $V = \mathbb{Z}_{57} \times \{0, 1\}$. The first parallel class is $\{(3i, j) : i = 0, 1, \ldots, 18\}$ for $j = 0, 1$ and their translates. The second parallel class is $\{(19i, j) : i = 0, 1, 2; j = 0, 1\}$ and its translates. Base blocks are

{(0,0),(8,0),(28,0),(2,1)}, {(0,0),(10,0),(26,0),(6,1)}, {(0,0),(1,1),(9,1),(35,1)},
{(0,0),(10,1),(15,1),(32,1)}, {(0,0),(11,0),(25,0),(4,1)}, {(0,0),(3,1),(5,1),(16,1)},
{(0,0),(1,0),(13,1),(56,1)}, {(0,0),(2,0),(22,1),(42,1)}, {(0,0),(4,0),(28,1),(29,1)},
{(0,0),(5,0),(44,1),(54,1)}, {(0,0),(7,0),(18,1),(34,1)}, {(0,0),(13,0),(21,1),(46,1)},
{(0,0),(17,0),(43,1),(47,1)}, {(0,0),(22,0),(17,1),(45,1)}, {(0,0),(23,0),(7,1),(14,1)}.

Develop the blocks under $\mathbb{Z}_{57}$. □

Lemma 128 *There is a* 4*-MGDD of type* 6^{31}*.*

Proof: Let $V = \mathbb{Z}_{93} \times \{0, 1\}$. The first parallel class consists of the translates of $\{(0, 0), (31, 0), (62, 0), (0, 1), (31, 1), (62, 1)\}$. The second parallel class is $\{(3i, j) : i = 0, 1, \ldots, 30\}$ for $j = 0, 1$ and their translates. Base blocks are

{(0,0),(1,0),(8,0),(87,1)}, {(0,1),(1,1),(8,1),(3,0)}, {(0,0),(5,0),(14,1),(27,1)},
{(0,0),(10,0),(17,1),(67,1)}, {(0,0),(14,0),(43,1),(53,1)}.

Multiply the first coordinate of each block by 16^i for $i = 1, 2, 3, 4$ to obtain 20 further blocks. Develop them over $\mathbb{Z}_{93}$. □

Lemma 129 *There is a 4-MGDD of type* 6^{37}.

Proof: Let $V = \mathbb{Z}_{222}$. The first parallel class is $\{37i : i = 0, 1, \ldots, 5\}$ and the second parallel class is $\{6i : i = 0, 1, \ldots, 36\}$. The base blocks are {0,1,8,21}, {0,25,56,117}, {0,43,128,28}, {0,49,182,196}, {0,67,129,70}. Multiply each of them by 211 and 121 to obtain 10 more blocks. Develop these 15 blocks over $\mathbb{Z}_{222}$. □

Here is the first recursive construction.

Lemma 130 *Suppose there exists a 4-MGDD of type* 6^r *and there exists a 4-IGDD of type* $(6r; r, r, \ldots, r)^h$, *then there is a 4-MGDD of type* 6^{rh}.

Proof: Align the h copies of 4-MGDD of type 6^r on the h groups of the IGDD so that the block of size r coincides with the hole. Use each hole to form a new block of size rh. □

Let $I_n = \{1, 2, \ldots, n\}$ be an index set on n elements.

Lemma 131 *Suppose there exists a TD*$(7, m)$ *and a 4-MGDD of type* $(3a + 1)^6$ *where* $0 \leq a \leq m - 1$. *Then there exists a 4-MGDD of type* $(6m + 3a + 1)^6$.

Proof: Truncate a group of a TD$(7, m)$ to $a+1$ points, $s_0, s_1, \ldots .s_a$. We construct a 4-MGDD of type $(6m+3a+1)^6$ on the point set $V \times I_6 \cup s_0 \times I_6 \cup \{s_i : i = 1, 2, \ldots, a\} \times I_3 \times I_6$. Let $G_1, G_2, \ldots, G_6$ be the six other groups in the TD$(7, m)$. The new groups on the 4-MGDD are $G_i \times I_6 \cup \{s_0\} \times \{i\} \cup \{s_i : i = 1, 2, \ldots, a\} \times I_3 \times \{i\}$. For every block of size seven in the original design, if it hits the point s_0, we put a 4-MGDD of type 6^7 on $B \times I_6$ so that the blocks of size six align on $B \times \{i\}$ where $i \in I_n$ omitting the block of size six on $s_0 \times I_6$ where B is the set of all points in the block. For every other block of size seven, put a 4-IGDD of type $(9,3)^6$ on $(B \setminus \{s_i\}) \times I_6 \cup s_i \times I_3 \times I_6$ so that the hole aligns on $s_i \times I_3 \times I_6$ and the groups align on $a_i \times I_6 \cup s_i \times I_3 \times \{i\}$ where a_i is $B \cap G_i$, with G_i being the i^{th} groups in the original design. For every block of size six, put a 4-GDD of type 6^6 on the set $B \times I_6$. Finally, put a 4-MGDD of type $(3a+1)^6$ on the set $s_0 \times I_6 \cup \{s_i : i = 1, 2, \ldots, a\} \times I_3 \times I_6$. This gives a 4-MGDD of type $(6m+3a+1)^6$. □

With the two recursions, we are now in a position to close the spectrum of 4-MGDDs of type 6^r.

Lemma 132 *If $g \equiv 1 \pmod 6$, $g \geq 43$, there exists a 4-MGDD of type 6^g.*

Proof: When m is odd and $m \geq 7$, there exists a TD$(7, m)$ with the possible exceptions of $m = 15, 39$ [3]. Apply Lemma 131 with $a = 0, 2, 4, 6$ to obtain a 4-MGDD of type $(6m+1)^6$, $(6m+7)^6$, $(6m+13)^6$ and $(6m+19)^6$. □

Combining Lemmas 124, 126, 127, 128, 129 and 132, we obtain:

Lemma 133 *If $g \equiv 1 \pmod 6$, $g \neq 25$, there exists a 4-MGDD of type 6^g.*

Lemma 134 *There are 4-MGDDs of type 6^{28} and 6^{40}.*

Proof: There exists a 4-HTD of type 7^6 and 10^6 [3]. Apply Lemma 130. □

Lemma 135 *If $m \geq 388$ and $m \equiv 4 \pmod 6$, there exists a 4-MGDD of type $(6m+10)^6$.*

Proof: A TD$(7, m)$ exists for all $m \geq 63$ [3]. Apply Lemma 131 with $a = 3$ to obtain a 4-MGDD of type $(6m+10)^6$, using the 4-MGDD of type 6^{10} from Lemma 125. □

Lemma 136 *If $g \notin \{70, 94, 100, 118, 130, 142, 166, 190, 214, 238, 244, 286, 334, 370, 382\}$ and $g \geq 52$, then there exists a 4-MGDD of type g^6.*

Proof: Lemma 135 handles all cases when $g > 382$. Now apply Lemma 131 with $a = 3$ and values of $m \leq 62$ for which a TD$(7,m)$ exists [3]. □

Lemma 137 *If $g \geq 52$ and $g \neq 70, 118$, then there is a 4-MGDD of type g^6.*

Proof: First apply Lemma 136. Then use Lemma 131 with $a = 9$ and values of m =11, 12, 17, 19, 21, 27, 31, 35, 36, 43, 51, 57, and 59. The 4-MGDD of type 6^{28} exists by Lemma 134. □

Lemma 138 *There is a 4-MGDD of type 6^{46}.*

Proof: Give weight nine to all points in a block of a TD(6,7), and give weight six to all other points. Append a new column of six points. Take a parallel class of blocks of size six including the block in which all points have weight nine. For every block in the parallel class, put a 4-MGDD of type $(k+1)^6$ $(k = 6, 9)$ on the corresponding points together with the new adjoined points. For every other block, put a 4-GDD of type 6^6 or $6^5 9^1$ [97]. This gives a 4-MGDD of type 6^{46}. □

Lemma 139 *There exists a 4-MGDD of type 6^{70}.*

Proof: Take a 4-MGDD of type 7^6 (Lemma 124) and give every point weight 10. For every block of size six, put a 4-MGDD of type 10^6 (Lemma 125) on the 60 points. For every block of size four, put a 4-GDD of type 10^4. This gives a 4-MGDD of type 6^{70}. □

Lemma 140 *There exists a 4-MGDD of type* 6^{118}.

Proof: Take a 4-MGDD of type 13^6 (Lemma 126). Give every point weight nine and append a new column of six points. For every block of size 6, employ a 4-MGDD of type 10^6 (Lemma 125). For every other block of size four, employ with a 4-GDD of type 9^4 [97]. This gives a 4-MGDD of type 6^{118}. □

Combining Lemmas 134, 136, 137, 138, 139 and 140, we have the following result.

Lemma 141 *If* $g \equiv 4 \pmod 6$, $g \neq 16, 22, 34$, *there exists a 4-MGDD of type* 6^g.

Finally, we combine Lemmas 133 and 141 to yield:

Theorem 47 *There is a 4-MGDD of type* 6^n *for all* $n \notin \{16, 22, 25, 34\}$, $n \equiv 1 \pmod 3$ *and* $n \geq 7$.

In addition, we update the theorem of Assaf and Wei [14].

Lemma 142 *There is a 4-MGDD of type* 10^8.

Proof: Let $V = \mathbb{Z}_{10} \times (\mathbb{Z}_7 \cup \{\infty\})$. The first parallel class is $\{\{i\} \times (\mathbb{Z}_7 \cup \{\infty\}) : i \in \mathbb{Z}_{10}$. The second parallel class is $\{\mathbb{Z}_{10} \times \{j\} : j \in \mathbb{Z}_7 \cup \{\infty\}\}$. Base blocks are:

$$\{0,0),(1,1),(3,3),(9,2\},\ \{0,0),(4,4),(5,1),(8,6\},\ \{0,0),(5,5),(7,3),(1,6\},$$
$$\{0,\infty),(1,1),(7,3),(8,6\},\ \{0,\infty),(2,2),(5,1),(3,4\},\ \{0,\infty),(4,4),(9,1),(6,0\}.$$

□

Lemma 143 *There is a 4-MGDD of type* 10^{23}.

Proof: Let $V = \mathbb{Z}_5 \times \{0,1\} \times \mathbb{Z}_{23}$. The two parallel classes are $\{(0,0,i),(0,1,i) : i \in \mathbb{Z}_{23}\}$ and $\{(i,0,0),(i,1,0) : i \in \mathbb{Z}_5$. The base blocks are

$$\{(0,0,0),(1,0,1),(4,0,2),(0,1,3)\},\ \{(0,0,0),(0,1,5),(2,1,1),(3,1,2)\},$$
$$\{(1,0,0),(4,0,5),(2,1,7),(3,1,22)\}.$$

Multiply each block by $(-,-,2^i)$ for $i = 1,2,\ldots,10$ to obtain the remaining base blocks. □

Lemma 144 *There is a 4-MGDD of type* 19^{11}.

Proof: Let $V = \mathbb{Z}_{19} \times \mathbb{Z}_{11}$. The two parallel classes are $\{(0,i) : i \in \mathbb{Z}_{19}\}$ and $\{(i,0) : i \in \mathbb{Z}_{11}\}$ together with their translates over $\mathbb{Z}_{19} \times \mathbb{Z}_{11}$. The base blocks are

$$\{(0,1),(1,1),(3,2),(12,3)\},\ \{(0,0),(1,2),(5,1),(13,8)\},\ \{(0,0),(4,1),(6,7),(9,8)\}.$$

Multiply each block by $(1,4)^i$ for $i = 1,2,\ldots,4$ to obtain 12 more blocks. Develop these blocks over $\mathbb{Z}_{19} \times \mathbb{Z}_{11}$. □

Lemma 145 *There is a 4-MGDD of type* 19^{12}.

Proof: Take a 5-MGDD of type 6^{13} [102] and remove a group of size six to obtain a $\{4,5\}$-MGDD of type 6^{12}. Give weight three to each point and append a new column of 12 points. Employ 4-GDDs of type 3^4 and 3^5 and a 4-MGDD of type 4^{12}. $\square$

Lemma 146 *There is a* 4*-MGDD of type* 19^{14}.

Proof: Let $V = \mathbb{Z}_{19} \times (\mathbb{Z}_{13} \cup \{\infty\})$. The first parallel class is $\{\{i\} \times (\mathbb{Z}_{13} \cup \{\infty\}) : i \in \mathbb{Z}_{19}$. The second parallel class is $\{\mathbb{Z}_{19} \times \{j\} : j \in \mathbb{Z}_{13} \cup \{\infty\}\}$. Take the blocks

$$\{(0,0),(1,1),(3,3),(7,7)\},\ \{(0,0),(5,5),(14,1),(11,4)\},\ \{(0,0),(8,8),(18,5),(16,9)\},$$
$$\{(0,0),(11,11),(15,8),(7,10)\},\ \{(0,0),(15,2),(9,7),(3,5)\},$$
$$\{(0,\infty),(1,1),(15,8),(12,10)\},\ \{(0,\infty),(2,2),(16,8),(4,1)\}$$

and multiply each by $(11,1)^i$ for $i = 0,1,2$ to obtain 21 base blocks. Develop these under the action of the group. $\square$

Lemma 147 *There is a* 4*-MGDD of type* 19^{15}.

Proof: Let $V = \mathbb{Z}_{19} \times \mathbb{Z}_{15}$. The two parallel classes are $\{(i,0) : i \in \mathbb{Z}_{19}\}$ and $\{(0,i) : i \in \mathbb{Z}_{15}\}$ together with their translates over $\mathbb{Z}_{19} \times \mathbb{Z}_{15}$. Take the blocks

$$\{(0,0),(1,1),(3,3),(7,7)\},\ \{(0,0),(5,5),(14,14),(6,10)\},\ \{(0,0),(10,10),(3,7),(1,9)\},$$
$$\{(0,0),(13,13),(12,1),(16,9)\},\ \{(0,0),(9,13),(8,9),(11,2)\},$$
$$\{(0,0),(15,4),(3,12),(18,5)\},\ \{(0,0),(17,6),(15,1),(4,3)\}$$

and multiply each by $(11,1)^i$ for $i = 0,1,2$ to obtain 21 base blocks. Develop these under the action of the group. $\square$

Lemma 148 *There is a 4-MGDD of type* 19^{18}.

Proof: Let $V = \mathbb{Z}_{19} \times (\mathbb{Z}_{17} \cup \{\infty\})$. The first parallel class is $\{\{i\} \times (\mathbb{Z}_{17} \cup \{\infty\}) : i \in \mathbb{Z}_{19}$. The second parallel class is $\{\mathbb{Z}_{19} \times \{j\} : j \in \mathbb{Z}_{17} \cup \{\infty\}\}$. Take the blocks

$$\{(0,0),(1,1),(3,3),(7,7)\},\ \{(0,0),(5,5),(14,14),(6,8)\},\ \{(0,0),(8,8),(18,1),(11,13)\},$$
$$\{(0,0),(15,15),(17,2),(13,4)\},\ \{(0,0),(16,16),(2,6),(4,14)\},$$
$$\{(0,0),(9,11),(12,7),(15,16)\},\ \{(0,0),(14,16),(18,9),(7,12)\},$$
$$\{(0,\infty),(1,1),(4,8),(8,7)\},\ \{(0,\infty),(2,2),(13,4),(17,16)\}$$

and multiply each by $(7,1)^i$ for $i = 0, 1, 2$ to obtain 27 base blocks. Develop these under the action of the group. □

Lemma 149 *There is a 4-MGDD of type* 19^{23}.

Proof: Let $V = \mathbb{Z}_{19} \times \mathbb{Z}_{23}$. The two parallel classes are $\{(0,i) : i \in \mathbb{Z}_{23}\}$ and $\{(i,0) : i \in \mathbb{Z}_{19}\}$ together with its translate over $\mathbb{Z}_{19} \times \mathbb{Z}_{23}$. The base blocks are

$$\{(0,1),(1,1),(3,2),(12,3)\},\ \{(0,0),(1,5),(5,1),(13,2)\},\ \{(0,0),(4,1),(6,6),(9,11)\}.$$

Multiply each block by $(1,2)^i$ for $i = 1, 2, \ldots, 10$ to obtain 30 more blocks. Develop these blocks over $\mathbb{Z}_{19} \times \mathbb{Z}_{23}$. □

With these lemmas, we can restate the theorem.

Let $F = \{\{6,16\}, \{6,22\}, \{6,25\}, \{6,34\}, \{10,15\}\}$.

Theorem 48 *If* $\{m,n\} \neq \{6,4\}$, *then there exists a 4-MGDD of type* m^n *if and only if* $(m-1)(n-1) \equiv 0 \pmod 3$ *with the possible exceptions of* $\{m,n\} \in F$.

3.5.2 Higher Index

Next, we examine the existence of 4-MGDDs with index greater than one. Simple counting establishes that for a 4-MGDD of type m^n and index λ to exist, one requires that $\lambda(m-1)(n-1) \equiv 0 \pmod 3$ and $m, n \geq 4$. Hence when $\lambda \equiv 0$ (mod 3), the basic necessary condition reduces to $m, n \geq 4$. When $\lambda \not\equiv 0 \pmod 3$, the basic necessary condition is the same as for index one. Now the union of two 4-MGDDs of type m^n, one of index λ_1 and the other of index λ_2, is a 4-MGDD of type m^n and index $\lambda_1 + \lambda_2$. Hence it suffices to examine cases with $\lambda \in \{2, 3\}$ when the 4-MGDD of index one and type m^n is nonexistent or unknown although the basic necessary condition is met, and cases with $\lambda = 3$ when $m, n \equiv 0, 2 \pmod 3$ and $m, n \geq 4$.

First we treat the cases with $\lambda = 3$.

Lemma 150 *If whenever $n, m \in S = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}$ there is a 4-MGDD of type n^m and index 3, then whenever $n, m \geq 4$, there is a 4-MGDD of type n^m and index 3.*

Proof: There exist PBDs with block sizes from S of order n and m [19]. Let $(V, \mathcal{B})$ be such a PBD of order m, and $(W, \mathcal{D})$ be such a PBD of order n. We form the required 4-MGDD on the point set $V \times W$. For $B \in \mathcal{B}$ and $D \in \mathcal{D}$, place a 4-MGDD of index 3 on $B \times D$, omitting the parallel classes on $\{b\} \times D$ for $b \in B$, and on $B \times \{d\}$ for $d \in D$. □

Lemma 151 *Let $K \subseteq \{4, 7, 10, 13, 19\}$. If a K-PBD of order m and index 3 exists, and $n \in S$, then a 4-MGDD of type n^m and index 3 exists except possibly when $4 \in K$ and $n = 6$, or when $10 \in K$ and $n = 15$.*

Proof: Let $(V,\mathcal{B})$ be the K-PBD of order m and index 3. Let W be an n-set. We form the required 4-MGDD on the point set $V \times W$. For $B \in \mathcal{B}$, place a 4-MGDD of index 1 on $B \times W$, omitting the parallel classes on $\{b\} \times W$ for $b \in B$, and on $B \times \{w\}$ for $w \in W$. □

In view of Lemma 150, useful ingredients for Lemma 151 have $m \in S$.

Lemma 152 *There is a {4}-PBD of index 3 and order m whenever $m \equiv 0, 1 \pmod 4$. There is a {7}-PBD of index 3 and order 15. There is a {4,10}-PBD of index 3 and order 11. There are {4,7}-PBDs of index 3 and orders 14, 18, and 23.*

Proof: For the first two statements, see [63]. For order 11, employ base blocks $\{0,1,5,7\}$ and $\{\infty,0,1,3\}$ over $\mathbb{Z}_{10} \cup \{\infty\}$, together with $\mathbb{Z}_{10}$ as a block of size 10. For order 14, on $\mathbb{Z}_7 \times \{0,1\}$, take base blocks

$$\{0_0,1_0,0_1,3_1\},\ \{0_0,2_0,0_1,6_1\},\ \{0_0,4_0,0_1,5_1\},\ \{0_0,1_1,2_1,4_1\},\ \{0_1,1_0,2_0,4_0\},\ \{0_1,3_0,5_0,6_0\},$$

together with the single block $\mathbb{Z}_7 \times \{1\}$ of size 7.

For order 18, on $\mathbb{Z}_9 \times \{0,1\}$, form the base blocks

$$\{(0,0),(1,0),(2,0),(4,0),(0,1),(1,1),(3,1)\}\ \{(0,0),(1,0),(4,0),(4,1)\}$$
$$\{(0,0),(2,0),(5,0),(7,1)\}\ \{(0,0),(1,1),(4,1),(5,1)\}\ \{(0,0),(2,1),(4,1),(6,1)\}$$
$$\{(0,0),(3,1),(6,1),(7,1)\}$$

For order 23, on $\mathbb{Z}_{16} \cup \{\infty_i : 0 \le i \le 6\}$, form the starter blocks

$$\{\infty_0,0,1,3\},\ \{\infty_1,0,1,5\},\ \{\infty_2,0,1,8\},\ \{\infty_3,0,2,7\},\ \{\infty_4,0,2,5\},\ \{\infty_5,0,3,9\},\ \{\infty_6,0,4,10\}$$

with the short orbit $\{0, 4, 8, 12\}$, and a block of size 7 on the infinite points included three times. □

We must treat cases when $n = 6$ and $m \in \{4, 5, 6, 8, 9, 11, 12, 14, 18, 23\}$ to complete the solution for index 3.

Lemma 153 *Whenever $m \in \{4, 5, 6, 8, 9, 11, 12, 14, 18, 23\}$, a 4-MGDD of index three and type 6^m exists.*

Proof: For $m = 4$, the point set is $(\mathbb{Z}_5 \cup \{\infty\}) \times \{0, 1, 2, 3\}$. Base blocks are:

$$\{(\infty,0),(i,1),(2i,2),(3i,3)\},\ \{(0,0),(\infty,1),(2i,2),(3i,3)\},\ \{(0,0),(i,1),(\infty,2),(3i,3)\},$$
$$\{(0,0),(i,1),(2i,2),(\infty,3)\}$$

for $i = 1, 2, 3$, and three copies of the base block $\{(0, 0), (4, 1), (3, 2), (2, 3)\}$.

For $m = 5$, the point set is $\mathbb{Z}_{30}$, and base blocks are

$$\{0,1,2,3\},\ \{0,2,9,16\},\ \{0,3,7,16\},\ \{0,3,11,22\},\ \{0,4,8,17\}.$$

For $m = 6$, the point set is $(\mathbb{Z}_5 \cup \{\infty\}) \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(2,3),(3,2)\},\ \{(3,0),(4,1),(1,3),(0,\infty)\},\ \{(2,0),(3,1),(4,3),(0,\infty)\},$$
$$\{(0,3),(1,1),(2,4),(\infty,0)\},\ \{(0,4),(2,3),(4,2),(\infty,0)\},\ \{(1,1),(2,4),(0,\infty),(\infty,0)\},$$
$$\{(2,1),(4,2),(0,\infty),(\infty,0)\},\ \{(1,3),(3,2),(0,\infty),(\infty,0)\},\ \{(0,0),(1,4),(3,1),(\infty,\infty)\}$$

For $m = 8$, the point set is $(\mathbb{Z}_7 \cup \{\infty\}) \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(3,3),(5,2)\},\ \{(0,0),(4,4),(6,3),(1,2)\},\ \{(\infty,\infty),(0,0),(6,1),(5,2)\},$$
$$\{(0,\infty),(\infty,0),(1,1),(2,2)\},\ \{(0,\infty),(\infty,0),(3,2),(5,4)\},\ \{(0,\infty),(\infty,0),(4,3),(1,4)\},$$
$$\{(0,\infty),(4,0),(5,3),(6,1)\},\ \{(0,\infty),(1,0),(4,3),(6,4)\},\ \{(0,\infty),(2,2),(3,4),(6,0)\},$$
$$\{(0,\infty),(2,4),(3,2),(5,0)\},\ \{(\infty,0),(0,1),(6,2),(3,3)\},\ \{(\infty,0),(0,3),(1,4),(3,1)\}$$

For $m = 9$, the point set is $\mathbb{Z}_9 \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(2,2),(3,3)\}\ \{(0,0),(2,2),(6,1),(5,4)\}\ \{(0,\infty),(1,1),(4,4),(3,2)\}$$
$$\{(0,\infty),(1,1),(5,0),(6,3)\}\ \{(0,\infty),(1,1),(7,2),(3,0)\}\ \{(0,\infty),(2,2),(4,3),(7,4)\}$$

Multiply each by $(8,1)^i$ for $i = 0,1$ to obtain 12 base blocks, and develop over the group.

For $m = 11$, the point set is $\mathbb{Z}_{11} \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(2,2),(3,3)\}\ \{(0,\infty),(1,1),(5,0),(9,4)\}\ \{(0,\infty),(2,2),(8,0),(6,4)\}$$

Multiply each by $(4,1)^i$ for $i = 0,1,2,3,4$ to obtain 15 base blocks, and develop over the group.

For $m = 12$, there is a 5-MGDD of type 6^{13} [102] and hence a $\{4,5\}$-MGDD of type 6^{12}. Triplicate each block of size 4, and replace each 5-block by a $\{4\}$-PBD of order 5 and index 3.

For $m = 14$, the point set is $(\mathbb{Z}_{13} \cup \{\infty\}) \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(6,1),(1,4),(10,3)\},\ \{(0,0),(7,2),(6,4),(10,1)\},\ \{(0,0),(11,1),(5,3),(6,2)\},$$
$$\{(0,0),(3,3),(6,4),(1,2)\},\ \{(0,0),(9,2),(2,3),(5,1)\},\ \{(\infty,\infty),(0,0),(2,2),(11,1)\},$$
$$\{(0,\infty),(\infty,0),(3,1),(11,4)\},\ \{(0,\infty),(\infty,0),(5,3),(9,2)\},\ \{(0,\infty),(\infty,0),(4,1),(8,3)\},$$
$$\{(\infty,0),(0,2),(1,3),(2,4)\},\ \{(\infty,0),(0,4),(12,1),(2,2)\},\ \{(0,\infty),(1,0),(12,1),(7,2)\},$$
$$\{(0,\infty),(1,0),(2,4),(7,2)\},\ \{(0,\infty),(1,0),(4,1),(9,4)\},\ \{(0,\infty),(3,0),(2,2),(11,4)\},$$
$$\{(0,\infty),(2,0),(6,2),(12,1)\},\ \{(0,\infty),(8,0),(5,3),(6,4)\},\ \{(0,\infty),(10,0),(4,2),(12,3)\},$$
$$\{(0,\infty),(6,0),(10,4),(7,2)\},\ \{(0,\infty),(11,0),(8,2),(10,3)\},\ \{(0,\infty),(3,0),(5,2),(9,1)\}$$

For $m = 18$, the point set is $(\mathbb{Z}_{17} \cup \{\infty\}) \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(2,2),(3,3)\}\ \{(0,0),(4,4),(13,3),(10,2)\}\ \{(0,0),(6,1),(12,4),(4,3)\}$$
$$\{(0,0),(7,2),(1,3),(14,1)\}\ \{(0,0),(8,3),(2,4),(7,1)\}\ \{(0,0),(9,4),(4,1),(3,2)\}$$
$$\{(0,0),(6,3),(2,1),(8,2)\}\ \{(\infty,\infty),(0,0),(7,4),(5,1)\}\ \{(0,\infty),(\infty,0),(5,4),(10,1)\}$$
$$\{(0,\infty),(\infty,0),(1,2),(8,1)\}\ \{(0,\infty),(\infty,0),(7,2),(2,3)\}\ \{(\infty,0),(0,1),(12,3),(14,4)\}$$
$$\{(\infty,0),(0,4),(1,2),(15,3)\}\ \{(0,\infty),(1,0),(3,2),(13,1)\}\ \{(0,\infty),(6,0),(9,3),(8,4)\}$$
$$\{(0,\infty),(2,0),(5,3),(9,1)\}\ \{(0,\infty),(10,0),(14,4),(11,2)\}\ \{(0,\infty),(12,0),(8,1),(15,4)\}$$
$$\{(0,\infty),(1,0),(7,1),(11,2)\}\ \{(0,\infty),(16,0),(2,1),(6,2)\}\ \{(0,\infty),(4,0),(11,2),(16,3)\}$$
$$\{(0,\infty),(4,0),(12,3),(16,1)\}\ \{(0,\infty),(14,0),(5,3),(13,2)\}\ \{(0,\infty),(13,0),(7,1),(15,3)\}$$
$$\{(0,\infty),(15,0),(9,1),(12,3)\}\ \{(0,\infty),(3,0),(12,1),(4,2)\}\ \{(0,\infty),(6,0),(14,2),(3,4)\}$$

For $m = 23$, the point set is $\mathbb{Z}_{23} \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(2,2),(3,3)\}\ \{(0,\infty),(1,1),(3,3),(10,0)\}\ \{(0,\infty),(1,1),(5,0),(14,4)\}$$

Multiply each by $(2,1)^i$ for $i = 0, 1, \ldots, 10$ to obtain 33 base blocks, and develop over the group. □

Theorem 49 *A 4-MGDD of index 3 and type n^m exists whenever $n, m \geq 4$.*

Proof: If $m, n \notin S$, apply Lemma 150. If $m \in S\backslash\{6\}$, apply Lemma 151 using the PBDs from Lemma 152. This handles all cases except when $n = 6$, or $m \in \{10, 11\}$ and $n = 15$. The latter cases, and 6^{15}, are treated by using $m = 15$ in Lemma 151. When $m = 6$ and $n \in \{7, 10, 19\}$, triplicate a 4-MGDD of index one. The remaining cases arise when $m = 6$, and these are treated in Lemma 153. □

Now we turn to index 2. The only cases to treat are those missing when $\lambda = 1$. For type 10^{15}, employ a $\{4\}$-PBD of order 10 and index 2 together with a 4-MGDD of type 4^{15} to handle index 2 for type 10^{15}.

For 6^4, the point set is $(\mathbb{Z}_5 \cup \{\infty\}) \times \{0, 1, 2, 3\}$. Base blocks are:

$$\{(\infty,0),(i,1),(2i,2),(3i,3)\},\ \{(0,0),(\infty,1),(2i,2),(3i,3)\},\ \{(0,0),(i,1),(\infty,2),(3i,3)\},$$
$$\{(0,0),(i,1),(2i,2),(\infty,3)\}$$

for $i = 1,2$, and two copies of the base blocks $\{(0,0),(4,1),(3,2),(2,3)\}$ and $\{(0,0),(3,1),(1,2),(4,3)\}$. Since {4,7,10}-PBDs of order 16, 22, 25, and 34 all exist, this settles the remaining cases for index 2.

Putting the pieces together, we obtain:

Theorem 50 *A 4-MGDD of type n^m and index λ exists whenever $\lambda(m-1)(n-1) \equiv 0 \pmod 3$ and $m,n \geq 4$, except when $\lambda = 1$ and $\{m,n\} = \{6,4\}$, and possibly when $\lambda = 1$ and $\{m,n\} \in \{\{6,16\},\{6,22\},\{6,34\},\{6,25\},\{10,15\}\}$.*

Chapter 4

Related Codes

In this chapter, we study codes that are related to combinatorial designs.

4.1 Erasure Codes

In this section, we study erasure codes. First of all, we need to know what erasure codes are.

In order to enhance the performance of disk access in a computer system, records to be stored on disk are often partitioned into small packets, and each packet is stored on a separate disk. This permits the storage and subsequent retrieval(s) of the record to proceed by parallel access to all disks upon which the packets are stored. More parallelism in the read/write operation in the cost of disks promotes the use of large collections of physically independent disks.

By itself, each disk may be quite reliable. However, as disk arrays become large, failure of one or more disks becomes likely. Failure of a disk can take many different forms; here we are concerned with failure resulting in unavailability of the disk (e.g., its total erasure, physical removal, or power loss). One's primary goal in the event of catastrophic failure of one ore more disks is to reconstruct their content; in most transaction-processing systems, a second major requirement is to remain "on the air" during such reconstruction.

Rabin [100] proposed an efficient scheme. In his *information dispersal algorithm* (IDA), there are two parameters, m and n. Each record is partitioned into n packets of (approximately) equal length l, and from these a list of m *images* and each of length l are computed. The critical feature of the encoding is that from any n of the m images, one can recover the n packets of the record (in IDA, this is done by solving a small linear system of equations). Rabin observes that one can make $\frac{m}{n}$ as close to 1 as desired, so that the overhead in redundant storage is relatively small. Three basic operations must be supported: read, write, and reconstruction. Of course, reconstruction is a combination of read and write operations. Rabin's IDA is particularly well-suited to applications in which loss data is frequent, since it has no preference for one set of n surviving disks over another set. A read operation must always be accompanied by a recovery of the packets from the images.

In a typical disk systems, however, one expects that most read operations performed will not encounter a disk failure. Hence, it is beneficial to design schemes in which, in the absence of a failure, no recovery of packets from images is needed, but rather packets are simply read from disk. Hellerstein *et al.* [67] consider this situation, and our investigation follows on from theirs.

The essential features of the schemes considered are as follows. Disks are assigned to be either *information disks* which contain packets of data records in plain text, or *check disks* which contain redundant information to cope with disk loss. The *check-disk overhead* is the ratio of check disks to information disks, and reflects the cost of redundant storage.

Each check disk is associated with a subset of the information disks, and its content is determined by parity encoding. To be precise, its content is the modulo 2 sum of its information disks. The *check group size* of a check disk is one more than the number of information disks with which it is associated. Check group size indicates the number of disks to be accessed during reconstruction of a failed disk; for load balancing reasons, uniform check group size is desirable, and cost of reconstruction makes small check group size desirable.

Dually, each information disk has an associated set of check disks. An update on the information disk requires an update on each of its check disks. The *update penalty* for an information disk is the number of associated check disks. Again, it is desirable for update penalties to be uniform and small. Since updates of data are taken to be much more frequent than reconstruction of lost disks, the update penalty is of more concern typically than the check group sizes.

Let $C_1, C_2, \ldots, C_c$ be the set of c check disks, and $F_1, F_2, \ldots, F_f$ be the set of f information disks. A *scheme* is a $c \times f$ binary matrix $A = (a_{ij})$ in which $a_{ij} = 1$ if and only if F_j is in the check group C_i. Adjoining a $c \times c$ identity matrix to form $[A|I]$ gives a $c \times (f + c)$ *parity check matrix* of a binary linear code, in which the columns are indexed by the information *and* the check disks. Binary linear codes have been very extensively studied in connection with error detection and correction when binary data is corrupted by bit inversions rather than data loss [89]; one essential difference is that the positions of the errors are known in the case of erasure.

Now let us consider the loss of k disks (both information and check disks can fail). If $[A|I]$ has a set of k or fewer linearly independent columns, loss of the corresponding disks cannot be corrected; however, as observed in [67], when the failed disks induce a set of linearly independent columns, their erasure can be corrected. Thus a scheme A is called *k-erasure correcting* whenever every set of k columns of $[A|I]$ contains no nonempty set of dependent columns. Precisely the same condition determines when the parity check matrix $[A|I]$ gives a k-error detecting code [89], but the study of codes for error detection has not focussed on update penalties.

The magnitude of the update penalties in a k-erasure correcting code are of paramount importance. Evidently, if k erasures are to be survived, every update must affect the content of at least $k+1$ disks (one information disk and k check disks, in our setting). Hence the update penalties must all be at least k. Henceforth we consider only those codes in which all update penalties are equal to k, the minimum possible.

It is convenient to recast some of the prior discussion in alternate language. A *set system* $(V, \mathcal{B})$ is a set V of *elements* (or *points*), and a collection $\mathcal{B}$ of subsets of V called *blocks*. Associated with a scheme A is a set system

$$(\{C_1, C_2, \ldots, C_e\}, \{\{C_i : a_{ij} = 1\} : 1 \leq j \leq f\}).$$

In this language, the check-disk overhead is $|V|/b$ where $b = |\mathcal{B}|$, the update penalties are the block sizes, and the check group sizes are the *replication numbers* which specify in how many blocks each elements is contained.

A *configuration* in a set system $(V, \mathcal{B})$ is a set system $(W, \mathcal{C})$ with $W \subseteq V$ and $\mathcal{C} \subseteq \mathcal{B}$. In a configuration, an element is *even* if it occurs on an even number of blocks, *odd* otherwise. When a scheme is k-erasure correcting, this translates to a requirement that certain configurations not appear in the associated set system.

Lemma 154 *A set system is associated with a k-erasure correcting code if every configuration of $t \leq k$ blocks in set system has at least $k+1-t$ odd elements.*

We have noted already that a set system with a block of size k cannot be associated with a $(k+1)$-erasure correcting code. Indeed an uncorrectable $(k+1)$-erasure corresponds to the single block of size k and its k check disks. Following [67], such a $(k+1)$-erasure is called *bad*. They observe that, with update penalty k, one can nonetheless hope to correct all $(k+1)$-erasures except for bad ones. In fact, when all blocks have size k, it can happen that all t-erasures for $t \leq 2k+2$ are correctable except for bad $(k+1)$-erasures. With this in mind, we call a scheme (code, or set system) (k,l)-*erasure correcting* if all update penalties are k, it is k-erasure correcting, and in addition corrects all t-erasures for $k+1 \leq t \leq l$ except for bad $(k+1)-$erasures.

In a (k,l)-erasure correcting code, an erasure is *unacceptable* if it is a t-erasure for $t \leq l$ which cannot be corrected, and is not a bad $(k+1)$-erasure.

Lemma 155 *A set system is associated with a (k,l)-erasure correcting code if every configuration of $2 \leq t \leq l$ blocks has at least $l+1-t$ odd elements.*

Proof: An unacceptable erasure corresponds precisely to such a configuration, along with the check disks for the odd elements. □

4.2 Anti-Pasch STSs

A *Steiner triple system* $S = (V, \mathcal{B})$ of *order* v, briefly STS(v), is a collection $\mathcal{B}$ of triples (3-elements subsets) on a set V, $|V| = v$, such that each unordered pair of elements of V is contained in exactly one triple from $\mathcal{B}$. It is well known that an STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$; such orders are *admissible.*

A (k, ℓ)-*configuration* in an STS($V, \mathcal{B}$) is a subset of ℓ triples of $\mathcal{B}$ whose union is a k-element subset of V. Two particular configurations are of interest here. The *Pasch configuration* or *quadrilateral*, P, is the $(6, 4)$-configuration on elements (say) a, b, c, e, d, f with the triples $\{a, b, c\}$, $\{a, d, e\}$,$\{f, b, d\}$ and $\{f, c, e\}$. The *mitre*, M, is the $(7, 5)$-configuration on 7 elements a, b, c, d, e, f, g with the 5 triples $\{a, b, e\}$, $\{a, c, f\}$, $\{a, d, g\}$, $\{b, c, d\}$ and $\{e, f, g\}$; a is the *centre* or *central element* of the mitre. An STS is *anti-Pasch* (or *anti-mitre*) if it does not contain P (or M, respectively). For instance, the unique STS of order 7 and one of the two STS (the cyclic system) of order 13 are anti-mitre but contain P, whereas the unique STS of order 9 is anti-Pasch but contains mitres. Of the eighty STS(15), four (Nos. 1,2,3, and 16 in [90]) are anti-mitre, while one (No, 80) is anti-Pasch.

The problem of characterizing those v for which there exists an anti-Pasch STS of order v and anti-mitre STS of order v appears to be difficult. For every $v \equiv 3 \pmod 6$, an anti-Pasch STS(v) is known to exist [29]. There is no anti-Pasch STS of order 7 or 13; while it has been conjectured that an anti-Pasch STS(v) exists for all other $v \equiv 1 \pmod 6$. This remains far from settled. Nevertheless, substantial progress towards settling this conjecture has been made [29, 60, 112]. Also, progress has been made on anti-mitre STS [47].

It has been long known that all affine spaces over $\mathbf{F}_3$ are anti-Pasch STS; meanwhile all projective spaces over $\mathbf{F}_2$ contain a maximum number of Pasch configuration of a given order [112]. It is natural to ask for what orders v there exists a anti-Pasch STS. In another context, Erdős conjectures that for every positive integer l, there exists a STS(v) such that it is free of any $(l + 2, l)$ configuration. In the case of $l = 4$, this coincides with anti-Pasch STS.

Before we continue, we give the following connection to erasure codes.

Lemma 156 *There exists an anti-Pasch STS(v) if and only if there exists a* $(3,5)$*-erasure correcting code with* v *check disks and update penalty 3.*

Proof: Trivial. □

Next, we present three recursive constructions of anti-Pasch STS.

4.2.1 Stinson and Wei's Construction

In this subsection, we extend the second recursive construction of Stinson and Wei [112]. This is a singular direct product construction. It employs latin squares with certain properties. A *subsquare* of a latin square is a square subarray that is itself a latin square. A latin square is an *N_2-latin square* if it contains no subsquare of order 2. An N_2-latin Square of order n exists for all $n \geq 3$ and $n \neq 4$ [73, 74, 93].

We need N_2-latin squares with additional properties, similar to (but weaker than) the "special" latin squares in [112]. An *oneroan square* of order $2w$ is an N_2-latin square L of order $2w$ with rows, columns and symbols indexed by $\{0,1,\ldots,n-1\}$, and enjoying three properties:

1. $\{L(2i,s), L(2i+1,s)\} \neq \{2j, 2j+1\}$ for $0 \leq i,j < w$;
2. $\{L(s,2i), L(s,2i+1)\} \neq \{2j, 2j+1\}$ for $0 \leq i,j < w$;
3. $L(2i,2j)$, $L(2i,2j+1)$, $L(2i+1,2j)$, and $L(2i+1,2j+1)$ are all distinct when $0 \leq i,j < w$.

Stinson and Wei used similar N_2-latin squares to prove:

Theorem 51 *[112] If there is a QFSTS(u) and $u \equiv 1 \pmod 4$, and $u-1$ has an odd divisor exceeding three, then there is a QFSTS($3(u-1)+1$).*

We extend Theorem 51 to relax the condition that $u \equiv 1 \pmod 4$, and the condition on divisors.

Lemma 157 *There is an oneroan square of order $2w$ whenever $w \geq 4$ except possibly when $w = 6$.*

Proof: First we prove that whenever an oneroan square of order $2w$ exists, one of order $4w$ also exists. Let π be the permutation of rows which interchanges rows $2i$ and $2i+1$ for $0 \leq i < w$. For a latin square L, denote by $L + a$ the latin square obtained by adding a to each entry. Then when N is an oneroan square of order $2w$ ($w > 1$), the square $N' = \begin{bmatrix} N & N+2w \\ \pi(N)+2w & N \end{bmatrix}$ is an oneroan square of order $4w$. That N' is latin and satisfies properties (1), (2), and (3) is immediate. To verify that it is an N_2-latin square, observe that a putative subsquare of order two selects one entry from each quadrant in N', but the application of π destroys each 2×2 subsquare which would otherwise be formed.

Next we treat cases when $w \geq 5$ is odd, which is essentially the case treated in [112]. Form a $2w \times 2w$ array L by setting

$$L(2i+a, 2j+b) = \begin{cases} i+j \bmod w & \text{if } a=b=0 \\ i+j-2 \bmod w & \text{if } a=b=1 \\ (i+j \bmod w)+w & \text{if } a=0, b=1 \\ (i+j+2 \bmod w)+w & \text{if } a=1, b=0 \end{cases}$$

for $0 \leq i,j < w$. That L is a latin square with properties (1), (2), and (3) is immediate. That L is an N_2-latin square follows from a consideration of the possible positions of subsquares of order 2.

Now an oneroan square of order 8 is:

$$\begin{array}{cccccccc}
0 & 2 & 1 & 3 & 4 & 6 & 5 & 7 \\
5 & 7 & 4 & 6 & 2 & 0 & 3 & 1 \\
1 & 3 & 5 & 7 & 0 & 4 & 6 & 2 \\
4 & 6 & 0 & 2 & 3 & 7 & 1 & 5 \\
2 & 4 & 3 & 5 & 7 & 1 & 0 & 6 \\
7 & 1 & 6 & 0 & 5 & 3 & 2 & 4 \\
3 & 5 & 7 & 1 & 6 & 2 & 4 & 0 \\
6 & 0 & 2 & 4 & 1 & 5 & 7 & 3
\end{array}$$

Letting A be the oneroan square of order 8, the square $\begin{bmatrix} A & A+8 & A+16 \\ A+8 & A+16 & A \\ A+16 & A & A+8 \end{bmatrix}$ is an oneroan square of order 24. □

Theorem 52 *If there is a QFSTS*(u)*, then there exists a QFSTS*$(3(u-1)+1)$.

Proof: Let $u-1=2w$. Let X, Y and Z be disjoint sets of cardinality $2w$, and let $\infty \notin X \cup Y \cup Z$. Denote the elements of X, Y and Z by $X = \{x_i : 0 \le i < 2w\}$, $Y = \{y_i : 0 \le i < 2w\}$ and $Z = \{z_i : 0 \le i < 2w\}$.

Let $(X \cup \{\infty\}, \mathcal{A})$, $(Y \cup \{\infty\}, \mathcal{B})$ and $(Z \cup \{\infty\}, \mathcal{C})$ be QFSTS(u). Without loss of generality, we can stipulate that the STSs contain the blocks $\{\infty, m_{2i}, m_{2i+1}\}$ for $0 \le i < w$, and $m = x$, y, or z as appropriate.

Let L be an oneroan square of order $2w$. Then define a set of blocks $\mathcal{D} = \{\{x_i, y_j, z_{L(i,j)}\} : 0 \le i < 2w, 0 \le j < 2w\}$.

Now $(\{\infty\} \cup X \cup Y \cup Z, \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D})$ is a STS$(3(u-1)+1)$. We prove that it is quadrilateral-free.

Let Q denote the four blocks in a hypothetical quadrilateral. There are the following possible distributions of the four blocks to consider:

(i) $Q \in \mathcal{A}$, $Q \in \mathcal{B}$ and $Q \in \mathcal{C}$. There are no quadrilaterals contained in $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$, since the STS(u)s are quadrilateral-free.

(ii) $Q \in \mathcal{D}$. Such a quadrilateral must look like

$$\{x_i, y_j, z_k\}, \{x_i, y_h, z_g\}, \{x_f, y_j, z_g\}, \{x_f, y_h, z_k\}.$$

Then $L(i,j) = L(f,h) = k$ and $L(f,j) = L(i,h) = g$, so L has a subsquare of order two, a contradiction.

(iii) $|Q \cap \mathcal{A}| = 1$, $|Q \cap \mathcal{B}| = 1$ and $|Q \cap \mathcal{D}| = 2$. Then Q has the form

$$\{\infty, x_i, x_j\}, \{\infty, y_g, y_h\}, \{x_i, y_g, z_k\} \text{ and } \{x_j, y_h, z_k\}$$

so that $\{i,j\} = \{2a, 2a+1\}$ and $\{g,h\} = \{2b, 2b+1\}$. But $L(i,g) = L(j,h)$, contradicting property (3).

(iv) $|Q \cap \mathcal{A}| = 1$, $|Q \cap \mathcal{C}| = 1$ and $|Q \cap \mathcal{D}| = 2$. Then Q has the form

$$\{\infty, x_i, x_j\}, \{\infty, z_g, z_h\}, \{x_i, y_k, z_g\} \text{ and } \{x_j, y_k, z_h\}$$

so that $\{i,j\} = \{2a, 2a+1\}$ and $\{g,h\} = \{2b, 2b+1\}$. Then $\{L(i,k), L(j,k)\} = \{2b, 2b+1\}$, contradicting property (1).

(v) $|Q \cap \mathcal{B}| = 1$, $|Q \cap \mathcal{C}| = 1$ and $|Q \cap \mathcal{D}| = 2$. Then Q has the form

$$\{\infty, y_i, y_j\}, \{\infty, z_g, z_h\}, \{x_k, y_i, z_g\} \text{ and } \{x_k, y_j, z_h\}$$

so that $\{i,j\} = \{2a, 2a+1\}$ and $\{g,h\} = \{2b, 2b+1\}$. Then $\{L(k,i), L(k,j)\} = \{2b, 2b+1\}$, contradicting property (2).

No other possible distributions of Q need to be considered. Hence, the STS$(3(u-1)+1)$ is quadrilateral-free. □

Our next construction generalizes this.

Theorem 53 *If there exists a QFSTS(v) and QFSTS(u), and $u > 3$, then there exists a QFSTS($v(u-1)+1$).*

Proof: Suppose there exists a QFSTS(v) on V. Let $I_{u-1} = \{0, 1, \ldots, u-2\}$. For every block $\{a,b,c\}$, put the TD$(3, u-1)$ on $\{a,b,c\} \times I_{u-1}$ which arises from an oneroan square of order $u-1$. For every $v \in V$, put a QFSTS(u) on $\{\infty\} \cup (\{v\} \times I_{u-1})$. We claim that this produces a QFSTS($v(u-1)+1$). First of all, if there is a Pasch configuration in the STS which involves the point ∞, then the Pasch configuration must lie in one of the TD$(3, u-1)$s together with ∞, which is a contradiction. Suppose the Pasch configuration involves a block in the subsystem QFSTS(u); then the other three points in the Pasch configuration must come from distinct points in QFSTS(v). Projecting the Pasch configuration back to the QFSTS(v) yields a pair of points appearing in more than one block, a contradiction. So the points in the putative Pasch configuration must arise from distinct points in the QFSTS(v). Projecting the Pasch configuration back to the QFSTS(v) yields a Pasch configuration in the QFSTS(v), the final contradiction needed. □

4.2.2 Lu's Construction

We employ a construction of Lu [88] to obtain a construction of anti-Pasch STS.

Theorem 54 *Suppose there exists a QFSTS*$(n+2)$ *and a QFSTS*$(m+2)$. *Then there exists a QFSTS*$(mn+2)$.

Proof: Let $(\{a,b\} \cup \mathbb{Z}_n, \mathcal{B})$ be a QFSTS$(2+n)$, and let $(\mathbb{Z}_m \cup \{a,b\}, \mathcal{A})$ be a QFSTS$(2+m)$ with $\{a,b,0\} \in \mathcal{A}$. Let $N_{ab} = \{\{x_i, x_j\} : m \in \{a,b\}$ and $\{m, x_i, x_j\} \in \mathcal{A}\}$. N_{ab} is a set of pairs on $\mathbb{Z}_m \backslash \{0\}$ with every element appearing in two pairs. Each pair can then be ordered so that each element is the first element of one pair, and the second element of another; call this set of ordered pairs Q_{ab}. Define a permutation π on $\mathbb{Z}_m \backslash \{0\}$ by setting $\pi(i) = j$ whenever $(i,j) \in Q_{ab}$. Subsequently, it is crucial that since $\{a,b,0\}$, $\{a,i,\pi(i)\}$ and $\{b,\pi(i),\pi^2(i)\}$ appear in $\mathcal{A}$ (or the three blocks obtained by interchanging a and b appear in $\mathcal{A}$), no block of the form $\{0,i,\pi^2(i)\}$ can appear in $\mathcal{A}$ since it is anti-Pasch.

We construct a STS$(2+mn)$ on the point set $(\mathbb{Z}_m \times \mathbb{Z}_n) \cup \{a,b\}$ with triples of the following forms where $x_1, x_2, x_3 \in \mathbb{Z}_m$, $y_1, y_2, y_3 \in \mathbb{Z}_n$.

(i) $\{(0,y_1),(0,y_2),(0,y_3)\}$ whenever $\{y_1,y_2,y_3\} \in \mathcal{B}$, and $\{\ell,(0,y_2),(0,y_3)\}$ whenever $\{\ell,y_2,y_3\} \in \mathcal{B}$ and $\ell \in \{a,b\}$, and $\{a,b,(0,y_3)\}$ when $\{a,b,y_3\} \in \mathcal{B}$;

(ii) $\{(x_1,y_1),(x_1,y_2),(x_2,y_3)\}$ where $(x_1,x_2) \in Q_{ab}$ and $y_1+y_2 \equiv 2y_3 \pmod n$.

(iii) $\{m,(x_1,y_1),(x_2,y_1)\}$ where $m = a$ or b, $\{m,x_1,x_2\} \in \mathcal{A}$.

(iv) $\{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$ where $\{x_1,x_2,x_3\} \in \mathcal{A}$, $x_1 < x_2 < x_3$ and $y_1+y_2+y_3 \equiv 0 \pmod n$.

First of all, we prove that the construction gives a STS$(2+mn)$. The number of type (i) blocks is $(n+2)(n+1)/6$. The number of type (ii) blocks is $(m-1)n(n-1)/2$, the number of type (iii) blocks is $(m-1)n$. The number of type (iv) is $[((m+2)(m+1)/6)-m)]n^2$. So the total number of blocks is $(mn+2)(mn+1)/6$ as expected. Therefore, it suffices to show that every pair of S is contained in a triple. All possibilities are exhausted as follows:

(1) Pairs $\{a,b\}$, $\{a,(0,y_1)\}$, $\{b,(0,y_2)\}$ and $\{(0,y_1),(0,y_2)\}$ are contained in some type (i) triple.

(2) When $x \in \mathbb{Z}_m \backslash \{0\}$, $\{(x,y_1),(x,y_2)\}$ is contained in some type (ii) triple, since x is the first element of some pair in Q_{ab}. Since n must be odd, the equation $y_1+y_2 \equiv 2y_3 \pmod{n}$ has a solution.

(3) Pairs $\{a,(x,y)\}$ and $\{b,(x,y)\}$ are contained in some type (iii) triple.

(4) If $\{x,x'\} \in N_{ab}$, then $\{(x,y_1),(x',y_2)\}$ for $x \neq x'$ is contained in a type (ii) or (iii) triple; if $\{x,x'\} \notin N_{ab}$, then $\{(x,y_1),(x',y_2)\}$ for $x \neq x'$ is contained in some type (iv) triple.

Next, we show that the STS$(2+mn)$ is anti-Pasch. Assume to the contrary that there exists a Pasch configuration in the STS. We treat all of the cases.

(a) Suppose a block in the Pasch configuration contains the block $\{a,b,(0,0)\}$. There must be blocks of the form $\{a,(i,r),(\pi(i),r)\}$ and $\{b,(\pi(i),r),(\pi^2(i),r)\}$. Since no blocks of the form $\{0,i,\pi^2(i)\}$ appear in $\mathcal{A}$, this is a contradiction.

(b) Suppose the Pasch configuration contains the block $\{p,(0,m),(0,n)\}$ for $p \in \{a,b\}$. It must also contain a block of the form $\{p,(i,r),(\pi(i),r)\}$. Without loss of generality, the remaining blocks are $\{(0,m),(i,r),(x,y)\}$ and $\{(0,n),(\pi(i),r),(x,y)\}$. This implies that both $\{0,i,x\}$ and $\{0,\pi(i),x\}$ are blocks in $\mathcal{A}$, a contradiction.

(c) Suppose there exists a block of the form $\{(0,x),(0,y),(0,z)\}$. Then the other blocks must be of the form $\{(0,x),(r_1,r_2),(s_1,s_2)\}$, $\{(0,y),(r_1,r_2),(t_1,t_2)\}$ and $\{(0,z),(s_1,s_2),(t_1,t_2)\}$. We obtain a contradiction by restricting to the first coordinates.

(d) Suppose there exists a block of the form $\{m,(i,r),(\pi(i),r)\}$ where $m \in \{a,b\}$. There must be another block of the form $\{m,(j,s),(\pi(j),s)\}$. If $i = j$, then the first coordinates of the third point containing the points (i,r) and (j,s) is $\pi(i)$. Meanwhile, the first coordinates of the third point containing the points $(\pi(i),r)$ and $(\pi(j),s)$ is $\pi^2(i)$. This is impossible. The third point on the block joining (i,r) and $(\pi(j),s)$ is $(i,2s-r)$, while the third point on the block joining (j,r) and $(\pi(i),s)$ is $(i,2r-s)$. We must have $2s-r \equiv 2r-s \pmod{n}$, so $3r \equiv 3s \pmod{n}$, but $n+2 \equiv 1,3 \pmod{6}$ and thus $r \equiv s \pmod{n}$. So the case when $i = j$ is impossible. Next, we consider the case when $j = \pi^2(i)$. If the last point in the Pasch configuration does not have first coordinate in $\{i,\pi(i),\pi^2(i),\pi^3(i)\}$, then projecting the design on its first coordinates gives us a Pasch configuration in the STS$(2+m)$. So, the only possibility is that we have $\{(\pi(i),r),(\pi^2(i),s),(\pi(i),2s-r)\}$ and $\{(i,r),(\pi(i),2s-r),(\pi^3(i),s)\}$. But the last block is impossible, because $\{\{m,i,\pi(i)\},\{i,\pi(i),\pi^3(i)\}\} \subset \mathcal{A}$. For the remaining cases, the sixth point in the Pasch configuration must have first coordinate different from $\{i,\pi(i),j,\pi(j)\}$, giving a Pasch configuration in the STS$(2+m)$.

(e) Suppose there is a block of the form $\{(i,r),(i,2s-r),(\pi(i),s)\}$. Suppose (i,r) is also on a block $\{(i,r),(i,2t-r),(\pi(i),t)\}$. We consider the possible first coordinates of the sixth point in the Pasch configuration. If $(i,2s-r)$ and $(i,2t-r)$ are joined, then the first coordinate must by $\pi(i)$. But no block has all three points with first coordinate $\pi(i)$. If $(i, 2s-r)$ is joined to $(\pi(i), s)$, then the point must be $(i, 2t-2s+r)$. Similarly, the remaining pair of points force the final point to be of the form $(i, 2s-2t+r)$. To form a Pasch configuration , we must have $2t-2s+r \equiv 2s-2t+r \pmod{n}$, and so $t = s$ and a contradiction. Next, if (i, r) is also on a block of the form $\{(i,r),(j,u),(k,-r-u)\}$, a block is needed containing $(i, 2s-r)$ and (j, u) and hence the first coordinates must be k. The first coordinates in the last block must be k, k and $\pi(i)$. But $k \neq i$ as we have a block $\{(i,r),(j,u),(k,-r-u)\}$, a contradiction. Hence, in any Pasch, no block has two first coordinates the same.

(f) Suppose there is a block of the form $\{(i,r),(j,s),(k,-s-r)\}$. Suppose the other block through (i, r) is of the form $\{(i,r),(j_1,s_1),(k_1,-s_1-r)\}$ where $\{j_1,k_1\} \neq \{j,k\}$. Then the last must must have different first coordinates, and corresponds a Pasch configuration in STS$(2+m)$. So the block must be of the form $\{(i,r),(j,t),(k,-r-t)\}$. To form a Pasch configuration, we must have $x+s-r-t \equiv 0 \pmod{n}$ and $x+t-s-r \equiv 0 \pmod{n}$, so $s = t$, a contradiction.

□

4.2.3 GDD Constructions

A TD$(3,n)$ without any-sub TD$(3,2)$ is equivalent to a N_2-latin square of order n. We call such a TD$(3,n)$ a N_2-TD$(3,n)$.

Theorem 55 *If there exists a QFSTS$(2v+1)$ and a QFSTS$(2n+1)$, and $n > 4$, then there exists a QFSTS$(2vn+1)$.*

Proof: Delete a point from the QFSTS($2v+1$) to form a 3-GDD of type 2^v. Give weight n using an N_2-TD($3,n$) to produce a 3-GDD of type $(2n)^v$. Add one infinite point ∞, and on each group together with ∞, place a copy of the QFSTS($2n+1$) so that when $\{\infty, a, b\}$ is a triple, a and b arise from different points of the 3-GDD of type 2^v. Call the triples of the 3-GDD of type $(2n)^v$ *vertical*, and the triples of the STS($2n+1$)s *horizontal*. The result is an STS($2vn+1$), which we prove is anti-Pasch.

Suppose to the contrary that a Pasch configuration is present. If it contains ∞, it contains two horizontal and two vertical triples, since the STS($2n+1$) used is anti-Pasch. The placement of the blocks containing ∞, and the fact that the STS($2v+1$) is anti-Pasch, ensures that the two vertical blocks are disjoint and hence not in a Pasch configuration. Hence any Pasch configuration must involve six points other than ∞. Then there cannot be two horizontal triples (since they are either disjoint or from the same QFSTS($2n+1$). If there is one horizontal triple, the three vertical triples cannot involve only three further points. So all triples are vertical. However, at most one can arise from each N_2-TD($3,n$) used, and hence any Pasch configuration would correspond to a Pasch configuration in the QFSTS($2v+1$), which is a contradiction. □

We have one more recursive construction using GDDs.

Theorem 56 *Let $t > 4$, $w, n > 0$, $wn > 8$, and $wn \equiv 0 \pmod 2$. If there exist QFSTS($2v+1$), QFSTS($2wn+1$), and QFSTS($wn(v-1)+1$), then there exists a QFSTS($wn(3v-1)+1$).*

Proof: Take an N_2-TD($3,v$), and delete a point to obtain a $\{3,v\}$-GDD of type $2^v(v-1)^1$. Give weight wn to each point, using an N_2-TD($3,wn$) for the blocks of size three. For blocks of size v, start with the QFSTS($2v+1$) and delete a point to form a $\{3\}$-GDD of type 2^v; then inflate using an N_2-TD$(3,(wn)/2)$ to obtain a $\{3\}$-GDD of type $(wn)^v$ to use in the inflation of blocks of size v. The result is a $\{3\}$-GDD of type $(2wn)^v(wn(v-1))^1$. Add an infinite point ∞, and fill groups using QFSTS$(2wn+1)$ and QFSTS$(wn(v-1)+1)$, so that blocks containing ∞ have points arising from two different points of the $\{3,v\}$-GDD of type $2^v(v-1)^1$. The proof of this theorem is a special case of a general construction in next subsection and thus omitted □

4.2.4 Summary

In summary, we state:

Theorem 57 *If a QFSTS(v) exists whenever $v > 100$ and*

1. $v = 7^1 13^2$;
2. $v = p$ *and* $p \equiv 13, 29 \pmod{72}$ *is a prime;*
3. $v = 7p$ *and* $p \equiv 25, 43, 61 \pmod{72}$ *is a prime;*
4. $v = 13p$ *and* $p \equiv 1, 19, 55 \pmod{72}$ *is a prime;*
5. $v = pq$ *where* $p, q \equiv 5 \pmod 6$ *are primes and* $pq \equiv 13, 31, 67 \pmod{72}$*;*
6. $v = 7pq$ *where* $p, q \equiv 5 \pmod 6$ *are primes and* $pq \equiv 25, 43, 61 \pmod{72}$*; or*
7. $v = 13pq$ *where* $p, q \equiv 5 \pmod 6$ *are primes and* $pq \equiv 1, 19, 55 \pmod{72}$*,*

and

1. $v - 1 = 6p$ *for* $p \equiv 5 \pmod 6$ *a prime; or*

2. *$v-1=12p$ for $p \equiv 1 \pmod 6$ a prime,*

and

1. *$v-2=5^1 11^2$;*

2. *$v-2=p$ and $p \equiv 11, 29, 65 \pmod{72}$ is a prime, and the multiplicative order of $-2 \pmod p$ is not singly even;*

3. *$v-2=5p$ and $p \equiv 13, 31, 49 \pmod{72}$ is a prime; or*

4. *$v-2=11p$ and $p \equiv 1, 19, 55 \pmod{72}$ is a prime,*

then a QFSTS(v) exists whenever $v \equiv 1, 3 \pmod 6$ except when $v \in \{7, 13\}$.

Proof: If $v < 100$, see [47] and references therein. If $v \equiv 3 \pmod 6$, see [60]. For the remaining cases, we proceed inductively. If $v \equiv 1, 7 \pmod{18}$, write $u = \frac{v+2}{3}$. Apply Theorem 52, observing that $v = 3(u-1)+1$. If $v \equiv 49 \pmod{72}$, apply Theorem 55 with $2 \cdot 4 + 1$ and $2 \cdot \frac{v-1}{8} + 1$. It remains to treat $v \equiv 13, 31, 67 \pmod{72}$.

If $v = q_1 q_2$ with $q_i \equiv 1 \pmod 6$ and $19 \leq q_i$ for $i = 1, 2$, then direct product produces the $QFSTS(v)$. If $v = 7^i 13^{3-i}$, the only case with $v \equiv 13 \pmod{18}$ is $v = 7^1 13^2$. If $v \equiv 67 \pmod{72}$ and v is prime, the Netto triple system is a QFSTS(v) (see [47]).

Now when $v \equiv 13 \pmod{18}$, $\frac{v-1}{6} \equiv 2 \pmod 3$; and since $v \not\equiv 49 \pmod{72}$, $v - 1 \not\equiv 0 \pmod 8$. So $v - 1 = 6p_1 p_2 \cdots p_k$ or $v - 1 = 12 p_1 p_2 \cdots p_k$, where each p_i is a prime at least 5; in the first case, an odd number of these primes satisfy $p_i \equiv 5 \pmod 6$, and in the second case an even number do. If $p_k \equiv 5 \pmod 6$, $k \geq 2$, apply singular direct product to $2 p_1 p_2 \cdots p_{k-1} + 1$ and $3 \cdot p_k$ to obtain the QFSTS(v). We may suppose then that each $p_i \equiv 1 \pmod 6$, and hence that $v = 12 p_1 p_2 \cdots p_k$.

If $k \geq 2$, and $p_k \geq 19$, apply singular direct product with $12p_1p_2 \cdots p_{k-1}+1$ and p_k. It remains to treat cases with $p_i \in \{7,13\}$ for $1 \leq i \leq k$. If $k \geq 3$, apply singular direct product with $12p_1p_2 \cdots p_{k-2}+1$ and $p_{k-1}p_k$. If $k=2$, apply Theorem 55 with $6p_1+1$ and $2p_2+1$.

Finally, write $v-2 = 5^i 11^j p_1 \cdot \dots \cdot p_k$ where each p_i is a prime other than 2, 3, 5, and 11. Now if $k \geq 2$, apply Theorem 54 with $5^i 11^j p_1 \cdot \dots \cdot p_{k-1}+2$ and p_k+2. If $i,j>1$, or $i,j \geq 1$ and $k=1$, apply Theorem 54 with 57 and $\frac{v-2}{55}+2$. If $i>2$, or $i \geq 2$ and $k=1$, apply Theorem 54 with 27 and $\frac{v-2}{25}+2$. If $j>2$, or $j \geq 2$ and $k=1$, apply Theorem 54 with 123 and $\frac{v-2}{121}+2$. In the cases that remain, $v-2 \equiv 5 \pmod 6$. Now if $k=0$, $v-2=5^i 11^j$ for $i+j \leq 3$, but $i+j$ is odd. If $i+j=1$, these are the two nonexistent cases ($v=7,13$). If $i+j=3$, the only case in which $v \equiv 13 \pmod{18}$ is $v-2=605=5 \cdot 11^2$. So suppose that $k=1$, so that $i+j \leq 1$. If $p_1 \equiv 1 \pmod 6$ then $i+j=1$, and if $p_1 \equiv 5 \pmod 6$ then $i=j=0$. In the latter case, $p_1 \equiv 11, 29, 65 \pmod{72}$. In the former, $v-2$ is either $5p_1$ or $11p_1$. If $5p_1 \equiv 11, 29, 65 \pmod{72}$, then $p_1 \equiv 13, 31, 49 \pmod{72}$. If $11p_1 \equiv 11, 29, 65 \pmod{72}$, then $p_1 \equiv 1, 19, 55 \pmod{72}$. Some of the remaining cases are handled by a theorem of Grannell, Griggs, and Phelan [57] that when, for every prime divisor p of $v-2$, -2 has singly even order modulo p, a QFSTS(v) exists. □

4.3 Anti-Pasch Packings

Let $(V, \mathcal{B})$ be a PBD(v, K). We say that $(V, \mathcal{B})$ is a QFPBD if for every four blocks, the six intersection points do not induce a Pasch configuration. If a certain set of blocks forms a parallel class, we denote a QFPBD by a QFGDD with the corresponding group type.

A QFPBD($v, K \cup \{k^\star\}$) is a PBD($v, K \cup \{k^\star\}$) and a QFPBD($v, K \cup \{l\}$).

QFPBDs are of special interest because of the following lemma.

Lemma 158 *If there exists a K-QFPBD on v points, and for every $k \in K$ there exists a QFSTS on k points, then there exists a QFSTS(v).*

Proof: Fill in each block of size k by a QFSTS(k). This gives a QFSTS(v). □

We present a singular indirect product type construction for QFPBDs.

Let $(\mathbb{Z}_m \cup \{a, b\}, \mathcal{A})$ be a QFSTS($2+m$) with $\{a, b, 0\} \in \mathcal{A}$. Let $N_{ab} = \{\{x_i, x_j\} : m \in \{a, b\}$ and $\{m, x_i, x_j\} \in \mathcal{A}\}$. N_{ab} is a set of pairs on $\mathbb{Z}_m \backslash \{0\}$ with every element appearing in two pairs. Each pair can then be ordered so that each element is the first element of one pair, and the second element of another; call this set of ordered pairs Q_{ab}. Define a permutation π on $\mathbb{Z}_m \backslash \{0\}$ by setting $\pi(i) = j$ whenever $(i, j) \in Q_{ab}$. By permuting the elements, we can assume that π satisfies $\pi(i) = i + 1$ or $\pi(i) = i + 1 - d$ where d is the length of the cycle containing i.

Let D be the set of possible cycle lengths of π. Let $x + yD$ be the set $\{x + yd : d \in D\}$. If A and B are two sets, define $AB^{-1} = \{ab^{-1} : a \in A$ and $b \in B\}$.

A QFSTS(v) *admits* a set D if there exists two points in the QFSTS so that the all cycle lengths of the induced permutation π is are the set D.

Let $D^\star = D \cup -D \cup D - 1 \cup 1 - D \cup (1 - D)(1 - D)^{-1} \cup (D - 1)(1 - D)^{-1}$.

Theorem 58 *If there exist three elements, $M = \{\alpha, \beta, \gamma\}$ for which MM^{-1} is disjoint from $D^\star$, and each element in M is relatively prime to m, then there exists a QFPBD($3m + 2, \{3, 5^\star\}$).*

Proof: Let $V = \mathbb{Z}_m \times \{0,1,2\} \cup \{a,b\}$. We construct a QFPBD on V. Let $\alpha \in \mathbb{Z}_m$. We define a function $f_\alpha : V \to V$ as $f_\alpha(x) = x$ if $x \in \{a,b\}$ and $f_\alpha(x) = \alpha x$ if $x \in \mathbb{Z}_m$. If $(\alpha, m) = 1$, then f_α is a bijection. We construct a QFPBD$(3m+2, \{3, 5^*\})$ as follows:

(i) For any block $\{x,y,z\} \in \mathcal{A}$ and $\{x,y,z\} \neq \{a,b,0\}$, we have three blocks $\{(f_{m_i}(x), i), (f_{m_i}(y), i), (f_{m_i}(z), i)\}$ where $(m_i, i) \in \{(\alpha, 0), (\beta, 1), (\gamma, 2)\}$.

(ii) For any $a, b, c \in \mathbb{Z}_m$, construct a block $\{(a,0),(b,1),(c,2)\}$ if $a + b + c \equiv 0 \pmod{m}$.

(iii) Construct a block of size five by taking $\{(0,0),(0,1),(0,2),a,b\}$.

We claim that this construction gives a QFPBD.

(i) If none of four blocks in the Pasch configuration involves either a or b, then it is impossible for them to form a Pasch configuration as the blocks all come from a TD$(3,n)$ which is free of the Pasch configuration.

(ii) If a block in the Pasch configuration is of the form $\{a, b, (0,i)\}$ for some i, then all three other points in the Pasch configuration have a second coordinate i. Hence, this corresponds to a Pasch configuration in S.

(iii) It cannot involve a block of form $\{m, (0,i), (0,j)\}$ where $m \in \{a,b\}$ by considering the other block containing the point m.

(iv) Suppose it has a block in a Pasch configuration of the form $\{m, (tx,i), (t\pi(x), i)\}$ and another block of form $\{m, (sy,j), (s\pi(y), j)\}$. Then $i \neq j$, since otherwise all four blocks are from a subsystem of order $2+m$. Hence $s \neq t$. We either have $sy + tx = s\pi(y) + t\pi(x)$ or $sy + t\pi(x) = s\pi(y) + tx$. In the former case, three subcases arise. The first one has $\pi(x) = x + 1$ and $\pi(y) = y + 1$, so $st^{-1} = -1$. The second one has $\pi(x) = x + 1$ and $\pi(y) = y + 1 - d_1$ where $d_1 \in D$, so $ts^{-1} = d_1 - 1$. The third case has $\pi(x) = x + 1 - d_2$ and $\pi(y) = y + 1 - d_1$, so $ts^{-1} = (d_1 - 1)(1 - d_2)^{-1}$. In the remaining case, it is similar to check that $ts^{-1} = 1, 1 - d_1, (d_1 - 1)(d_2 - 1)^{-1}$ as appropriate. By our condition on α, β, γ, we avoid all possible Pasch configurations in the QFPBD.

In fact, this is a 3-QFGDD of type $3^{m-1}5^1$ because the TD used in the construction is resolvable as m is odd. □

We define an optimal anti-Pasch packing on v points to be an optimal packing on v points with block size three in which no four blocks form a Pasch configuration. Theorem 58 is of particular interest because of the following lemma. The reason that anti-Pasch packing is of interest because it gives erasure with v check disks, update penalty three with the maximum number of information disks.

Lemma 159 *If there exists a QFPBD($v, \{3, 5^*\}$), then there exists an optimal anti-Pasch packing on v points.*

Proof: Replace the block of size five, $\{a, b, c, e, d\}$, by two blocks of size three $\{a, b, c\}$ and $\{a, d, e\}$. If there exists a Pasch configuration containing both blocks, then all blocks must be contained in $\{a, b, c, d, e\}$. Otherwise, any other Pasch configurations contradict the definition of QFPBD. □

No construction is useful unless we find an application. Hence, we want to find certain class of QFSTS with the corresponding permutation having only a small number of cycle lengths. In the sequel, we discuss the possible cycle lengths from various existing construction for anti-Pasch STS.

Theorem 59 *[57] If for every prime divisor p of $v-2$, -2 has singly even order modulo p, a QFSTS(v) exists.*

For completeness, we restate the construction in [57].

Let $V = \{a, b\} \cup \mathbb{Z}_{v-2}$. We construct the following three collections of blocks. Define a permutation π so that $\pi(i) = -2i$ where all elements are reduced modulo $v-2$. Since -2 has singly even order modulo p for every prime divisor p of $v-2$, then each cycle of π must have even length.

(i) $\{x, y, z\}$ if $x + y + z \equiv 0 \pmod{v-2}$ where x, y, z are distinct.

(ii) A block of the form $\{a, b, 0\}$.

(iii) For every cycle in π, pick a representative element $x \in \mathbb{Z}_{v-2}$, and construct blocks of type $\{a, \pi^{2i}(x), \pi^{2i+1}(x)\}$ and $\{b, \pi^{2i+1}(x), \pi^{2i+2}(x)\}$ for all i.

This give an anti-Pasch STS($2+v$) if for every prime divisor p of v, -2 has singly even order modulo p.

Corollary 17 *Let D be the set of orders of -2 modulo p, a divisor of $v-2$. If all elements of D are singly even, then there exists a QFSTS(v) admitting D.*

To illustrate how the construction works, consider the following. If v is a prime, and -2 is of singly even order modulo p, then all cycles in π must have the same length. If $v = 23$, then the order of -2 is 22 modulo 23. Hence $D = \{23\}$. A simple computation shows that $D^{\star} = \{1, 2, 3, 22, 23, 24\}$, so $M = \{1, 4, 6\}$ satisfies $MM^{-1} \cap D^{\star} = \emptyset$. This gives a QFPBD($71, \{3, 5^{\star}\}$).

Next, we look at the Lu's construction in Theorem 54.

Corollary 18 *If there exists a QFSTS*$(2+m)$ *admitting* M *and a QFSTS*$(2+n)$ *admitting* N*, then there exists a QFSTS*$(2+mn)$ *admitting* $M \cup N$*.*

Lemma 160 *If there exists a QFSTS*(v) *admitting* D *and if* p *is a divisor of* v*, then there exists a QFSTS*$(3v)$ *admitting* $D \cup \{2p\}$*.*

Proof: We use a standard inflation construction. Let the two points that induce D in the QFSTS(v), $(V, \mathcal{B})$, be a and b. Write the point set of the QFSTS(v) as $\mathbb{Z}_v$, identifying a with 0, and identifying b with $m = v/p$. We construct a QFSTS$(3v)$ on $\mathbb{Z}_v \times \{0, 1, 2\}$ as follows:

(i) For every block $\{x, y, z\} \in \mathcal{B}$, construct three blocks $\{(x, i), (y, i), (z, i)\}$ for $i = 0, 1, 2$.

(ii) For every distinct $x, y, z \in \mathbb{Z}_v$ with $x + y + z \equiv 0 \pmod{v}$, construct a block $\{(x, 0), (y, 1), (z, 2)\}$.

We now look at the corresponding permutation induced by points $(0, 0)$ and $(0, m)$. In $\mathbb{Z}_v \times \{0\}$, it gives a set of cycles whose lengths are in D. Now, for every path starting from $(0, 0)$, the cycle must be of the form $(x, 1), (-x, 2), (x - m, 1), (-x + m, 2), (x - 2m, 1), \ldots, (x - pm, 1)$ but $pm \equiv 0 \pmod{v}$. Hence, this gives a cycle of length $2p$. □

QFPBDs are interesting, not only for their importance in constructing optimal anti-Pasch packings, but also as ingredients for inflation type techniques for anti-Pasch STS.

Theorem 60 *Let* $(V, \mathcal{G}, \mathcal{B})$ *be a QFGDD (the* master *QFGDD) with groups* $G_1, G_2, \ldots, G_t$*. Suppose there exists a function* $w : V \to \mathbb{Z}^+ \cup \{0\}$ *(a weight function)* *which has the property that for each block $B = \{x_1, x_2, \ldots, x_k\} \in \mathcal{B}$ there exists a K-QFGDD of type $(w(x_1), w(x_2), \ldots, w(x_k))$ (such a QFGDD is an* ingredient *QFGDD). Then there exists a K-QFGDD of type*

$$(\sum_{x \in G_1} w(x), \sum_{x \in G_2} w(x), \ldots, \sum_{x \in G_t} w(x)).$$

Proof: The construction is a simple modification of Wilson's Fundamental Construction. All we have to prove is that this results in a QFGDD. Suppose there exist four blocks in the design which form a Pasch configuration. If any two blocks are from the same subdesign, then all four blocks must be from the same subdesign. However, the subdesign is a QFGDD, and hence it is impossible. If all four blocks are from different subdesigns, then they cannot form a Pasch configuration since projecting all blocks to the master GDD would give a Pasch configuration there. If the Pasch configuration involves some groups, then it cannot involve more than one group. If the three subsets from a group do not come from the same point, by projecting back to the original design, we have a Pasch in configuration the master GDD. If two points in a group correspond to the same point, then it contradicts the fact that $\lambda = 1$. □

The construction in general form is of limited use as it is very difficult to find designs which are QFGDD with block sizes at least four. Whenever there exists a block of size three, we can only inflate by a constant factor.

The following provide the main ingredients for the construction.

Lemma 161 *There exists a 3-QFGDD of type m^3 for all $m \neq 2, 4$.*

Proof: A N_2-latin square of order m gives a 3-QFGDD of type m^3. Such latin squares are known to exist [73, 74]. □

Lemma 162 *If there exists a 3-QFGDD of type 3^r5^1 and a 3-QFGDD if type 3^s5^1, then there exists a QFSTS$((3r+5)(3s+5))$.*

Proof: Take a 3-QFGDD of type 3^r5^1, give weight five and inflate by a 3-QFGDD of type 5^3, to get a 3-QFGDD of type 15^r25^1. Filling in the hole with a QFSTS(15) and a QFSTS(25), we obtain a QFSTS$(15r+25)$. Now, take a 3-QFGDD of type 3^s5^1 and inflate it by $3r+5$, to get a 3-QFGDD of type $(9r+15)^s(15r+25)$. Fill in the holes with a QFSTS$(9r+15)$ or a QFSTS$(15r+25)$. □

No corresponding analog for the product construction was known when $v = pq$ where both $p, q \equiv 5 \pmod 6$.

It is well known that deleting a point from a PBD gives a GDD. Using this simple observation, we can obtain a further construction.

Theorem 61 *Suppose (V, B) is a QFPBD on $v+1$ points for which removal of a point x gives a K-QFGDD of type T. If, for every $k \in K$, there exists a M-QFGDD of type m^k and there exists a QFPBD$(mt+1, M)$ so that there exists a point in the QFPBD not on any block of size greater than t, then there exists a QFPBD$(mv+1, M)$.*

Proof: Take a K-QFGDD of type T and give every point weight m to get a M-QFGDD of type mT. Now, add a point ∞ to each group, and for every group of size m, put a QFPBD$(mt+1, M)$ with an extra point y and identify y with the new point that we adjoin to the QFGDD. Elements $\{a, b\}$ contained in a block with y in the QFPBD$(mt+1, M)$ are identified with different points if we project the QFGDD of type mT to the QFGDD of type T. We claim that this gives a QFPBD$(mv+1, M)$. It is a PBD$(mv+1, M)$ as the construction is just a singular direct product. To show that it is a QFPBD$(mv+1, M)$, all we need to prove is that there is no Pasch configuration involving the point ∞. If there were a Pasch configuration involving ∞, project all points other than ∞ to the points from the QFGDD of type T and project ∞ to the deleted point in the original QFPBD. □

The following is a generalization of the above theorem. Since, this is of no use in this thesis. We just state this without proof.

We say a QFPBD$(v, K \cup \{l^*\})$ *(m,l)-colorable* if there exists a function such that it maps each point not in the block of size l to $\{1, 2, \ldots, m\}$ so that for any block b containing a point from the block of size l, all other points receive different function values.

Theorem 62 *If there exists a (m,l)-colorable QFPBD$(mt + l, K \cup \{l^*\})$ and a K-QFGDD of type m^r, then there exists a K-QFPBD$(mrt + l, K \cup \{l^*\})$.*

4.4 Anti-Pasch KTSs

In this section, we study anti-Pasch Kirkman triple systems. As shown in the previous section, anti-Pasch STS(v) corresponds to $(3, 5)$-erasure codes with the maximum number of information disks subject to v check disks.

Erasure codes coming from the affine spaces of order 3^n have a 1-balanced ordering ([67]). Chee [33] observed that the problem of constructing $(3, 5)$-erasure codes with optimal check disk overhead having a 1-balanced ordering is equivalent to the existence of the anti-Pasch KTS. A 3-GDD is 3-QFRGDD of the same type if the 3-GDD is a 3-QFGDD, and the QFGDD is resolvable.

In order to state the construction in this section, we need the following notion of resolvability. A set of blocks is called an *α-parallel class* if for every point x is contained in exactly α blocks. A GDD$(X, \mathcal{G}, \mathcal{B})$ is called *A-resolvable* where A is a multiset of positive integers of r elements and if its block set $\mathcal{B}$ admits a partition into subsets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r$ where for each $i = 1, 2, \ldots, r$, there is an $\alpha \in A$ such that $\mathcal{B}_i$ is an α-parallel class. The case when $A = [1^r]$ corresponds to the case of the usual notion of resolvability.

4.4.1 Direct Constructions

In this subsection, we present some direct constructions of anti-Pasch KTS.

The basic necessary condition for the existence of anti-Pasch KTS(v) is $v \equiv 3 \pmod 6$. A first infinite class of the anti-Pasch KTS(v) are of the form $v = 3^n$ which come from the affine spaces. There does not exist an anti-Pasch KTS(15) as the only anti-Pasch STS(15) is no.80 of [90] which is not resolvable. Hence, the smallest open case is when $v = 21$. In [91], 30 nonisomorphic Kirkman triple systems of order 21 are found. However, each of them contains a sub-system of order 7. Hence, none can be anti-Pasch.

Lemma 163 *There exists an anti-Pasch KTS*(33).

Proof: Consider the following KTS(33) taken from [116]. Let $V = \mathbb{Z}_{33}$.

$$\{1,3,6\},\ \{17,19,32\},\ \{9,11,24\},\ \{22,25,13\},$$
$$\{5,8,29\},\ \{27,30,18\},\ \{31,4,23\},\ \{14,20,6\},$$
$$\{15,21,7\},\ \{28,2,12\},\ \{26,0,10\}.$$
$$\{3,10,20\},\ \{1,2,6\},\ \{2,3,7\},$$
$$\{3,4,8\},\ \{1,12,23\}.$$

Let $\pi(x) = x + 3$. The design is generated by letting π act on the set of blocks. The first set of eleven blocks is a parallel class, hence the action of π gives eleven parallel classes; each of the remaining base blocks generate a parallel class. $\square$

Lemma 164 *There exists an anti-Pasch KTS*(39).

Proof: Let $V = \mathbb{Z}_{39}$. Consider

$$\{0,7,16\},\ \{4,10,25\},\ \{1,6,18\},\ \{8,9,11\}$$
$$\{0,8,19\},\ \{0,4,14\},\ \{2,15,28\}.$$

These form the base blocks of an anti-Pasch STS(39) over $\mathbb{Z}_{39}$. The 12 points in the first four starter blocks are distinct $\pmod{13}$. Adding 13 and 26 to each block and appending the block $\{2,15,28\}$ gives a parallel class. Develop to obtain 11 parallel classes. Each of the two remaining starter blocks generates three parallel classes as the points in each block are distinct $\pmod 3$. $\square$

4.4.2 Cyclic Anti-Pasch STSs with Mutually Disjoint Base Blocks

In this subsection, we present a simple construction for anti-Pasch KTS.

Theorem 63 *Suppose that* $v \equiv 1 \pmod 6$, *and there exists a cyclic anti-Pasch STS*(v) *over* V *with mutually disjoint base blocks. Then there exists an anti-Pasch KTS*($3v$).

Proof: This construction is a simple modification of a construction in [56]. We present it here for completeness. Let $V' = V \times \{0,1,2\}$. We construct the following set of blocks.

(i) For every block $\{a, b, c\}$ in the STS(v), we construct blocks $\{(a, 0), (b, 0), (c, 0)\}$, $\{(2a, 1), (2b, 1), (2c, 1)\}$ and $\{(3a, 2), (3b, 2), (3c, 2)\}$.

(ii) $\{(i, 0), (i + 2j, 1), (i + 3j, 2)\}$ for $i, j \in V$.

This results in a KTS($3v$) so we only have to prove that this is anti-Pasch. The TD($3, n$) that is used in the construction is free of subsquares of order two. Hence, this is a 3-QFGDD of type v^3. Also, if any Pasch configuration involves a block of type (ii), then all blocks in the Pasch configuration must come from the STS(v). This must result in a QFKTS($3v$). □

By way of example, a cyclic anti-Pasch of order 19 is presented in [47] with base blocks $\{0, 1, 8\}$, $\{0, 2, 5\}$, $\{0, 4, 13\}$. By adding 2 to the second block and 5 to the third block, we obtain a cyclic anti-Pasch STS of order 19 with mutually disjoint base blocks. By Theorem 63, we obtain a anti-Pasch KTS(57).

It is therefore of great interest to determine when a cyclic anti-Pasch STS exists whose base blocks can be made mutually disjoint. In fact, a well known conjecture of Novak [99] asserts that for every $v \equiv 1 \pmod 6$, every cyclic STS(v) can be made to have disjoint base blocks. This is widely believed to be true but not much progress has been made toward settling this conjecture.

The only known infinite class of cyclic anti-Pasch STS(v) when $v \equiv 1 \pmod 6$ is the Netto triple systems. Let $q = p^n$ where p is a prime such that $p \equiv 7 \pmod{12}$. Take two primitive sixth roots of unity ϵ_1 and ϵ_2 in $\mathbf{F}_q$; they both are non-squares and satisfy the equation $x^2 - x + 1 = 0$. It follows that $\epsilon_1 + \epsilon_2 = \epsilon_1\epsilon_2 = 1$, $\epsilon_1^2 = -\epsilon_2$ and $\epsilon_2^2 = -\epsilon_1$. For any two distinct elements $a, b \in \mathbf{F}_q$ define $a \to b$ if and only if $b - a$ is a non-zero square in $\mathbf{F}_q$. This relation has the property that exactly one of $a \to b$ and $b \to a$ is true for $a \neq b$, since -1 is not a square in $\mathbf{F}_q$. Now, on the set of all ordered pairs (a, b) such that $a \to b$. Define a function f by $f(a, b) = a\epsilon_1 + b\epsilon_2$.

Now if $c = f(a,b)$, then also $b \to c$ with $f(b,c) = a$ and $c \to a$ with $f(c,a) = b$. The *Netto system* $N(q)$ is the STS(V,$\mathcal{B}$) where $V = \mathbb{F}_q$ and $\mathcal{B} = \{\{a,b,c\}; a \to b$ and $c = f(a,b)\}$.

Theorem 64 *[104] If* $p \equiv 19 \pmod{24}$, *then* $N(q)$ *is anti-Pasch.*

Indeed, when $v = p^n$ and $p \equiv 19 \pmod{24}$, then $N(v)$ is 5-sparse [47].

Lemma 165 $N(q)$ *is cyclic over* $\mathbb{F}_q$.

Proof: Let $\{a, b, a\epsilon_1 + b\epsilon_2\}$ be a block. If $\alpha \in \mathbb{F}_q$, then we claim that $\{a+\alpha, b+\alpha, a\epsilon_1 + b\epsilon_2 + \alpha\}$ is a block. Note that $b - a$ is a non-zero square if and only if $b + \alpha - (a+\alpha)$ is a non-zero square. Also $a\epsilon_1 + b\epsilon_2 + \alpha = (a+\alpha)\epsilon_1 + (b+\alpha)\epsilon_2$ since $1 = \epsilon_1 + \epsilon_2$. □

Lemma 166 *If* $\{a,b,c\}$ *is a block in Netto triple system, so is* $\{\omega^2 a, \omega^2 b, \omega^2 c\}$ *for any* $\omega \in \mathbb{F}_q$

Theorem 65 *If* q *is a prime power congruent to* $1 \pmod 6$, ω *is a primitive root over* $\mathbb{F}_q$, *and* A *is a block of size three so that* $\{\omega^{6i}A : i = 0, 1, \ldots, \frac{q-1}{6}\}$ *is the set of base blocks for the cyclic STS(*q*), then the STS(*q*) can be made to have disjoint base blocks.*

Proof: If A is the base block then $\{\omega^{6i}A : i = 0, 1, \ldots, \frac{q-1}{6} - 1\}$ is the set of base blocks for the cyclic STS(q). Define a mapping from $f : V \to \{\infty\} \cup \mathbb{Z}_3$ by $f(0) = \infty$ and $f(\omega^i) = i \pmod 3$ for all $i = 1, 2, \ldots, q-1$. Next, look at the set of translates of A under the mapping of f. It has q blocks, if there exists a block of the form $\{0,1,2\}$, then let $B = A + \alpha$ is the block that maps to it. Then

$\{\omega^{6i}B : i = 0, 1, \ldots, \frac{q-1}{6}\}$ is a set of mutually disjoint base blocks which generate the cyclic STS(v). Hence, we can assume that there is no block of the form $\{0, 1, 2\}$. The number of pairs involving ∞ and i is 2. The number of pairs involving i and i is $\frac{q-4}{3}$ for each $i = 0, 1, 2$ since the number of pairs involving ω^m and ω^n where $m \equiv n \pmod 3$ are $\frac{q-4}{3}$. The number of pairs involving i and j when $i \neq j$ is $\frac{2(q-1)}{3}$. Since there exists no block of form $\{0, 1, 2\}$, all $q-3$ blocks of size three not involving the point ∞ must involve at least one pair of the form i and i for some i. Hence, there are at least $q-3$ pairs of type i and i. However, we can only have $q-4$ pairs of them which is impossible. Therefore, it can always be made to base block disjoint. □

Combining Theorem 63 and 65 together with Netto triple systems, we obtain:

Corollary 19 *If $v = 3q$, $q = p^\alpha$ and $p \equiv 19 \pmod{24}$ a prime, then there exists a anti-Pasch KTS(v).*

Next, we present some base block disjoint anti-Pasch STS(v) where $v \equiv 1 \pmod 6$.

19: {1,2,9}, {3,5,8}, {0,6,10}

25: {1,2,4}, {3,7,14}, {6,12,21}

31: {1,2,4}, {3,7,14}, {5,10,18}, {6,12,24}, {8,16,25}

37: {1,2,4}, {3,7,29}, {5,10,19}, {6,12,31}, {8,15,25}, {9,17,30}

43: {1,2,4}, {3,7,12}, {5,11,33}, {6,13,29}, {8,16,41}, {9,20,39}, {10,22,36}

49: {1,2,4}, {3,7,12}, {5,11,22}, {6,13,29}, {8,16,38}, {9,19,40}, {10,23,35}, {14,38,48}

55: {1,2,4}, {3,7,12}, {5,11,21}, {6,13,38}, {8,16,37}, {9,20,51}, {14,26,41}, {10,24,46}, {15,32,52}

61: {1,2,4}, {3,7,12}, {5,11,18}, {6,14,31}, {9,19,42}, {10,21,40}, {13,25,45}, {8,22,48}, {15,30,52}, {16,32,50}

67: {1,2,4}, {3,7,12}, {5,11,18}, {6,14,29}, {9,19,47}, {10,21,43}, {8,20,50}, {13,27,45}, {15,31,58}, {16,33,52}, {17,37,63}

73: {3,7,13}, {5,10,40}, {9,16,41}, {6,14,30}, {8,17,63}, {11,22,64}, {15,27,67}, {18, 32,47}, {0,19,36}, {20,42,65}, {1,2,4}

79: {9,15,31}, {12,19,64}, {11,20,66}, {14,24,67}, {16,28,75}, {17,32,71}, {25,43,73}, {0,23,44}, {1,2,30}, {3,5,22}, {4,7,18}, {6,10,48}, {8,13,21}

85: {1,2,60}, {3,5,52}, {4,7,50}, {6,10,67}, {8,13,27}, {9,15,77}, {11,18,29}, {12,20,68}, {14,23,83}, {16,26,48}, {19,31,64}, {17,30,61}, {21,36,56}, {24,45,79}

91: {1,2,4}, {22,43,62}, {23,46,72}, {3,7,14}, {5,10,39}, {6,12,65}, {8,16,44}, {9,18,36}, {11,21,82}, {13,25,71}, {15,28,63}, {17,31,56}, {19,34,88}, {24,40,84}, {20,37,87}

97: {1,2,4}, {10,21,82}, {14,26,56}, {17,30,74}, {15,29,62}, {18,33,50}, {3,7,12}, {5,11,31}, {6,13,72}, {8,16,43}, {9,19,67}, {25,48, 77}, {20,36,57}, {22,40,91}, {23,42,96}, {24,46,80}

All designs are taken from [47] and are made base block disjoint.

4.4.3 Rees's Construction

In this section, we employ Rees's construction [101] on resolvable group divisible designs to obtain some new anti-Pasch KTSs.

A *partial transversal design* $\mathrm{PlTD}_\lambda(k,n)$ is a triple $(X,\mathcal{C},\mathcal{B})$ where X is a kn-set, $\mathcal{B}$ is a collection of k-subsets of X (blocks) so that any pair of distinct points from X is contained in at most λ blocks, and $\mathcal{C}$ is a strong k-vertex-colouring of X (i.e., each block receives k different colours) so that $|C| = n$ for each $C \in \mathcal{C}$. Any transversal design is a PlTD (just take each group as a colour class). Similarly, a *partial group divisible design* K-PlGD_λ of type T is a triple $(X,\mathcal{C},\mathcal{B})$ where X is a v-set, $\mathcal{B}$ is a collection of subsets of C (blocks) each having same size from the set K so that any pair of distinct points from X is contained in at most λ blocks, and $\mathcal{C}$ is a strong colouring of X.

A group $\mathcal{H}$ of automorphism son a set V is acting *sharply transitively* on V if for every two elements $x, y \in V$, there exists $h \in \mathcal{H}$ so that $xh = y$ where the group action is written as left multiplication.

A *block-partition* of a transversal design $(X, G, \mathcal{B})$ is a partition P of its block set $\mathcal{B}$; we refer to the members of P as *aggregates*. If each member of P is a clear set (i.e., composed of mutually disjoint blocks) then we refer to P by the usual term *block-coloring*.

Theorem 66 *[101] Let $(X, G, \mathcal{B})$ be an A-resolvable K-$PlGD_\lambda$ of type T in which for each $\alpha_i \in A$, there are r_i α_i-parallel classes of blocks. Suppose that there is a $TD_\lambda(u,h)$ admitting $\mathcal{H}$ as a group of automorphism acting transitively on the points of each group where $u = |G|$. Let H_j be a collection of subsets of $\mathcal{H}$, there being r_i such subset of size α_i for each $\alpha_i \in A$, and suppose that the collection*

$\{H_i * r : r \in \mathcal{H}, j = 1, 2, \ldots, \sum_i r_i\}$ is Γ-resolvable on $\mathcal{H}$. Then there is a Γ-resolvable K-PIGD$_{\lambda_1\lambda_2}$ of type hT.

Theorem 67 *[101] Let $(X, G, \mathcal{B})$ be a K-PIGD$_{\lambda_1}$ of type T whose block set $\mathcal{B}$ forms an α-parallel class, and let $u = |G|$. Suppose that there is a TD$_{\lambda_2}(u, h)$ each of whose groups $J_1, J_2, \ldots, J_u$ is written on the symbols of a group $\mathcal{H}$, and let $H^1, H^2, \ldots, H^u$ be a sequence of subsets of $\mathcal{H}$ each of size α. Let C be a block-partition of the TD with the following property: for each aggregate $C \in C$ and each $i = 1, 2, \ldots, u$, the set $\{H^i * r : r \in J_i \cap (\bigcup_{b \in C} b)\}$ form a γ-parallel class on J_i. Then there is a K-PIGD$_{\lambda_1\lambda_2}$ of type hT whose block set is γ-resolvable.*

These two constructions are complicated and very powerful. In our case, if we begin with a anti-Pasch GDD, we can inflate to get anti-Pasch resolvable GDD. The proof of this theorem is involved, and we do not include it here. However, this construction works as if we inflate the GDD is such a way that for every block of size k, we put the TD(k, h) that corresponds to the groups of of the k points. Hence, in the case of all blocks having size three, if the TD(u, h) has the extra property that any latin square induces by three rows is an anti-Pasch GDD, then we produce an anti-Pasch GDD.

Therefore, it is important to know if such TD(u, h) exists.

Lemma 167 *If $h = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, where p_i are odd prime powers and α_i are positive integers, and $m = min_i(p_i^{\alpha_i})$, then there exists a TD$(m - 1, h)$ admitting $\mathcal{H} = \mathbb{F}_{p_1}^{\alpha_1} \times \mathbb{F}_{p_2}^{\alpha_2} \times \ldots \times \mathbb{F}_{p_n}^{\alpha_n}$ acting sharply transitively on the points of each group. In addition, the TD$(3, h)$ that is defined by any three groups is free of a subsquare of order two.*

Proof: Let $V = \mathbb{F}_{p_1}^{\alpha_1} \times \mathbb{F}_{p_2}^{\alpha_2} \times \ldots \times \mathbb{F}_{p_n}^{\alpha_n}$. There exists $m-1$ elements $t_1, t_2, \ldots, t_{m-1} \in V$ so that the difference between any two of them are invertible over the ring V. We can construct a TD$(m-1,h)$ over $V \times I_{m-1}$ by taking the blocks $\{(at_1+b,1),(at_2+b,2),\ldots,(at_{m-1}+b,m-1)\}$ for $a,b \in V$. This is a TD$(m-1,h)$ which V act sharply transitively on the points of each group. To see that the TD$(3,h)$ that is defined by any three groups is free of a subsquare of order two, if there exists a subsquare of order two, then by either a simple computation or by projecting into certain TD$(3,p)$ where p is a odd prime power to obtain a subsquare of order two in the TD$(3,p)$. This implies that the desarguesian projective plane of order p contains a projective subplane of order two [24]. □

In order to apply Rees's technique, we begin with an anti-Pasch GDD which admits a certain resolution. A large class of examples comes from Bose's construction.

Theorem 68 *[60] If $v = 3n$ where n is odd and $(n,7) = 1$, then there exist an anti-Pasch STS$(3v)$.*

Proof: We state the construction; see [60] for a proof. The anti-Pasch STS$(3n)$ is constructed over $V = \mathbb{Z}_n \times \mathbb{Z}_3$. For every $a,b,c \in \mathbb{Z}_n$, we construct a block of form $\{(a,i),(b,i),(c,i+1)\}$ if $a+b=2c$ and $i \in \mathbb{Z}_3$. Also, we take n blocks of form $\{(x,0),(x,1),(x,2)\}$ for $x \in \mathbb{Z}_n$. □

In the above construction, if $\{(a,i),(b,i),(c,i+1)\}$ is a block then so does $\{(a+1,i),(b+1,i),(c+1,i+1)\}$ and $\{(a,i+1),(b,i+1),(c,i+2)\}$. Hence, this design is transitive over $\mathbb{Z}_n \times \mathbb{Z}_3$. In fact, the starter blocks are $\{(0,0),(2,0),(1,1)\}$, $\{(0,0),(4,0),(2,1)\},\ldots,\{(0,0),(n-1,0),((n-1)/2,1)\}$ together with a short orbit $\{(0,0),(0,1),(0,2)\}$. Each starter blocks form a 3-resolution class. Hence, Bose's construction gives a 3-resolvable anti-Pasch GDD of type 3^n.

Before we present a general construction, we first illustrate with an example.

Take a STS(15) that is constructed by Bose's construction. The starter blocks are $\{(0,0),(2,0),(1,1)\}$, $\{(0,0),(4,0),(2,1)\}$ and $\{(0,0),(0,1),(0,2)\}$. The starter block $\{(0,0),(4,0),(2,1)\}$ generate a 3-parallel class when it is developed over $\mathbb{Z}_5 \times \mathbb{Z}_3$. Let $A = \{(0,0),(2,0),(1,1)\}$ and $B = \{(0,0),(0,1),(0,2)\}$. Now $A+(0,0), A+(0,1), A+(0,2), B+(3,0), B+(4,0)$ and $A+(2,0), A+(2,1), A+(2,2), B+(0,0), B+(1,0)$ give two 1-parallel classes. The remaining blocks form a 2-parallel class. We can treat this as a $[1,2,3]$-resolvable anti-Pasch 3-GDD of type 3^5. We take $\mathcal{H}= \mathbb{Z}_p$ where p is a prime, $(p,5)=1$ and apply Theorem 66. Let $H_1 = \{0,1\}$, $H_2 = \{0,1,2\}$ and $H_3 = \{0\}$. We consider two cases.

$p \equiv 5 \pmod 6$: Six parallel classes

$$\{H_1 + k - 2\} \cup \{H_2 + 3i + k : i = 0,1,\dots,\frac{1}{3}(p-5)\} : k = 0,1,2,$$

$$\{H_1 + 2i + k + 1 : i = 0,1,\dots,\frac{1}{2}(p-5)\} \cup \{H_2 + k - 2\} : k = 0,1.$$

and

$$\{H_3 + i : i = 0,1,\dots,p\}.$$

$p \equiv 1 \pmod 6$ $(p \geq 13)$: Six parallel classes

$$\{H_1+k+3, H_1+k+8, H_2+k, H_2+k+5\} \cup \{H_2+3i+k+10 :$$
$$i = 0,1,\dots,\frac{1}{3}(p-13)\} : k = 0,2,4;$$

and

$$\{H_1+9, H_2+3, H_2+6, H_2+11\} \cup \{H_1+2i+14 : i = 0,1,\dots,\frac{1}{2}(p-13)\};$$

and

$$\{H_1 + 2i + 11 : i = 0, 1, \ldots, \frac{1}{2}(p-5)\} \cup \{H_2 + 8\}.$$

and

$$\{H_3 + i : i = 0, 1, \ldots, p\}.$$

When $h = 7$, we simply take $\{H_1 + i, H_1 + 3 + i, H_3 + 6 + i\}$ for $i = 0, 1, 2$, $\{H_1 + 6, H_2 + 2, H_2 + 4\}$, $\{H_2, H_2 + 5, H_2 + 3, H_3 + 2\}$ and $\{H_2 + 1, H_2 + 6, H_3 + 3, H_3 + 4, H_3 + 5\}$.

Hence, Theorem 66 yields a 3-QFRGDD of type $(3p)^5$ for all $(p, 5) = 1$ and p an odd prime. The general pattern is extracted from [101].

If $h = 25$, we use a $\mathbb{F}_{25} = \{a + b\alpha : a, b \in \mathbb{Z}_5\}$. Take $H_1 = \{0, 1, 2\}$, $H_2 = \{3, 4\}$ and $H_3 = \{0\}$. It is clear that $\{H_1 + k + a\alpha : a = 0, 1, \ldots, 4\} \cup \{H_2 + k + a\alpha\} : a = 0, 1, \ldots, 4\}\}$ for $k \in \mathbb{Z}_5$ gives 5 parallel classes. Also, $\{H_3 + i : i \in \mathbb{F}_{25}\}$ gives the last parallel class. Hence, by Theorem 66 gives a 3-QFRGDD of type $(75)^5$.

Filling holes in QFRGDD by anti-Pasch KTS yields anti-Pasch KTS of bigger order. For example, we fill the holes of a a 3-QFRGDD of type 57^5 using an anti-Pasch KTS(57) to obtain an anti-Pasch KTS(285). Also, we can inflate a QFRGDD by a QFRTD$(3, n)$ to obtain a QFRGDD of bigger order.

The most natural way to extend this example is to find a A-resolvable QFGDD and use Rees's construction. First of all, we need to understand more about the resolvability of the QFSTS given by Bose's construction.

We first look at the case of the QFSTS$(3(6n + 1))$ with $n \geq 2$. In particular, we prove that it can always be resolved into ten parallel classes and the remaining blocks into 3-parallel classes.

Lemma 168 *Let* $\mathcal{B}=\{\{0,b,2b\}+a\} : a \in \mathbb{Z}_{6n+1},\ b \in \mathbb{Z}_{6n+1}\}$ *and* $(b, 6n+1) = 1$. *Then* $\mathcal{B}\backslash\{\{0,b,2b\}+a\}$ *can be partitioned into three sets of size* $2n$ *so that any two blocks in each class are block disjoint for any* $a \in \mathbb{Z}_{6n+1}$.

Proof: If $b = 1$, let $B = \{a, a+1, a+2\}$, then we consider the following partition $\{\{B + 3i + j + a\} : i = 0, 1, \ldots, 2n-1\}$ for $j = 1, 2, 3$. Since $(b, 6n+1) = 1$, we just multiply the partition by b to obtain a solution for the general case. □

Lemma 169 *There exists a QFSTS*$(3(6n+1))$ $(n \geq 2)$ *with at least ten parallel classes and the remaining blocks can be partitioned into 3-parallel classes.*

Proof: Let $A_1 = \{0, 1, 2\}$, $A_2 = \{3, 5, 7\}$ and $A_4 = \{4, 8, 12\}$. Form the starter blocks $\{(0,0), (2a, 0), (a, 1)\}$ and develop them over $(-, \mathbb{Z}_3)$. From Lemma 168, we know that each of the starter blocks, when developed over $\mathbb{Z}_{6n+1}$ can be partitioned into three almost parallel classes (missing one point with respect to first component). By taking the starter blocks $\{(0,0), (2a, 0), (a, 1)\}$ for $a = 1, 2, 4$, we can put the leftover block block in A_a. For each almost parallel class, we can add a block $\{(i, 0), (i, 1), (i, 2)\}$ to form a parallel class. Hence, $\{(0,0), (2a, 0), (a, 1)\}$ for $a = 1, 2, 3$ together with $\{(0,0), (0,1), (0,2)\}$ generates ten parallel classes when developed over the group. □

Lemma 170 *There exists a 3-QFRGDD of type* $(3p)^{6n+1}$ *for all prime* $p \geq 6n+1$ *and* $(6n+1, 7) = 1$ *and* $p \equiv 1 \pmod 6$.

Proof: From Lemma 169, there exists a 3-QFGDD of type 3^{6n+1} with nine parallel classes. We use Theorem 66 by taking $H_i = \{0, 1, 2\}$ for $i = 1, 2, \ldots, 3n-3$ and $G_i = \{0\}$ for $i = 1, 2, \ldots, 9$ since there are $3n-3$ 3-parallel classes and 9 1-parallel classes. In order to apply Theorem 66, we have to partition $\{H_i + a\}$ and $\{G_i + a\}$ for $i = 1, 2, \ldots, 3n - 3$, $j = 1, 2, \ldots, 9$ and $a \in \mathbb{Z}_p$ into 1-parallel classes on $\mathbb{Z}_p$. Now for every $\{H_i + a\}$, one can obtain 3 almost parallel class on $\mathbb{Z}_p$ together with a leftover block $\{a, a+1, a+2\}$ for any $a \in \mathbb{Z}_p$ (Lemma 168). For each almost parallel class, we complete it by adding an extra block $G_j + k$ to obtain a parallel class on $\mathbb{Z}_p$. Since a is arbitrary, we can force the leftover block in $\{H_1 + b\}$ for $b \in \mathbb{Z}_p$ be $\{0, 1, 2\}$, the leftover block in $\{H_2 + b\}$ for $b \in \mathbb{Z}_p$ be $\{3, 4, 5\}$ and so on. So $\{H_i + b\}$ for $b \in \mathbb{Z}_p$ for $i = 1, 2, 3, \ldots, 2p$ and G_1 can together produce $6p + 1$ parallel classes; three from each H_i and the last parallel is obtained by taking all leftover blocks in each set of $\{H_i + b\}$ for $i = 1, 2, \ldots, 2p$ together with the remaining block in $\{G_1 + b\}$. When we exhausted all $\{H_i + b\}$ for $b \in \mathbb{Z}_p$ and $i = 1, 2, \ldots, 3n - 3$, then everything else must be able to partition to 1-parallel classes since all $\{G_i + b\}$ for $b \in \mathbb{Z}_p$ are singleton. □

We just illustrate the technique by using the above example; however, we can obtain a much stronger result by using other constructions in case of anti-Pasch KTS($3u$) for $(u, 3) = 1$.

In the case of anti-Pasch KTS($9u$), we obtain an excellent solution using Rees's construction. We need the following technical lemma.

Lemma 171 *Let $V = \mathbb{Z}_v$, $v \geq 3$ odd and $\mathcal{B} = \{\{0, 1, 2\} + a\} : a \in \mathbb{Z}_v\}$. If $v \neq 5$, then there exists a strong vertex colouring on V with at most 4 colour classes.*

Proof: If $v = 3m$, then let $C_1 = \{3i : i = 0, 1, \ldots, m - 1\}$, $C_2 = C_1 + 1$ and $C_3 = C_1 + 2$. If $v = 6m + 1$, then let $C_1 = \{3i : i = 0, 1, \ldots, 2m - 1\}$, $C_2 = C_1 + 1$, $C_3 = C_1 + 2$ and $C_4 = \{6m\}$. If $v = 6m + 5$, let $C_1 = \{3i : i = 0, 1, 2, \ldots, 2m - 1\} \cup \{6m + 1\}$, $C_2 = C_1 + 1$, $C_3 = C_1 + 2$ and $C_4 = \{6m, 6m + 4\}$.

For any block $\{a, a+1, a+2\}$, the three points are in three different colour classes.
$\square$

Lemma 172 *Let $V = \mathbb{Z}_{2n+1} \times \mathbb{Z}_3$, $n \neq 2$ and $B = \{\{(0,0),(2a,0),(a,1)\} + b : b \in V\}$ for $a = 1, 2, \ldots, n$. There exists a strong vertex colouring on V with at most 4 colour classes for every $a = 1, 2, \ldots, n$.*

Proof: First of all, use Lemma 171 by taking $v = 2n+1$ to obtain C_i for $i = 1,2,3,4$. If $(a, 2n+1) = 1$, then we can construct $D_i = aC_i \times \mathbb{Z}_3$ which is the appropriate vertex-colouring. If $(a, 2n+1) = c$, let $(\frac{a}{c}, \frac{2n+1}{c}) = 1$ and apply Lemma 171 by taking $v = \frac{2n+1}{c}$ to obtain C_i for $i = 1, 2, \ldots, 4$. Then define $T_i = \frac{a}{c}C_i$ for $i = 1,2,3,4$. For every $x = 0, 1, \ldots, 2n$, define $T_i^* = \{x : x = qc + r, q \in T_i\}$. Finally, define $D_i = T_i^* \times \mathbb{Z}_3$, which is a strong vertex colouring. $\square$

We can now prove the following theorem using Rees's Theorem.

Theorem 69 *If $v = 9n$ where n is odd, $v \neq 45$ and $(n, 7) = 1$, then there exists an anti-Pasch KTS(v).*

Proof: From the given condition, Bose's construction constructs a 3-resolvable QFGDD of type 3^n. For every 3-parallel class, there exists a strong 4-vertex colouring. Hence, we can regard this as a PIGD with block size three and four groups. Apply Theorem 67 with a TD(4, 3), taking each $H^i = \mathbb{Z}_3$ and C be the block set of the TD. This gives a 3-QFRGDD of type 9^n. Fill in the hole with the QFKTS(9) to get the desired result. $\square$

4.4.4 Zhu, Du and Zhang's Construction

In this subsection, we use a technique introduced by Zhu, Du and Zhang [123] and later extended by Rees and Stinson [102].

A design $\mathcal{D}$ is said to be *s-block-colourable* if its blocks can be coloured with s colours in such a way that any two blocks of the same colour do not intersect. Such an assignment of s colours is said to be an *s-colouring*. If $\mathcal{D}$ is *s-block-colourable* but not *$(s-1)$-block-colourable*, we say that the *chromatic index* of $\mathcal{D}$ is s. In a sense, the chromatic index is a measurement of how close the design is to being resolvable.

Theorem 70 *[123] Suppose there exists an RBIBD$(u,k,1)$, a $B(v,k,1)$ which is s-block-colourable, and a RTD(k,v). If $s \le r_u + r_v$ where $r_u = \frac{u-1}{k-1}$ and $r_v = \frac{v-1}{k-1}$, then there exists an RBIBD$(uv,k,1)$.*

Theorem 71 *[102] Suppose there exists a k-RGDD of type g^u, a k-GDD of type $(mg)^v$ with the property that there is an s-colouring of its blocks such that each color class precisely covers some subset of its groups, and a RTD(k,mv). If $s \le r_u + r_v$ where $r_u = \frac{g(u-1)}{k-1}$ and $r_v = \frac{mg(v-1)}{k-1}$, then there exists a k-RGDD of type $(mg)^{uv}$.*

In both of these constructions, we just take a RGDD and inflate it by a RTD and fill in the hole with GDD and we obtain the resolution by using the colour classes. If we can replace all ingredient by a QFKTS, QFGDD and QFRTD, then we can obtain a similar result for the construction of QFRGDD. A QFRTD$(3,n)$ exists for all n odd. More results on QFRTD are proved in next subsection. We therefore need some QFGDD with small number of colour classes. Again, we can obtain some from Bose's construction.

Again, we need a technical lemma.

Lemma 173 *Let $n = 6k+5$ and $C_i = \{i, i+1, i+2\}$ for $i = 0,1,\ldots,n-1$, arithmetic over $\mathbb{Z}_n$. If $C = \{C_i : i = 0,1,\ldots,n-1\}$, then for any $a,b \in \mathbb{Z}_n$*

$C \backslash (C_a \cup C_b)$ can be partitioned into three sets of $2k+1$ blocks so that any two blocks in the same set are disjoint.

Proof: We sort the blocks in increasing order of i, then we put the i^{th} blocks in the $i \pmod 3$ set. This gives the required partition. □

Theorem 72 *There exists a 3-QFGDD of type 3^{2k+1} which is $3k+6$ colorable and each colour class misses a subset of the groups when $k \equiv 0,2 \pmod 3$. and $(2k+1,7)=1$.*

Proof: We use the QFGDD of type 3^{2k+1} from Bose's construction where the groups are formed by taking $\{(i,0),(i,1),(i,2)\}$ where $i \in \mathbb{Z}_{2k+1}$. We construct a graph $G=(V,E)$ as follows: $V = \mathbb{Z}_{2k+1} \backslash \{0\}$ and $(a,b) \in E$ if $\{a,2a,3a\} \cap \{b,2b,3b\} \neq \emptyset$. Each vertex has degree at most six so by Brooks's Theorem in vertex colouring [25], this graph is 6-colourable. If $2k+1 \equiv 1 \pmod 6$, for every colour class, we consider a subset of the vertex induce by the vertex $\{1,2,\ldots,k\}$. In each of the colour classes, we can obtain a partial parallel class missing a subset of group as follows: for C a colour class, take $\{(c,i),(3c,i),(2c,i+1)\}$ for $i \in \mathbb{Z}_3$ and $c \in C$. This gives a partial parallel class missing a subset of groups. Hence, we obtain six partial parallel classes. For any starter blocks $\{(0,0),(2a,0),(a,1)\}$ over $\mathbb{Z}_{2k+1} \times \mathbb{Z}_3$, we have used up the translates $\{(a,i),(3a,i),(2a,i+1)\}$. The remaining blocks can be partitioned into 3 partial parallel classes, each missing one group. In the case when $2k+1 \equiv 5 \pmod 6$, we observe that the vertices for a and $-a$ correspond to two distinct translates of the starter block $\{0,a,2a\}$, for every starter block $\{(0,0),(2a,0),(a,1)\}$ in the QFSTS by Bose's Construction, six blocks are used up to obtain 6 partial parallel classes. The remaining blocks for every starter block form 3 partial parallel classes. □

Theorem 73 *Suppose there exists an anti-Pasch KTS(v) where $v \geq 15$, and $w \equiv 3, 15 \pmod{18}$, then there exists an anti-Pasch KTS($\frac{vw}{3}$).*

Proof; Take a 3-QFGDD of type $3^{\frac{w}{3}}$ from Bose's Construction which is $\frac{w-3}{2} + 6$ colourable. Apply Theorem 71 to obtain the result. □

Lemma 174 *There exists a 14-colourable 3-QFGDD of type 3^7 so that each colour class misses a subset of groups.*

Proof: A QFSTS(21) exists by taking $V = \mathbb{Z}_7 \times \mathbb{Z}_3$ with the starter blocks $\{(0,0),(0,1),(0,2)\}$, $\{(0,0),(1,1),(3,0)\}$, $\{(5,0),(2,2),(4,0)\}$ and $\{(0,0),(4,1),(5,0)\}$. The first starter block generates a parallel class for the STS. The second and third starter blocks generates seven partial parallel classes when developed over $\mathbb{Z}_7 \times \mathbb{Z}_3$ since each mod 7 component is distinct. The last block generates another seven partial parallel classes. □

Corollary 20 *If $v = 9n$, $v \neq 45, 63$, then there exists a QFKTS(v).*

Proof: If $(v,7) = 1$, then we obtain the conclusion of the corollary by Theorem 69. If $v = 7w$ where $(w,7) = 1$ and $w \neq 9, 45$, then take a 14-colourable 3-QFGDD of type 3^7 Lemma 174, a QFRGDD of type $3^{\frac{w}{3}}$ apply Theorem 71 to obtain a QFRGDD of type $3^{\frac{7w}{3}}$. When $w = 45$, a QFKTS(105) is constructed in next subsection. Inflate it by 3 to obtain a QFKTS(7(45)). If $v = 7^a w$ where $(w,7) = 1$ and $a \geq 2$, we can apply Theorem 69 by taking a QFRGDD $3^{\frac{w}{7}}$ and a 14-colourable 3-QFGDD of type 3^7, the case when $v = 7^2(9)$ can be obtained by taking a QFKTS(3(49)) and inflate by 3. □

4.4.5 A GDD Construction

In this section, we present a GDD construction for anti-Pasch KTS.

Theorem 74 *If there exists a QFKTS*$(2v+1)$*, a QFKTS*$(2w+1)$ *and a QFRTD*$(3, w)$*, then there exists a QFKTS*$(2vw+1)$*.*

Proof: The proof is similar to Theorem 55 and thus omitted. □

In the remainder of the subsection, we prove some existence results concerning QFRTD$(3, n)$.

Lemma 175 *If n is odd, then there exists a QFRTD*$(3, n)$*.*

Proof: Construct the TD$(3, n)$ by taking $V = \mathbb{Z}_n \times \{0, 1, 2\}$. The block set is $\{\{(a, 0), (b, 1), (a + b, 2)\} : a, b \in \mathbb{Z}_n\}$. □

Corollary 21 *There exists a QFKTS*(105)*.*

Proof: Apply Theorem 74 with $v = 4$ and $w = 13$. □

We only have to deal the case when n is even.

Lemma 176 *There is no QFRTD*$(3, n)$ *for* $n = 2, 4, 6, 8$*.*

Proof: When $n = 2, 6$, there do not exist two MOLS of order n. When $n = 4$, there is no QFTD$(3, 4)$ [93]. All QFTD$(3, 8)$ were enumerated in [50], none of which is resolvable. □

Lemma 177 *There exists a QFRTD*$(3, n^2)$ *when* $n \neq 2, 4$*.*

Proof: This is a simple consequence from Lemma 2.1 in [101] by starting off with a QFTD$(3, n)$. □

Lemma 178 *If there exists a QFRTD$(3, n)$ and a QFRTD$(3, m)$, then there exists a QFRTD$(3, mn)$.*

Proof: This is just a simple inflation and the proof is thus omitted. □

In order to apply Rees's techniques for constructing QFRTD, we need to construct QFTD$(3, n)$ with some type of resolutions. We give one example here.

Lemma 179 *There exists a N_2-square of order eight with six disjoint transversals.*

Proof: We consider the following latin square of order 8.

$$\begin{array}{cccccccc}
\ddot{1} & \hat{2} & \vec{3} & \dot{4} & \bar{5} & 6 & 7 & \breve{0} \\
\breve{2} & \bar{3} & \hat{1} & 5 & \dot{6} & \vec{7} & \ddot{0} & 4 \\
\dot{3} & \vec{6} & \breve{4} & \bar{7} & \ddot{2} & 0 & 1 & \hat{5} \\
\vec{4} & \dot{0} & \ddot{7} & 2 & 3 & \breve{5} & \hat{6} & \bar{1} \\
5 & 4 & \bar{0} & \vec{1} & \breve{7} & \hat{3} & \dot{2} & \ddot{6} \\
\bar{6} & 7 & \dot{5} & \hat{0} & 1 & \ddot{4} & \breve{3} & \vec{2} \\
\hat{7} & \ddot{5} & 2 & \breve{6} & \vec{0} & \dot{1} & \bar{4} & 3 \\
0 & \breve{1} & 6 & \ddot{3} & \hat{4} & \bar{2} & \vec{5} & \dot{7}
\end{array}$$

Each of the six different types of accents corresponds to a transversal. □

Lemma 180 *If $n \equiv 8 \pmod{16}$ and $n \neq 8$, then there exists a QFRTD$(3, n)$.*

Proof: Since $(n, 3) = 1$, then we can use Theorem 66 since a point regular QFTD$(3, n)$ exists. If $n = 3$, take a transversal and a 2-resolution class and apply

Theorem 67 with a RTD(3, 3). For any other parallel classes, we simply take a direct product with a RTD(3, 3). □

Next, we have a non-trivial application of Rees's theorem.

Lemma 181 *If m and n are odd number at least one, then there exists a QFRTD$(3, 4mn)$.*

Proof: We recall a construction in [73]. If n is odd, we can construct a QFTD$(3, 2n)$ by taking the square $N = \begin{bmatrix} A & B+n \\ C+n & A \end{bmatrix}$ where $A_{ij} = i - j + 1 \pmod{n}$, $B_{ij} = i + j - 1 \pmod{n}$ and $C_{ij} = i + j - 2 \pmod{n}$. We can treat this square with elements in $\mathbb{Z}_{2n}$. Since n is odd, then all A, B and C are resolvable. For any transversal, l, $T \in \{A, B, C\}$, we have $\{\{0, n\} + T_{ij} : (i, j) \in l\}$ forming a 1-parallel class on $\mathbb{Z}_n$. Also, the TD$(3, 2n)$ that is constructed here is 2-resolvable. We apply Theorem 67 as follows: take a QFTD$(3, 2m)$ which is 2-resolvable and take a QFTD$(3, 2n)$ that is arisen from the above construction. Let $H^i = \{0, n\}$ for $i = 1, 2, 3$, apply Theorem 67 to obtain a QFRTD$(3, 4mn)$. □

Theorem 75 *If $8|v$, then there exists a QFRTD$(3, v)$ for all $v \neq 2^a$ for $a = 1, 2, 3, 4, 5$.*

Proof: If $n = 8m$, $m \geq 2$ and m odd, then it is proved in Lemma 180 that we can obtain a QFRTD$(8, n)$. If $n = 16m$ where $m \geq 2$ and m odd, we take a [2,2,2,2]-resolvable QFTD$(3, 8)$ and apply the technique in Lemma 181 to obtain a QFRTD$(16, n)$. If $n = 32m$ where $m \geq 3$ and m odd, we use a similar technique by treating the QFTD$(3, 8)$ as a [4,4]-resolvable QFTD$(3, 8)$ and multiply it by a QFTD$(3, 4m)$ coming from a non-uniform direct product of N_2-latin square [73]. We take the point set of the QFTD$(3, 4m)$ as $\mathbb{Z}_{4m}$ and each

$H^i = \{0, m, 2m, 3m\}$ for $i = 1, 2$. In general, when $n = 2^a m$ where $a \geq 6$, $m \geq 2$ and m odd, if $a \equiv 0 \pmod 2$, then we take a QFTD$(3, 2^{a/2})$ to obtain a QFRTD$(3, 2^a)$ by Lemma 177. This gives a QFRTD$(3, n)$ by a simple direct product with a QFRTD$(3, n)$. If $a \equiv 1 \pmod 2$, we can construct a 2-resolvable QFTD$(3, 2^a)$ by doubling a QFRTD$(3, 2^{a-1})$ (Lemma 157), multiplying by a QFTD$(3, 2m)$ as in Lemma 181. If $n = 2^a$, take a $[4, 4]$-resolvable QFTD$(3, 8)$ and doubling gives a [8,8]-resolvable QFTD$(3, 16)$. Apply Theorem 67 to obtain a QFRTD$(3, 128)$. Since a QFRTD$(3, 2^{2a})$ exists for all $a \geq 3$, a simple direct product gives a QFRTD$(3, 2^b)$ for all $b \geq 13$. We can also obtain a [16,16]-resolvable QFTD$(3, 32)$ and a [32,32]-resolvable QFTD$(3, 64)$ by taking a non-uniform direct product. Multiplying it by a QFTD$(3, 32)$ and a QFTD$(3, 64)$ gives a QFRTD$(3, 2^a)$ for $a = 9$ and $a = 11$. □

It is of great interest to settle the problem QFRTD, both of its interest in QFKTS and it is also an extension of the N_2-latin squares problem.

4.5 5-sparse Triple Systems

As mentioned in Section 4.2, the problem of determining those v for which there exists an anti-Pasch STS of order v and anti-mitre STS of order v appears to be difficult.

One might ask for the stronger property that an STS(v) be both anti-Pasch and anti-mitre. No such system exists for $v \leq 15$. More generally, call an STS(v) *r-sparse* if every set of $r+2$ elements carries fewer than r triples. Every STS(v) is 3-sparse, and every r-sparse STS(v) is also $(r-1)$-sparse. Erdős (see [76]) conjectures that for every r, there exists a finite r-sparse STS(v). An STS(v) is 4-sparse if and only if it is anti-Pasch; and it is 5-sparse if and only if it is both anti-Pasch and anti-mitre. It appears that the only known class of 5-sparse triple system is a special class of Netto triple systems (see [76]).

Let G be an abelian group. An STS(v) is *transitive* over G if $V = G$ and for every $\alpha \in G$ and $\{a, b, c\} \in \mathcal{B}$, $\{a + \alpha, b + \alpha, c + \alpha\} \in \mathcal{B}$. When G is the cyclic group, the STS is also called *cyclic*.

4.5.1 Main Construction

Let $S = (V, \mathcal{B})$ be a transitive 5-sparse triple system on $\mathbb{Z}_v$ or $\mathbb{F}_v$, where $|V| = v$. Let $\mathcal{C} = \{\{-a, -b, -c\} : \{a, b, c\} \in \mathcal{B}\}$. Let $S' = (V, \mathcal{C})$. S' and S are isomorphic, and hence S' is also a transitive 5-sparse triple system. When $v \equiv 1 \pmod 6$, one can verify that S and S' are block-disjoint.

Theorem 76 *If $v \equiv 1 \pmod 6$ and a transitive 5-sparse STS(v) exists, then a 5-sparse STS($3v$) exists.*

Proof: We construct an STS($3v$) on $V \times \{0, 1, 2\}$:

(1) For any block $\{a, b, c\} \in \mathcal{B}$, construct two blocks $\{(a, i), (b, i), (c, i)\}$ for $i = 0, 1$.

(2) For any block $\{a, b, c\} \in \mathcal{C}$, construct a block $\{(a, 2), (b, 2), (c, 2)\}$.

(3) Construct the blocks $\{(i, 0), (j, 1), (i + j, 2)\}$ for $i, j \in \mathbb{Z}_v$ or $\mathbb{F}_v$.

We show that this is anti-mitre. We call blocks of types 1 and 2 *horizontal* and blocks of type 3 *vertical.* Suppose to the contrary that there exists a mitre in the STS($3v$). Let a be the centre in the mitre. We distinguish two cases:

Case 1: There exists a block in the mitre through a which is horizontal. We assume the block is of the form $\{(a,i),(b,i),(k,i)\}$. Next, we have to distinguish into three sub-cases according to the value of i.

Subcase 1.1: When $i = 0$, we have a block of the form $\{(a,0),(b,0),(c,0)\}$. If there is one more block in the mitre through a in horizontal, then all blocks in the mitre are horizontal. Hence, this gives a mitre in S, a contradiction. Hence, we can assume the other two blocks through a are vertical. We assume they are of the form $\{(a,0),(e,1),(a+e,2)\}$ and $\{(a,0),(f,1),(a+f,2)\}$. We can force the remaining two blocks are of the form $\{(b,0),(e,1),(a+f,2)\}$ and $\{(c,0),(f,1),(a+e,2)\}$. We must have $b+e=a+f$ and $c+f=a+e$. Adding gives $b+c=2a$ which is the same as $b-a=a-c$. But S is cyclic, a contradiction.

Subcase 1.2: When $i = 2$, we have a block of the form $\{(a,2),(b,2),(c,2)\}$. Again all other blocks through a must be vertical blocks. We can assume that they are of the form $\{(e,0),(a-e,1),(a,2)\}$ and $\{(f,0),(a-f,1),(a,2)\}$. Hence the remaining blocks must be of the form $\{(e,0),(a-f,1),(b,2)\}$ and $\{(f,0),(a-e,1),(c,2)\}$. This gives $e+a=b+f$ and $f+a=c+e$. Adding gives $2a=b+c$, a contradiction.

Subcase 1.3: When $i = 1$, it reduces to Subcase 1.1 by symmetry.

Case 2: All blocks through a are vertical. So the remaining two blocks must be horizontal. We break down in to three cases again.

Subcase 2.1: When a is in level 0, without loss of generality, the two horizontal blocks are $\{(p,1),(q,1),(r,1)\}$ and $\{(x,2),(y,2),(z,2)\}$. We must have $a+p=x$, $a+q=y$ and $a+r=z$. Since $\{p,q,r\} \in \mathcal{B}$, then $\{a+p,a+q,a+r\} \in \mathcal{B}$, that is, $\{x,y,z\} \in \mathcal{B}$. But $\{x,y,z\} \in \mathcal{C}$. This is a contradiction.

Subcase 2.2: The case when a is in level 1 reduces by symmetry to Subcase 2.1.

Subcase 2.3: When a is in level 2. Assume the two horizontal blocks are $\{(p,0),(q,0),(r,0)\}$ and $\{(x,1),(y,1),(z,1)\}$. This gives $p+x=q+y=r+z=a$. We know $\{p,q,r\},\{x,y,z\} = \{a-p,a-q,q-r\} \in \mathcal{B}$. Since S is cyclic, $\{-p,-q,-r\} \in \mathcal{B}$. But then $\{p,q,r\} \in \mathcal{C}$, which contradicts the observation that $\mathcal{B}$ and $\mathcal{C}$ are disjoint.

That the STS($3v$) is anti-Pasch follows from the argument in [60]. Hence, we obtain a 5-sparse triple system of order $3v$.

□

Corollary 22 *If $v = 3m$ where $m = p^n$ where p is a prime and $p \equiv 19 \pmod{24}$, then there exists a 5-sparse triple system of order $3m$.*

Proof: The Netto triple system of order p^n [104] is transitive over $\mathbb{F}_{p^n}$ and 5-sparse [47]. □

Corollary 23 *If $v \equiv 3 \pmod{18}$ and $99 \le v \le 291$, then there exists a 5-sparse triple system of order v.*

Proof: In [47], cyclic 5-sparse triple systems are constructed for all $33 \le v \le 97$ and $v \equiv 1,3 \pmod 6$. □

4.5.2 An Extension

In this section, we extend the construction in previous construction to give a product-type construction of 5-sparse triple systems.

In the construction in the subsection 1, it is possible to permute the points so that we put the same copy of STS in $V \times \{2\}$ as in $V \times \{1\}$ and $V \times \{0\}$. In fact, all we need to do is to replace the vertical blocks by $\{(i,0),(j,1),(-i-j,2)\}$. Using this simple observation, we can obtain a product construction.

Theorem 77 *If there exist a transitive 5-sparse STS(v) $S=(V,\mathcal{B})$ over $\mathbb{F}_v$ or $\mathbb{Z}_v$, $v \equiv 1 \pmod 6$ and a 5-sparse STS(w) $S'=(W,\mathcal{C})$, then there exists a 5-sparse STS(vw).*

Proof: We construct a STS(vw) on $V \times W$. For every block $\{a,b,c\} \in \mathcal{C}$, we construct v^2 blocks of form $\{(i,a),(j,b),(-i-j,c)\}$. For every block of form $\{a,b,c\} \in \mathcal{B}$, we construct w blocks of the form $\{(a,w),(b,w),(c,w)\}$ where $w \in W$. These form a STS(vw). We show that it is anti-mitre. If the mitre involves a block of form $\{(a,w),(b,w),(c,w)\}$, then either all blocks in the mitre has second coordinates w or all blocks form the mitre are from a TD$(3,v)$ together with the blocks from the three STS(v). In this case, we have a contradiction by Theorem 76. Since none of the blocks involved can come from a block of the form $\{(a,w),(b,w),(c,w)\}$, all points in the mitre must have distinct second coordinates. (Consider the central element of the mitre and its neighbours.) Hence, if we project all points to their second coordinate, we obtain a mitre in STS(w), a contradiction.

Next, we show that the STS(vw) is anti-Pasch. Suppose there exists a Pasch in the STS(vw). If the Pasch involves a block of form $\{(a,w),(b,w),(c,w)\}$, then it is easy to check that the three remaining points in the Pasch must correspond to different points in STS(w). By projecting the points to STS(w), this gives a pair of point appearing in two blocks in the STS(w), a contradiction. Otherwise, it is easy to check that the six points of the Pasch must correspond to either 3 points or 6 points in STS(w). In the former case, this reduces to a Pasch in the TD$(3,v)$ that we constructed, a contradiction. In the latter case, this gives a Pasch in STS(w), a contradiction. □

This product construction is different from those for anti-Pasch systems in [60] and [112]. In fact, using this product construction together with the techniques in [60], one can easily see that there exists a 3-resolvable anti-Pasch STS$(6m+3)$ for any m.

The following construction is easily seen to be embedded from the construction shown in this section.

Theorem 78 *If there exists a transitive anti-mitre STS*(v), $v \equiv 1 \pmod 6$ *and an anti-mitre STS*(w), *then there exists an anti-mitre STS*(vw).

4.6 Update Penalty Four

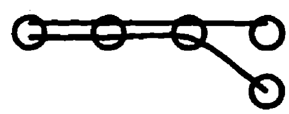
In this section, we examine codes in which each information disk has exactly four check disks. Let us remark at the outset that the *full 4-code* consisting of all distinct columns with four 1's fails to correct all 4-erasures. Indeed if any two columns have 1 entries in three common rows, an unacceptable 4-erasure consists of the two corresponding information disks, and the two check disks required to obtain zero sum.

Lemma 182 *A 4-erasure correcting code with c check disks and minimum update penalty has at most* $\frac{c(c-1)(c-2)}{24}$ *information disks.*

Proof: We remark earlier that any two rows cannot have 3 common entries. Hence, a simple computation reveals the result. □

If $(V, \mathcal{B})$ is a BIBD, we call the design *simple* (*super-simple*) if $|B_1 \cap B_2| < k$ ($|B_1 \cap B_2| \leq 2$, respectively), for all choices of $B_1, B_2 \in \mathcal{B}$. Thus 4-erasure codes arising from BIBDs come from super-simple designs. In fact, if the bound of Lemma 182 is met, one can easily check that every triple occurs in exactly one block, and hence the set system is a *Steiner quadruple system* (see [65] for a comprehensive survey of these designs).

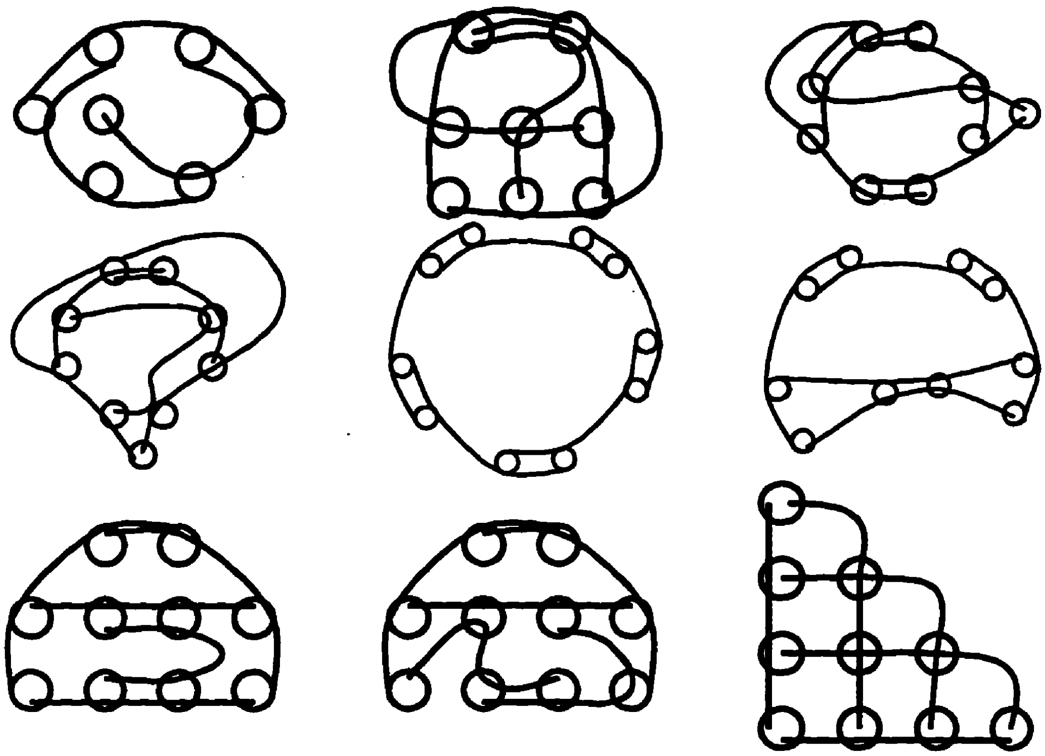
However, not all super-simple designs yield 4-erasure correcting codes. Super-simple designs avoid the configurations

A set system in which no union of t blocks contains another is called *t-cover-free*; one in which unions of t blocks are all distinct is *t-union-free.* Under the constraint of super-simplicity, the exclusion of the first configuration ensures that a 4-erasure correcting code arises from a 2-cover-free set system; the exclusion of the second and third requires in addition a 2-union-free set system. Set systems that are simple, super-simple, 2-cover-free or 2-union-free have all been studied to varying degrees. However, set systems avoiding the six required configurations have not been studied. Later (in Theorem 79), we establish a cubic lower bound on the number of blocks avoiding these six (and other) configurations.

4.6.1 (4,5)-Erasure Correcting Codes

Here we address the more difficult problem of finding 4-erasure correcting codes with update penalty four, which correct all 5-erasures except for bad 5-erasures (1 information disk and its four check disks). Naturally, any set system giving such a code must avoid the six configurations shown in the previous subsection. A tedious calculation (best done by computer) demonstrates that there are precisely nine other configurations that must be avoided to ensure that no unacceptable 5-erasures occur. These configurations are shown next.



Theorem 79 *Let q be an odd prime or prime power, and let n be an integer satisfying $1 \leq n \leq \frac{q-1}{2}$. Then there exists a (4,5)-erasure correcting code having $3q-1+n$ check disks and $nq(q-1)$ information disks.*

Proof: Let ω be a primitive element of the finite field $GF(q)$. We will define a code $[A \mid I]$ with rows indexed by

$$(GF(q) \times \{r\}) \cup ((GF(q)\backslash\{0\}) \times \{c\}) \cup (GF(q) \times \{s\}) \cup \{d_1, d_2, \ldots, d_n\}.$$

Columns are defined as follows. For $x \in GF(q)$, $y \in (GF(q)\backslash\{0\})$, and $1 \leq i \leq n$, there is a column containing '1' entries in rows (x, r), (y, c), $(x + \omega^i y, s)$ and d_i. This defines $nq(q-1)$ columns for A. We must verify that the code so defined corrects all 4-erasures, and all 5-erasures except bad ones. Partition A into n matrices $A_1, \ldots, A_n$, each $(3q-1+n) \times q(q-1)$, by placing all columns having $d_i = 1$ in A_i. Now suppose that $t \leq 5$ columns are selected; we must ensure that they do not form an unacceptable erasure. Let t_i be the number of columns in the selected set that are from A_i. If $t_i = 5$ for any i, the sum in row d_i is odd and hence the erasure is correctable. If $t_i = 4$, then all of $\{t_1, \ldots, t_n\} \backslash \{t_i\}$ are zero. Moreover, within A_i there are columns with '1' entries in positions $\{(r_1, r), (c_1, c), (s_1, s)\}$, $\{(r_1, r), (c_2, c), (s_2, s)\}$, $\{(r_2, r), (c_1, c), (s_2, s)\}$, and $\{(r_2, r), (c_2, c), (s_1, s)\}$. But then $r_1 + \omega^i c_1 = s_1 = r_2 + \omega^i c_2$ and $r_1 + \omega^i c_2 = s_2 = r_2 + \omega^i c_1$. It follows that $s_1 = s_2$, $r_1 = r_2$ and $c_1 = c_2$, and hence four distinct columns have not been chosen.

If $t_i = t_j = 1$, $i \neq j$, then none of $\{t_1, \ldots, t_n\} \backslash \{t_i, t_j\}$ are nonzero (otherwise, $t \geq 3$ but at least 3 rows have odd sum in the chosen columns, so the erasure is correctable). Now no column of A_i agrees with a column of A_j in the position of three '1' entries, so at least four rows have odd sum, and the erasure is correctable. If $t_i = 1$ and all of $\{t_1, \ldots, t_n\} \backslash \{t_i\}$ are zero, this is precisely a bad 5-erasure. If $t_i = 2$, since no two columns of A_i agree in the position of three '1' entries, these two columns have odd sum in at least four rows (consisting of zero or two each among the 'r', 'c', and 's' groups). Now no column in A contains '1' entries in more than two of these rows, and hence the only case to consider is when $t_i = t_j = 2$, $i \neq j$, then none of $\{t_1, \ldots, t_n\} \backslash \{t_i, t_j\}$. In this case, an unacceptable erasure must consist of columns with '1' entries in positions $\{(r_1, r), (c_1, c), (s_1, s), d_{a_1}\}$, $\{(r_1, r), (c_2, c), (s_2, s), d_{a_2}\}$, $\{(r_2, r), (c_1, c), (s_2, s), d_{a_3}\}$, and $\{(r_2, r), (c_2, c), (s_1, s), d_{a_4}\}$. Without loss of generality, (a_1, a_2, a_3, a_4) is one of (i, i, j, j), (i, j, i, j), or (i, j, j, i). If it is (i, i, j, j), we have the equations $r_1 + \omega^i c_1 = r_2 + \omega^j c_2$ and $r_1 + \omega^i c_2 = r_2 + \omega^j c_1$. These are satisfied only when $\omega^i = -\omega^j$ or $c_1 = c_2$. But $-\omega^j = \omega^{j+(q-1)/2}$ since q is odd, and

$i \not\equiv j \pmod{(q-1)/2}$ since $1 \le i, j \le (q-1)/2$. Hence $c_1 = c_2$, from which it follows that $s_1 = s_2$ and $r_1 = r_2$, which is impossible.

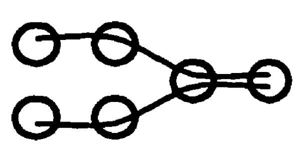
If $(a_1, a_2, a_3, a_4) = (i, j, i, j)$, one obtains $r_1 - s_1 = r_2 - s_2$ and $r_1 - s_2 = r_2 - s_1$, which forces $s_1 = s_2$ and $r_1 = r_2$ since q is odd. But then $c_1 = c_2$ as well, which is impossible. If $(a_1, a_2, a_3, a_4) = (i, j, j, i)$, we have the equations $r_1 + \omega^i c_1 = r_2 + \omega^i c_2$ and $r_1 + \omega^j c_2 = r_2 + \omega^j c_1$; the argument proceeds as in the first case. □

Taking n as large as possible in Theorem 79, when $c = 3q - 1 + \frac{q-1}{2}$ is the number of check disks, the number of information disks is $(4c^3 - 10c^2 - 8c + 24)/343$. Hence the check disk overhead approaches $\frac{343}{4c^2}$ for large q.

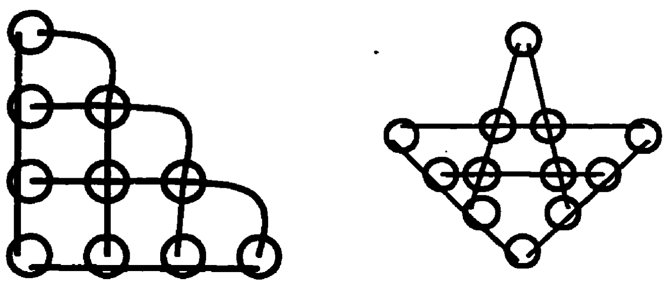
One drawback of the codes produced in Theorem 79 is that the row sums are large and not uniform. Among the $3q - 1 + n$ rows, $2q$ have sum $n(q-1)$, $q - 1$ have sum nq, and the remaining n have sum $q(q-1)$. When $n = \frac{q-1}{2}$, all groups have size $\Theta(q^2)$, but the largest group remains twice the size of the smallest. One could, however, split each of the rows $d_1, \ldots, d_n$ into two rows, assigning arbitrarily half the '1' entries to each. This yields a code with $4q - 2$ rows, and all groups of size $q^2/2$ or $q(q-1)/2$.

4.6.2 (4,6)- and (4,7)-Erasure Correcting Codes

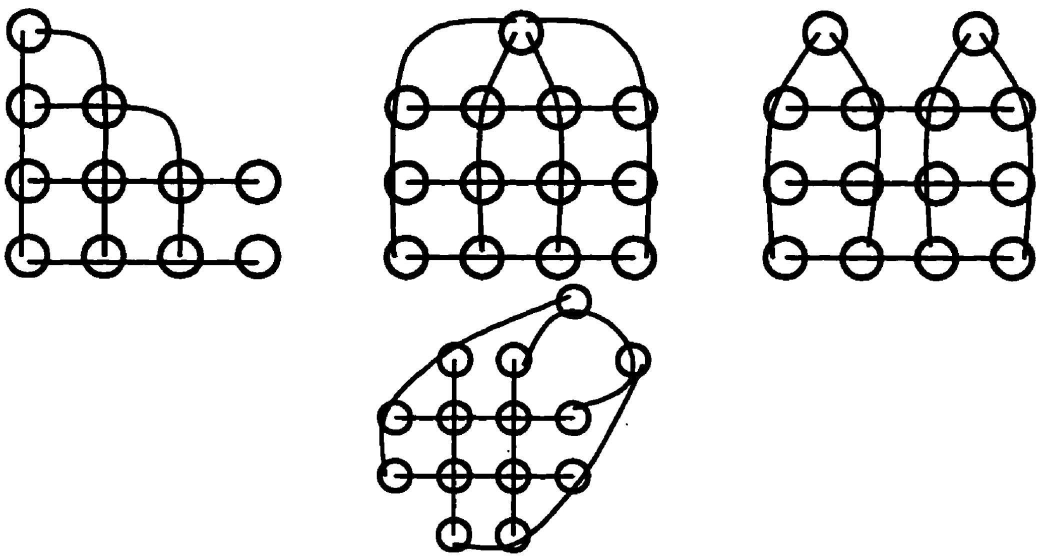
In order to correct 6-erasures, one must avoid the configuration shown next:



Hence the corresponding set system must be a packing with block size four. This obviates the need to consider many of the configurations treated earlier. Indeed, a packing with block size 4 is always 4-erasure correcting; it is (4,5)-erasure correcting if and only if it avoids the first configuration (the *dual arc* or *darc*) shown next, and it is (4,6)-erasure correcting if and only if it avoids both configurations shown next:



Finally, it is (4,7)-erasure correcting if and only if it avoids in addition the four configurations depicted here:



In a $(4,6)$ or $(4,7)$-erasure correcting codes, the number of information is at most $\frac{v(v-1)}{6}$. The equality occurs when the codes correspond to a BIBD$(v,4,1)$. In the case of $(4,7)$-erasure correcting correction codes, we have found some codes which correspond to BIBDs

Lemma 183 *There exists a BIBD*$(13,4,1)$ *avoiding all six configurations.*

Proof: The unique BIBD$(13, 4, 1)$ corresponds to a projective plane of order 3. Hence, any two lines intersect in exactly one point. Therefore, it misses the last five configurations. If the first configuration sits in the BIBD$(13, 4, 1)$, dualize the design to obtain a hyperoval in a projective plane of order three, a contradiction. □

Lemma 184 *There exists a BIBD$(v, 4, 1)$ avoiding all six configurations for $v =$ 40, 49, 52, 61, 64.*

Proof: All of them are found over the group $\mathbb{Z}_v$.

40: $\{0, 10, 20, 30\}$, $\{0, 1, 4, 13\}$, $\{0, 2, 17, 24\}$, $\{0, 5, 26, 34\}$.

49: $\{0, 1, 3, 9\}$, $\{0, 4, 18, 37\}$, $\{0, 5, 25, 32\}$, $\{0, 10, 21, 36\}$.

52: $\{0, 13, 26, 39\}$, $\{0, 1, 3, 11\}$, $\{0, 4, 16, 37\}$, $\{0, 5, 14, 32\}$, $\{0, 6, 23, 30\}$.

61: $\{0, 1, 3, 8\}$, $\{0, 4, 13, 36\}$, $\{0, 6, 28, 49\}$, $\{0, 10, 27, 47\}$, $\{0, 11, 30, 46\}$.

64: $\{0, 16, 32, 48\}$, $\{0, 1, 3, 9\}$, $\{0, 4, 18, 39\}$, $\{0, 5, 15, 41\}$, $\{0, 7, 20, 47\}$, $\{0, 11, 30, 42\}$.

In fact, there exists 1, 4, 4, 218 and 125 cyclic $(v, 4, 1)$-design over $\mathbb{Z}_v$ avoiding all six configurations for $v = 40, 49, 52, 61, 64$ respectively. □

We have a recursive construction for $(4, 7)$-erasure code.

Theorem 80 *Suppose there exists a $(4, 7)$-erasure code on b information disks and v check disks, then there exists a $(4, 7)$-erasure code on $9b + v$ information disks and $3v + 1$ check disks. In particular, if the $(4, 7)$-erasure code is a BIBD$(v, 4, 1)$, then there exists a $(4, 7)$-erasure code which is a BIBD$(3v + 1, 4, 1)$.*

Proof: We use the standard $v \to 3v+1$ construction. We can regard the $(4,7)$-erasure code as a 4-PIGD of type 1^v on V. We construct a packing design on $V \times \{0,1,2\} \cup \{\infty\}$. For every block of size 4, B, we put a TD$(4,3)$ on $B \times \{1,2,3\}$. For every $v \in V$, we add a block $\{\infty, (v,1), (v,2), (v,3)\}$.

We claim that this results in a $(4,7)$-erasure code. Suppose to the contrary, there exists a bad configuration in the packing. If it involves the point ∞ and ∞ is on at least two blocks of size four, $\{\infty, (v,1), (v,2), (v,3)\}$ and $\{\infty, (w,1), (w,2), (w,3)\}$, then all other blocks must be in the sub-TD$(4,3)$ including the points $(v,1)$ and $(w,1)$, however this is impossible. If it involves the point ∞, then ∞ can only on one block of size four, suppose the block is of the form $\{\infty, (v,1), (v,2), (v,3)\}$. Let $\{w,1),(x,1),(y,1),(v,1)\}$ be another block of size four, since $(w,1)$ is also on a block with $(v,2)$, it is clear that all blocks are from the same sub-TD$(4,3)$, it is impossible.

Next, we claim that if such a configuration exists, then all blocks must be from different TD$(4,3)$. Suppose to the contrary, if $\{(x,1),(y,1),(z,1),(w,1)\}$ and $\{(x,i),(y,2),(z,2),(w,2)\}$ be two such blocks where $i \in \{1,2\}$. By examining all six configurations, we see that for any pair of blocks, there must be a third block intersects both blocks and the point of intersection is not the ponint $(x,1)$. Hence, all other blocks must be from the same sub-TD$(4,3)$. Hence, the configuration must sit inside a TD$(4,3)$, a contradiction.

If all blocks are from different sub-TD$(4,3)$, then by projecting the configuration on V, we can obtain a contridiction. We note that if we project the "near darc" configuration (the unique configuration having exactly two points of degree one" of $(4,7)$-erasure code, we may get a darc configuration.

The remaining of the theorem follows from a simple counting argument. □

It cannot be (4,8)-erasure correcting unless no two blocks intersect (in which case the code is a simple 1-dimensional parity code).

4.7 Weakly Union-Free Twofold Triple Systems

A GDD $(X, \mathcal{G}, \mathcal{B})$ with block size three is a *weakly union-free* GDD (*wuf* GDD) if

1. whenever $\{\{a,b,x\},\{a,b,y\}\} \subseteq \mathcal{B}$, the points x and y are in different groups; and

2. whenever four distinct blocks B_1, B_2, B_3, B_4 are chosen from $\mathcal{B}$, it does not happen that $B_1 \cup B_2 = B_3 \cup B_4$.

The second condition can be made more explicit: there cannot exist four blocks of any of the following four forms:

$$\begin{aligned}
\text{C1} &: \{\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\},\\
\text{C2} &: \{\{x,a,b\},\{x,a,c\},\{x,b,d\},\{x,c,d\}\},\\
\text{C3} &: \{\{x,a,b\},\{x,a,c\},\{x,b,d\},\{a,c,d\}\},\\
\text{C4} &: \{\{x,a,b\},\{x,c,d\},\{y,a,b\},\{y,c,d\}\}.
\end{aligned}$$

These forms correspond, respectively, to the hypergraphs depicted below.

C1 C2 C3 C4

Our interest is in the construction of wuf 3-GDDs, and in particular those of type 1^n and index two. A uniform GDD with group size 1 is a *balanced incomplete* *block design*; those with $k = 3$ and $\lambda = 2$ are called *twofold triple systems* of order n, or TTS(n). Frankl and Füredi [55] began the study of wuf TTS(n) in the study of an old problem of Erdős [54]. In 1938, Erdős [53] asked what the maximum number of edges a graph can have and have no 3-cycle, no 4-cycle, and no repeated edges. In 1977, he [54] asked the more general question: How many hyperedges can a k-uniform hypergraph have, so that whenever four hyperedges A, B, C, D satisfy $A \cup B = C \cup D$, we find $\{A, B\} = \{C, D\}$. Such a family is *union-free.* Frankl and Füredi [55] settled this question when $k = 3$, showing that a class of designs, the Steiner triple systems, realize the maximum.

They also addressed the related question of enforcing the union-free condition only for sets of four distinct blocks A, B, C, D. This gives the notion of weakly union-free, already defined. Frankl and Füredi [55] established an important bound, and showed that it is realized infinitely often:

Theorem 81 (Frankl and Füredi [55]) *A weakly union-free 3-uniform hypergraph on n vertices has at most $\left\lfloor \frac{n(n-1)}{3} \right\rfloor$ hyperedges. Equality occurs when all, or all but one, pair of vertices occur in two hyperedges each.*

They established that this bound is met whenever $n \equiv 1 \pmod 6$, and either n is a prime power at least 13 or n is sufficiently large. In this section, we establish that equality is met for all $n \equiv 0, 1 \pmod 3$, with a small number of definite, and a small number of possible, exceptions. Theorem 81 also admits the possibility that $n \equiv 2 \pmod 3$. While we have also found small designs sufficient to obtain a closure in this class, we concentrate on the twofold triple system case here.

The difficulty of this problem appears initially to be that, while catalogues of twofold triple systems for small orders are available (see [40], for example), no

TTS(n) is weakly union-free when $n \in \{3,4,6,7,9,10\}$. Moreover, when a wuf 3-GDD of type T can be decomposed into two 3-GDDs of index one and type T, condition (1) together with the exclusion of C4 ensure that these two index one 3-GDDs are "orthogonal" (see [44]). The existence of orthogonal uniform 3-GDDs with group size 1, the so-called orthogonal Steiner triple systems, remained open for thirty years until its recent solution [45]. The exclusion of further configurations adds to the difficulty of the problem for wuf TTS.

4.7.1 Direct Constructions

In this subsection, we develop a direct construction technique that is used to construct both wuf TTS(n) and, more generally, wuf 3-GDDs of index two. The general framework follows. We aim to construct a 3-GDD of index two on point set $\mathbb{Z}_{gu} \cup \{\infty_1, \ldots, \infty_k\}$, whose type is $g^u k^1$. Naturally, we chose $\mathbb{Z}_{gu}$ for a portion of the point set to suggest the cyclic action of the cyclic group on these points. Indeed our goal is to construct 3-GDDs that have $\mathbb{Z}_{gu}$ as an automorphism group.

Let $X = \mathbb{Z}_{gu} \cup \{\infty_1, \ldots, \infty_k\}$, and let σ be a permutation mapping $i \mapsto i + 1 \bmod gu$ for $i \in \mathbb{Z}_{gu}$, and fixing $\{\infty_1, \ldots, \infty_k\}$. Let $\mathcal{B}$ be the blocks of a 3-GDD of type $g^u k^1$ on X that admits σ as an automorphism. The action of σ partitions $\mathcal{B}$ into orbits of size gu or, when $gu \equiv 0 \pmod 3$, possibly $\frac{gu}{3}$. A set of representatives of these orbits forms a *set of starter blocks* for the 3-GDD. Starter blocks of the form $\{0, a, b\} \subset \mathbb{Z}_{gu}$ may generate orbits of length gu under σ, in which case the starter block is said to *cover* the differences $\pm a$, $\pm b$, $\pm(b-a)$ with arithmetic in $\mathbb{Z}_{gu}$ (if repetitions occur, such differences are covered the number of times that they occur). When $gu \equiv 0 \pmod 3$, a starter block of the form $\{0, \frac{gu}{3}, 2\frac{gu}{3}\}$ generates only $\frac{gu}{3}$ distinct blocks (a *short orbit*), and is said therefore to *cover* the differences

$\pm\frac{gu}{3}$ once each. Finally, a starter block may have the form $\{\infty_i, 0, d_i\}$; again, gu blocks appear in the orbit generated, but here only the differences $\pm d_i$ are covered, once each.

A set $\mathcal{D}$ is a set of starter blocks for a 3-GDD of index two and type $g^u k^1$ (under the action of σ) if

1. for $1 \leq i \leq k$, there is exactly one starter block containing ∞_i; and
2. each $d \in \mathbb{Z}_{gu}$ is covered twice as a difference, unless $d \equiv 0 \pmod{u}$, in which case the difference is not covered.

The reader can quickly verify that these conditions on starter blocks are equivalent to the existence of a 3-GDD of index two and type $g^u k^1$ admitting σ.

In order to be a wuf 3-GDD, further conditions are imposed. Suppose that $\mathcal{D}$ is the set of starter blocks for a 3-GDD of index two and type $g^u k^1$. Partition $\mathcal{D}$ into the blocks $\mathcal{A}$ which contain one of the infinite points, and the blocks $\mathcal{B}$ which do not. Evidently, $\mathcal{A}$ contains exactly k blocks, one for each of the infinite points. In addition, in order to meet the first wuf condition, we have:

(1) If $\{\infty_i, 0, a\}, \{\infty_j, 0, b\} \in \mathcal{A}$, then $a \not\equiv \pm b \pmod{gu}$.

Call a difference *external* if it is covered once in $\mathcal{A}$ and once in $\mathcal{B}$, and *internal* if it is covered twice in $\mathcal{B}$. For each external difference d, define $\alpha(d) = \min(\pm 2d)$. For each internal difference d, when blocks $\{0, d, x\}$ and $\{0, d, y\}$ appear in the orbits of blocks of $\mathcal{B}$, define $\alpha(d) = \min(\pm(x-y))$.

First we examine constraints resulting from prohibiting the appearance of one of the infinite points in one of the configurations C1, C2, C3, or C4. In order to ensure that no infinite point occurs in a C1 configuration, we require that

(2) If $gu \equiv 0 \pmod 3$ and $\frac{gu}{3}$ is an external difference, then $\mathcal{B}$ does not contain $\{0, \frac{gu}{3}, 2\frac{gu}{3}\}$.

In order to ensure that no infinite point occurs in a C2 configuration, we require that

(3) If d is an external difference, then $4d \not\equiv 0 \pmod{gu}$.

In order to ensure that no infinite point occurs in a C3 configuration, we require that

(4) If d is an external difference and $\{0, d, x\}$ is a block in an orbit of a starter block of $\mathcal{B}$, then $2x \not\equiv d \pmod{gu}$ and none of $\{0, d, 3d\}$, $\{0, 2d, 3d\}$, $\{0, 2d, d+x\}$, $\{0, 2d, x\}$, or $\{0, d, \frac{gu}{2}\}$ when $gu \equiv 0 \pmod 2$, appear in the orbits of the starter blocks in $\mathcal{B}$.

In order to ensure that no infinite point occurs in a C4 configuration, we require that

(5) If d and d' are external differences, or if d is external and d' is internal, then $\alpha(d) = \alpha(d')$ only if $d = d'$.

Once conditions (1)–(5) are met, any violation of the wuf conditions occurs entirely among the blocks on $\mathbb{Z}_{gu}$.

In order to check that none of the conditions are violated on the blocks involving no infinite points, we first observe that the first wuf condition is equivalent to:

(6) If d is an internal difference then $\alpha(d) \not\equiv 0 \pmod u$.

Lemma 45 (Frankl and Füredi [55]) *A 3-GDD of index two and type 1^q exists whenever $q \equiv 1 \pmod 6$ is a prime power, except when $q = 7$.*

It is essential that ingredients for other congruence classes modulo 6 be found as well. We employed a combination of backtracking and hillclimbing techniques to produce a large number of wuf GDDs.

Numerous 3-GDDs of type $1^u x^1$ over $\mathbb{Z}_u$ are given in order to establish the statement:

Lemma 46 *A wuf 3-GDD of type 1^n exists for $n = 21, 24, 27, 28, 30, 33, 34, 36, 39, 40, 42, 45,$ and 46.*

Proof: For each pair $\{a, b\}$ presented in the table to follow, $\{0, a, b\}$ is a starter block. In addition, if $u \equiv 0 \pmod 3$ and $x \equiv 1 \pmod 3$, then $\{0, \frac{u}{3}, \frac{2u}{3}\}$ is a starter block. Finally, each difference covered only once in the starter blocks so produced is also in a starter block with an infinite point.

GDD	Internal Starter Blocks
$1^{20}1^1$	{1,7} {1,9} {2,4} {3,8} {3,13} {4,9}
$1^{23}1^1$	{1,6} {2,13} {2,16} {3,12} {3,18} {4,8} {6,16}
$1^{26}1^1$	{1,6} {2,12} {2,23} {3,19} {4,13} {4,18} {6,17} {7,18}

GDD	Internal Starter Blocks
$1^{28}0^1$	{1,2} {2,13} {3,7} {3,12} {4,12} {5,11} {5,19} {6,13} {8,18}
$1^{29}1^1$	{1,7} {2,15} {2,18} {3,8} {3,12} {4,22} {4,23} {5,19} {8,17}
$1^{32}1^1$	{1,3} {1,8} {3,10} {4,19} {4,20} {5,18} {5,26} {6,23} {8,18} {9,21}
$1^{33}1^1$	{1,4} {2,8} {2,21} {3,16} {4,26} {5,15} {5,24} {6,24} {7,23} {8,21}
$1^{35}1^1$	{1,4} {2,6} {2,19} {3,20} {5,12} {5,29} {7,18} {8,16} {9,22} {9,23} {10,20}
$1^{38}1^1$	{1,4} {2,7} {2,17} {3,15} {4,18} {5,13} {6,12} {7,27} {8,22} {9,22} {9,28} {10,27}
$1^{40}0^1$	{1,2} {2,5} {3,7} {4,27} {5,15} {6,24} {7,29} {8,21} {8,28} {9,21} {9,26} {11,26}
$1^{41}1^1$	{1,4} {2,6} {2,14} {3,30} {5,24} {5,31} {6,21} {7,18} {7,32} {8,20} {8,25} {9,28} {10,28}
$1^{44}1^1$	{1,2} {2,5} {3,7} {5,13} {6,20} {6,33} {7,23} {8,24} {9,26} {9,19} {10,34} {11,31} {12,30} {13,28}
$1^{45}1^1$	{1,4} {2,6} {2,7} {3,29} {5,35} {6,34} {7,25} {8,22} {8,29} {9,20} {9,31} {10,27} {12,24} {13,26}

□

Lemma 47 *A wuf 3-GDD of type* 1^n *exists for* $n = 48$, 51, 52, 54, 55, 57, 58, 60, 63, 64, 66, 69, 70, 72, 75, 76, 78, 81, 82, 84, 85, 87, 88, 90, 91, 93, 94, 96, 99, 100, 102, 105, 108, 111, 112, 114, 115, 117, 118, 120, 123, 124, 126, 129, 130, 132, 133, 135, 136, 138, 141, 142, 144, 145, 148, 150, 154, 156, 159, 160, 161, 165, 166, 171, 177, 178, 184, 195, 201, 207, 213, 219, *and* 243.

Proof: See Appendix B. □

The remaining small values do not appear to be able to be handled by this general approach. However, we have succeeded in one more case:

Lemma 48 *A wuf 3-GDD of type 1^{16} exists.*

Proof: Let $X = \mathbb{Z}_8 \times \{0,1\}$. For succinctness, we write $(x,i) \in X$ as x_i. Let $\sigma : X \to X$ be the permutation such that $\sigma : x_i \mapsto (x+1 \pmod 8)_i$. Developing the following set of starter blocks by σ gives a wuf 3-GDD of type 1^{16} on X:

$$\{0_0,1_0,3_1\}\ \{0_0,4_0,0_1\}\ \{0_0,2_0,5_0\}\ \{0_0,2_0,1_1\}\ \{3_0,0_1,1_1\}$$
$$\{0_0,1_1,3_1\}\ \{0_0,1_0,5_1\}\ \{0_1,2_1,5_1\}\ \{0_0,2_1,6_1\}\ \{0_0,0_1,7_1\}$$

□

4.7.2 Recursive Constructions

We employ two well known constructions.

Theorem 82 (Wilson's Fundamental Construction [117]) *Let $(X,\mathcal{G},\mathcal{B})$ be a GDD (the* master *GDD) with groups $G_1, G_2 \ldots G_t$. Suppose there exists a function $w : X \to \mathbb{Z}^+ \cup \{0\}$ (a* weight function*) which has the property that for each block $B = \{x_1, x_2, \ldots, x_k\} \in \mathcal{B}$ there exists a K-GDD of type $[w(x_1), w(x_2) \ldots, w(x_k)]$ (such a GDD is an* "ingredient" *GDD). Then there exists a K-GDD of type*

$$\left[\sum_{x\in G_1} w(x), \sum_{x\in G_2} w(x), \ldots, \sum_{x\in G_t} w(x)\right].$$

We leave as an easy exercise that when all of the ingredient GDDs are wuf, so also is the GDD constructed. In general, our desire is to produce GDDs with group size 1, so we need to fill in the holes in some way.

Theorem 83 (Filling in Holes, variant of [97]) *If there exists a wuf GDD of type $g_1g_2\ldots g_n$, and for $2 \le i \le n$ a wuf GDD of type $1^{g_i}h^i$ exists, then there exists a wuf GDD of type $1^{\sum_{i=2}^{n} g_i}(g_1+h)^1$.*

In Theorem 83, both $g_1 = 0$ and $h = 0$ correspond to useful special cases. Filling in holes preserves the wuf property primarily as a consequence of the first requirement, since none of the forbidden configurations can have both a block from the wuf GDD of type $g_1g_2\ldots g_n$ and one from a wuf GDD of type $1^{g_i}h^i$. Normally, we do not comment on applications of Theorem 83, leaving this to the diligent reader. Typically, Theorem 82 is applied using suitable ingredients, and Theorem 83 is then applied to extract useful consequences for group size 1.

Now we give some applications of Theorem 82.

Lemma 49 *If a TD$(6,n)$ exists, then a wuf 3-GDD of type $(3n)^5(6n)^1$ exists. Moreover, there exist wuf TTS of orders* 106, 147, 168, 189, *and* 231.

Proof: A wuf 3-GDD of type 3^56^1 exists with presentation $\{\{1,12\},\{2,9\}\}$. Use the TD$(6,n)$ as a master design and the 3-GDD of type 3^56^1 as an ingredient design in Theorem 82. Apply with $n = 5,7,8,9,11$ and fill in holes using wuf 3-GDDs of types $1^{15}1^1$ and $1^{30}1^1$ when $n = 5$, and of types 1^{3n} and 1^{6n} for the remaining values of n. □

Lemma 50 *If a TD$(7,n)$ exists, then a wuf 3-GDD of type $(2n)^7$ exists. Hence wuf TTS of orders* 112, 183, *and* 225 *exist.*

Proof: A wuf 3-GDD of type 2^7 exists with presentation $\{\{1,4\},\{1,6\},\{2,6\},\{2,11\}\}$; Theorem 82 gives the wuf 3-GDD of type $(2n)^7$. Applying with $n = 8,13,16,$ and filling holes with wuf 3-GDDs of types 1^{16}, $1^{26}1^1$, and $1^{32}1^1$ gives the required consequences. □

Lemma 51 *If a TD(8, n) exists, and $0 \le x \le n$, then a wuf 3-GDD of type $(3n)^7(3n+6x)^1$ exists. Hence there exist wuf TTS of orders* 174, 180, 186, 192, 198, 204, 210, 216, 222, 228, *and* 187.

Proof: A wuf 3-GDD of type $3^8 = 3^7 3^1$ exists over $\mathbb{Z}_{24}$ with presentation

$$\{\{1,3\},\{1,20\},\{2,12\},\{3,10\},\{4,11\},\{5,18\},\{6,15\}\}.$$

A wuf 3-GDD of type $3^7 9^1$ exists with presentation $\{\{1,13\},\{2,5\},\{4,10\}\}$. Apply Theorem 82 using weight 3 in seven groups and weights 3 or 9 in the eighth, to produce a wuf 3-GDD of type $(3n)^7(3n+6x)^1$. Apply with $n = 7, 8$ and fill in holes to obtain the stated consequences. For the final value, apply with $n = 7$ and employ a wuf 3-GDD of type $1^{21}7^1$ to fill holes. It has presentation $\{\{1,4\},\{1,6\},\{2,9\},\{2,13\}\}$. □

Lemma 52 *If a TD(14, q) exists, and $0 \le x \le 6q$ satisfies $x \equiv 0 \pmod 3$, then a wuf 3-GDD of type $q^{13}x^1$ exists. If, in addition, a wuf 3-GDD of type $1^q h^1$ exists, so also does a wuf GDD of type $1^{13q}(x+h)^1$.*

Proof: Use as ingredient wuf 3-GDDs the ones of type $1^{13}0^1$ from Lemma 45, of type $1^{13}3^1$ presented as $\{\{1,4\},\{3,4\},\{2,8\}\}$, and the one of type $1^{13}6^1$ presented as $\{\{1,4\},\{2,8\}\}$. Give all points in thirteen groups of the TD(14, q) weight one, and points in the final group weights 0, 3, or 6 so that the total weight in the final group is x. Theorem 82 then gives a wuf 3-GDD of type $q^{13}x^1$. Filling in holes with a 3-GDD of type $1^q h^1$ (when one exists) gives a wuf GDD of type $1^{13q}(x+h)^1$. □

1. $r_0 = 19$ and $s_0 = 23$;

2. $r_{i+1} > r_i$ and $s_{i+1} > s_i$;

3. $13r_{i+1} + 21 \leq 19r_i$ and $13s_{i+1} + 25 \geq 19s_i + 1$;

4. $r_i \equiv 1 \pmod 3$ and $s_i \equiv 2 \pmod 3$; and

5. $TD(14, r_i)$ and $TD(14, s_i)$ exist.

A TD$(14, n)$ exists whenever n is relatively prime to 2, 3, 5, 7, and 11 (by MacNeish's theorem; see [3]). Among the integers congruent to 1 modulo 3, considering the sequence of those relatively prime to 2, 3, 5, 7, and 11, we find a largest difference between consecutive values of 24. Choose the r_is to be the sequence of numbers congruent to 1 modulo 3 and relatively prime to 2, 3, 5, 7, and 11, beginning with 19, in addition to the number 25. It is now an easy verification that we have the specified properties. In the same way, the s_is are the sequence of numbers congruent to 2 modulo 3 and relatively prime to 2, 3, 5, 7, and 11, beginning with 23, in addition to the number 32.

To prove the theorem, we proceed inductively. In general, we suppose that wuf TTS have been produced for all orders less than n, where $n \equiv 0, 1 \pmod 3$, and we establish that a wuf TTS(n) exists. By assumption, wuf TTS(n) exist whenever $24 \leq n \leq 304$. Now if $n \equiv 1 \pmod 3$, find the largest i for which $13r_i + 24 \leq n \leq 19r_i$; such a choice exists by the definition of the sequence. Then a TD$(14, r_i)$ exists. Wuf 3-GDDs of type 1^{r_i} and 1^{n-13r_i} exist by the inductive hypothesis. Apply Theorem 52 to obtain the wuf TTS(n). In the same way, if $n \equiv 0 \pmod 3$, find the largest s_i for which $13s_i + 25 \leq n \leq 19s_i + 1$; such a choice exists by the definition of the sequence. Then a TD$(14, s_i)$ exists. Wuf 3-GDDs of type $1^{s_i}1^1$ and 1^{n-13s_i} exist by the inductive hypothesis. Apply Theorem 52 to obtain the wuf TTS(n). □

Now we can prove the main theorem.

Theorem 85 *A wuf TTS(n) exists whenever $n \equiv 0, 1 \pmod 3$ except when $n \in \{3, 4, 6, 7, 9, 10\}$ and possibly when $n \in \{12, 15, 18, 22\}$.*

Proof: The definite exceptions can all be verified by an exhaustive search. Now if n is a prime or prime power, apply Lemma 45. Otherwise, apply Lemmas 46, 47, and 48 to treat most small orders, and Lemmas 49, 50, 51, 53 and Corollaries 24 and 25 to treat $n = 21$ and all remaining values satisfying $24 \le n \le 304$. Then apply Theorem 84 to complete the proof. □

4.7.3 An Application to Group Testing

Let Ω be a population of items, where each item is in exactly one of the states 0, 1. Furthermore, at most r items are in state 1. The problem is to determine the state of each item (or equivalently, to determine the set of all items in state 1) through some tests. A test can be performed on any subset $P \subseteq \Omega$, called a *pool*. The *feedback* to a test on pool P, denoted $f(P)$, is defined by $f(P) = \max_{\omega \in P}\{\text{state of } \omega\}$. This problem, known as the *group testing problem*, has numerous real-world applications ranging from multiple access communications [22] to DNA clone isolation [32], and its study constitutes an important part of combinatorial search theory [52]. In some applications, it is desirable to have each item involved in exactly k pools. We call the resulting problem *k-restricted*. For simplicity, we denote the k-restricted group testing problem, with at most r items in state 1, by $\mathrm{GTP}_k(r)$.

An algorithm for the group testing problem is said to be an *α-approximation algorithm* if it returns a set S of at most αr items, so that S contains all items of Ω that are in state 1.

There are two well-known classes of algorithms for solving group testing problems: *sequential* and *nonadaptive* algorithms. In a sequential algorithm, the decision of which pool to test next can depend on the feedbacks to previous tests. On the other hand, a nonadaptive algorithm must specify all the pools to be tested at the very beginning, without receiving any feedbacks. The *complexity* of a group testing algorithm is defined to be the number of tests conducted (hence, also the number of pools). It is obvious that the best sequential algorithm has a complexity no higher than any nonadaptive algorithm. However, the advent of massively parallel computers have prompted Hwang and Sós [70] to make a case for the study of nonadaptive algorithms. Further support of this case is given by Knill and Muthukrishnan [72] who observed that certain features in the screening of clone libraries with hybridization probes strongly encourage nonadaptive algorithms.

Our focus in this section is on nonadaptive $\frac{3}{2}$-approximation algorithms for $\mathrm{GTP}_3(2)$. Any nonadaptive algorithm $\mathcal{A}$ for $\mathrm{GTP}_3(2)$ corresponds to a 3-uniform hypergraph $\mathcal{H}(\mathcal{A}) = (X, \mathcal{B})$ as follows:

1. $X = \{x_P : P \text{ is a pool of } \mathcal{A}\}$.
2. $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$.
3. $x_P \in B_\omega$ if and only if $\omega \in P$.

We call $\mathcal{H}(\mathcal{A})$ the *hypergraph of $\mathcal{A}$*. We make the following useful observation concerning $\mathcal{H}(\mathcal{A})$. Let $\emptyset$ be the set of all state 1 items in Ω. Then $x_P \in \bigcup_{\omega \in \emptyset} B_\omega$ if and only if P is a pool of $\mathcal{A}$ such that $f(P) = 1$. Hence, if we know that one of $\emptyset$ or $\emptyset'$ contains the set of all state 1 items in Ω, then a necessary and sufficient condition which allows us to distinguish them is

$$\bigcup_{\omega \in \emptyset} B_\omega \neq \bigcup_{\omega \in \emptyset'} B_\omega.$$

Lemma 54 *If $\mathcal{A}$ is a nonadaptive $\frac{3}{2}$-approximation algorithm for $GTP_3(2)$, then $\mathcal{H}(\mathcal{A}) = (X, \mathcal{B})$ is weakly union-free.*

Proof: Assume on the contrary that there are four distinct hyperedges $B_{\omega_i} \in \mathcal{B}$, $1 \leq i \leq 4$, such that $B_{\omega_1} \cup B_{\omega_2} = B_{\omega_3} \cup B_{\omega_4}$. Hence, if one of $\{\omega_1, \omega_2\}$ or $\{\omega_3, \omega_4\}$ is the pair of state 1 items, then $\mathcal{A}$ cannot distinguish them. The best $\mathcal{A}$ can do is then to conclude that $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ contains all the state 1 items of Ω. But this violates the condition that $\mathcal{A}$ is a $\frac{3}{2}$-approximation algorithm. □

Corollary 26 *The complexity of any nonadaptive $\frac{3}{2}$-approximation algorithm for $GTP_3(2)$ with a population of n items is at least $\left\lceil \sqrt{3n} + \frac{1}{2} \right\rceil$.*

Lemma 55 *Any wuf $TTS(n)$ is the hypergraph of a nonadaptive $\frac{3}{2}$-approximation algorithm for $GTP_3(2)$.*

Proof: Let $\mathcal{A}$ be the nonadaptive algorithm specified by a wuf $TTS(n)$, $\mathcal{H}(\mathcal{A}) = (X, \mathcal{B})$. Let $\omega_1, \omega_2, \omega_3 \in \Omega$ be any three distinct items. Then $B_{\omega_1} \neq B_{\omega_2}$ since $H(\mathcal{A})$ contains no repeated hyperedges, and $B_{\omega_1} \neq B_{\omega_2} \cup B_{\omega_3}$ since the union of two distinct hyperedges contains at least four vertices. Hence if Ω contains only one item in state 1, then $\mathcal{A}$ can identify that item precisely. We are thus left with the task of considering the case with two items in state 1.

It suffices to show that for any three distinct hyperedges $B_{\omega_1}, B_{\omega_2}, B_{\omega_3} \in \mathcal{B}$ such that $B_{\omega_1} \cup B_{\omega_2} = B_{\omega_1} \cup B_{\omega_3} = F$, we have $\{B, B'\} \subseteq \{B_{\omega_1}, B_{\omega_2}, B_{\omega_3}\}$ whenever $B \cup B' = F$. So let $B \cup B' = F$. Suppose that at least one of B or B' is not B_{ω_1}, B_{ω_2}, or B_{ω_3}, for otherwise we are done. Therefore we must have $\{B, B'\} = \{B_{\omega_1}, B_{\omega_4}\}$, for some $\omega_4 \notin \{\omega_1, \omega_2, \omega_3\}$ since $(X, \mathcal{B})$ is weakly union-free. We know that $|B_{\omega_1} \cap B_{\omega_2}| \neq 0$ or 3 because $\mathcal{B}$ contains no repeated hyperedges. If $|B_{\omega_1} \cap B_{\omega_2}| = 2$, then $|F| = 4$, implying that $\{B_{\omega_1}, B_{\omega_2}, B_{\omega_3}, B_{\omega_4}\}$ is the complete 3-uniform hypergraph on four vertices, which is not weakly union-free. It follows that $|B_{\omega_1} \cap B_{\omega_2}| = 1$. But then $B_{\omega_2} \setminus B_{\omega_1}$ is a 2-subset that must also be contained in the blocks B_{ω_3} and B_{ω_4}. This contradicts the assumption that $(X, \mathcal{B})$ is a twofold triple system. $\square$

Corollary 27 *For any $n \equiv 0, 1 \pmod 3$, and $n > 22$, there exists a nonadaptive $\frac{3}{2}$-approximation algorithm of (optimal) complexity n for $GTP_3(2)$ with a population of $n(n-1)/3$ items.*

Chapter 5

Conclusion

In this thesis, we have studied pairwise balanced designs, group divisible designs and related codes. We conclude with a short discussion of the main themes that have been explored, and an outline of the extensive collaborations reported here.

5.1 Collaborations

A large amount of the research reported here has been done in collaboration with others. In this section, these collaborations are made clear. Section 2.1 is joint work with Colbourn, and appears in [84]. Sections 2.2 and 2.3 are in collaboration with Colbourn, Mullin and Zhu and appears in [87]. Section 2.4, with Colbourn, appear in [46]. Section 2.6 is with Chee, Colbourn and Gallant and appears in [34]. For further PBD closure results of the author, not included in this thesis, see [98] reporting collaborative work of the author with Mullin, Abel and Bennett. Some results on the generating sets of the author with Colbourn, not included in this thesis, see [83].

Section 3.1 and 3.2 report joint work with Abel, Colbourn, and Yin which appears in [122]. Section 3.5 concerns research with Colbourn appearing in [85]. Some results on the existence of GDDs with block sizes 3 and n of the author with Chee, see [36].

Section 4.1 introduces research with Chee and Colbourn; see also [33]. Section 4.2 is a joint work with Colbourn. Section 4.5 appears in [81]. Section 4.7 is a joint work with Chee and Colbourn [35].

5.2 Some Themes

Finite projective planes are used extensively in this thesis in constructing new combinatorial designs. We are able to obtain some new pairwise balanced designs by deleting various line configurations from finite projective planes. We believe that there are many more interesting configurations in the finite projective plane which lead to interesting combinatorial objects.

We have studied the existence of 5-GDDs of uniform group size. Many direct constructions are developed in order to obtain a strong existence result. Unlike most papers in the literature, we have obtained a strong result by a large set of direct constructions. In most of the direct constructions, certain automorphism groups and underlying structure are assumed in order to make the search feasible. Identifying a potential automorphism group and implementation are key factors to succeed in finding the design. In term of identifying a potential automorphism group, there is a trade off between the size of search space and the flexibility of the existence of the design under a given group. For example, many attempts were made to find a 5-GDD of type 10^7. We cannot find it with a group of order 35, and cannot complete the search with a smaller automorphism group. Although, this theme arises frequently throughout the course of this research, due to space limitations, we have not explicitly stated why and how we chose the automorphism group that we have used for constructing the designs.

Although we have a large number of direct constructions, it is noteworthy that if $g \equiv 2 \pmod 4$ and $u \equiv 11, 15 \pmod{20}$, we do not have a good set of techniques for constructing 5-GDD of type g^u. Such a design can not exist with a cyclic group of order gu. Due to the limitation of the direct construction method, we have circumvented it by applying a new recursive construction using modified group divisible designs. The required modified group divisible designs are often much easier to construct. By combining both direct and new recursive constructions, we are able to show that 5-GDD of type g^u exists for but possibly finite number of pairs (g, u).

Finally, we have obtained some new connections between coding theory and design theory. The interaction between coding theory and design theory had been known for a long time. It is a pleasant surprise that designs with certain forbidden sub-configurations can be used to obtain some practical codes arising from computer science. In particular, it relates a well known open problem in design theory is related in a useful way to coding theory. Despite numerous effort, it is disappointing that we have not been able to settle the existence of anti-Pasch STSs completely. However, several new constructions are presented and they can obtained new infinite classes of anti-Pasch STSs.

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Appendix A

In this appendix, we construct a $\{5,6\}$-GDD of type type $5^8 6^1$. We take design no. 33 from [92].

Block No.	Block
0	$\{(8,1),(8,2),(8,3),(8,4),(8,5)\}$
1	$\{(0,1),(0,2),(0,3),(0,4),(0,5)\}$
2	$\{(1,1),(1,2),(1,3),(1,4),(1,5)\}$
3	$\{(2,1),(2,2),(2,3),(2,4),(2,5)\}$
4	$\{(3,1),(3,2),(3,3),(3,4),(3,5)\}$
5	$\{(8,1),(0,2),(2,2),(0,7),(2,7)\}$
6	$\{(8,1),(1,2),(3,2),(1,7),(3,7)\}$
7	$\{(8,2),(0,3),(2,3),(0,8),(2,8)\}$
8	$\{(8,2),(1,3),(3,3),(1,8),(3,8)\}$
9	$\{(8,3),(0,4),(2,4),(0,9),(2,9)\}$
10	$\{(8,3),(1,4),(3,4),(1,9),(3,9)\}$
11	$\{(8,4),(0,5),(2,5),(0,0),(2,0)\}$
12	$\{(8,4),(1,5),(3,5),(1,0),(3,0)\}$

Block No.	Block
13	$\{(8,5),(0,1),(2,1),(0,6),(2,6)\}$
14	$\{(8,5),(1,1),(3,1),(1,6),(3,6)\}$
15	$\{(8,1),(0,1),(1,6),(0,9),(3,0)\}$
16	$\{(8,1),(1,1),(2,6),(1,9),(0,0)\}$
17	$\{(8,1),(2,1),(3,6),(2,9),(1,0)\}$
18	$\{(8,1),(3,1),(0,6),(3,9),(2,0)\}$
19	$\{(8,2),(0,2),(3,6),(1,7),(0,0)\}$
20	$\{(8,2),(1,2),(0,6),(2,7),(1,0)\}$
21	$\{(8,2),(2,2),(1,6),(3,7),(2,0)\}$
22	$\{(8,2),(3,2),(2,6),(0,7),(3,0)\}$
23	$\{(8,3),(0,3),(0,6),(3,7),(1,8)\}$
24	$\{(8,3),(1,3),(1,6),(0,7),(2,8)\}$
25	$\{(8,3),(2,3),(2,6),(1,7),(3,8)\}$
26	$\{(8,3),(3,3),(3,6),(2,7),(0,8)\}$
27	$\{(8,4),(0,4),(0,7),(3,8),(1,9)\}$
28	$\{(8,4),(1,4),(1,7),(0,8),(2,9)\}$
29	$\{(8,4),(2,4),(2,7),(1,8),(3,9)\}$
30	$\{(8,4),(3,4),(3,7),(2,8),(0,9)\}$
31	$\{(8,5),(0,5),(0,8),(3,9),(1,0)\}$
32	$\{(8,5),(1,5),(1,8),(0,9),(2,0)\}$
33	$\{(8,5),(2,5),(2,8),(1,9),(3,0)\}$
34	$\{(8,5),(3,5),(3,8),(2,9),(0,0)\}$
35	$\{(8,1),(0,3),(2,4),(1,5),(3,8)\}$

Block No.	Block
36	$\{(8,1),(1,3),(3,4),(2,5),(0,8)\}$
37	$\{(8,1),(2,3),(0,4),(3,5),(1,8)\}$
38	$\{(8,1),(3,3),(1,4),(0,5),(2,8)\}$
39	$\{(8,2),(0,1),(3,4),(1,5),(2,9)\}$
40	$\{(8,2),(1,1),(0,4),(2,5),(3,9)\}$
41	$\{(8,2),(2,1),(1,4),(3,5),(0,9)\}$
42	$\{(8,2),(3,1),(2,4),(0,5),(1,9)\}$
43	$\{(8,3),(0,1),(3,2),(2,5),(1,0)\}$
44	$\{(8,3),(1,1),(0,2),(3,5),(2,0)\}$
45	$\{(8,3),(2,1),(1,2),(0,5),(3,0)\}$
46	$\{(8,3),(3,1),(2,2),(1,5),(0,0)\}$
47	$\{(8,4),(0,1),(2,2),(1,3),(3,6)\}$
48	$\{(8,4),(1,1),(3,2),(2,3),(0,6)\}$
49	$\{(8,4),(2,1),(0,2),(3,3),(1,6)\}$
50	$\{(8,4),(3,1),(1,2),(0,3),(2,6)\}$
51	$\{(8,5),(0,2),(2,3),(1,4),(3,7)\}$
52	$\{(8,5),(1,2),(3,3),(2,4),(0,7)\}$
53	$\{(8,5),(2,2),(0,3),(3,4),(1,7)\}$
54	$\{(8,5),(3,2),(1,3),(0,4),(2,7)\}$
55	$\{(0,1),(1,1),(2,4),(1,7),(2,8)\}$
56	$\{(1,1),(2,1),(3,4),(2,7),(3,8)\}$
57	$\{(2,1),(3,1),(0,4),(3,7),(0,8)\}$
58	$\{(3,1),(0,1),(1,4),(0,7),(1,8)\}$

Block No.	Block
59	$\{(0,2),(1,2),(2,5),(1,8),(2,9)\}$
60	$\{(1,2),(2,2),(3,5),(2,8),(3,9)\}$
61	$\{(2,2),(3,2),(0,5),(3,8),(0,9)\}$
62	$\{(3,2),(0,2),(1,5),(0,8),(1,9)\}$
63	$\{(0,1),(2,3),(3,3),(3,9),(0,0)\}$
64	$\{(1,1),(3,3),(0,3),(0,9),(1,0)\}$
65	$\{(2,1),(0,3),(1,3),(1,9),(2,0)\}$
66	$\{(3,1),(1,3),(2,3),(2,9),(3,0)\}$
67	$\{(0,2),(2,4),(3,4),(0,6),(3,0)\}$
68	$\{(1,2),(3,4),(0,4),(1,6),(0,0)\}$
69	$\{(2,2),(0,4),(1,4),(2,6),(1,0)\}$
70	$\{(3,2),(1,4),(2,4),(3,6),(2,0)\}$
71	$\{(0,3),(2,5),(3,5),(3,6),(0,7)\}$
72	$\{(1,3),(3,5),(0,5),(0,6),(1,7)\}$
73	$\{(2,3),(0,5),(1,5),(1,6),(2,7)\}$
74	$\{(3,3),(1,5),(2,5),(2,6),(3,7)\}$
75	$\{(0,4),(1,5),(0,6),(3,6),(2,8)\}$
76	$\{(1,4),(2,5),(1,6),(0,6),(3,8)\}$
77	$\{(2,4),(3,5),(2,6),(1,6),(0,8)\}$
78	$\{(3,4),(0,5),(3,6),(2,6),(1,8)\}$
79	$\{(0,1),(3,5),(2,7),(3,7),(1,9)\}$
80	$\{(1,1),(0,5),(3,7),(0,7),(2,9)\}$
81	$\{(2,1),(1,5),(0,7),(1,7),(3,9)\}$

Block No.	Block
82	$\{(3,1),(2,5),(1,7),(2,7),(0,9)\}$
83	$\{(0,1),(1,2),(0,8),(3,8),(2,0)\}$
84	$\{(1,1),(2,2),(1,8),(0,8),(3,0)\}$
85	$\{(2,1),(3,2),(2,8),(1,8),(0,0)\}$
86	$\{(3,1),(0,2),(3,8),(2,8),(1,0)\}$
87	$\{(0,2),(1,3),(2,6),(0,9),(3,9)\}$
88	$\{(1,2),(2,3),(3,6),(1,9),(0,9)\}$
89	$\{(2,2),(3,3),(0,6),(2,9),(1,9)\}$
90	$\{(3,2),(0,3),(1,6),(3,9),(2,9)\}$
91	$\{(0,3),(1,4),(2,7),(0,0),(3,0)\}$
92	$\{(1,3),(2,4),(3,7),(1,0),(0,0)\}$
93	$\{(2,3),(3,4),(0,7),(2,0),(1,0)\}$
94	$\{(3,3),(0,4),(1,7),(3,0),(2,0)\}$
95	$\{(0,6),(0,7),(0,8),(0,9),(0,0)\}$
96	$\{(1,6),(1,7),(1,8),(1,9),(1,0)\}$
97	$\{(2,6),(2,7),(2,8),(2,9),(2,0)\}$
98	$\{(3,6),(3,7),(3,8),(3,9),(3,0)\}$

We note that block no. $0,1,4,76,81,84,88,92,72$ form a parallel class. Further, block no. $0,2,3,78,79,86,90,94,95$ form another parallel class. The two parallel classes have exactly one block in common. Add an infinite point to obtain a $\{5,6\}$-GDD of type $5^8 6^1$.

Appendix B . Computer Constructions of weakly union-free $TTS(n)$

Block No.	Block
$1^{35}13^{1}$	{1,3} {1,7} {2,26} {3,30} {4,21} {4,19} {10,22}
$1^{35}16^{1}$	{1,8} {1,10} {2,16} {3,15} {4,17} {5,11}
$1^{39}13^{1}$	{1,34} {1,32} {2,30} {2,23} {3,13} {3,17} {4,27} {4,19}
$1^{41}13^{1}$	{1,2} {2,7} {3,21} {3,25} {4,33} {4,31} {5,30} {6,19} {9,26}
$1^{39}16^{1}$	{1,4} {1,6} {2,13} {2,17} {7,19} {8,18} {9,23}
$1^{41}16^{1}$	{1,3} {1,11} {2,38} {4,22} {4,16} {6,33} {7,20} {9,26}
$1^{45}13^{1}$	{1,2} {2,5} {3,10} {4,9} {4,23} {6,20} {6,30} {7,34} {8,32} {12,28}
$1^{41}19^{1}$	{1,4} {1,6} {2,19} {7,15} {9,29} {10,23} {11,25}
$1^{47}16^{1}$	{1,2} {2,5} {3,37} {4,23} {4,29} {5,12} {6,22} {6,20} {8,17} {11,32}
$1^{43}21^{1}$	{1,6} {2,32} {3,18} {4,26} {7,34} {8,20} {10,24}
$1^{47}19^{1}$	{1,3} {1,5} {5,7} {3,11} {6,26} {9,25} {10,33} {12,29} {13,28}
$1^{53}16^{1}$	{1,2} {2,5} {3,7} {4,13} {5,12} {6,22} {6,28} {8,35} {8,33} {10,24} {11,30} {15,32}
$1^{49}21^{1}$	{1,3} {1,9} {3,5} {4,15} {6,30} {7,23} {10,28} {12,29} {13,27}
$1^{53}19^{1}$	{1,3} {1,5} {5,7} {3,11} {9,13} {17,25} {7,21} {6,29} {10,37} {12,34} {15,33}
$1^{59}16^{1}$	{1,2} {2,5} {3,7} {4,9} {6,34} {6,18} {7,23} {8,44} {8,34} {9,22} {10,42} {10,29} {11,31} {14,35}
$1^{55}21^{1}$	{1,2} {2,5} {3,47} {4,18} {4,26} {5,12} {6,26} {9,34} {10,38} {13,36} {15,31}
$1^{53}25^{1}$	{1,4} {1,6} {2,11} {7,36} {8,27} {10,31} {12,40} {14,30} {15,35}
$1^{65}16^{1}$	{1,2} {2,5} {3,7} {4,9} {6,12} {7,31} {8,42} {8,53} {9,26} {10,28} {10,38} {11,25} {11,33} {13,27} {15,44} {16,35}

Block No.	Block
$1^{55}27^{1}$	{1,40} {2,10} {3,44} {4,35} {5,32} {6,19} {7,37} {9,38} {12,33}
$1^{59}25^{1}$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,30} {7,39} {11,26} {12,28} {13,34} {17,36}
$1^{61}24^{1}$	{1,3} {1,9} {3,5} {6,18} {6,48} {18,30} {4,41} {7,29} {10,25} {11,34} {14,40} {16,44}
$1^{59}28^{1}$	{1,4} {1,6} {2,16} {7,41} {8,28} {9,47} {10,36} {11,30} {13,35} {15,32}
$1^{61}27^{1}$	{1,3} {1,9} {3,5} {4,20} {6,46} {7,33} {10,39} {11,38} {12,25} {14,31} {18,37}
$1^{65}25^{1}$	{1,2} {2,5} {3,55} {4,18} {4,20} {5,12} {6,37} {6,31} {8,44} {9,33} {11,30} {15,38} {17,39}
$1^{61}30^{1}$	{1,6} {2,31} {3,17} {4,26} {7,19} {8,48} {9,24} {10,43} {11,34} {16,36}
$1^{65}28^{1}$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,46} {7,38} {11,44} {12,35} {13,29} {15,39} {17,37}
$1^{67}27^{1}$	{1,3} {1,9} {3,5} {6,18} {6,48} {18,30} {4,20} {7,43} {10,45} {11,34} {13,28} {14,40} {17,38}
$1^{65}31^{1}$	{1,4} {1,6} {2,21} {7,49} {8,36} {9,34} {10,22} {11,50} {13,45} {14,41} {17,47}
$1^{71}28^{1}$	{1,3} {1,5} {5,7} {3,11} {9,13} {17,25} {7,21} {6,36} {10,43} {12,51} {15,42} {16,47} {18,37} {22,45}
$1^{67}33^{1}$	{1,12} {2,49} {3,16} {4,14} {5,39} {6,37} {7,24} {8,46} {9,44} {15,42} {19,41}
$1^{71}31^{1}$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,40} {7,30} {11,47} {12,28} {13,32} {15,44} {17,50} {20,45}

Block No.	Block
$1^{71}34^{1}$	{1,4} {1,6} {2,54} {7,51} {8,33} {9,56} {10,39} {11,45} {12,35} {13,31} {14,55} {21,43}
$1^{77}31^{1}$	{1,3} {1,5} {5,7} {3,11} {9,13} {17,25} {7,21} {6,47} {10,34} {12,40} {15,44} {16,55} {18,45} {19,42} {20,46}
$1^{77}34^{1}$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,46} {7,57} {11,28} {12,48} {13,43} {15,38} {16,42} {19,44} {21,45}
$1^{79}33^{1}$	{1,3} {1,9} {3,5} {6,18} {6,48} {18,30} {4,24} {7,34} {10,68} {13,51} {14,33} {15,44} {16,39} {17,43} {22,47}
$1^{77}37^{1}$	{1,4} {1,6} {2,62} {7,18} {8,28} {9,50} {10,33} {12,42} {13,38} {14,45} {16,37} {19,53} {22,51}
$1^{79}36^{1}$	{1,3} {1,9} {3,5} {4,25} {6,49} {7,64} {10,37} {11,24} {12,41} {14,46} {16,56} {17,35} {19,53} {20,48}
$1^{83}34^{1}$	{1,7} {7,13} {2,10} {10,18} {1,5} {15,19} {3,14} {3,34} {9,51} {12,40} {16,45} {17,44} {20,50} {21,57} {22,46} {23,48}
$1^{79}39^{1}$	{1,42} {2,25} {3,32} {4,11} {5,14} {6,64} {8,60} {10,46} {12,51} {13,62} {16,34} {20,55} {22,48}
$1^{83}37^{1}$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,43} {7,19} {11,56} {13,44} {15,35} {16,49} {17,42} {21,53} {23,59} {26,55}
$1^{83}40^{1}$	{1,4} {1,6} {2,31} {7,17} {8,56} {9,32} {11,26} {12,53} {13,47} {14,39} {16,37} {18,38} {19,43} {22,55}
$1^{85}39^{1}$	{1,3} {1,9} {3,5} {4,45} {6,13} {10,34} {11,71} {12,42} {15,63} {16,47} {17,35} {19,39} {21,49} {23,52} {26,53}
$1^{89}37^{1}$	{1,7} {7,13} {2,10} {10,18} {1,5} {15,19} {3,14} {3,34} {9,45} {12,52} {16,59} {17,42} {20,41} {22,51} {23,62} {24,56} {26,61}

Block No.	Block
$1^{89}40^1$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,32} {7,56} {11,35} {12,60} {13,55} {15,66} {16,46} {17,44} {19,39} {21,52} {25,61}
$1^{91}39^1$	{1,3} {1,9} {3,5} {6,18} {6,48} {18,30} {4,56} {7,64} {10,72} {11,37} {13,46} {14,31} {15,40} {16,36} {21,59} {22,63} {23,47}
$1^{89}43^1$	{1,4} {1,6} {2,13} {7,62} {8,36} {9,75} {10,64} {12,58} {15,56} {16,42} {17,57} {18,70} {20,59} {21,65} {22,51}
$1^{91}42^1$	{1,3} {1,9} {3,5} {4,81} {6,46} {7,73} {11,60} {12,39} {13,32} {15,62} {16,37} {17,53} {20,43} {22,56} {24,50} {28,58}
$1^{95}40^1$	{1,7} {7,13} {2,10} {10,18} {1,5} {15,19} {3,14} {3,34} {9,53} {12,57} {16,36} {17,52} {21,54} {22,48} {23,55} {24,49} {27,66} {28,58}
$1^{93}43^1$	{1,4} {1,6} {2,29} {2,33} {7,78} {8,25} {9,70} {10,69} {11,39} {12,55} {13,57} {14,51} {16,46} {18,53} {19,45} {20,41}
$1^{95}43^1$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,58} {7,61} {11,57} {12,25} {15,31} {17,44} {19,47} {20,50} {21,60} {23,55} {24,53} {26,59}
$1^{95}46^1$	{1,4} {1,6} {2,24} {7,42} {8,63} {9,50} {10,28} {11,27} {12,61} {13,70} {14,80} {17,48} {19,58} {20,43} {21,51} {26,59}
$1^{99}43^1$	{1,2} {2,7} {3,31} {3,33} {4,55} {4,51} {6,14} {9,62} {10,36} {11,81} {12,25} {15,64} {16,40} {17,60} {19,42} {20,41} {22,54} {27,65}
$1^{101}43^1$	{1,7} {7,13} {2,10} {10,18} {1,5} {15,19} {3,14} {3,34} {9,38} {12,52} {16,55} {17,47} {20,73} {21,56} {22,58} {23,60} {24,51} {25,69} {26,59}
$1^{97}48^1$	{1,83} {2,24} {3,60} {4,43} {5,84} {6,51} {7,87} {8,74} {9,28} {11,47} {12,41} {16,42} {20,64} {21,48} {25,59} {30,65}
$1^{105}43^1$	{1,2} {2,5} {3,10} {4,45} {4,39} {5,12} {6,57} {6,65} {8,51} {9,77} {11,42} {13,84} {14,89} {15,32} {18,38} {19,72} {22,78} {23,81} {25,61} {26,76}

Block No.	Block
$1^{101}49^1$	{1,4} {1,6} {2,38} {7,88} {8,78} {9,48} {10,57} {11,25} {12,86} {16,46} {17,41} {18,58} {19,64} {21,49} {22,51} {26,59} {32,66}
$1^{103}51^1$	{1,73} {2,54} {3,10} {4,50} {5,89} {6,69} {8,35} {9,37} {11,85} {12,60} {13,83} {15,77} {16,39} {17,42} {21,65} {22,58} {24,71}
$1^{107}49^1$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,94} {7,56} {11,84} {12,42} {15,68} {16,79} {17,69} {20,47} {21,57} {24,64} {25,62} {26,59} {29,75} {31,72}
$1^{107}52^1$	{1,4} {1,6} {2,22} {7,66} {8,36} {9,19} {11,53} {12,50} {13,37} {14,72} {15,75} {16,55} {17,40} {18,80} {21,46} {26,77} {29,63} {31,64}
$1^{109}51^1$	{1,3} {1,9} {3,5} {4,65} {6,32} {7,74} {10,40} {11,23} {13,41} {14,78} {15,58} {16,71} {17,39} {18,52} {19,46} {20,53} {21,50} {24,73} {25,62}
$1^{113}49^1$	{1,7} {7,13} {2,10} {10,18} {1,5} {15,19} {3,14} {3,34} {9,78} {12,65} {16,77} {17,55} {20,47} {21,84} {22,62} {23,64} {24,54} {25,57} {26,71} {28,74} {33,70}
$1^{113}52^1$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,94} {7,45} {11,39} {12,59} {13,57} {15,63} {16,33} {20,90} {21,72} {24,76} {26,58} {27,67} {29,82} {30,64} {35,77}
$1^{115}51^1$	{1,3} {1,9} {3,5} {6,18} {6,48} {18,30} {4,45} {7,27} {10,32} {11,77} {13,75} {14,82} {15,84} {16,81} {17,61} {19,91} {21,58} {23,59} {25,76} {26,86} {28,80}
$1^{119}52^1$	{1,7} {7,13} {2,10} {10,18} {1,5} {15,19} {3,14} {3,34} {9,77} {12,65} {16,40} {17,62} {20,81} {21,80} {22,48} {23,70} {25,52} {28,69} {29,84} {30,63} {32,75} {36,73}

Block No.	Block
$1^{119}58^1$	{1,4} {1,6} {2,19} {7,76} {8,47} {9,25} {10,66} {11,107} {13,40} {14,34} {15,64} {18,59} {21,51} {22,54} {24,82} {26,71} {28,90} {31,67} {33,77} {35,73}
$1^{121}57^1$	{1,3} {1,9} {3,5} {4,73} {6,111} {7,40} {11,32} {12,43} {13,57} {14,76} {15,80} {17,63} {18,102} {20,49} {22,47} {23,94} {24,66} {26,61} {28,67} {30,68} {34,70}
$1^{127}57^1$	{1,3} {1,9} {3,5} {6,18} {6,48} {18,30} {4,84} {7,20} {10,103} {11,68} {14,53} {15,91} {16,106} {17,78} {19,75} {22,50} {23,87} {25,94} {26,72} {27,92} {29,96} {32,73} {38,82}
$1^{131}64^1$	{1,4} {1,6} {2,107} {7,64} {8,68} {9,45} {10,51} {11,32} {12,46} {13,79} {14,58} {15,103} {16,47} {17,39} {18,93} {19,78} {20,50} {23,106} {27,82} {29,69} {33,70} {35,77}
$1^{137}64^1$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,52} {7,73} {11,49} {12,122} {13,83} {16,111} {17,36} {20,44} {21,61} {23,82} {25,62} {28,69} {29,77} {30,81} {31,74} {32,90} {33,98} {34,84} {35,80}
$1^{143}64^1$	{1,7} {7,13} {2,10} {10,18} {1,5} {15,19} {3,14} {3,34} {9,52} {12,65} {16,96} {17,72} {20,68} {21,57} {22,51} {23,81} {24,61} {25,104} {26,59} {27,77} {28,97} {30,70} {32,67} {38,94} {41,101} {44,98}
$1^{143}70^1$	{1,4} {1,6} {2,61} {7,55} {8,117} {9,31} {10,93} {11,47} {12,44} {13,62} {14,29} {16,69} {17,106} {18,43} {19,58} {20,87} {21,92} {23,91} {24,66} {27,65} {28,63} {30,103} {33,79} {41,86}
$1^{149}70^1$	{1,3} {3,5} {1,9} {4,14} {4,22} {6,75} {7,48} {11,111} {12,42} {13,37} {15,130} {16,77} {17,93} {20,124} {21,81} {23,78} {26,79} {27,92} {28,64} {29,62} {31,114} {32,90} {39,82} {40,86} {44,95} {47,97}

Block No.	**Block**
$1^{167}76^{1}$	{1,7} {7,13} {2,10} {10,18} {1,5} {15,19} {3,14} {3,34} {9,98} {12,54} {16,45} {17,39} {20,91} {21,93} {23,85} {24,97} {25,132} {26,58} {27,88} {28,120} {30,130} {33,116} {36,102} {38,115} {40,81} {43,87} {46,110} {48,104} {49,99} {53,108}