Automating Radiotherapy: Parameterizations of Sensor Time Delay Compensators and the Separation Principle

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Motivated by recent research to automate radiotherapy, this thesis looks into feedback control problems where the feedback sensor imposes considerable time delay. The use of an asymptotic estimator is considered as a method to compensate for the time delay. Properties and parameterizations of asymptotic estimators are analyzed. It is shown that if such a delay compensation scheme is adopted, a separation principle holds, which allows for independent design of the feedback controller and the time delay compensator. The radiotherapy problem is used as a case study to show how asymptotic estimators may be designed, exploiting the separation principle. Lastly, the thesis considers multivariable versions of asymptotic estimators.

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Chapter 1

Introduction

1.1 Sensor Time Delay

Sensors detect signals or stimuli and generate measurable outputs. Sensors may be mechanical, electrical, or chemical and are used to detect various physical quantities such as motion, level, pressure, temperature, or flow [2]. In feedback control, sensors are mandatory to measure the desired regulated parameters and supply the information to the controller. Sensor characteristics and performance can play an important role in determining the achievable control system performance [3].

Sensor time delay or sensor lag is the delay in the change of the sensor output with respect to a corresponding change in the measured quantity [2]. Certain sensors used in control applications have considerable delays. Examples include an oxygen sensor in [4] and an ultrasonic distance sensor in [5]. When the sensor time delay is significant, it must be considered explicitly in the design process of the control system. This is precisely the case that arises in recent work where image feedback is used to automate radiotherapy.

1.2 Automating Radiotherapy

This work is motivated by a control engineering problem in which radiotherapy, for the treatment of cancer using external high-energy x-ray beams, is automated [6, 7, 8, 9]. In

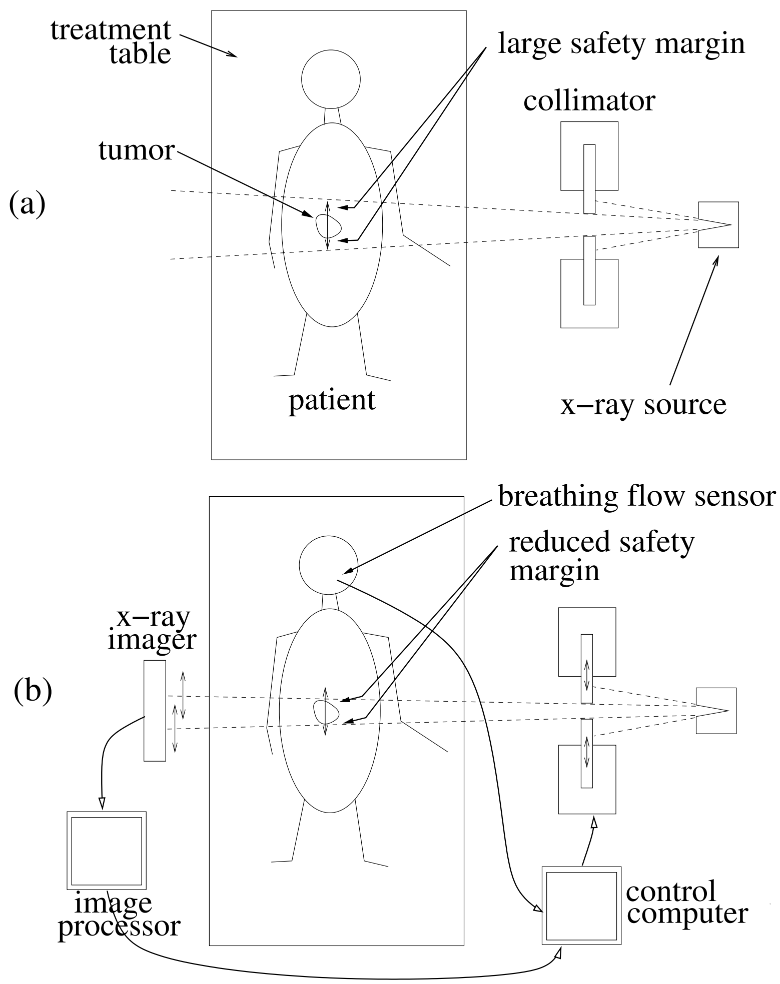

Figure 1.1: (a) Traditional radiotherapy; (b) Radiotherapy with tumor-tracking capability [6, 7].

external radiotherapy, radiation (usually in the form of x-rays) is generated by a machine. A device known as a collimator is used to shape the radiation beam into a desired profile, and to direct the beam to a desired location on the patient's body (Figure 1.1(a)). For many tumors (especially abdominal tumors, e.g., prostate cancer), patient breathing causes the tumor to move significantly. To compensate for such motion, typically the area of irradiation is enlarged by a "safety margin" to guarantee that the tumor is actually targeted by the x-ray beam. The unfortunate result is that the healthy tissues surrounding the tumor are also irradiated and this leads to (typically harmful) side-effects.

In [6, 7], it is proposed that feedforward and feedback control be used to adjust the collimator in real time so that the x-ray beam tracks the position of the tumor (Figure 1.1(b)). An x-ray imager is used to observe movement of the tumor as well as the movement of the leafs to provide feedback to the collimator leaf controller. With this scheme, the "safety margins" can be made smaller, resulting in fewer side effects. Other attempts to integrate imaging and radiation delivery are reported in [10, 11, 12].

Figure 1.2 shows a simplified block diagram of the scheme proposed in [6, 7]. Although the problem is inherently multivariable, for simplicity we consider only one degree of freedom for tumor movement and the control of only one collimator leaf. (A multi-leaf collimator is made up many opposing independently moving slats of metal called leaves.) In the diagram, $q[k]$ is the breathing flow rate of the patient. We assume the existence of a model relating $q[k]$ to the tumor position ($y_{tumor}[k]$); this model has a linear dynamic component, $M_{breathing}[z]$, and an unknown bias component, $w_{tumor}[k]$. The image processing dynamics are modeled as a pure time delay. The collimator leaf is also modelled by a linear dynamic component, $M_{leaf}[z]$, and an unknown bias component, $w_{leaf}[k]$.

As reported in [6, 7], the x-ray image processing time is significant and cannot be ignored. As shown in Figure 1.2, observers are implemented to generate tumor and leaf position estimates to feed the collimator leaf controller. This use of the observer-based sensor time delay compensation scheme is discussed in the next section.

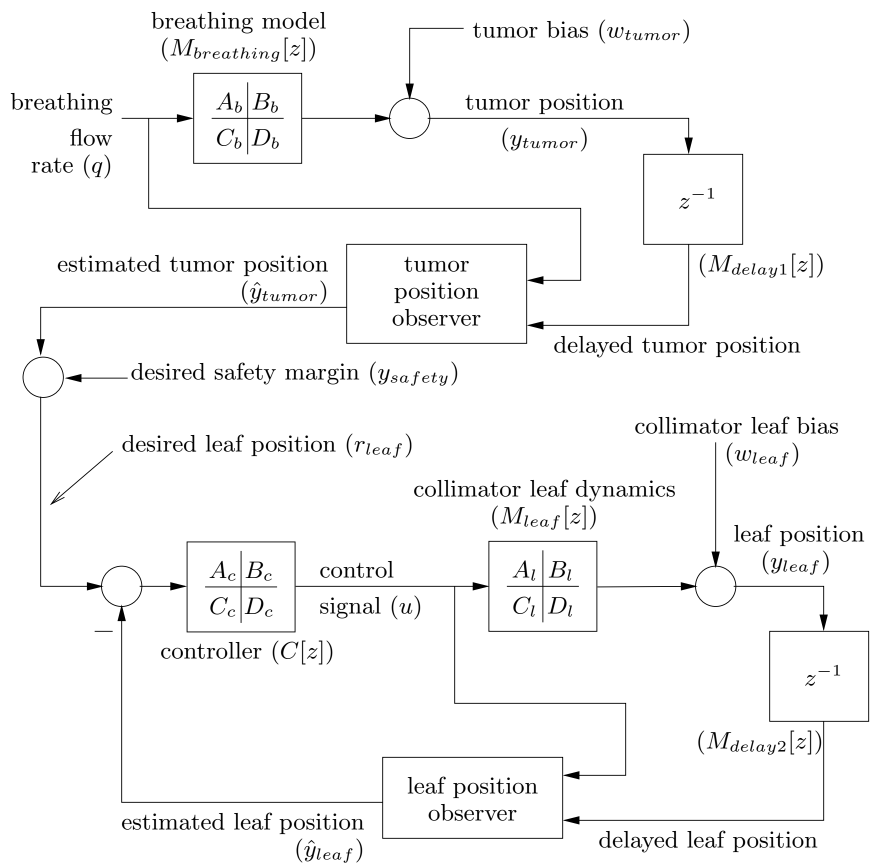

Figure 1.2: *The essential components of the radiotherapy tumor-tracking control scheme proposed in [6, 7].*

1.3 Sensor Time Delay Compensation

The authors of [7] study the two systems in Figure 1.3. In Figure 1.3(a), it is assumed that a controller, C, has been designed so that the feedback system with the plant P, which does not have a sensor time delay, exhibits good closed-loop performance. Then it is imagined that a sensor time delay, $H[z] = 1/z^n$, is introduced; consequently, an observer-based estimator is incorporated into the feedback system, as shown in Figure 1.3(b), to compensate for the time delay. The main result in [7] is that a *separation principle* holds, in the sense that

- the set of closed-loop poles of the system in Figure 1.3(b) equals the union of the set of closed-loop poles of the unity-feedback system in Figure 1.3(a), the set of poles of H, and the set of poles of the observer; and
- the closed-loop transfer functions from r to y are identical in the two block diagrams, implying that the observer-based estimator really does "cancel out" the sensor time delay.

This separation theory in [7] is similar in spirit to the more familiar separation principle associated with constant-gain state-feedback. Figure 1.4(a) shows the usual state feedback control where the poles of the closed-loop system are given by the eigenvalues of the matrix $(A + BK)$. Figure 1.4(b) shows the observer-based state-feedback design where the observer generates an estimate of the state. The closed-loop poles are given by the union of $\mathrm{eig}(A + BK)$ and $\mathrm{eig}(A + LC)$. The constant state-feedback control gain K can be designed separately from the observer gain L. It can also be shown that the closed-loop transfer function from r to y is the same in both the plain state-feedback design and the observer-based state-feedback design. Separation principles have also been reported in other control schemes [13, 14, 15].

Although a separation principle holds, the sensor time delay in Figure 1.3(b) can significantly degrade the closed-loop performance using other measures of performance (e.g., disturbance rejection or closed-loop sensitivity). Performance limitation results along these lines are reported in [16, 17].

In [1], the observer-based estimation scheme in Figure 1.3(b) is extended to the general (linear) asymptotic estimator scheme shown in Figure 2.1 (see page 10). In this scheme,

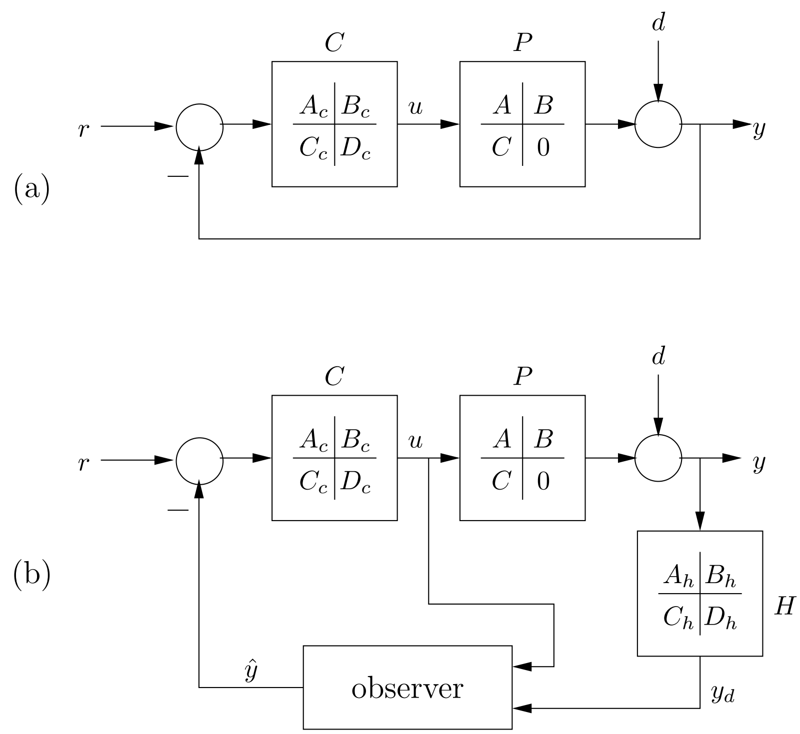

Figure 1.3: *The two systems studied in [7]: (a) unity-feedback control with plant P and controller C; (b) introduction of a sensor time delay, H, and an observer-based estimator to compensate for the delay.*

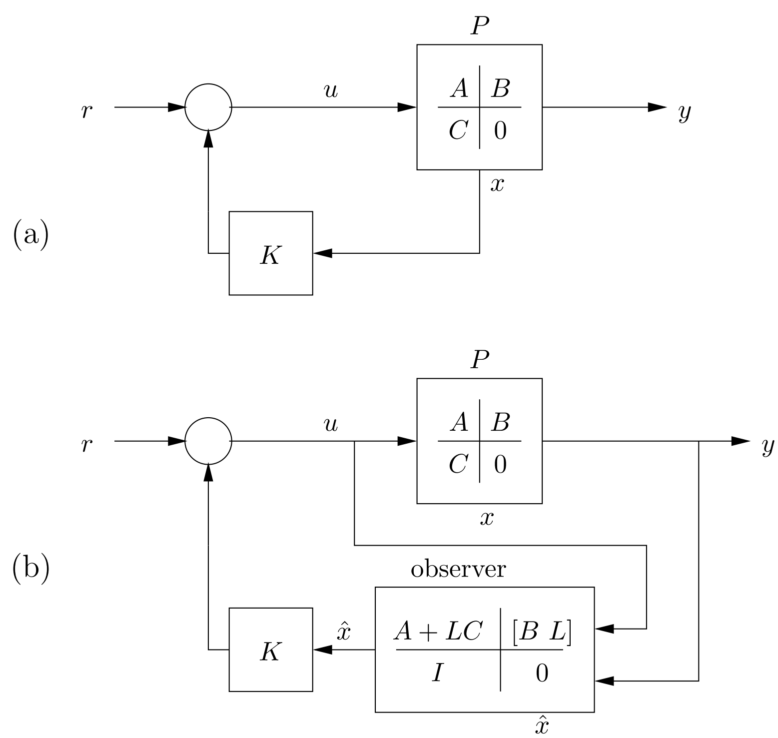

Figure 1.4: *(a) Plain constant-gain state-feedback control; (b) observer-based equivalent.*

which includes all observer-based estimators as a special case, G_1 and G_2 are arbitrary transfer functions subject only to the condition that $\hat{y}$ asymptotically approach y. Paper [1] parameterizes all such estimators and looks at performance limitations associated with estimation. The paper does not, however, go one step further and consider what happens when the feedback loop is closed with the general estimator, as shown in Figure 3.1(b) (see page 21).

The general asymptotic estimator can be considered a special case of the "linear observers" reported in [18], where the "observer" estimates a linear functional of the plant state vector. Other "observers" for discrete-time time-delays systems are reported in [19, 20].

1.4 Overview of Thesis

This thesis aims to look further into the properties of the asymptotic estimators and to consider the implications when a general asymptotic estimator is used as in Figure 3.1(b) (see page 21). Chapter 2 reviews, and extends, the properties of the asymptotic estimators formulated in [1]. Chapter 3 states and proves that a separation principle still holds when a general asymptotic estimator is used. Chapter 4 considers some design strategies exploiting the results in Chapter 2 and 3, and applies the strategies to the radiotherapy control problem. Chapter 5 extends some of the results in previous chapters to multivariable systems. Chapter 6 suggests future directions out of this work. Appendix A contains some mathematical background that is essential to the understanding of the thesis. Appendix B contains a number of lemmas that are used in the proofs in the main text.

In terms of notation, $\mathbb{R}[z]$ denotes all the rational transfer functions, $\mathcal{S}$ denotes the set of all stable proper transfer functions, $M(\mathbb{R}[z])$ denotes matrices with entries in $\mathbb{R}[z]$, and $M(\mathcal{S})$ denotes matrices with entries in $\mathcal{S}$. The transfer function from α to β is denoted by $T_{\alpha\beta}$, but when confusion may arise, superscripts are used to distinguish block diagrams, e.g., T_{ry}^{a} refers to the transfer function from r to y in Figure 3.1(a). Throughout the thesis, a discrete-time framework is assumed.

Chapter 2

Asymptotic Estimators

Asymptotic estimators with the structure shown in Figure 2.1 are studied in [1]. An asymptotic estimator is defined to be a pair of proper transfer functions (G_1, G_2) such that, for $d = w = 0$,

$$\lim_{k\to\infty} (\hat{y}[k] - y[k]) = 0 \quad \forall\, u, \forall \text{ initial conditions.} \tag{2.1}$$

Several ways of characterizing all asymptotic estimators are provided in [1]. This chapter aims to extend the results in [1]. We first review the parameterizations of asymptotic estimators outlined in [1], then we introduce two additional parameterizations. Lastly, we study a special class of asymptotic estimators that rejects step disturbances. Most material in this chapter, except Section 2.2, also appears in [21].

2.1 Parameterizations of Asymptotic Estimators in [1]

The following theorem characterizes all asymptotic estimators in Figure 2.1:

Theorem 2.1 [1] Consider the estimation scheme in Figure 2.1 with $d = w = 0$. Let $p_1, \ldots, p_m$ denote the unstable poles of P, if any exist. Perform a coprime factorization of the system PH over $\mathcal{S}$, i.e., find $N, M, X, Y \in \mathcal{S}$ such that

$$PH = \frac{N}{M}, \quad NX + MY = 1.$$

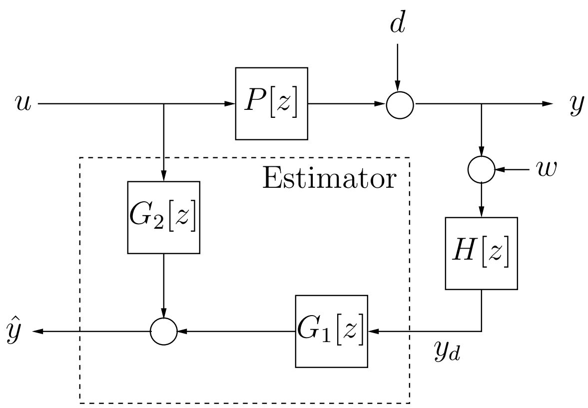

Figure 2.1: *Asymptotic estimator compensates for the delay H and predicts the signal y [1].*

Define the following four sets:

$$\mathcal{A} := \{(G_1, G_2) : (2.1) \text{ is satisfied}\} \tag{2.2}$$

$$\mathcal{B} := \{(G_1, G_2) : G_1 \in \mathcal{S}, G_2 \in \mathcal{S}, T_{u\hat{y}} = T_{uy}\} \tag{2.3}$$

$$\mathcal{C} := \{(G_1, G_2) : G_1 \in \mathcal{S} \text{ with } G_1[p_i] = \frac{1}{H[p_i]}, \quad i=1,\ldots,m, \text{and } G_2 = (1 - G_1 H)P\} \tag{2.4}$$

$$\mathcal{D} := \{(G_1, G_2) : G_1 = PMX + QM \text{ and } G_2 = PMY - QN \text{ for } Q \in \mathcal{S}\}. \tag{2.5}$$

Then, $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{D}$. □

In [1], the authors only consider the case when $H[z]$ is a pure time delay. Actually, the parameterizations still apply when $H[z]$ is extended to include any proper stable transfer function that does not have zeros coinciding with the unstable poles of $P[z]$. (However, the class of $H[z]$ for which Theorem 2.1 holds cannot be extended any further. To see this, suppose p is an unstable pole of $P[z]$ that is also a zero of $H[z]$. According to (2.3), G_1 and G_2 should be stable and $G_2 + G_1HP = P$. If G_1 is stable, then $G_1[p] < \infty$. But

$$G_2[p] = (1 - G_1[p]H[p])P[p] = P[p].$$

Therefore G_2 is not stable and thus an asymptotic estimator does not exist.)

Note also that parameterization $\mathcal{C}$ in (2.4) applies only if the unstable poles of P are distinct; this constraint was not mentioned in [1].

2.2 Extension to the Repeated Pole Case

This section aims to address the deficiency of parameterization $\mathcal{C}$ in (2.4) above by proposing a parameterization that correctly accounts for repeated unstable plant poles.

Let $p_1, \ldots, p_m$ denote the unstable poles of P, with multiplicities $r_1, \ldots, r_m$ respectively. The condition $G_1[p_i] = \frac{1}{H[p_i]}$ in (2.4) now becomes

$$\begin{aligned}
&\text{for } i = 1, \ldots, m, \\
&\quad \text{for } j = 0, \ldots, r_i - 1, \\
&\qquad \left.\frac{d^j}{dz^j} G_1[z]\right|_{z=p_i} = \left.\frac{d^j}{dz^j}\frac{1}{H[z]}\right|_{z=p_i}.
\end{aligned} \tag{2.6}$$

That is, Theorem 2.1 is extended as follows:

Theorem 2.2 Define

$$\begin{aligned}
\mathcal{A} &:= \{(G_1, G_2) : (2.1) \text{ is satisfied}\} \\
\mathcal{B} &:= \{(G_1, G_2) : G_1 \in \mathcal{S}, G_2 \in \mathcal{S}, T_{u\hat{y}} = T_{uy}\} \\
\mathcal{C}' &:= \{(G_1, G_2) : G_1 \in \mathcal{S} \text{ and satisfies (2.6) and } G_2 = (1 - G_1 H)P\} \\
\mathcal{D} &:= \{(G_1, G_2) : G_1 = PMX + QM \text{ and } G_2 = PMY - QN \text{ for } Q \in \mathcal{S}\}.
\end{aligned}$$

Then $\mathcal{A} = \mathcal{B} = \mathcal{C}' = \mathcal{D}$. □

Proof: The proof that $\mathcal{A} = \mathcal{B} = \mathcal{D}$ provided in [1] is correct in the repeated pole case. Hence we show that $\mathcal{B} = \mathcal{C}'$. First we prove $\mathcal{B} \subseteq \mathcal{C}'$. The condition $T_{u\hat{y}} = T_{uy}$ implies $G_2 = (1 - G_1 H)P$. The stability of G_2 implies that $(1 - G_1 H)$ has zeros at $p_1, \ldots, p_m$ with multiplicities $r_1, \ldots, r_m$, which means

$$\begin{aligned}
&\text{for } i = 1, \ldots, m, \\
&\quad \text{for } j = 0, \ldots, r_i - 1, \\
&\quad\quad \frac{d^j}{dz^j}(1 - G_1[z]H[z])\bigg|_{z=p_i} = 0 \qquad \text{(by (A.1))}
\end{aligned}$$

or

$$\begin{aligned}
&\text{for } i = 1, \ldots, m, \\
&\quad G_1[p_i]H[p_i] = 1 \text{ and} \\
&\quad \frac{d^j}{dz^j}G_1[z]H[z]\bigg|_{z=p_i} = 0, \text{ for } j = 1, \ldots, r_i - 1.
\end{aligned} \tag{2.7}$$

By Lemma B.1, (2.7) implies (2.6). Therefore $\mathcal{B} \subseteq \mathcal{C}'$.

Now we prove $\mathcal{C}' \subseteq \mathcal{B}$. The condition $G_2 = (1 - G_1 H)P$ implies $T_{u\hat{y}} = T_{uy}$. By Lemma B.1, (2.6) implies (2.7); by (A.1), (2.7) implies that $(1 - G_1 H)$ has zeros at $p_1, \ldots, p_m$ with multiplicities $r_1, \ldots, r_m$. Therefore $G_2 = (1 - G_1 H)P$ is stable. Therefore $\mathcal{C}' \subseteq \mathcal{B}$. $\square$

Example 2.1 Suppose

$$P[z] = \frac{1}{(z-2)^2}, \quad H[z] = \frac{1}{z}.$$

Then condition (2.6) requires that

$$G_1[2] = \frac{1}{H[z]}\bigg|_{z=2} = 2, \tag{2.8}$$

$$G_1'[2] = -\frac{H'[z]}{(H[z])^2}\bigg|_{z=2} = -\frac{-1/z^2}{1/z^2}\bigg|_{z=2} = 1. \tag{2.9}$$

We can take G_1 of the form

$$G_1[z] = \frac{az + b}{z - \frac{1}{2}},$$

which is a member of $\mathcal{S}$ with two parameters, a and b. Forcing conditions (2.8) and (2.9) leads to

$$G_1[z] = \frac{\frac{7}{2}z - 4}{z - \frac{1}{2}},$$

and also

$$G_2[z] = (1 - G_1[z]H[z])P[z] = \frac{1}{z(z - \frac{1}{2})},$$

which is also a member of $\mathcal{S}$, as required by (2.3).

$\square$

2.3 Alternative Q-Parameterization

Parameterization $\mathcal{D}$ in (2.5) is correct, but has the disadvantage that the expressions for G_1 and G_2 contain P, which is possibly unstable. Here we present an alternative parameterization involving only stable terms:

Theorem 2.3 [21] Let

$$P = \frac{N_P}{M_P}, \quad H = \frac{N_H}{M_H} \tag{2.10}$$

where $N_P, M_P \in \mathcal{S}$ are coprime and $N_H, M_H \in \mathcal{S}$ are coprime. By the assumption that $H[z]$ does not have zeros coinciding with unstable poles of $P[z]$, $N_H N_P$, $M_H M_P$ are coprime in $\mathcal{S}$ and therefore there exist $X, Y \in \mathcal{S}$ such that

$$N_H N_P X + M_H M_P Y = 1. \tag{2.11}$$

The following two sets are equal to set $\mathcal{B}$ (and therefore sets $\mathcal{A}$, $\mathcal{C}'$, and $\mathcal{D}$) in Theorem 2.2:

$$\mathcal{E} := \{(G_1, G_2) : G_1 \in \mathcal{S}, G_2 \in \mathcal{S}, G_2 = (1 - G_1 H)P\}, \tag{2.12}$$

$$\begin{aligned} \mathcal{F} := \{(G_1, G_2) : G_1 &= M_H N_P X + M_H M_P Q \text{ and} \\ G_2 &= M_H N_P Y - N_H N_P Q \text{ for } Q \in \mathcal{S}\}. \end{aligned} \tag{2.13}$$

$\square$

Proof: First, it is not difficult to see that $\mathcal{E} = \mathcal{B}$ since $G_2 = (1 - G_1 H)P$ is equivalent to $T_{u\hat{y}} = T_{uy}$. Second, we show $\mathcal{F} \subseteq \mathcal{E}$. Choose $(G_1, G_2) \in \mathcal{F}$. Clearly $G_1, G_2 \in \mathcal{S}$.

Moreover,

$$\begin{aligned}
(1 - G_1 H)P &= \left(1 - (M_H N_P X + M_H M_P Q)\frac{N_H}{M_H}\right)\frac{N_P}{M_P} && \text{(by (2.10), (2.13))} \\
&= (1 - N_H N_P X - N_H M_P Q)\frac{N_P}{M_P} \\
&= (M_H M_P Y - N_H M_P Q)\frac{N_P}{M_P} && \text{(by (2.11))} \\
&= M_H N_P Y - N_H N_P Q \\
&= G_2, && \text{(by (2.13))}
\end{aligned}$$

and therefore $(G_1, G_2) \in \mathcal{E}$. Finally, show $\mathcal{E} \subseteq \mathcal{F}$ by choosing $(G_1, G_2) \in \mathcal{E}$. Let

$$Q_0 := \frac{G_1 - M_H N_P X}{M_H M_P}. \tag{2.14}$$

Then

$$\begin{aligned}
Q_0 &= \frac{1}{M_H M_P}\left(\frac{1}{H} - \frac{G_2}{PH} - M_H N_P X\right) && \text{(by (2.12))} \\
&= \frac{1}{M_H M_P}\left(\frac{M_H}{N_H} - \frac{G_2 M_H M_P}{N_H N_P} - M_H N_P X\right) && \text{(by (2.10))} \\
&= \frac{1}{N_H N_P M_P}(N_P(1 - N_H N_P X) - G_2 M_P) \\
&= \frac{1}{N_H N_P M_P}(N_P M_H M_P Y - G_2 M_P) && \text{(by (2.11))} \\
&= \frac{1}{N_H N_P}(N_P M_H Y - G_2). && (2.15)
\end{aligned}$$

Therefore

$$\begin{aligned}
Q_0 &= (N_H N_P X + M_H M_P Y)Q_0 && \text{(by (2.11))} \\
&= (N_P M_H Y - G_2)X + (G_1 - M_H N_P X)Y && \text{(by (2.14), (2.15))} \\
&= -G_2 X + G_1 Y \in \mathcal{S}.
\end{aligned}$$

The last step is to recognize from (2.14) and (2.15) that

$$\begin{aligned}
G_1 &= M_H N_P X + M_H M_P Q_0 \\
G_2 &= M_H N_P Y - N_H N_P Q_0,
\end{aligned}$$

so $(G_1, G_2) \in \mathcal{F}$. $\square$

Example 2.2 Suppose

$$P[z] = \frac{1}{(z-2)^2}, \quad H[z] = \frac{1}{z}.$$

Coprime factorizations of $P[z]$ and $H[z]$ can be performed such that

$$N_P[z] = \frac{1}{(z-\frac{1}{2})^2},\ M_P[z] = \frac{(z-2)^2}{(z-\frac{1}{2})^2},\ N_H[z] = \frac{1}{z},\ M_H[z] = 1.$$

Then $X[z]$ and $Y[z]$ are found to be

$$X[z] = \frac{32.1(z-1.68)}{(z-\frac{1}{2})^2},\ Y[z] = \frac{(z+2.24)(z^2-0.238z+6.03)}{z(z-\frac{1}{2})^2}.$$

By (2.13), the set of asymptotic estimators for $P[z]$ and $H[z]$ is given by

$$\begin{aligned}\{(G_1, G_2) : G_1[z] &= \frac{32.1(z-1.68)}{(z-\frac{1}{2})^4} + \frac{(z-2)^2}{(z-\frac{1}{2})^2}Q[z],\\ G_2[z] &= \frac{(z+2.24)(z^2-0.238z+6.03)}{z(z-\frac{1}{2})^4} + \frac{1}{z(z-\frac{1}{2})^2}Q[z],\ \text{for } Q[z] \in \mathcal{S}\}.\end{aligned}$$

□

2.4 "Sensitivity" Functions of Asymptotic Estimators

It is convenient to introduce "sensitivity" and "complementary sensitivity" functions for the estimator in Figure 2.1:

$$\begin{aligned}S_E &:= 1 - G_1 H\\ T_E &:= G_1 H.\end{aligned}$$

Note that $S_E + T_E = 1$. The transfer functions S_E and T_E are related to the performance of the asymptotic estimator. Specifically, define the estimation error $e := \hat{y} - y$ in Figure 2.1. Then the disturbance rejection of the estimator is

$$T_{de} = G_1 H - 1 = -S_E \tag{2.16}$$

and the sensor noise rejection of the estimator is

$$T_{we} = G_1 H = T_E.$$

Using the parameterization $\mathcal{F}$ in Theorem 2.3, S_E and T_E can be expressed in terms of Q:

$$\begin{aligned} S_E &= M_H M_P Y - N_H M_P Q \\ T_E &= N_H N_P X + N_H M_P Q. \end{aligned}$$

The functions S_E and T_E will be used in Chapter 4, when design strategies are considered.

2.5 Special Classes of Asymptotic Estimators

A subset of asymptotic estimators that is of particular interest to us is those that achieve perfect steady-state rejection of step disturbances. It turns out that a parameterization for this subset exists. Indeed, for an asymptotic estimator to achieve asymptotic rejection of step disturbances, (2.16) and parameterization $\mathcal{E}$ (in (2.12)) imply that

$$\lim_{z\to 1} T_{de}[z] = \lim_{z\to 1}(G_1[z]H[z] - 1) = \lim_{z\to 1} \frac{-G_2[z]}{P[z]} = 0. \tag{2.17}$$

The last equality in (2.17) implies that $P[z]$ must have a pole at 1 or $G_2[z]$ must have a zero at 1. The parameterizations introduced below characterize the subset of asymptotic estimators for which $G_2[z]$ has a zero at 1:

Theorem 2.4 [21] Assume that $P[z]$ does not have a pole or zero at 1. Define $I[z] = z - 1$ and let

$$\frac{1}{I} = \frac{N_I}{M_I} \tag{2.18}$$

where $N_I, M_I \in \mathcal{S}$ are coprime. Assume that $N_H N_P N_I$, $M_H M_P M_I$ are coprime in $\mathcal{S}$. Then there exist $X, Y \in \mathcal{S}$ such that

$$N_H N_P N_I X + M_H M_P M_I Y = 1, \tag{2.19}$$

and the following two equal sets both characterize the subset of asymptotic estimators that achieve perfect steady-state step disturbance rejection:

$$\begin{aligned}\mathcal{E}_I := \{(G_1, G_2) : G_1 \in \mathcal{S}, G_2 \in \mathcal{S}, G_2 = (1 - G_1 H)P,& \\ M_I \text{ divides } G_2 \text{ in } \mathcal{S}\}& \qquad (2.20)\\ \mathcal{F}_I := \{(G_1, G_2) : G_1 = M_H N_P N_I X + M_H M_P M_I Q \text{ and}& \\ G_2 = M_H N_P M_I Y - N_H N_P M_I Q \text{ for } Q \in \mathcal{S}\}.& \qquad (2.21)\end{aligned}$$

□

Proof: First, we show $\mathcal{F}_I \subseteq \mathcal{E}_I$. Choose $(G_1, G_2) \in \mathcal{F}_I$. Clearly $G_1, G_2 \in \mathcal{S}$. Moreover,

$$\begin{aligned}(1 - G_1 H)P &= \left(1 - (M_H N_P N_I X + M_H M_P M_I Q)\frac{N_H}{M_H}\right)\frac{N_P}{M_P} && \text{(by (2.10), (2.21))}\\ &= (1 - N_H N_P N_I X - N_H M_P M_I Q)\frac{N_P}{M_P}\\ &= (M_H M_P M_I Y - N_H M_P M_I Q)\frac{N_P}{M_P} && \text{(by (2.19))}\\ &= M_H N_P M_I Y - N_H N_P M_I Q\\ &= G_2. && \text{(by (2.21))}\end{aligned}$$

Since M_I also divides G_2 in $\mathcal{S}$, $(G_1, G_2) \in \mathcal{E}_I$.

As for $\mathcal{E}_I \subseteq \mathcal{F}_I$, observe that $\frac{G_2}{M_I} \in \mathcal{S}$ since M_I divides G_2 in $\mathcal{S}$. Let

$$Q_0 := \frac{G_1 - M_H N_P N_I X}{M_H M_P M_I}. \qquad (2.22)$$

Then

$$\begin{aligned}Q_0 &= \frac{1}{M_H M_P M_I}\left(\frac{1}{H} - \frac{G_2}{PH} - M_H N_P N_I X\right) && \text{(by (2.20))}\\ &= \frac{1}{M_H M_P M_I}\left(\frac{M_H}{N_H} - \frac{G_2 M_H M_P}{N_H N_P} - M_H N_P N_I X\right) && \text{(by (2.10))}\\ &= \frac{1}{N_H N_P M_P M_I}(N_P(1 - N_H N_P N_I X) - G_2 M_P)\\ &= \frac{1}{N_H N_P M_P M_I}(N_P M_H M_P M_I Y - G_2 M_P) && \text{(by (2.19))}\\ &= \frac{1}{N_H N_P N_I}\left(M_H N_P N_I Y - \frac{G_2}{M_I} N_I\right). && (2.23)\end{aligned}$$

Therefore

$$\begin{aligned}
Q_0 &= (N_H N_P N_I X + M_H M_P M_I Y) Q_0 && \text{(by (2.19))} \\
&= (M_H N_P N_I Y - \frac{G_2}{M_I} N_I) X + (G_1 - M_H N_P N_I X) Y && \text{(by (2.22), (2.23))} \\
&= -(\frac{G_2}{M_I}) N_I X + G_1 Y \in \mathcal{S}.
\end{aligned}$$

The next step is to recognize from (2.22) and (2.23) that

$$\begin{aligned}
G_1 &= M_H N_P N_I X + M_H M_P M_I Q_0 \\
G_2 &= M_H N_P M_I Y - N_H N_P M_I Q_0,
\end{aligned}$$

so $(G_1, G_2) \in \mathcal{F}_I$. Finally, $\mathcal{E}_I$ is the subset of $\mathcal{E}$ that achieve perfect steady-state step disturbance rejection since $\mathcal{E}_I$ captures all the members of $\mathcal{E}$ that have $(z-1)$ in the numerator of G_2. □

Note that Theorem 2.4 can be easily generalized to other types of disturbances. For example, the subset of asymptotic estimators that obtain perfect asymptotic rejection of *ramp* disturbances is characterized by $\mathcal{E}_I$ and $\mathcal{F}_I$ using $I[z] = (z-1)^2$ instead of $I[z] = z-1$.

Using parameterization $\mathcal{F}_I$, S_E and T_E can be expressed in terms of Q:

$$S_E = 1 - G_1 H = M_H M_P M_I Y - N_H M_P M_I Q \tag{2.24}$$

$$T_E = G_1 H = N_H N_P N_I X + N_H M_P M_I Q. \tag{2.25}$$

Example 2.3 Suppose

$$P[z] = \frac{1}{(z-2)^2}, \quad H[z] = \frac{1}{z}, \quad I[z] = z - 1.$$

Coprime factorizations of P, H, and $\frac{1}{I}$ can be performed such that

$$N_P[z] = \frac{1}{(z-\frac{1}{2})^2},\ M_P[z] = \frac{(z-2)^2}{(z-\frac{1}{2})^2},\ N_H[z] = \frac{1}{z},\ M_H[z] = 1,\ N_I[z] = \frac{1}{z-\frac{1}{2}},\ M_I[z] = \frac{z-1}{z-\frac{1}{2}}.$$

Then $X[z]$ and $Y[z]$ are found to be

$$X[z] = \frac{79.8(z-1)(z-1.71)}{(z-\frac{1}{2})^3},\ Y[z] = \frac{(z^2+3.64z+5.39)(z^2-1.64z+6.34)}{z(z-\frac{1}{2})^3}.$$

By (2.21), the set of asymptotic estimators for $P[z]$ and $H[z]$ that reject step disturbance is given by

$$\{(G_1, G_2) : G_1[z] = \frac{79.8(z-1)(z-1.71)}{(z-\frac{1}{2})^6} + \frac{(z-1)(z-2)^2}{(z-\frac{1}{2})^3} Q[z],$$
$$G_2[z] = \frac{(z-1)(z^2+3.64z+5.39)(z^2-1.64z+6.34)}{z(z-\frac{1}{2})^6} + \frac{z-1}{z(z-\frac{1}{2})^3} Q[z], \text{ for } Q[z] \in \mathcal{S}\}.$$

$\square$

Chapter 3

Separation Principle

The purpose of this section is to show that a separation principle exists when an asymptotic estimator is put into a feedback control loop involving a sensor time delay. With this separation principle, we can design the feedback controller and the asymptotic estimator independently, as explained in Chapter 4. The 1-DOF result of the separation principle also appears in [21].

To prove the separation principle, we need to be able to characterize the closed-loop poles of an interconnected system. A theorem proved in [22] relates the poles of an interconnected system to the system determinant Δ used in Mason's Gain Rule. Recall that, for an arbitrary block diagram with n blocks, the system determinant Δ is defined to be

$$\Delta = 1 - \sum_i F_{1i} + \sum_j F_{2j} - \sum_k F_{3k} + \cdots ,$$

where F_{1i} are the loop gains, F_{2j} are the products of two nontouching loop gains, F_{3k} are products of three nontouching loop gains, and so on. Denote the transfer functions of the individual n blocks in the interconnected system by $G_i[z], i = 1, 2, \ldots, n$, and let $p_i(z)$ denote the characteristic polynomial of $G_i[z]$ (i.e., the denominator polynomial). Then the closed-loop poles can be computed as follows:

Theorem 3.1 [22] The poles of a linear time-invariant interconnected system with scalar

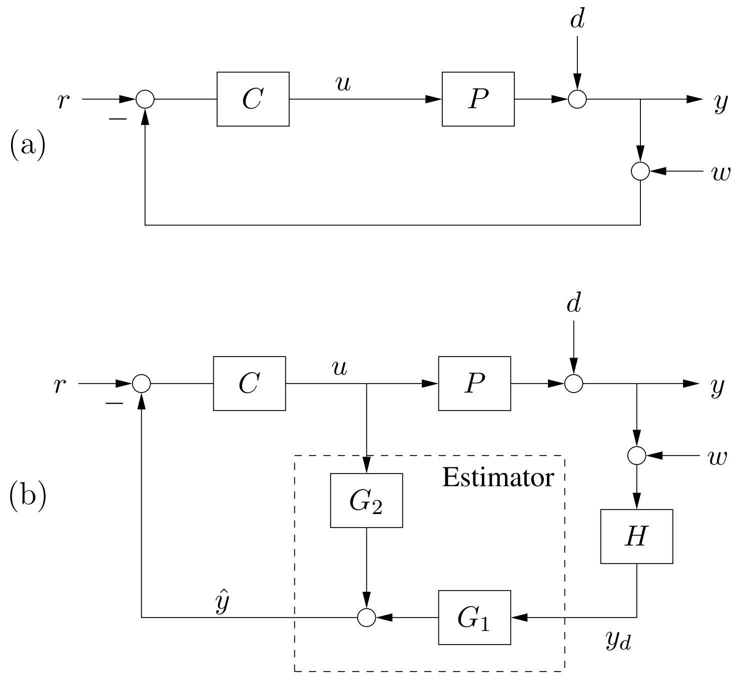

Figure 3.1: *(a) Basic 1-DOF control loop with plant P and controller C; (b) feedback control with the sensor time delay and asymptotic estimator in place.*

signals are the roots of the polynomial

$$p_c(z) := \Delta[z] \cdot \prod_{i=1}^{n} p_i(z).$$

□

3.1 The 1-DOF (Degree-of-Freedom) Result

For the system in Figure 3.1(b), what properties constitute a separation principle? We will say that a separation principle holds if the following two conditions hold:

- the set of poles of the system in Figure 3.1(b) equals the union of the poles of H, the poles of G_1, the poles of G_2, and the set of poles of the system in Figure 3.1(a); and

- the transfer function from r to y is the same for the system in Figure 3.1(b) as it is for the system in Figure 3.1(a).

The following theorem shows that the separation principle holds for any asymptotic estimator (G_1, G_2):

Theorem 3.2 [21] Consider the system in Figure 3.1(b). Assume that PC is not identically zero and that C stabilizes P in the sense that the feedback system in Figure 3.1(a) is stable. Then:

(a) The separation principle holds if and only if G_1 and G_2 satisfy

$$G_2 = (1 - G_1 H)P. \tag{3.1}$$

(b) A necessary and sufficient condition for the system in Figure 3.1(b) to be stable *and* for the separation principle to hold is that (G_1, G_2) be an asymptotic estimator.

$\square$

Proof: The transfer function from r to y in Figure 3.1(a) is

$$T_{ry}^a = \frac{PC}{1 + PC} \tag{3.2}$$

and the transfer function from r to y in Figure 3.1(b) is

$$T_{ry}^b = \frac{PC}{1 + CG_2 + CG_1 HP}. \tag{3.3}$$

Let $p_C(z), p_P(z), p_1(z), p_2(z), p_H(z)$ denote the characteristic polynomials of $C[z]$, $P[z]$, $G_1[z]$, $G_2[z]$, $H[z]$ respectively. By Theorem 3.1, the characteristic polynomial of the closed-loop system in Figure 3.1(a) is

$$p_c^a = (1 + PC)p_C p_P, \tag{3.4}$$

and that of the closed-loop system in Figure 3.1(b) is

$$p_c^b = (1 + CG_2 + CG_1 HP)p_C p_P p_1 p_2 p_H. \tag{3.5}$$

To prove sufficiency in (a), assume $G_2 = (1 - G_1 H)P$. Then (3.3) simplifies to (3.2) and (3.5) simplifies to $p_c^b = p_c^a p_1 p_2 p_H$, showing that the separation principle holds. To prove necessity in (a), assume that the separation principle holds, i.e., $T_{ry}^a = T_{ry}^b$ and $p_c^b = p_c^a p_1 p_2 p_H$. Equating (3.2) and (3.3) yields (3.1).

To prove (b), recall parameterization $\mathcal{E}$ in (2.12) which states that (G_1, G_2) is an asymptotic estimator if and only if G_1 and G_2 are both stable and $G_2 = (1 - G_1 H)P$, or, equivalently (by part (a)), G_1 and G_2 are both stable and the separation principle holds. Since H and the system in Figure 3.1(a) are both stable, equivalent conditions are that the system in Figure 3.1(b) is stable and the separation principle holds. $\square$

Example 3.1 Suppose

$$P[z] = \frac{1}{(z-2)^2}, \quad H[z] = \frac{1}{z}.$$

It is easy to show that

$$C[z] = \frac{13.5(z - 1.625)}{z^2 + 2z + 5.5}$$

stabilizes $P[z]$ in a unity-feedback loop (Figure 3.1(a)), and results in the following characteristic polynomial

$$\begin{aligned} p_c^a(z) &= \left(1 + \frac{1}{(z-2)^2} \frac{13.5(z - 1.625)}{z^2 + 2z + 5.5}\right)(z-2)^2(z^2 + 2z + 5.5) \qquad \text{(by (3.4))} \\ &= (z - \frac{1}{2})^4, \end{aligned}$$

and tracking transfer function

$$T_{ry}^a[z] = \frac{13.5(z - 1.625)}{(z - 0.5)^4}. \qquad \text{(by (3.2))}$$

In Example 2.1, we computed an asymptotic estimator for $P[z]$ and $H[z]$:

$$G_1[z] = \frac{\frac{7}{2}z - 4}{z - \frac{1}{2}}, \quad G_2[z] = \frac{1}{z(z - \frac{1}{2})}.$$

By (3.5), we obtain

$$\begin{aligned}
p_c^b(z) &= \left(1 + \frac{13.5(z-1.625)}{z^2+2z+5.5}\frac{1}{z(z-\frac{1}{2})} + \frac{13.5(z-1.625)}{z^2+2z+5.5}\frac{\frac{7}{2}z-4}{z-\frac{1}{2}}\frac{1}{z}\frac{1}{(z-2)^2}\right)\\
&\quad \cdot (z^2+2z+5.5)(z-2)^2(z-\frac{1}{2})z(z-\frac{1}{2})z\\
&= (z-\frac{1}{2})^4(z-\frac{1}{2})z(z-\frac{1}{2})z\\
&= p_c^a(z)p_1(z)p_2(z)p_H(z),
\end{aligned}$$

as suggested by Theorem 3.2. Moreover, by (3.3) we obtain

$$\begin{aligned}
T_{ry}^b[z] &= \frac{\frac{1}{(z-2)^2}\frac{13.5(z-1.625)}{z^2+2z+5.5}}{1 + \frac{13.5(z-1.625)}{z^2+2z+5.5}\frac{1}{z(z-\frac{1}{2})} + \frac{13.5(z-1.625)}{z^2+2z+5.5}\frac{\frac{7}{2}z-4}{z-\frac{1}{2}}\frac{1}{z}\frac{1}{(z-2)^2}}\\
&= \frac{13.5(z-1.625)}{(z-0.5)^4} = T_{ry}^a[z].
\end{aligned}$$

Consequently, the separation principle holds. Suppose (G_1, G_2) is not an asymptotic estimator but still satisfies (3.1), for example,

$$G_1[z] = 1,\ G_2[z] = (1 - G_1[z]H[z])P[z] = \frac{z-1}{z(z-2)^2}.$$

In this case, $G_2[z]$ is unstable and therefore (G_1, G_2) is not an asymptotic estimator. Then we get from (3.5)

$$\begin{aligned}
p_c^b(z) &= \left(1 + \frac{13.5(z-1.625)}{z^2+2z+5.5}\frac{z-1}{z(z-2)^2} + \frac{13.5(z-1.625)}{z^2+2z+5.5}\frac{1}{z}\frac{1}{(z-2)^2}\right)\\
&\quad \cdot (z^2+2z+5.5)(z-2)^2z(z-2)^2z\\
&= (z-\frac{1}{2})^4z(z-2)^2z\\
&= p_c^a(z)p_1(z)p_2(z)p_H(z).
\end{aligned}$$

The separation principle still holds, as per Theorem 3.2(a). However, Theorem 3.2(b) tells us that the closed-loop will not be stable since (G_1, G_2) is not an asymptotic estimator.

□

In the next subsection we show that this separation principle result also applies when a more general 2-DOF control scheme is used.

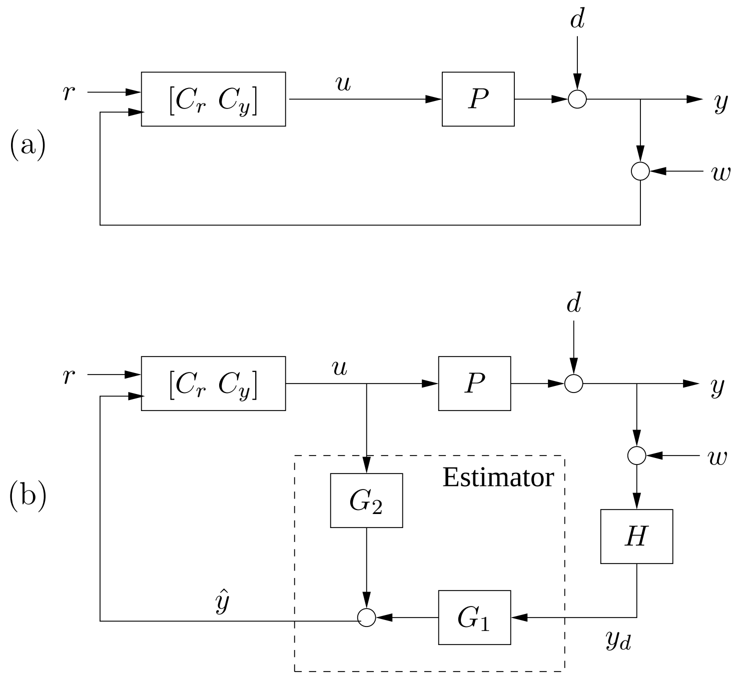

Figure 3.2: *(a) 2-DOF control loop with plant P and controller (C_r, C_y); (b) feedback control with the sensor time delay and asymptotic estimator in place.*

3.2 The 2-DOF Result

The 2-DOF controller can be considered a two-input-one-output block such that the control signal is generated as follows:

$$U[z] = C_r[z]R[z] + C_y[z](Y[z] + W[z])$$

where $U[z], R[z], Y[z], W[z]$ are Z-transforms of $u[k], r[k], y[k], w[k]$ respectively. Note that the 1-DOF controller is a special case of the 2-DOF controller, with $C_y = -C_r$. The *separation principle* for the 2-DOF configuration is the same as that of the 1-DOF configuration, except in the two conditions listed in the beginning of Section 3.1, references to Figure 3.1 are replaced by references to Figure 3.2. The 2-DOF version of Theorem 3.2 is as follows:

Theorem 3.3 Consider the system in Figure 3.2(b). Assume that PC_rC_y is not identically zero and that (C_r, C_y) stabilizes P in the sense that the feedback system in Figure 3.2(a) is stable. Then:

(a) The separation principle holds if and only if G_1 and G_2 satisfy

$$G_2 = (1 - G_1 H)P.$$

(b) A necessary and sufficient condition for the system in Figure 3.2(b) to be stable *and* for the separation principle to hold is that (G_1, G_2) be an asymptotic estimator.

□

Proof: The transfer function from r to y in Figure 3.2(a) is

$$T_{ry}^a = \frac{PC_r}{1 - PC_y} \tag{3.6}$$

and the transfer function from r to y in Figure 3.2(b) is

$$T_{ry}^b = \frac{PC_r}{1 - C_y G_2 - C_y G_1 H P}. \tag{3.7}$$

Let $p_C(z)$, $p_P(z)$, $p_1(z)$, $p_2(z)$, $p_H(z)$ denote the characteristic polynomials of $(C_r[z], C_y[z])$, $P[z]$, $G_1[z]$, $G_2[z]$, $H[z]$ respectively. By Theorem 3.1, the characteristic polynomial of the closed-loop system in Figure 3.2(a) is

$$p_c^a = (1 - PC_y) p_C p_P, \tag{3.8}$$

and that of the closed-loop system in Figure 3.2(b) is

$$p_c^b = (1 - C_y G_2 - C_y G_1 H P) p_C p_1 p_2 p_H. \tag{3.9}$$

To prove sufficiency in (a), assume $G_2 = (1 - G_1 H)P$. Then (3.7) simplifies to (3.6) and (3.9) simplifies to $p_c^b = p_c^a p_1 p_2 p_H$, showing that the separation principle holds. To prove necessity in (a), assume that the separation principle holds, i.e., $T_{ry}^a = T_{ry}^b$ and $p_c^b = p_c^a p_1 p_2 p_H$. Equating (3.6) and (3.7) yields (3.1).

To prove (b), recall parameterization $\mathcal{E}$ in (2.12) which states that (G_1, G_2) is an asymptotic estimator if and only if G_1 and G_2 are both stable and $G_2 = (1 - G_1 H)P$, or, equivalently (by part (a)), G_1 and G_2 are both stable and the separation principle holds. Since H and the system in Figure 3.2(a) are both stable, equivalent conditions are that the system in Figure 3.2(b) is stable and the separation principle holds. $\square$

Chapter 4

Design Strategies

The previous chapter shows that a separation principle exists when an asymptotic estimator is put into a feedback control loop involving a sensor time delay. This separation principle implies that we can design the feedback controller and the asymptotic estimator independently. This chapter aims to outline strategies to design the asymptotic estimator. The first goals are stability and tracking. The next step involves addressing additional closed-loop properties such as disturbance rejection. Then we will apply the design strategies to the radiotherapy problem. The material in this chapter also appears in [21].

4.1 Design for Stability and Tracking

Assume that C has been designed so that tracking performance in Figure 3.1(a) is good. Now consider Figure 3.1(b), where a sensor time delay has been included. By Theorem 3.2(b), using *any* asymptotic estimator (G_1, G_2) in Figure 3.1(b) will guarantee closed-loop stability and recover the good tracking performance (as measured by the transfer function from r to y) of Figure 3.1(a). It is emphasized that the designer is free to choose any $Q \in \mathcal{S}$ in parameterization $\mathcal{F}$ (in (2.13))to obtain an asymptotic estimator. A valid choice, for example, is $Q = 0$, i.e.,

$$G_1 = M_H N_P X, \quad G_2 = M_H N_P Y.$$

If it is desired that the asymptotic estimator asymptotically reject step disturbances, then parameterization $\mathcal{F}_I$ (in (2.21)) can be used instead of parameterization $\mathcal{F}$.

Even though any asymptotic estimator recovers tracking performance, according to Theorem 3.2 the poles of G_1 and G_2 are among the poles of the closed-loop system in Figure 3.1(b). Hence, there exist some closed-loop transfer functions in Figure 3.1(b) that the poles of G_1 and G_2 affect. Hence, there is motivation to place the poles of G_1 and G_2 in certain "nice" locations to get good settling time, damping, etc. Pole placement can be done by choosing coprime factorizations of P, H, I so that the poles of $N_P, M_P, N_H, M_H, N_I, M_I$ all lie in the "nice" region. Similarly, X, Y can be chosen as such. The example presented in Section 4.3.1 takes this approach.

4.2 Design for Additional Closed-Loop Properties

Instead of simply placing the poles of G_1 and G_2 in "nice" locations, it is reasonable to exploit the extra degree of freedom associated with the asymptotic estimator to satisfy some additional specifications. Here we focus on disturbance rejection and sensor noise rejection. The disturbance rejection of the combined system in Figure 3.1(b) is given by

$$\begin{aligned} T^b_{dy} = \frac{1+CG_2}{1+PC} &= \frac{1}{1+PC} + \frac{PC}{1+PC}(1-G_1H) \\ &= T^a_{dy} - T^a_{wy}S_E \end{aligned} \tag{4.1}$$

while the sensor noise rejection is given by

$$T^b_{wy} = \frac{-PC}{1+PC}G_1H = T^a_{wy}T_E. \tag{4.2}$$

Based on (4.1) and (4.2), the designer may consider a *sequential* or a *parallel* approach to design the controller C and the asymptotic estimator (G_1, G_2). In the *sequential* approach, the controller C may be designed first to attain certain T^a_{dy} and T^a_{wy}. Then the asymptotic estimator (G_1, G_2) is designed to achieve acceptable T^b_{dy} and T^b_{wy}, taking T^a_{dy} and T^a_{wy} into account. The sequence may be reversed to design (G_1, G_2) first.

In the *parallel* approach, the controller C and the asymptotic estimator (G_1, G_2) are designed independently from each other, without taking the other into account. A design strategy based on the parallel approach is as follows: choose C so that $|T^a_{dy}|$ and $|T^a_{wy}|$ satisfy the desired performance goals for disturbance rejection and sensor noise rejection, and design separately the asymptotic estimator so that $|S_E| \approx 0$ and $|T_E| \approx 1$. Designing the asymptotic estimator this way implies, from (4.1) and (4.2), $|T^b_{dy}| \approx |T^a_{dy}|$ and $|T^b_{wy}| \approx |T^a_{wy}|$. In practice, the asymptotic estimator can be designed by solving the mixed-sensitivity problem

$$\left\| \begin{bmatrix} W_1 S_E \\ T_E \end{bmatrix} \right\|_\infty < 1, \tag{4.3}$$

where W_1 describes the relative weighting of frequencies that are significant for disturbance rejection. This approach is pursued in Section 4.3.2.

4.3 Example

Now we apply the above strategies to the radiotherapy control problem. Simplified models of the components in the feedback loop of Figure 1.2 are as follows [7]: the sampling period is 0.3 seconds, the collimator leaf model is $M_{leaf}[z] = \frac{2.851}{z-0.0498}$ (i.e., a first-order system with a time constant of 0.1 seconds), and $M_{delay2}[z] = \frac{1}{z}$ models the image processing delay. Note that $M_{leaf}[z]$ equals $P[z]$ in Figure 3.1(b) and $M_{delay2}[z]$ equals $H[z]$.

4.3.1 Design for Stability and Tracking

The paper [7] introduces the controller

$$C[z] = 0.4 \frac{(z-0.4)}{(z-1)} \frac{(z^2 - 2\cos(\pi/10)(0.7)z + 0.7^2)}{(z^2 - 2\cos(\pi/10)z + 1)}, \tag{4.4}$$

which stabilizes P in Figure 3.1(a), has a pole at 1 (guaranteeing asymptotic step rejection), and has poles on the unit circle (guaranteeing perfect asymptotic tracking at discrete-time frequency $\pi/10$, corresponding to a 1/6 Hz breathing rate). Let's now consider the design of the asymptotic estimator in Figure 3.1(b). According to the discussion in Section 4.1, the separation principle implies that *any* asymptotic estimator guarantees stability of the combined system and recovers the tracking performance of the 1-DOF system in Figure 3.1(a).

Suppose we would like to place all the poles of G_1, G_2 at $z = 0.5$. We can choose

$$\begin{aligned} N_P[z] &= \frac{2.851}{z-0.5}, & M_P[z] &= \frac{z-0.0498}{z-0.5}, \\ N_H[z] &= \frac{1}{z-0.5}, & M_H[z] &= \frac{z}{z-0.5}, \\ X[z] &= \frac{-0.15087(z-0.1453)}{(z-0.5)^2}, & Y[z] &= \frac{z^2-1.95z+1.403}{(z-0.5)^2}. \end{aligned}$$

With $Q = 0$, we obtain, from parameterization $\mathcal{F}$ in (2.13), the asymptotic estimator

$$\begin{aligned} G_1[z] &= \frac{-0.43014z(z-0.1453)}{(z-0.5)^4} \\ G_2[z] &= \frac{2.851z(z^2-1.95z+1.403)}{(z-0.5)^4}, \end{aligned}$$

which guarantees closed-loop stability and $T_{ry}^b = T_{ry}^a$.

4.3.2 Design for Additional Closed-Loop Properties

Here we adopt the *parallel* approach mentioned in Section 4.2. We continue to use $C[z]$ in (4.4) as the controller. As for the asymptotic estimator, rather than placing the poles of G_1 and G_2 somewhat arbitrarily at $z = 0.5$, let's instead design (G_1, G_2) with the following goals in mind:

1. asymptotic rejection of step disturbances,
2. attenuation of disturbances up to 1/6 Hz (the nominal breathing rate)[1], and
3. no more than 5% worsening of sensor noise performance compared to Figure 3.1(a).

These goals are translated into the requirements

1. $I[z] = z - 1$ (to guarantee $S_E[1] = 0$),
2. $\|W_1 S_E\|_\infty \leq 1$, and

[1]In the radiotherapy problem, only step disturbances are expected. This design goal is not very relevant, but is added to illustrate the method.

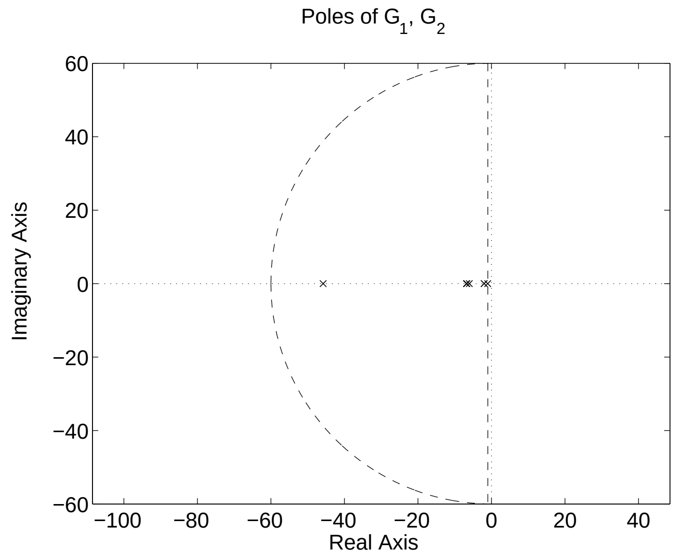

Figure 4.1: *The asymptotic estimator in* (4.5) *is designed to place the poles (denoted by crosses) in the desired region: to the left of* $s = -1$ *and inside a disk of radius 60.*

3. $\|T_E\|_\infty \leq 1.05$,

where W_1 is a first-order low-pass filter with cut-off frequency at 1/6 Hz. Substitute (2.24) and (2.25) into (4.3) to get the following mixed-sensitivity problem:

$$\left\| \begin{bmatrix} W_1(M_H M_P M_I Y - N_H M_P M_I Q) \\ N_H N_P N_I X + N_H M_P M_I Q \end{bmatrix} \right\|_\infty < 1.05.$$

The methods in [23] (conveniently implemented in the “hinfmax” Matlab routine) were used to solve this problem. To get good settling time and reasonable bandwidth, the poles (when mapped to the s-plane) were restricted to lie to the left of $s = -1$ and inside a disk of radius 60 (see Figure 4.1). The resulting asymptotic estimator is

$$G_1[z] = \frac{1.01(z + 0.752)(z - 0.733)(z - 0.532)(z - 0.0504)(z + 0.000232)}{(z + 0.746)(z - 0.731)(z - 0.539)^3(z - 0.0498)} \cdot \frac{(z^2 - 1.08z + 0.293)}{(z + 0.0101)}$$

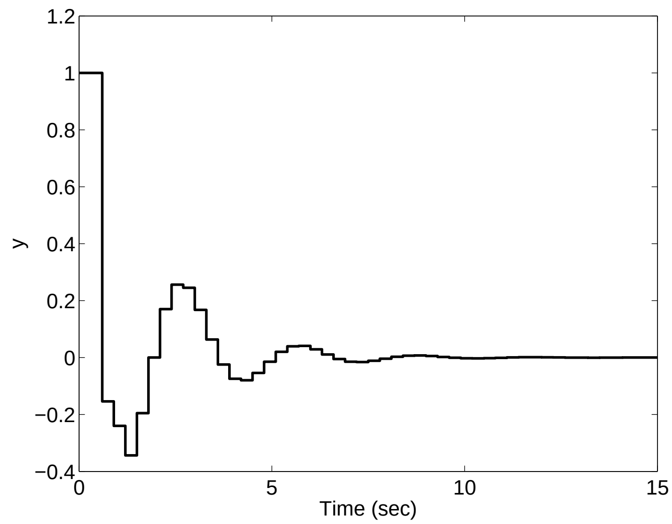

Figure 4.2: *Response of the system in Figure 3.1(b) to a unit step disturbance, showing perfect steady-state rejection and good settling time.*

$$G_2[z] = \frac{-7.96 \times 10^{-9}(z - 3.58 \times 10^{8})(z-1)(z-0.738)(z+0.749)(z-0.0504)}{(z+0.746)(z-0.731)(z-0.0504)(z-0.0492)(z+0.0101)} \cdot \frac{(z+0.000233)(z^2 - 1.08z + 0.289)}{(z - 5.95 \times 10^{-6})(z^2 - 1.08z + 0.290)}. \quad (4.5)$$

Figure 4.2 and Figure 4.3 show that the asymptotic estimator achieves satisfactory transient and frequency responses for the combined system in Figure 3.1(b).

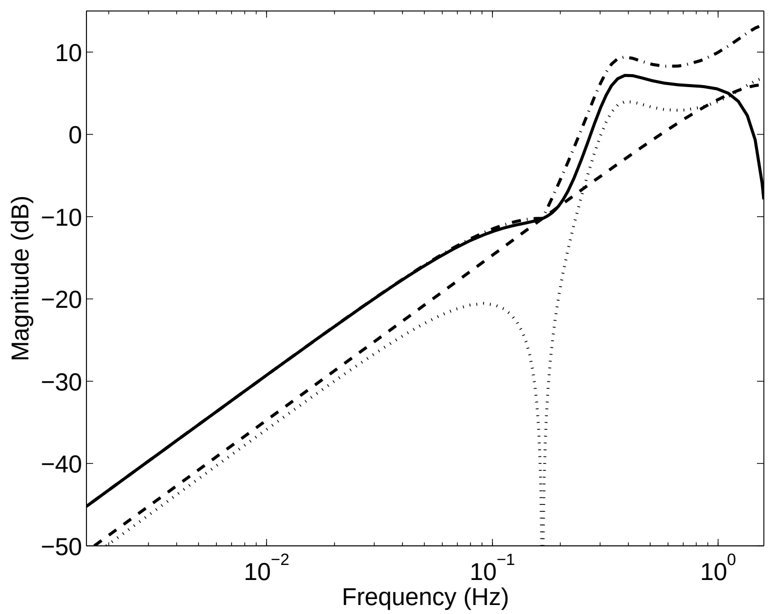

Figure 4.3: *Frequency responses:* $|T^a_{dy}|$ *(dot),* $|S_E|$ *(dashed),* $|T^b_{dy}|$ *(solid),* $|T^a_{dy}| + |T^a_{wy}||S_E|$ *(dash-dot), showing* $|T^b_{dy}| \leq |T^a_{dy}| + |T^a_{wy}||S_E|$ *as predicted by* (4.1).

Chapter 5

Multivariable Extensions

The results presented in the previous chapters apply to SISO systems. The chapter extends some of the SISO results to MIMO systems.

5.1 Parameterizations of Asymptotic Estimators

This section develops the MIMO version of the parameterizations in Theorems 2.1 and 2.2. In the MIMO setting, an asymptotic estimator is defined to be a pair of proper transfer matrices (G_1, G_2) such that, for $d = w = 0$,

$$\lim_{k\to\infty} (\hat{y}[k] - y[k]) = 0 \quad \forall\, u, \forall \,\text{initial conditions.} \tag{5.1}$$

Parameterizations $\mathcal{A}$ and $\mathcal{B}$ are basically the same as their SISO counterparts:

Lemma 5.1 The following sets are equal

$$\mathcal{A} := \{(G_1, G_2) : (5.1) \text{ is satisfied}\} \tag{5.2}$$

$$\mathcal{B} := \{(G_1, G_2) : G_1 \in M(\mathcal{S}), G_2 \in M(\mathcal{S}), T_{u\hat{y}} = T_{uy}\} \tag{5.3}$$

$\square$

Proof: See Section B.2 for details. $\square$

Parameterizations $\mathcal{C}$ and $\mathcal{D}$ are more complicated and are addressed below.

5.1.1 Parameterization with Interpolation Constraints

Section 2.2 has shown a parameterization that involves interpolation constraints on G_1 (and its derivatives) at the unstable poles of P. This section extends the concept to MIMO systems.

Definition 5.1 [24] For a pole of $P[z] \in M(\mathbb{R}[z])$, a, its *multiplicity* is the minimum integer r such that $(z-a)^r P[z]$ is analytic at a. $\square$

If r_{ij} is the multiplicity of a for the ij-entry of $P[z]$, then $r = \max_{i,j} r_{ij}$. Note that this definition of pole multiplicity is different from the one based on the Smith-Macmillan form.

Let the poles of $P[z]$ be $a_1, \ldots, a_m$ with multiplicities $r_1, \ldots r_m$ as defined by Definition 5.1. Now, $G_1[z]$ being stable implies that $G_1[z]$ is analytic at a_i, for $i = 1, \ldots, m$. Thus we can obtain a (matrix) Taylor expansion of $G_1[z]$ at a_i:

$$G_1[z] = \sum_{j=0}^{\infty} g_j^i (z-a_i)^j,$$

where $g_j^i = \frac{1}{j!} \frac{d^j}{dz^j} G_1[z] \Big|_{z=a_i}$ (obtained via entry-by-entry differentiation). Similarly, $H[z]$ is assumed to be stable and thus we can obtain a (matrix) Taylor expansion of $H[z]$ at a_i:

$$H[z] = \sum_{j=0}^{\infty} h_j^i (z-a_i)^j,$$

where $h_j^i = \frac{1}{j!} \frac{d^j}{dz^j} H[z] \Big|_{z=a_i}$. Finally, since a_i is a pole of $P[z]$ with multiplicity r_i, we can obtain a (matrix) Laurent expansion of $P[z]$ at a_i:

$$P[z] = \sum_{j=-r_i}^{\infty} p_j^i (z-a_i)^j,$$

where $p_j^i = \frac{1}{(j+r_i)!} \frac{d^{j+r_i}}{dz^{j+r_i}} [(z-a_i)^{r_i} P[z]] \Big|_{z=a_i}$.

One can observe that product of two (matrix) polynomials can be represented by the product of two Toeplitz matrices [24, 25]. For example, we may express the negative

Laurent coefficients of $V[z] := H[z]P[z]$ at a_i by

$$\begin{bmatrix} h_0^i & h_1^i & \cdots & h_{r_i-1}^i \\ & h_0^i & \cdots & h_{r_i-2}^i \\ & & \ddots & \vdots \\ & & & h_0^i \end{bmatrix} \begin{bmatrix} p_{-r_i}^i & p_{-r_i+1}^i & \cdots & p_{-1}^i \\ & p_{-r_i}^i & \cdots & p_{-2}^i \\ & & \ddots & \vdots \\ & & & p_{-r_i}^i \end{bmatrix} = \begin{bmatrix} v_{-r_i} & v_{-r_i+1} & \cdots & v_{-1} \\ & v_{-r_i} & \cdots & v_{-2} \\ & & \ddots & \vdots \\ & & & v_{-r_i} \end{bmatrix}, \tag{5.4}$$

where $V[z] = \sum_{j=-r_i}^{\infty} v_j^i (z-a_i)^j$. Because of the structure of the Toeplitz matrices, the first row or the last column contains all the information of a polynomial. Therefore (5.4) can also be expressed in the following two forms:

$$\begin{bmatrix} h_0^i & h_1^i & \cdots & h_{r_i-1}^i \end{bmatrix} \begin{bmatrix} p_{-r_i}^i & p_{-r_i+1}^i & \cdots & p_{-1}^i \\ & p_{-r_i}^i & \cdots & p_{-2}^i \\ & & \ddots & \vdots \\ & & & p_{-r_i}^i \end{bmatrix} = \begin{bmatrix} v_{-r_i} & v_{-r_i+1} & \cdots & v_{-1} \end{bmatrix},$$

$$\begin{bmatrix} h_0^i & h_1^i & \cdots & h_{r_i-1}^i \\ & h_0^i & \cdots & h_{r_i-2}^i \\ & & \ddots & \vdots \\ & & & h_0^i \end{bmatrix} \begin{bmatrix} p_{-1}^i \\ \vdots \\ p_{-r_i}^i \end{bmatrix} = \begin{bmatrix} v_{-1} \\ \vdots \\ v_{-r_i} \end{bmatrix}.$$

For $G_2[z] = P[z] - G_1[z]H[z]P[z]$ to be analytic at a_i, the negative coefficients of the Laurent series of $F[z] := G_1[z]H[z]P[z]$ should match those of $P[z]$, i.e., $f_k = p_k$, for $k = -1, -2, \ldots, -r_i$, which means that

$$\begin{bmatrix} g_0^i & g_1^i & \cdots & g_{r_i-1}^i \\ & g_0^i & \cdots & g_{r_i-2}^i \\ & & \ddots & \vdots \\ & & & g_0^i \end{bmatrix} \begin{bmatrix} h_0^i & h_1^i & \cdots & h_{r_i-1}^i \\ & h_0^i & \cdots & h_{r_i-2}^i \\ & & \ddots & \vdots \\ & & & h_0^i \end{bmatrix} \begin{bmatrix} p_{-1}^i \\ \vdots \\ p_{-r_i}^i \end{bmatrix} = \begin{bmatrix} p_{-1}^i \\ \vdots \\ p_{-r_i}^i \end{bmatrix} \tag{5.5}$$

or

$$\begin{bmatrix} g_0^i & \cdots & g_{r_i-1}^i \end{bmatrix} \begin{bmatrix} h_0^i & h_1^i & \cdots & h_{r_i-1}^i \\ & h_0^i & \cdots & h_{r_i-2}^i \\ & & \ddots & \vdots \\ & & & h_0^i \end{bmatrix} \begin{bmatrix} p_{-r_i}^i & p_{-r_i+1}^i & \cdots & p_{-1}^i \\ & p_{-r_i}^i & \cdots & p_{-2}^i \\ & & \ddots & \vdots \\ & & & p_{-r_i}^i \end{bmatrix} = \begin{bmatrix} p_{-r_i}^i & \cdots & p_{-1}^i \end{bmatrix}. \tag{5.6}$$

From the above discussions, we can conclude the following:

Lemma 5.2 The following two sets are equal:

$$\begin{aligned} \mathcal{B} &:= \{(G_1, G_2) : G_1 \in M(\mathcal{S}), G_2 \in M(\mathcal{S}), T_{u\hat{y}} = T_{uy}\} \\ \mathcal{C} &:= \{(G_1, G_2) : G_1 \in M(\mathcal{S}), (5.5) \text{ or } (5.6) \text{ is satisfied}, G_2 = (I - G_1 H)P\}. \end{aligned} \tag{5.7}$$

□

Example 5.1 Suppose

$$P[z] = \begin{bmatrix} \frac{1}{(z-2)^2} & \frac{1}{z-\frac{1}{2}} \\ \frac{1}{z-\frac{1}{2}} & \frac{1}{z-2} \end{bmatrix}, \quad H[z] = \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{z} \end{bmatrix}.$$

According to Definition 5.1, $P[z]$ has one unstable pole, 2, with multiplicity of 2. We can obtain the following Laurent and Taylor coefficients:

$$p_{-2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, p_{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, h_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, h_1 = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}.$$

Therefore (5.6) becomes

$$\begin{bmatrix} g_0 & g_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where one possible (non-unique) solution is

$$g_0 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.8}$$

One $G_1[z]$ that satisfies (5.8) is

$$G_1[z] = \begin{bmatrix} \frac{\frac{7}{2}z-4}{z-\frac{1}{2}} & 0 \\ 0 & 2 \end{bmatrix},$$

and the corresponding $G_2[z]$ is given by

$$G_2[z] = (I - G_1[z]H[z])P[z] = \begin{bmatrix} \frac{1}{z(z-\frac{1}{2})} & \frac{(z-2)^2}{z(z-\frac{1}{2})} \\ \frac{-2}{z-\frac{1}{2}} & \frac{1}{z} \end{bmatrix},$$

which is a member of $M(\mathcal{S})$, as required by (5.3).

$\square$

Simplification of (5.6) is possible when further assumptions are made, for example p_{-r_i} is invertible.

The authors of [25] provide techniques that can be used to derive, from condition (5.5), a Q-parameterization of G_1. This Q-parameterization is equivalent to that obtained using coprime factorization methods, as presented in the next subsection.

5.1.2 Q-Parameterization

Let $P[z] = (A_P, B_P, C_P, D_P)$, $H[z] = (A_H, B_H, C_H, D_H)$ be minimal realizations. Assume that the following realization of $H[z]P[z]$ is stabilizable and detectable:

$$\begin{aligned} H[z]P[z] &= (A, B, C, D) \\ &= \left(\begin{bmatrix} A_P & 0 \\ B_H C_P & A_H \end{bmatrix}, \begin{bmatrix} B_P \\ B_H D_P \end{bmatrix}, \begin{bmatrix} D_H C_P & C_H \end{bmatrix}, \begin{bmatrix} D_H D_P \end{bmatrix} \right). \end{aligned}$$

Therefore there exist K, L such that $A + BK$ and $A + LC$ are Hurwitz.

A doubly coprime factorization of $H[z]P[z]$ is given by [26]

$$H[z]P[z] = N[z]M^{-1}[z] = \tilde{M}^{-1}[z]\tilde{N}[z] \tag{5.9}$$

where

$$\begin{aligned} M[z] &= (A + BK, B, K, I), & N[z] &= (A + BK, B, C + DK, D), \\ \tilde{M}[z] &= (A + LC, L, C, I), & \tilde{N}[z] &= (A + LC, B + LD, C, D). \end{aligned}$$

Since $A+BK$ and $A+LC$ are Hurwitz, $N[z]$, $M[z]$, $\tilde{M}[z]$, $\tilde{N}[z]$ are all members of $M(\mathcal{S})$. Moreover,

$$\begin{aligned} Y[z] &= (A+LC, B+LD, -K, I), & X[z] &= (A+LC, L, K, 0), \\ \tilde{Y}[z] &= (A+BK, -L, C+DK, I), & \tilde{X}[z] &= (A+BK, L, K, 0) \end{aligned}$$

satisfy the Bezout identity,

$$\begin{bmatrix} Y[z] & X[z] \\ -\tilde{N}[z] & \tilde{M}[z] \end{bmatrix} \begin{bmatrix} M[z] & -\tilde{X}[z] \\ N[z] & \tilde{Y}[z] \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{5.10}$$

Likewise, $Y[z]$, $X[z]$, $\tilde{Y}[z]$, $\tilde{X}[z]$ are all members of $M(\mathcal{S})$.

With the above definitions, we can state the following Q-parameterization of all asymptotic estimators:

Lemma 5.3 The following sets are equal:

$$\begin{aligned} \mathcal{B} &:= \{(G_1, G_2) : G_1 \in M(\mathcal{S}), G_2 \in M(\mathcal{S}), T_{u\hat{y}} = T_{uy}\} \\ \mathcal{D} &:= \{(G_1, G_2) : G_1 = PMX + Q\tilde{M} \text{ and } G_2 = PMY - Q\tilde{N} \text{ for } Q \in M(\mathcal{S})\}. \end{aligned} \tag{5.11}$$

□

Proof: See Section B.3 for details. □

5.1.3 Main Result

Putting together the above lemmas, we arrive at the main result:

Theorem 5.1 The following four sets are equal:

$$\begin{aligned} \mathcal{A} &:= \{(G_1, G_2) : (5.1) \text{ is satisfied}\} \\ \mathcal{B} &:= \{(G_1, G_2) : G_1 \in M(\mathcal{S}), G_2 \in M(\mathcal{S}), T_{u\hat{y}} = T_{uy}\} \\ \mathcal{C} &:= \{(G_1, G_2) : G_1 \in M(\mathcal{S}), (5.6) \text{ is satisfied}, G_2 = (I - G_1 H)P\} \\ \mathcal{D} &:= \{(G_1, G_2) : G_1 = PMX + Q\tilde{M} \text{ and } G_2 = PMY - Q\tilde{N} \text{ for } Q \in M(\mathcal{S})\}. \end{aligned}$$

□

Proof: The results follow from Lemmas 5.1, 5.2, and 5.3. □

5.2 Separation Principle

The MIMO version of the separation principle is similar to the SISO version. Namely, for the separation principle to hold in Figure 3.1(b), two conditions must hold:

1. the set of poles of the system in Figure 3.1(b) equals the union of the poles of H, the poles of G_1, the poles of G_2, and the set of poles of the system in Figure 3.1(a); and

2. the transfer function (matrix) from r to y is the same for the system in Figure 3.1(b) as it is for the system in Figure 3.1(a).

Theorem 5.3 below shows that the separation principle holds for *any* MIMO asymptotic estimator (G_1, G_2). The proof requires the MIMO version of Theorem 3.1:

Theorem 5.2 [22] Let $G_i[z]$, $i = 1, 2, \ldots, n$ denote the transfer function matrices of the n plants of an interconnected system. Let $p_i(z)$ denote the characteristic polynomial of $G_i[z]$. The system interconnection matrix $W[z]$ is defined such that

$$W_{ij} = \begin{cases} -I, & i = j \\ G_i, & i \neq j, \text{ and } G_j\text{'s output is an input to } G_i \\ 0, & i \neq j, \text{ and } G_j\text{'s output is not an input to } G_i \end{cases}.$$

Then the poles of a linear time-invariant interconnected system with vector signals are the roots of the polynomial

$$p_c(z) := \det(W[z]) \prod_{i=1}^{n} p_i(z).$$

□

We can now state the MIMO version of Theorem 3.2. Note that the sufficiency result is identical to the SISO case, but the necessity result is stated only under additional assumptions.

Theorem 5.3 Consider the system in Figure 3.1(b). Assume that C stabilizes P in the sense that the feedback system in Figure 3.1(a) is stable. Then:

(a) The separation principle holds if G_1 and G_2 satisfy

$$G_2 = (I - G_1 H)P. \tag{5.12}$$

(b) If (G_1, G_2) is an asymptotic estimator (according to Theorem 5.1), then the system in Figure 3.1(b) is stable *and* the separation principle holds.

(c) If P is square, and PC has full normal rank[1], then

  (i) if the separation principle holds, then G_1 and G_2 satisfy (5.12);

  (ii) if the system in Figure 3.1(b) is stable *and* the separation principle holds, then (G_1, G_2) is an asymptotic estimator.

□

Proof:

By Theorem 5.2, the characteristic polynomial of the closed-loop system in Figure 3.1(a) is

$$p_c^a = \det(W^a) p_C p_P \tag{5.13}$$

where

$$\det(W^a) = \begin{vmatrix} -I & -C \\ P & -I \end{vmatrix} = \begin{vmatrix} -I - CP & 0 \\ P & -I \end{vmatrix} = (-1)^{k_a} |I + CP|,$$

where k_a is an integer constant. By Theorem 5.2, the characteristic polynomial of the closed-loop system in Figure 3.1(b) is

$$p_c^b = \det(W^b) p_C p_P p_H p_1 p_2 \tag{5.14}$$

[1] Normal rank of a matrix $G[z]$ is the rank for "almost all" values of z.

where $\det(W^b)$

$$
\begin{aligned}
&= \begin{vmatrix} -I & 0 & 0 & C & C \\ P & -I & 0 & 0 & 0 \\ 0 & H & -I & 0 & 0 \\ -G_2 & 0 & 0 & -I & 0 \\ 0 & 0 & -G_1 & 0 & -I \end{vmatrix} \\
&= \begin{vmatrix} -I & 0 & -CG_1 & C & 0 \\ P & -I & 0 & 0 & 0 \\ 0 & H & -I & 0 & 0 \\ -G_2 & 0 & 0 & -I & 0 \\ 0 & 0 & -G_1 & 0 & -I \end{vmatrix} && \left(\begin{array}{l}\text{by adding } C \text{ times block-row 5} \\ \text{to block-row 1}\end{array}\right) \\
&= \begin{vmatrix} -I - CG_2 & 0 & -CG_1 & 0 & 0 \\ P & -I & 0 & 0 & 0 \\ 0 & H & -I & 0 & 0 \\ -G_2 & 0 & 0 & -I & 0 \\ 0 & 0 & -G_1 & 0 & -I \end{vmatrix} && \left(\begin{array}{l}\text{by adding } C \text{ times block-row 4} \\ \text{to block-row 1}\end{array}\right) \\
&= \begin{vmatrix} -I - CG_2 & -CG_1H & 0 & 0 & 0 \\ P & -I & 0 & 0 & 0 \\ 0 & H & -I & 0 & 0 \\ -G_2 & 0 & 0 & -I & 0 \\ 0 & 0 & -G_1 & 0 & -I \end{vmatrix} && \left(\begin{array}{l}\text{by adding } -CG_1 \text{ times block-} \\ \text{row 3 to block-row 1}\end{array}\right) \\
&= \begin{vmatrix} -I - CG_2 - CG_1HP & 0 & 0 & 0 & 0 \\ P & -I & 0 & 0 & 0 \\ 0 & H & -I & 0 & 0 \\ -G_2 & 0 & 0 & -I & 0 \\ 0 & 0 & -G_1 & 0 & -I \end{vmatrix} && \left(\begin{array}{l}\text{by adding } -CG_1H \text{ times block-} \\ \text{row 2 to block-row 1}\end{array}\right) \\
&= (-1)^{k_b} |I + C(G_2 + G_1HP)|,
\end{aligned}
$$

where k_b is an integer constant. When $G_2 + G_1HP = P$, we have

$$\det(W^b) = (-1)^{k_b - k_a} \det(W^a) \quad \text{and so}$$

$$p_c^b = (-1)^{k_b - k_a} p_c^a p_H p_1 p_2, \tag{5.15}$$

which establishes condition 1 of the separation principle.

The transfer function from r to y in Figure 3.1(a) is

$$T_{ry}^a = P[I + CP]^{-1}C \tag{5.16}$$

and the transfer function from r to y in Figure 3.1(b) is

$$T_{ry}^b = P[I + C(G_2 + G_1 HP)]^{-1}C \tag{5.17}$$

When $G_2 + G_1 HP = P$, then $T_{ry}^b = T_{ry}^a$, which establishes condition 2 of the separation principle. So we have established part (a) of the Theorem.

If (G_1, G_2) is an asymptotic estimator, Theorem 5.1 guarantees that (5.12) is satisfied and thus the separation principle holds according to (a). Since the separation principle holds, the closed-loop characteristic polynomial is given by (5.15). Since (G_1, G_2) is an asymptotic estimator, G_1, G_2 are stable. Therefore the closed-loop is stable since C stabilizes P and since H is stable. So we have established part (b) of the Theorem.

All that remains is to prove (c). If P is square, C is square also. Then PC having full normal rank implies that both P and C also have full normal rank. Therefore we can invert both P and C. Now assume that the separation principle holds. Hence, $T_{ry}^b = T_{ry}^a$, i.e., $P[I + C(G_2 + G_1 HP)]^{-1}C = P[I + CP]^{-1}C$. We can use the facts that P and C are invertible to conclude that $G_2 + G_1 HP = P$. This proves (c)(i). Now assume that, in addition to the separation principle holding, the system in Fig 3.1(b) is stable. This additional assumption implies, from (5.15), that G_1 and G_2 are stable. Hence, G_1 and G_2 are stable and satisfy $G_2 + G_1 HP = P$; from Theorem 5.1, we conclude that (G_1, G_2) is an asymptotic estimator. This proves (c)(ii). □

5.3 Relating Closed-Loop Properties to Asymptotic Estimator Properties

The "sensitivity" functions discussed in Section 2.4 can be extended to MIMO asymptotic estimators. The "sensitivity" and "complementary sensitivity" functions for the estimator in Figure 3.1(b) are defined as:

$$\begin{aligned} S_E &:= I - G_1 H \\ T_E &:= G_1 H. \end{aligned}$$

Note that $S_E + T_E = I$. The transfer functions S_E and T_E are related to the performance of the asymptotic estimator. Specifically, define the estimation error $e := \hat{y} - y$. Then the disturbance rejection of the estimator is

$$T_{de} = G_1 H - I = -S_E$$

and the sensor noise rejection of the estimator is

$$T_{we} = G_1 H = T_E.$$

Section 4.2 relates the closed-loop disturbance rejection and sensor noise rejection to the "sensitivity" functions of the SISO asymptotic estimator. We are going to see that similar results are available for MIMO systems.

The disturbance rejection of the 1-DOF system in Figure 3.1(a) is given by

$$T_{dy}^a = (I + PC)^{-1},$$

while the disturbance rejection of the combined system in Figure 3.1(b) is given by

$$T_{dy}^b = I - P(I + CG_2 + CG_1 HP)^{-1} CG_1 H.$$

With $G_2 = (I - G_1 H)P$, we have

$$\begin{aligned} T_{dy}^b &= I - P(I + CP)^{-1} CG_1 H \\ &= I - (I + PC)^{-1} PCG_1 H \\ &= (I + PC)^{-1}(I + PC - PCG_1 H) \\ &= (I + PC)^{-1} + [(I + PC)^{-1} PC](I - G_1 H) \\ &= T_{dy}^a - T_{wy}^a S_E, \end{aligned}$$

which is the same as (4.1) for the SISO case.

We obtain similar MIMO extension for sensor noise rejection. The sensor noise rejection of the 1-DOF system in Figure 3.1(a) is given by

$$T_{wy}^{a} = -(I + PC)^{-1} PC.$$

while the sensor noise rejection of the combined system in Figure 3.1(b) is given by

$$T_{wy}^{b} = -P(I + CG_2 + CG_1 HP)^{-1} CG_1 H.$$

With $G_2 = (I - G_1 H)P$, we have

$$\begin{aligned} T_{wy}^{b} &= -P(I + CP)^{-1} CG_1 H \\ &= -(I + PC)^{-1} PCG_1 H \\ &= T_{wy}^{a} T_E \end{aligned}$$

which is the same as (4.2) for the SISO case.

Chapter 6

Conclusions

In this thesis we have considered feedback control systems that have sensor time delays. The focus has been on the use of an asymptotic estimator to compensate for the time delay, and the main result is that a separation principle holds. We then suggested two design strategies that exploit the separation principle. Lastly we extended some of SISO results to MIMO systems.

The following areas warrant further investigations:

- One area of future work is to investigate the robustness of the compensation scheme with respect to plant uncertainty. When the plant diverges from the model used in designing the asymptotic estimator, how will various control performance metrics be affected?

- Another area of future work is to utilize the Q-parameterization of the MIMO asymptotic estimators to compute performance limitations associated with sensor time delay compensation. The authors of [1] have used the "Model Matching" method [27] in conjunction with the Q-parameterization of the SISO asymptotic estimators to derive SISO performance limitations.

- Yet another area worth looking into is performance limitations under constraints on pole locations. Good pole locations result in good closed-loop behavior including good transient responses. It is worth knowing whether there are tradeoffs between pole assignments and other performance metrics in the presence of sensor time delay. In the SISO case, [1, 16, 17] demonstrate tradeoffs between bandwidth and disturbance rejection in the presence of sensor time delay.

- The mixed-sensitivity approach is adopted in Section 4.2 to design asymptotic estimators with desirable S_E and T_E. Admittedly, the mixed-sensitivity approach is sub-optimal, especially in the MIMO case. It is desirable to consider alternatives to the mixed-sensitivity approach.

- Theorem 2.4 outlines a parameterization of SISO asymptotic estimators that reject step disturbance. We have not yet come up with a parameterization for MIMO asymptotic estimators.

Appendix A

Mathematical Background

This chapter contains results that are used in the proofs found in the thesis.

A.1 Rings

Definition A.1 [26] A *ring* is a set R, together with two binary operations $+$ and $\cdot$ on R satisfying the following axioms. For all $a, b, c, \in R$,

1. $(a + b) + c = a + (b + c)$.
2. $a + b = b + a$.
3. there exists $0 \in R$ such that $a + 0 = a$.
4. there exists $(-a) \in R$ such that $a + (-a) = 0$.
5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
6. $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot b + a \cdot c$.

$\square$

A ring R is called a *commutative ring* if

$$a \cdot b = b \cdot a \quad \text{for all } a, b \in R.$$

A ring R is said to have an *identity* if there exists $1 \in R$ such that

$$1 \cdot a = a \cdot 1 = a.$$

Under the above definitions, $\mathcal{S}$, the set of all proper stable transfer functions, is a commutative ring with identity.

For $a, b \in R$, we say a *divides* b, if there is an element $c \in R$ such that $b = ca$.

Two members of $\mathcal{S}$, N, M, are coprime if and only if they have no common zeros in $|z| \geq 1$, and at least one of them has relative degree zero.

Two members of $\mathcal{S}$, N, M, are coprime if and only if there exist $X, Y \in S$ such that $NX + MY = 1$ [26, 28].

Let $G[z]$ be a proper transfer function, and (A, B, C, D) a stabilizable and detectable state space realization of $G[z]$. Therefore there exists real matrices K, L such that $A+BK$ and $A+LC$ are Hurwitz. Then [28]

$$G[z] = \frac{N[z]}{M[z]}, \quad N[z] \text{ and } M[z] \text{ are coprime in } \mathcal{S},$$

where

$$\begin{aligned} M[z] &= (A+BK, B, K, 1), \qquad & N[z] &= (A+BK, B, C+DK, D) \\ X[z] &= (A+LC, L, K, 0), \qquad & Y[z] &= (A+LC, B+LD, -K, 1) \end{aligned}$$

satisfy

$$N[z]X[z] + M[z]Y[z] = 1.$$

Matrices with entries in $\mathcal{S}$, $M(\mathcal{S})$, form a ring, though not a commutative ring since matrix multiplication is not commutative. One way to obtain coprime factorization of a proper transfer matrix $G[z]$ over $M(\mathcal{S})$ is given in Section 5.1.2.

A.2 Calculus

Claim: Suppose $f[z] \in \mathbb{R}[z]$ is analytic at b. Then $f[z]$ has a zero at b with multiplicity at least r $(r \geq 1)$ if and only if

$$\left. \frac{d^j}{dz^j} f[z] \right|_{z=b} = 0, \quad \text{for } j = 0, \ldots, r-1. \tag{A.1}$$

Proof: $(\Rightarrow)$: $f[z]$ can be expressed as

$$f[z] = (z-b)^r g[z]$$

where $g[z]$ does not have a pole at b. Clearly $f[b] = 0$. And for $j = 1, \dots, r-1$,

$$\begin{aligned} \left.\frac{d^j}{dz^j} f[z]\right|_{z=b} &= \sum_{k=0}^{j} \binom{j}{k} \left[\frac{d^{j-k}}{dz^{j-k}} (z-b)^r\right]_{z=b} g^{(k)}[b] \\ &= \sum_{k=0}^{j} \binom{j}{k} \left[\frac{r!}{(r-j+k)!} (z-b)^{r-j+k}\right]_{z=b} g^{(k)}[b] \\ &= 0. \end{aligned}$$

So (A.1) holds.

($\Leftarrow$): Suppose $f[z]$ has a zero at b with multiplicity $m < r$. So we can write $f[z]$ as

$$f[z] = (z-b)^m g[z] \tag{A.2}$$

where b is neither a pole nor zero of $g[z]$ (so $g[b] \neq 0$). From (A.2), we get

$$\begin{aligned} \left.\frac{d^m}{dz^m} f[z]\right|_{z=b} &= \sum_{k=0}^{m} \binom{m}{k} \left[\frac{d^{m-k}}{dz^{m-k}} (z-b)^m\right]_{z=b} g^{(k)}[b] \\ &= \sum_{k=0}^{m} \binom{m}{k} \left[\frac{m!}{(m-(m-k))!} (z-b)^{m-(m-k)}\right]_{z=b} g^{(k)}[b] \\ &= \sum_{k=0}^{m} \binom{m}{k} \left[\frac{m!}{k!} (z-b)^{k}\right]_{z=b} g^{(k)}[b] \\ &= \binom{m}{0} \left[\frac{m!}{0!} (z-b)^{0}\right]_{z=b} g[b] + \sum_{k=1}^{m} \binom{m}{k} \left[\frac{m!}{k!} (z-b)^{k}\right]_{z=b} g^{(k)}[b] \\ &= m! g[b] + 0 \\ &\neq 0, \end{aligned}$$

which means (A.1) is false.

$\square$

A.3 Linear Algebra

The equality $(I + AB)^{-1} A = A(I + BA)^{-1}$ holds for any matrices A and B (assuming compatible dimensions and the inverses exist).

When A, B are square, $\det(AB) = \det(A)\det(B)$.

When A, B are square, $\det \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} = \det A \det B$.

A.4 Linear System Theory

Let $G_1[z] = (A_1, B_1, C_1, D_1)$ and $G_2[z] = (A_2, B_2, C_2, D_2)$. One realization for $G_2[z]G_1[z]$ is

$$\left(\begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}, \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix}, \begin{bmatrix} D_2D_1 \end{bmatrix} \right). \tag{A.3}$$

One realization for $G_1[z] + G_2[z]$ is

$$\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \begin{bmatrix} D_1 + D_2 \end{bmatrix} \right).$$

Given any state space realization (A, B, C, D), there always exists state transformation, T, such that

$$TAT^{-1} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix}, TB = \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix},$$

$$CT^{-1} = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix}.$$

The transfer function of (A, B, C, D) is given by

$$\begin{aligned} G[z] &= C(zI - A)^{-1}B + D \\ &= CT^{-1}(zI - TAT^{-1})^{-1}TB + D \\ &= C_{co}(zI - A_{co})^{-1}B_{co} + D. \end{aligned}$$

The poles of $G[z]$ are the eigenvalues of A_{co}, the controllable and observable modes of A.

The characteristic polynomial of a proper rational matrix $G[z]$ is the least common denominator of all minors of $G[z]$. Suppose $G[z]$ can be factored into a left coprime polynomial matrix fraction [29]:

$$G[z] = M^{-1}[z]N[z].$$

Then the characteristic polynomial of $G[z]$ is $\det(M[z])$.

Appendix B

Technical Lemmas

B.1 Lemma B.1 for Proving Theorem 2.2

Lemma B.1 Let $p_1, \ldots, p_m$ denote the unstable poles of P, with multiplicities $r_1, \ldots, r_m$ respectively. Then

$$\begin{aligned}
&\text{for } i = 1, \ldots, m, \\
&\quad \text{for } j = 0, \ldots, r_i - 1, \\
&\qquad \frac{d^j}{dz^j} G_1[z]\Big|_{z=p_i} = \frac{d^j}{dz^j} \frac{1}{H[z]}\Big|_{z=p_i}
\end{aligned} \tag{2.6}$$

is equivalent to

$$\begin{aligned}
&\text{for } i = 1, \ldots, m, \\
&\quad G_1[p_i] H[p_i] = 1 \text{ and} \\
&\quad \frac{d^j}{dz^j} G_1[z] H[z]\Big|_{z=p_i} = 0, \text{ for } j = 1, \ldots, r_i - 1.
\end{aligned} \tag{2.7}$$

□

Proof: $G_1[p_i]H[p_i] = 1$ is trivially equivalent to $G_1[p_i] = \frac{1}{H[p_i]}$.

Let $J[z] = \frac{1}{H[z]}$.

($\Rightarrow$) For $j = 1, \ldots, r_i - 1$,

$$\begin{aligned}
\left.\frac{d^j}{dz^j} G_1[z]H[z]\right|_{z=p_i} &= \sum_{k=0}^{j} \binom{j}{k} G_1^{(j-k)}[p_i]H^{(k)}[p_i] \\
&= \sum_{k=0}^{j} \binom{j}{k} J^{(j-k)}[p_i]H^{(k)}[p_i] \qquad \text{(by (2.6))} \\
&= \left.\frac{d^j}{dz^j} J[z]H[z]\right|_{z=p_i} = \left.\frac{d^j}{dz^j} 1\right|_{z=p_i} = 0.
\end{aligned}$$

($\Leftarrow$) For $j = 1, \ldots, r_i - 1$,

$$\begin{aligned}
0 = \frac{d^j}{dz^j} J[z]H[z] &= \sum_{k=0}^{j} \binom{j}{k} J^{(j-k)}[z]H^{(k)}[z] \\
&= J^{(j)}[z]H[z] + \sum_{k=1}^{j} \binom{j}{k} J^{(j-k)}[z]H^{(k)}[z] \\
\text{or} \quad J^{(j)}[z] &= \frac{1}{H[z]} \left[-\sum_{k=1}^{j} \binom{j}{k} J^{(j-k)}[z]H^{(k)}[z] \right]. \qquad \text{(B.1)}
\end{aligned}$$

We are going to prove by induction. First prove (2.7) implies (2.6) for $j = 1$:

$$\begin{aligned}
0 &= \left.\frac{d}{dz} G_1[z]H[z]\right|_{z=p_i} \qquad \text{(by (2.7))} \\
&= G_1^{(1)}[p_i]H[p_i] + G_1[p_i]H^{(1)}[p_i] \\
&= G_1^{(1)}[p_i]H[p_i] + \frac{1}{H[p_i]} H^{(1)}[p_i] \\
\Rightarrow \quad G_1^{(1)}[p_i] &= -\frac{H^{(1)}[p_i]}{H[p_i]^2} = \left.\frac{d}{dz}\frac{1}{H[z]}\right|_{z=p_i} = J^{(1)}[p_i].
\end{aligned}$$

Therefore (2.6) is true for $j = 1$. Now assume (2.6) is true for $j = 1, \ldots, l$ where $l < r_i - 1$, i.e.,

$$G_1^{(j)}[p_i] = J^{(j)}[p_i], \text{ for } j = 1, \ldots, l. \qquad \text{(B.2)}$$

We are to prove that (2.6) is true for $j = l + 1$. We have

$$\begin{aligned}
0 &= \frac{d^{(l+1)}}{dz^{(l+1)}} G_1[z]H[z]\Big|_{z=p_i} && \text{(by (2.7))} \\
&= \sum_{k=0}^{l+1} \binom{l+1}{k} G_1^{(l+1-k)}[p_i]H^{(k)}[p_i] \\
&= G_1^{(l+1)}[p_i]H[p_i] + \sum_{k=1}^{l+1} \binom{l+1}{k} G_1^{(l+1-k)}[p_i]H^{(k)}[p_i]
\end{aligned}$$

or

$$\begin{aligned}
G_1^{(l+1)}[p_i] &= \frac{1}{H[p_i]} \left[-\sum_{k=1}^{l+1} \binom{l+1}{k} G_1^{(l+1-k)}[p_i]H^{(k)}[p_i] \right] \\
&= \frac{1}{H[p_i]} \left[-\sum_{k=1}^{l+1} \binom{l+1}{k} J^{(l+1-k)}[p_i]H^{(k)}[p_i] \right] && \text{(by (B.2))} \\
&= J^{(l+1)}[p_i]. && \text{(by (B.1))}
\end{aligned}$$

Therefore (2.6) is true for $j = l + 1$. □

B.2 Proof of Lemma 5.1 – Equivalence of Parameterizations $\mathcal{A}$ and $\mathcal{B}$

This proof is very similar to the corresponding SISO proof in [1]. The difference lies with Lemma B.2, given below, which proves a MIMO result that has been proven only for the SISO case in [1].

($\mathcal{B} \subseteq \mathcal{A}$): Fix $(G_1, G_2) \in \mathcal{B}$. Introduce minimal state-space realizations for each element in Figure 2.1: $P[z] = (A_P, B_P, C_P, D_P)$ (with initial state x_{P0}), $H[z] = (A_H, B_H, C_H, D_H)$ (with initial state x_{H0}), $G_1[z] = (A_1, B_1, C_1, D_1)$ (with initial state x_{10}), and $G_2[z] = (A_2, B_2, C_2, D_2)$ (with initial state x_{20}).

Taking into the effects of the initial states, we can write

$$\begin{aligned}
y[k] &= \mathcal{Z}^{-1}\{P[z]U[z] + C_P(zI - A_P)^{-1}x_{P0}\} \\
\hat{y}[k] &= \mathcal{Z}^{-1}\{(G_2[z] + G_1[z]H[z]P[z])U[z] \\
&\qquad + C_2(zI - A_2)^{-1}x_{20} + C_1(zI - A_1)^{-1}x_{10} \\
&\qquad + G_1[z]C_H(zI - A_H)^{-1}x_{H0} + G_1[z]H[z]C_P(zI - A_P)^{-1}x_{P0}\},
\end{aligned}$$

implying

$$\begin{aligned}
e[k] &= \hat{y}[k] - y[k] \\
&= \mathcal{Z}^{-1}\{(G_2[z] + G_1[z]H[z]P[z] - P[z])U[z] \\
&\qquad + C_2(zI - A_2)^{-1}x_{20} + C_1(zI - A_1)^{-1}x_{10} \\
&\qquad + G_1[z]C_H(zI - A_H)^{-1}x_{H0} + (G_1[z]H[z] - I)C_P(zI - A_P)^{-1}x_{P0}\}. \qquad \text{(B.3)}
\end{aligned}$$

By assumption, $G_2[z] + G_1[z]H[z]P[z] = P[z]$ and A_1, A_2, A_H are all stable. Since $G_1[z]$ is stable, $G_1[z]C_H(zI - A_H)^{-1}$ is stable. The following lemma shows that $(G_1[z]H[z] - I)C_P(zI - A_P)^{-1}$ is also stable.

Lemma B.2 Assume $H[z]$ is stable. If $G_1[z] \in M(\mathcal{S}), G_2[z] \in M(\mathcal{S}), G_2[z] = P[z] - G_1[z]H[z]P[z]$, then $V[z] = (G_1[z]H[z] - I)C_P(zI - A_P)^{-1}$ is stable. □

Proof: Let $P[z] = (A_P, B_P, C_P, D_P)$, $H[z] = (A_H, B_H, C_H, D_H)$, and $G_1[z] = (A_1, B_1, C_1, D_1)$, all minimal realizations. A state-space realization of $G_2[z] = P[z] - G_1[z]H[z]P[z]$ is

$$(A_2, B_2, C_2, D_2) = \left(\begin{bmatrix} A_P & 0 & 0 \\ B_H C_P & A_H & 0 \\ D_H C_P & B_1 C_H & A_1 \end{bmatrix}, \begin{bmatrix} B_P \\ B_H D_P \\ B_1 D_H D_P \end{bmatrix}, \begin{bmatrix} C_P - D_1 D_H C_P & -D_1 C_H & -C_1 \end{bmatrix}, \begin{bmatrix} D_P - D_1 D_H D_P \end{bmatrix} \right). \qquad \text{(B.4)}$$

Since A_2 is block-diagonal and A_H, A_1 are Hurwitz, all the unstable modes of A_2 come from A_P and are in fact the unstable poles of $P[z]$. Let p be any unstable pole of $P[z]$. The stability of $G_2[z]$ implies that all the poles of $G_2[z]$ are stable and so are the controllable and observable modes of A_2. Therefore p must be among the uncontrollable or unobservable modes of A_2. Since (A_P, B_P, C_P, D_P) is assumed to be minimal, (A_P, B_P) is controllable or $\begin{bmatrix} A_P - pI & B_P \end{bmatrix}$ has full rank. So

$$\begin{bmatrix} A_2 - pI & B_2 \end{bmatrix} = \begin{bmatrix} A_P - pI & 0 & 0 & B_P \\ B_H C_P & A_H - pI & 0 & B_H D_P \\ D_H C_P & B_1 C_H & A_1 - pI & B_1 D_H D_P \end{bmatrix}$$

has full rank since p is not a mode of the Hurwitz A_H or A_1. $\begin{bmatrix} A_2 - pI & B_2 \end{bmatrix}$ has full rank implies that p is among the controllable modes of A_2. Therefore we can conclude that p must be among the unobservable modes of A_2 and so

$$\begin{bmatrix} A_2 - pI \\ C_2 \end{bmatrix} = \begin{bmatrix} A_P - pI & 0 & 0 \\ B_H C_P & A_H - pI & 0 \\ D_H C_P & B_1 C_H & A_1 - pI \\ C_P - D_1 D_H C_P & -D_1 C_H & -C_1 \end{bmatrix}$$

does not have full rank.

A state-space realization of $V[z] = (G_1[z]H[z] - I)C_P(zI - A_P)^{-1}$ is

$$(A_V, B_V, C_V, D_V) = \left(\begin{bmatrix} A_P & 0 & 0 \\ B_H C_P & A_H & 0 \\ D_H C_P & B_1 C_H & A_1 \end{bmatrix}, \begin{bmatrix} I \\ B_H D_P \\ B_1 D_H D_P \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} C_P - D_1 D_H C_P & -D_1 C_H & -C_1 \end{bmatrix}, \begin{bmatrix} D_P - D_1 D_H D_P \end{bmatrix} \right).$$

which is identical to (B.4), except with B_P replaced by I. So we can also conclude p is also an unobservable mode of A_V. Hence, all the unstable modes of A_V are unobservable and so $V[z]$ is stable. □

We conclude that each term in (B.3) is stable. Thus for any x_{P0}, x_{H0}, x_{10}, x_{20}, and any signal u, $e[k] \to 0$ as $k \to \infty$. Hence $(G_1, G_2) \in \mathcal{A}$.

$(\mathcal{A} \subseteq \mathcal{B})$: Fix $(G_1, G_2) \in \mathcal{A}$. The expression (B.3) for $e[k]$ still holds. Since for any x_{P0}, x_{H0}, x_{10}, x_{20}, and any signal u, $e[k] \to 0$ as $k \to \infty$, the following must be true:

- $G_2[z] + G_1[z]H[z]P[z] = P[z]$ (to set the coefficient of $U[z]$ in (B.3) zero),
- $G_1[z]$ is stable (to ensure the coefficient of x_{10} in (B.3) stable),
- $G_2[z]$ is stable (to ensure the coefficient of x_{20} in (B.3) stable).

These three conclusions imply that $(G_1, G_2) \in \mathcal{B}$. □

B.3 Proof of Lemma 5.3 – Equivalence of Parameterizations $\mathcal{B}$ and $\mathcal{D}$

The following lemma is needed:

Lemma B.3 $P[z]M[z]$ is stable. □

Proof: One realization of $P[z]M[z]$ is

$$(A_{PM}, B_{PM}, C_{PM}, D_{PM}) = \left(\begin{bmatrix} A + BK & 0 \\ B_P K & A_P \end{bmatrix}, \begin{bmatrix} B \\ B_P \end{bmatrix}, \begin{bmatrix} D_P K & C_P \end{bmatrix}, D_P \right) \qquad \text{(by (A.3))}$$

$$= \left(\begin{bmatrix} A_P + B_P K_1 & B_P K_2 & 0 \\ B_H C_P + B_H D_P K_1 & A_H + B_H D_P K_2 & 0 \\ B_P K_1 & B_P K_2 & A_P \end{bmatrix}, \begin{bmatrix} B_P \\ B_H D_P \\ B_P \end{bmatrix}, \begin{bmatrix} D_P K_1 & D_P K_2 & C_P \end{bmatrix}, D_P \right)$$

where $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$. We can apply the state-transformation

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -I & 0 & I \end{bmatrix} \quad \left(\text{with } T^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & I \end{bmatrix} \right)$$

to obtain a new realization

$$(TA_{PM}T^{-1}, TB_{PM}, C_{PM}T^{-1}, D_{PM})$$

where

$$\begin{aligned}
TA_{PM}T^{-1} &= T \begin{bmatrix} A_P + B_P K_1 & B_P K_2 & 0 \\ B_H C_P + B_H D_P K_1 & A_H + B_H D_P K_2 & 0 \\ B_P K_1 & B_P K_2 & A_P \end{bmatrix} T^{-1} \\
&= \begin{bmatrix} A_P + B_P K_1 & B_P K_2 & 0 \\ B_H C_P + B_H D_P K_1 & A_H + B_H D_P K_2 & 0 \\ -A_P & 0 & A_P \end{bmatrix} T^{-1} \\
&= \begin{bmatrix} A_P + B_P K_1 & B_P K_2 & 0 \\ B_H C_P + B_H D_P K_1 & A_H + B_H D_P K_2 & 0 \\ 0 & 0 & A_P \end{bmatrix} = \begin{bmatrix} A + BK & 0 \\ 0 & A_P \end{bmatrix}, \\
TB_{PM} &= T \begin{bmatrix} B_P \\ B_H D_P \\ B_P \end{bmatrix} = \begin{bmatrix} B_P \\ B_H D_P \\ 0 \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \\
C_{PM}T^{-1} &= \begin{bmatrix} D_P K_1 & D_P K_2 & C_P \end{bmatrix} T^{-1} = \begin{bmatrix} D_P K_1 + C_P & D_P K_2 & C_P \end{bmatrix} \\
&= \begin{bmatrix} D_P K + \begin{bmatrix} C_P & 0 \end{bmatrix} & C_P \end{bmatrix}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
P[z]M[z] &= C_{PM}T^{-1}(zI - TA_{PM}T^{-1})^{-1}TB_{PM} + D_{PM} \\
&= \begin{bmatrix} D_P K + \begin{bmatrix} C_P & 0 \end{bmatrix} & C_P \end{bmatrix} \begin{bmatrix} zI - (A + BK) & 0 \\ 0 & zI - A_P \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} + D_P \\
&= \begin{bmatrix} D_P K + \begin{bmatrix} C_P & 0 \end{bmatrix} & C_P \end{bmatrix} \begin{bmatrix} (zI - (A + BK))^{-1} & 0 \\ 0 & (zI - A_P)^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} + D_P \\
&= \begin{bmatrix} D_P K + \begin{bmatrix} C_P & 0 \end{bmatrix} \end{bmatrix} (zI - (A + BK))^{-1} B + D_P \\
&\in M(\mathcal{S})
\end{aligned}$$

since $A + BK$ is Hurwitz. □

The machinery is now in place to prove Lemma 5.3:

($\mathcal{D} \subseteq \mathcal{B}$): Fix $(G_1, G_2) \in \mathcal{D}$. By Lemma B.3, $PM \in M(\mathcal{S})$. Therefore $G_1 \in M(\mathcal{S})$ and $G_2 \in M(\mathcal{S})$. Moreover

$$\begin{aligned}
T_{u\hat{y}} &= G_2 + G_1 HP \\
&= PMY - Q\tilde{N} + (PMX + Q\tilde{M})HP && \text{(by (5.11))} \\
&= PMY - Q\tilde{N} + PMXHP + Q\tilde{M}HP \\
&= PMY - Q\tilde{N} + PMXNM^{-1} + Q\tilde{M}\tilde{M}^{-1}\tilde{N} && \text{(by (5.9))} \\
&= PM(YM + XN)M^{-1} \\
&= PMIM^{-1} = P && \text{(by (5.10))} \\
&= T_{uy}.
\end{aligned}$$

Therefore $(G_1, G_2) \in \mathcal{B}$

($\mathcal{B} \subseteq \mathcal{D}$) [30]: Fix $(G_1, G_2) \in \mathcal{B}$. Let

$$Q = \begin{bmatrix} G_2 & G_1 \end{bmatrix} \begin{bmatrix} -\tilde{X} \\ \tilde{Y} \end{bmatrix}. \tag{B.5}$$

So $Q \in M(\mathcal{S})$. From

$$G_2 + G_1 HP = P,$$

we can see that

$$\begin{aligned}
G_2 + G_1 NM^{-1} &= P && \text{(by (5.9))} \\
\text{or} \quad G_2 M + G_1 N &= PM. && \text{(B.6)}
\end{aligned}$$

Therefore combining (B.5) and (B.6), we have

$$\begin{bmatrix} PM & Q \end{bmatrix} = \begin{bmatrix} G_2 & G_1 \end{bmatrix} \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix}$$

and so

$$\begin{aligned}
\begin{bmatrix} G_2 & G_1 \end{bmatrix} &= \begin{bmatrix} PM & Q \end{bmatrix} \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} PM & Q \end{bmatrix} \begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} && \text{(by (5.10))} \\
&= \begin{bmatrix} PMY - Q\tilde{N} & PMX + Q\tilde{M} \end{bmatrix}.
\end{aligned}$$

Hence $(G_1, G_2) \in \mathcal{D}$.

$\square$

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