Near Optimal $\mathcal{H}_\infty$ Performance in the Decentralized Setting

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis we consider the use of a linear periodic controller (LPC) for the control of linear time-invariant (LTI) plants in the decentralized setting with an H_∞-performance criterion in mind. If a plant has an unstable decentralized fixed mode (DFM), it is well known that no decentralized LTI controller can stabilize it, let alone provide good performance, which is why we turn to more complicated controllers. Here we show that if the graph associated with the plant is strongly connected and certain technical conditions on the relative degree hold, then we can design a decentralized LPC to provide a level of H_∞ performance as close as desired to the centralized H_∞-optimal performance; this will be the case even if the plant has an unstable decentralized fixed mode (DFM). The proposed controller in each channel consists of a sampler, a zero-order-hold, and a discrete-time linear periodic compensator, which makes it easy to implement.

Acknowledgements

I would like to express my sincere gratitude to my supervisor Professor Daniel Miller for the continuous support for my Masters study and the related research, for his unending patience, motivation, and immense knowledge and without whom this research would not have been possible. His guidance helped me in all the time of research and writing of this thesis. I could not have asked for a better arrangement.

I would like to thank my family for their unlimited love, never ending support and the motivation. Also I am grateful for my supportive friends, who made this journey enjoyable.

Dedication

This is dedicated to my family and friends.

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Chapter 1

Introduction

1.1 Background

The information flow of a system plays a vital role in the design of a feedback controller. If the information flow constraints imposed that each disjoint subset of inputs of the system has access to only one subset of the outputs and that the information fed back through the local input only depends on the corresponding subset of outputs, then the system is said to be in a decentralized setting and the overall feedback strategy is called decentralized control. The information flow constraints may arise due to a geographical separation of sensors and actuators, such as in a large power grid, or in a large chemical plant. Of course, there are more complicated examples of restrictions on information flow, e.g., the input u_i depends not only on y_i but also on a delayed version of y_{i-1}, which may arise in specific application problems such as controlling a platoon of vehicles or in networked control, e.g. see [27], [23], and some of the cases considered in [24]. However here we are considering only the classical constraint on information flow: input u_i depends solely on output y_i.

In the context of this classical decentralized control problem, a linear time invariant (LTI) plant can be stabilized using a decentralized LTI controller if and only if the system does not possess any unstable DFMs [7]; however, it has long been known that this is not the case when using time-varying controllers. In [3] it has been proven that certain time-invariant systems that cannot be stablized by a LTI decentralized controller can be stabilized by a decentralized time-varying controller. Moreover the author of [28] showed that sample-and-hold feedback strategy can be applied to eliminate some unstable-DFMs with certain properties; the author of [29] expands the idea presented in [28] and [22] and proposed a methodology to precisely identify the DFMs that cannot be moved by sampling.

In [11], the authors propose decentralized control strategies to stabilize linear-periodically time varying plants and it turns out that these decentralized controllers are linear periodic in nature.

It is worth noting that the structure of the directed graph associated with the system plays a significant role in designing a decentralized controller. In [1] the authors show that by applying a decentralized generalized sample data hold functions (GSHF)(See [10]) to a LTI continuous time plant, the associated graph can be suitably modified to form a hierarchical system model, for which the design of the controller is substantially simplified. Indeed, [9] provides an approach to classify DFMs into those which are truly fixed and those which can be moved using a sufficiently sophisticated controller by inspecting the graph associated with the system; an associated 'quotient system' is defined and it is proved that its DFMs, labelled QDFMs, are exactly those DFMs of the original system which are immoveable by any form of nonlinear time-varying (NLTV) feedback.

In this document we are interested not only in closed-loop stability but also in H_∞-optimal (or near-optimal) performance. Carrying out optimal controller design in the decentralized setting is difficult, and has only been solved in special situations. These include:

(i) that of [5], in which the the plant has no unstable DFMs and is minimum phase;

(ii) that of [23], where a variety of cases are considered, including a classical one in which there is a 'triangular constraint' on information flow;

(iii) that of [24], which discusses the central notion of quadratic invariance, and considers which provides a detailed historical account of work on decentralized optimal control, and where it is argued that the underlying concept in most of these approaches is that of 'quadratic invariance', and computational techniques are provided; a variety of cases, including some of the classical kind, with followup work by the same author given in [26], [13];

(iv) that of [25], in which a general class of systems having a poset structure is considered, and an H_2-optimal decentralized LTI controller is provided;

(v) that of [18], wherein a centrally controllable and observable plant with an associated strongly connected graph is considered, and a linear periodic controller is designed which provides a level of performance as close as desired to the centralized optimal LQR performance.

We conclude that (a) the optimal decentralized control problem is difficult, and (b) in all of the cases listed above except the last one, either the optimal performance is achieved by an LTI controller or only LTI controllers are considered.

In this document our goal is to extend the approach of [18] on providing (near) LQR-optimal performance discussed in (v) above to providing the more demanding goal of providing (near) H_∞-optimal performance. In [18] it was shown, under reasonably general conditions, that LQR-optimal centralized performance can be recovered by a decentralized linear periodic controller. In that paper the assumptions are that the plant is centrally controllable and observable and the graph associated with the system is strongly connected. It is important to note that the class of systems considered there does not have the quadratic invariance property of [24] nor the poset stucture of [25]; however, it turns out that the three required system properties are generic, which demonstrates that the result is typical and not atypical (see Proposition 1 of [18]). The H_∞ performance objective is much more demanding than the LQR performance objective. Of course, since we will be using a linear periodic controller we can no longer use the frequency domain interpretation of the H_∞ performance index; instead we adopt the natural time-domain interpretation of the system gain in the induced $2-$norm sense. In contrast to [18] but in accordance with a typical H_∞ optimal problem, both inputs and outputs of the plant are partitioned into two different classes: a set of control inputs, a set of disturbance(reference) inputs, a set of outputs to be controlled and a set of measured outputs. This added complexity of the problem requires some additional assumptions, e.g., for our approach to work, certain relative degree conditions are imposed.

Our approach is motivated by the earlier work on the robust control of linear (possibly time-varying) centralized systems using linear periodic controllers [20], [15], and [19] and [21] as well as the earlier work on decentralized LQR performance [18]. The first step is to compute an LTI centralized controller K_{cen} which provides a level of performance as close as desired to the centralized H_∞-optimal performance. Next, we construct a linear periodic sampled-data decentralized controller which emulates the behaviour of the afore-mentioned LTI centralized controller K_{cen}; we impose a relative degree constraint to ensure that the approach works. We make use of the strongly connected assumption to pass information between channels, and we end up with a linear periodic decentralized controller parametrized by the period $T > 0$; we show that as $T \to 0$, the closed loop performance tends toward that provided by the centralized LTI controller K_{cen}.

The periodic controller works as follows: with p channels, we place a copy of K_{cen} in one of the channels - we somewhat arbitrarily choose the last one. In each channel i we maintain an estimate of the corresponding output of the copy of K_{cen}, which we label $\hat{\sqcap}_i$. While at all times applying our estimate $\hat{\sqcap}_i$ in channel i, during the first part of each period we probe the system with y_1, $y_2, ..., y_{p-1}$, so that in channel p we can estimate these quantities, which are then coupled with the measurement of y_p to yield an estimate of y, which can then be used to drive the copy of K_{cen} to update the 'near-optimal control signal'. In the last part of the period we probe the system in the p^{th} channel to pass this information to the first $p-1$ channels. The relative degree assumption plays a critical role in ensuring that the probing is successful.

We emphasize that the proof for the H_∞ problem considered here is significantly different from the earlier work on the LQR problem [18]: (i) first of all, the centralized controller K_{cen} is now dynamic output feedback rather than static state feedback, (ii) second of all, now the problem is tracking rather than stability, which significantly complicates the probing and estimation, which gives rise to the need for relative degree assumptions, and (iii) now we need to use input-output operator norms rather than the norm of signals.

1.2 Mathematical Notation and Preliminaries

Before proceeding further, we will provide an overview of the mathematical tools and the notations that are being used throughout the document. The natural numbers, real numbers, non-negative real numbers, integers and non-negative integers are denoted by $\mathbf{N}, \mathbf{R}, \mathbf{R}^+, \mathbf{Z}$ and $\mathbf{Z}^+$ respectively. The Euclidean norm is used for the vectors and the corresponding induced norm is used for the matrices. The norm of a vector or matrix is denoted by $\|.\|$. The Lebesgue space $\mathcal{L}_2(\mathbf{R}^n)$ denotes the set of $\mathbf{R}^n$ valued, Lebesgue measurable square-integrable signals x on $[0\ \infty)$ for which

$$\|x\|_2 := \left[\int_0^\infty \|x(\tau)\|^2 \,\mathrm{d}\tau\right]^{\frac{1}{2}} < \infty.$$

On occasion we wish to measure the size of a signal on an interval: with $x \in \mathcal{L}_2(\mathbf{R}^n)$ and $t_2 > t_1 \geq 0$, we define

$$\left\|x_{[t_1,t_2]}\right\|_2 := [\int_{t_1}^{t_2} \|x(\tau)\|^2 \,\mathrm{d}\tau]^{\frac{1}{2}}.$$

Throughout the thesis, we often use the size of the signal measured over a period of $T > 0$; with $x \in \mathcal{L}_2(\mathbf{R}^n)$ and $k \in \mathbf{Z}$ we define

$$\|x_k\|_2 := [\int_{kT}^{(k+1)T} \|x(\tau)\|^2 \,\mathrm{d}\tau]^{\frac{1}{2}}.$$

A function $f : \mathbf{R}^+ \to \mathbf{R}^{n\times m}$ is said have an order of T^j and is written as $f = \mathcal{O}(T^j)$, if there exist constants $c_1 > 0$ and $T_1 > 0$ such that

$$\|f(T)\| \leq c_1 T^j, \;\; T \in (0, T_1).$$

Here we consider the order notation for small $T \in (0, T_1)$. The order notation has several properties that will be used in later sections. If functions $f_1, f_2 : \mathbf{R}^+ \to \mathbf{R}^{n\times m}$ are orders of T^j and T^i respectively, i.e., $f_1 = \mathcal{O}(T^j)$ and $f_2 = \mathcal{O}(T^i)$, then the sum property implies that

$$f_1 + f_2 = \mathcal{O}(T^j + T^i)$$

and the product property implies that

$$f_1 f_2 = \mathcal{O}(T^{j+i}).$$

Moreover, the order notation is used in conjunction with arithmetic operations. For example if we write $f_1 = f_2 + \mathcal{O}(T^k)$ for some $k \in \mathbf{R}$, then we mean that there exist constants $c_k > 0$ and $T_k > 0$ such that

$$\|f_1(T) - f_2(T)\| \leq c_k T^k, \; T \in (0, T_k).$$

Positive definite matrices are very useful when applying Lyaponov's direct method of stability analysis. Let $M \in \mathbf{R}^{n\times n}$ be a symmetric matrix partitioned as $M := \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ with $A \in \mathbf{R}^{p\times p}$, $B \in \mathbf{R}^{p\times q}$, and $C \in \mathbf{R}^{q\times q}$. The Schur complement S_A of block matrix A is written as

$$S_A = C - B^T A^{-1} B,$$

and the Schur complement S_C of block matrix C in M is given by

$$S_C = A - BC^{-1}B^T.$$

From [31] the matrix M is positive definite if and only if A and its Schur complement S_A are positive definite:

$$M \succ 0 \Leftrightarrow A \succ 0, S_A \succ 0.$$

Similarly both C and its Schur complement S_C are positive definite if and only if M is positive definite.

Let $\mathcal{G} \subset \mathbf{R}^n$ be a euclidean subset. A continuous function $h : \mathcal{G} \to \mathcal{G}$ has a fixed point if there exist $x \in \mathcal{G}$ such that $h(x) = x$. The Brouwer fixed-point theorem states that any continuous function h will always have at least one fixed point if $\mathcal{G}$ is convex and compact. We invoke this idea to find a root of a function in a later chapter.

A single-input-single-output(SISO) system with a state space representation of

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned}$$

said to have a relative degree ($rel.deg[c(sI-A)^{-1}b]$) of η, if

$$cA^ib = 0 \ \ for \ i = 0, 1, \cdots, \eta - 2$$

and

$$cA^{\eta-1}b \neq 0.$$

We assign a relative degree of ∞ if the transfer function is identically zero. We extend the idea to a multi-input-multi-output(MIMO) system by defining its relative degree to be the smallest relative degree of all the possible SISO subsystems.

1.3 Organization of the Thesis

The thesis is organized as follows. Chapter 2 is partitioned into three sub-sections. In the first sub-section we formulate the decentralized setup and discuss the assumptions. In the second sub-section, we discuss the control problem and provide a high level description of the approach. Finally in the last sub-section we discuss a regularization step which simplifies the controller description. We partition Chapter 3 into two sub-sections. First we present a preliminary result on estimation, which is used to motivate the controller description presented in the next subsection. Chapter 4 provides the detailed proof in threefold: first we start off with a lengthy Preamble followed by a stability analysis for the the closed loop system and finally we tie in our objective by analysing the performance. Chapter 5 illustrates the simulation results using the proposed controller in a simple plant with a DFM and a non-minimum phase zero. In Chapter 6 we provide a comprehensive summary of this thesis and we recommend potential future expansions for this research. In the Appendix we provide proofs for the preliminary Lemmas that are been used in the main proof.

Chapter 2

Problem Formulation

2.1 The Setup

The plant model P is modelled as follows:

$$\left.\begin{array}{rcl} \dot{x} &=& Ax + \sum_{i=1}^{p} B_i u_i + Er, \quad x(t_0) = x_0 \\ z &=& C_1 x + \sum_{i=1}^{p} D_{11}^i u_i + D_{12} r \\ y_i &=& C_2^i x, \quad i = 1, ..., p, \end{array}\right\} \tag{2.1}$$

with $x(t) \in \mathbf{R}^n$ the state, $u_i(t) \in \mathbf{R}^{m_i}$ the i^{th} control input, $r(t) \in R^{\mu}$ the disturbance (or reference) signal, $z(t) \in \mathbf{R}^{\rho}$ the output to be controlled and $y_i(t) \in \mathbf{R}^{l_i}$ the measured i^{th} output for $i = 1, ..., p$; we set $m = \sum_{i=1}^{p} m_i$, and $l = \sum_{i=1}^{p} l_i$. Associated with this model are

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}, \quad C_2 := \begin{bmatrix} C_2^1 \\ \vdots \\ C_2^p \end{bmatrix}$$

$$B := \begin{bmatrix} B_1 & \cdots & B_p \end{bmatrix}, \quad D_{11} := \begin{bmatrix} D_{11}^1 & \cdots & D_{11}^p \end{bmatrix}.$$

Notice that it is implicitly assumed that the disturbance $r(t)$ does not appear directly in the measured output $u(t)$ ($D_{21} = 0$ in the centralized context). This assumption is not overly restrictive as with any sample-data controller, it is the norm to pass the signal through an anti-aliasing filter prior to sampling. On the other hand this assumption is crucial for the implementation of the controller.

Another implicit assumption is that in the decentralized context the channels are localized, i.e. u_i depends solely on y_i, regardless of the type (LTI, linear time varying (LTV), or non-linear time varying (NTLV)) of the controller used. This is assumption is not too restrictive as it possible to reconfigure input-output map so that the resulting channels are localized. For example, consider a plant with 4 channels and the following information flow characteristics.

- u_1 has access to y_1 and y_3
- u_2 has access to y_2 and y_3
- u_3 has access to y_3
- u_4 has access to y_1, y_2 and y_4.

A set of new outputs can be implemented to ensure the system is fully decentralized. Consider the outputs $\bar{y}_1 := \begin{bmatrix} y_1 \\ y_3 \end{bmatrix}$, $\bar{y}_2 := \begin{bmatrix} y_2 \\ y_3 \end{bmatrix}$, $\bar{y}_3 := y_3$ and $\bar{y}_4 := \begin{bmatrix} y_1 \\ y_2 \\ y_4 \end{bmatrix}$. With these outputs the plant is completely decentralized as all four channels are localized.

The overall objective of this thesis to design a decentralized controller to closely match the performance of the centralized $\mathcal{H}_\infty$ optimal controller. In order to design a $\mathcal{H}_\infty$ optimal controller using the method of [8], it is required that

Assumption 1: (A, B) is stablizable and (C_2, A) is detectable;

Assumption 2: (C_1, A) is detectable.

These technical assumptions play a pivotal role in designing the centralized optimal controller.

Before we talk about performance, it is crucial to discuss the issues surrounding closed-loop stability in the decentralized setting. The notion of DFM was introduced in [7]; the goal is to identify those eigenvalues that are immovable using LTI feedback which respects the information flow constraints. It has been shown in [7] that an eigenvalue is movable using dynamic output feedback if and only if it is movable using static output feedback. In the decentralized setting the set of static output feedback gains is given by:

$$\mathcal{K}_{dec}(A; B_1, \cdots, B_p; C_2^1, \cdots, C_2^p) := \{K \in \mathbf{R}^{m \times l} :$$

$$K = diag\{K_1, \cdots, K_p\}|K_i \in \mathbf{R}^{m_i \times l_i}\},$$

or simply $\mathcal{K}_{dec}$ for short is used to specify the unmovable eigenvalues known as DFM.

Definition 1 The decentralized fixed modes of (2.1) are given by

$$\cap_{K \in \mathcal{K}_{dec}} \sigma(A + BKC_2).$$

Remark 1 *As mentioned in the Introduction, for some systems, some of the DFMs are moveable using non-LTI control laws, e.g. [3], [28], [22], [29] and [11]. The DFMs which are immoveable by any NLTV control law are the QDFMs - see [9] and [17].*

Graph theory can be used to study decentralized systems, and was used in [9] to identify QDFMs. Following [4], one can build a directed graph of the plant (2.1) as follows: there are p nodes representing the p control agents and p sensor agents, with an edge from node i to node j iff $C_2^j(sI - A)^{-1}B_i \neq 0$. Recall that a directed graph is said to be **strongly connected** if there is path from every node to every other node in the graph.

Remark 2 *It was proven in [9] that if the plant is centrally controllable and observable and the associated graph is strongly connected, then the system has no QDFMs (though it may have DFMs).*

Hence, we can be assured that the system will not have any unstable QDFMs if we insist that the directed graph associated with the plant is strongly connected. Hence we impose

Assumption 3: The directed graph corresponding to (2.1) is strongly connected.

We will be using ideas from the paper on decentralized control in the LQR setting [18] as well as the earlier work on simultaneous stabilization in the H_∞ context [16]. We will be carrying out probing at each plant input u_i and measuring the response at each plant output y_j; for this to work, we need to ensure that the effect of the probe overwhelms the effect of the disturbance signal r. We do so by imposing a relative degree assumption. The relative degree of a non-zero siso transfer function is the degree of the denominator less the degree of the numerator, with the understanding that if the transfer function is identically zero then the relative degree is defined to be ∞; for a multivariable transfer function the relative degree is the smallest relative degree of the scalar elements, with the understanding that it is infinity if the multivariable transfer function is identically zero. For our set-up we are interested in two distinct relative degrees, one associated with the control input and one associated with the disturbance input:

$$\eta_1 := \max_{i,j \in \{1, \cdots p\}} rel.deg.(C_2^i (sI - A)^{-1} B_j),$$
$$\eta_2 := rel.deg.(C_2 (sI - A)^{-1} E).$$

Assumption 4: $\eta_2 > \eta_1$.

The motivation of this assumption is as follows. Suppose for simplicity that $l_i = m_i = 1$, and that at time 0 we probe the input u_i with a small test signal ϕ on a small interval $[0, h]$, with all other inputs zero and with the plant initial condition equal to zero. With $\hat{\eta}_1$ the relative degree of $C_2^j (sI - A)^{-1} B_i$ and $\hat{\eta}_2$ the relative degree of $C_2^j (sI - A)^{-1} E$, after the small time period h we have

$$\begin{aligned} y_j(h) &= \int_0^h C_2^j e^{A(h-\tau)} [B_i \phi \; + E r(\tau)] \, d\tau \\ &\approx C_2^j A^{\hat{\eta}_1 - 1} B_i \frac{h^{\hat{\eta}_1}}{\hat{\eta}_1 !} \phi + \mathcal{O}(h^{\hat{\eta}_2 - 0.5}) (\int_0^h \|r(\tau)\|^2 d\tau)^{1/2}. \end{aligned}$$

From the definition of $\hat{\eta}_1$ we have that $C_2^j A^{\hat{\eta}_1 - 1} B_i \neq 0$, so

$$\frac{1}{C_2^j A^{\hat{\eta}_1 - 1} B_i} \frac{\hat{\eta}_1 !}{h^{\hat{\eta}_1}} y_j(h) \approx \phi + \mathcal{O}(h^{\hat{\eta}_2 - \hat{\eta}_1 - 1/2}) (\int_0^h \|r(\tau)\|^2 d\tau)^{1/2}; \tag{2.2}$$

but $\hat{\eta}_1 \leq \eta_1$ and $\hat{\eta}_2 \geq \eta_2$, which means that

$$\hat{\eta}_2 - \hat{\eta}_1 - 1/2 \geq \eta_2 - \eta_1 - 1/2 \geq 1/2.$$

This means that the LHS of (2.2) provides a good estimate of ϕ, even in the presence of the disturbance r. Of course, the above is more complicated in the case of non-scalar inputs and output, non-zero initial conditions, and when one is trying to carry out control at the same time as estimation. These issues will be dealt with in due course.

2.2 The Problem

The objective of this thesis is to design a linear periodic controller that not only provides closed loop stability, but also guarantees near optimal $\mathcal{H}_\infty$ performance in the decentralized setting. To proceed, we need to make precise the notion of stability and $\mathcal{H}_\infty$ performance. We first consider a centralized LTI controller K_{cen} described by

$$\begin{array}{rcl} \dot{v} & = & Fv + Gy,\ v(t_0) = v_0 \in \mathbf{R}^\ell \\ u & = & Hv + Jy. \end{array} \tag{2.3}$$

By closed-loop stability we mean that, if $r(t)$ is identically zero, then for every t_0 and every set of initial conditions x_0 and v_0, we have that $\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \to 0$ as $t \to \infty$. With $t_0 = 0$, $x_0 = 0$, and $v_0 = 0$, we let $\mathcal{F}(P, K_{cen})$ denote the closed-loop map from $r \in \mathcal{L}_2(R^\mu)$ to $z \in \mathcal{L}_2(R^\rho)$. The classical H_∞-optimal control problem is to find the LTI controller K_{cen} which stabilizes P and minimizes the cost $\|\mathcal{F}(P, K_{cen})\|$. In general the minimizing controller does not exist, but one can obtain an LTI controller which provides a level of performance as close to optimality as desired [8]. So given a near optimal **centralized LTI controller** K_{cen}, our goal here is to obtain a **decentralized controller** K_{dec} which provides a level of performance close to this; to achieve this we use a linear periodic sampled-data controller.

Here we consider decentralized sampled-data controllers K_{dec} of the form

$$\left.\begin{array}{rcl} \psi_i[k+1] & = & L_i[k]\psi_i[k] + M_i[k]y_i(kh), \\ & & \psi_i[0] = \psi_{i_0} \in \mathbf{R}^{\bar{l}_i}, \\ u_i(kh+\tau) & = & Q_i[k]\psi_i[k] + R_i[k]y_i(kh), \\ & & \tau \in [0, h) \end{array}\right\} \tag{2.4}$$

with the controller gains L_i, M_i, Q_i, and R_i periodic of period $q \in \mathbf{N}$ for every $i \in \{1, 2, ..., p\}$; the period of the overall controller is $T := qh$, and we associate this system with $((L_i, M_i, Q_i, R_i), i = 1, ..., p; T; q)$. Note that for each i, (2.4) can be implemented with a sampler, a zero-order-hold, and an $\bar{l}_i^{th}$ order periodically time-varying discrete-time system of period q. We define the augmented controller state as $\psi[k] := \begin{bmatrix} \psi_1[k] \\ \vdots \\ \psi_p[k] \end{bmatrix}$ and $\psi[0] := \psi_0$.

The state of the closed loop-system is a combination of discrete and continuous states, defined by

$$x_{sd}(t) := \begin{bmatrix} x(t) \\ \psi[k] \end{bmatrix}, \quad t \in [kh, (k+1)h);$$

the dimension of ψ is $\bar{l} := \bar{l}_1 + \bar{l}_2 + \cdots + \bar{l}_p$. Now we make precise our notion of stability.

Definition 2 The sampled-data controller (2.4) **exponentially stabilizes** (2.1) if there exist constants $\gamma > 0$ and $\lambda < 0$ so that, with $t_0 = 0$ and $r = 0$, for every $x_0 \in \mathbf{R}^n$ and $\psi_0 \in \mathbf{R}^{\bar{l}}$, we have

$$\|x_{sd}(t)\| \leq \gamma e^{\lambda t}\|x_{sd}(0)\|, \ \ t \geq 0.$$

Suppose that the sampled-data controller (2.4) labelled K_{dec} exponentially stabilizes (2.1); then with $t_0 = 0$, $x(0) = 0$ and $\psi[0] = 0$, we let $\mathcal{F}(P, K_{dec})$ denote the closed-loop map from $r \in \mathcal{L}_2(R^\mu)$ to $z \in \mathcal{L}_2(R^\rho)$. The goal is to design a stabilizing controller K_{dec} so that $\|\mathcal{F}(P, K_{dec})\|$ is as close as desired to $\|\mathcal{F}(P, K_{cen})\|$.

Before we proceed to the formal controller design, we will provide some intuition on how to design such a controller. The centralized controller K_{cen} has access to all of y, whereas in the decentralized case the controller in the i^{th} channel can only measure y_i. We place a discretized version of K_{cen} in one of the channels - we somewhat arbitrarily choose the p^{th} one - which we drive with a running estimate $\hat{y}$ of y and which generates a running estimate of the control signal:[1]

$$\left.\begin{array}{rcl} \nu[k+1] &=& \mathcal{F}\nu[k] + \mathcal{G}\hat{y}(kT) \\ \sqcap[k+1] &=& H\nu[k+1] + J\hat{y}(kT), \end{array}\right\} \tag{2.5}$$

where $\mathcal{F} = \mathrm{e}^{FT}$ and $\mathcal{G} = (\int_0^T \mathrm{e}^{F\tau} d\tau)G$. On each period $[kT, (k+1)T)$, we apply an estimate $\hat{\sqcap}[k]$ of $\sqcap[k]$, at the same time doing a small amount of probing to obtain a better estimate of this quantity for use during the next period. We make use of the fact that the graph associated with the plant is complete to pass information amongst the channels. More specifically, first we carry out probing in channels $1, ..., p-1$, using scaled versions of $y_1(kT), ..., y_{p-1}(kT)$ so that an estimate of $y_1(kT), ..., y_{p-1}(kT)$ can be constructed in channel p, which is combined with the local measurement of $y_p(kT)$ to construct an estimate $\hat{y}(kT)$ of $y(kT)$ which can be used to drive (2.5) to generate $v[k+1]$ and hence $\sqcap[k+1]$. Second of all, we then probe from channel p with elements of $\sqcap[k+1]$ to provide an estimate $\hat{\sqcap}[k+1]$ of the updated control signal for use in channels $1, ..., p-1$ during the next period. It turns out that we can do this in a linear periodic fashion, and end up with an overall controller of the form (2.4). The ensuing control signal is of the form displayed in Figure 1. With this controller implemented, we will show not only that the controller exponentially stabilizes (2.1), but also that we can make the closed-loop performance as close as desired to the level of performance provided by K_{cen}.

[1] In order to use the available information in constructing the control signal, we use $\hat{y}(kT)$ rather than $\hat{y}((k+1)T)$ in constructing the control signal $\sqcap[k+1]$.

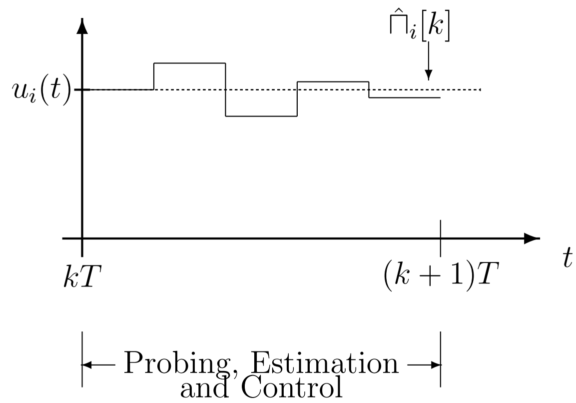


Figure 2.1: A typical control signal over a period for the proposed controller.

2.3 Regularization

At this point it is convenient to put our system into a form which is amenable to analysis. In particular it is desirable to have the graph associated with the system not only strongly connected, but also complete; recalling that a graph is complete if and only if there exist an edge between every two nodes, i.e. in our case

$$C_2^j(sI-A)^{-1}B_i \neq 0,$$

for all $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, p$ and $i \neq j$. As proven in [12] (and used in the earlier work [18]), most static decentralized output feedback control laws of the form $u = Ky + u^0$ will result in the graph associated with the new system

$$\left.\begin{array}{rcl} \dot{x} & = & (A+BKC_2)x + \sum_{i=1}^{p} B_i u_i^0 + Er, \\ & & \quad x(t_0) = x_0 \\ z & = & (C_1 + D_{11}KC_2)x + \sum_{i=1}^{p} D_{11}^i u_i^0 + D_{12}r \\ y_i & = & C_2^i x, \quad i = 1, ..., p, \end{array}\right\} \tag{2.6}$$

being complete. To avoid cumbersome new notation, instead of assuming that the graph is strongly connected, we may as well assume that it is complete, since it can easily adjusted if need be.

Assumption 5: The directed graph corresponding to (2.1) is complete.

In order to implement the idea of the previous section, we will be passing information from one channel to the rest. In order to do so, it is particularly convenient to convert the system to one with single-input single-output (siso) channels. To this end, consider vectors $v \in \mathbf{R}^l$ and $w \in \mathbf{R}^m$ partitioned in a natural way as

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix}, \quad v_i \in \mathbf{R}^{l_i}, \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_p \end{bmatrix}, \quad w_i \in \mathbf{R}^{m_i}.$$

Proposition 1 *[18] For almost all $(v, w) \in \mathbf{R}^l \times \mathbf{R}^m$, we have that for every $i, j \in \{1, 2, ..., p\}$, the transfer function $v_i^T C_2^i (sI - A)^{-1} B_j w_j$ is not identically zero.*

It turns out that we can strengthen Proposition 1: there exist $(v, w) \in \mathbf{R}^l \times \mathbf{R}^m$ so that for every $i, j \in \{1, 2, ..., p\}$, the transfer function $v_i^T C_i (sI - A)^{-1} B_j w_j$ is not only non-zero but also has the same relative degree as $C_i (sI - A)^{-1} B_j$; so freeze such a v and w. We now introduce the natural notation

$$\bar{C}_2^i := v_i^T C_2^i, \quad \bar{B}_i := B_i w_i,$$

$$\bar{y}_i = \bar{C}_2^i x = v_i^T C_2^i x, \quad i = 1, ..., p.$$

During probing and estimation, we will carry out probing on one channel at a time, so the following notation will prove useful: $\bar{w}_j := \begin{bmatrix} 0 & w_j^T & 0 \end{bmatrix}^T \in \mathbf{R}^m$.

At this point we are ready to construct a controller to achieve our objective. This will be carried out in the next chapter.

Chapter 3

Controller Design

In this chapter we design a controller achieve our objective.

3.1 Estimation

The underlying idea of the proposed controller is to apply an estimate $\hat{\sqcap}[k]$ of $\sqcap[k]$, while at the same time constructing a new estimate for use during the next time period. [1] The method we use in this thesis for estimation is similar to the one developed in [15] and [18] with some modifications. To this end we choose $\bar{n} \in \{1, ..., n\}$ and define two $(\bar{n}+1) \times (\bar{n}+1)$ matrices and a vector of samples of $\bar{y}_i$:

$$S := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^{\bar{n}} \\ & & \vdots & & \\ 1 & \bar{n} & \bar{n}^2 & \cdots & \bar{n}^{\bar{n}} \end{bmatrix},$$

$$H(h) := \mathrm{diag}\{1, h, \frac{h^2}{2!}, ..., \frac{h^{\bar{n}}}{\bar{n}!}\},$$

[1] We partition $\hat{\sqcap}[k] = \begin{bmatrix} \hat{\sqcap}_1[k] \\ \vdots \\ \hat{\sqcap}_p[k] \end{bmatrix}$ and $\sqcap[k] = \begin{bmatrix} \sqcap_1[k] \\ \vdots \\ \sqcap_p[k] \end{bmatrix}$.

$$\bar{\mathcal{Y}}_i(t) := \begin{bmatrix} \bar{y}_i(t) \\ \bar{y}_i(t+h) \\ \cdots \\ \bar{y}_i(t+\bar{n}h) \end{bmatrix}.$$

The following result illustrates how information can be passed from one channel to another.

Lemma 1 *(***Key Estimation Lemma***) For every $\tilde{h} \in (0,1)$ there exists a constant $\gamma > 0$ so that for every $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^n$, $h \in (0,\tilde{h})$, $\bar{u} \in \mathbf{R}^m$ and $\phi \in \mathbf{R}$, the solution of (2.1) with*

$$u(t) = \begin{cases} \bar{u} + \bar{w}_j\phi & t \in [t_0, t_0+\bar{n}h) \\ \bar{u} - \bar{w}_j\phi & t \in [t_0+\bar{n}h, t_0+2\bar{n}h) \end{cases}$$

satisfies, for $i = 1,...,p$ and $j = 1,...,p$:

$$\|H(h)^{-1}S^{-1}[\bar{\mathcal{Y}}_i(t_0) - \bar{\mathcal{Y}}_i(t_0+\bar{n}h)] - 2\underbrace{\begin{bmatrix} 0 \\ \bar{C}_2^i\bar{B}_j \\ \vdots \\ \bar{C}_2^i A^{\bar{n}-1}\bar{B}_j \end{bmatrix}}_{=:M_{i,j}}\phi\|$$

$$\le \gamma h(\|x_0\| + \|\bar{u}\| + |\phi|) + \gamma h^{\eta_2-\bar{n}-\frac{1}{2}}(\int_{t_0}^{t_0+2\bar{n}h}\|r(\tau)\|^2 d\tau)^{\frac{1}{2}}$$
$$+ \gamma h^{\frac{1}{2}}(\int_{t_0}^{t_0+2\bar{n}h}\|r(\tau)\|^2 d\tau)^{\frac{1}{2}}$$

and

$$\|x(t) - x_0\| \le \gamma h(\|x_0\| + \|\bar{u}\| + |\phi|) + \gamma h^{\frac{1}{2}}(\int_{t_0}^{t_0+2\bar{n}h}\|r(\tau)\|^2 d\tau)^{\frac{1}{2}},$$
$$t \in [t_0, t_0+2\bar{n}h].$$

Proof: See the Appendix (A.1).

Here we scale our probing signal by a factor of T^δ with $\delta \in (0,\frac{1}{2})$; the controller period T is an integer multiple of the base sampling period h, so $\mathcal{O}(h) = \mathcal{O}(T)$. To see how this lemma can be used, first consider the control signal

$$u(t) = \hat{\sqcap}[k] + \begin{cases} T^\delta\bar{w}_j[y_j(kT)]_1 & t \in [kT, kT+\bar{n}h) \\ -T^\delta\bar{w}_j[y_j(kT)]_1 & t \in [kT+\bar{n}h, kT+2\bar{n}h), \end{cases}$$

where $[y_j(kT)]_1$ is the first element of output signal $y_j(kT)$. Using Lemma 1 (and the definition of $M_{i,j}$ provided there), observe that

$$\frac{T^{-\delta}}{2}H(h)^{-1}S[\bar{\mathcal{Y}}_i(kT)-\bar{\mathcal{Y}}_i(kT+\bar{n}h)] =$$

$$\underbrace{\begin{bmatrix} 0 \\ \bar{C}_2^i\bar{B}_i \\ \vdots \\ \bar{C}_2^iA^{\bar{n}-1}\bar{B}_i \end{bmatrix}}_{M_{i,j}}[y_j(kT)]_1 + \mathcal{O}(T^{1-\delta})(\|x(kT)\| + \|\hat{\sqcap}[k]\|)+$$

$$\mathcal{O}(T)\|y_j(kT)\| + [\mathcal{O}(T^{\frac{1}{2}-\delta}) + \mathcal{O}(T^{\eta_2-\bar{n}-\delta-\frac{1}{2}})](\int_{kT}^{kT+2\bar{n}h}\|r(\tau)\|^2 d\tau)^{\frac{1}{2}}.$$

Now fix $\bar{n}=\eta_1$; then the quantity

$$\eta_2-\bar{n}-\delta-\frac{1}{2} = \eta_2-\eta_1-\delta-\frac{1}{2} \geq \frac{1}{2}-\delta > 0,$$

which means that the last three terms of the above equation tend to zero as T tends to zero; furthermore, this means that $M_{i,j}\neq 0$ for every $i,j\in\{1,2,\cdots,p\}$, in which case we can construct an estimate of $[y_j(kT)]_1$ in channel i as follows:

$$\underbrace{\frac{T^{-\delta}}{2}(M_{i,j}^{\mathrm{T}}M_{i,j})^{-1}M_{i,j}^{\mathrm{T}}H(h)^{-1}S^{-1}}_{=:\bar{M}_{i,j}}\times$$

$$[\bar{\mathcal{Y}}_i(kT)-\bar{\mathcal{Y}}_i(kT+\bar{n}h)] =: [\hat{y}_j(kT)]_1,$$

Of course, we can adopt the same procedure to estimate $[y_j(kT)]_2$, $[y_j(kT)]_3$, and so on, in channel p, and we can use the same technique to probe in channel p to pass information about $\sqcap_i[k+1]$ to channel i for $i=1,...,p-1$. Therefore, **at this point we freeze the value of $\bar{n}$ to be** η_1. Now observe that the error bound on $[\hat{y}_j(kT)]_1$ is given by:

$$[\hat{y}_j(kT)]_1-[y_j(kT)]_1 = \mathcal{O}(T^{1-\delta})(\|x(kT)\|+\left\|\hat{\sqcap}[k]\right\|)+\mathcal{O}(T^{\frac{1}{2}-\delta})(\int_{kT}^{kT+2\bar{n}h}\|r(\tau)\|^2 d\tau)^{\frac{1}{2}}. \quad (3.1)$$

Since our objective is to obtain an estimate of $y_i(k)$ of dimension l_i, $i=1,...,p-1$, in channel p, and an estimate of $\sqcap_i[k]$ of dimension m_i in channels $1,...,p-1$, it is convenient to define some new block diagonal matrices containing multiple copies of $\bar{M}_{i,j}$:

$$\hat{M}_{p,i} := \mathrm{diag}\{\underbrace{\bar{M}_{p,i},...,\bar{M}_{p,i}}_{l_i \text{ copies}}\}, \quad i=1,...,p-1,$$

$$\hat{M}_{i,p} := \mathrm{diag}\{\underbrace{\bar{M}_{i,p}, ..., \bar{M}_{i,p}}_{m_i \text{ copies}}\}, \quad i = 1, ..., p-1.$$

3.2 The Controller

Recall that we have fixed $\bar{n} = \hat{\eta}_1$ and $\delta \in (0, \frac{1}{2})$. Let $h > 0$ and define $\bar{h} := \bar{n}h$. Now we set the controller period to be $T := \underbrace{(2\bar{n}(l + m - l_p - m_p) + 1)}_{=:q} h$. As discussed above, we designate channel p to be the channel where a copy of K_{cen} is placed. With $\nu[0] \in \mathbf{R}^{\ell}$ and $\hat{\sqcap}[0] \in \mathbf{R}^m$, we define the controller in three steps - for $k \in \mathbf{Z}^+$:

(i) Estimate the output signals $y_i(kT)$, $i = 1, ..., p-1$, on $[kT, kT + \underbrace{2(l - l_p)\bar{n}}_{=:q_p} h)$. We probe $y_j(kT)$ in channels $j = 1, 2, ..., p-1$, in the following manner: with

$$\begin{array}{rcl} T_1 & := & kT \\ T_2 & := & T_1 + 2l_1\bar{h} \\ T_3 & := & T_2 + 2l_2\bar{h} \\ \vdots & & \\ T_p & := & T_{p-1} + 2l_{p-1}\bar{h}, \end{array}$$

we apply $\hat{\sqcap}[k]$ while at the same time probing with weighted elements of $y_1(kT), ..., y_{p-1}(kT)$ in sequence:

$$u(t) = \hat{\sqcap}[k] + \begin{cases} T^{\delta}\bar{w}_1[y_1(kT)]_1 & t \in [T_1, T_1 + \bar{h}) \\ -T^{\delta}\bar{w}_1[y_1(kT)]_1 & t \in [T_1 + \bar{h}, T_1 + 2\bar{h}) \\ \vdots & \\ T^{\delta}\bar{w}_{p-1}[y_{p-1}(kT)]_{l_{p-1}} & t \in [T_p - 2\bar{h}, T_p - \bar{h}) \\ -T^{\delta}\bar{w}_{p-1}[y_{p-1}(kT)]_{l_{p-1}} & t \in [T_p - \bar{h}, T_p). \end{cases} \tag{3.2}$$

Now we form the estimate $\hat{y}_i(kT)$ in the channel p :

$$\hat{y}_i(kT) := \hat{M}_{p,i} \begin{bmatrix} \bar{\mathcal{Y}}_i(T_i) - \bar{\mathcal{Y}}_i(T_i + \bar{h}) \\ \vdots \\ \bar{\mathcal{Y}}_i(T_i + (2l_i - 2)\bar{h}) - \bar{\mathcal{Y}}_i(T_i + (2l_i - 1)\bar{h}) \end{bmatrix}$$

for $i = 1, ..., p-1$, and then form an overall estimate of $y(kT)$ in channel p:

$$\hat{y}(kT) := \begin{bmatrix} \hat{y}_1(kT) \\ \vdots \\ \hat{y}_{p-1}(kT) \\ y_p(kT) \end{bmatrix}.$$

(ii) Update the control signal $\sqcap$ on $[T_p, T_p + h)$ using (2.5):

$$\left.\begin{array}{rcl} \nu[k+1] & = & \mathcal{F}\nu[k] + \mathcal{G}\hat{y}(kT) \\ \sqcap[k+1] & = & H\nu[k+1] + J\hat{y}(kT) \end{array}\right\} \tag{3.3}$$

while applying the present estimate:

$$u(t) = \hat{\sqcap}[k], \quad t \in [T_p, T_p + h). \tag{3.4}$$

(iii) Estimate the updated control signal $\sqcap[k+1]$ on channels $1, ..., p-1$ during the time period $[T_p + h, T_p + 2\bar{n}(m - m_p)h + h)$: with

$$\begin{array}{rcl} \tilde{T}_1 & = & T_p + h \\ \tilde{T}_2 & = & \tilde{T}_1 + 2m_1\bar{h} \\ \tilde{T}_3 & = & \tilde{T}_2 + 2m_2\bar{h} \\ \vdots & & \\ \tilde{T}_p & = & \tilde{T}_{p-1} + 2m_{p-1}\bar{h}, \end{array}$$

we apply $\hat{\sqcap}[k]$ while at the same time probing with weighted elements of the first $m - m_p$ elements of $\sqcap[k+1]$ in sequence:

$$u(t) = \hat{\sqcap}[k] + \begin{cases} T^\delta \bar{w}_p[\sqcap_1[k+1]]_1 & t \in [\tilde{T}_1, \tilde{T}_1 + \bar{h}) \\ -T^\delta \bar{w}_p[\sqcap_1[k+1]]_1 & t \in [\tilde{T}_1 + \bar{h}, \tilde{T}_1 + 2\bar{h}) \\ \quad\vdots & \\ T^\delta \bar{w}_p[\sqcap_{p-1}[k+1]]_{m_{p-1}} & t \in [\tilde{T}_p - 2\bar{h}, \tilde{T}_p - \bar{h}) \\ -T^\delta \bar{w}_p[\sqcap_{p-1}[k+1]]_{m_{p-1}} & t \in [\tilde{T}_p - \bar{h}, \tilde{T}_p). \end{cases} \tag{3.5}$$

Now we form the estimate $\hat{\sqcap}_i[k+1]$ of $\sqcap_i[k+1]$ in channel i: for $i = 1, ..., p-1$ we define

$$\hat{\sqcap}_i[k+1] :=$$

$$\hat{M}_{i,p}\begin{bmatrix} \bar{\mathcal{Y}}_i(\tilde{T}_i) - \bar{\mathcal{Y}}_i(\tilde{T}_i+\bar{h}) \\ \bar{\mathcal{Y}}_i(\tilde{T}_i+2\bar{h}) - \bar{\mathcal{Y}}_i(\tilde{T}_i+3\bar{h}) \\ \vdots \\ \bar{\mathcal{Y}}_i(\tilde{T}_i+(2m_i-2)\bar{h}) - \bar{\mathcal{Y}}_i(\tilde{T}_i+(2m_i-1)\bar{h}) \end{bmatrix}$$

and then we define

$$\hat{\sqcap}_p[k+1] := \sqcap_p[k+1]. \tag{3.6}$$

The above controller given by (3.2)-(3.5) is a description of its behaviour on each period of length T. It turns out that it has a desireable state-space representation of the form (2.4), which we label $K_{dec}(T)$:

Lemma 2 *There exists an LPC of the form (2.4) given by*

$$(L_i, M_i, Q_i, R_i), i = 1, \cdots, p; T; q)$$

with the parameters and the state partitioned as

$$\begin{aligned} \psi_i[j] &= \begin{bmatrix} \psi_i^1[j] \\ \psi_i^2[j] \\ \psi_i^3[j] \end{bmatrix}, \; M_i[j] = \begin{bmatrix} M_i^1[j] \\ M_i^2[j] \\ M_i^3[j] \end{bmatrix} \\ L_i[j] &= \begin{bmatrix} L_i^{11}[j] & 0 & 0 \\ L_i^{21}[j] & L_i^{22}[j] & 0 \\ L_i^{31}[j] & 0 & L_i^{33}[j] \end{bmatrix} \end{aligned} \tag{3.7}$$

with the following properties:
(i) $L_i^{11}[0] = 0, L_i^{21}[0] = 0, L_i^{31}[0] = 0.$
(ii) $L_i^{33}[j] = 0, M_i^3[j] = 0$ *for all* $i \in \{1, 2, \cdots, p-1\}$ *and* $\forall j \in \mathbf{Z}^+$.
(iii) With $\psi_i^2[0] = \hat{\sqcap}_i[0]$ *and* $\psi_p^3[0] = \nu[0]$ *the behaviour of this LPC is identical to that of (3.2)-(3.5). Moreover,*

$$\psi_i^2[j] = \hat{\sqcap}_i[k], \; j = kq, \cdots, (k+1)q-1$$

and

$$\psi_p^3[j] = \begin{cases} \nu[k], \; j = kq, \cdots, kq+q_p \\ \nu[k+1], j = kq+q_p+1, \cdots, (k+1)q-1 \end{cases}$$

Proof: See the Appendix (A.2).

Remark 3 *The initial conditions of (3.2)-(3.5), namely* $\nu[0] \in \mathbf{R}^\ell$ *and* $\hat{\sqcap}[0] \in \mathbf{R}^m$*, are connected to the initial conditions of* $\psi_i[0] = \begin{bmatrix} \psi_i^1[0] \\ \psi_i^2[0] \\ \psi_i^3[0] \end{bmatrix}$ *via*

$$\nu[0] = \psi_p^3[0]$$

and

$$\hat{\sqcap}_i[0] = \psi_i^2[0], \quad i = 1, ..., p.$$

The remaining elements of $\psi_i[0]$ *are irrelevant, and play no role in the controller output.*

Chapter 4

Analysis

To prove that the proposed controller achieves our objective, we first analyse the the closed-loop behaviour over a single period of T time units. More precisely, we prove that the desired estimation of the plant output y and the passing of the desired control signal from channel p to the rest of the channels works well while the inter-sampler behaviour of the plant state x and the control signal u have acceptable upper bounds:

Lemma 3 *(One Period Lemma): For every $\delta \in (0, \frac{1}{2})$ there exist constants $\gamma > 0$ and $\bar{T} > 0$ so that for every $T \in (0, \bar{T})$, $x_0 \in \mathbf{R}^n$ and $k \in \mathbf{Z}^+$, when the controller (3.2)-(3.5) is applied to the plant (2.1), the closed loop system satisfies the following:*

$$\begin{aligned} \|x(t) - x(kT)\| \le \gamma T(\|x(kT)\| + \left\|\hat{\sqcap}[k]\right\|) + \gamma T^{1+\delta} \|\nu[k]\| + \gamma T^{\frac{1}{2}} \|r_k\|_2 \,, \\ t \in [kT, (k+1)T), \end{aligned} \tag{4.1}$$

$$\begin{aligned} \left\|u(t) - \hat{\sqcap}[k]\right\| \le \gamma T^{\delta}(\|x(kT)\| + \|\nu[k]\|) + \gamma T \left\|\hat{\sqcap}[k]\right\| + \gamma T^{\frac{1}{2}} \|r_k\|_2 \,, \\ t \in [kT, (k+1)T), \end{aligned} \tag{4.2}$$

$$\begin{aligned} \left\|\hat{\sqcap}[k+1] - \sqcap[k+1]\right\| \le \gamma T^{1-\delta}(\|x(kT)\| + \left\|\hat{\sqcap}[k]\right\|) + \gamma T \|\nu[k+1]\| \\ + \gamma T^{\frac{1}{2}-\delta} \|r_k\|_2 \,, \end{aligned} \tag{4.3}$$

$$\|\hat{y}(kT) - y(kT)\| \le \gamma T^{1-\delta}(\|x(kT)\| + \left\|\hat{\sqcap}[k]\right\|) + \gamma T^{\frac{1}{2}-\delta} \|r_k\|_2 \,. \tag{4.4}$$

Proof: See the Appendix(A.3).

Lemma 3 provides a comprehensive outlook of the behaviour of the closed-loop system over a single period. This result can be leveraged to prove that the proposed decentralized linear periodic controller works very much like the centralized LTI controller:

Theorem 1 *There exists a $\tilde{T} > 0$ so that for every $T \in (0, \tilde{T})$, the linear periodic controller $K_{dec}(T)$ exponentially stabilizes the plant (2.1) and satisfies*

$$\lim_{T \to 0} \|\mathcal{F}(P, K_{cen}) - \mathcal{F}(P, K_{dec}(T))\| = 0.$$

Proof:

We will carry out the proof in three parts: we start with a lengthy preamble, then move onto exponential stability and finish up with an analysis of the performance. We let $x(0) \in \mathbf{R}^n$, $\psi_i[0] \in \mathbf{R}^{\bar{\ell}_i}$, $r(t) \in \mathcal{L}_2(\mathbf{R}^{\mu})$ and $T > 0$ be arbitrary and set $h = T/q$.

Step 1: Preamble

Here we apply the controller (3.2)-(3.5), suitably rewritten in the form of (2.4) courtesy of Lemma 2. Observe that $\psi_i[0] = \begin{bmatrix} \psi_i^1[0] \\ \psi_i^2[0] \\ \psi_i^3[0] \end{bmatrix}$. Using Lemma 2 we can connect the intial conditions of (3.2)-(3.6), namely $x(0) \in \mathbf{R}^n$, $\nu[0] \in \mathbf{R}^\ell$ and $\hat{\sqcap}[0] \in \mathbf{R}^m$ to that of the LPC (2.4) via

$$\begin{aligned} \nu[0] &= \psi_p^3[0] \\ \hat{\sqcap}_i[0] &= \psi_i^2[0],\ i = 1, 2, \cdots, p. \end{aligned}$$

Before we proceed further let us introduce intermediate states

$$\begin{aligned} \zeta[k] &:= \hat{\sqcap}[k] - H\nu[k] - J\hat{y}((k-1)T) \in \mathbf{R}^m, \\ \xi[k] &:= \hat{y}((k-1)T) - C_2 x(kT) \in \mathbf{R}^l, \end{aligned}$$

and define $x_d[k] := \begin{bmatrix} x(kT) \\ \nu[k] \\ \zeta[k] \\ \xi[k] \end{bmatrix}$. We define $\hat{y}(-T) := 0$. Observe that we can write $\hat{\sqcap}[k]$ in terms of these new variables:

$$\hat{\sqcap}[k] = \zeta[k] + H\nu[k] + J\xi[k] + JC_2 x(kT). \tag{4.5}$$

We first prove that $x_d[k]$ is bounded by a decaying exponential and we use this to prove the same for $x_{sd}(t)$. The first step is to obtain a bound on each sub-vector of $x_d[k+1]$. We start with plant state. Solving the plant equation 2.1 yields

$$\begin{aligned} &x((k+1)T) \\ &= e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)\mathrm{d}\tau + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Er(\tau)\mathrm{d}\tau \\ &= (I + AT)x(kT) + \mathcal{O}(T^2)x(kT) + \int_{kT}^{(k+1)T} Bu(\tau)\mathrm{d}\tau + \int_{kT}^{(k+1)T} Er(\tau)\mathrm{d}\tau \\ &\quad + \int_{kT}^{(k+1)T} (e^{A((k+1)T-\tau)} - I)Bu(\tau)\mathrm{d}\tau + \int_{kT}^{(k+1)T} (e^{A((k+1)T-\tau)} - I)Er(\tau)\mathrm{d}\tau. \end{aligned} \tag{4.6}$$

The probing part of the control signal averages out on $[kT,(k+1)T)$, so from its definition and equation (4.5) we have

$$\int_{kT}^{(k+1)T} Bu(\tau)\mathrm{d}\tau = TB\hat{\sqcap}[k]$$
$$= TB[\zeta[k] + H\nu[k] + J\xi[k] + JC_2x(kT)].$$

Using the second equation (4.2) of Lemma 3 to bound $\|u(t)\|$ on $[kT,(k+1)T)$ we see that

$$\left\|\int_{kT}^{(k+1)T}(e^{A((k+1)T-\tau)} - I)Bu(\tau)\mathrm{d}\tau\right\| = \mathcal{O}(T^2)\left\|\hat{\sqcap}[k]\right\| + \mathcal{O}(T^{2+\delta})(\|x(kT)\| + \|\nu[k]\|)$$
$$+ \mathcal{O}(T^{2\frac{1}{2}})\left\|r_k\right\|_2;$$

using equation (4.5) for another representation of $\hat{\sqcap}[k]$, we have

$$\left\|\int_{kT}^{(k+1)T}(e^{A((k+1)T-\tau)} - I)Bu(\tau)\mathrm{d}\tau\right\| = \mathcal{O}(T^2)(\|x(kT)\| + \|\nu[k]\| + \|\zeta[k]\| + \|\xi[k]\|)$$
$$+ \mathcal{O}(T^{2\frac{1}{2}})\left\|r_k\right\|_2.$$

Using the Cauchy Schwarz inequality, we have

$$\left\|\int_{kT}^{(k+1)T}(e^{A((k+1)T-\tau)} - I)Er(\tau)\mathrm{d}\tau\right\| = \mathcal{O}(T^{\frac{3}{2}})\left\|r_k\right\|_2.$$

We conclude that there is a constant $\gamma_1 > 0$ and a function $\mu_1(T)$ so that (4.6) can be rewritten as

$$x((k+1)T) = (I + AT)x(kT) + TBH\nu[k] + TBJC_2x(kT)$$
$$+ TB\zeta[k] + TBJ\xi[k] + \int_{kT}^{(k+1)T} Er(\tau)\mathrm{d}\tau + \mu_1(T) \tag{4.7}$$

with $\mu_1(T)$ satisfying

$$\|\mu_1(T)\| \le +\gamma_1T^2(\|x(kT)\| + \|\nu[k]\| + \|\zeta[k]\| + \|\xi[k]\|) + \gamma_1T^{3/2}\left\|r_k\right\|_2 \tag{4.8}$$

for small T.

Now we turn to the controller state. By expanding the discrete controller states in (2.5), we obtain

$$\begin{aligned}\nu[k+1] &= e^{FT}v[k] + \int_0^T e^{F(\tau)}G\hat{y}(kT)\mathrm{d}\tau \\ &= (I+FT)\nu[k] + TG\hat{y}(kT) + \mathcal{O}(T^2)\nu[k] + \mathcal{O}(T^2)\hat{y}(kT).\end{aligned}$$

We now use the bound from (4.4) of the One Period Lemma to rewrite the quantity $\hat{y}(kT)$ and equation (4.5) to obtain another representation of $\hat{\sqcap}[k]$ we conclude that there exists a constant γ_2 and a function $\mu(T)$ so that

$$\nu[k+1] = (I+FT)\nu[k] + TGC_2x(kT) + \mu_2(T), \tag{4.9}$$

with $\mu_2(T)$ satisfying

$$\|\mu_2(T)\| \leq \gamma_2 T^{2-\delta}(\|x(kT)\| + \|\nu[k]\| + \|\zeta[k]\| + \|\xi[k]\|) + \gamma_2 T^{\frac{3}{2}-\delta}\left\|r_k\right\|_2. \tag{4.10}$$

Now we obtain a bound on $\|\zeta[k+1]\|$. From the definition of $\zeta[k]$ and the control update law (2.5) we see that

$$\zeta[k+1] = \hat{\sqcap}[k+1] - \sqcap[k+1].$$

Using this fact and the equation (4.3) from the One Period Lemma, we can write

$$\|\zeta[k+1]\| = \mathcal{O}(T^{1-\delta})(\|x(kT)\| + \left\|\hat{\sqcap}[k]\right\|) + \mathcal{O}(T)(\|\nu[k+1]\|) + \mathcal{O}(T^{\frac{1}{2}-\delta})\left\|r_k\right\|_2.$$

From equation (4.5) we see that

$$\hat{\sqcap}[k] = \mathcal{O}(1)x(kT) + \mathcal{O}(1)\nu[k] + \mathcal{O}(1)\zeta[k] + \mathcal{O}(1)\xi[k]),$$

from equation (2.5) observe that

$$\nu[k+1] = \mathcal{O}(1)\nu[k] + \mathcal{O}(T)\hat{y}(kT)$$

and from Lemma 3 observe that

$$\|\hat{y}(kT)\| = \mathcal{O}(1)\left\|x(kT)\right\| + \mathcal{O}(T^{1-\delta})\left\|\hat{\sqcap}\right\| + \mathcal{O}(T^{\frac{1}{2}-\delta})\left\|r_k\right\|_2.$$

So we conclude that

$$\|\zeta[k+1]\| = \mathcal{O}(T^{1-\delta})(\|x(kT)\| + \|\nu[k]\| + \|\zeta[k]\| + \|\xi[k]\|) + \mathcal{O}(T^{\frac{1}{2}-\delta})\left\|r_k\right\|_2. \tag{4.11}$$

Now we derive the bound on $\|\xi[k+1]\|$. Using the definition of $\xi[k+1]$ it follows immediately that

$$\hat{y}(kT) - C_2 x(kT) - C_2(x(k+1)T - x(kT)).$$

Using (4.4) and (4.5) we obtain

$$\|\hat{y}(kT) - C_2 x(kT)\| = \mathcal{O}(T^{1-\delta})(\|x(kT)\| + \|\nu[k]\| + \|\zeta[k]\| + \|\xi[k]\|) + \mathcal{O}(T^{\frac{1}{2}-\delta})\left\|r_k\right\|_2.$$

We use (4.7) and (4.8) to find a bound on $C_2(x(k+1)T - x(kT))$:

$$\|C_2(x(k+1)T - x(kT))\| = \mathcal{O}(T)(\|x(kT)\| + \|\nu[k]\| + \|\zeta[k]\| + \|\xi[k]\|) + \mathcal{O}(T^{\frac{1}{2}})\left\|r_k\right\|_2.$$

Now combining these two bounds yields

$$\|\xi[k+1]\| = \mathcal{O}(T^{1-\delta})(\|x(kT)\| + \|\nu[k]\| + \|\zeta[k]\| + \|\xi[k]\|) + \mathcal{O}(T^{\frac{1}{2}-\delta})\left\|r_k\right\|_2. \tag{4.12}$$

Now we combine (4.7)-(4.12) to obtain the update equation for $x_d[k]$. Since the controller can be written as an LPC of period T, it follows that the map from $(x_d[k], r_{[kT,(k+1)T)})$ to $x_d[k+1]$ is linear, so we can combine (4.7)-(4.12) and conclude that there exist functions $\mu_x(T), \mu_\nu(T), \mu_\zeta(T), \mu_\xi(T)$ and a constant γ_3 so that

$$\left[\begin{array}{c} x((k+1)T) \\ \nu[k+1] \\ \hline \zeta[k+1] \\ \xi[k+1] \end{array}\right] = \left[\begin{array}{cc|cc} I + (A + BJC_2)T & TBH & \mathcal{O}(T) & \mathcal{O}(T) \\ TGC_2 & I + FT & 0 & 0 \\ \hline \mathcal{O}(T^{1-\delta}) & \mathcal{O}(T^{1-\delta}) & \mathcal{O}(T^{1-\delta}) & \mathcal{O}(T^{1-\delta}) \\ \mathcal{O}(T^{1-\delta}) & \mathcal{O}(T^{1-\delta}) & \mathcal{O}(T^{1-\delta}) & \mathcal{O}(T^{1-\delta}) \end{array}\right] \left[\begin{array}{c} x(kT) \\ \nu[k] \\ \hline \zeta[k] \\ \xi[k] \end{array}\right]$$
$$+ \left[\begin{array}{c} \int_{kT}^{(k+1)T} Er(\tau)\mathrm{d}\tau \\ 0 \\ \hline 0 \\ 0 \end{array}\right] + \left[\begin{array}{c} \mu_x(T) \\ \mu_\nu(T) \\ \mu_\zeta(T) \\ \mu_\xi(T) \end{array}\right] + \mathcal{O}(T^{2-\delta}) x_d[k]. \tag{4.13}$$

with

$$\left.\begin{array}{rcl} \|\mu_x(T)\| & \leq & \gamma_3 T^{\frac{3}{2}} \left\|r_k\right\|_2, \\ \|\mu_\nu(T)\| & \leq & \gamma_3 T^{\frac{3}{2}-\delta} \left\|r_k\right\|_2, \\ \|\mu_\zeta(T)\| & \leq & \gamma_3 T^{\frac{1}{2}-\delta} \left\|r_k\right\|_2, \\ \|\mu_\xi(T)\| & \leq & \gamma_3 T^{\frac{1}{2}-\delta} \left\|r_k\right\|_2. \end{array}\right\} \tag{4.14}$$

Before we proceed further, let us examine what the centralized closed loop system looks like; we use the subscript 'c' to differentiate them from the decentralized counterparts

$$\left.\begin{aligned} \begin{bmatrix} \dot{x}_c \\ \dot{v}_c \end{bmatrix} &= \underbrace{\begin{bmatrix} A+BJC_2 & BH \\ GC_2 & F \end{bmatrix}}_{=:A_c} \begin{bmatrix} x_c \\ v_c \end{bmatrix} + \underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix}}_{=:E_c} r, \\ z_c &= \begin{bmatrix} C_1 + D_{11}JC_2 & D_{11}H \end{bmatrix} \underbrace{\begin{bmatrix} x_c \\ v_c \end{bmatrix}}_{\bar{x}_c} + D_{12} r. \end{aligned}\right\} \tag{4.15}$$

Now we return to the decentralized case, but now we will adopt some of the notation of the centralized case. A careful examination of the closed loop system given by (4.13) reveals that it has two time-scales: a slow one and a fast one. Hence, we partition the states accordingly: the slow sub-system state is $\bar{x}[k] := \begin{bmatrix} x(kT) \\ \nu[k] \end{bmatrix}$ while the fast sub-system state is $e[k] := \begin{bmatrix} \zeta[k] \\ \xi[k] \end{bmatrix}$. With this notation, (4.13) can be rewritten as

$$\begin{bmatrix} \bar{x}[k+1] \\ e[k+1] \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^{A_c T} + \mathcal{O}(T^{2-\delta}) & \mathcal{O}(T) \\ \mathcal{O}(T^{1-\delta}) & \mathcal{O}(T^{1-\delta}) \end{bmatrix}}_{=:A_d(T)} \begin{bmatrix} \bar{x}[k] \\ e[k] \end{bmatrix} + \begin{bmatrix} \int_{kT}^{(k+1)T} E_c r(\tau) \mathrm{d}\tau \\ 0 \end{bmatrix} + \begin{bmatrix} \mu_x(T) \\ \mu_\nu(T) \\ \mu_\zeta(T) \\ \mu_\xi(T) \end{bmatrix}. \tag{4.16}$$

Step 2: Exponential Stability

In this part of the proof we assume that $r = 0$.

> **Claim 1** : There exist constants $\bar{T} > 0$, $\bar{\gamma}_0 > 0$ and $\bar{\lambda}_0 < 0$ so that for every $T \in (0, \bar{T})$ with $r(t) = 0$ we have
>
> $$\|x_d[k]\| \leq \bar{\gamma}_0 e^{\bar{\lambda}_0 kT} \|x_d[0]\|, k \geq 0$$

Proof: See the Appendix (A.4).

We leverage Claim 1 to prove that the sample-data controller (2.4) exponentially stabilizes the system. At this point we will prove that $x(t)$ is well-behaved on $[kT, (k+1)T)$. To this end, from the One Period Lemma we see that there exists a $T_1 \in (0, \bar{T})$ and a $\gamma_1 > 0$ so that for $T \in (0, T_1)$:

$$\|x(t) - x(kT)\| \leq \gamma_1 T(\|x(kT)\| + \left\|\hat{\sqcap}[k]\right\| + \|\nu[k]\|),$$
$$t \in [kT, (k+1)T).$$

Using (4.5) to provide a bound on $\left\|\hat{\sqcap}[k]\right\|$, we conclude that there exists a constant γ_2 such that for $T \in (0, T)$:

$$\|x(t) - x(kT)\| \leq \gamma_2 T \|x_d[k]\|, \ t \in [kT, (k+1)T). \tag{4.17}$$

The next step is to prove that $x_{sd}(t)$ decays exponentially to zero. First of all, from Lemma 2(iii), we see that

$$\|\nu[0]\| = \left\|\psi^3[0]\right\|. \tag{4.18}$$

From the definition of $\zeta[0]$, we have

$$\zeta[0] = \hat{\sqcap}[0] - H\nu[0] - J\hat{y}(-T); \tag{4.19}$$

but by Lemma 2(iii),

$$\hat{\sqcap}[0] = \psi^2[0]$$

and by definition $\hat{y}(-T) = 0$, so combining (4.19) and (4.18) yields

$$\|\zeta[0]\| \leq \left\|\psi^2[0]\right\| + \|H\| \cdot \left\|\psi^3[0]\right\|.$$

Similarly, by the definition of $xi[0]$, we have

$$\|\xi[0]\| = \|\hat{y}(-T) - C_2 x(kT)\| \leq \|C_2\| \cdot \|x(0)\|.$$

We conclude that

$$\begin{aligned}
&\|x_d[0]\| \\
&\leq \|x(0)\| + \|\nu[0]\| + \|\zeta[0])\| + \|\xi[0]\| \\
&\leq \|x(0)\| + \left\|\psi^3[0]\right\| + \left\|\psi^2[0]\right\| \\
&\quad + \|H\| \cdot \left\|\psi^3[0]\right\| + \|C_2\| \cdot \|x(0)\| \\
&\leq \underbrace{(3 + \|H\| + \|C_2\|}_{=:\gamma_3} \|x_{sd}[0]\|.
\end{aligned}$$

Hence if we combine this with (4.17) and Claim 1, we see that there exists a constant γ_4 so that for $T \in (0, T_1)$:

$$\|x(t)\| \leq \gamma_4 e^{\bar{\lambda}_0 t} \|x_{sd}(0)\|, \ t > 0, \tag{4.20}$$

$$\|x_d[k]\| \le \gamma_4 e^{\bar{\lambda}_0 kT} \|x_{sd}(0)\|,\ k \ge 0. \tag{4.21}$$

It remains to find a bound of the form (4.21) for ψ. Fix $T \in (0, T_1)$. We start with ψ^1: by Lemma 2 (i), we see that ψ^1 provides a weighted sum over at most q samples of y; clearly there exists a constant γ_5 so that

$$\left\|\psi^1[kq+j]\right\| \le \gamma_5 \max_{\tau\in[kT,(k+1)T)} \|y(t)\|, j = 1, 2, \cdots, q,$$

so

$$\left\|\psi^1[kq+j]\right\| \le \gamma_5 \|C_2\| \max_{\tau\in[kT,(k+1)T)} \|x(t)\|,$$
$$j = 1, 2, \cdots, q.$$

Using (4.20) we see that

$$\left\|\psi^1[kq+j]\right\| \le \gamma_5 \|C_2\| \gamma_4 e^{\bar{\lambda}_0 kT} \|x_{sd}(0)\|,$$
$$k \ge 0,\ j = 1, 2, \cdots, q$$

so

$$\left\|\psi^1[kq+j]\right\| \le \gamma_5 \|C_2\| \gamma_4 e^{-\bar{\lambda}_0 T} e^{\bar{\lambda}_0 (kT+jh)} \|x_{sd}(0)\|,$$
$$k \ge 0,\ j = 1, 2, \cdots, q$$

If we set $\gamma_6 := max\{1, \gamma_5\gamma_4 \|C_2\| e^{-\bar{\lambda}_0 T}\}$, we see that

$$\left\|\psi^1[j]\right\| \le \gamma_6 e^{\bar{\lambda}_0 jh} \|x_{sd}[0]\|,\ j \in \mathbf{Z}^+. \tag{4.22}$$

Now we turn to ψ^2. We see from Lemma 2 (iii) that

$$\psi^2[kq+j] = \hat{\sqcap}[k],\ j = 0, 1, \cdots, q-1.$$

But (4.5) yields a formula for $\hat{\sqcap}[k]$ as a linear function of $x_d[k]$, so using (4.21) we conclude that there exist a constant γ_7 so that

$$\left\|\psi^2[kq+j]\right\| \le \gamma_7 e^{\bar{\lambda}_0 kT} \|x_{sd}(0)\|,$$
$$k \ge 0,\ j = 1, 2, \cdots, q,$$

so

$$\left\|\psi^2[kq+j]\right\| \le \gamma_7 e^{-\bar{\lambda}_0 T} e^{\bar{\lambda}_0 (kT+jh)} \|x_{sd}(0)\|.$$
$$k \ge 0,\ j = 1, 2, \cdots, q,$$

setting $\gamma_8 = max\{1,\ \gamma_7 e^{-\bar{\lambda}_0 T}\}$ yields

$$\left\|\psi^2[j]\right\| \leq \gamma_8 e^{\bar{\lambda}_0 jh} \left\|x_{sd}[0]\right\|,\ j \in \mathbf{Z}^+. \tag{4.23}$$

Finally we examine ψ^3. We see from Lemma 2 (ii) that

$$\psi_i^3[j] = 0,\ j \geq 1,\ i = 1, \cdots, p-1.$$

From Lemma 2 (iii), we see that

$$\begin{aligned}\left\|\psi_p^3[kq+j]\right\| &\leq \max\{\left\|\nu[k]\right\|, \left\|\nu[k+1]\right\|\}\\ &\leq \max\{\left\|x_d[k]\right\|, \left\|x_d[k+1]\right\|\},\\ &\quad k \geq 0,\ j = 0, 1, \cdots, q-1.\end{aligned}$$

Using (4.21) we see that

$$\begin{aligned}\left\|\psi_p^3[kq+j]\right\| &\leq \gamma_4 e^{\bar{\lambda}_0 kT} \left\|x_{sd}(0)\right\|,\\ &\quad k \geq 0,\ j = 0, 1, \cdots, q-1,\\ \left\|\psi_p^3[kq+j]\right\| &\leq \underbrace{\gamma_4 e^{-\bar{\lambda}_0 T}}_{=:\gamma_9} e^{\bar{\lambda}_0 (kT+jh)} \left\|x_{sd}(0)\right\|,\\ &\quad k \geq 0,\ j = 1, 2, \cdots, q.\end{aligned}$$

If we define $\gamma_{10} := \max\{1, \gamma_9\}$, we have

$$\left\|\psi^3[j]\right\| \leq \gamma_{10} e^{\bar{\lambda}_0 jh} \left\|x_{sd}[0]\right\|,\ j \in \mathbf{Z}^+. \tag{4.24}$$

If we combine (4.20),(4.22),(4.23) and (4.24), we conclude that

$$\begin{aligned}\left\|x_{sd}(t)\right\| \leq [\gamma_4 + e^{-\bar{\lambda}_0 h}(\gamma_6 + \gamma_8 + \gamma_1 0)] e^{\bar{\lambda}_0 t} \left\|x_{sd}(0)\right\|,\\ k \geq 0,\ j = 0, 1, \cdots, q-1.\end{aligned}$$

Hence, the sampled-data controller provides exponential stability for $T \in (0, T_1)$.

Step 3: Performance

Here we set the plant initial condition x_0 and the controller initial condition $\psi[0]$ to zero. To facilitate our analysis we apply a similarity transformation to decouple the slow-moving sub-state $\bar{x}[k]$ from the fast-moving sub-state $e[k]$. To this end, we would like to define the matrix $W \in \mathbf{R}^{(m+l)\times(n+\ell)}$ so that

$$\begin{bmatrix} I & 0 \\ W & I \end{bmatrix} A_d(T) \begin{bmatrix} I & 0 \\ -W & I \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

For this to hold, we need

$$\begin{aligned} W &= \left[e^{A_c T} + \mathcal{O}(T^{2-\delta})\right]^{-1} \left[W\mathcal{O}(T)W + \mathcal{O}(T^{1-\delta})W + \mathcal{O}(T^{1-\delta})\right] \\ &=: f(W). \end{aligned} \tag{4.25}$$

It is easy to see that for small T the continuous function f maps the closed unit ball in $\mathbf{R}^{(m+l)\times(n+\ell)}$ to itself. Brouwer's Fixed Point Theorem guarantees the existence of a solution to (4.25); since $f(W)$ is $\mathcal{O}(T^{1-\delta})$ for all W in the closed -unit ball, it follows that this solution (which we simply label W) is also $\mathcal{O}(T^{1-\delta})$. So for $T \in (0, T_2)$, define

$$\begin{bmatrix} x^\star[k] \\ e^\star[k] \end{bmatrix} := \begin{bmatrix} I & 0 \\ W & I \end{bmatrix} \begin{bmatrix} \bar{x}[k] \\ e[k] \end{bmatrix};$$

then (4.16) can be rewritten as

$$\begin{aligned} \begin{bmatrix} x^\star[k+1] \\ e^\star[k+1] \end{bmatrix} &= \begin{bmatrix} e^{A_c T} + \mathcal{O}(T^{2-\delta}) & \mathcal{O}(T) \\ 0 & \mathcal{O}(T^{1-\delta}) \end{bmatrix} \begin{bmatrix} x^\star[k] \\ e^\star[k] \end{bmatrix} + \begin{bmatrix} \int_{kT}^{(k+1)T} E_c r(\tau)\mathrm{d}\tau \\ \int_{kT}^{(k+1)T} \mathcal{O}(T^{1-\delta}) r(\tau)\mathrm{d}\tau \end{bmatrix} \\ &\quad + \begin{bmatrix} \begin{bmatrix} \mu_x(T) \\ \mu_v(T) \end{bmatrix} \\ W \begin{bmatrix} \mu_x(T) \\ \mu_v(T) \end{bmatrix} + \begin{bmatrix} \mu_\zeta(T) \\ \mu_\xi(T) \end{bmatrix} \end{bmatrix}. \end{aligned} \tag{4.26}$$

It is convenient to define

$$\mu_{x^\star}(T) := \begin{bmatrix} \mu_x(T) \\ \mu_v(T) \end{bmatrix}$$

and

$$\mu_{e^\star}(T) := \begin{bmatrix} \mu_\zeta(T) \\ \mu_\xi(T) \end{bmatrix} + W \begin{bmatrix} \mu_x(T) \\ \mu_v(T) \end{bmatrix};$$

it follows from (4.14) that there exists a constant γ_{11} so that for $T \in (0, T_2)$:

$$\begin{aligned} \|\mu_{x^\star}\| &\le \gamma_{11} T^{\frac{3}{2}-\delta} \|r_k\|_2 \,, \\ \|\mu_{e^\star}\| &\le \gamma_{11} T^{\frac{1}{2}-\delta} \|r_k\|_2 \,. \end{aligned} \tag{4.27}$$

In the analysis we will occasionally measure the size of a discrete signal. Rather than introducing new notation, which differs from the continuous time notation, we will simply define the norm of such signal $x[k]$ as $\|x\|_2 = (\sum_{k=0}^{\infty} \|x[k]\|^2)^{\frac{1}{2}}$. Now that the states are decoupled, we can obtain bounds on the size of the states. First of all, using frequency domain analysis

$$\begin{aligned}\|e^\star\|_2 &= (\sum_{k=0}^{\infty}\|e^\star[k]\|^2)^{\frac{1}{2}}\\ &= \sup_{\omega\in\mathbf{R}}\left\|\left(e^{j\omega}-\mathcal{O}(T^{1-\delta})\right)^{-1}\right\|\mathcal{O}(T^{\frac{1}{2}-\delta})(\sum_{k=0}^{\infty}\|r_k\|^2)^{\frac{1}{2}}.\end{aligned}$$

Observe that

$$(\sum_{k=0}^{\infty}\|r_k\|^2)^{\frac{1}{2}} = \|r\|_2\,;$$

hence it follows that

$$\|e^\star\|_2 = \mathcal{O}(T^{\frac{1}{2}-\delta})\,\|r\|_2\,. \tag{4.28}$$

Using this bound and the same approach, we can obtain bound on $\|x^\star\|_2$:

$$\begin{aligned}\|x^\star\|_2 = \sup_{\omega\in\mathbf{R}}&\left\|\left(e^{j\omega}-e^{A_cT}-\mathcal{O}(T^{2-\delta})\right)^{-1}\right\|\\ &[\mathcal{O}(T)\,\|e^\star\|_2+\mathcal{O}(T^{\frac{1}{2}})\,\|r\|_2+\mathcal{O}(T^{\frac{3}{2}-\delta})\,\|r\|_2].\end{aligned} \tag{4.29}$$

Using standard Lyapunov stability arguments, it is straight forward to verify that there exist $T_3\in(0,T_2)$, $\beta^\star>0$ and $\lambda^\star<0$ such that for all k and $T\in(0,T_3)$, we have

$$\left\|\left(e^{A_cT}+\mathcal{O}(T^{2-\delta})\right)^k\right\|\leq\beta^\star(e^{\lambda^\star T})^k,\quad k\geq 0.$$

Using this fact with the matrix power series expansion of $[e^{j\omega}-e^{A_cT}-\mathcal{O}(T^{2-\delta})]^{-1}$ we obtain,

$$\begin{aligned}\sup_{\omega\in\mathbf{R}}\left\|\left(e^{j\omega}-e^{A_cT}-\mathcal{O}(T^{2-\delta})\right)^{-1}\right\| &\leq \beta^\star+\beta^\star e^{\lambda^\star T}+\beta^\star(e^{\lambda^\star T})^2+\cdots\\ &= \frac{\beta^\star}{1-e^{\lambda^\star T}}\\ &= \mathcal{O}(T^{-1}).\end{aligned} \tag{4.30}$$

Substituting (4.30) and (4.28) into (4.29) results in

$$\|x^\star\|_2 = \mathcal{O}(T^{-\frac{1}{2}})\,\|r\|_2\,. \tag{4.31}$$

From (4.28) and (4.31) we can find bounds on the original variables:

$$\begin{aligned}\|\bar{x}\|_2 &= \|x^\star\|_2 \\ &= \mathcal{O}(T^{-\frac{1}{2}})\,\|r\|_2\,.\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}\|e\|_2 &\le \|Wx^\star\| + \|e^\star\| \\ &= \mathcal{O}(T^{\frac{1}{2}-\delta})\,\|r\|_2\,.\end{aligned} \tag{4.33}$$

Now we wish to compare the performance provided by this controller with the centralized behaviour provided in (4.16). With

$$\bar{x}_c[k] := \begin{bmatrix} x_c(kT) \\ v_c(kT) \end{bmatrix},$$

we see from (4.16) that

$$\begin{aligned}\bar{x}_c[k+1] &= e^{A_cT}\bar{x}_c[k] + \int_{kT}^{(k+1)T} e^{A_c[(k+1)T-\tau]}E_c r(\tau)\mathrm{d}\tau \\ &= e^{A_cT}\bar{x}_c[k] + \int_{kT}^{(k+1)T} E_c r(\tau)\mathrm{d}\tau + \underbrace{\int_{kT}^{(k+1)T}[e^{A_c[(k+1)T-\tau]} - I]E_c r(\tau)\mathrm{d}\tau}_{=:\mu_{\bar{x}_c}(T)};\end{aligned} \tag{4.34}$$

it easy to see that

$$\mu_{\bar{x}_c} = \mathcal{O}(T^{\frac{3}{2}})\,\|r_k\|_2\,. \tag{4.35}$$

If we define $\tilde{x}[k] := \bar{x}_c[k] - \bar{x}[k] = \bar{x}_c[k] - x^\star[k]$ and combine the above with (4.26), we obtain

$$\tilde{x}[k+1] = e^{A_cT}\tilde{x}[k] + \mathcal{O}(T^{2-\delta})x^\star[k] + \mathcal{O}(T)e\star[k] - \mu_{x^\star}(T) + \mu_{\bar{x}_c}(T).$$

But as proven above,

$$\sup_{\omega\in R}\left\|(e^{j\omega}I - e^{A_cT})^{-1}\right\| = \mathcal{O}(T^{-1}),$$

so using frequency domain analysis together with the bound on $\|x^\star\|_2$ given in (4.31), the bound on $\|e\|_2$ given in (4.28), the bound on $\mu_x(T)$ given in (4.27) and the bound on $\|\mu_{\bar{x}_c}(T)\|$ given in (4.35), we obtain

$$
\begin{aligned}
\|\tilde{x}\|_2 &= \mathcal{O}(T^{-1})[\mathcal{O}(T^{2-\delta})\mathcal{O}(T^{-\frac{1}{2}})\|r\|_2 + \mathcal{O}(T)\mathcal{O}(T^{\frac{1}{2}-\delta})\|r\|_2 + \mathcal{O}(T^{\frac{3}{2}-\delta})\|r\|_2 \\
&\quad + \mathcal{O}(T^{\frac{3}{2}})\|r\|_2] \\
&= \mathcal{O}(T^{\frac{1}{2}-\delta})\|r\|_2 . \qquad (4.36)
\end{aligned}
$$

Now we will shift our focus to the output z. First we will find a bound on $z(t)$ for $t \in [kT, (k+1)T$; from the plant equation (2.1), we have

$$z(t) = C_1 x(t) + D_{11} u(t) + D_{12} r(t).$$

Using (4.1) to bound $\|x(t) - x(kT)\|$ and (4.2) to bound $\left\|u(t) - \hat{\sqcap}[k]\right\|$, we see that there exist a constant γ_{12} and a function $\mu_d(t,T)$ so that

$$z(t) = C_1 x(kT) + D_{11}\hat{\sqcap}[k] + D_{12} r(t) + \mu_d(t,T),\ t \in [kT, (k+1)T),$$

with $\mu_d(t,T)$ satisfying

$$
\begin{aligned}
\mu_d(t,T) \le \gamma_{12} T^{\delta} \left\| \begin{bmatrix} x(kT) \\ \nu[k] \end{bmatrix} \right\| + \gamma_{12} T \left\| \begin{bmatrix} \zeta[k] \\ \xi[k] \end{bmatrix} \right\| + \gamma T^{\frac{1}{2}} \|r_k\|_2 , \\
t \in [kT, (k+1)T). \qquad (4.37)
\end{aligned}
$$

Using the expression for $\hat{\sqcap}[k]$ given in (4.5), we can rewrite the equation as

$$
\begin{aligned}
z(t) = C_1 x(kT) + D_{11}[\zeta[k] + H\nu[k] + J\xi[k] + JC_2 x(kT)] + D_{12} r(t) + \mu_d(t,T), \\
t \in [kT, (k+1)T).
\end{aligned}
$$

Now we stack the sub-states in a suitable manner to obtain

$$
\begin{aligned}
z(t) = \begin{bmatrix} C_1 + D_{11} J C_2 & D_{11} H \end{bmatrix} \begin{bmatrix} x(kT) \\ \nu[k] \end{bmatrix} + \begin{bmatrix} D_{11} & D_{11} J \end{bmatrix} \begin{bmatrix} \zeta[k] \\ \xi[k] \end{bmatrix} \\
+ D_{12} r(t) + \mu_d(t,T),\ t \in [kT, (k+1)T). \qquad (4.38)
\end{aligned}
$$

Now we analyse the nominal centralized output. From (4.15) we have,

$$z_c(t) = \begin{bmatrix} C_1 + D_{11} J C_2 & D_{11} H \end{bmatrix} \underbrace{\begin{bmatrix} x_c(t) \\ v_c(t) \end{bmatrix}}_{=\bar{x}_c(t)} + D_{12} r(t).$$

If we sample $\bar{x}_c(t)$ every T units of time we end up with a constant γ_{13} and a function $\mu_c(t,T)$ so that

$$
\begin{aligned}
z_c(t) = \left[\begin{array}{cc} C_1 + D_{11}JC_2 & D_{11}H \end{array}\right] \left[\begin{array}{c} x_c(kT) \\ v_c(kT) \end{array}\right] + D_{12}r(t) + \mu_c(t,T),& \\
t \in [kT,(k+1)T),&
\end{aligned}
\tag{4.39}
$$

with

$$
\mu_c(t,T) \leq \gamma_{13}T \left\| \left[\begin{array}{c} x_c(kT) \\ v_c(kT) \end{array}\right] \right\| + \gamma_{13}T^{\frac{1}{2}} \left\| r_k \right\|_2, \qquad t \in [kT,(k+1)T). \tag{4.40}
$$

Next we form the output error $\tilde{z} := z_c(t) - z(t)$ for $t \in [kT,(k+1)T$. By subtracting (4.39) from (4.38) we obtain

$$
\begin{aligned}
\tilde{z}(t) = \left[\begin{array}{cc} C_1 + D_{11}JC_2 & D_{11}H \end{array}\right] \tilde{x}[k] - \left[\begin{array}{cc} D_{11} & D_{11}J \end{array}\right] e[k] + \mu_c(t,T) - \mu_d(t,T),& \\
t \in [kT,(k+1)T).&
\end{aligned}
\tag{4.41}
$$

By taking 2-norm on both sides of equation(4.41) we end up with

$$
\begin{aligned}
&\|\tilde{z}\|_2^2 = \\
&\int_0^\infty \|\tilde{z}(\tau)\|^2 \,\mathrm{d}\tau = \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \|\tilde{z}(\tau)\|^2 \\
&\leq 2 \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \left\| \left[\begin{array}{cc} C_1 + D_{11}JC_2 & D_{11}H \end{array}\right] \tilde{x} \right\|^2 \mathrm{d}\tau \\
&\quad + 2 \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \left\| \left[\begin{array}{cc} -D_{11} & -D_{11}J \end{array}\right] e[k] + \mu_c(\tau,T) - \mu_d(\tau,T) \right\|^2 \mathrm{d}\tau \\
&\leq 2 \left\| \left[\begin{array}{cc} C_1 + D_{11}JC_2 & D_{11}H \end{array}\right] \right\|^2 T \sum_{k=0}^{\infty} \|\tilde{x}[k]\|^2 \\
&\quad + 4 \left\| \left[\begin{array}{cc} D_{11} & D_{11}J \end{array}\right] \right\|^2 T \sum_{k=0}^{\infty} \|e[k]\|^2 + 4 \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \|\mu_c(\tau,T) - \mu_d(\tau,T)\|^2 \mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq \underbrace{2\left\|\begin{bmatrix} C_1 + D_{11}JC_2 & D_{11}H \end{bmatrix}\right\|^2 T\sum_{k=0}^{\infty}\|\tilde{x}[k]\|^2}_{=:t_1(T)} \quad \underbrace{+\,4\left\|\begin{bmatrix} D_{11} & D_{11}J \end{bmatrix}\right\|^2 T\sum_{k=0}^{\infty}\|e[k]\|^2}_{=:t_2(T)} \\
&\quad \underbrace{+\,8\sum_{k=0}^{\infty}\int_{kT}^{(k+1)T}\|\mu_c(\tau,T)\|^2\,\mathrm{d}\tau}_{=:t_3(T)} \underbrace{+\,8\sum_{k=0}^{\infty}\int_{kT}^{(k+1)T}\|\mu_d(\tau,T)\|^2\,\mathrm{d}\tau}_{=:t_4(T)}.
\end{aligned}
\tag{4.42}
$$

From (4.36) we see that

$$
\begin{aligned}
t_1(T) &= 2\left\|\begin{bmatrix} C_1 + D_{11}JC_2 & D_{11}H \end{bmatrix}\right\|^2 T\mathcal{O}(T^{1-2\delta})\left\|r\right\|_2^2 \\
&= \mathcal{O}(T^{2-2\delta})\left\|r\right\|_2^2.
\end{aligned}
$$

From (4.33) we see that

$$
\begin{aligned}
t_2(T) &= 4\left\|\begin{bmatrix} D_{11} & D_{11}J \end{bmatrix}\right\|^2 T\mathcal{O}(T^{1-2\delta})\left\|r\right\|_2^2 \\
&= \mathcal{O}(T^{2-2\delta})\left\|r\right\|_2^2.
\end{aligned}
$$

From (4.40) we see that

$$
\begin{aligned}
t_3(T) &\leq 8T\sum_{k=0}^{\infty}[2\gamma_{13}T^2\left\|\bar{x}_c[k]\right\|^2 + 2\gamma_{13}T\left\|r_k\right\|^2 \\
&= \mathcal{O}(T^3)\underbrace{\sum_{k=0}^{\infty}\left\|\bar{x}_c[k]\right\|^2}_{=\|\bar{x}_c\|_2^2} + \mathcal{O}(T^2)\left\|r\right\|_2^2.
\end{aligned}
$$

But from (4.34) we see that

$$
\bar{x}_c[k+1] = e^{A_cT}\bar{x}_c[k] + \mathcal{O}(T^{\frac{1}{2}})\left\|r_k\right\|_2.
$$

By using frequency domain analysis we obtain

$$
\left\|\bar{x}_c\right\|_2 = \mathcal{O}(T^{-1})\mathcal{O}(T^{\frac{1}{2}})\left\|r\right\|_2
$$

so

$$
\left\|\bar{x}_c\right\|_2^2 = \mathcal{O}(T^{-1})\left\|r\right\|_2.
$$

Hence,

$$
t_3(T) = \mathcal{O}(T^2)\left\|r\right\|_2^2.
$$

From (4.37) we see that

$$
\begin{aligned}
t_4(T) &\leq 32T\gamma_{12}^2 \sum_{k=0}^{\infty} \left[T^{2\delta} \left\|\bar{x}[k]\right\|^2 + T^2 \left\|e[k]\right\|^2 + T \left\|r_k\right\|^2\right] \\
&= \mathcal{O}(T^{2\delta+1}) \left\|\bar{x}\right\|_2^2 + \mathcal{O}(T^3) \left\|e\right\|_2^2 + \mathcal{O}(T^2) \left\|r\right\|_2^2 .
\end{aligned}
$$

Using (4.32) to provide a bound on $\|\bar{x}\|_2^2$ and (4.33) to provide a bound on $\|e\|_2^2$, we conclude that

$$
\begin{aligned}
t_4(T) &= \mathcal{O}(T^{2\delta+1})\mathcal{O}(T^{-1}) \left\|r\right\|_2^2 + \mathcal{O}(T^3)\mathcal{O}(T^{1-2\delta}) \left\|r\right\|_2^2 + \mathcal{O}(T^2) \left\|r\right\|_2^2 \\
&= \mathcal{O}(T^{2\delta}) \left\|r\right\|_2^2 .
\end{aligned}
$$

If we substitute the bounds on $t_1(T)$, $t_2(T)$, $t_3(T)$ and $t_4(T)$ into (4.42), we obtain

$$
\left\|\tilde{z}\right\|_2 = \mathcal{O}(T^{\delta}) \left\|r\right\|_2^2
$$

We conclude that

$$
\left\|\mathcal{F}(P, K_{cen}) - \mathcal{F}(P, K_{dec})\right\| = \mathcal{O}(T^{\delta}),
$$

and the right hand side tends to zero as T goes to zero as required. $\square$

Remark 4 *It turns out that our closed loop system is noise tolerant. By this we mean that if we inject noise at the plant-controller interfaces, then the map from the noise to the plant inputs and outputs are bounded in the induced $\mathcal{L}_\infty$-norm sense.*

Chapter 5

An Illustrative Example

Here we consider the closed-loop configuration given below, where P_0 plays the role of the physical plant model and W plays the role of a filter shaping the reference signal r.

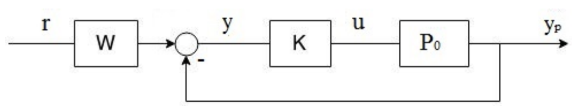


Figure 5.1: A typical tracking problem

The model of P_0 is given by

$$\dot{x}_p = \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix}}_{=:A_p} x_p + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{=:b_1} u_1 + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{=:b_2} u_2$$

$$y_{p1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{=:c_1} x_p$$

$$y_{p2} = \underbrace{\begin{bmatrix} 0 & -1 & 1 \end{bmatrix}}_{=:c_2} x_p.$$

Here $u = [u_1 \; u_2]$ and W is simply a low-pass filter with a transfer function of $\text{diag}\{\frac{1}{(s+1)^3}, \frac{1}{(s+1)^3}\}$; with a minimal realization of $\frac{1}{(s+1)^3}$ be given by $C_w(sI - A_w)^{-1}B_w$ and a minimal realization of W is given by

$$\begin{aligned}
\dot{x}_w &= \underbrace{\begin{bmatrix} A_w & 0 \\ 0 & A_w \end{bmatrix}}_{=:\bar{A}_w} x_w + \underbrace{\begin{bmatrix} B_w & 0 \\ 0 & B_w \end{bmatrix}}_{=:\bar{B}_w} r \\
r_w &= \underbrace{\begin{bmatrix} C_w & 0 \\ 0 & C_w \end{bmatrix}}_{=:\bar{C}_w} x_w.
\end{aligned}$$

Last of all, the output signal to be controlled is $z = \begin{bmatrix} y \\ 0.1u \end{bmatrix}$. This yields an overall state-space model of the plant P:

$$\begin{aligned}
\begin{bmatrix} \dot{x}_p \\ \dot{x}_w \end{bmatrix} &= \begin{bmatrix} A_p & 0 \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x_p \\ x_w \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} b_2 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ B_w \end{bmatrix} r \\
z &= \begin{bmatrix} \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix} & C_w \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_w \end{bmatrix} + \begin{bmatrix} 0 \\ .1 \end{bmatrix} u \\
y_i &= \begin{bmatrix} c_i & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_w \end{bmatrix}, \quad i = 1, 2.
\end{aligned}$$

The system has a decentralized fixed mode at 1, which means that no LTI controller can stabilize it, let alone provide good performance; furthermore, Assumptions 1, 2 and 3 hold. Our objective here is to design a controller which not only achieves closed loop stability but also provides a desireable level of $\mathcal{H}_\infty$ performance.

The optimal LTI centralized controller performance is 0.21 and the optimal controller uses extremely large gains. To avoid this, we have designed a reasonable sub-optimal centralized controller K_{cen} which yields a closed loop performance of $\|\mathcal{F}(P, K_{cen})\| = 0.73$. In transfer function form, K_{cen} is described by

$$\begin{bmatrix} 1 & K_1(s) \\ K_2(s) & 1 \end{bmatrix}$$

with

$$K_1(s) = 10$$

and

$$K_2(s) = k \prod_{i=1}^{6} \frac{(s - b_i)}{(s - a_i)}$$

with $k = -15.43$, $a_i \in \{-24.47, -2.104 \pm 1.772j, -0.083, -1.001, -0.999\}$ and $b_i \in \{-2.00 \pm 1.732j, -1.0058 \pm 0.010j, -1.000, -0.988\}$.

Let us construct the controller outlined in this paper. We set $\bar{n} = \eta_1 = 2$, $\delta = 0.25$; q turns out to be equal to nine, so that the controller period is $T = 9h$. For our simulation, we choose $h = 0.005$. For the case of

$$r_1(t) = sin(t), \;\; r_2(t) = sin(0.5t),$$

$x_p(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, $x_w(0) = 0$, $\nu(0) = 0$, and $\hat{\sqcap}[0] = 0$, we carried out a simulation and display the results in Figure 5.2-Figure 5.7. We compare the results with that provided by the centralized controller K_{cen} (the variables are denoted using a superscript of 'c'). Figure 5.2 shows behaviour of the original plant states for both decentralized and centralized settings. While Figure 5.3 shows the overall control signal through out the simulation, Figure 5.4 shows a close-up of the control signal and clearly it is equal to its centralized counterpart but with some dither added. Figure 5.5 and Figure 5.6 illustrates the similarity between the centralized and decentralized outputs of the original plant. Observe that the decentralized behaviour is nearly identical to the centralized performance as illustrated in Figure 5.7.

Of course it will not be surprising that there are trade-offs for this exceptional behaviour. First of all, as proven before the smaller the sampling period T, the closer the performance provided by the decentralised controller compared to the performance provided by the centralized controller. Consequently, the sample-data controller not only requires sensors with fast sampling, but also potentially high-band width actuators for better performance. Secondly, due to the large gains involved during probing we may have poor noise tolerance; in Figure(5.9) we redo the simulation when the measured output is corrupted by noise of the form

$$n(t) = 5 \times 10^{-4} sin(30t);$$

we see that even though the performance is degraded from the nominal and it is still quite acceptable.

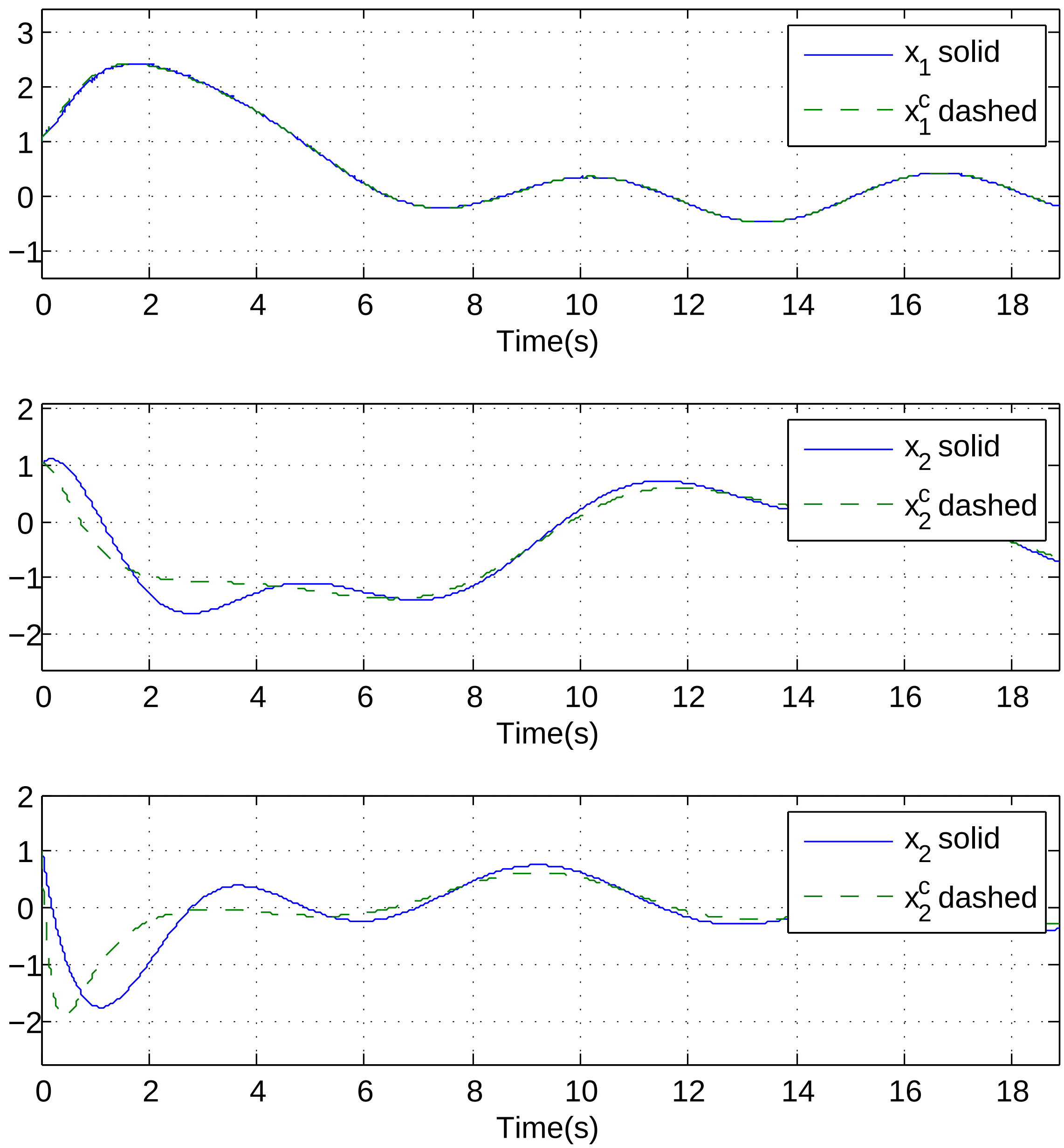


Figure 5.2: Plant state $x(t)$.

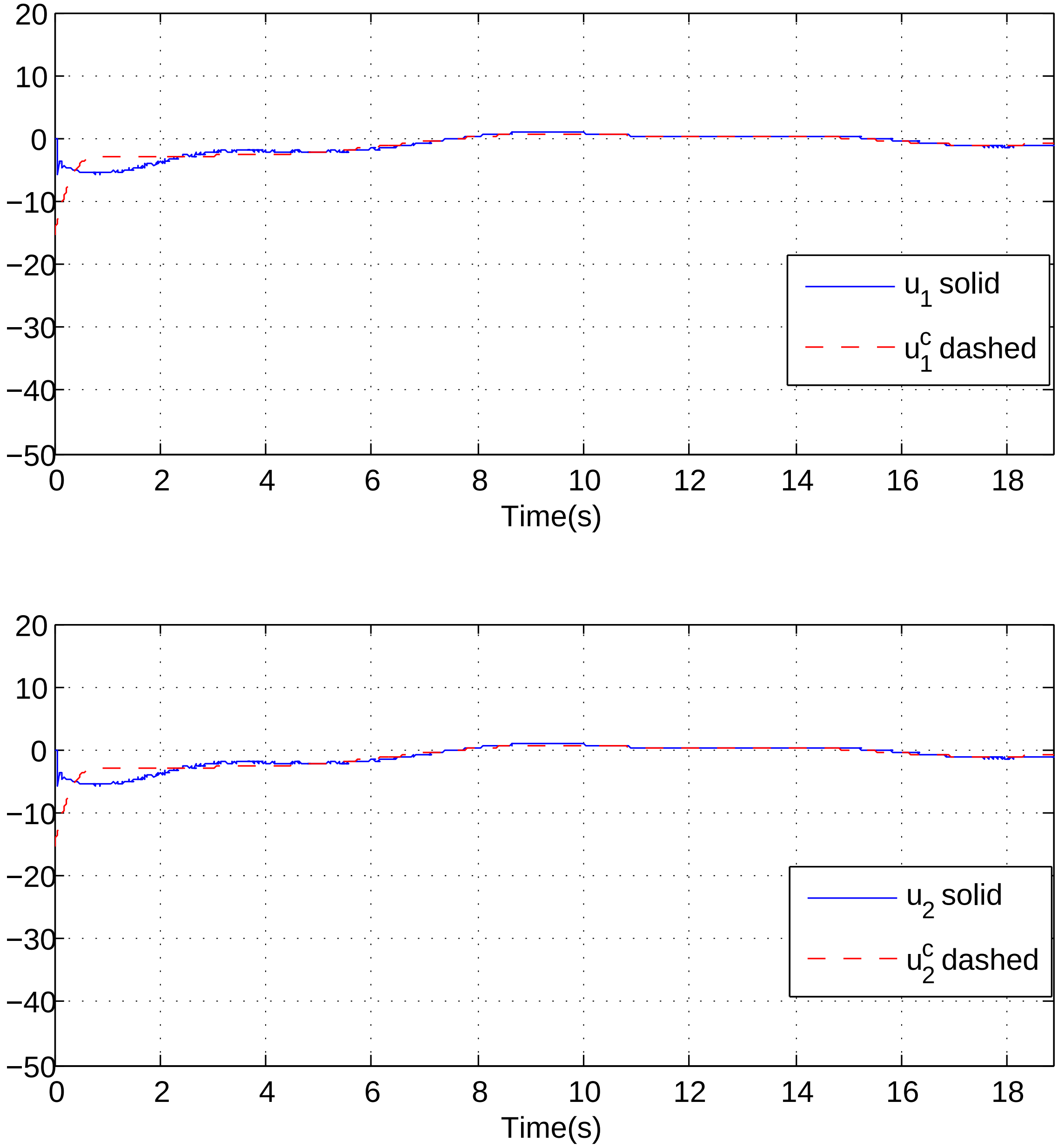


Figure 5.3: The control signal u.

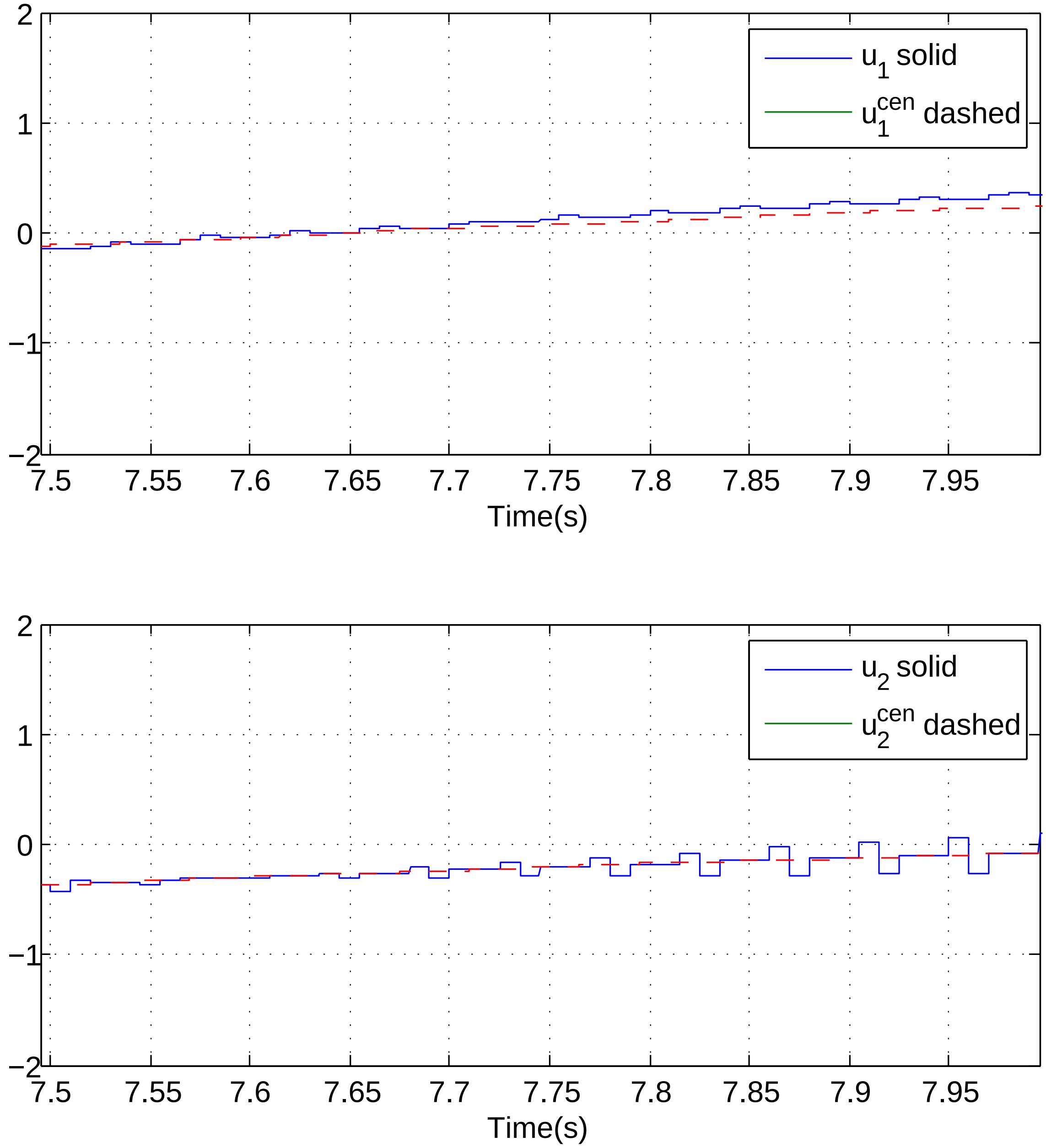


Figure 5.4: A close-up of the control signal u.

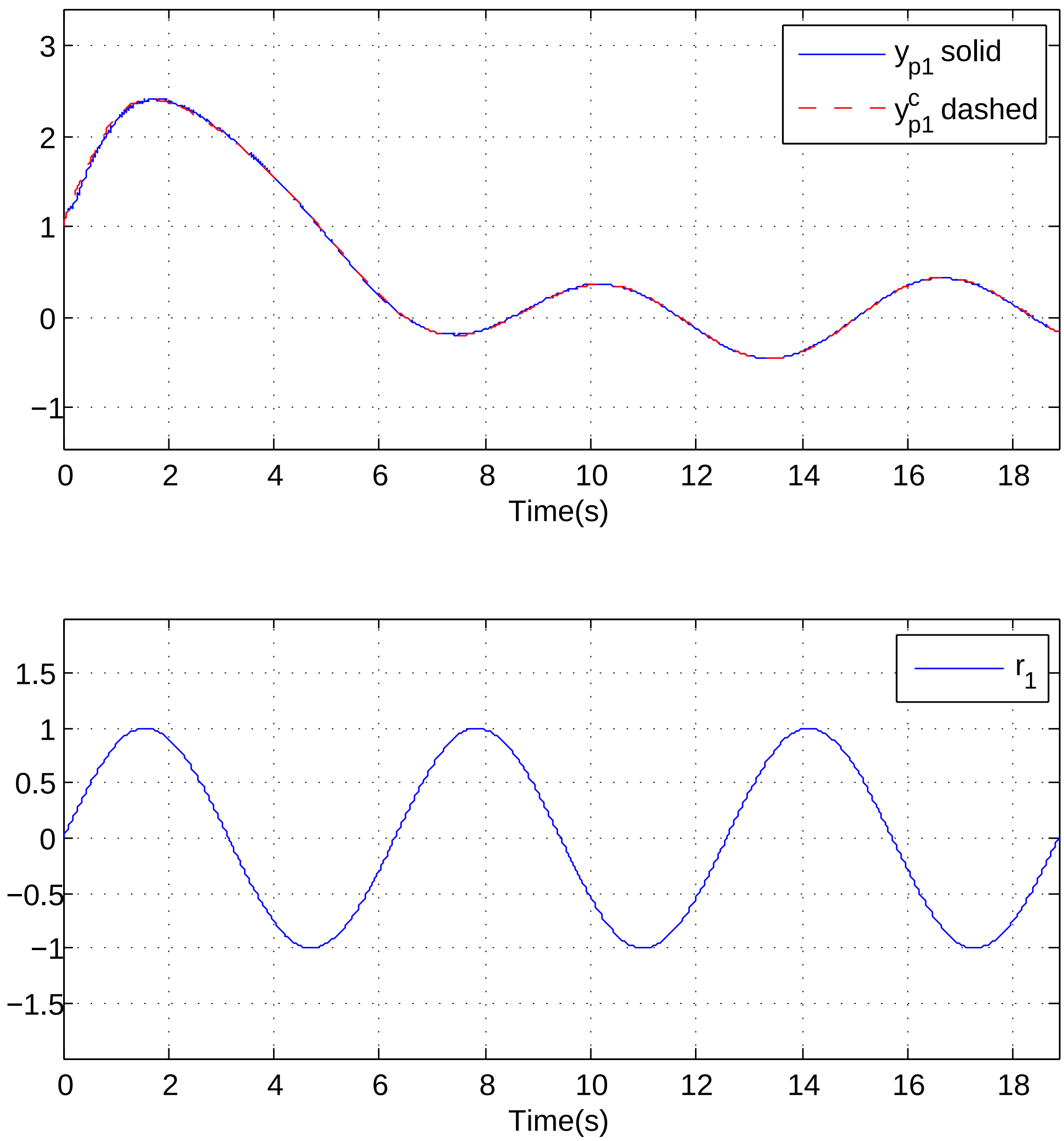


Figure 5.5: The original plant output y_{p1}.

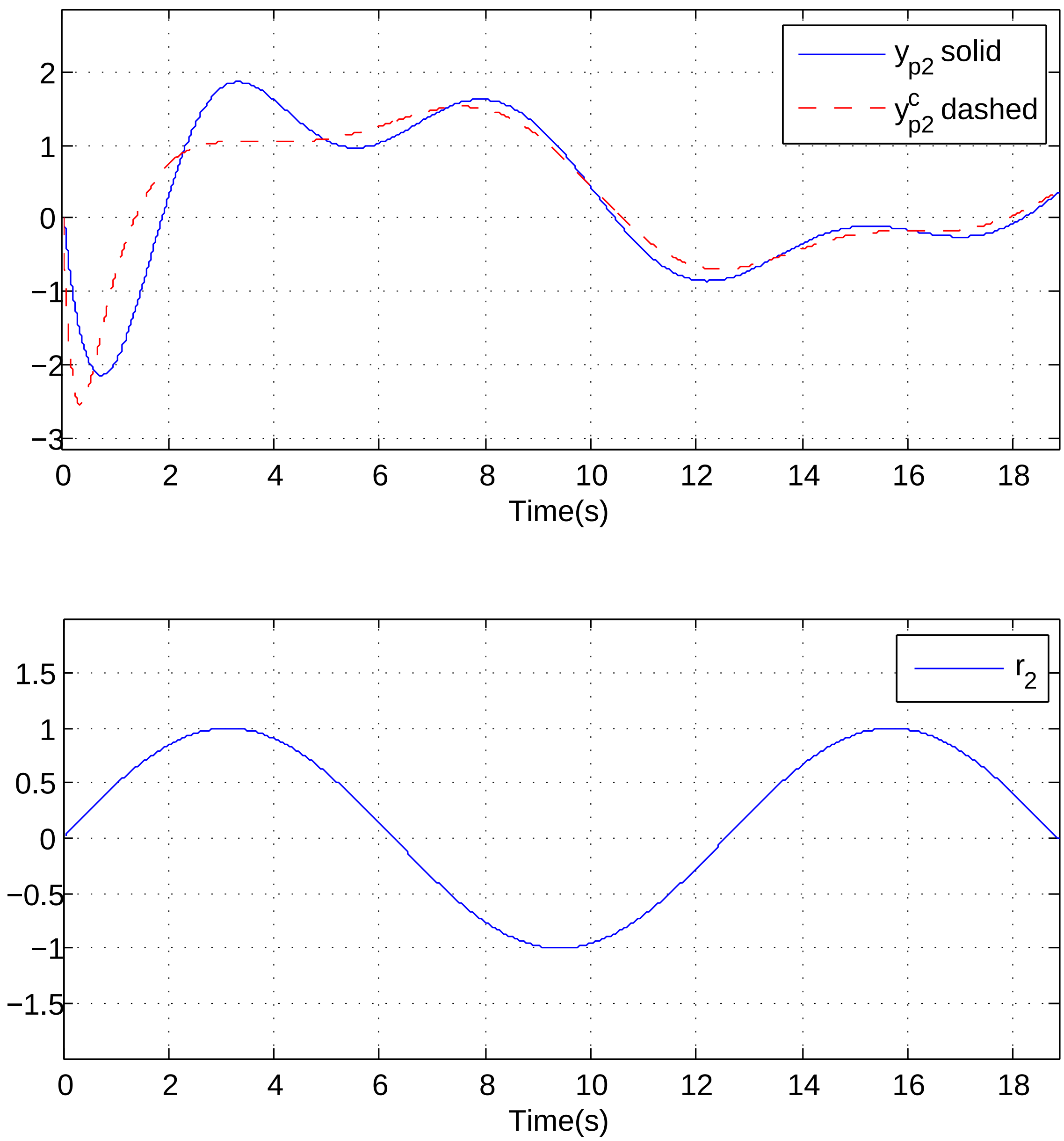


Figure 5.6: The original plant output y_{p2}.

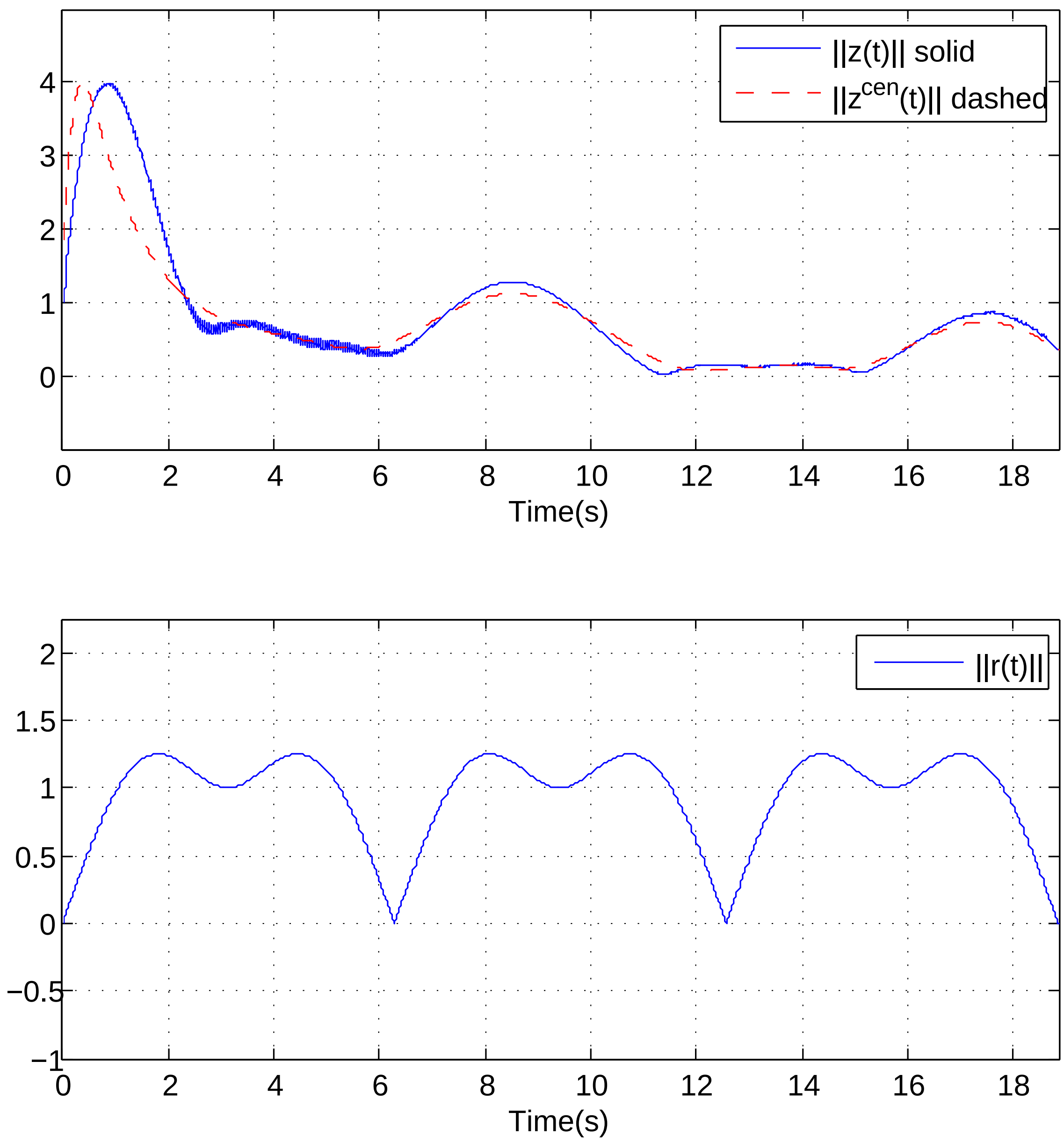
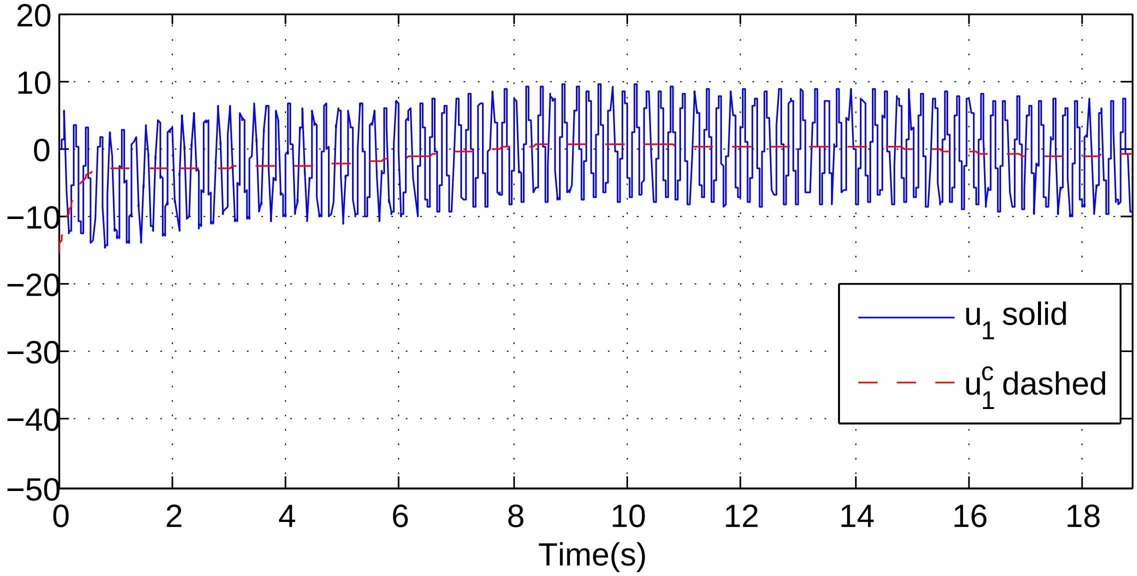


Figure 5.7: The 2-norm of the output $z(t)$ and the 2-norm of $r(t)$



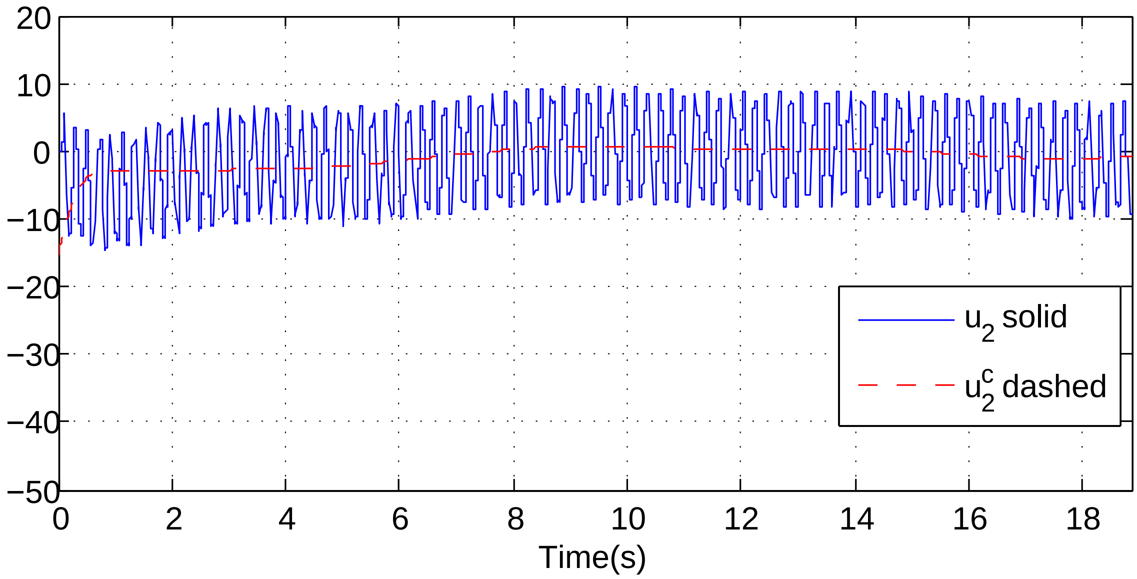


Figure 5.8: The control signal u for the system with noise.

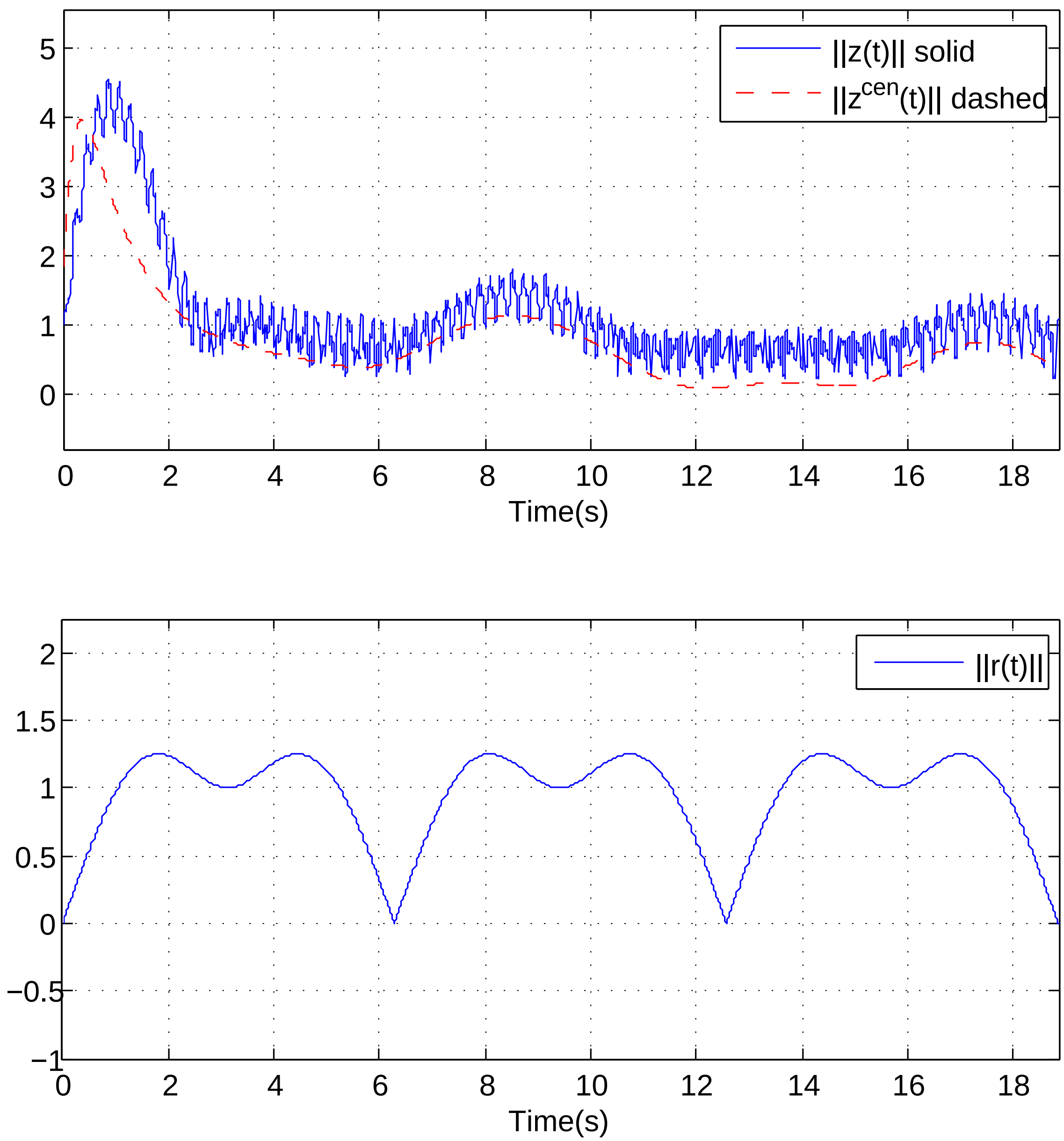


Figure 5.9: The 2-norm of the output $z(t)$ and the 2-norm of $r(t)$ for the system with noise.

Chapter 6

Summary and Conclusion

In this thesis we consider the problem of designing a controller to provide (near) optimal centralized H_∞ performance in the decentralized context. It is well known fact that designing a controller in a decentralized setting is a challenging task. Indeed in the presence of an unstable DFM, there is no LTI controller that stabilizes the plant, let alone provides good performance. In [18] the authors showed that with the use of an LPC, near-optimal LQR-type performance is achievable even at the presence of a DFM. Here we present a methodology to design a decentralized LPC that achieves the stability and the near-optimal centralized H_∞ performance.

Since we are considering a more complex plant with the addition of an external reference signal, it is expected that there will be some conditions imposed on the plant for this approach to work. We prove that if the graph associated with the plant is strongly connected and certain technical conditions on the relative degree hold, then we can design a decentralized LPC to achieve this objective. The approach works even in situations in which the plant has an unstable decentralized fixed mode (DFM). The controller guarantees stability for small sampling period; the centralized performance is recovered as the sampling period tends to zero. This exceptional behaviour has its drawbacks; as we have seen in the Example chapter, due to probing, the control signal has high frequency components and to capture these high frequency components accurately we may need fast actuators. Although the controller tolerates noisy measurements - it is actually Bounded-Input-Bounded-Output (BIBO) stable - the gain on the noise may be large.

We would like to extend this approach to the situation in which the plant no longer has a strongly connected graph. A possible starting point is make use of [4], where it is proven that every decentralized system can be partitioned into a set of strongly connected subsets, inter-connected in a hierarchy. Perhaps recent work on handling hierarchical decentralized systems could be of use. Furthermore it will be interesting to see what can be achieved, instead of the H_∞ performance measure if we consider optimal H_2 performance in a decentralized setting.

Appendices

Appendix A

A.1 Proof of Lemma 1

Fix $\bar{n} \in \mathbf{N}$ and $\bar{h} \in (0,1)$. Let $\tilde{h} \in (0,1)$; $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}$, $h \in (0,\tilde{h})$,$\bar{u} \in \mathbf{R}$ and $\phi \in \mathbf{R}$ be arbitrary. The state of the plant(2.1) satisfies

$$x(t) = \mathrm{e}^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau) \int_{t_0}^{t} e^{A(t-\tau)}Er(\tau)\mathrm{d}\tau.$$

For the interval $t \in [t_0, t_0 + 2\bar{n}h)$, using the Cauchy Schwarz inequality we can easily form a bound

$$\begin{aligned}\|x(t) - x_0\| \leq &\left\|(\mathrm{e}^{A(t-t_0)} - I)x_0\right\| + \int_{t_0}^{t} \left\|e^{A(t-\tau)}B\right\| (\|\bar{u}\| + |\phi|)(\tau)\mathrm{d}\tau) \\ &+ \left(\int_{t_0}^{t} \left\|e^{A(t-\tau)}E\right\|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \underbrace{\left(\int_{t_0}^{t} \|r(\tau)\|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}}_{\left\|r_{[t_0,t]}\right\|_2}.\end{aligned}$$

Clearly there exist a constant $\gamma_1 > 0$ such that,

$$\begin{aligned}\|x(t) - x_0\| \leq \gamma_1 h(\|x_0\| + \|\bar{u}\| + |\phi|) + \gamma_1 h^{\frac{1}{2}} \left\|r_{[t_0,t_0+2\bar{n}h]}\right\|_2, \\ t \in [t_0, t_0 + 2\bar{n}h).\end{aligned} \tag{A.1}$$

Using the fact that $\bar{y}_i(t) = v_i^{\mathrm{T}} y_i(t)$ for $t \in [t_0, t_0 + \bar{n}h)$ we have

$$
\begin{aligned}
\bar{y}_i(t) &= v_i^{\mathrm{T}} C_2^i(\mathrm{e}^{A(t-t_0)} x_0 + \int_0^{t-t_0} e^{A(\tau)} Bu(t-\tau)\mathrm{d}\tau + \int_{t_0}^{t} e^{A(t-\tau)} Er(\tau)\mathrm{d}\tau)) \\
&= \sum_{k=0}^{\bar{n}} \frac{\bar{C}_2^i A^k (t-t_0)^k}{k!} x_0 + \mathcal{O}(h^{\bar{n}+1}) x_0 + \sum_{k=0}^{\bar{n}-1} \frac{\bar{C}_2^i A^k (t-t_0)^{k+1} B}{(k+1)!} (\bar{u} + \bar{w}_j \phi) \\
&\quad + \mathcal{O}(h^{\bar{n}+1})(\bar{u} + \phi) + \underbrace{\bar{C}_2^i \int_{t_0}^{t} e^{A(t-\tau)} Er(\tau)\mathrm{d}\tau)}_{\mu_1(t)} \\
&= \begin{bmatrix} 1 & t-t_0 & \cdots & (t-t_0)^{\bar{n}}/\bar{n}! \end{bmatrix} \begin{bmatrix} \bar{C}_2^i \\ \bar{C}_2^i A \\ \vdots \\ \bar{C}_2^i A^{\bar{n}} \end{bmatrix} x_0 \\
&\quad + \begin{bmatrix} 1 & t-t_0 & \cdots & (t-t_0)^{\bar{n}}/\bar{n}! \end{bmatrix} \begin{bmatrix} 0 \\ \bar{C}_2^i B \\ \vdots \\ \bar{C}_2^i A^{\bar{n}-1} B \end{bmatrix} (\bar{u} + \bar{w}_j \phi) \\
&\quad + \mathcal{O}(h^{\bar{n}+1}) x_0 + \mathcal{O}(h^{\bar{n}+1})(\bar{u} + \phi) + \mu_1(t).
\end{aligned}
$$

Before we proceed further, let us get a bound on the term $\mu_1(t)$. Using the Cauchy-Schwarz inequality $\mu_1(t)$ can be written in following manner:

$$
\begin{aligned}
\|\mu_1(t)\| &\leq \left(\int_{t_0}^{t} \|\bar{C}_2^i e^{A(t-\tau)} E\|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \left(\int_{t_0}^{t} \|r(\tau)\|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^{\bar{n}h} \left\| \sum_{k=0}^{\infty} \frac{\bar{C}_2^i A^k E \tau^k}{k!} \right\|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \|r_{[t_0, t_0+\bar{n}h]}\|_2
\end{aligned}
$$

Note that $\bar{C}_2^i A^k E$ is identically zero for all $k < rel.deg(\bar{C}_2^i (sI-A)^{-1} E) - 1$. However

$$
\begin{aligned}
rel.deg(\bar{C}_2^i (sI-A)^{-1} E) &\geq rel.deg(C_2 (sI-A)^{-1} E) \\
&= \eta_2,
\end{aligned}
$$

so $\bar{C}_2^i A^k E$ is zero for all $k < \eta_2 - 1$. Using this fact we can easily show that

$$
\|\mu_1(t)\| = \mathcal{O}(h)^{\eta_2 - \frac{1}{2}} \|r_{[t_0, t_0+\bar{n}h}\|_2.
$$

If we sample $\bar{y}_i$ in the interval of $[t_0, t_0 + \bar{n}h)$ and form $\bar{\mathcal{Y}}_i(t_0)$ we see that there exists a constant γ_2 and a function $\mu_2(h)$ so that

$$\bar{\mathcal{Y}}_i(t_0) = SH(h)[\begin{bmatrix} \bar{C}_2^i \\ \bar{C}_2^i A \\ \vdots \\ \bar{C}_2^i A^{\bar{n}} \end{bmatrix} x_0 + \begin{bmatrix} 0 \\ \bar{C}_2^i B \\ \vdots \\ \bar{C}_2^i A^{\bar{n}-1} B \end{bmatrix} (\bar{u} + \bar{w}_j \phi)] + \mu_2(h),$$

with

$$\|\mu_2(h)\| \leq \gamma_2 h^{\bar{n}+1}(\|x_0\| + \|\bar{u}\| + |\phi|) + \gamma_2 h^{\eta_2 - \frac{1}{2}} \left\| r_{[t_0, t_0+\bar{n}h]} \right\|_2 . \tag{A.2}$$

If we analyse $\bar{y}_i(t)$ for $t \in [t_0 + \bar{n}h, t_0 + 2\bar{n}h)$ in a similar fashion and form $\bar{\mathcal{Y}}_i(t_0 + \bar{n}h)$ we see that there exists a constant γ_3 and a function $\mu_3(h)$ so that

$$\bar{\mathcal{Y}}_i(t_0 + \bar{n}h) = SH(h)[\begin{bmatrix} \bar{C}_2^i \\ \bar{C}_2^i A \\ \vdots \\ \bar{C}_2^i A^{\bar{n}} \end{bmatrix} x(t_0 + \bar{n}h) + \begin{bmatrix} 0 \\ \bar{C}_2^i B \\ \vdots \\ \bar{C}_2^i A^{\bar{n}-1} B \end{bmatrix} (\bar{u} - \bar{w}_j \phi)] + \mu_3(h)$$

with

$$\|\mu_3(h)\| \leq \gamma_3 h^{\bar{n}+1}(\|x(t_0 + \bar{n}h)\| + \|\bar{u}\| + |\phi|) + \gamma_3 h^{\eta_2 - \frac{1}{2}} \|r_{[t_0+\bar{n}h, t_0+2\bar{n}h]}\|_2 . \tag{A.3}$$

Using the fact that $\bar{B}_j = B\bar{w}_j$ and subtracting $\bar{\mathcal{Y}}_i(t_0 + \bar{n}h)$ from $\bar{y}_i$ we obtain

$$\begin{aligned} &\bar{\mathcal{Y}}_i(t_0) - \bar{\mathcal{Y}}_i(t_0 + \bar{n}h) - 2SH(h) \begin{bmatrix} 0 \\ \bar{C}_2^i \bar{B}_j \\ \vdots \\ \bar{C}_2^i A^{\bar{n}-1} \bar{B}_j \end{bmatrix} \phi \\ &= SH(h) \begin{bmatrix} \bar{C}_2^i \\ \bar{C}_2^i A \\ \vdots \\ \bar{C}_2^i A^{\bar{n}} \end{bmatrix} (x(t_0 + \bar{n}h) - x_0) + \mu_2(h) - \mu_3(h) \\ &=: \mu_4(h). \end{aligned}$$

We can form a bound on $\|x(t_0 + \bar{n}h) - x_0\|$ using (A.1) and bounds on $\mu_2(h), \mu_3(h)$ using (A.2) and (A.3) respectively. It follows that there exist a constant γ_4 such that

$$\|\mu_4(h)\| \leq \gamma_4 h^{\bar{n}+1}(\|x_0\| + \|\bar{u}\| + |\phi|) + \gamma_4 h^{\eta_2 - \frac{1}{2}} \left\| r_{[t_0, t_0+2\bar{n}h]} \right\|_2 + \gamma_4 h^{\bar{n}+\frac{1}{2}} \left\| r_{[t_0, t_0+2\bar{n}h]} \right\|_2 .$$

Using the fact that $H(h)^{-1} = \mathcal{O}(h^{-\bar{n}})$ we see that there exist a γ_5 so that

$$\|H_{\bar{n}}(h)^{-1}S_{\bar{n}}^{-1}[\bar{\mathcal{Y}}_i(t_0) - \bar{\mathcal{Y}}_i(t_0+\bar{n}h)] - 2\begin{bmatrix} 0 \\ \bar{C}_2^i\bar{B}_j \\ \vdots \\ \bar{C}_2^i A^{\bar{n}-1}\bar{B}_j \end{bmatrix}\phi\|$$
$$\leq \gamma_5 h(\|x_0\| + \|\bar{u}\| + |\phi|) + \gamma_5 h^{\frac{1}{2}}\left\|r_{[t_0,t_0+2\bar{n}h]}\right\|_2 + \gamma_5 h^{\eta_2-\bar{n}-\frac{1}{2}}\left\|r_{[t_0,t_0+2\bar{n}h]}\right\|_2,$$

as required. $\square$

A.2 Proof of Lemma 2

We will model the proof on that of Lemma 2 of to [18]. Here our objective is to show the existence of the controller outlined and we will not pursue the lowest order representation of the controller.

Since the controller is LPC, it is sufficient to look at what happens on the interval $[kT, (k+1)T)$. In channel i we partition the state into three sub-states: ψ_i^1 ,ψ_i^2 and ψ_i^3. We will use ψ_i^1 of dimension ql_i to store $\{y_i(kT), y_i(kT+h), \cdots, y_i(kT+(q-1)h)\}$. The second sub-state$\psi_i^2$ of dimension m_i stores the estimation of the control signal ($\hat{\sqcap}_i[k]$). The last sub-state ψ_i^3 of dimension ℓ only comes comes to play in channel p. ψ_i^3 stores $\nu[k]$ in channel p.

We will limit our analysis to the interval of $[0, T)$ as it can easily be extended due to the periodic nature of the controller. Let e_i denote the i^{th} normal vector. We set

$$(L_i^{11}, M_i^1)[j] = \begin{cases} (0, e_1 \otimes I_{r_i}) & j = 0 \\ (I, e_{j+1} \otimes I_{r_i}) & j = 1, \cdots, q-1; \end{cases}$$

It is clear that the vector $\begin{bmatrix} \psi_i^1[j] \\ y_i(jh) \end{bmatrix}$ for $j = 0, 1, \cdots, q-1$ contains all the elements $\{y_i(0), y_i(h), \cdots, y_i(jh)\}$. Next we set

$$(L_i^{21}, L_i^{22}, M_i^2)[j] = \begin{cases} (0, I, 0) & j = 0, 1, \cdots, q-2 \\ (L_i^{21}[q-1], 0, M_i^2[q-1]) & j = q-1 \end{cases}$$

Since $\hat{\sqcap}_i[1]$ is a linear function of $\begin{bmatrix} \psi_i^1[j] \\ y_i(jh) \end{bmatrix}$ (See Key Estimation Lemma) we can choose $L_i^{21}[q-1], M_i^2[q-1]$, such that $\psi_i^2[q] = \hat{\sqcap}_i[1]$. With $q_p = 2\bar{n}(l - l_p)$ we set

$$(L_i^{31}, L_i^{33}, M_i^3)[j] = (0, 0, 0)$$
$$i = \{1, 2, \cdots, p-1\},\ j = \{0, 1, \cdots\}$$

$$(L_p^{31}, L_p^{33}, M_p^3)[j] = \begin{cases} (0, I, 0) & j = 0, 1, \cdots, q_p - 1 \\ (L_p^{31}[q_p], \mathcal{F}, M_p^3[q_p]) & j = q_p \\ (0, I, 0) & j = q_p + 1, \cdots, q - 1 \end{cases}$$

Since $\hat{y}(0)$ is a linear function of $\begin{bmatrix} \psi_i^1[j] \\ y_i(jh) \end{bmatrix}$ (See Key Estimation Lemma), we can choose $L_p^{31}[q_p]$ and $M_p^3[q_p]$ such that

$$L_p^{31}[q_p]\psi_p^1[q_p] + M_p^3[q_p]y_p(q_p h) = \hat{y}(0).$$

So if we initialize $\psi_p^3 = \nu[0]$, we see that

$$\psi_p^3[j] = \begin{cases} \nu[0], & j = 0, 1, \cdots, q_p - 1 \\ \nu[1], & j = q_p, \cdots, q - 1. \end{cases}$$

At this point we define

$$\phi_i[j] = \begin{bmatrix} \psi_i[j] \\ y_i(jh) \end{bmatrix}.$$

It is straightforward to verify that $\phi_i[j]$ contains $\{\hat{\sqcap}_i[0], y_i(0), \cdots, y_i((jh)\}$; for the case of $i = p$, $\phi_i[j]$ also contains

$$\begin{cases} \nu[0], & j = 0, 1, \cdots, q_p \\ \nu[1], & j = q_p + 1, \cdots, q - 1. \end{cases}$$

At this point we have defined L_i and M_i of K_{dec}. It remains to define the time varying matrices Q_i and R_i related to the outputs.

(i) For $j = 0, 1, \cdots, q_p - 1$ the control signal $u_i(jh)$ equals $\hat{\sqcap}_i[0]$ plus a linear combination of the elements of $y_i(0)$. Since $\hat{\sqcap}_i[0]$ is contained in $\psi_i[j]$ and $y_i(0)$ is contained in $\begin{bmatrix} \psi_i[j] \\ y_i(jh) \end{bmatrix}$; we see that $u_i(jh)$ is a linear combination of $\phi_i[j]$ for $j = 0, 1, \cdots, q_p - 1$.

(ii) At $j = q_p$ $u_i(j) = \hat{\sqcap}_i[0]$ and $\hat{\sqcap}_i[0]$ is contained in $\psi_i[j]$, hence it is a linear combination of elements in $\phi_i(j)$.

(iii) Now consider $j = q_p + 1, \cdots, q - 1$, the control signal $u_i(jh), i = 1, \cdots, p - 1$ equals $\hat{\sqcap}_i[0]$, which is contained in $\psi_i[j]$, so it is a linear combination of elements in $\phi_i[j]$. The control signal $u_p(jh)$ equals $\hat{\sqcap}_i[0]$ plus a scaled quantity of $\sqcap[1]$, both of which are contained in $\psi_p[j]$, so it is a linear combination of $\phi_p[j]$.

Using the fact that

$$u(t) = u(jh), \quad t \in [jh, (j+1)h), \quad k \in \mathbf{Z}^+$$

we conclude for $j = 0, 1, \cdots, q-1$ the control signal $u_i(jh)$ is a linear combination of $\phi_i[j]$, which means that Q_i and R_i can be defined so that the controller (2.4) is identical to that of (3.2)-(3.5).□

A.3 Proof of Lemma 3

Suppose that the controller (3.2)-(3.5) is applied to the plant (2.1), and let $x_0 \in \mathbf{R}^n$, $\nu[0] \in \mathbf{R}^\ell$, $\hat{\sqcap}[0] \in \mathbf{R}^m$, $k \in \mathbf{Z}^+$ and $T > 0$ be arbitrary. With the estimate $\hat{y}(kT)$ of $y(kT)$ given by (3.1) we see that

$$\hat{y}(kT) - y(kT) = \begin{bmatrix} \hat{y}_1(kT) - y_1(kT) \\ \hat{y}_2(kT) - y_2(kT) \\ \vdots \\ \hat{y}_{p-1}(kT) - y_{p-1}(kT) \\ 0 \end{bmatrix}.$$

We can use Lemma 1 to derive the estimation error of each individual quantity $y_i[k]_j$ for $i \in \{1, 2, \ldots, p-1\}$ and $j \in \{1, 2, \ldots, l_i\}$. More specifically, extending the error bound provided in (3.1) to the general case, we have

$$\begin{aligned} \|\hat{y}(kT) - y(kT)\| = \mathcal{O}(T^{1-\delta}) \sup_{\tau \in [kT, kT + 2\bar{n}(l - l_p)h)} \|x(\tau)\| + \mathcal{O}(T^{1-\delta}) \left\|\hat{\sqcap}[k]\right\| \\ + \mathcal{O}(T^{\frac{1}{2}-\delta}) \left\|r_{[kT, kT+2\bar{n}(l-l_p)h)}\right\|_2. \end{aligned}$$

For $t \in [kT, kT + 2\bar{n}(l - l_p)h)$,

$$x(t) = \mathrm{e}^{A(t-kT)}x(kT) + \int_{kT}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + \int_{kT}^{t} e^{A(t-\tau)}Er(\tau)\mathrm{d}\tau).$$

Therefore we can show that

$$\begin{aligned} \|x(t)\| &= \mathcal{O}(1) \|x(kT)\| + \int_{kT}^{t} [\mathcal{O}(1) \left\|\hat{\sqcap}[k]\right\| + \mathcal{O}(T^\delta) \|x(kT)\|] \mathrm{d}\tau \\ &\quad + \left(\int_{kT}^{t} \mathcal{O}(1)^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \|r_k\|_2 \\ &= \mathcal{O}(1) \|x(kT)\| + \mathcal{O}(T) \left\|\hat{\sqcap}[k]\right\| + \mathcal{O}(T^{\frac{1}{2}}) \|r_k\|_2. \end{aligned}$$

Using this we can simplify the output estimation error as

$$\begin{aligned}\|\hat{y}(kT) - y(kT)\| &= \mathcal{O}(T^{1-\delta}) \|x(kT)\| + \mathcal{O}(T^{2-\delta}) \left\|\hat{\sqcap}[k]\right\| \\ &\quad + \mathcal{O}(T^{\frac{3}{2}-\delta}) \|r_k\|_2 + \mathcal{O}(T^{1-\delta}) \left\|\hat{\sqcap}[k]\right\| + \mathcal{O}(T^{\frac{1}{2}-\delta}) \|r_k\|_2 .\end{aligned}$$

It is clear that there exist constant $\gamma_1 > 0$ such that

$$\|\hat{y}(kT) - y(kT)\| \leq \gamma_1 T^{1-\delta}(\|x(kT)\| + \left\|\hat{\sqcap}[k]\right\|) + \gamma_1 T^{\frac{1}{2}-\delta} \|r_k\|_2 , \tag{A.4}$$

which yields (4.4).

Now let us find a conservative bound for the signal $u(t)$ in the interval $t \in [kT, (k+1)T)$. Observe that

$$\begin{aligned}\left\|u(t) - \hat{\sqcap}[k]\right\| &= \mathcal{O}(T^{\delta}) \|y(kT)\| + \mathcal{O}(T^{\delta}) \|\sqcap[k+1]\| \\ &= \mathcal{O}(T^{\delta}) \|x(kT)\| + \mathcal{O}(T^{\delta}) \|\nu[k+1]\| + \mathcal{O}(T^{\delta}) \|\hat{y}(kT)\| , \\ &\qquad t \in [kT, (k+1)T).\end{aligned}$$

Using (A.4) to bound $\hat{y}(kT)$ and using the fact that $\nu[k+1] = \nu[k] + \mathcal{O}(T)\nu[k] + \mathcal{O}(T)\hat{y}(kT)$, we can simplify this to

$$\begin{aligned}\left\|u(t) - \hat{\sqcap}[k]\right\| &= \mathcal{O}(T^{\delta}) \|x(kT)\| + \mathcal{O}(T^{\delta}) \|\nu[k]\| + \mathcal{O}(T^{1+\delta}) \|\nu[k]\| + \mathcal{O}(T^{1+\delta}) \|\hat{y}(kT)\| \\ &\quad + \mathcal{O}(T^{\delta})\hat{y}(kT) \\ &= \mathcal{O}(T^{\delta}) \|x(kT)\| + \mathcal{O}(T^{\delta}) \|\nu[k]\| + \mathcal{O}(T) \left\|\hat{\sqcap}[k]\right\| + \mathcal{O}(T^{\frac{1}{2}}) \|r_k\|_2 , \\ &\qquad t \in [kT, (k+1)T),\end{aligned} \tag{A.5}$$

which yields (4.2).

Solving the state equation (2.1) yields

$$\begin{aligned}x(t) = \mathrm{e}^{A(t-kT)}x(kT) + \int_{kT}^{t} e^{A(t-\tau)} Bu(\tau)\mathrm{d}\tau + \int_{kT}^{t} e^{A(t-\tau)} Er(\tau)\mathrm{d}\tau), \\ t \in [kT, (k+1)T).\end{aligned}$$

Using (A.5) we can form a bound on

$$\begin{aligned}\|x(t) - x(kT)\| &= \mathcal{O}(T)x(kT) + \int_{kT}^{t} \mathcal{O}(1)(\|\hat{\sqcap\|}[k] + \mathcal{O}(T^{\delta})(\|x(kT)\| + \|\nu[k]\|) \\ &\quad + \mathcal{O}(T) \left\|\hat{\sqcap}[k]\right\| + \mathcal{O}(T^{\frac{1}{2}}) \|r_k\|_2)\mathrm{d}\tau + \left(\int_{kT}^{t} \mathcal{O}(1)\mathrm{d}\tau\right)^{\frac{1}{2}} \|r_k\|_2 \\ &= \mathcal{O}(T)(\|x(kT)\| + \left\|\hat{\sqcap}[k]\right\|) + \mathcal{O}(T^{1+\delta}) \|\nu[k]\| + \mathcal{O}(T^{\frac{1}{2}}) \|r_k\|_2\end{aligned} \tag{A.6}$$

which yields equation (4.1). We can perform similar analysis as before to obtain a bound on the quantity $\hat{\sqcap}[k+1] - \sqcap[k+1]$.We can use lemma 1 to obtain the estimation error of each element of $\hat{\sqcap}_i[k+1]_j$ for $i \in \{1, 2, \cdots, p-1\}$ and $j \in \{1, 2, \cdots, m_i\}$. More specifically, we use the bound in (3.1) with a minor change reflecting the fact that we are now probing with $T^\delta \hat{w}_p \sqcap_i[k+1]_j$ rather than $T^\delta \hat{w}_i[y_i(kT)]_j$. So (3.1) becomes (for $i = j = 1$):

$$\begin{aligned}&\mathcal{O}(T^{1-\delta})(\|x(kT + (2\bar{n}(l - l_p) + 1)h)\| + \left\|\hat{\sqcap}[k]\right\|)\\ &+ \mathcal{O}(T)|\sqcap_1[k+1]_1| + \mathcal{O}(T^{\frac{1}{2}-\delta})\left\|r_k\right\|_2.\end{aligned}$$

As a result we end up with

$$\begin{aligned}\left\|\hat{\sqcap}[k+1] - \sqcap[k+1]\right\| =&\mathcal{O}(T^{1-\delta}) \sup_{\tau \in [kT + (2\bar{n}(l-l_p)+1)h, (k+1)T)} \|x(\tau)\| \\ &+ \mathcal{O}(T^{1-\delta})\left\|\hat{\sqcap}[k]\right\| + \mathcal{O}(T)\left\|\sqcap[k+1]\right\| + \mathcal{O}(T^{\frac{1}{2}-\delta})\left\|r_k\right\|_2.\end{aligned} \tag{A.7}$$

It is easy to show that

$$\sqcap[k+1] = \mathcal{O}(1)v[k] + \mathcal{O}(1)\hat{y}(kT).$$

After we use the bound on $\hat{y}(kT)$ given by (A.4) to simplify this, we substitute the resulting expression for $\sqcap[k+1]$ into (A.7) and use (A.6) to obtain a bound on $\sup_{\tau \in [kT, (k+1)T)} \|x(\tau)\|$ yielding (4.4):

$$\left\|\hat{\sqcap}[k+1] - \sqcap[k+1]\right\| =\mathcal{O}(T^{1-\delta})(\|x(kT)\| + \left\|\hat{\sqcap}[k]\right\|) + \mathcal{O}(T)\left\|\nu[k]\right\| + \mathcal{O}(T^{\frac{1}{2}-\delta})\left\|r_k\right\|_2.$$

□

A.4 Proof of Claim 1

Let $k \in \mathbf{Z}^+$ be arbitrary. By definition, the near optimal centralized controller guarantees closed loop stability. Thus there exist constants $\gamma_0 > 0$ and $\lambda_0 < 0$ such that

$$\left\|e^{A_c t}\right\| \leq \gamma_0 e^{\lambda_0 t},\ t \geq 0. \tag{A.8}$$

To this end, freeze $\bar{\lambda}_0 \in (\lambda_0, 0)$ and consider the unique positive definite solution P_1 of the Lyapunov equation

$$(A_c - \bar{\lambda}_0 I)^T P_1 + P_1(A_c - \bar{\lambda}_0 I) = -I \tag{A.9}$$

Using an expanded positive definite matrix $P := \begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix}$ we can analyse the Lyapunov stability of the decentralized system. First define $\omega[k] := e^{\bar{\lambda}_0 kT} x_d[k]$; it follows that

$$\omega[k+1] = \underbrace{A_d(T)e^{-\bar{\lambda}_0 T}}_{\tilde{A}_d(T)} \omega[k], \quad k \geq 0$$

Given the Lyapunov candidate function

$$V(\omega[k]) := \omega[k]^{\mathrm{T}} P \omega[k]$$

we would like to show

$$\begin{aligned} \Delta V[k] &:= V(\omega[k+1]) - V(\omega[k]) \\ &= \omega[k]^{\mathrm{T}} (\tilde{A}_d(T)^{\mathrm{T}} P \tilde{A}_d(T) - P) \omega[k] \end{aligned}$$

is negative definite. After some simplifications, we see that

$$\bar{P}(T) := P - \tilde{A}_d(T)^{\mathrm{T}} P \tilde{A}_d(T)$$

becomes

$$\begin{bmatrix} IT + \mathcal{O}(T^{2-\delta}) & \mathcal{O}(T) \\ \mathcal{O}(T) & I + \mathcal{O}(T^{2-2\delta}) \end{bmatrix}.$$

We partition $\bar{P}(T)$ as $\begin{bmatrix} \bar{P}_{11}(T) & \bar{P}_{12}(T) \\ \bar{P}_{21}(T) & \bar{P}_{22}(T) \end{bmatrix}$ in a natural way.

For sufficiently small $\bar{T}$, the term $\bar{P}_{11}$ is positive definite. Moreover the Schur complement of $\bar{P}_{11}(T)$ in $\bar{P}(T)$ is

$$\bar{P}_{22}(T) - \bar{P}_{21}(T)\bar{P}_{11}(T)^{-1}\bar{P}_{12}(T) = I + \mathcal{O}(T)$$

is clearly positive definite. It follows that $-\bar{P}(T)$ is negative definite. Using standard arguments, it follows that for small T there exists a $\bar{\gamma}_0 > 0$ such that

$$\|\omega[k]\| \leq \bar{\gamma}_0 \|\omega[0]\|, \quad k \geq 0.$$

Hence, it follows immediately that for small T,

$$\|x_d[k]\| \leq \bar{\gamma}_0 e^{\bar{\lambda}_0 kT} \|x_d[0]\|, \quad k \geq 0. \tag{A.10}$$

$\square$

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