The Algebraic Kirchberg–Phillips Conjecture for Leavitt Path Algebras

by

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Author's declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This essay is meant to be an exposition of the theory of Leavitt path algebras and graph C*-algebras, with an aim to discuss some current classification questions. These two classes of algebras sit on opposite sides of a mirror, each reflecting aspects of the other. The majority of these notes is taken to describe the basic properties of Leavitt path algebras and graph C*-algebras, the main theme being the translation of graph-theoretic properties into exclusively (C*-)algebraic properties.

A pair of well-known results in the classification of C*-algebras, due to Elliott and Kirchberg–Phillips, state that the classes of approximately finite-dimensional (af) C*-algebras and purely infinite simple C*-algebras can be classified, up to isomorphism or Morita equivalence, by a pair of functors K_0, K_1 from the category of C*-algebras to category of abelian groups. Since simple graph C*-algebras must either be AF or purely infinite, combining the Elliott and Kirchberg–Phillips theorems yields a full classification of simple graph C*-algebras.

On the other side of the mirror, the corresponding Kirchberg–Phillips type question can be asked of Leavitt path algebras: can purely infinite simple Leavitt path algebras be classified by the algebraic K-theory functors $K_0, K_1 : \mathrm{Ring} \to \mathrm{Ab}$? This rift between graph C*-algebras and Leavitt path algebras has been partially bridged, the only remaining gap being the sign of $\det(I - A)$ where A is the adjacency matrix of the underlying graph. The "mixed sign" case remains unresolved.

The first section of this essay is dedicated to providing the definitions of the C*-algebras and Leavitt path algebras associated to graphs, and basic examples. The second section covers a collection of powerful theorems used to establish injectivity of homomorphisms whose domain is graph C*-algebra or Leavitt path algebra. The third section is a list of theorems characterizing various algebraic and C*-algebraic properties in completely graph-theoretic terms. The final section is an exposition of the state-of-affairs in the Kirchberg–Phillips classification of Leavitt path algebras.

Acknowledgments

A few conventions are maintained throughout the essay, unless specified otherwise.

- All rings are associative but not necessarily unital.

- For a ring k and k-algebra R, we assume $\alpha r = r\alpha$ for all $r \in R$ and $\alpha \in k$.

- If R is unital, then for a right R-module M we expect $m1 = m$ for all $m \in M$.

- E usually denotes a row-finite directed graph.

Contents

Note on numbering: Proposition 1.2.3 refers to the unique proposition in section 1.2.3. If there is more than one proposition there, then Proposition 1.2.3(4) will be used to refer to Proposition 4 in that section.

§1 The algebras associated to graphs

To a row-finite directed graph E we associate two algebras: the graph C*-algebra $C^*(E)$, and the Leavitt path algebra $L_k(E)$ when k is a field. These are defined using nearly identical generators and relations, and $L_{\mathbf{C}}(E)$ turns out to be to a dense subalgebra of $C^*(E)$. Defining an algebra by generators and relations is quite easy — take a quotient of a free algebra — but it is more difficult to construct universal C*-algebras. Thus the construction of $C^*(E)$ is specialized. Our first goal is to define $C^*(E)$ and establish some basic properties, and then we use that construction to motivate the definition of $L_k(E)$.

1.1 Motivation: Leavitt and Cuntz algebras

In [18], Leavitt introduced rings $L = L_n$, defined by generators and relations, with the malfunction that $L \simeq L^n$ as right L-modules but $L, L^2, \ldots, L^{n-1}$ are pairwise nonisomorphic. Independently, Cuntz [10] studied the C*-algebras $\mathcal{O}_n$ generated by isometries satisfying similar relations as in Leavitt's algebras L_n. Nowadays it is well-known that L_n is naturally a dense subalgebra of $\mathcal{O}_n$. In this section we highlight Leavitt's reasons for inventing L_n, and how L_n fits into the more general class of Leavitt path algebras.

1.1.1 Invariant basis number and module type. A unital ring R has **invariant basis number** (or **ibn**) if, whenever $R^m \simeq R^n$ as right R-modules, necessarily $m = n$. For such R, the *rank* of a free R-module can be defined to be the cardinality of a basis. Any field has ibn, since a vector space over a field has a uniquely determined dimension. In fact all division rings have ibn for the same reason.

Any commutative (unital) ring R has ibn: if m is a maximal ideal of R and R^n is any free R-module, then
$$R^n \otimes_R (R/m) \simeq (R/m)^n$$
as (R/m)-modules. But (R/m) is a field, so the dimension n is uniquely determined. More generally, whenever we have a unital ring homomorphism $R \to k$ and k has ibn, then R must also have ibn (same proof as above). Thus the class of ibn rings is quite large — but not all rings have ibn.

Example. [16] Let V be an infinite-dimensional vector space over a field and let $R = \mathrm{End}(V)$ be the ring of linear endomorphisms $V \to V$. Then any isomorphism $\varphi : V \simeq V \oplus V$ induces an isomorphism of vector spaces

$$R = \mathrm{Hom}(V, V) \simeq \mathrm{Hom}(V, V \oplus V) \simeq \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(V, V) = R^2.$$

But in fact one checks easily that it is an isomorphism of right R-modules, and so we see $R \simeq R^2 \simeq R^3 \simeq \cdots$. So R spectacularly fails to have ibn.

We've seen two extremes: ibn rings, versus rings with $R \simeq R^2$. Is there a middle-ground? For instance, is it possible that $R \simeq R^2 \not\simeq R^3$? Of course, once $R \simeq R^3$ we necessarily have $R^2 \simeq R^4 \simeq R^6 \simeq \cdots$, so we can't expect to have arbitrarily wild isomorphisms between free modules. In [18], Leavitt introduces the *module type* of a ring R which fails ibn. Let m be the first integer such that $R^m \simeq R^n$ for some $n > m$, and let n be the smallest with this property; the pair (m, n) is called the **module type** of R. Thus in the ring $R = \mathrm{End}(V)$ in the above example has module type $(1, 2)$.

We note a necessary condition for module type $(1, n)$.

Proposition. *Let R be a unital ring. Then $R \simeq R^n$ if and only if R has elements $x_1, \ldots, x_n, x_1^*, \ldots, x_n^*$ satisfying the relations*

    (CK1) $x_i^* x_i = 1$, *and* $x_i^* x_j = 0$ *for* $i \neq j$,

    (CK2) $1 = x_1 x_1^* + \cdots + x_n x_n^*$.

Proof. If $R \simeq R^n$, then R has a basis $\{x_1, \ldots, x_n\}$ as a right R-module. Then we may write $1 := x_1 x_1^* + \cdots + x_n x_n^*$ for some $x_i^* \in R$. Note that multiplying on the left by x_j gives

$$x_j = x_1(x_1^* x_j) + \cdots + x_n(x_n^* x_j)$$

which, by linear independence of the x_i's, implies $x_i^* x_j = 0$ if $i \neq j$ and $x_j^* x_j = 1$.

    Conversely, suppose x_i, x_i^* are elements of R satisfying (CK1) and (CK2). Then (CK2) implies that $R = x_1 R + \cdots + x_n R$, and (CK1) implies linear independence of the x_i's. So $x_1, \ldots, x_n$ is a basis for R as a right R-module, and so $R \simeq R^n$. ∎

    The two relations observed above are a special case of the **Cuntz–Krieger relations**, which can be defined via the structure of a directed graph E. The version of the CK relations above arises when E is the graph consisting of n loops at a single vertex.

1.1.2 Leavitt's algebras. Let k be a field. Following the ideas in 1.1.1 we define **Leavitt's algebra** $L = L_n$ as the free unital k-algebra on $2n$ generators $x_1, \ldots, x_n$, $x_1^*, \ldots, x_n^*$ satisfying the Cuntz–Krieger relations:

    (CK1) $x_i^* x_j = \delta_{ij}$,

    (CK2) $1 = x_1 x_1^* + \cdots + x_n x_n^*$.

By Proposition 1.1.1 we see that $L^n \simeq L$ as right L-modules. In [18], it is shown that, in fact, all the modules $L, L^2, \ldots, L^{n-1}$ are pairwise nonisomorphic, and therefore L has module type $(1, n)$. A few years later Leavitt proved that his algebras were simple in a strong sense, now known as *purely infinite simple*.

Theorem (Leavitt). *For $x \in L_n$, $x \neq 0$, there exist $a, b \in L_n$ such that $axb = 1$. In particular L_n is simple.*

    In fact, for each pair (m, n) with $m < n$, Leavitt constructs k-algebras $L(m, n)$ which have module type (m, n); moreover, $L_n = L(1, n)$.

1.1.3 Cuntz's algebras. Independently from Leavitt's construction, in [10] Cuntz defines C*-algebras $\mathcal{O}_n$ which look very similar to L_n. Let H be a separable Hilbert space and let $s_1, \ldots, s_n \in \mathcal{B}(H)$ be isometries satisfying the Cuntz–Krieger relations:

    (CK1) $s_i^* s_j = \delta_{ij}$,

    (CK2) $1 = s_1 s_1^* + \cdots + s_n s_n^*$.

These generate a unital C*-subalgebra $C^*(s_1, \ldots, s_n)$ of $\mathcal{B}(H)$. Cuntz showed that the $*$-isomorphism type of this C*-algebra is independent of the choice of the s_i's, and therefore we may uniquely discern a C*-algebra $\mathcal{O}_n$, called the **Cuntz algebra**. For this algebra he was able to show a result which is Leavitt's result verbatim, although through vastly different means. This was originally stated in [10].

Theorem (Cuntz). *For $x \in \mathcal{O}_n$, $x \neq 0$, there exist $a, b \in \mathcal{O}_n$ such that $axb = 1$. In particular $\mathcal{O}_n$ is simple.*

We will see that $\mathcal{O}_n$ is the graph C*-algebra arising from the graph consisting of n loops at a single vertex, exactly as with L_n, and it will be obvious from the graph that both of these algebras are purely infinite simple.

1.2 Preliminaries on directed graphs

In this section we fix the terminology and notation of directed graphs to be used throughout these notes. Instead of overwhelming the reader with technicalities now, more pertinent definitions will be given as they come up.

1.2.1 Directed graphs. A **directed graph** is a quadruple

$$E = (E^0, E^1, r, s)$$

where E^0 and E^1 are sets called the **vertex set** and **edge set** respectively, and $r, s : E^1 \to E^0$ are called the **range** and **source** functions. In other words, each edge e is an arrow from a vertex $s(e)$ to a vertex $r(e)$.
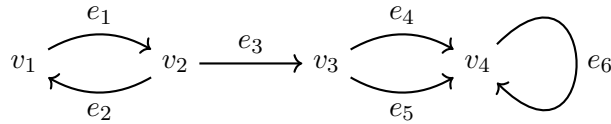
In the above picture we have $v_1 = s(e_1) = r(e_2)$, $v_2 = r(e_1) = s(e_2) = s(e_3)$, $v_3 = r(e_3) = s(e_4) = s(e_5)$, and $v_4 = r(e_4) = r(e_5) = r(e_6) = s(e_6)$.

Some extra terminology: for a vertex $v \in E^0$, we say that v is a **source** if it receives no edges, *i.e.* $r^{-1}(v) = \varnothing$; we say that v is a **sink** if it emits no edges, *i.e.* $s^{-1}(v) = \varnothing$.

a sink   a source
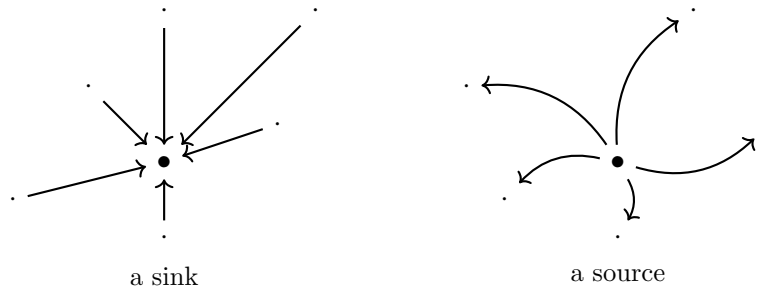
1.2.2 Row-finite graphs. Now we introduce the main type of directed graphs we are interested in: we say E is **row-finite** if $s^{-1}(v) = \{e \in E^1 : s(e) = v\}$ is a finite set for all vertices $v \in E^0$. Put simply, this means that no vertex emits infinitely many edges. The terminology comes from the equivalent property that the adjacency "matrix" A_E, defined by

$$A_E(v, w) = \# \text{ of edges from } v \text{ to } w$$
$$= \left| e \in E^1 : s(e) = v, \, r(e) = w \right|,$$

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has finite row sums. For technical reasons, we will only define Leavitt path algebras of row-finite directed graphs, but most of the theory can be bootstrapped to work for arbitrary directed graphs.

We remark that if E is a row-finite graph with finitely many vertices, then E must have finitely many edges too. Since the only graphs under consideration in these notes are row-finite, we thus speak only of "finite graphs" as opposed to "graphs with finitely many vertices".

1.2.3 Paths. Let E be a directed graph. A **path** in E is a sequence of edges $\mu = e_1 \cdots e_n$, such that $r(e_i) = s(e_{i+1})$ for each i. The **length** of μ is $|\mu| := n$; by convention, a path of length 0 is a vertex. We denote

$$E^n := \{\text{paths of length } n\}.$$

This is consistent with our notation so far: E^0 consists of vertices (paths of length 0) and E^1 consists of edges (paths of length 1). We also let $\mu^0 := \{s(e_1), \ldots, s(e_n), r(e_n)\}$ be the set of vertices on μ, and $\mu^1 := \{e_1, \ldots, e_n\}$ be the set of edges on μ.

Given a path $\mu \in E^n$, its **source** is $s(\mu) := s(e_1)$ and its **range** is $r(\mu) := r(e_n)$. Thus we can think of the source and range functions as functions $s, r : E^\bullet \to E^0$. If $\nu = f_1 \cdots f_m \in E^m$ is another path with $r(\mu) = s(\nu)$, we define the **concatenation** of μ and ν intuitively:

$$\mu\nu = e_1 \cdots e_n f_1 \cdots f_m.$$

This is a path of length $n + m$.

For two vertices $v, w \in E^0$, we write $v \to w$ to indicate the existence of a path μ such that $s(\mu) = v$ and $r(\mu) = w$. Thus $v \to w$ can be read "v leads to w".

A **cycle** in E is a path $\xi = e_1 \cdots e_n$ of length $n \geq 1$ such that $s(\xi) = r(\xi)$. We say that ξ is a **simple cycle** if it does not visit the same vertex twice, *i.e.* $r(e_1), \ldots, r(e_n)$ are all distinct. Finally we say that E is **acyclic** if E has no cycles.

1.2.4 Subgraphs. Let $E = (E^0, E^1, r_E, s_E)$ be a directed graph. A **subgraph** of E is a directed graph $F = (F^0, F^1, r_F, s_F)$ such that $F^0 \subseteq E^0$, $F^1 \subseteq E^0$, $r_F = r_E|_{F^1}$, and $s_F = s_E|_{F^1}$. In other words, we require a pair (F^0, F^1) such that if $f \in F^1$ then $s_E(f), r_E(f) \in F^0$. In this case we denote $F \hookrightarrow E$ and call this a **graph inclusion**. A graph inclusion $F \hookrightarrow E$ is **complete** if $s_F^{-1}(v) = s_E^{-1}(v)$ whenever $v \in F^0$ is not a sink in F; that is, every nonsink in F emits the same edges in F that it emits in E.
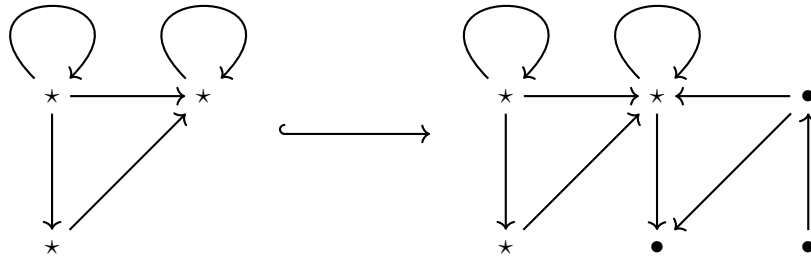
A graph inclusion which is not complete
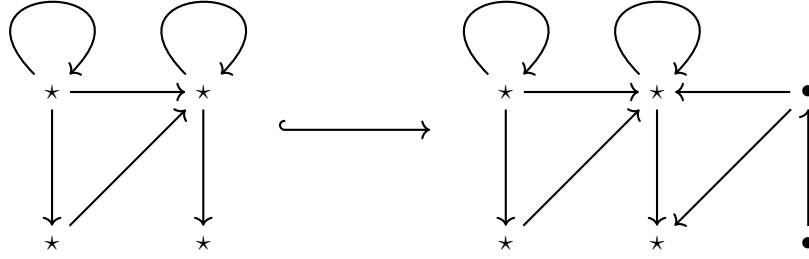
A complete graph inclusion

Later in these notes the following fact will be useful.

Proposition. *Every row-finite graph E is a direct union of finite complete subgraphs.*

By "direct" it is meant that the subgraphs involved in the union form a *directed set* under inclusion: given two such complete subgraphs F_1, F_2, there is a third F so that $F_1, F_2 \hookrightarrow F$.

Proof. For each finite set of vertices $X \subseteq E^0$, let $F^1 = F_X^1$ consist of all edges with $s(e) \in X$ and let $F^0 = F_X^0 := X \cup r(F^1)$. Thus F^0 consists of all vertices of X and all their outneighbors, and F^1 consists of all edges from vertices of X to their outneighbors. Now let
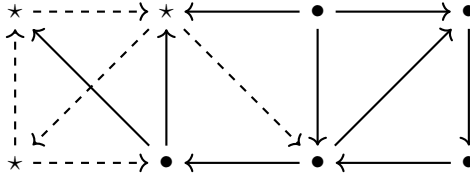
$$F = F_X := \left(F^0, F^1, r_F := r|_{F^1}, s_F := s|_{F^1} \right).$$

For example,

If X denotes the starred vertices then F_X^1 consists of the dashed edges, and F_X^0 consists of all starred edges in addition to the ranges of the dashed edges.

Notice that each $v \in F^0 \setminus X$ is a sink in F, and each $v \in X$ has $s_F^{-1}(v) = s_E^{-1}(v)$. Therefore $F \hookrightarrow E$ is a complete graph inclusion.

Since E is row-finite, F is finite by construction. For finite sets $X \subseteq Y \subseteq E^0$ we have complete graph inclusions $F_X \hookrightarrow F_Y$, so $\{F_X\}_{X \subseteq E^0 \text{ finite}}$ is a directed system of finite subgraphs of E. Finally, since $E = \bigcup_{X \subseteq E^0} F_X$ the proposition is established. ∎

1.3 Definition of a graph C*-algebra

The goal in this section is to define the C*-algebra $C^*(E)$ associated to a row-finite directed graph E. First we define C*-algebras generated by so-called Cuntz–Krieger E-families, and then define $C^*(E)$ as the universal such algebra.

This is done following the exposition in [23], except we have reversed graph-theoretic conventions from there: for example, in these notes "row-finite" means no infinite emitters, while in [23] it means no infinite receivers. There are very good reasons to maintain the convention in [23], but we deviate from it in order to stay consistent with the Leavitt path algebra literature. The main advantage is that paths are read *left to right*.

1.3.1 Cuntz–Krieger families. Let E be a row-finite directed graph. For technical reasons, we further assume that E^0 and E^1 are countable sets. The **graph C*-algebra** of E is the *universal* C*-algebra $C^*(E)$ generated by two families $S = \{s_e : e \in E^1\}$ and $P = \{p_v : v \in E^0\}$, subject to the following conditions:

(a) The p_v's are *mutually orthogonal* projections, *i.e.* $p_v^2 = p_v^* = p_v$ for all $v \in E^0$, and $p_v p_w = 0$ if $v \neq w$; and

(b) (S, P) satisfies the two **Cuntz–Krieger relations**:

$$(\text{CK1}) \ p_{r(e)} = s_e^* s_e \text{ for each edge } e \in E^1,$$

$$(\text{CK2}) \ p_v = \sum_{s(e)=v} s_e s_e^* \text{ whenever } v \in E^0 \text{ is not a sink.}$$

Note that the sum in (CK2) is nonempty since v is a not sink, and it's finite since E is row-finite. In general, a family (S, P) of elements in a C*-algebra A which satisfy conditions (a), (b) above is called a **Cuntz–Krieger E-family**.

By saying that $C^*(E)$ is *universal* we mean that it has the following universal property: for any C*-algebra A containing a CK E-family $(T = \{t_e\}_{e \in E^1}, Q = \{q_v\}_{v \in E^0})$, there exists a unique $*$-homomorphism $\varphi : C^*(E) \to A$ such that $\varphi(p_v) = q_v$ and $\varphi(s_e) = t_e$.

$$
\begin{array}{ccc}
(s_e, p_v) & \lhook\joinrel\longrightarrow & C^*(E) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi} \\
(t_e, q_v) & \lhook\joinrel\longrightarrow & A
\end{array}
$$

As is standard, it's easy to see that a C*-algebra with the universal property is determined uniquely up to isomorphism. But it's not clear that such a universal C*-algebra exists. We will first examine the structure of C*-algebras generated by CK E-families, and reverse-engineer this to construct $C^*(E)$.

An element s of a C*-algebra A such that $s^* s$ is idempotent is called a **partial isometry**; it is equivalent to require $ss^* s = s$. If $A = \mathcal{B}(H)$ is the algebra of bounded operators on a Hilbert space, s is a partial isometry if and only if its restriction to $(\ker s)^\perp$ is isometric, and in this case **initial projection** $s^* s$ is the orthogonal projection onto $(\ker s)^\perp$, and the **range projection** ss^* is the orthogonal projection onto the image of s. (For more information on partial isometries, see section A.1 in [23].) Thus each s_e in a CK E-family is a partial isometry, and (CK1) says that the initial projection of s_e is $p_{r(e)}$.

$$\bullet \xrightarrow{ s_e } \bullet \ s_e^* s_e$$

(CK2) says that to recover the projection p_v one can sum the range projections of all partial isometries s_e where e is an edge starting at v — but this only works if v emits at least one edge.
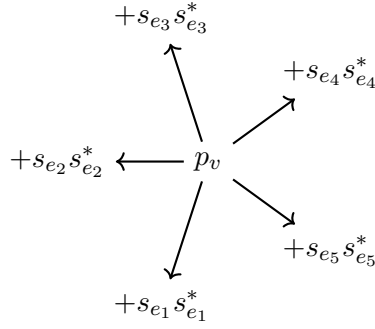
1.3.2 Examples. It turns out that many familiar C*-algebras can be generated by CK E-families for various directed graphs E.

Example 1. Here's a graph with two loops based at one vertex:

A CK family for this graph consists of one projection p_v and two partial isometries s_e, s_f, subject to $s_e^* s_e = p_v = s_f^* s_f$ and $p_v = s_e s_e^* + s_f s_f^*$. Taking $p_v := 1$ (as is necessary, we'll see later), we see that the CK relations for $\{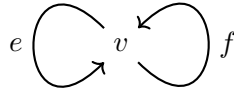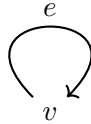p_v, s_e, s_f\}$ are exactly the relations satisfied by partial isometries generating the Cuntz algebra $\mathcal{O}_2$. Since $\mathcal{O}_2$ is the universal C*-algebra containing such generators with these relations, we conclude $C^*(E) \simeq \mathcal{O}_2$. More generally, if E is the graph consisting of n loops at a single vertex, then $C^*(E) \simeq \mathcal{O}_n$.

Example 2. If we take a graph consisting of a single loop,

then the CK relations amount to $s_e^* s_e = p_v = s_e s_e^*$; in particular $C^*(E)$ is commutative. For an example of a CK E-family, in the algebra $\mathcal{C}(\mathbf{T})$ of continuous functions $\mathbf{T} \to \mathbf{C}$ where $\mathbf{T}$ is the unit circle in the complex plane, let t_e be the identity function $z \mapsto z$ and let q_v be the constant function $z \mapsto 1$. These satisfy $t_e^* t_e = q_v = t_e t_e^*$, so by the universal property there is a $*$-homomorphism $\varphi : C^*(E) \to \mathcal{C}(\mathbf{T})$ sending $p_v \mapsto 1$ and $s_e \mapsto z$. Note that $\{1, z\}$ generates $\mathcal{C}(\mathbf{T})$ as a C*-algebra (due to the Stone–Weierstrass Theorem), so φ is surjective.

We'll prove that φ is injective, so that $C^*(E) \simeq \mathcal{C}(\mathbf{T})$. Observe that if $\sigma(s_e)$ is the spectrum of s_e, the continuous functional calculus (see Corollary I.3.2 in [11])

$$\mathcal{C}(\sigma(s_e)) \to C^*(E), \qquad f \mapsto f(s_e)$$

is a $*$-homomorphism with $z \mapsto s_e$ and $1 \mapsto 1 = p_v$, and is thus equal to φ^{-1} if we show $\sigma(s_e) = \mathbf{T}$. To this end, observing that the spectrum of z in $\mathcal{C}(\mathbf{T})$ is $\mathbf{T}$, we get

$$\mathbf{T} = \sigma(z) = \sigma(\varphi(s_e)) \subseteq \sigma(s_e) \subseteq \mathbf{T}$$

so $\sigma(s_e) = \mathbf{T}$ as required. In conclusion $C^*(E) \simeq \mathcal{C}(\mathbf{T})$.

Example 3. Now let's do the easiest example without a loop: a straight line.

$$v_1 \xrightarrow{\ e_1\ } v_2 \xrightarrow{\ e_2\ } v_3$$

The CK relations for this graph are

$$s_{e_1}^* s_{e_1} = p_{v_2}, \qquad s_{e_2}^* s_{e_2} = p_{v_3}, \qquad p_{v_1} = s_{e_1} s_{e_1}^*, \qquad p_{v_2} = s_{e_2} s_{e_2}^*.$$

We can cook up a nice CK family in $M_3(\mathbf{C})$ — if we let ε_{ij} denote the ij^{th} matrix unit, recall that $\varepsilon_{ij}\varepsilon_{jk} = \varepsilon_{ik}$ and $\varepsilon_{ij}^* = \varepsilon_{ji}$. So

$$p_{v_i} \mapsto \varepsilon_{ii}, \qquad s_{e_i} \mapsto \varepsilon_{i,i+1}$$

induces a surjective $*$-homomorphism $\varphi : C^*(E) \to M_3(\mathbf{C})$. In fact φ is injective, because the linear mapping $M_3(\mathbf{C}) \to C^*(E)$ given by

$$
\begin{array}{lll}
\varepsilon_{11} \mapsto p_{v_1} & \varepsilon_{12} \mapsto s_{e_1} & \varepsilon_{13} \mapsto s_{e_1} s_{e_2} \\
\varepsilon_{21} \mapsto s_{e_1}^* & \varepsilon_{22} \mapsto p_{v_2} & \varepsilon_{23} \mapsto s_{e_2} \\
\varepsilon_{31} \mapsto s_{e_2}^* s_{e_1}^* & \varepsilon_{32} \mapsto s_{e_2} & \varepsilon_{33} \mapsto p_{v_3}
\end{array}
$$

serves as a left inverse for φ. Of course, we could have done this if E was a line with n vertices, in which case $C^*(E) \simeq M_n(\mathbf{C})$.

Example 4. Next we'll do the easiest infinite graph: a straight line with infinitely many vertices.

$$v_1 \xrightarrow{\ e_1\ } v_2 \xrightarrow{\ e_2\ } v_3 \xrightarrow{\ e_3\ } \cdots$$

The CK relations for this graph are an infinite version of Example 3: for each i let

$$s_{e_i}^* s_{e_i} = p_{v_{i+1}}, \qquad p_{v_i} = s_{e_i} s_{e_i}^*.$$

We can realize a CK family for this graph inside $\mathcal{B}(H)$ by

$$p_{v_i} := \varepsilon_{ii}, \qquad s_{e_i} := \varepsilon_{i,i+1}$$

where ε_{ij} is the standard matrix unit in $\mathcal{B}(H)$. By the exact same calculation as in Example 4, these form a CK family. Each "finite truncation" of this graph yields a matrix algebra (as in the previous example), and so we have a chain

$$M_1(\mathbf{C}) \subseteq M_2(\mathbf{C}) \subseteq M_3(\mathbf{C}) \subseteq \cdots$$

where $M_n(\mathbf{C})$ embeds in $M_{n+1}(\mathbf{C})$ as the upper-left block. Thus $C^*(E)$ is the closure of the union $\bigcup_{n=1}^{\infty} M_n(\mathbf{C})$, which is well-known to be the C*-algebra $\mathcal{K}(H)$ of *compact* operators on H. Thus $C^*(E) \simeq \mathcal{K}(H)$.

Example 5. Let E be a cycle on n vertices: for example, if $n = 3$ then E can be drawn as follows.
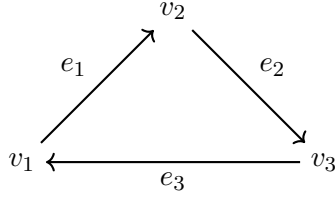
Again we can use matrix units to describe $C^*(E)$ as a familiar C*-algebra: in the matrix algebra $M_n(\mathcal{C}(\mathbf{T}))$, consider setting

$$p_{v_i} := \varepsilon_{ii}, \qquad s_{e_i} := \varepsilon_{i,i+1},\ 1 \le i < n, \quad \text{and} \quad s_{e_n} := z\varepsilon_{n1}.$$

Similar calculations as in the previous examples yield that these form a CK E-family in $M_n(\mathcal{C}(\mathbf{T}))$ — note $s_{e_n}^* s_{e_n} = \overline{z}z\varepsilon_{n1}^*\varepsilon_{n1} = \varepsilon_{11} = p_{v_1}$ since $\overline{z}z = |z|^2 = 1$. By a Stone–Weierstrass argument, observe that (s_{e_i}, p_{v_i}) generates $M_n(\mathcal{C}(\mathbf{T}))$ as a C*-algebra. The universal property therefore furnishes a surjective homomorphism $C^*(E) \to M_n(\mathcal{C}(\mathbf{T}))$. In 2.1.2 we will see a way to a general result which verifies the injectivity of this homomorphism, so we may conclude

$$C^*(E) \simeq M_n(\mathcal{C}(\mathbf{T})).$$

1.3.3 First consequences of the CK relations. Fix a row-finite graph E and let (S, P) be a CK E-family in $\mathcal{B}(H)$, where H is a Hilbert space. (Note that any C*-algebra is isometrically $*$-isomorphic to a subalgebra of $\mathcal{B}(H)$ by the Gelfand–Naimark Theorem, see Theorem I.9.12 in [11]; so this is the most general scenario.) Recall (CK2): if $v \in E^0$ is not a sink, then

$$p_v = \sum_{s(e)=v} s_e s_e^*.$$

From this it's clear that $\operatorname{im} s_e = \operatorname{im} s_e s_e^* \subseteq \operatorname{im} p_{s(e)}$, and s_e maps $\operatorname{im} p_{r(e)}$ isometrically into $\operatorname{im} p_{s(e)}$. Algebraically this means $s_e = p_{s(e)} s_e$. Since s_e is a partial isometry, we also have $s_e = s_e(s_e^* s_e) = s_e p_{r(e)}$; in total we have proven

$$s_e = p_{s(e)} s_e = s_e p_{r(e)}. \tag{$\spadesuit$}$$

Another consequence of (CK2) is that, for each vertex $v \in E^0$, the set $\{s_e s_e^* : s(e) = v\}$ is a *finite* collection of projections whose sum $\sum s_e s_e^* = p_v$ is a projection. This is only possible if the projections are mutually orthogonal (by Corollary A.3 in [23]), *i.e.* $(s_e s_e^*)(s_f s_f^*) = 0$ if $s(e) = v = s(f)$ and $e \ne f$. But even more is true: even if $s(e) \ne s(f)$, the same conclusion holds because the p_v's are mutually orthogonal:

$$\begin{aligned}
(s_e s_e^*)(s_f s_f^*) &= s_e(p_{s(e)} s_e)^*(p_{s(f)} s_f) s_f^* && \text{by } (\spadesuit) \\
&= s_e s_e^*(p_{s(e)} p_{s(f)}) s_f s_f^* \\
&= 0.
\end{aligned}$$

It also follows that $s_e^* s_f = 0$ if $e \ne f$, because

$$s_e^* s_f = s_e^*(s_e s_e^*)(s_f s_f^*) s_f = 0.$$

Lastly, another application of $(\spadesuit)$ yields

$$s_e s_f = s_e p_{r(e)} p_{s(f)} s_f = 0$$

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when $r(e) \neq s(f)$. In other words, if ef is not a sensible concatenation of edges in E, then the product is zero.

We summarize the above findings in the following proposition.

Proposition 1. *Let (S,P) be a CK E-family. Then:*

(i) $s_e = p_{s(e)}s_e = s_e p_{r(e)}$ *for all $e \in E^1$.*

(ii) $s_e s_f = 0$ *if $r(e) \neq s(f)$.*

(iii) $s_e^* s_f = 0$ *if $e \neq f$.*

These properties may be extended to paths: for a path $\mu = e_1 \cdots e_n$ in E, we define

$$s_\mu := s_{e_1} \cdots s_{e_n}.$$

If $\mu = v$ is a vertex (*i.e.* a path of length 0), we set $s_\mu := p_v$. Notice that in this notation we have $s_\mu^* s_\mu = p_{r(\mu)}$, reinforcing (CK1). The multiplicative properties of these path elements are derived immediately from the above proposition.

Proposition 2. *Let (S,P) be a CK E-family and let $\mu, \nu \in E^\bullet$. Then:*

(i) $s_\mu s_\nu = \begin{cases} s_{\mu\nu} & \text{if } r(\mu) = s(\nu), \\ 0 & \text{otherwise.} \end{cases}$

(ii) $s_\mu^* s_\nu = \begin{cases} s_{\mu'}^* & \text{if } \mu = \nu\mu' \text{ for some } \mu' \in E^\bullet, \\ s_{\nu'} & \text{if } \nu = \mu\nu' \text{ for some } \nu' \in E^\bullet, \\ 0 & \text{otherwise.} \end{cases}$

Part (i) says that multiplication corresponds to concatenation of paths, and part (ii) gives a formula for $s_\mu^* s_\nu$ when μ is an initial segment of ν or vice-versa.

Proof. (i) is obvious from Proposition 1, so we prove (ii). Suppose we have factored $\mu = \nu\mu'$, where $r(\nu) = s(\mu')$. Then

$$s_\mu^* s_\nu = (s_\nu^* s_\mu)^* = (s_\nu^* s_\nu s_{\mu'})^* = (p_{r(\nu)} s_{\mu'})^* = (p_{s(\mu')} s_{\mu'})^* = s_{\mu'}^*.$$

This proves the first case, and the second case is similar. For the third case, assume μ, ν are not initial segments of one another. Then if we write $\mu = e_1 \cdots e_n$ and $\nu = f_1 \cdots f_m$, we may choose the first i for which $e_i \neq f_i$. Thus $\nu = e_1 \cdots e_{i-1} f_i \cdots f_m$, and so repeatedly applying Proposition 1(i) we get

$$
\begin{aligned}
s_\mu^* s_\nu &= (s_{e_n}^* \cdots s_{e_1}^*)(s_{e_1} \cdots s_{e_{i-1}} s_{f_i} \cdots s_{f_m}) \\
&= s_{e_n}^* \cdots s_{e_2}^* (s_{e_1}^* s_{e_1}) s_{e_2} \cdots s_{e_{i-1}} s_{f_i} \cdots s_{f_m} \\
&= s_{e_n}^* \cdots s_{e_2}^* p_{r(e_1)} s_{e_2} \cdots s_{e_{i-1}} s_{f_i} \cdots s_{f_m} \\
&= s_{e_n}^* \cdots (s_{e_2}^* s_{e_2}) \cdots s_{e_{i-1}} s_{f_i} \cdots s_{f_m} \\
&\qquad \vdots \\
&= s_{e_n}^* \cdots s_{e_i}^* s_{f_i} \cdots s_{f_m} \\
&= 0
\end{aligned}
$$

since $s_{e_i}^* s_{f_i} = 0$ by Proposition 1(iii). $\blacksquare$

We use the last proposition to provide an insightful description of the algebra $C^*(S, P)$: it is densely spanned by elements of the form $s_\mu s_\nu^*$. Moreover we can give a formula for the product of two such elements.

Corollary 3. *Let (S, P) be a CK E-family. Then*

$$C^*(S, P) = \overline{\mathrm{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^\bullet\},$$

and we have the following formula for products: if $\mu, \nu, \alpha, \beta \in E^\bullet$, then

$$(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \begin{cases} s_{\mu\alpha'} s_\beta^*, & \text{if } \alpha = \nu\alpha' \\ s_\mu s_{\beta\nu'}^*, & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The formula for the product is immediate from Proposition 2, and it shows that $B := \overline{\mathrm{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^\bullet\}$ is closed under multiplication. Since B is clearly $*$-closed and topologically closed, it is a C*-subalgebra of $C^*(S, P)$. On the other hand, notice that $s_e = s_e s_{r(e)}^*$ and $p_v = s_v s_v^*$ are both in B for any $e \in E^1$ and $v \in E^0$. Thus $C^*(S, P) \subseteq B$. ∎

Using Corollary 3, we can characterize when $C^*(E)$ is unital. Moreover, even in the nonunital case we can explicitly describe an *approximate unit* for $C^*(E)$: if A is a separable C*-algebra and $\{e_n\}_{n=1}^\infty$ is a sequence in A, we say that $\{e_n\}$ is a **(countable) approximate unit** if $e_n a \to a$ and $a e_n \to a$ for all $a \in A$. It is a standard theorem that every separable C*-algebra admits an approximate unit (Theorem I.9.16 in [11]).

Corollary 4. *Let E be a (countable) row-finite graph.*

(a) *Let $E^0 = \{v_1, v_2, v_3, \dots\}$ be an enumeration of the vertices of E and let $\xi_n := \sum_{i=1}^n p_{v_n}$. Then $\{\xi_n\}_{n=1}^\infty$ constitutes an approximate unit for $C^*(E)$.*

(b) *$C^*(E)$ is unital if and only if E has finitely many vertices; in this case*

$$1 = \sum_{v \in E^0} v.$$

Note that the approximate unit $\{\xi_n\}$ described in (a) consists of *strictly increasing projections*; that is, each $\xi_n = \xi_n^2 = \xi_n^*$ and $\xi_1 < \xi_2 < \xi_3 < \cdots$.

Proof. (a) Let $a \in C^*(E)$; we must check that $\xi_n a \to a$ and $a\xi_n \to a$. By Corollary (3) it suffices to check for $a = s_\mu s_\nu^*$; in this case, if $s(\mu) = v_n$ then $v_i s_\mu = 0$ for all $i \neq n$ and $v_n s_\mu = s_\mu$, so

$$a\xi_N = \sum_{i=1}^N p_{v_i} s_\mu s_\nu^* = p_{v_n} s_\mu s_\nu^* = s_\mu s_\nu^* = a$$

for all $N \geq n$. Thus $a\xi_n \to a$ and similarly $\xi_n a \to a$.

(b) If E^0 is finite then so is E^1 by row-finiteness. In this case let $q := \sum_{v \in E^0} p_v$. We check that $q s_e = s_e q = s_e$ and $q p_w = p_w q = p_w$ for all $e \in E^1$ and $w \in E^0$: indeed,

$$q s_e = \sum_{v \in E^0} p_v s_e e = p_{s(e)} s_e = s_e = s_e p_{r(e)} = \sum_{v \in E^0} s_e p_v = s_e q$$

and

$$qp_w = \sum_{v \in E^0} p_v p_w = p_v p_v = p_v = \sum_{v \in E^0} p_w p_v = wq.$$

So q acts as the identity on all generators, hence $q = 1$.

Conversely let $\{\xi_n\}$ be as in part (a). If $1 \in C^*(E)$ then necessarily $\xi_n \to 1$ in norm. Since the group of units of a C*-algebra is open and contains 1, we must have that ξ_n is invertible for sufficiently large n. But then $\xi_n p_{v_{n+1}} = \sum_{i=1}^n p_{v_i} p_{v_{n+1}} = 0$ implies $p_{v_{n+1}} = 0$ — a contradiction. ∎

Here is a simple example to illustrate the calculations done in this section.

Example. Consider the graph
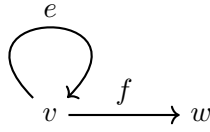
The CK relations for this graph are

$$p_v = s_e s_e^* + s_f s_f^*, \quad s_e^* s_e = p_v, \quad s_f^* s_f = p_w.$$

These say that the initial projection of s_e is p_v, the initial projection of s_f is p_w, and the image of p_v is the direct sum of the images of s_e and s_f. We will prove that $C^*(E)$ is generated by a nonunitary isometry.

Indeed, $u := s_e + s_f$. Using Proposition 1, we get

$$u^* u = s_e^* s_e + s_e^* s_f + s_f^* s_e + s_f^* s_f = p_v + p_w = 1,$$

where the last equality is by Corollary 4(b). But a similar calculation gives $uu^* = s_e s_e^* + s_f s_f^* = p_v \neq 1$. So u is a nonunitary isometry. To see that u generates $C^*(E)$ as a C*-algebra, observe $p_v = uu^*$, $p_w = u^*u - uu^*$, $s_e = s_e s_e^* s_e = s_e p_v = p_v = uu^*$, and $s_f = u - s_e = u - uu^*$. Thus $C^*(E) = C^*(u)$.

By Coburn's Theorem, up to $*$-isomorphism there is a unique C*-algebra generated by a nonunitary isometry, called the **Toeplitz algebra** $\mathcal{T}$ (see Theorem 3.5.18 in [20]). We therefore conclude $C^*(E) \simeq \mathcal{T}$ in this case.

1.3.4 The universal graph C*-algebra. In this section we will give an explicit construction of the universal C*-algebra $C^*(E)$. For set-theoretic reasons, we assume that E is countable (*i.e.* E^0 and E^1 are countable sets) so that we need only deal with separable C*-algebras and Hilbert spaces.

For the construction, we draw inspiration from the results of 1.3.3: $C^*(E)$ should be densely spanned by $\{s_\mu s_\nu^*\}_{\mu,\nu \in E^\bullet}$. But note that $s_\mu s_\nu^* = 0$ unless $r(\mu) = r(\nu)$, because

$$s_\mu s_\nu^* = s_\mu p_{r(\mu)} p_{r(\nu)} s_\nu^* = 0$$

if $r(\mu) \neq r(\nu)$. Thus we start by introducing formal symbols $s_{\mu,\nu}$ whenever $r(\mu) = r(\nu)$, and let them span a complex algebra

$$\mathcal{L} := \bigoplus_{\substack{\mu,\nu \in E^\bullet \\ r(\mu)=r(\nu)}} \mathbf{C} s_{\mu,\nu}$$

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whose multiplication mimics 1.3.3:

$$(s_{\mu,\nu})(s_{\alpha,\beta}) = \begin{cases} s_{\mu\alpha',\beta}, & \text{if } \alpha = \nu\alpha' \\ s_{\mu,\beta\nu'} & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases}$$

This gives rise to an associative multiplication on $\mathcal{L}$. We also introduce an involution on $\mathcal{L}$ by setting $s_{\mu,\nu}^* := s_{\nu,\mu}$ and extending conjugate-linearly to all of $\mathcal{L}$. So $\mathcal{L}$ is a complex $*$-algebra. To make it a C*-algebra we wish to define a positive-definite C*-norm. First, remark that any CK E-family must factor through $\mathcal{L}$ — $i.e.$ whenever we have a CK E-family (S, P), there is a surjective, $*$-preserving algebra homomorphism

$$\pi : \mathcal{L} \to C^*(S, P), \qquad s_{\mu,\nu} \mapsto s_\mu s_\nu^*.$$

This is due to 1.3.3. Since E is countable, the C*-algebra $C^*(S, P)$ is *separable*; thus we may restrict our attention to CK E-families (S, P) in $\mathcal{B}(H)$ where H is a fixed infinite-dimensional separable Hilbert space[1]. Define a norm on $\mathcal{L}$ as follows: for $a \in \mathcal{L}$,

$$\|a\| := \sup_\pi \|\pi(a)\|$$

where the sup is taken over all $*$-homomorphisms $\pi : \mathcal{L} \to \mathcal{B}(H)$. (By the above remark this is essentially a sup over the set of all CK E-families in $\mathcal{B}(H)$.) This sup is finite because, for any $a = \sum \lambda_{\mu,\nu} s_{\mu,\nu}$ in $\mathcal{L}$, we have

$$\|\pi_{(S,P)}(a)\| \leq \sum |\lambda_{\mu,\nu}| \underbrace{\|s_\mu\|\|s_\nu^*\|}_{\leq 1} \leq \sum |\lambda_{\mu,\nu}|$$

(we've used the fact that every partial isometry has norm ≤ 1). The function $\|\cdot\|$, thus defined, is a seminorm satisfying $\|ab\| \leq \|a\|\|b\|$ and the C*-identity $\|a^*a\| = \|a\|^2$, making $\mathcal{L}$ into a seminormed $*$-algebra. To make it a C*-algebra we use a standard trick: mod out by the kernel of $\|\cdot\|$, then take the metric space completion.

The set $N := \{n \in \mathcal{L} : \|n\| = 0\}$ is a topologically closed ideal which is $*$-closed, and so we may form a new $*$-algebra

$$\mathcal{L}_0 := \mathcal{L}/N, \quad \|a + N\| := \inf_{n \in N} \|a + n\|.$$

The quotient norm defined here is indeed positive definite by construction, and the submultiplicativity and C*-identity are maintained through this process. Now we take completion of $\mathcal{L}_0$ with respect to the quotient norm:

$$C^*(E) := \overline{\mathcal{L}_0}.$$

This is a C*-algebra, called the **graph C*-algebra** associated to E. From now on we make no effort to distinguish the elements $s_{\mu,\nu}$ in $\mathcal{L}$ with their images in $C^*(E)$.

It turns out that $C^*(E)$, thus defined, is generated by an appropriate CK E-family. Set $s_e := s_{e,r(e)}$ and $p_v := s_{v,v}$ in $C^*(E)$. Then $S := \{s_e\}$, $P := \{p_v\}$ constitute a CK E-family in $C^*(E)$, as is easily checked using the definition of multiplication. Moreover,

---

[1]We are using the fact that any separable C*-algebra is $*$-isomorphic to a subalgebra of $\mathcal{B}(H)$, where H is an infinite-dimensional separable Hilbert space.

in this notation we have $s_{\mu,\nu} = s_\mu s_\nu^*$, and so $C^*(E) = C^*(S, P)$. In particular $C^*(E)$ is separable. From now on we will use s_e, p_v, etc. to denote the CK E-family generating $C^*(E)$. For generic CK E-families we reserve t_e, q_v; we deviate from this convention at times, but the use will be clear from context.

Now we state the universal property possessed by $C^*(E)$, which is a byproduct of the construction.

Proposition (universal property). *Let E be a countable, row-finite graph. Then for any CK E-family (T, Q), there exists a unique $*$-homomorphism $\varphi : C^*(E) \to C^*(T, Q)$ such that $s_e \mapsto t_e$ and $p_v \mapsto q_v$.*

$$
\begin{array}{ccc}
C^*(E) & \longrightarrow & C^*(S, P) \\
s_\mu s_\nu^* & \longmapsto & t_\mu t_\nu^*.
\end{array}
$$

Proof. Fix a CK E-family (T, Q). Then the map $\pi : \mathcal{L} \to C^*(T, W)$, $s_{\mu,\nu} \mapsto t_\mu t_\nu^*$ descends to a bounded $*$-homomorphism $\mathcal{L}_0 \to C^*(T, Q)$ because $N \subseteq \ker \pi_{(T,Q)}$. By continuity this extends to the completion $\overline{\mathcal{L}_0} = C^*(E)$, and we denote the resulting map by $\varphi = \varphi_{(T,Q)}$.

$$
\begin{array}{ccc}
\mathcal{L} & \longrightarrow \mathcal{L}_0 \hookrightarrow & C^*(E) \\
& {\searrow}_{\pi} & \downarrow{\varphi} \\
& & C^*(S, P)
\end{array}
$$

Now φ is surjective because $\varphi(s_e) = t_e$ and $\varphi(p_v) = q_v$, and uniqueness of φ is clear since it is determined on the generators. ∎

1.4 Definition of a Leavitt path algebra

The reader undoubtedly realizes that almost none of the development in section 1.3 required the topological structure of C*-algebras — with the exception of the identity (♠) in 1.3.3, all arguments were "element-wise". So the only difference in defining the Leavitt path algebra is that we must impose (♠) in the defining relations.

Let E be a row-finite graph and let k be any field. Following the work in 1.3.3(2), the multiplication in the Leavitt path algebra $L_k(E)$ will be defined by concatenation of paths.

1.4.1 The Leavitt path algebra. Assume that E is row-finite. We define the **Leavitt path algebra**, or **Lpa**, denoted $L = L(E) = L_k(E)$, as the k-algebra generated by $\{v : v \in E^0\}$, $\{e, e^* : e \in E^1\}$ subject to the concatenation relations

- $s(e)e = er(e) = e$;

- $r(e)e^* = e^* s(e) = e^*$;

- $v^2 = v$, and $vw = 0$ if $v \neq w$;

in addition to the **Cuntz–Krieger relations**

(CK1) $e^* e = r(e)$ for all $e \in E^1$, while $e^* f = 0$ if $e \neq f$; and

(CK2) $v = \sum_{s(e)=v} ee^*$ whenever $v \in E^0$ is not a sink.

Formally, $L(E)$ is defined as the free (nonunital) k-algebra $k\langle v, e, e^* \rangle$ modulo the ideal generated by the elements defining the above relations. It's useful to think of the first three relations as simply concatenation of edges: if $r(e) \neq s(f)$, then $ef = (er(e))(s(f)f) = e(r(e)s(f))f = 0$ via the first and third relations. The elements e^* represent "ghost edges": if e points from v to w, then e^* points from w to v. These relations are enforced in order to "algebra-ify" Proposition 1.3.3(1).

We remark that the Leavitt path algebra of a not-necessarily-row-finite graph E can be defined by only imposing (CK2) at vertices v which are neither sinks nor infinite emitters; it turns out that even if E is not row-finite, $L(E)$ is sometimes Morita equivalent to $L(E')$ for some row-finite graph E'; see [2] for details.

The Leavitt path algebra $L(E)$ can be characterized by the following universal property, which is clear from the definition.

Proposition (universal property)**.** *Let E be a row-finite graph, and let R be a k-algebra containing elements $\{s_e, s_e^*\}_{e \in E^1}$, $\{p_v\}_{v \in E^0}$ which satisfy the following relations:*

- $p_{s(e)} s_e = s_e p_{r(e)} = s_e$;

- $p_{r(e)} s_e^* = s_e^* p_{s(e)} = s_e^*$;

- $p_v^2 = p_v$ *and* $p_v p_w = 0$ *if* $v \neq w$;

- $s_e^* s_e = p_{r(e)}$;

- $p_v = \sum_{s(e)=v} s_e s_e^*$ *when* $v \in E^0$ *is not a sink.*

Then there exists a unique k-algebra homomorphism $L_k(E) \to R$ such that $\mu\nu^* \mapsto s_\mu s_\nu^*$.

Note that the graph C*-algebra $C^*(E)$ contains a family of elements as in the above proposition, and therefore we have a map $L_{\mathbf{C}}(E) \to C^*(E)$. We will see section 2.4.2 that this map is injective and its image is dense in $C^*(E)$.

1.4.2 Examples. Same as for graph C*-algebras, we can realize some familiar rings as Leavitt path algebras. In each of the examples to follow, we will note that $L(E)$ sits naturally as a dense subalgebra of $C^*(E)$.

Example 1. Let E consist of n loops at one vertex:
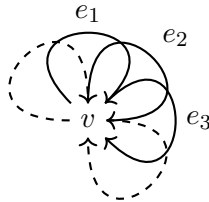
Notice that necessarily $v = 1$. The CK relations for this graph are $e_i^* e_i = v$ and $e_1 e_1^* + \cdots + e_n e_n^* = v$, which are exactly the relations in Leavitt's algebra L_n — thus $L(E) \simeq L_n$, which is dense in $\mathcal{O}_n \simeq C^*(E)$.

Example 2. Let E be a single loop:

$$\overset{e}{\underset{v}{\circlearrowright}}$$

The Leavitt path algebra $L(E)$ is generated by $\{v,e\}$ satisfying $e^*e = v = ee^*$; in particular $L(E)$ is commutative. The key observation is that $L(E)$ is unital and the identity is necessarily v, because $ve = s(e)e = e = er(e) = ev$ and $vv = v^2 = v$. So $v = 1$ in $L(E)$, and e is a unit with $e^{-1} = e^*$. We may thus identify $L(E) \simeq k[x, x^{-1}]$ via

$$
\begin{array}{ccc}
L(E) & \to & k[x, x^{-1}] \\
v & \mapsto & 1 \\
e & \mapsto & x
\end{array}
$$

In the case where $k = \mathbf{C}$ we may view Laurent polynomials as functions $\mathbf{T} \to \mathbf{C}$, and as such $z^{-1} = \bar{z}$. So $L_{\mathbf{C}}(E) \simeq \mathbf{C}[z, \bar{z}]$ is a conjugate-closed, unital subalgebra of $\mathcal{C}(\mathbf{T})$, hence by the Stone–Weierstrass Theorem it is dense in $\mathcal{C}(\mathbf{T}) \simeq C^*(E)$.

Example 3. By the same argument as in Example 1.3.2(3), the Leavitt path algebra of a straight line

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} v_n$$

is a full matrix algebra $L(E) \simeq M_n(k)$, via the following isomorphism:

$$L_k(E) \to M_n(k), \qquad v_i \mapsto \varepsilon_{ii}, \quad e_i \mapsto \varepsilon_{i,i+1}$$

where $\{\varepsilon_{ij}\}$ denotes the standard $n \times n$ matrix units in $M_n(k)$. Note in this case $L(E) \simeq C^*(E)$.

Example 4. Let $\varinjlim M_n(k)$ be the direct limit of the matrix algebras $M_n(k)$, where $M_n(k) \hookrightarrow M_{n+1}(k)$ is given by $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$. If E is the infinite line

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} \cdots$$

then, as in Example 1.3.2(4), the map $L(E) \to \varinjlim M_n(k)$ given by $v_i \mapsto \varepsilon_{ii}$, $e_i \mapsto \varepsilon_{i,i+1}$ is an isomorphism. We can realize $L_{\mathbf{C}}(E)$ as the ideal of finite rank operators in $\mathcal{B}(H)$ where H is separable and infinite-dimensional, and this is dense in the compact operators $\mathcal{K}(H) \simeq C^*(E)$.

Example 5. Let E be a loop with an exit:

$$\overset{e}{\underset{v}{\circlearrowright}} \xrightarrow{f} w$$
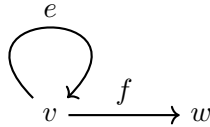
In Example 1.3.3 we saw that $C^*(E)$ is the Toeplitz algebra — the unique C*-algebra generated by a nonunitary isometry. Thus the Leavitt path algebra $L_k(E)$ is called the **algebraic Toeplitz algebra**. It is universally generated by the element $u := e + f$, which is left-invertible but not a unit.

Notice that in all the above examples of graphs E (except Example 5), we saw that $L(E)$ sits naturally as a dense subalgebra of $C^*(E)$.

1.4.3 Unit properties in Lpa's. Similar to graph C*-algebras, $L(E)$ is unital if and only if E is finite; we'll prove this by first establishing an analogue of Corollary 1.3.3(3) for $L(E)$. Even in the nonunital case, we will see that $L(E)$ is always *locally unital*. This is analogous to the fact that every C*-algebra has an approximate unit.

First we note the following formula, proven in the exact same way as Proposition 1.3.3(2)(ii): for paths μ, ν in E,

$$\mu^* \nu = \begin{cases} \gamma^* & \text{if } \mu = \nu\gamma, \\ \gamma & \text{if } \nu = \mu\gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{$\clubsuit$}$$

Proposition 1. *Let E be a row-finite graph. Then*

$$L(E) = \operatorname{span}\{\mu\nu^* : \mu, \nu \in E^\bullet\}.$$

Proof. The argument is practically identical to that of Corollary 1.3.3(3). Observe that the span on the right-hand side is closed under multiplication: for paths μ, ν, α, β, we have

$$(\mu\nu^*)(\alpha\beta^*) = \begin{cases} (\mu\alpha')\beta^*, & \text{if } \alpha = \nu\alpha', \\ \mu(\beta\nu')^*, & \text{if } \nu = \alpha\nu', \\ 0 & \text{otherwise} \end{cases}$$

which is immediate from ($\clubsuit$). So the RHS is a subalgebra of $L(E)$. But it contains the generators: $v = vv^*$, $e = er(e)$, and $e^* = r(e)e^*$. The result follows. ∎

Corollary. *Let E be a row-finite graph. Then $L(E)$ is unital if and only if E has finitely many vertices, and in this case the identity is*

$$1 = \sum_{v \in E^0} v.$$

Proof. If E^0 is finite then it is shown that $\sum_{v \in E^0} v$ is a unit for $L(E)$ with the same argument as Corollary 1.3.4. Conversely, suppose $1 \in L(E)$. Then by Proposition 1 we can write $1 = \sum_{i=1}^n c_i \mu_i \nu_i^*$, where μ_i, ν_i are paths with $r(\mu_i) = r(\nu_i)$ and $c_i \in k$ are scalars. If E^0 is infinite we may choose a vertex $v \notin \{s(\nu_1), \ldots, s(\nu_n)\}$. But then $\nu_i^* v = \nu_i^* s(\nu_i)v = 0$ for all i, and so we calculate

$$v = 1v = \sum_{i=1}^n c_i \mu_i \nu_i^* v = 0$$

which is not the case. ∎

Even if a ring R is nonunital, it is possible for subrings to have a unit; for example, if p is a (nonzero) idempotent in R then the corner pRp is always a unital subring of R with unit p. A ring R is called **locally unital** if each finite subset of R is contained in some unital subring S; equivalently, if each finite subset of R is contained in a corner pRp with $p \in R$ idempotent. Thus for every finite set $F = \{x_1, \ldots, x_n\}$, there exists an idempotent $p = p_F$ such that $x_i p = px_i = x_i$ for all i. Such a collection $\{p_F : F \subseteq R \text{ finite}\}$ is called a **family of local units** for R.

Proposition 2. *Every Lpa $L(E)$ is locally unital, with a family of local units given by*

$$\left\{ p_V := \sum_{v \in V} v : V \subseteq E^0 \text{ finite} \right\}.$$

Proof. Let $x_1, \ldots, x_n$ be finitely many elements in $L(E)$. By Proposition 1 we may choose a finite collection of paths $\mu_1, \nu_1, \ldots, \mu_m, \nu_m$ such that each x_i can be written as a linear combination $x_i = \sum_{j=1}^{m} c_{ij} \mu_j \nu_j^*$ where c_{ij} are scalars. Letting $V := \{s(\mu_j), s(\nu_j) : 1 \leq j \leq m\}$ and $p := \sum_{v \in V} v$, we see that $p = p^2$ since it is a sum of orthogonal idempotents, and for each $i = 1, \ldots, n$ we have

$$x_i p = \sum_{j=1}^{m} \sum_{v \in V} c_{ij} \mu_j \nu_j^* v = \sum_{j=1}^{m} c_{ij} \mu_j \nu_j^* = x_i$$

since $\nu_j^* v = 0$ unless $v = s(\nu_j)$, in which case $\nu_j^* v = \nu_j^*$. Similarly $p x_i = x_i$, and therefore each $x_i = p x_i p$ is contained in the corner $p L(E) p$. ∎

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§2 Uniqueness theorems

Suppose E is a row-finite and (S, P) is a CK E-family in some C*-algebra A. How can it be determined whether the C*-algebra $C^*(S, P)$ generated by (S, P) is in fact that *universal* graph C*-algebra? Since (S, P) is a CK E-family, the universal property automatically furnishes a *-homomorphism $\varphi : C^*(E) \to A$ mapping onto $C^*(S, P)$, but in general it can be difficult to verify that φ is injective. In the past examples we have used such a method, and each time there was an ad-hoc method used to prove injectivity. The goal of this section is to introduce theorems which give *automatic* injectivity in common cases — these are called *uniqueness theorems*. There are also such theorems for Leavitt path algebras.

There are essentially two flavors of uniqueness theorems. The first requires that φ preserve some "extra structure" of $C^*(E)$ or $L(E)$, and as it turns out, these structures are "dual" to one another. For this reason there are two separate algebraic uniqueness theorems, detailed in the first two sections. The second flavor requires a geometric property on E, which is the same for $C^*(E)$ and $L(E)$, and is described in the third section. The fourth section consists of two applications of graded uniqueness: we use it to show that certain graph homomorphisms $F \to E$ induce embeddings $L(F) \hookrightarrow L(E)$, and that $L_{\mathbf{C}}(E)$ is a dense *-subalgebra of $C^*(E)$. No uniqueness theorems will be proven, so the sections will be brief.

2.1 Gauge-invariant uniqueness theorem

The gauge-invariant uniqueness theorem states that if $\varphi : C^*(E) \to A$ is a *-homomorphism such that $\varphi(p_v) \neq 0$ for all $v \in E^0$, then φ is injective in case it preserves a certain family of automorphisms on $C^*(E)$, called the *gauge action*. We thus require a similar action on A for this to make sense.

2.1.1 The gauge action. Let G be a topological group and A any C*-algebra. An **action** of G on A is nothing but a group homomorphism $\gamma : G \to \mathrm{Aut}^*(A)$, where $\mathrm{Aut}^*(A)$ is the group of *-automorphisms of A. We denote $\gamma_g := \gamma(g)$. The action γ is **strongly continuous** if, for each fixed $a_0 \in A$, the map $g \mapsto \gamma_g(a_0)$ is a continuous map $G \to A$.

We now describe a strongly continuous action carried by all graph C*-algebras.

Proposition (gauge action)**.** *If E is a row-finite graph, there is a natural strongly continuous action γ of $\mathbf{T}$ on $C^*(E)$, given on the generators as $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = z s_e$.*

The action γ given in this proposition is called the **gauge action** on $C^*(E)$.

Proof. To verify that this γ well-defined, first note that for each fixed $z \in \mathbf{T}$ we have a CK E-family $(z s_e, p_v)$ in $C^*(E)$. Indeed, it satisfies (CK1) because

$$(z s_e)^*(z s_e) = \bar{z} z s_e^* s_e = s_e^* s_e = p_{r(e)}$$

and it satisfies (CK2) because, for each nonsink v,

$$\sum_{s(e)=v} (z s_e)(z s_e)^* = \sum_{s(e)=v} z \bar{z} s_e s_e^* = \sum_{s(e)=v} s_e s_e^* = p_v.$$

Thus (zs_e, p_v) is a CK E-family in $C^*(E)$, so by the universal property there exists a unique $*$-homomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(s_e) = zs_e$ and $\gamma(p_v) = p_v$. It is easily checked that γ_z is invertible with $\gamma_z^{-1} = \gamma_{\bar{z}}$, and that $\gamma : \mathbf{T} \to \mathrm{Aut}^*(C^*(E))$ is a group homomorphism.

It remains to check that γ is strongly continuous: that is, for each $a_0 \in C^*(E)$, the map $z \mapsto \gamma_z(a_0)$ is continuous. It suffices to verify this for a_0 in the dense subalgebra $\mathrm{span}\{s_\mu s_\nu^*\}$, and by linearity it suffices to check for a single "monomial" $a_0 = s_\mu s_\nu^*$. But then we have $\gamma_z(a_0) = z^{|\mu|-|\nu|} a_0$, which is clearly continuous in z. This completes the proof. ∎

The below examples illustrate how the gauge action looks in familiar examples.

Example. Recall that $\mathcal{C}(\mathbf{T})$ is the C*-algebra of a single loop based at a vertex; the vertex corresponds to the constant function 1, and the edge corresponds to the identity function $\iota(w) := w$. Thus the gauge action is given by $\gamma_z(1) = 1$ and $\gamma_z(\iota) = z\iota$. On a finite Laurent series $f(w) = \sum_{n \in \mathbf{Z}} a_n w^n$ we have

$$\gamma_z(f)(w) = \sum_{n \in \mathbf{Z}} (a_n z^n) w^n = f(zw).$$

Example. $M_n(\mathbf{C})$ is the C*-algebra of a straight line on n vertices; the vertices correspond to ε_{ii} and the edges to $\varepsilon_{i,i+1}$, so $\gamma_z(\varepsilon_{ii}) = \varepsilon_{ii}$ and $\gamma_z(\varepsilon_{i,i+1}) = z\varepsilon_{i,i+1}$. On an arbitrary matrix unit ε_{ij} we have $\gamma_z(\varepsilon_{ij}) = \gamma_z(\varepsilon_{i,i+1} \cdots \varepsilon_{j-1,j}) = z^{j-i}\varepsilon_{ij}$, and on a matrix $a = (a_{ij})_{ij} \in M_n(\mathbf{C})$ we have

$$\gamma_z(a) = \sum_{i,j} a_{ij} \gamma_z(\varepsilon_{ij}) = (z^{j-i} a_{ij})_{ij}.$$

2.1.2 The gauge-invariant uniqueness theorem. We now provide a powerful criterion to check that a $*$-homomorphism is injective, given that its domain is a graph C*-algebra equipped with its gauge action. To guarantee this we need γ to "intertwine" the gauge action with another strongly continuous action.

Theorem (gauge-invariant uniqueness). *Let A be a C*-algebra equipped with a strongly continuous action σ of $\mathbf{T}$. Suppose that $\varphi : C^*(E) \to A$ is $*$-homomorphism which intertwines σ with the gauge action:*

$$\sigma_z \circ \varphi = \varphi \circ \gamma_z \quad \text{for all } z \in \mathbf{T}.$$

If $\varphi(p_v) \neq 0$ for all $v \in E^0$, then φ is injective.

Any map φ satisfying the intertwining property as in the above theorem is called a **gauge-invariant** map. The proof of this theorem is the topic of Chapter 3 in [23].

Example. Let E be the following graph:
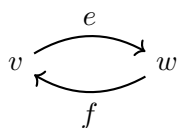
In Example 1.3.2(5), we used the universal property of $C^*(E)$ to establish a surjective $*$-homomorphism

$$\varphi : C^*(E) \to M_2(\mathcal{C}(\mathbf{T})), \qquad \begin{cases} p_v \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & p_w \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ s_e \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & s_f \mapsto \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix}. \end{cases}$$

where u denotes the identity function in $\mathcal{C}(\mathbf{T})$. Equipped with the gauge-invariant uniqueness theorem we can prove that φ is injective. Consider the strongly continuous action σ of $\mathbf{T}$ on $M_2(\mathcal{C}(\mathbf{T}))$ induced by

$$\sigma_z(\varepsilon_{ij}) = \varepsilon_{ij}, \qquad \sigma_z(u\varepsilon_{ij}) := zu\varepsilon_{ij}.$$

Then it is easy to show that φ intertwines σ with the gauge action: $\varphi(\gamma_z(x)) = \sigma_z(\varphi(x))$ for all $x \in C^*(E)$. Since $\varphi(p_v) \neq 0 \neq \varphi(p_w)$ we conclude that φ is injective by the gauge-invariant uniqueness theorem.

2.2 Graded uniqueness theorem

In this section we provide a theorem which is similar to the gauge-invariant uniqueness theorem, except instead of requiring φ to preserve the gauge action on $C^*(E)$, we require it to preserve a *grading* on $L(E)$. First we introduce graded rings in general, then describe the grading on $L(E)$.

2.2.1 Graded rings and algebras. Let Γ be an abelian group and let R be a ring. We say that R is Γ-**graded** if R has additive subgroups $\{R_g\}_{g \in \Gamma}$ such that

$$R = \bigoplus_{g \in \Gamma} R_g$$

as abelian groups, and $R_g R_h \subseteq R_{gh}$ for $g, h \in \Gamma$. We say R is a Γ-**graded ring**. Elements $x \in R_g$ are called **homogeneous**, and we define the **degree** by $\deg(x) := g$. By definition, if x, y are homogeneous elements then $\deg(xy) = \deg(x)\deg(y)$.

Example. The ring $R := k[x, x^{-1}]$ admits a natural $\mathbf{Z}$-grading by $R_n := \text{span}\{x^n\}$; in other words, x has degree 1 and x^{-1} has degree -1. Since $x^n x^m = x^{n+m}$ this is easily seen to be legitimate $\mathbf{Z}$-grading.

Let $R = \bigoplus_{g \in \Gamma} R_g$ be a graded ring. An ideal I in R is a **graded ideal** if

$$I = \bigoplus_{g \in \Gamma} I \cap R_g;$$

that is, if $x = x_1 + \cdots + x_n \in I$ is a sum of homogeneous elements x_i then each $x_i \in I$. If R and S are both Γ-graded rings, we say that a ring homomorphism $\varphi : R \to S$ is a **graded homomorphism** if $\varphi(R_g) \subseteq S_g$ for all $g \in \Gamma$. Thus a graded homomorphism preserves the degree of a homogeneous element.

It turns out that quotients of Γ-graded rings by graded ideals remain Γ-graded: this is the content of the following standard fact.

Proposition. *Let R be a Γ-graded ring.*

(a) *An ideal I is graded if and only if I is generated by homogeneous elements.*

(b) *If $\varphi : R \to S$ is a graded homomorphism, then $\ker \varphi$ is a graded ideal.*

(c) *If I is a graded ideal of R, then R/I admits a natural Γ-grading via*

$$(R/I)_g := R_g/(I \cap R_g).$$

Proof. (a) If I is graded and $X = \bigcup_{g \in \Gamma} I \cap R_g$ is the set of all homogeneous elements in I, then clearly the ideal generated by X is $\sum I \cap R_g = I$. So I is generated by its homogeneous elements. Conversely assume I is generated by a set S of homogeneous elements; then $S \subseteq \bigcup_{g \in \Gamma} I \cap R_g$, so

$$\begin{aligned} I &= \text{ideal generated by } S \\ &\subseteq \text{ideal generated by } \bigcup_{g \in \Gamma} I \cap R_g \\ &= \sum_{g \in \Gamma} I \cap R_g. \end{aligned}$$

The reverse inclusion is obvious, hence $I = \sum_{g \in \Gamma} I \cap R_g$ and I is graded.

(b) Let $x \in \ker \varphi$. If $x = x_1 + \cdots + x_n$ is the decomposition of x into homogeneous elements, then in S we have

$$0 = \varphi(x) = \varphi(x_1) + \cdots + \varphi(x_n).$$

As $\varphi(x_1), \ldots, \varphi(x_n)$ are homogeneous of distinct degrees (since φ is graded), this is only possible if $\varphi(x_i) = 0$ for all i — that is, $x_i \in \ker \varphi$. Therefore $\ker \varphi \subseteq \sum_{g \in \Gamma} \ker \varphi \cap R_g$.

(c) Let $\pi : R \to R/I$, $x \mapsto \overline{x}$ be the quotient map; the fact that I is a graded ideal means precisely that π is a graded homomorphism. We'll show that

$$S_g := R_g/(I \cap R_g) = \pi(R_g)$$

are the homogeneous components for a Γ-grading on R/I. Clearly $R/I = \sum_{g \in \Gamma} S_g$. We can see that the sum is direct as follows: if $\overline{x} \in S_g \cap S_h$ then $x \in R_g \cap R_h$ since π is graded, but $R_g \cap R_h = 0$. Finally, if $\overline{x} \in S_g$ and $\overline{y} \in S_h$, then $x \in R_g$ and $y \in R_h$, so

$$\overline{x} \cdot \overline{y} = \overline{xy} \in \pi(R_{gh}) = S_{gh}.$$

So $R/I = \bigoplus_{g \in \Gamma} S_g$ is a Γ-grading, as required. ∎

2.2.2 The Z-grading on an Lpa. For a row-finite graph E, the Lpa $L(E)$ is **Z**-graded. To see this, first we grade the free algebra $k\langle v, e, e^* \rangle$ as follows: define the **degree** of a generator by $\deg(v) := 0$, $\deg(e) := 1$, and $\deg(e^*) := -1$, and for a monomial $w = x_1 \cdots x_n$ with $x_i \in \{v, e, e^*\}$, define $\deg(w) := \sum_{i=1}^n \deg(x_i)$. Now the homogeneous components of the free algebra are defined to be

$$k\langle v, e, e^* \rangle_n := \text{span}\{\text{monomials of degree } n\}.$$

Since deg is an additive function on the semigroup of monomials, these are easily seen to be the homogeneous components of a **Z**-grading on $k\langle v, e, e^* \rangle$. Now, the Leavitt

path algebra $L(E)$ is precisely the quotient of $k\langle v, e, e^*\rangle$ by the ideal I generated by the elements

$$e - s(e)e, \quad e - er(e), \quad e^* - r(e)e^*, \quad e^* - e^*s(e),$$

$$v - v^2, \quad vw, \quad r(e) - e^*e, \quad v - \sum_{s(e)=v} ee^*$$

all of which are homogeneous. So I is a graded ideal by Proposition 2.2.1(a), and so by (c) we see that $L(E) = k\langle v, e, e^*\rangle/I$ is also $\mathbf{Z}$-graded as described in that proposition, and the homogeneous components are simply the images of the components of $k\langle v, e, e^*\rangle$ in the quotient.

There is a more natural interpretation of the homogeneous components of $L(E)$.

Proposition. *The graded components of the Leavitt path algebra $L(E)$ are precisely*

$$L(E)_n = \mathrm{span}\{\mu\nu^* : |\mu| - |\nu| = n\}.$$

Proof. First note that $\deg(\mu\nu^*) = |\mu| - |\nu|$ by definition of the degree function, so "$\supseteq$" is easily seen. Conversely, let $x \in L(E)_n$. By Proposition 1.4.3(1) we may write $x = \sum_{i=1}^n c_i \mu_i \nu^i$. But each $\mu_i \nu_i^*$ has degree $|\mu_i| - |\nu_i|$, so $c_i = 0$ unless $|\mu_i| - |\nu_i| = n$ since we assumed x was in the nth component. Therefore x is in the right-hand side. $\blacksquare$

To summarize this section, we've shown that an Lpa $L(E)$ admits a natural grading

$$L(E) = \bigoplus_{n \in \mathbf{Z}} L(E)_n.$$

The following examples illustrate this grading on familiar examples.

Example. $k[x, x^{-1}]$ is the Lpa of a single loop, whose vertex is 1 and whose edge is x. Thus if $\mu\nu^*$ is a monomial with $|\mu| - |\nu| = n$, then $\mu\nu^* = x^{|\mu|}(x^*)^{|\nu|} = x^{|\mu|-|\nu|} = x^n$ since $x^* = x^{-1}$. From this it's easy to see that the natural grading of $k[x, x^{-1}]$ agrees with its grading as a Leavitt path algebra.

Example. Recall that $M_d(k) \simeq L(E)$ where E is a straight line.

$$v_1 \xrightarrow{\ e_1\ } v_2 \xrightarrow{\ e_2\ } \ \cdot \ \cdot \ \cdot \ \xrightarrow{\ e_{d-1}\ } v_d$$

The identification $L(E) \simeq M_d(k)$ is given by $v_i \mapsto \varepsilon_{ii}$ and $e_i \mapsto \varepsilon_{i,i+1}$, where ε_{ij} are the standard matrix units. Now, the only paths in E are $\mu_{ij} := e_i \cdots e_j$ with $1 \leq i \leq j \leq d$, which has length $|\mu_{ij}| = j - i + 1$, so a typical monomial $\mu_{ij}\mu_{k\ell}^*$ looks like

$$\begin{aligned}
\mu_{ij}\mu_{k\ell}^* &= e_i \cdots e_j (e_k \cdots e_\ell)^* \\
&= \varepsilon_{i,i+1} \cdots \varepsilon_{j,j+1}(\varepsilon_{k,k+1} \cdots \varepsilon_{\ell,\ell+1})^* \\
&= \varepsilon_{i,j+1}\varepsilon_{k,\ell+1}^* \\
&= \varepsilon_{i,j+1}\varepsilon_{\ell+1,k} \\
&= \delta_{j\ell}\varepsilon_{ik}
\end{aligned}$$

which, in case $j = \ell$, has degree $n = |\mu_{ij}| - |\mu_{k\ell}| = (j - i + 1) - (\ell - k + 1) = k - i$. Therefore the nth graded component of $M_d(k)$ is the subspace spanned by $\{\varepsilon_{ik} : k - i = n\}$; in particular the nth component is nonzero only for $|n| \leq d - 1$.

Example. Recall that the dth Leavitt algebra is $L_d = L(E)$, where E consists of one vertex v and d edges $e_1, \ldots, e_d$. Thus it satisfies the relations $e_i^* e_j = \delta_{ij}$ and $\sum e_i e_i^* = 1$. The paths in E all have the form $\mu = e_{i_1} \cdots e_{i_k}$ for some $i_1, \ldots, i_k \geq 1$, and so a monomial $\mu\nu^*$ looks like

$$\mu\nu^* = e_{i_1} \cdots e_{i_k} e_{j_\ell}^* \cdots e_{j_1}^*$$

**2.2.3 The graded uniqueness theorem.** Now we can state the graded uniqueness theorem.

Theorem (graded uniqueness)**.** *Let R be a $\mathbf{Z}$-graded ring and let $\varphi : L(E) \to R$ be a graded ring homomorphism such that $\varphi(v) \neq 0$ for all $v \in E^0$. Then φ is injective.*

Note that we do not require φ to be an *algebra* homomorphism, *i.e.* perhaps not k-linear; also note that R can have *any* $\mathbf{Z}$-grading. For a proof of the graded uniqueness theorem, see Theorem 4.8 in [31].

2.3 Cuntz–Krieger uniqueness theorem

The third uniqueness theorem has a different flavor from the first two. Instead of requiring φ to preserve extra structure, we require E to satisfy a geometric condition, called Condition (L). As long as E satisfies this, any homomorphism $\varphi : L(E) \to R$ (or $\varphi : C^*(E) \to A$) will be injective so long as $\varphi(v) \neq 0$ (or $\varphi(p_v) \neq 0$) for all $v \in E^0$.

2.3.1 Condition (L). Recall that a **cycle** in a graph E is nothing but a path $\xi = e_1 \cdots e_n$, $n \geq 1$, such that $s(\xi) = r(\xi)$. We say that ξ is **simple** if it has no repeated vertices: that is, $r(e_i) \neq r(e_j)$ for $i \neq j$. Finally, an **exit** for a cycle ξ is an edge e, distinct from $e_1, \ldots, e_n$, such that $s(e) = r(e_i)$ for some i.
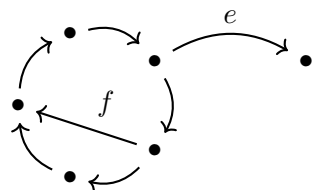
Both e and f are exits for this cycle.

The graph E satisfies **Condition (L)** if every simple cycle in E has an exit.

satisfies Condition (L)   satisfies Condition (L)   fails Condition (L)

Warning: in much of the literature, the word "loop" means what we have called here a "cycle", and "cycle" means what we have called here a "simple cycle". Thus Condition (L) is frequently described as "every cycle has an exit". In the middle graph of the above three, not every cycle has an exit — but every *simple* cycle has an exit, so it satisfies Condition (L).

2.3.2 The CK uniqueness theorem for C*-algebras. Now we state the Cuntz–Krieger uniqueness theorem for C*-algebras.

Theorem (CK uniqueness). *Let E be a row-finite graph satisfying Condition (L). Assume that $\varphi : C^*(E) \to A$ is a $*$-homomorphism such that $\varphi(p_v) \neq 0$ for all $v \in E^0$. Then φ is injective.*

The proof of this theorem can be found in Chapter 3 of [23].

2.3.3 The CK Uniqueness Theorem for Lpa's. Unlike the relationship between the Gauge-Invariant and Graded Uniqueness Theorems, the CK Uniqueness Theorem for Lpa's is verbatim Theorem 2.3.2 — but the proof is different. The proof is given in Theorem 6.8 in [31].

Theorem (CK uniqueness). *Let E be a row-finite graph satisfying Condition (L). Assume that $\varphi : L(E) \to R$ is a ring homomorphism such that $\varphi(v) \neq 0$ for all $v \in E^0$. Then φ is injective.*

For later purposes, we record the following lemma which immediately implies the CK Uniqueness Theorem.

Lemma. *Let E satisfy Condition (L) and suppose $x \in L(E)$, $x \neq 0$. Then there exist paths $\mu, \nu \in E^\bullet$ so that $\mu^* x \nu = cv$ for some $c \in k^\times$ and vertex $v \in E^0$.*

Obviously this implies that every ideal of $L(E)$ contains a vertex, which is equivalent to the CK Uniqueness Theorem. For its proof, see Proposition 6 in [1].

2.4 Applications

We give two applications of the uniqueness theorems: the first is that every Leavitt path algebra is a direct limit of *unital* Leavitt path algebras, and ditto for graph C*-algebras; the second is that $L_{\mathbf{C}}(E)$ is a dense subalgebra of $C^*(E)$.

2.4.1 Direct limits. Complete graph inclusions were introduced in 1.2.4, where we saw that every row-finite graph is a directed union of finite complete subgraphs. First we prove that complete graph inclusions induce embeddings of Leavitt path algebras and C*-algebras.

Proposition. *Let E, F be row-finite graphs and let $F \hookrightarrow E$ be a complete graph inclusion.*

(a) *There is a natural embedding of algebras $L(F) \hookrightarrow L(E)$.*

(b) *There is a natural embedding of C*-algebras $C^*(F) \hookrightarrow C^*(E)$.*

Proof. By definition we have $F^0 \subseteq E^0$ and $F^1 \subseteq E^1$. Let R be the subalgebra of $L(E)$ generated by $F^0 \cup F^1 \cup (F^1)^*$; we will show that $R \simeq L(F)$. First we verify that (F^0, F^1) is a CK F-family in R: indeed (CK1) is free, and the fact that F is a complete subgraph of E means $s_F^{-1}(v) = s_E^{-1}(v)$ for all nonsinks $v \in F^0$, so

$$v = \sum_{s_E(e)=v} ee^* = \sum_{s_F(e)=v} ee^*$$

verifying (CK2). Therefore, the universal property of $L(F)$ implies that there is a surjection $\varphi : L(F) \to R$. On the other hand, φ is a graded homomorphism and

$\varphi(v) = v \neq 0$ for all $v \in F^0$. Thus φ is an isomorphism by the graded uniqueness theorem, so $L(F) \simeq R \hookrightarrow L(E)$, proving (a). For (b), use the same argument but instead apply the gauge-invariant uniqueness theorem at the end. ∎

Since every row-finite graph is a direct limit of finite complete subgraphs by Proposition 1.2.4, we deduce the following.

Corollary. *Let E be a row-finite graph. Then there is a direct system $\{F_i\}$ of finite complete subgraphs of E such that $L(E) = \varinjlim L(F_i)$ and $C^*(E) = \varinjlim C^*(F_i)$.*

2.4.2 Density of $L_{\mathbf{C}}(E)$ in $C^*(E)$. In this section we work only over $k = \mathbf{C}$. We will show that there a natural embedding $L(E) \hookrightarrow C^*(E)$ realizing $L(E)$ as a *dense* complex subalgebra of $C^*(E)$.

Using the Lpa $L(E)$, we can revisit the construction of $C^*(E)$ from 1.3.4: there we started with a complex algebra $\mathcal{L}$, which in retrospect is *exactly* the Leavitt path algebra $L(E)$. Fixing an infinite-dimensional separable Hilbert space H, we then defined a seminorm on $L(E)$ by

$$\|a\| := \sup_\pi \|\pi(a)\|$$

where the sup is taken over all $*$-homomorphisms $\pi : L(E) \to \mathcal{B}(H)$. This is a seminorm, but it may not be nondegenerate: the kernel $N := \{a \in L(E) : \|a\| = 0\}$ measures the nondegeneracy of $\|\cdot\|$. The $*$-algebra $L(E)/N$ is thus a normed algebra. Finally we defined $C^*(E)$ to be the norm-completion of the $*$-algebra $L(E)/N$:

$$C^*(E) := \overline{L(E)/N}.$$

By construction there is a natural map

$$
\begin{array}{rccc}
\varphi : & L(E) & \longrightarrow & C^*(E) \\
& v & \longmapsto & p_v \\
& e & \longmapsto & s_e
\end{array}
$$

whose kernel is exactly $\ker \varphi = N$. Notice that the image of φ is exactly the $*$-subalgebra $\mathcal{L}_0 := \mathrm{span}\{s_\mu s_\nu^*\}$, which is dense by Corollary 1.3.3(3). To prove the dense embedding result we must verify that $N = 0$, *i.e.* that $\|\cdot\|$ was already nondegenerate to begin with, and so there was no reason to quotient by N after all. This is done using the graded uniqueness theorem. The first written proof of this fact can be found as Theorem 7.3 in [31].

Theorem (dense embedding theorem)**.** *Let E be a row finite graph. Then the map $\varphi : L(E) \to C^*(E)$ given by $\mu\nu^* \mapsto s_\mu s_\nu^*$ realizes $L(E)$ as a dense $*$-subalgebra of $C^*(E)$. Thus $C^*(E)$ is the norm-completion of $L(E)$.*

Proof. Given the above exposition, it remains to show that φ is injective. The image of φ is $\mathcal{L}_0 := \mathrm{span}\{s_\mu s_\nu^*\}$, and this is $\mathbf{Z}$-graded because

$$\mathcal{L}_0 = \bigoplus_{n \in \mathbf{Z}} \mathcal{A}_n$$

where $\mathcal{A}_n := \mathrm{span}\{s_\mu s_\nu^* : |\mu| - |\nu| = n\}$. Clearly $\varphi : L(E) \to \mathcal{L}_0$ is a graded homomorphism with respect to this $\mathbf{Z}$-grading, and $\varphi(v) = p_v \neq 0$ for all $v \in E^0$. So φ is injective by the graded uniqueness theorem. ∎

§3 Graphical-algebraic properties

There are plenty of geometric conditions on a directed graph E which guarantee that $L(E)$ has a particular algebraic structure. For example, we will show that if E is finite and acyclic, then $L(E)$ is a direct sum of matrix rings, and the dimension of each summand can be predicted. Other properties we can characterize "graphically" include: simplicity, primitivity, primality, and pure infiniteness, each of which will be introduced in their respective sections. Most theorems studied in this section will be of the form

"E has graphical property (X) $\iff$ $L(E)$ has algebraic property (Y)."

Many of these properties also hold for C*-algebras. One point of distinction is that "prime $\implies$ primitive" for separable C*-algebras, but not Lpa's. Another highlight of this section is the dichotomy of simple Lpa's: if $L(E)$ is simple, then it is either locally matricial or purely infinite. An analogous dichotomy holds for graph C*-algebras.

3.1 Ideal structure

Let $L(E)$ be a Leavitt path algebra. Recall that an ideal I of $L(E)$ is **graded** if $I = \bigoplus_{n \in \mathbf{Z}} I \cap L(E)_n$, where $L(E)_n$ denote the homogeneous components of $L(E)$. In this section we will derive a bijection between the graded ideals of $L(E)$ and certain subsets of E^0, called **hereditary saturated sets**. In case E satisfies "Condition (K)", it turns out all ideals are graded, thus entailing a necessary-and-sufficient graphical condition for simplicity of an Lpa. As a consequence of all this, we will be able to prove that the Jacobson radical of $L(E)$ is zero, *i.e.* all Lpa's are semiprimitive.

For C*-algebras, the aforementioned bijection is between hereditary saturated sets and **gauge-invariant ideals** of $C^*(E)$ — ideals which are invariant under the gauge-action γ. The remark about Condition (K) still holds for C*-algebras. Since all C*-algebras are semiprimitive, the corresponding property for Lpa's is more notable. All the proofs for C*-algebras are the same as in the Lpa case, mutatis mutandis, the only differences being the application of the gauge-invariant uniqueness theorem versus the graded uniqueness theorem. Thus we only establish the results for Leavitt path algebras, and merely mention how to adjust them for C*-algebras.

3.1.1 Hereditary saturated sets. Let E be a directed graph. We define two terminologies: a set $H \subseteq E^0$ is

- **hereditary** if: whenever $e \in E^1$ and $s(e) \in H$, then $r(e) \in H$.

- **saturated** if: whenever $v \in E^0$ is a nonsink such that $r(e) \in H$ for all $e \in s^{-1}(v)$, we must have $v \in H$.

Heredity can be interpreted as saying that once you are in H, you cannot get out. Saturation is the converse: if you only emit edges leading into H, then you must have been in H to begin with. So a hereditary saturated set can be thought of as being trapped in an exclusive zombie night club — if all your friends are in, the bouncers will let you in too; however, once you're in you will never escape.

Example. Consider the following graph:
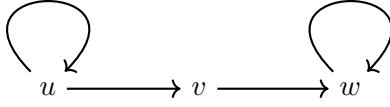
The set $\{v, w\}$ is hereditary and saturated; note that u has an outneighbor in $\{v, w\}$, but not *all* outneighbors are in $\{v, w\}$, so the saturation condition does not fail at u. The set $\{u\}$ is saturated but not hereditary, the set $\{w\}$ is hereditary but not saturated, and the set $\{u, w\}$ is neither hereditary nor saturated. Punchline: "hereditary" and "saturated" are mutually exclusive conditions.

Hereditary sets are important because they generate subgraphs (see 1.2.4): if H is hereditary, we define a subgraph $E \setminus H$, called the **complementary subgraph**, by deleting H and all edges leading into it.

$$E \setminus H := \left(E^0 \setminus H, \, r^{-1}(E^0 \setminus H), \, r, \, s \right).$$

Since any edge of $E \setminus H$ must end outside H, it must also start outside H by heredity. So $E \setminus H$ is indeed a subgraph in the sense of 1.2.4.

Example. In the graph of the previous example, we have the following subgraphs obtained by removing the sets $\{w\}$ and $\{v, w\}$.

$$E \setminus \{w\} \qquad\qquad E \setminus \{v, w\}$$
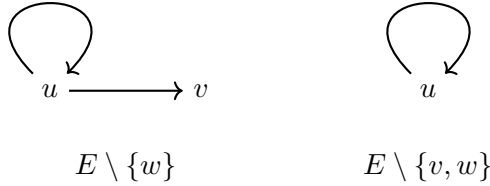
3.1.2 Ideals $\to$ hereditary saturated sets. Now we will begin to correspond ideals to hereditary saturated sets of vertices. For an ideal I of $L(E)$, consider the set

$$I \cap E^0 = \{v \in E^0 : v \in I\}.$$

Proposition. *Let I be an ideal. Then $I \cap E^0$ is hereditary and saturated.*

Proof. For heredity, suppose e is an edge with $s(e) \in I$.

$$r(e) = e^* e = e^* s(e) e \in I$$

so $I \cap E^0$ is hereditary. For saturation, suppose v is a nonsink such that $r(e) \in I$ whenever $s(e) = v$. We must check $v \in I$. Indeed, by (CK2) we have

$$v = \sum_{s(e)=v} e e^* = \sum_{s(e)=v} e r(e) e^* \in I$$

as required. So $I \cap E^0$ is saturated. $\blacksquare$

The upshot of this proposition is that if I is an ideal, we can form the complementary subgraph associated to the hereditary set $I \cap E^0$:

$$E_I := E \setminus (I \cap E^0).$$

This is row-finite since E is, so we can form the Leavitt path algebra $L(E_I)$ over the same field. As we will now show, there is a nice CK E_I-family in the quotient algebra $L(E)/I$.

Corollary. *The map $L(E_I) \to L(E)/I$ given by $e \mapsto e + I$ and $v \mapsto v + I$ induces a surjective algebra homomorphism. Consequently, $L(E_I) \simeq L(E)/I$ if one of the following conditions holds:*

  (i) *E_I satisfies Condition (L); or*

  (ii) *I is graded.*

We'll see later that (i) implies (ii).

Proof. Let $x \mapsto \overline{x}$ denote passage to the quotient $L(E)/I$; first we check that $\{\overline{e}\}_{e \in E_I}$, $\{\overline{v}\}_{v \in E_I}$ satisfy the CK relations. (CK1) is automatic: $\overline{e^*e} = \overline{r(e)}$ since the preimages satisfy (CK1) in $L(E)$. For (CK2), note that if v is a nonsink in E_I then it is a nonsink in E, so (CK2) in E implies

$$
\begin{aligned}
\overline{v} &= \sum_{\substack{e \in E^1 \\ s(e)=v}} \overline{e}\,\overline{e^*} && \text{by (CK2) in } E \\
&= \sum_{\substack{r(e) \notin H \\ s(e)=v}} \overline{e}\,\overline{e^*} + \sum_{\substack{r(e) \in H \\ s(e)=v}} \overline{e}\,\overline{e^*} && \\
&= \sum_{\substack{r(e) \notin H \\ s(e)=v}} \overline{e}\,\overline{e^*} && \text{since } r(e) \in H \implies ee^* = er(e)e^* \in I \\
&= \sum_{\substack{e \in (E_I)^1 \\ s(e)=v}} \overline{e}\,\overline{e^*} &&
\end{aligned}
$$

which is (CK2) in E_I. Therefore $\{\overline{e}\}_{e \in E_I}$, $\{\overline{v}\}_{v \in E_I}$ satisfy the CK relations, and so by the universal property of $L(E_I)$ there is an algebra homomorphism $\varphi : L(E_I) \to L(E)/I$ sending $e \mapsto \overline{e}$ and $v \mapsto \overline{v}$. This is surjective since this CK E_I-family clearly generates $L(E)/I$.

Note that if $v \in E_I$, then $v \notin H$ and so $\overline{v} \neq 0$. Thus if E_I has Condition (L) then φ is an isomorphism by the Cuntz–Krieger Uniqueness Theorem; if I is graded then φ is a graded homomorphism, hence injective by the Graded Uniqueness Theorem. ∎

Example. It is possible for $I \cap E^0$ to be empty if I is not graded: for example, in $k[x, x^{-1}]$ take $I := (1 + x)$. This is a proper, nongraded ideal. Since the only vertex in $k[x, x^{-1}]$ is $v = 1$, we necessarily have $I \cap E^0 = \varnothing$. Similarly if $J := (1 - x)$ then $J \cap E^0 = \varnothing$; in particular, notice that the correspondence $I \mapsto I \cap E^0$ is not injective.

We will show that the correspondence *is* injective when restricted to graded ideals.

3.1.3 Condition (K). In Corollary 3.1.2 we saw that $L(E_I) \simeq L(E)/I$ if E_I has Condition (L). We now introduce a condition on directed graphs which guarantees this is always the case.

A directed graph E satisfies **Condition (K)** if, for every hereditary saturated set $H \subseteq E^0$, the complementary subgraph $E \setminus H$ satisfies Condition (L). Taking $H = \varnothing$ we see that Condition (K) implies Condition (L), but the converse is not true.

Example 1. The following graph satisfies Condition (L):
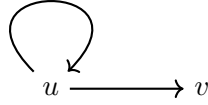
But if we remove the hereditary saturated set, $\{v\}$, the resulting subgraph $E \setminus \{v\}$ fails Condition (L) because it is a single loop. Therefore E fails Condition (K). On the other hand, the graph
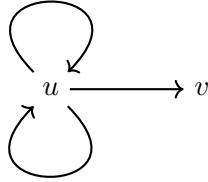
passes Condition (L), and the subgraph $E \setminus \{v\}$ still satisfies Condition (L) because u is on *two* distinct loops. Now it's clear that E satisfies Condition (K).

In the above example we noted that a vertex on a single loop can provide an obstruction to Condition (K). We will make this formal. If $\xi = e_1 \cdots e_n$ is a cycle with $s(\xi) =: v$, we say that ξ is a **taboo cycle** if $s(e_i) \neq v$ for all $2 \leq i \leq n$. So the source v is "tabooed" from appearing on ξ more than once. For example a simple cycle is a taboo cycle based at its source, but simplicity isn't necessary.

Example 2. In the following graph,
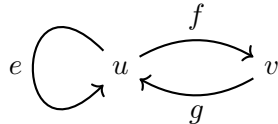
the cycle $geef$ visits u twice, but it is still a taboo cycle since its source is v and it only visits v once. So it is a nonsimple taboo cycle. Some simple taboo cycles are e and fg. On the other hand, ee is not a taboo cycle.

In Example 1, note that having two taboo cycles based at u saved the graph from failing Condition (K). In general we have the following characterization.

Proposition. *Let E be a directed graph. Then the following are equivalent:*

(a) *E satisfies Condition (K); and*

(b) *for every vertex $v \in E^0$, either v is not on a cycle, or there are at least two distinct taboo cycles ξ such that $s(\xi) = v$.*

The graph of Example 2 satisfies (b): two taboo cycles at u are e and fg, and two taboo cycles at v are gf and gef.

Proof. Assume (b), and let H be a hereditary saturated set. We must show $E \setminus H$ satisfies Condition (L). Let $\xi = e_1 \cdots e_n$ be a cycle in $E \setminus H$ and $v := s(\xi)$; wlog ξ is a simple cycle, hence it is a taboo cycle based at v. By (b) there is a taboo cycle $\zeta = f_1 \cdots f_m$ based at v, distinct from ξ. Choose the first i such that $e_i \neq f_i$ (possible since ξ and ζ are taboo cycles); then clearly f_i is an exit for ξ in E. But $v \notin H$ and $r(f_i) \to v$ implies $r(f_i) \notin H$ by heredity, and so f_i is an exit for ξ in $E \setminus H$. Therefore $E \setminus H$ satisfies Condition (L).

Conversely, assume $E \setminus H$ satisfies Condition (L) for all hereditary saturated sets H in E. Let $\xi = e_1 \cdots e_n$ be a simple cycle in E with $v := s(\xi)$; we must show that there is a taboo cycle ζ based at v distinct from ξ. The set $H := \{w \in E^0 : w \not\to v\}$, consisting of vertices which do not lead to v, is hereditary and saturated, so by Condition (K) the subgraph $E \setminus H$ satisfies Condition (L). Now ξ is a cycle in $E \setminus H$, so it has an exit, say $e \in E^1$ with $s(e) \in \xi^0$, $e \notin \xi^1$, and $r(e) \notin H$. By definition of H, it must be the case that $r(e)$ leads to v, say by some path μ. Choose μ of minimal length. Finally we can obtain a taboo cycle ζ based at v by following along ξ, exiting at e, and following μ back to v — formally, if $s(e) = r(e_i)$ for some i, let $\zeta := e_1 \cdots e_{i-1} e \mu$. Then ζ is distinct from ξ since $e \in \zeta^1$ but $e \notin \xi^1$, and ζ does not visit v more than once by choice of μ. ∎

We record now that the conclusion of Corollary 3.1.2 is automatic when E satisfies Condition (K).

Corollary. *Let E satisfy Condition (K). Then for any ideal I,*

$$L(E)/I \simeq L\big(E \setminus I \cap E^0\big).$$

3.1.4 Hereditary saturated sets $\leftrightarrow$ ideals. Now we complete the correspondence between hereditary saturated sets and ideals. If $H \subseteq E^0$ is a hereditary saturated set, we correspond it to an ideal I of $L(E)$ in the naive way, which turns out to work: simply let I be the ideal generated by H.

$$I := \langle v : v \in H \rangle.$$

Since this is generated by homogeneous elements, in fact I is a graded ideal. It turns out that this is the *unique* graded ideal with $I \cap E^0 = H$.

Theorem. *Let $L(E)$ be a Leavitt path algebra. There is a bijective, order-preserving correspondence between graded ideals of $L(E)$ and hereditary saturated sets in E:*

$$\begin{array}{ccc}
\{\text{Graded ideals}\} & \longrightarrow & \{\text{Hereditary saturated sets}\} \\
I & \longmapsto & I \cap E^0 \\
\langle v \in H \rangle & \longleftarrow\!\shortmid & H
\end{array}$$

Moreover,

(a) $L(E)/I \simeq L\big(E \setminus (I \cap E^0)\big)$ *for all graded ideals I;*

(b) *if E satisfies Condition (K), then all ideals are graded.*

Proof. The following proof is adapted from Theorem 4.9 in [23]. First we show that $H \mapsto \langle v \in H \rangle$ is surjective. Let I be any graded ideal; we know $I \cap E^0$ is hereditary and saturated by Proposition 3.1.2. Let $J := \langle I \cap E^0 \rangle$ so that J is graded and $J \subseteq I$, and we want to show $I = J$. We have the following diagram of quotients of $L(E)$:

Notice $\ker q_{I/J} = I/J$, so we're done if we show $q_{I/J}$ is injective.

If E_I denotes the subgraph $E \setminus (I \cap E^0)$, then the quotient map $\varphi : L(E_I) \to L(E)/I$ is an isomorphism by Corollary 3.1.2. On the other hand, $\{q_J(e), q_J(v)\}$ is a CK E_I-family generating $L(E)/J$ — same proof as Corollary 3.1.2 since $J \subseteq I$ — so by the universal property of $L(E_I)$ we have a surjection $\pi : L(E_I) \to L(E)/J$ sending $e \mapsto q_J(e)$ and $v \mapsto q_J(v)$. But then $q_{I/J} \circ \pi$ and φ both send $e \mapsto q_I(e)$ and $v \mapsto q_I(v)$, hence they are equal and the following diagram commutes.

Thus $\varphi = q_{I/J} \circ \pi$. But φ is an isomorphism by Corollary 3.1.2 and π is surjective, so this is only possible if $q_{I/J}$ is injective, *i.e.* $I = J$ as claimed.

Secondly, we must show that $H \mapsto \langle v \in H \rangle$ is injective; equivalently, if $I := \langle v \in H \rangle$ then $H = I \cap E^0$. Clearly "$\subseteq$" holds, so we have to prove that $H \supseteq I \cap E^0$.

In the Lpa $L(E \setminus H)$ associated to the subgraph $E \setminus H$, define the following elements for each $e \in E^1$ and $v \in E^0$:

$$\widehat{e} := \begin{cases} e & \text{if } r(e) \notin H \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \widehat{v} := \begin{cases} v & \text{if } v \notin H \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\{\widehat{e}\}_{e \in E^1}$, $\{\widehat{v}\}_{v \in E^0}$ form a CK E-family in $L(E \setminus H)$ — indeed, this is readily verified; note that if $v \in E^1$ is not a sink in E then it is not a sink in $E \setminus H$ because H is saturated, so there is no issue in verifying (CK2). Thus by the universal property of $L(E)$ there is a surjective homomorphism $\varphi : L(E) \to L(E \setminus H)$ sending $e \mapsto \widehat{e}$ and $v \mapsto \widehat{v}$, and notice $\ker \varphi \cap E^0 = H$, so $I := \langle v \in H \rangle \subseteq \ker \varphi$.

Now if $v \in I \cap E^0$, then $\widehat{v} = \varphi(v) = 0$, which is only possible if $v \in H$. Therefore $I \cap E^0 \subseteq H$, as required.

It remains to verify the claims (a) and (b) in the statement of the theorem. Claim (a) was done in Corollary 3.1.2. For (b), note that if $E_I = E \setminus I \cap E^0$ satisfies Condition (L) then the map $\varphi : L(E_I) \to L(E)/I$, defined in the second paragraph of this proof, is still an isomorphism by Corollary 3.1.2. The rest of the argument runs verbatim to show that $I = \langle I \cap E^0 \rangle$, and so I is graded. By Proposition 3.1.3, this holds for all ideals if E satisfies Condition (K). $\blacksquare$

The above proof shows that every nonzero graded ideal is generated by vertices. It is also interesting to note that the argument in the last paragraph shows that (i) implies (ii) in Corollary 3.1.2.

3.1.5 Simplicity. Now it is clear that E has no nontrivial hereditary saturated sets if and only if $L(E)$ is **graded-simple**, *i.e.* has no nontrivial *graded* ideals. For example $R = k[x, x^{-1}]$ is graded-simple, which can be seen either directly or from its structure as a Leavitt path algebra. But it is not simple, because *e.g.* $(1 - x)R$ is a proper nonzero ideal; we will see that this is because $k[x, x^{-1}]$ fails Condition (L) as a Leavitt path algebra.

We can give graphical conditions characterizing when an Lpa is simple, but we have to introduce some graph-theoretic terminology. First, an **infinite path** in a directed graph E is a sequence of edges $\mu = e_1 e_2 e_3 \cdots$ such that $r(e_i) = s(e_{i+1})$ for all i; we denote by E^∞ the set of infinite paths in E. We say E is **cofinal** if, for all $\mu \in E^\infty$, all sinks $w \in E^0$ and all vertices $v \in E^0$, there are paths $v \to \mu$ and $v \to w$. Succinctly, this means any vertex in E can reach any infinite path and any sink; in particular every vertex can reach every cycle ξ, because $\xi\xi\xi\cdots$ is an infinite path.

cofinal   not cofinal
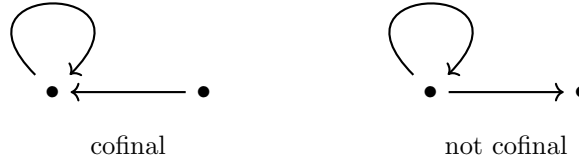
If E is finite, the only infinite paths are ones containing cycles — so in this case, cofinality is equivalent to saying that any vertex can reach any cycle and any sink. Clearly, then, a cofinal graph has at most one sink: if it has two sinks, they can't reach other.

The following fact reveals the importance of cofinality: essentially E is cofinal if and only if $L(E)$ is graded-simple.

Lemma 1. *Let E be a directed graph.*

(a) *E is cofinal if and only if it has no nontrivial hereditary saturated sets.*

(b) *If E is cofinal, then E has Condition (L) if and only if E has Condition (K).*

Proof. (b) is obvious from (a), as Condition (K) means that $E \setminus H$ satisfies Condition (L) for all hereditary saturated subsets H. This is vacuous for $H = E^0$, so if E is cofinal then this only needs to be checked for $H = \varnothing$, which is the same as checking that E satisfies Condition (L).

We thus focus on proving (a). Suppose E is cofinal. If H is a proper hereditary saturated set in E, choose a vertex $v_1 \in E^0 \setminus H$.

> **Claim:** There is a path μ, either infinite or ending in a sink, which starts at v_1 and $\mu^0 \cap H = \varnothing$.

Indeed, if v_1 is a sink let $\mu := v_1$. Otherwise, saturation of H implies that there is an edge e_1 starting at v_1 such that $v_2 := r(e_1) \notin H$. If v_2 is a sink let $\mu := e_1$; otherwise continue this process on v_2. If the process terminates then μ ends in a sink, and otherwise $\mu \in E^\infty$ — either way, μ is as claimed. Now if H is nonempty, say $w \in H$, then cofinality implies that $w \to v_i \in \mu^0$ for some i. But heredity of H implies $v_i \in H$, contradicting the construction of μ. Thus H must be empty, as required.

Conversely, assume that E has no nontrivial hereditary saturated sets. To see that E is cofinal, let μ be an infinite path or a path ending in a sink, and consider the set

H of vertices which *do not* lead to μ; we want to show $H = \varnothing$. But H is clearly a hereditary saturated set, so by our assumption on E we must have $H = \varnothing$ or $H = E^0$. Clearly $H \neq E^0$ — e.g. $s(\mu) \notin H$ — so the only possibility is $H = \varnothing$ which is what we wanted to show. ∎

Before proceeding to the simplicity theorem, we will state a trusty lemma: essentially, if E fails Condition (L) it allows us to construct a commutative corner of $L(E)$ which isomorphic to a Laurent polynomial ring. This will be useful throughout these notes.

Lemma 2. *Let ξ be a (simple) cycle with no exits, with base vertex $v = s(\xi)$. Then*

$$vL(E)v \simeq k[x, x^{-1}]$$

via $\xi \mapsto x$, $\xi^* \mapsto x^{-1}$, $v \mapsto 1$.

Proof. Fix $L := L(E)$. The corner vLv is a subalgebra of L with identity v, and note $\xi = v\xi v \in vLv$. We claim that ξ is a unit in vLv: indeed (CK1) gives $\xi^*\xi = r(\xi) = v$. For (CK2), note that since ξ has no exits, v emits exactly one edge and so (CK2) at v reduces to $v = \xi\xi^*$. So $\xi^* = \xi^{-1}$ in vLv. Thus we have a well-defined algebra homomorphism

$$\varphi : k[x, x^{-1}] \to vLv, \qquad 1 \mapsto v, \ x \mapsto \xi, \ x^{-1} \mapsto \xi^*.$$

Our aim is to prove that φ is an algebra isomorphism.

First we show that it is surjective. Let $\xi = e_1 \cdots e_n$. Since $L = \mathrm{span}\{\mu\nu^*\}$, we see that vLv is spanned by terms of the form $v\mu\nu^*v$. But such a term is zero unless $v = s(\mu) = s(\nu)$ and $r(\mu) = r(\nu)$, in which case $v\mu\nu^*v = \mu\nu^*$. But since ξ is a simple cycle without exits, the only paths starting at v are ones obtained by traversing ξ — in symbols, this means any path μ with $s(\mu) = v$ must have the form $\mu = \xi^s e_1 \cdots e_t$, where $s \geq 0$ and $0 \leq t \leq n$.
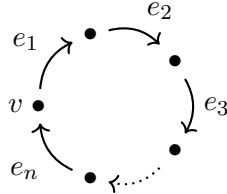
Writing $\mu = \xi^s e_1 \cdots e_t$ and $\nu = \xi^p e_1 \cdots e_q$, it is easy to see that $v\mu\nu^*v = \mu\nu^* = \xi^{s-p}$ (if it is nonzero). Here $\xi^{-1} = \xi^*$ is the inverse of ξ in vLv. We therefore conclude that the corner vLv looks like

$$vLv = \mathrm{span}\{v\mu\nu^*v\} = \mathrm{span}\{\mu\nu^* : s(\mu) = v = r(\nu)\} = \mathrm{span}\{\xi^d : d \in \mathbf{Z}\}$$

which is exactly the image of φ. So φ is surjective.

To see that φ is injective, observe that we can give vLv a $\mathbf{Z}$-grading via

$$vLv = \bigoplus_{d \in \mathbf{Z}} \mathrm{span}\{\xi^d\},$$

with respect to which $\varphi : k[x, x^{-1}] \to vLv$ is a graded homomorphism. Since $\varphi(1) = v \neq 0$, injectivity follows from the Graded Uniqueness Theorem. ∎

Corollary. *Let E be a row-finite graph. Then $L(E)$ is simple if and only if E is cofinal and satisfies Condition (L).*

Proof. Assume $L(E)$ is simple. In particular it is graded-simple, so E has no nontrivial hereditary saturated sets by Theorem 3.1.4; this is the same as saying E is cofinal as per the preceding lemma. If E fails Condition (L), then Lemma 2 implies that $L(E)$ contains $k[x, x^{-1}]$ as a corner. On one hand, a corner of a simple ring is simple — $I \mapsto RIR$ is an injective map from ideals of eRe to ideals of R, with left inverse $J \mapsto eJe$ — but on the other hand $k[x, x^{-1}]$ is not simple. This contradiction implies that E must pass Condition (L).

Conversely assume E is cofinal and satisfies Condition (L). Then E has no hereditary saturated sets so $L(E)$ is graded-simple. But by Lemma 1, Condition (L) implies Condition (K) in the presence of cofinality, so all ideals are graded by the theorem. Therefore $L(E)$ is simple. ∎

Example. Using the above graph-theoretic conditions, we see that $\varinjlim M_n(k)$ and the Leavitt algebra L_n ($n \geq 1$) are both simple.

3.1.6 The case of graph C*-algebras. All the results established for Leavitt path algebras in this section can be made to work for graph C*-algebras, with very little adjustment. Instead of graded ideals, we consider **gauge-invariant ideals**: that is, closed ideals I of $C^*(E)$ such that $\gamma_z(I) \subseteq I$ for all $z \in \mathbf{T}$, where γ denotes the gauge action defined in 2.1.1. For a set X, let us denote by $\overline{\langle X \rangle}$ the *closed* two-sided ideal generated by X. Replacing all invocations of the graded uniqueness theorem by the gauge-invariant uniqueness theorem, the following theorem can be proven in the exact same way as Theorem 3.1.4:

Theorem. *Let $C^*(E)$ be a graph C*-algebra. There is a bijective, order-preserving correspondence between gauge-invariant ideals of $C^*(E)$ and hereditary saturated sets in E:*

$$
\begin{array}{ccc}
\{\text{Gauge-invariant ideals}\} & \longrightarrow & \{\text{Hereditary saturated sets}\} \\
I & \longmapsto & \{v \in E^0 : p_v \in I\} =: H_I \\
\overline{\langle p_v : v \in H \rangle} & \longleftarrow\!\shortmid & H
\end{array}
$$

Moreover,

(a) *$C^*(E)/I \simeq L\big(E \setminus H_I\big)$ for all gauge-invariant ideals I;*

(b) *if E satisfies Condition (K), then all ideals are gauge-invariant.*

The simplicity theorem carries over verbatim as well, with the following caveat: for C*-algebras, simplicity means no nontrivial *closed* ideals. (For example, the algebra $\mathcal{K}(H)$ of compact operators on a Hilbert space H is simple as a C*-algebra, but not simple as a ring since the algebra of finite rank operators is a proper nonclosed ideal.)

We should also point out the usage of Lemma 3.1.5(2), which allowed us to construct a corner of $L(E)$ isomorphic to $k[x, x^{-1}]$. The analogous lemma holds for C*-algebras, with continuous functions $\mathcal{C}(\mathbf{T})$ replacing Laurent polynomials.

Lemma. *Let ξ be a (simple) cycle with no exits, with base vertex $v = s(\xi)$. Then*

$$
p_v C^*(E) p_v \simeq \mathcal{C}(\mathbf{T})
$$

via $\xi \mapsto x$, $\xi^* \mapsto x^{-1}$, $v \mapsto 1$.

From this one deduces the following simplicity result for C*-algebras.

Theorem. *Let E be a row-finite graph. Then $C^*(E)$ is simple if and only if E is cofinal and satisfies Condition (L).*

Example. From the above result it is clear that the Cuntz algebra $\mathcal{O}_n$ and the algebra of compact operators $\mathcal{K}(H)$ are both simple.

3.2 Prime and primitive ideals

The section is structured as follows: first we investigate prime and primitive Lpa's, then prime and primitive C*-algebras. For each, we can find graphical conditions to determine which graded ideals are prime and/or primitive. We will see that $C^*(E)$ is prime if and only if it is primitive, if and only if $L(E)$ is primitive — but it is possible for $L(E)$ to be prime while $C^*(E)$ is not. This will be our first example of an Lpa property which does not perfectly reflect C*-algebras.

3.2.1 Prime rings. Let R be a ring. An ideal P of R is **prime** if, whenever I, J are ideals with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. It is equivalent to require that for all $x, y \in R$, $xRy \subseteq P$ implies $x \in P$ or $y \in P$ (see Proposition 10.2 in [15]).

R is a **prime ring** if $\{0\}$ is a prime ideal; that is, if $x, y \in R$ and $xry = 0$ for all $r \in R$, then either $x = 0$ or $y = 0$. If R is commutative and unital, this reduces to the condition that R has no zero divisors, *i.e.* R is an integral domain. For noncommutative rings it is sufficient but not necessary that R has no zero divisors, *e.g.* $M_n(\mathbf{C})$ is a prime ring with many zero divisors.

We now investigate prime Leavitt path algebras.

Proposition. *Let $L(E)$ be a Leavitt path algebra. Then $L(E)$ is a prime ring if and only if E is **downward directed**: that is, for all vertices $v, w \in E^0$, there exists $x \in E^0$ such that $v \to x$ and $w \to x$.*

Recall that "$v \to w$" just means that there is a path μ from v to w, *i.e.* $s(\mu) = v$ and $r(\mu) = w$.

downward directed   not downward directed
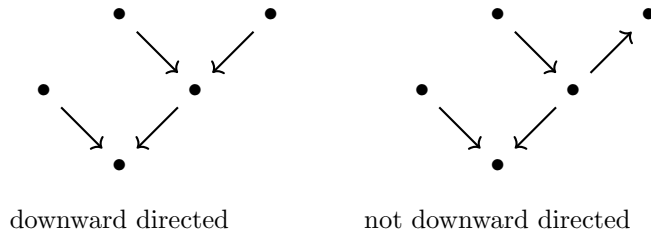
Proof. Assume $L(E)$ is prime, and let $v, w \in E^0$. Then v and w are nonzero when viewed as elements of $L(E)$, and so by primality there exists $r \in L(E)$ so that $vrw \neq 0$. Since we can write r as a linear combination of $\mu\nu^*$, it follows that there exists a monomial $\mu\nu^*$ such that $v\mu\nu^*w \neq 0$. The fact that $\mu\nu^* \neq 0$ implies $r(\mu) = r(\nu)$ — call this vertex x. Now $v\mu \neq 0$ implies $v = s(\mu)$, and so $v \xrightarrow{\mu} x$; similarly $w \xrightarrow{\nu} x$. This proves that E is downward directed.

Conversely, let E be downward directed. We must check that if I, J are ideals with $IJ = 0$, then $I = 0$ or $J = 0$. By Proposition 5.2.6(1) in [21], it suffices to check this for I, J graded. If I, J are nonzero, then by Proposition 3.1.4 we may choose vertices

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$v \in I$ and $w \in J$. Now since E is downward directed, we can find a vertex x so that $v \to x$ and $w \to x$. But then $x \in I \cap J$ by heredity, so

$$x = xx \in IJ = 0,$$

which is impossible. ∎

Since an ideal P is prime if and only if R/P is a prime ring, we note a graphical characterization of prime ideals.

Corollary. *Let P be a graded ideal of $L(E)$. Then P is prime if and only if $E \setminus P \cap E^0$ is downward directed.*

Proof. P is prime if and only if $L(E)/P$ is a prime ring. But $L(E)/P \simeq L(E \setminus P \cap E^0)$ by Corollary 3.1.2, and by the last proposition, this is a prime ring if and only if $E \setminus P \cap E^0$ is downward directed. ∎

Thus to detect graded prime ideals, one must detect those hereditary saturated sets H for which the complementary subgraph $E \setminus H$ is downward directed. Identifying the nongraded prime ideals is a different story which will not be covered in these notes; see [5] for more information.

3.2.2 Primitive rings. Let R be a ring. If M is a right R-module, its **kernel** (or **annihilator**) is the two-sided ideal $\ker M := \{r \in R : Mr = 0\}$. A proper ideal P of R is **(right) primitive** if $P = \ker M$ for some nonzero, *simple* right R-module M. If $\{0\}$ is a primitive ideal we say that R is a **(right) primitive ring**; equivalently, R admits a *faithful* simple right R-module. Thus an ideal P is primitive if and only if R/P is a primitive ring. We note that in commutative rings, the only primitive rings are fields, and the primitive ideals are the maximal ideals.

Here is a standard fact.

Proposition. *Primitive ideals are prime.*

Proof. Taking quotients, it suffices to show all primitive rings are prime. Let I, J be two nonzero ideals of R; we want to show $IJ \neq 0$. If M is a nonzero faithful simple right R-module, then $MI = \{\sum mr : m \in M, r \in I\}$ is a nonzero submodule of M by faithfulness, so $MI = M$ by simplicity. Similarly $MJ = M$. Thus

$$M(IJ) = (MI)J = MJ = M$$

which implies $IJ \neq 0$. ∎

Thus if $L(E)$ is primitive, in particular it is prime and so E is downward directed. We will prove that, conversely, if $L(E)$ is downward directed *and* satisfies Condition (L), then $L(E)$ is primitive. First we record some general ring-theoretic facts: they will help determine when a prime ring is primitive.

Lemma. *Let R be a k-algebra where k is a commutative unital domain.*

(a) *Let R be unital. Then R is right primitive if and only if it has a proper right ideal $\mathfrak{m}$ of R which is comaximal with every nonzero two-sided ideal I of R, i.e. $\mathfrak{m} + I = R$.*

(b) *Let R be prime and let I be an ideal of R. Then R is primitive if and only if I is primitive (when viewed as a subring of R).*

(c) *Let R be prime. Then there is a unital prime k-algebra S containing R as a two-sided ideal.*

Proof. For (a), if we use Zorn's Lemma to select such $\mathfrak{m}$ maximally, then $R/\mathfrak{m}$ is a faithful simple right module. Conversely, if M is a faithful simple right module then $M \simeq R/\mathfrak{m}$ for some maximal right ideal $\mathfrak{m}$ which is as required. For (b) and (c), see [17]. ∎

Armed with the lemma, we can prove the main theorem of this section.

Theorem. *An Lpa $L(E)$ is primitive if and only if E is downward directed and satisfies Condition (L).*

Proof. Assume that $L = L(E)$ is primitive. In particular $L(E)$ is prime, hence E is downward directed by Proposition 3.2.1. To see that E satisfies Condition (L), suppose otherwise. Then by Lemma 3.1.5(2), there is a vertex v so that the corner $vLv \simeq k[x, x^{-1}]$ is commutative. On the other hand, corners of primitive rings should be primitive — if M is a faithful simple R-module then Me is a faithful simple eRe module — but $k[x, x^{-1}]$ is not primitive since it is commutative but not a field. This contradiction shows that E must satisfy Condition (L), as required.

Now assume that E is downward directed and satisfies Condition (L). Then L is prime, so by Lemma (c) there is a prime algebra S containing L as a two-sided ideal; by Lemma (b), S is primitive if and only if L is. Thus the goal is to show S is primitive. This will be done using Lemma (a): we must find a proper right ideal $\mathfrak{m}$ of S such that $\mathfrak{m} + I = S$ for all nonzero two-sided ideals I of S.

Fix any vertex $v_0 \in E$. Since E is row-finite, v may only lead to at most countably many vertices: say $\{v_0, v_1, v_2, \ldots\}$.

Claim: There is an infinite path λ so that $v_i \to \lambda$ for all i.

To see this, we repeatedly use the fact that E is downward directed. First choose a vertex w_1 so that $v_0, v_1 \to w_1$ and let λ_1 be a path from v_0 to w_1; then $v_1 \to w_1 = r(\lambda_1)$. Next choose a vertex w_2 so that $w_1, v_2 \to w_2$ and let λ_2 be a path from w_1 to w_2; then $v_2 \to w_2 = r(\lambda_2)$. Next choose a vertex w_3 so that $w_2, v_3 \to w_3$ and let λ_3 be a path from w_2 to w_3; then $v_3 \to w_3 = r(\lambda_3)$. Continuing this, the infinite path $\lambda := \lambda_1 \lambda_2 \lambda_3 \cdots$ is as required.

Keeping the notation as above, let $\mu_n := \lambda_1 \cdots \lambda_n$ so that v_n leads to $r(\mu_n)$. Define the following right ideal of S:

$$\mathfrak{m} := \sum_{n \geq 1} (1 - \mu_n \mu_n^*) S.$$

We will show that $\mathfrak{m}$ is a proper right ideal of S which is comaximal with every ideal, from which we may conclude that S is primitive by Lemma (a). To see that $\mathfrak{m}$ is proper, if $1 \in \mathfrak{m}$ then we can write

$$1 = \sum_{n=1}^{N} (1 - \mu_n \mu_n^*) s_n$$

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for some $s_n \in S$. Multiplying on the left by $\mu_N \mu_N^*$ gives

$$
\begin{aligned}
\mu_N \mu_N^* &= \sum_{n=1}^{N} \mu_N \mu_N^* (1 - \mu_n \mu_n^*) s_n \\
&= \sum_{n=1}^{N} (\mu_N \mu_N^* - \mu_N \mu_N^* \mu_n \mu_n^*) s_n \\
&= \sum_{n=1}^{N} (\mu_N \mu_N^* - \mu_N \mu_N^*) s_n \qquad\qquad \text{by formula 1.4.3(\clubsuit)} \\
&= 0,
\end{aligned}
$$

which is impossible because it implies $r(\mu_N) = r(\mu_N) r(\mu_N) = \mu_N^* \mu_N \mu_N^* \mu_N = 0$. We therefore conclude $\mathfrak{m} \subsetneq S$.

Now we show that $\mathfrak{m} + I = S$ whenever I is a nonzero two-sided ideal of S. Since I is nonzero and L is a prime subalgebra of S embedded as an ideal, $I \cap L$ must be a *nonzero* ideal of L. By the Cuntz–Krieger Uniqueness Theorem, Condition (L) on E implies that every nonzero ideal of L contains a vertex, so we can choose $w \in (I \cap L) \cap E^0$. Using the downward directed condition on w and the vertex $v_0 = s(\lambda)$ from earlier, there exists $u \in E^0$ such that $w, v_0 \to u$. But $u = v_n$ for some $n \geq 0$ (since the v_n's are all vertices which can be reached from v_0), so $w \to u = v_n \to r(\mu_n)$. By heredity of $I \cap E^0$ this implies $r(\mu_n) \in I$, so

$$
1 = (1 - \mu_n \mu_n^*) + \mu_n \mu_n^* = (1 - \mu_n \mu_n^*) + \mu_n r(\mu_n) \mu_n^* \in \mathfrak{m} + I.
$$

Therefore $\mathfrak{m} + I = S$ as required. ∎

By taking quotients we can use the above theorem to characterize the primitive ideals. Recall that we only gave a graphical identification for *graded* primes, but for primitive ideals we gradedness is automatic from Condition (L).

Corollary. *Let P be an ideal in $L(E)$. Then P is a graded primitive ideal if and only if $E \setminus P \cap E^0$ is downward directed and satisfies Condition (L).*

Proof. If P is graded and primitive, then $L(E)/P \simeq L(E \setminus P \cap E^0)$ is a primitive ring, so $E \setminus P \cap E^0$ has the required properties due to the preceding proposition. Conversely, if $E \setminus P \cap E^0$ has Condition (L) then P is graded — see the remark following Proposition 3.1.4 — and so $L(E)/P \simeq L(E \setminus P \cap E^0)$ is a primitive ring by the preceding theorem. ∎

To summarize the last two sections, we have established the motto

$$
\text{Primitive} = \text{Prime} + \text{Condition (L)},
$$

which reflects the algebraic fact that Primitive $\implies$ Prime but not conversely. For example, if E is a loop based at one vertex then E is downward directed but fails Condition (L), and indeed $L(E) \simeq k[x, x^{-1}]$ is prime but not primitive.

3.2.3 Prime and primitive C*-algebras. Having determined precisely when $L(E)$ is prime and/or primitive, we now focus on doing the same for C*-algebras. A C*-algebra A is **C*-prime** if, whenever I, J are topologically closed two-sided ideals of A, we have $IJ = 0$ implies $I = 0$ or $J = 0$. Our first remark is that this is equivalent to the algebraic primeness defined previously, ignoring the topological structure of A.

Proposition. *Let A be a C*-algebra. Then A is a C*-prime if and only if A is prime as a ring.*

Proof. Clearly if A is prime then it is C*-prime. Conversely suppose A is C*-prime and let I, J be not-necessarily-closed ideals with $IJ = 0$. Then $\overline{I} \cdot \overline{J} = \overline{IJ} = 0$ and so either $\overline{I} = 0$ or $\overline{J} = 0$ by C*-primeness. As $I \subseteq \overline{I}$ and $J \subseteq \overline{J}$ we have either $I = 0$ or $J = 0$. ∎

It is less trivial to deduce the analogous conclusion for primitivity. First we provide the germane definitions. Let A be a C*-algebra. A **∗-representation** of A on a Hilbert space H is nothing but a ∗-homomorphism $\pi : A \to \mathcal{B}(H)$. A closed subspace K of H is π-**invariant** if $\pi(A)K \subseteq K$; we say π is **irreducible** if the only π-invariant closed subspaces of H are 0 and H. Finally, A is **C*-primitive** if it admits a faithful irreducible ∗-representation π; note "faithful" just means that π is injective.

Theorem. *Let A be a C*-algebra. Then A is C*-primitive if and only if A is (right or left) primitive as a ring.*

For a proof, see [12]. By this result there is no effort to distinguish "prime" and "primitive" from their C*-counterparts, and it is thus easy to deduce that primitive C*-algebras are prime. More surprisingly, the converse is true for *separable* C*-algebras. This is originally a result of Dixmier.

Theorem (Dixmier). *Every primitive C*-algebra is prime. Conversely, every separable prime C*-algebra is primitive.*

For a proof of Dixmier's theorem, see Theorem A.49 in [25]. Considering that $C^*(E)$ is separable if E is countable — which is our standing assumption during any mention of graph C*-algebras — we can deduce the following remarkable property: if $C^*(E)$ is prime then it is primitive. We now seek to establish graph-theoretic conditions under which this is the case.

Example. Let E consist of a single vertex and edge. Then $L(E) \simeq k[x, x^{-1}]$ is prime while $C^*(E) \simeq \mathcal{C}(\mathbf{T})$ is not. To see the latter, observe that if $X, Y \subseteq \mathbf{T}$ are proper closed sets with $\mathbf{T} = X \cup Y$ and I, J are the ideals of functions vanishing on X, Y respectively, then $IJ = 0$ but $I \neq 0 \neq J$. Thus we see that for $L(E)$ to be prime it is sufficient for E to be downward directed, but this is *not* sufficient for $C^*(E)$ to be prime.

Theorem. *Let E be a (countable) row-finite graph. Then $C^*(E)$ is primitive if and only if E is downward directed and satisfies Condition (L).*

Proof. Since $C^*(E)$ is separable, we freely use the equivalence of primeness and primitivity given by Dixmier's Theorem.

Assume E is downward directed and satisfies Condition (L); we will prove that $C^*(E)$ is C*-prime. Note that Condition (L) and the Cuntz–Krieger Uniqueness Theorem imply that any closed ideal of $C^*(E)$ contains a vertex projection: thus if I, J

are two nonzero closed ideals, we can choose vertices $v, w \in E^0$ so that $p_v \in I$ and $p_w \in J$. The remainder of the argument is identical to Proposition 3.2.1: since E is downward directed we can find a vertex $x \in E^0$ so that $v, w \to x$. But then $p_x \in I \cap J$ by heredity, so

$$p_x = p_x p_x \in IJ = 0$$

which is impossible.

Conversely, assume that $C^*(E)$ is primitive; equivalently, prime by Dixmier's Theorem. We'll show E has Condition (L). This is done the same way as in Theorem 3.2.2: if E fails Condition (L), then by Lemma 3.1.6, $C^*(E)$ has a corner which is $*$-isomorphic to $\mathcal{C}(\mathbf{T})$. Now primeness should pass to corners, but on the other hand $\mathcal{C}(\mathbf{T})$ is not prime. So E must satisfy Condition (L).

Finally we'll show E is downward directed. Let $v, w \in E^0$. Then $p_v \neq 0 \neq p_w$, so by algebraic primeness of $C^*(E)$ we have $p_v C^*(E) p_w \neq 0$. This implies that $p_v L_{\mathbf{C}}(E) p_w \neq 0$ by density of $L_{\mathbf{C}}(E)$ in $C^*(E)$, so $p_v r p_w \neq 0$ for some $r \in L_{\mathbf{C}}(E)$ (see Theorem 2.4.2). The rest of the argument follows Proposition 3.2.1 verbatim: writing r as a linear combination of monomials $s_\mu s_\nu^*$, we see that $p_v s_\mu s_\nu^* p_w \neq 0$ for some paths $\mu, \nu \in E^\bullet$; it follows that $v \xrightarrow{\mu} x$ and $w \xrightarrow{\nu} x$ where $x := r(\mu) = r(\nu)$. ∎

In conclusion, $C^*(E)$ is prime if and only if the Lpa $L_k(E)$ is primitive for any field k, but it may happen that $L(E)$ is prime while $C^*(E)$ is not. For an easy example, note $k[x, x^{-1}]$ is prime nonprimitive Lpa whose corresponding C*-algebra $\mathcal{C}(\mathbf{T})$ is not prime.

3.3 Semisimplicity

Semisimple rings play a central role in noncommutative algebra. If R is a finite-dimensional unital algebra then $R/J(R)$ is always semisimple, where $J(R)$ denotes the Jacobson radical of R, so the Artin–Wedderburn Theorem says that it must be a sum of matrix algebras over finite-dimensional division algebras. Our goal in this section is to show that $L(E)$ is semisimple if and only if E is finite and acyclic — in fact, we will see that $J(L(E)) = 0$ no matter what, and semisimplicity is only possible if $L(E)$ is finite-dimensional. This mirrors a similar property for C*-algebras. In this case we can determine the structure of matrix algebras given in the Artin–Wedderburn Theorem using graphical data from E. If E is *infinite* and acyclic, we can prove that $L(E)$ is a direct limit of finite-dimensional Lpa's.

3.3.1 Semisimple rings and C*-algebras. A unital ring R is **semisimple** if, for every right ideal $\mathfrak{a}$, there is a right ideal $\mathfrak{b}$ such that $R = \mathfrak{a} \oplus \mathfrak{b}$ as right R-modules. An equivalent formulation is that $R = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$ is a direct sum of right ideals $\mathfrak{a}_i$, each one simple as a right R-module. Although this is a homological property in nature — each short exact sequence $0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$ must split — it turns out to force very stringent structure.

The classical theorem of Artin–Wedderburn is a complete structure theorem for semisimple rings. To state it, recall that a ring R is **right** (resp. **left**) **artinian** if it has no infinite decreasing chain of right (resp. left) ideals, and **artinian** if it is both left and right artinian. If R is unital, its **Jacobson radical** $J(R)$ is defined to be the intersection of all maximal right ideals of R.

Theorem 1. *Let R be a unital ring. Then the following are equivalent:*

(a) *R is semisimple.*

(b) *R is artinian and $J(R) = 0$.*

(c) [**Artin–Wedderburn**]. *R is isomorphic to a direct sum of matrix rings*

$$R \simeq M_{d_1}(\Delta_1) \oplus \cdots \oplus M_{d_n}(\Delta_n)$$

where the Δ_i's are division rings. Moreover, $(d_1, \ldots, d_n)$ and $(\Delta_1, \ldots, \Delta_n)$ are uniquely determined by R.

See Theorem 3.5 in [15]. Note that a finite-dimensional algebra over a field must be artinian, and indeed Wedderburn's original structure theorem was for finite-dimensional algebras with zero Jacobson radical.

A similar result holds for C*-algebras. It turns out that every C*-algebra has $J(A) = 0$, even in the nonunital case, and that the only artinian C*-algebras are the finite-dimensional ones.

Theorem 2. *Let A be a C*-algebra. Then the following are equivalent:*

(a) *A is finite-dimensional.*

(b) *A is artinian.*

(c) [**C*-Artin–Wedderburn**]. *A is $*$-isomorphic to a direct sum of full matrix algebras*

$$A \simeq M_{d_1}(\mathbf{C}) \oplus \cdots \oplus M_{d_n}(\mathbf{C}).$$

Moreover, $(d_1, \ldots, d_n)$ are uniquely determined by A.

For a proof of "(a)$\Longleftrightarrow$(b)", see [29]; to deduce (c), see Theorem III.1.1 in [11].

In this section we aim to discuss semisimplicity of Leavitt path algebras and graph C*-algebras. First we will show every Lpa has zero Jacobson radical, and that "artinian" and "finite-dimensional" are equivalent for Lpa's — this is another way in which Lpa's emulate C*-algebras. Finally we will determine the data $(d_1, \ldots, d_n)$, $(\Delta_1, \ldots, \Delta_n)$ appearing in the Artin–Wedderburn Theorem in the case where R is finite-dimensional Lpa or graph C*-algebra.

3.3.2 The Jacobson radical. In this section we define the Jacobson radical of a ring, separately for unital rings and for nonunital rings. This is not the most efficient method, but there are vast simplifications in the unital case and it is helpful to draw inspiration from there.

Let R be a unital ring. The **Jacobson radical** of R is defined to be the intersection of all maximal right ideals $\mathfrak{m}$ of R:

$$J(R) := \bigcap \mathfrak{m}.$$

Clearly this is a right ideal of R, but we will show that it is in fact a two-sided ideal.

Proposition 1. *Let R be a unital ring and $x \in R$. Then the following are equivalent:*

(a) *$x \in J(R)$;*

(b) *$Mx = 0$ for all simple right R-modules M; and*

(c) $1 - axb$ *is a unit for all* $a, b \in R$.

By (b), $J(R)$ is equal to the intersection of all annihilators of simple right R-modules. Since such annihilators are two-sided ideals, it follows that $J(R)$ is also two-sided. Also note that (c) is left-right symmetric — therefore the Jacobson radical can alternatively be defined as the intersection of all maximal *left* ideals, or the set of elements acting trivially in every simple *left* module.

Proof. Consider the following "one-sided" version of (c):

(c̃) $1 - xb$ *is right-invertible for all* $b \in R$.

We will prove (a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c̃), then deduce (c).

"(b)$\Longrightarrow$(a)": Suppose x annihilates every simple right R-module. In particular it annihilates $R/\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$, but then $\bar{x} = \overline{1x} = \bar{1}x = \bar{0}$ and so $x \in \mathfrak{m}$.

"(a)$\Longrightarrow$(c̃)": Suppose $x \in J(R)$ but $y := 1 - xb$ is not right-invertible. Then there is a maximal right ideal $\mathfrak{m}$ containing y. Now $xb \in J(R)$. In particular $xb \in \mathfrak{m}$, but then $1 = xb + y \in \mathfrak{m}$, contradicting maximality.

"(c̃)$\Longrightarrow$(b)": Let M be a simple right module and let $m \in M$. If $mx \neq 0$, then by simplicity we have $M = (mx)R$. In particular $m = mxb$ for some $b \in R$, and so $m(1 - xb) = 0$. But $1 - xb$ is right-invertible by (c̃), implying $m = 0$ — a clear contradiction. Thus $mx = 0$ and so x annihilates M.

Using the equivalence of (a), (b), (c̃), we see that $J(R)$ is a two-sided ideal. To prove (c), note $x \in J(R)$ implies $ax \in J(R)$, so $1 - axb$ is right-invertible by (c̃) — say $(1 - axb)u = 1$. Then u is left-invertible. But $u = 1 + axb = 1 - ax(-b)$ is right-invertible again by (c̃), so we conclude u is a unit and hence so is $1 - axb$. ∎

If R is a not-necessarily-unital ring, we may define its Jacobson radical in a similar way with some appropriate modifications. First, a right ideal $\mathfrak{m}$ is **modular** if there exists an element $e \in R$ such that $r - er \in \mathfrak{m}$ for all $r \in R$. So e acts as a "*left* identity mod $\mathfrak{m}$" (see Exercises 4.1–4.7 in [15]). One may also define modularity for left ideals; note that a two-sided ideal I is both left and right modular if and only if R/I is a unital ring. Now we define the **Jacobson radical** of R to be the intersection of all modular maximal right ideals $\mathfrak{m}$ of R:

$$J(R) := \bigcap \mathfrak{m}.$$

This agrees with the Jacobson radical defined earlier in the unital case, since if R is unital then all right ideals are modular (with $e := 1$).

Now we can prove a nonunital analog of Proposition 1.

Proposition 2. *Let R be a ring and $x \in R$. Then the following are equivalent:*

(a) $x \in J(R)$;

(b) $Mx = 0$ *for all simple right R-modules M; and*

(c) axb *is* **quasiregular** *for all* $a, b \in R$.

An element $x \in R$ is **right** (resp. **left**) **quasiregular** if there exists $y \in R$ such that $xy = x + y$ (resp. $yx = x + y$); and x is **quasiregular** if it is both left and right quasiregular. In this case y is uniquely determined by x. It's easy to verify that if R is unital then x is quasiregular if and only if $1 - x$ is a unit: if $xy = x + y$ then $(1 - x)(1 - y) = 1$. So condition (c) above is indeed generalizing condition (c) from Proposition 1.

Proof. The proof is similar to the unital case, and structured the same way — but it is carried out in detail here because I was not able to find a reference. first prove the following one-sided version of (c):

(c̃) *xb is right quasiregular for all $b \in R$.*

After proving (a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c̃), we deduce (c).

"(b)$\Longrightarrow$(a)": Suppose x annihilates every simple right R-module, and let $\mathfrak{m}$ be a modular maximal ideal. Thus choose $e \in R$ such that $r - er \in \mathfrak{m}$ for all $r \in R$. Now $R/\mathfrak{m}$ is simple, hence annihilated by x, but then $\overline{x} = \overline{ex} = \overline{e}x = \overline{0}$ in $R/\mathfrak{m}$. Therefore $x \in \mathfrak{m}$.

"(a)$\Longrightarrow$(c̃)": First we show every $x \in J(R)$ is right quasiregular; this suffices since $J(R)$ is a right ideal, so replacing x by $xb \in J(R)$ proves (c̃). Consider the right ideal $\mathfrak{a} := \{r - yr : r \in R\}$. This is modular with $e := y$ since, tautologically, $r - yr \in \mathfrak{a}$ for all $r \in R$. We claim that $\mathfrak{a} = R$. Indeed, if $\mathfrak{a} \subsetneq R$, then Zorn's Lemma furnishes a modular maximal right ideal $\mathfrak{m}$ containing $\mathfrak{a}$. Choose $r \notin \mathfrak{m}$. Then $yr \in J(R) \subseteq \mathfrak{m}$ and $r - yr \in \mathfrak{a} \subseteq \mathfrak{m}$, so $r = (r - yr) + yr \in \mathfrak{m}$ — a clear contradiction. We conclude that no modular maximal ideal contains $\mathfrak{a}$, which is only possible if $\mathfrak{a} = R$. Now this implies $-y \in R = \mathfrak{a}$, so $-y = r - yr$ for some $r \in R$. Equivalently $yr = y + r$, so y is right quasiregular.

"(c̃)$\Longrightarrow$(b)": Let M be a simple right module and let $m \in M$. If $mx \neq 0$, then by simplicity we have $M = (mx)R$. In particular $m = mxb$ for some $b \in R$. By (c̃) we have $(xb)y = xb + y$ for some $y \in R$, which implies

$$my = (mxb)y = m(xby) = mxb + my.$$

But then $m = mxb = 0$, which is absurd. Thus (b) must hold.

Using the equivalence of (a), (b), (c̃), we see that $J(R)$ is a two-sided ideal. To prove (c), note $x \in J(R)$ implies $ax \in J(R)$, so axb is right quasiregular by (c̃): say $(axb)y = axb + y$. Thus y is left quasiregular. But $y = (axb)y - axb \in J(R)$ so y is also right quasiregular. It's not hard to deduce from this that axb is also left quasiregular. ∎

There is some interesting interplay between Jacobson radicals in unital and nonunital cases, via unitization. The **unitization** of a ring R is the ring $R_1 := R \oplus \mathbf{Z}$, equipped with component-wise addition and with multiplication defined by the formula

$$(r \oplus m)(s \oplus n) := (rs + nr + ms) \oplus mn.$$

It's easy (though tedious) to check that R_1 is an associative ring, with unit $0 \oplus 1$. Moreover, R embeds as an ideal of R_1 via $r \mapsto r \oplus 0$. For this reason we identify $r + n := r \oplus n$.

Proposition 3. *For any ring R we have $J(R) = J(R_1)$.*

Proof. If $x \in J(R)$, then axb is quasiregular in R for all $a, b \in R$. We must check that in fact axb is quasiregular in R_1 for all $a, b \in R_1$. Writing $a = r + m$, $b = s + n$, where $r, s \in R$ and $m, n \in \mathbf{Z}$, we see that

$$axb = (r + m)x(s + n) = (rx + mx)(s + n) = rxs + mxs + nrx + mnx$$

is a sum of elements of $J(R)$, hence $axb \in J(R)$. So axb is quasiregular in R, *a fortiori* in R_1.

To show $J(R_1) \subseteq J(R)$, we first show $J(R_1) \subseteq R$. Note that the units in R_1 have the form $r \pm 1$ for some $r \in R$. So if $x = r + n \in J(R_1)$, then $1 \pm x = \pm r + (1 \pm n)$ is a unit in R_1, which implies $1 \pm n \in \{1, -1\}$. The only possibility is $n = 0$, so $x = r \in R$.

Now let $a, b \in R$. We must show axb is quasiregular in R. But since $x \in J(R_1)$ we know that axb axb is quasiregular as an element of R_1, say $(axb)y = axb + y$ for some $y \in R_1$, and we need only check $y \in R$. Indeed, $y = (axb)y - axb \in R$ since $x \in R$ and R is an ideal of R_1. ∎

Finally we prove a property which will be useful later.

Corollary 4. *For any ring R, the Jacobson radical $J(R)$ contains no nonzero idempotents.*

Proof. If e is an idempotent in $J(R)$, then working in the unitization we see that $1 - e$ is a unit in R_1 by Proposition 1(c). But then $e(1 - e) = 0$ implies $e = 0$. ∎

3.3.3 Semiprimitive rings. A ring R is **semiprimitive** if $J(R) = 0$. We will prove that every Leavitt path algebra is semiprimitive, thus giving a property of $L(E)$ which is independent of E. First, note that for graph C*-algebras this is free: all C*-algebras are semiprimitive.

Proposition. *Any C*-algebra A has $J(A) = 0$.*

Remark: it can be shown that every maximal modular right ideal in a C*-algebra is automatically closed; thus $J(A)$ is closed, so it is unnecessary to define a "topological" Jacobson radical.

Proof. The proof is conducted assuming some extra familiarity with C*-algebras. Note that for C*-algebras we can define a "C*-unitization" by $A_1 := A \oplus \mathbf{C}$, which is a unital C*-algebra with norm $\|(a, \lambda)\| := \sup_{\|b\| \leq 1} \|ab - \lambda b\|$ and multiplication

$$(a, \alpha)(b, \beta) := (ab + \alpha b + \beta a, \alpha\beta);$$

it is still true that $J(A) = J(A_1)$, thus we proceed assuming that A is unital. Recall that the **spectrum** of an element $a \in A$ can be defined as

$$\sigma(a) := \{\lambda \in \mathbf{C} : \lambda - a \text{ is not a unit}\}.$$

It is a standard theorem that this is a compact subset of $\mathbf{C}$, and we denote the **spectral radius** of a by $r(a) := \sup_{\lambda \in \sigma(a)} |\lambda|$. Clearly $\sigma(a) = 0$ for all $a \in J(A)$ by Proposition 3.3.2(1). Since $a^*a \in J(A)$ too we must also have $\sigma(a^*a) = 0$. But a^*a is self-adjoint,

so its norm is equal to its spectral radius by Proposition I.2.3 in [11], which allows us to conclude

$$\|a\|^2 = \|a^*a\| = \sup_{\lambda \in \sigma(a^*a)} |\lambda| = 0.$$

Thus $a = 0$. ∎

Now we aim to conclude the above C*-algebraic fact for Lpa's. If E has Condition (L) then this follows quickly: $J(L(E)) \neq 0$ implies that $J(L(E))$ contains a vertex by the Cuntz–Krieger Uniqueness Theorem, but the Jacobson radical never contains nonzero idempotents by Corollary 3.3.2(4). On the other hand, if E fails Condition (L), we can only guarantee that every *graded* ideal contains a vertex — thus our first step is to show that the Jacobson radical is graded. Indeed it is a classical theorem of Bergman that this is the case in any unital **Z**-graded ring; for a proof, see [8].

Theorem (Bergman's Theorem). *Let R be a **Z**-graded unital ring. Then $J(R)$ is a graded ideal.*

From this we may deduce the same conclusion in the locally unital case, and hence for any Leavitt path algebra.

Lemma (Bergman's Theorem, locally unital case). *Let R be a locally unital **Z**-graded ring, with a family of local units consisting of homogeneous elements. Then $J(R)$ is a graded ideal.*

Notice that the hypothesis is satisfied by every Leavitt path algebra, due to Corollary 1.4.3(2). I am grateful to Gene Abrams for providing me with the following argument.

Proof. Let $x \in J(R)$ and decompose x as a finite sum $x = \sum_{n \in \mathbf{Z}} x_n$ where each $x_n \in R_n$. We want to show $x_n \in J(R)$. By hypothesis it is possible to select a homogeneous idempotent u such that $uxu = x$. The corner uRu is a **Z**-graded unital ring, and we may decompose x as an element of uRu:

$$x = uxu = \sum_{n \in \mathbf{Z}} ux_n u$$

Since u is a homogeneous idempotent we necessarily have $\deg(u) = 0$, and so each $ux_n u$ is homogeneous of degree n. Thus uniqueness of graded decompositions implies $x_n = ux_n u$ for all n, *i.e.* each x_n is in the corner uRu.

Bergman's Theorem applies to the unital ring uRu: we conclude that $J(uRu) = uJ(R)u$ is a graded ideal. But $x = \sum_{n \in \mathbf{Z}} x_n \in J(uRu)$ is a graded decomposition of x in uRu, so all x_n must be in $J(uRu)$, hence in $J(R)$. ∎

It now follows from Theorem 3.1.4 that the Jacobson radical of an Lpa is zero.

Corollary. *Every Leavitt path algebra $L(E)$ is semiprimitive.*

Proof. $L(E)$ satisfies the hypothesis of Bergman's Theorem, and so $J := J(L(E))$ is graded. Thus Theorem 3.1.4 implies that if $J \neq 0$ then it contains a vertex, which is a nonzero idempotent, contradicting Corollary 3.3.2(4). The only possibility is $J = 0$. ∎

3.3.4 Finite-dimensional Leavitt path algebras. Now we consider conditions on E which guarantee that $L(E)$ is semisimple. In the last section we showed that $L(E)$ is always semiprimitive — so if it is artinian, then the Artin–Wedderburn Theorem implies that $L(E)$ is (uniquely) a direct sum of full matrix algebras over division rings:

$$L(E) \simeq M_{d_1}(\Delta_1) \oplus \cdots \oplus M_{d_n}(\Delta_n).$$

In fact we will show that $\Delta_i = k$, and that $(d_1, \ldots, d_n)$ can be obtained by graph-theoretical data from E.

Theorem. $L(E)$ *is finite-dimensional if and only if E is a finite acyclic graph. In this case, if $w_1, \ldots, w_n$ are the sinks in $L(E)$ and $d_i := |\{\mu \in E^\bullet : r(\mu) = w_i\}|$, then*

$$L(E) \simeq M_{d_1}(k) \oplus \cdots \oplus M_{d_n}(k).$$

Proof. If E is finite and acyclic, then E has at most finitely many paths, so $L(E) = \mathrm{span}\{\mu\nu^*\}_{\mu,\nu \in E^\bullet}$ is finite-dimensional. Conversely, if E has a cycle ξ, then $\xi, \xi^2, \xi^3, \ldots$ are linearly independent in $L(E)$ since they are homogeneous of distinct degree, so $L(E)$ is infinite-dimensional. Also note that vertices are linearly independent (being pairwise orthogonal idempotents), so if E has infinitely many vertices then $L(E)$ is again infinite-dimensional. This establishes the claimed equivalence.

Now assuming that E is finite and acyclic we derive the structure of $L(E)$ as a sum of matrix algebras. Let $w_1, \ldots, w_n$ be the distinct sinks of E, and for each sink w_i define a set $\mathcal{E}_i := \{\mu\nu^* : r(\mu) = r(\nu) = w_i\}$. Let $d_i := |\{\mu \in E^\bullet : r(\mu) = w_i\}|$. We will prove the following claims:

(i) $\mathcal{E}_i$ is a $d_i \times d_i$ family of matrix units over k;

(ii) $\mathrm{span}\,\mathcal{E}_i$, $1 \leq i \leq n$ are pairwise orthogonal; and

(iii) $L(E) = \sum \mathrm{span}\,\mathcal{E}_i$.

If we prove this we are done: (i) gives that $\mathrm{span}\,\mathcal{E}_i \simeq M_{d_i}(k)$ as k-algebras, and (ii) and (iii) together give $L(E) = \bigoplus \mathrm{span}\,\mathcal{E}_i \simeq \bigoplus M_{d_i}(k)$.

For (i), we must check the following relations:

$$(\mu\nu^*)(\alpha\beta^*) = \begin{cases} \mu\beta^* & \text{if } \nu = \alpha, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad (\mu\nu^*)^* = \nu\mu^*.$$

The latter is obvious. For the former, we recall formula (♣) from 1.4.3:

$$\nu^*\alpha = \begin{cases} \gamma^* & \text{if } \nu = \alpha\gamma, \\ \gamma & \text{if } \alpha = \nu\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

In the first case, notice that if $r(\nu) = r(\alpha) = w_i$ and $\nu = \alpha\gamma$, then $s(\gamma) = r(\alpha) = w_i$ and $r(\gamma) = r(\nu) = w_i$, so γ is a cycle based at w_i. Since we assumed E was acyclic, this is only possible if $\gamma = w_i$, which implies $\nu = \alpha$. Using the same argument in the second case, we thus see that $\nu^*\alpha = 0$ if $\nu \neq \alpha$, and $\nu^*\alpha = w_i$ otherwise. The required formula follows and we have thus established claim (i).

For claim (ii), we want to show that if $\mu\nu^* \in \mathcal{E}_i$ and $\alpha\beta^* \in \mathcal{E}_j$, then $(\mu\nu^*)(\alpha\beta^*) = 0$. We can use (♣) again: notice $\nu^*\alpha \neq 0$ if and only if $\nu = \alpha\gamma$ or $\alpha = \nu\gamma$ for some path γ.

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But α, ν end in sinks and thus cannot extend nontrivially, so both cases imply $\nu = \alpha$ — but this is impossible since $r(\nu) = w_i \neq w_j = r(\alpha)$. Thus the only possibility is $\nu^* \alpha = 0$, and so $(\mu \nu^*)(\alpha \beta^*) = 0$ as required.

Finally we prove claim (iii). Since $L(E) = \text{span}\{\mu\nu^*\}_{r(\mu)=r(\nu)}$ it suffices to show that each element of the form $\mu\nu^*$, $r(\mu) = r(\nu)$ can be expressed as a sum $\sum \mu_i \nu_i^*$ with each $r(\mu_i) = r(\nu_i)$ a sink. Let $v := r(\mu) = r(\nu)$; if v is a sink we're done. Otherwise, by (CK2) we may write

$$\mu\nu^* = \mu v \nu^* = \sum_{s(e)=v} \mu e e^* \nu^* = \sum_{s(e)=v} (\mu e)(\nu e)^*.$$

Thus we have extended μ, ν to longer paths $\mu e, \nu e$. Now examine each term $(\mu e)(\nu e)^*$ individually: if $r(e)$ is a sink we are done, otherwise repeat this process to get an even longer path. Eventually the obtained path will end in a sink since E has only finitely many paths. This establishes claim (iii) and completes the proof. ∎

We remark that the same conclusion holds for graph C*-algebras, with the same proof. Alternatively, one could note that if $L_{\mathbf{C}}(E)$ is finite-dimensional, then it is in particular a complete and hence closed subalgebra of $C^*(E)$. Since $L_{\mathbf{C}}(E)$ must be dense in $C^*(E)$ the only possibility is $L_{\mathbf{C}}(E) = C^*(E)$.

Corollary 1. *$C^*(E)$ is finite-dimensional if and only if E is a finite acyclic graph. In this case, if $w_1, \ldots, w_n$ are the sinks in $L(E)$ and $d_i := |\{\mu \in E^\bullet : r(\mu) = w_i\}|$, then*

$$C^*(E) \simeq M_{d_1}(\mathbf{C}) \oplus \cdots \oplus M_{d_n}(\mathbf{C}).$$

As another consequence of the finite-dimensionality theorem for Lpa's, we can deduce that the only artinian Lpa's are the finite-dimensional ones, which in turn must be sums of matrix algebras over k — another reflection of C*-algebras.

Corollary 2. *Every artinian Lpa is finite-dimensional.*

Proof. Assuming $L = L(E)$ is artinian, we will show that E is finite and acyclic so that L is finite-dimensional by the above theorem. We first note that by the Hopkins–Levitski Theorem (Theorem 4.15 in [15]), all artinian rings are noetherian. Now if $v_1, v_2, v_3, \ldots$ are infinitely many vertices in E, then the right ideals $I_k := \sum_{i=1}^{k} v_i L$ form an increasing chain $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ since $v_{k+1} \notin I_k$, contradicting noetherianness. Thus E must be finite.

To see that E is acyclic, assume instead that it has a cycle ξ. Let $v := s(\xi)$ and consider the corner ring $R := vLv$. Since L is artinian, so must be R. But if ξ has no exit, then by Lemma 3.1.5(2) we have $R \simeq k[x, x^{-1}]$ which is not artinian. If ξ has an exit, we can construct a descending chain of left ideals in R as follows:

$$R\xi\xi^* \supsetneq R\xi(\xi^*)^2 \supsetneq \cdots \supsetneq R\xi(\xi^*)^n \supsetneq \cdots$$

which is again contrary to the assumption that R is artinian. Let us verify the inclusions in the above chain. Indeed, $\xi(\xi^*)^{n+1} \in R\xi(\xi^*)^n$ because

$$\xi(\xi^*)^{n+1} = \xi\xi^* v(\xi^*)^n = \xi\xi^*(\xi^*\xi)(\xi^*)^n = [\xi\xi^*\xi^*]\xi(\xi^*)^n \in R\xi(\xi^*)^n$$

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where we note $v = \xi^*\xi$ by (CK1). To see that the inclusions are strict, note that if there is an equality at the nth step then $\xi(\xi^*)^n = r\xi(\xi^*)^{n+1}$ for some $r \in R$. Letting e be an exit for ξ at v, we obtain the following string of equalities:

$$
\begin{aligned}
0 \neq \xi e \qquad &\text{since } \xi e \text{ is a legitimate path in } E \\
= \xi v e \qquad &\text{since } e = v e \\
= \xi[(\xi^*)^n \xi^n] e \qquad &\text{since } \xi^*\xi = v \text{ by (CK1)} \\
= r\xi(\xi^*)^{n+1}\xi^n e \qquad &\text{since } \xi(\xi^*)^n = r\xi(\xi^*)^{n+1} \\
= r\xi\xi^* e \qquad &\text{since } \xi^*\xi = v \\
= 0 \qquad &\text{since } e \text{ is not on } \xi.
\end{aligned}
$$

This is a clear contradiction, so it must be the case that $R\xi(\xi^*)^n \supsetneq R\xi(\xi^*)^{n+1}$. ∎

3.3.5 Af C*-algebras. In the remainder of this section we deal with the algebras of *infinite* acyclic graphs. We begin by examining graph C*-algebras because the C*-techniques will motivate the proof methods for Leavitt path algebras.

A C*-algebra A is **approximately finite-dimensional**, or **af**, if the following condition holds: for every finite set $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$, there exists a finite-dimensional C*-subalgebra B of A and elements $b_1, \ldots, b_n \in B$ such that $\|a_i - b_i\| < \varepsilon$. In words, this means every finite collection of elements of A can be approximated by a finite-dimensional subalgebra.

Example. If H is a Hilbert space, the C*-algebra $\mathcal{K}(H)$ of compact operators is af. This is because we may view $M_n(\mathbf{C})$ as the algebra of operators $f \in \mathcal{B}(H)$ such that $\dim(\operatorname{im} f) \leq n$, so that $M_1(\mathbf{C}) \subseteq M_2(\mathbf{C}) \subseteq M_3(\mathbf{C}) \subseteq \cdots \subseteq \mathcal{K}(H)$ and

$$
\mathcal{K}(H) = \overline{\bigcup_{n \geq 1} M_n(\mathbf{C})}.
$$

Example. If X is a compact Hausdorff space, then $\mathcal{C}(X)$ is af if and only if X is *totally disconnected*; that is, its connected components are points. See *e.g.* Example III.2.5 in [11]. In particular the graph C*-algebra $\mathcal{C}(\mathbf{T})$ is not af.

Before giving our graphical characterization of af algebras, we state some well-known facts on af algebras. First let A be any C*-algebra. Recall that a **projection** in A is an element p such that $p^2 = p^* = p$. Two projections are **Murray–von Neumann equivalent**, denoted $p \sim q$, if there exists $s \in A$ so that $p = s^*s$ and $p = ss^*$. We denote $p \leq q$ if $pq = p = qp$, in which case p is a **subprojection** of q; we denote $p \lesssim q$ to mean $p \sim p' \leq q$ for some projection q; we denote $p \lneqq q$ to mean $p \sim_0 p' \leq q$. Finally we say p is an **infinite projection** if $p \lneqq p$.

A C*-algebra A is **finite** if it contains no infinite projections. Unraveling the definition, this can be described element-wise as follows: whenever $s \in A$ is an element such that $p = s^*s$ is a projection and $ss^* \leq p$, we must have $ss^* = p$.

Lemma. *Let A be a C*-algebra.*

(a) *If A is unital, then A is finite if and only if $s^*s = 1$ implies $ss^* = 1$.*

(b) *If A is af then A is finite.*

Proof. (a) Assume A is finite. If $s^*s = 1$ then ss^* is a projection, so $ss^* \le 1$; thus finiteness implies $ss^* = 1$. Conversely, assume that $s^*s = 1$ implies $ss^* = 1$ and let p be a projection in A. We must show that p is finite. Indeed, if $p = s^*s$ with $q := ss^* \le p$, then $u := s + (1 - p)$ satisfies $u^*u = 1$, so $uu^* = 1$ by assumption. On the other hand $uu^* = p + (1 - q)$; in total we get $p = q$ as required.

(b) We first prove that A is finite in case A is finite-dimensional. But then A is a sum of matrix algebras over $\mathbf{C}$, hence unital. Since $ab = 1$ implies $ba = 1$ in $M_n(\mathbf{C})$ (by linear algebra) we easily see that A is finite by using part (a).

Now let A be an af algebra. Since a subalgebra of a finite algebra is finite, replacing A with its unitization — which is still af — we may assume A is unital. Thus by (a) we must show that $s^*s = 1$ implies $ss^* = 1$. Since A is af we have $s = \lim s_n$, where s_n belongs to some finite-dimensional subalgebra A_n with $1 \in A_n$. Now $1 = s^*s = \lim s_n^* s_n$, so replacing (s_n) by a subsequence we may assume $\|1 - s_n^* s_n\| < 1$ for all n, which implies that $s_n^* s_n$ is invertible in A_n (see Theorem I.2.1 in [11]). In particular s_n^* is right-invertible, hence automatically left-invertible in A_n because A_n is finite-dimensional. Choosing $t_n \in A_n$ so that $t_n s_n^* = 1$, we see that $t_n \to s$ because

$$\|s - t_n\| \le \|s - s_n\| + \|s_n - t_n\| \le \|s - s_n\| + \|t_n\| \|s_n^* s_n - 1\|$$

and the right-hand side goes to zero since $\|t_n\|$ is bounded if we force $\|s_n^* s_n - 1\|$ to be small enough. Therefore $s = \lim t_n$, and

$$ss^* = (\lim t_n)(\lim s_n^*) = \lim t_n s_n^* = 1$$

as required. $\blacksquare$

Theorem. *Let E be a (countable) row-finite graph. Then $C^*(E)$ is af if and only if E is acyclic.*

Proof. First assume that E is acyclic. By Corollary 2.4.1, there is a directed system $\{F_i\}$ of finite subgraphs of E such that $C^*(E) = \overline{\bigcup_i C^*(F_i)}$. But since E is acyclic so must be each F_i, which implies each $C^*(F_i)$ is finite-dimensional. Thus $C^*(E)$ is a direct limit of finite-dimensional C*-algebras, which is the same as being af.

Conversely, if $C^*(E)$ is af we prove that E must be acyclic. Suppose $\xi = e_1 \cdots e_n$ is a cycle in E based at $v := s(\xi)$. If ξ has no exit, then $p_v C^*(E) p_v \simeq \mathcal{C}(\mathbf{T})$ by Lemma 3.1.6 — but a corner of an af algebra should be af, while $\mathcal{C}(\mathbf{T})$ is not. This proves that ξ must have an exit; without loss of generality we assume the exit is at v, so this is an edge $e \ne e_1$ such that $s(e) = v$. We will show that p_v is an infinite projection in $C^*(E)$, contradicting lemma (b) above. Indeed note that $s_e s_e^*$ and $s_{e_1} s_{e_1}^*$ are orthogonal projections since $e \ne e_1$ (see formula 1.3.3($\clubsuit$)), so

$$p_v = s_\xi^* s_\xi \sim s_\xi s_\xi^* \le s_{e_1} s_{e_1}^* < s_{e_1} s_{e_1}^* + s_e s_e^* \le \sum_{s(f)=v} s_f s_f^* = p_v$$

where the first equality is (CK1) and the last equality is (CK2). This shows that p_v is equivalent to a proper subprojection of itself, which is what it means for p_v to be infinite. $\blacksquare$

3.3.6 Locally matricial algebras. Having seen that the graph C*-algebra of an acyclic graph is af, we now seek to find an algebraic analogue for Leavitt path algebras.

A k-algebra R is **matricial** (over k) if it has the form $R \simeq M_{d_1}(k) \oplus \cdots \oplus M_{d_n}(k)$ for some positive integers $d_1, \ldots, d_k$. So matricial algebras are finite-dimensional and semiprimitive; in fact if k is algebraically closed then these are all the finite-dimensional semiprimitive algebras. The algebra R is **locally matricial** (over k) if, for each finite set $r_1, \ldots, r_n \in R$, there exists a matricial subalgebra $B \simeq M_{d_1}(k) \oplus \cdots \oplus M_{d_n}(k)$ such that $r_1, \ldots, r_n \in B$. It is equivalent to require that R is a direct limit of matricial algebras.

Example. We have inclusions $M_n(k) \hookrightarrow M_{n+1}(k)$ given by $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, so we may think of a chain $M_1(k) \subseteq M_2(k) \subseteq M_3(k) \subseteq \cdots$. Taking the direct limit with respect to these inclusions yields a locally matricial algebra:

$$M(k) := \varinjlim M_n(k) = \bigcup_{n \geq 1} M_n(k).$$

Example. $k[x, x^{-1}]$ is not locally matricial: since $k[x, x^{-1}]$ is a commutative domain, the only possible matricial subalgebras are fields. But, for example, $1 - x$ is not a unit and hence cannot belong to a field.

It turns out that locally matricial algebras are the appropriate ring-theoretic analogue of af algebras, and we have the following theorem for Lpa's of acyclic graphs.

Theorem. *Let E be row-finite. Then $L(E)$ is locally matricial if and only if E is acyclic.*

Proof. For "$\Longleftarrow$" ones argues as in Theorem 3.3.5, by using Corollary 2.4.1. For "$\Longrightarrow$", if ξ is a cycle then $\xi, \xi^2, \xi^3, \ldots$ are linearly independent since they are homogeneous of distinct degree, so ξ cannot be contained in any finite-dimensional subalgebra. ∎

3.4 Pure infiniteness

Having given a treatment of acyclic graphs in the previous section, now we turn to graphs which are not acyclic. We will see that if a graph C*-algebra $C^*(E)$ is simple and not af, then it is *purely infinite*: essentially this is a condition saying that it contains "enough" infinite projections. Under the same graphical conditions we will show that the Leavitt path algebra $L(E)$ is also purely infinite in a ring-theoretic sense which is closely related to the C*-algebraic property. We will see that the simple Lpa's are dichotomized: if $L(E)$ is simple, it must be either locally matricial (if E is acyclic) or purely infinite simple (if E has a cycle).

3.4.1 Purely infinite C*-algebras. Let A be a C*-algebra and let $p \in A$ be a projection. Recall that in 3.3.5, we defined what it meant for p to be **infinite**: in short, it means p is equivalent to a subprojection of itself. A is **purely infinite** if every closed right ideal of A contains an infinite projection.

Note that if A is unital and has no nontrivial right ideals, then A is a division algebra and hence $A \simeq \mathbf{C}$ (see Theorem 10.14 in [28]), which is not purely infinite because its unit ideal contains no infinite projections. Thus a unital purely infinite C*-algebra must have dimension at least 2 and at least one nontrivial right ideal, and hence at least one infinite projection.

Example. If H is a separable infinite-dimensional Hilbert space, the **Calkin algebra** $\mathcal{Q}(H) := \mathcal{B}(H)/\mathcal{K}(H)$ is purely infinite simple; see *e.g.* page 2 in [30].

Example. The Cuntz algebra $\mathcal{O}_n$ is purely infinite simple for $n \geq 2$, as will be made clear later from its structure as a graph C*-algebra.

Example. Since af algebras have no infinite projections, they cannot be purely infinite; for instance neither $M_n(\mathbf{C})$ nor $\mathcal{K}(H)$ are purely infinite. Commutative algebras cannot be purely infinite either, since equivalence of projections reduces to equality; so $\mathcal{C}(\mathbf{T})$ is not purely infinite. The reason for these that the former graphs are acyclic and the latter graph fails Condition (L).

Another example of a C*-algebra which is not purely infinite is $\mathcal{B}(H)$: even though it has many infinite projections, it contains the ideal $\mathcal{K}(H)$ which has no infinite projections.

Below is a neat characterization of when a simple C*-algebra is purely infinite. It is separated into two cases: unital and nonunital. The unital case is Theorem V.5.5 in [11], and the nonunital case is [19].

Theorem. *Let A be a simple C*-algebra, $\dim A \geq 2$.*

(a) *A is purely infinite if and only if, for all nonzero $x \in A$, there exist $a, b \in A$ so that axb is an infinite projection.*

(b) *Suppose A is unital. Then A is purely infinite if and only if, for all nonzero $x \in A$, there exist $a, b \in A$ so that $axb = 1$.*

Now we state the characterization of purely infinite graph C*-algebras. Since the proof uses some machinery not developed in these notes, we omit it; for details see Proposition 5.3 in [6].

Theorem. *Let E be a row-finite (countable) graph. Then $C^*(E)$ is purely infinite if and only if E satisfies Condition (L) and every vertex in E connects to a cycle.*

If $C^*(E)$ is simple, then E automatically satisfies Condition (L) and is cofinal. If E has a cycle ξ, then $\xi\xi\xi\cdots$ is an infinite path in E and so cofinality implies every vertex connects to ξ, and therefore $C^*(E)$ is purely infinite simple by the above quoted result. If E has no cycle, then all we can say is that $C^*(E)$ is an af algebra by Theorem 3.3.5. We conclude the following:

Corollary (dichotomy)**.** *Let $C^*(E)$ be a simple graph C*-algebra. Then $C^*(E)$ is either af (if E is acyclic) or purely infinite simple (if E has a cycle).*

3.4.2 Purely infinite rings. Now we focus on mirroring the purely infinite simplicity in C*-algebras in an exclusively algebraic setting. Since a general ring R is not necessarily equipped with a $*$ operation, instead of working with projections we work merely with idempotents.

Let R be a ring and $e, f \in R$ idempotents. Then e is a **subidempotent** of f if $ef = e = fe$, denoted $e \leq f$. We say e, f are **Murray–von Neumann equivalent**, written $e \sim f$, if $e = xy$ and $f = yx$ for some $x, y \in R$; it is not hard to see that $e \sim f$ if and only if $eR \simeq fR$ as right R-modules (see [4]). We write $e \lesssim f$ to indicate that $e \sim e' \leq f$ for some idempotent e', in which case e is **equivalent to a subidempotent**

of f; moreover $e \underset{\sim}{\lesssim} f$ if $e \sim e' \lneqq f$. An idempotent e is **infinite** in case $e \underset{\sim}{\lneqq} e$. In terms of modules, this means that the right R-module eR is isomorphic to a proper direct summand of itself. Finally, we say that R is a **purely infinite ring** if each nonzero right ideal of R contains an infinite idempotent.

Example. Let V be a vector space of countably-infinite dimension and consider the endomorphism ring $R := \operatorname{End} V$. This is a local ring whose maximal ideal M consists of those endomorphisms with finite-dimensional image. The quotient R/M is purely infinite simple; see [4].

Example. The classical Leavitt algebra L_n is purely infinite simple for $n \geq 2$ — simplicity is clear from Theorem 3.1.5, and we will prove below that L_n is purely infinite.

Example. Clearly commutative rings cannot have infinite idempotents, since in this case $\sim$ reduces to equality. No finite-dimensional algebra R contains an infinite idempotent: if $e \in R$ is an infinite idempotent, then $eR \simeq eR \oplus M$ for some nonzero right R-module M, and so eR contains an infinite decreasing chain of right ideals.

Now we can deduce that a locally matricial algebra R contains no infinite idempotents. Indeed, if $e \in R$ is an infinite idempotent then there is an idempotent $e' \leq e$ such that $e' \neq e$ but $e' \sim e$. Say $e = xy$ and $e' = yx$. Since R is locally matricial, there is a finite-dimensional subalgebra B containing all of e, e', x, y. But then e is an infinite idempotent in B, contradicting the fact that finite-dimensional algebras have no infinite idempotents.

The following is Theorem 1.6 in [4].

Theorem 1. *Let A be a simple unital ring which is not a division ring. Then A is purely infinite if and only if, for all nonzero $x \in R$, there exist $a, b \in R$ so that $axb = 1$.*

We observe that the condition that R not be a division ring mirrors the condition that $\dim A \geq 2$ in Theorem 3.4.1(1).

Finally we arrive at a graphical characterization for when $L(E)$ is purely infinite; it is interesting to observe that the conditions are exactly the same as those given in Theorem 3.4.1(2) for C*-algebras.

Theorem 2. *Let E be a row-finite graph. Then $L(E)$ is purely infinite if and only if E satisfies Condition (L) and every vertex in E connects to a cycle.*

To prove this we will repeatedly use the fact that corners of purely infinite rings contain infinite idempotents.

Lemma. *Let R be a purely infinite ring and let $e \in R$ be a nonzero idempotent. Then the corner eRe contains an infinite idempotent.*

Proof. Let p be an infinite idempotent in the right ideal eR; in particular $ep = p$, and infiniteness of p means that there is an idempotent p' so that $p \sim p' \lneqq p$. Letting $q := pe$ and $q' := p'e$, clearly q, q' are idempotents in eRe. We will show that $q \sim q' \lneqq q$ as idempotents in eRe.

Towards this, recall that $p \sim p'$ means that there exist elements $x, y \in R$ so that $p = xy$, $p' = yx$, and $p' \leq p$ means that $pp' = p' = p'p$. Note, then, that $p' = pp' \in eR$ and so $p' = ep'$. Replacing $x \mapsto pxp'$ and $y \mapsto p'yp$ we may assume $x = pxp'$ and

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$y = p'yp$. In particular $x \in eR$, so $x = ex$, and similarly $y = ey \in eR$. Finally set $u := xe$ and $v := ye$ so that u, v are elements of the corner eRe. Putting together all the above observations, we find that

$$uv = (xe)(ye) = xye = pe = q$$

and similarly $vu = q'$; so $q \sim q'$ in eRe. We also have $qq' = pep'e = pp'e = p'e = q'$ and similarly $q'q = q'$; so $q' \le q$ in eRe. To finish we must check that $q \ne q'$: indeed, if $q = q'$ then the fact that $p' \le p$ implies

$$p' = p'p = p'ep = q'p = qp = pep = pp = p$$

contradicting $p' \ne p$. ∎

Now we proceed to characterize the purely infinite Leavitt path algebras.

Proof of Theorem 2. First assume $L = L(E)$ is purely infinite. If E fails Condition (L), then by Lemma 3.1.5 the algebra L has a corner isomorphic to $k[x, x^{-1}]$, which contains no infinite idempotents — contradicting the lemma. To see that every vertex v connects to a cycle, we will show that otherwise the corner vLv has no infinite idempotents, again contradicting the lemma. Consider the subgraph H of E defined by $H^0 := \{w \in E^0 : v \to w\}$, $H^1 := \{e \in E^1 : s(e) \in H^0\}$. Then $H \hookrightarrow E$ is a complete graph inclusion in the sense of 1.2.4 — so Proposition 1.2.4 provides an inclusion of algebras $L_0 := L(H) \hookrightarrow L(E)$; in fact $L_0 = \mathrm{span}\{\mu\nu^* : \mu, \nu \in H^\bullet\}$. On one hand H is acyclic by choice of v, so L_0 is locally matricial by Theorem 3.3.6, hence contains no infinite idempotents — so the same must be true of the subalgebra vL_0v. On the other hand, $vL_0v = vLv$: this is because if $\mu\nu^* \in L$ is a typical monomial, then $v(\mu\nu^*)v = \mu\nu^*$ if $s(\mu) = v = s(\nu)$ and is otherwise zero, and in either case $\mu\nu^* \in L_0$. Therefore vLv is a corner of L containing no infinite idempotents.

Now we focus on proving that if E satisfies Condition (L) and every vertex connects to a cycle, then $L(E)$ is purely infinite. First:

Claim A: If $v \to w$ then $p_w \lesssim p_v$.

Here $p \lesssim q$ means that $p \sim p' \le q$ for some idempotent p'. To prove the claim, let $\mu = e_1 \cdots e_n$ be a path with $s(\mu) = v$ and $r(\mu) = w$. Then

$$w = \mu^*\mu \sim \mu\mu^* \le e_1 e_1^* \le \sum_{s(e)=v} ee^* = v$$

proving Claim #1. Assuming now that E satisfies Condition (L) and every vertex connects to a cycle, we may deduce the following.

Claim B: All $v \in E^0$ are infinite idempotents in $L(E)$.

Indeed if $v \in E^0$ is any vertex, then by assumption v connects to a cycle ξ and ξ must have an exit. Let $v_0 := s(\xi)$. The argument in the final paragraph in the proof of Theorem 3.3.5 can be easily adapted to show that v_0 is an infinite projection. By Claim A we have $v_0 \le v$ and so v must also be infinite. This proves Claim B.

Now let $\mathfrak{a}$ be an arbitrary nonzero right ideal of $L(E)$ and let $x \in \mathfrak{a}$ be nonzero. By Lemma 2.3.3 there exist paths $\alpha, \beta \in E^\bullet$ so that $\alpha^* x \beta = cv$ for some $c \in k^\times$ and vertex $v \in E^0$. Let $p := x(\frac{1}{c}\beta\alpha^*) \in \mathfrak{a}$; then p is nonzero because

$$0 \ne v = v^2 = \left(\frac{1}{c}\alpha^* x \beta\right)\left(\frac{1}{c}\alpha^* x \beta\right) = \frac{1}{c}\alpha^* px\beta.$$

To see that p is idempotent, note $0 \neq cv = cv^2 = \alpha^* x \beta v$ implies $\beta v \neq 0$, and this is only possible if $r(\beta) = v$, and so $\beta v = \beta$. Therefore,

$$p^2 = \left(\frac{1}{c} x \beta \alpha^*\right)\left(\frac{1}{c} x \beta \alpha^*\right) = \frac{1}{c^2} x \beta (\alpha^* x \beta) \alpha^* = \frac{1}{c^2} x \beta (cv) \alpha^* = \frac{1}{c} x \beta \alpha^* = p.$$

Hence p is a nonzero idempotent in $L(E)$. Finally, to see that p is infinite, let $s := \frac{1}{c}\alpha^*$ and $t := x\beta$. Then $st = v$ and $ts = p$, so $v \sim p$. Since v is infinite by Claim B it follows that p is infinite too, and therefore p is the required infinite idempotent in $\mathfrak{a}$. ∎

Just as with C*-algebras, we can deduce a dichotomy of simple Leavitt path algebras.

Corollary (dichotomy). *Let $L(E)$ be a simple Leavitt path algebra. Then either $L(E)$ is locally matricial (if E is acyclic) or purely infinite simple (if E has a cycle).*

Combining with 3.2.2 we make the following remark on primite Lpa's.1

**Corollary.** *Every purely infinite simple Lpa is primitive.*

Proof. By our graphical characterizations of purely infinite simplicity and primitivity (the above dichotomy, Theorem 3.1.5, and Theorem 3.2.2), it is equivalent to prove the following graph-theoretic statement: if E is a cofinal graph satisfying Condition (L) and which has at least one cycle, then E is downward directed.

Let $v, w \in E$ be two vertices and let ξ be a cycle in E. By cofinality we get that every vertex connects to every cycle, so $v \to x$ and $w \to x$ where $x := s(\xi)$. ∎

Naturally we pose the following question on general rings.

Question. *Is every purely infinite simple ring primitive?*

Without simplicity the above corollary may fail — simplicity implies every vertex connects to every cycle, but pure infiniteness alone only means every vertex connects to *some* cycle. Here is an example of a nonsimple purely infinite ring which is not primitive: the following graph E satisfies Condition (L) and every vertex connects to a cycle, but is not downward directed.
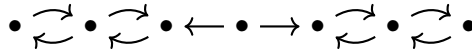
On the other hand, every purely infinite ring is at least *semi*primitive: otherwise the Jacobson radical is a nonzero right ideal, hence would contain an idempotent (contradicting Corollary 3.3.2(4)).

§4 K-theoretic classification

In this section we outline a conjecture in the theory of Leavitt path algebras which is motivated by a theorem from C*-algebras. This conjecture deals with the K-theoretic classification of rings and C*-algebras: given a C*-algebra A one can define two abelian groups $K_0(A)$ and $K_1(A)$, known as the K-theory of A, such that if A and B are $*$-isomorphic C*-algebras then $K_0(A) \simeq K_0(B)$ and $K_1(A) \simeq K_1(B)$. For simple (unital) graph C*-algebras the converse is true: if two graph C*-algebras have the same K-theory — in addition to some extra data — then they are $*$-isomorphic. Thus it is said that simple graph C*-algebras are classifiable by K-theory. On the other hand, one can also define K_0 and K_1 for rings, but the corresponding classification question for Leavitt path algebras remains unsolved. Even in the case of graph C*-algebras, it is not presently known how $*$-isomorphism translates into an equivalence relation on the underlying graphs.

First we define the K_0 group of a C*-algebra and of a ring — these are defined separately, but it turns out that if A is a C*-algebra then its C*-algebraic and ring-theoretic K_0 are isomorphic groups. The K_1 group will not be defined in these notes since we will not need it for the exposition. We will show a tidy description of $K_0(C^*(E))$ and $K_0(L(E))$ in terms of graph-theoretic data from E, then outline how one can deduce the classification of simple graph C*-algebras using some classical C*-theorems. Finally we provide an exposition of current lines of research towards solving the analogous classification conjecture in Leavitt path algebras.

4.1 The K_0 group

In this section, the K_0 groups of C*-algebras and rings are defined. We also provide a description of $K_0(L(E))$ purely in terms of E.

4.1.1 The Grothendieck construction. In this section we discuss the construction of the enveloping group associated to a monoid; this is necessary for both rings and C*-algebras. The definition of the enveloping group mimics the construction of the abelian group $\mathbf{Z}$ from the monoid $\mathbf{N}$. The idea is that $\mathbf{Z}$ consists of pairs $[a, b]$ with $a, b \in \mathbf{N}$, where $[a, b] = [a', b']$ in $\mathbf{Z}$ if and only if $a + b' = a' + b$ in $\mathbf{N}$. So $[a, b]$ formally represents the difference $a - b$. For noncancellative monoids — that is, $a + b = a + c$ while $b \neq c$ — this equivalence relation must be strengthened.

Let M be any abelian monoid written additively, and let $M \oplus M$ admit the usual monoid structure. Define a relation on $M \oplus M$ as follows:

$$(a, b) \sim (a', b') \iff a + b' + s = a' + b + s, \text{ for some } s \in M.$$

The addition of s on both sides is required to make $\sim$ a transitive relation; indeed, it is easily checked that this an equivalence relation. If M *is* cancellative, then $(a, b) \sim (a', b')$ reduces to $a + b' = a' + b$.

Denote the equivalence class of (a, b) by a formal difference $[a - b]$. We may define an addition on these equivalence classes via

$$[a - b] + [c - d] := [(a + c) - (b + d)].$$

This is essentially the effect of taking the quotient of $M \oplus M$ by the submonoid $N := [0 - 0]$. Finally, the **enveloping group** (also called the **groupification** or

Grothendieck group) of M is defined to be

$$\mathrm{Grp}(M) := \frac{M \oplus M}{\sim}$$

equipped with the above defined operation. It is indeed an abelian group: the identity is $[0 - 0]$, and additive inverses are given by $-[a - b] = [b - a]$. Thus we frequently simplify notation to $a - b = [a - b]$. Note, however, that the map $M \to \mathrm{Grp}(M)$ given by $m \mapsto m - 0$ is not generally injective; for example $\mathrm{Grp}(\mathbf{N} \sqcup \{0\}, \times) = 0$.

4.1.2 K_0 of a C*-algebra. Recall that if A is a C*-algebra and $p, q \in A$ are two projections, we define $p \sim q$ to mean that there exists $s \in A$ so that $p = s^*s$ and $q = ss^*$. We now extend this equivalence relation to matrices over A. First let $\mathrm{Proj}_n(A)$ denote the set of projections in the C*-algebra $M_n(A)$, and set $\mathrm{Proj}(A) := \bigsqcup_{n \geq 1} \mathrm{Proj}_n(A)$. If $p, q \in \mathrm{Proj}(A)$, we define $p \sim_0 q$ to mean that there exists a rectangular matrix s so that $p = s^*s$ and $q = ss^*$. Note that s need not be square if p and q are different sizes. We also use the following notation: for $a \in M_n(A)$ and $b \in M_m(A)$,

$$a \oplus b := \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_{n+m}(A).$$

This is obviously an associative operation.

Proposition 1. *Let A be a C*-algebra and consider projections in $\mathrm{Proj}(A)$.*

  (i) *If $p \sim_0 p'$ and $q \sim_0 q'$, then $p \oplus q \sim_0 p' \oplus q'$.*

  (ii) *$p \sim_0 p \oplus 0$, where 0 is a zero matrix of any size.*

  (iii) *$p \oplus q \sim_0 q \oplus p$.*

Proof. (i) We have $p = s^*s$, $p' = ss^*$, and $q = t^*t$, $q' = tt^*$ for some s, t of appropriate size. If we set $u := \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}$ then $p \oplus q = u^*u$ and $p' \oplus q' = uu^*$.

(ii) Let p be a projection and let $s := \begin{bmatrix} p \\ 0 \end{bmatrix}$. Then $p = s^*s$ and $p \oplus 0 = ss^*$.

(iii) Let $s := \begin{bmatrix} 0 & q \\ p & 0 \end{bmatrix}$. Then $p \oplus q = s^*s$ and $q \oplus p = ss^*$. ∎

In total, this proposition states that $\mathrm{Proj}(A)/\sim_0$ is a well-defined abelian monoid under the operation $[p]_0 \oplus [q]_0 := [p \oplus q]_0$, where $[-]_0$ denotes the equivalence class of a projection with respect to $\sim_0$. If A is unital, the **zeroth K-theory** of A is defined to be the enveloping group of this monoid:

$$K_0(A) := \mathrm{Grp}\left(\frac{\mathrm{Proj}(A)}{\sim_0}\right).$$

In other words, $K_0(A)$ consists of formal differences $[p]_0 - [q]_0$, with $[p]_0 - [q]_0 = [0]_0$ if and only if $p \oplus r \sim_0 q \oplus r$ for some $r \in \mathrm{Proj}(A)$.

Example. It is not hard to see that if $p, q \in M_n(\mathbf{C})$ are projections, then $p \sim_0 q$ if and only if $\mathrm{rank}(p) = \mathrm{rank}(q)$. Indeed, if $\mathrm{rank}(p) = \mathrm{rank}(q)$ then any isometry $s : \mathrm{im}(p) \to \mathrm{im}(q)$ extends to an element $s \in M_n(\mathbf{C})$ such that $p = s^*s$ and $q = ss^*$; conversely, if $p = s^*s$ and $q = ss^*$ then $s|_{\mathrm{im}(p)} : \mathrm{im}(p) \to \mathrm{im}(q)$ is an isometry. Thus $\mathrm{rank} : \mathrm{Proj}(\mathbf{C})/\sim_0 \to \mathbf{N} \sqcup \{0\}$ is a well-defined monoid isomorphism, and we conclude

$$K_0(\mathbf{C}) \simeq \mathrm{Grp}(\mathbf{N} \sqcup \{0\}) = \mathbf{Z}.$$

Since $\mathrm{Proj}_n(M_d(\mathbf{C})) = \mathrm{Proj}_{nd}(\mathbf{C})$, it can be easily deduced that $K_0(M_n(\mathbf{C})) \simeq \mathbf{Z}$ too.

Example. Let H be an infinite-dimensional separable Hilbert space and consider the C*-algebra $\mathcal{B}(H)$. Similar to the last example, two projections $p, q \in \mathrm{Proj}(\mathcal{B}(H))$ are equivalent if and only if $\mathrm{rank}(p) = \mathrm{rank}(q)$ — but now it's possible for a projection to have infinite rank. Thus $\mathrm{Proj}(\mathcal{B}(H))/\sim_0 \simeq \mathbf{N} \sqcup \{0, \infty\}$, where $n + \infty = \infty$ and $\infty + \infty = \infty$ — so ∞ is an "absorbing" element in this monoid. From this it is not hard to see that

$$K_0(\mathcal{B}(H)) \simeq \mathrm{Grp}(\mathbf{N} \sqcup \{0, \infty\}) = 0.$$

We record the following important property of $K_0(A)$: a $*$-homomorphism $A \to B$ induces a well-defined group homomorphism $K_0(A) \to K_0(B)$.

Proposition 2 (functoriality of K_0)**.** *Let A, B be two unital C*-algebras and let $\varphi : A \to B$ be a $*$-homomorphism. Then there is a uniquely determined group homomorphism $\varphi_* : K_0(A) \to K_0(B)$ such that $\varphi_*([p]_0) = [\varphi(p)]_0$ for all $p \in \mathrm{Proj}_1(A)$. Moreover, $\varphi \mapsto \varphi_*$ has the following properties:*

(i) *$(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$; and*

(ii) *$\mathrm{id}_* = \mathrm{id}$.*

Proof. First note φ induces a $*$-homomorphism $M_n(A) \to M_n(B)$ by applying φ entrywise, and hence a map $\mathrm{Proj}(A) \to \mathrm{Proj}(B)$ which preserves $\oplus$. It also preserves $\sim_0$ classes since φ is a $*$-homomorphism, so we have a monoid homomorphism $\overline{\varphi} : \mathrm{Proj}(A)/\sim_0 \to \mathrm{Proj}(B)/\sim_0$. Defining

$$\varphi_*([p]_0 - [q]_0) := \overline{\varphi}([p]_0) - \overline{\varphi}([q]_0)$$

gives a well-defined group homomorphism $\varphi_* : K_0(A) \to K_0(B)$ as required.

$$
\begin{array}{ccc}
\mathrm{Proj}(A)/\sim_0 & \longrightarrow & K_0(A) \\
\overline{\varphi} \downarrow & & \downarrow \varphi_* \\
\mathrm{Proj}(B)/\sim_0 & \longrightarrow & K_0(B)
\end{array}
$$

Claims (i) and (ii) are routinely verified. ∎

Now we can define $K_0(A)$ for A not-necessarily-unital. Let A_1 be the unitization of A — see the proof of Proposition 3.3.3. The canonical surjection $\varphi : A_1 \to \mathbf{C}$ induces a group homomorphism $\varphi : K_0(A_1) \to K_0(\mathbf{C}) \simeq \mathbf{Z}$ by Proposition 2, and we define the **zeroth K-theory of** A to be its kernel:

$$K_0(A) := \ker(K_0(A_1) \to \mathbf{Z}).$$

At first glance there seems to be a semantic issue, since if A is unital then we have defined $K_0(A)$ in two different ways. But in this case it's not hard to see (using Proposition 2) that $\varphi_* : K_0(A_1) \to \mathbf{Z}$ gives a splitting $K_0(A_1) \simeq K_0(A) \oplus \mathbf{Z}$, and the above kernel is exactly $K_0(A)$ as defined previously. For details see [26].

Example. Let H be an infinite-dimensional separable Hilbert space and let $\mathcal{K}(H) = \varinjlim M_n(\mathbf{C})$ be the algebra of compact operators on H. Recall $\mathcal{K}(H) = \varinjlim M_n(\mathbf{C})$, where $M_n(\mathbf{C}) \hookrightarrow M_{n+1}(\mathbf{C})$, $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ are the inclusions on the upper-left block. On the other hand we have $K_0(M_n(\mathbf{C})) \simeq \mathbf{Z}$ via an isomorphism sending $[1]_0 \mapsto n$. Thus the inclusions $M_n(\mathbf{C}) \hookrightarrow M_{n+1}(\mathbf{C})$ induce the identity map $\mathbf{Z} \simeq K_0(M_n(\mathbf{C})) \to K_0(M_{n+1}(\mathbf{C})) \simeq \mathbf{Z}$. Now continuity of K_0 (Theorem 6.3.2 in [26]) implies

$$K_0(\mathcal{K}(H)) \simeq \varinjlim K_0(M_n(\mathbf{C})) \simeq \varinjlim \mathbf{Z} = \mathbf{Z}.$$

4.1.3 K_0 of a ring using projective modules. Now we proceed to define $K_0(R)$ and $K_1(R)$ when R is a ring. There are two ways to define K_0: using projective modules or using idempotents; the latter construction makes it clear that this agrees with the K_0 defined for C*-algebras.

Let R be a ring. A finitely-generated (fg) right R-module P is **projective** if there exists a right R-module Q and $n \geq 1$ so that $P \oplus Q \simeq R^n$. Clearly, then, R^n is always projective. Let $\mathrm{Proj}(R)$ denote the set of isomorphism classes of (f.g.) projective right R-modules; this is an abelian monoid under the operation $[P] \oplus [Q] := [P \oplus Q]$, with identity $[0]$. If R is unital we define the **zeroth K-theory** of R to be the enveloping group of $\mathrm{Proj}(R)$ as in 4.1.1:

$$K_0(R) := \mathrm{Grp}(\mathrm{Proj}(R)).$$

Thus $K_0(R)$ consists of formal differences $[P] - [Q]$ where P, Q are fg projective modules. Note that $[P] = [Q]$ in $K_0(R)$ if and only if $P \oplus R^n \simeq Q \oplus R^n$ for some $n \geq 0$.

Example. Every projective $\mathbf{Z}$-module is free. To see this, consider a finitely-generated $\mathbf{Z}$-module P, *i.e.* an fg abelian group. By the structure theorem for fg abelian groups P must have the form $P \simeq \mathbf{Z}^d \oplus M$ where M is some finite torsion group. Now if P is projective, in particular it is a subgroup of some $\mathbf{Z}^n$ and hence torsion free — thus $M = 0$ and $P \simeq \mathbf{Z}^d$ is free. Therefore the map rank $: \mathrm{Proj}(R) \to \mathbf{N} \sqcup \{0\}$, $[P] \mapsto d$ is a monoid isomorphism, and we get

$$K_0(\mathbf{Z}) = \mathrm{Grp}(\mathrm{Proj}(R)) \simeq \mathrm{Grp}(\mathbf{N} \sqcup \{0\}) = \mathbf{Z}.$$

More generally, if R is a principal ideal domain then $K_0(R) \simeq \mathbf{Z}$ via rank by essentially the same argument as above.

As with C*-algebras, we point out that a ring homomorphism $R \to S$ induces a group homomorphism $K_0(R) \to K_0(S)$.

Proposition 1 (functoriality of K_0). *Let R, S be two unital rings and let $\varphi : R \to S$ be a ring homomorphism. Then there is a uniquely determined group homomorphism $\varphi_* : K_0(R) \to K_0(S)$ such that $\varphi_*([P]) = [P \otimes_R S]$ for all $[P] \in \mathrm{Proj}(R)$. Moreover, $\varphi \mapsto \varphi_*$ has the following properties:*

    (i) $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$; and

(ii) $\mathrm{id}_* = \mathrm{id}$.

Proof. Notice S becomes a left R-module via φ, and so we have a monoid homomorphism $\overline{\varphi} : \mathrm{Proj}(R) \to \mathrm{Proj}(S)$ given by $[P] \mapsto [P \otimes_R S]$. Via groupification this becomes a group homomorphism $\varphi_* : K_0(R) \to K_0(S)$ as required. Claims (i) and (ii) are routinely verified. ∎

If R is not-necessarily-unital, let R_1 be its unitization — more precisely, $R_1 := R \oplus \mathbf{Z}$ with the following operations: component-wise addition, and multiplication given by the formula

$$(r, m)(s, n) := (rs + ms + nr, mn).$$

Thus R_1 is a unital ring with identity $(0, 1)$ and which contains $R \simeq R \oplus 0$ as a two-sided ideal. The split surjection $R_1 \to \mathbf{Z}$ induces a group homomorphism $K_0(R_1) \to K_0(\mathbf{Z}) \simeq \mathbf{Z}$ by the above proposition, and we define the **zeroth K-theory** of R be the kernel of this map:

$$K_0(R) := \ker(K_0(R_1) \to \mathbf{Z}).$$

Like with C*-algebras, there is no ambiguity between this definition and the previous one: it is easily seen that if R is unital, then there is a splitting $K_0(R_1) \simeq K_0(R) \oplus \mathbf{Z}$ and $K_0(R) = \mathrm{Grp}(\mathrm{Proj}(R)) \simeq \ker(K_0(R_1) \to \mathbf{Z})$.

More pertinently, if R is a locally unital ring — such as a Leavitt path algebra — then it is still the case that $K_0(R) \simeq \mathrm{Grp}(\mathrm{Proj}(R))$.

Proposition 2. *Let R be a locally unital ring and let $\mathrm{Proj}(R)$ denote the monoid of fg projective right R-modules. Then*

$$K_0(R) \simeq \mathrm{Grp}(\mathrm{Proj}(R)).$$

Proof. Since R is locally unital, the corners $\{eRe\}_{e^2=e \in R}$ form a directed system of unital rings and $R = \varinjlim eRe$. On one hand, continuity of K_0 (Theorem 1.2.5 in [27]) implies $K_0(R) \simeq \varinjlim K_0(eRe)$; on the other hand, each eRe is a unital ring and so $K_0(eRe) = \mathrm{Grp}(\mathrm{Proj}(eRe))$. Now continuity of $S \mapsto \mathrm{Grp}(\mathrm{Proj}(S))$ implies

$$K_0(R) \simeq \varinjlim K_0(eRe) = \varinjlim \mathrm{Grp}(\mathrm{Proj}(eRe)) \simeq \mathrm{Grp}(\mathrm{Proj}(R))$$

as required. ∎

4.1.4 K_0 of a ring using idempotents. If A is a C*-algebra, then it is also a ring — so we can consider $K_0(A)$ as defined in either 4.1.2 or 4.1.3. In this section we will establish that these are the same. This will be accomplished by mimicking the construction of C*-algebraic K_0, using idempotents instead of projections.

First we make an observation on projective modules. Suppose that R is a unital ring. If P is an fg projective right R-module, say $P \oplus Q \simeq R^n$, then the composition $R^n \simeq P \oplus Q \twoheadrightarrow P \hookrightarrow R^n$ can be thought of as an idempotent endomorphism e of R^n — *i.e.* an idempotent $n \times n$ matrix over R — whose image is isomorphic to P. Conversely, if $e \in M_n(R) \simeq \mathrm{End}_R(R^n)$ is idempotent, then its image $P := eR^n$ is a projective R-module: indeed, letting $Q := (1-e)R^n$ be the complementary submodule we see that $P \oplus Q \simeq R^n$. In conclusion, projective modules correspond to idempotent matrices and vice-versa.

Let $\mathrm{Idem}_n(R)$ denote the set of idempotents in $M_n(R)$ and denote $\mathrm{Idem}(R) := \bigsqcup_{n \geq 1} \mathrm{Idem}_n(R)$. We think of these as idempotent endomorphisms of R^n. For $e, f \in$

$\mathrm{Idem}_n(R)$, let us write $e \sim_0 f$ if e, f have isomorphic images, *i.e.* $\mathrm{im}\, e \simeq \mathrm{im}\, f$ as right R-modules. An associative operation can be defined on $\mathrm{Idem}(R)$ as follows:

$$e \oplus f := \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$$

and it is easily verified (as in Proposition 4.1.2(1)) that this descends to a well-defined operation on $\mathrm{Idem}(R)/\sim_0$, making it a commutative monoid. Via the correspondence described in the previous paragraph, we have an isomorphism of monoids

$$\frac{\mathrm{Idem}(R)}{\sim_0} \simeq \mathrm{Proj}(R)$$

where $\mathrm{Proj}(R)$ is the monoid of projective modules. Now we will see that the equivalence relation $\sim_0$ is the "amplification" of the relation $\sim$ introduced in 3.4.2.

Proposition. *Let $e, f \in \mathrm{Idem}(R)$. Then $e \sim_0 f$ if and only if there exist rectangular matrices x, y so that $e = xy$ and $f = yx$.*

Proof. Let e and f be $n \times n$ and $m \times m$ respectively. If $e \sim_0 f$ then there is an isomorphism of right R-modules $\varphi : eR^n \to fR^m$. We may extend this to a map $y : R^n \to R^m$ given by $y(v) := \varphi(ev)$ for $v \in R^n$. Similarly, extend $\varphi^{-1} : fR^m \to eR^n$ to a map $x : R^m \to R^n$. One can think of $x \in M_{n \times m}(R)$ and $y \in M_{m \times n}(R)$, and we have $e = xy$ and $f = yx$.

Conversely, suppose $e = xy$ and $f = yx$ for some $x \in M_{n \times m}(R)$ and $y \in M_{m \times n}(R)$. These are maps $x : R^m \to R^n$ and $y : R^n \to R^m$. If we set $\varphi := y|_{eR^n}$ and $\psi := x|_{fR^m}$, it is straightforward to check that $\varphi : eR^n \to fR^m$ is an isomorphism with inverse ψ. ∎

If A is a unital C*-algebra we may still construct the monoid $\mathrm{Idem}(A)/\sim_0$ — but it turns out that the relation $\sim_0$ defined in this section is the same as the $\sim_0$ defined in 4.1.2, and that every equivalence class of idempotents in A contains a self-adjoint idempotent. For the proof see Proposition IV.1.1 in [11].

Lemma. *Let A be a C*-algebra and let $\mathrm{Proj}(A)$ denote the set projections in $M_n(A)$, for all $n \geq 1$.*

(a) *Let $p, q \in \mathrm{Proj}(A)$. Then $p = xy$, $q = yx$ for some rectangular x, y if and only if $p = s^*s$, $q = ss^*$ for some rectangular s.*

(b) *Let $e \in A$ be idempotent. Then there exists a projection $p \in A$ so that $e \sim_0 p$.*

We conclude from this lemma that $\mathrm{Proj}(A)/\sim_0 \simeq \mathrm{Idem}(A)/\sim_0$, and therefore:

Corollary. *Let A be any C*-algebra. Then the ring-theoretic and C*-algebraic K_0 groups of A are isomorphic.*

4.1.5 K_0 of $L(E)$ and $C^*(E)$. In this section we will determine $K_0(L(E))$ in terms of graph-theoretic data from E; for C*-algebras we will merely state the corresponding result without proof. By Proposition 4.1.3(2), the K_0 group of an Lpa $L(E)$ is the groupification of the monoid $\mathrm{Proj}(L(E))$. Thus our first goal is to determine this monoid.

Let E be a directed graph. The **graph monoid** of E is the abelian monoid M_E generated by $\{v \in E^0\}$ subject to the relations

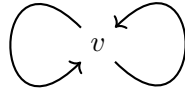$$v = \sum_{s(e)=v} r(e) \qquad \text{if } v \text{ is not a sink.}$$

We will be interested in the abelian group $\mathrm{Grp}(M_E)$.

Example. Let E be the following graph:

Then M_E is the abelian monoid generated by u, v, w subject to the relations $u = v$, $v = w$, and $w = v + x$. In $\mathrm{Grp}(M_E)$ we see that the last relation implies $x = 0$, so $\mathrm{Grp}(M_E) \simeq \langle u = v = w \rangle \simeq \mathbf{Z}$.

Example. Let E be the following graph:

Then $\mathrm{Grp}(M_E)$ is the abelian group generated by v, subject to the relation $v = v + v$. This implies $v = 0$, so $\mathrm{Grp}(M_E) = 0$.

We will prove that the graph monoid of E is isomorphic to $\mathrm{Proj}(L(E))$. Note that by 4.1.4 we will freely identify $\mathrm{Proj}(R)$ with the monoid $\mathrm{Idem}(R)/\sim_0$.

Theorem. *Let E be a row-finite graph. Then the map $\Phi : M_E \to \mathrm{Proj}(R)$ given by $v \mapsto [v]_0$ is a monoid isomorphism. Consequently*

$$K_0(L(E)) \simeq \mathrm{Grp}(M_E) = \bigoplus_{v \in E^0} \mathbf{Z}v \Big/ \Big\langle v - \sum_{s(e)=v} r(e) : v \text{ is not a sink} \Big\rangle.$$

The following result of Bergman will be key in the proof: it allows the construction of ring extensions $R \subseteq S$ such that $\mathrm{Proj}(S)$ is isomorphic to $\mathrm{Proj}(R)$ modulo a single relation. To state it, we first need a definition. Let R be a unital ring and let $P, Q \in \mathrm{Proj}(R)$; in [7] it is shown that there exists an R-algebra S — though R may not be central in S — and an isomorphism $\iota : P \otimes_R S \to Q \otimes_R S$ of right S-modules, such that (S, ι) is *universal* in the following sense: if S' is another ring containing R and $\iota' : P \otimes_R S' \xrightarrow{\cong} Q \otimes_R S'$ is a right S'-module isomorphism, then there is a unique ring homomorphism $\varphi : S \to S'$ such that $\iota' = \iota \otimes 1_{S'}$. Now S is (up to isomorphism) the unique ring with this property, and is denoted $S = R\langle [P] = [Q] \rangle$.

The following result is Theorem 5.2 in [7].

Proposition. *Let R be a unital ring, let $P, Q \in \mathrm{Proj}(R)$, and let S denote the universal ring $S := R\langle [P] = [Q] \rangle$. Then $\mathrm{Proj}(S) \simeq \mathrm{Proj}(R)/\langle [P] = [Q] \rangle$.*

Now we proceed to prove that $M_E \simeq \mathrm{Proj}(L(E))$.

Proof of the main theorem. First we verify that the map $\Phi : M_E \to \mathrm{Proj}(R)$ given by $v \mapsto [v]_0$ is well-defined. Observe that if v is a nonsink, then applying (CK2) and (CK1) gives

$$v = \sum_{s(e)=v} ee^* \sim_0 \sum_{s(e)=v} e^*e = \sum_{s(e)=v} r(e)$$

which implies $\Phi(v) = [v]_0 = \left[\sum_{s(e)=v} r(e)\right]_0 = \Phi\left(\sum_{s(e)=v} r(e)\right)$. So Φ respects the defining relations of M_E.

Now we show that Φ is an isomorphism in the case that E is finite. In this case, if $v_1, \ldots, v_m$ are the nonsinks of E then M_E has only the finite set of relations

$$v_1 = \sum_{s(e)=v_1} r(e), \qquad \ldots, \qquad v_m = \sum_{s(e)=v_m} r(e).$$

We proceed by induction on m, the number of nonsink vertices in E. If $m = 0$ then E is the graph on $|E^0|$ vertices and no edges, so $L(E) \simeq \bigoplus_{v \in E^0} k$; in this case it is easy to show that $M_E \simeq \bigoplus_{v \in E^0} \mathbf{N}v \simeq \mathrm{Proj}(L(E))$ via Φ.

Assume $m \geq 1$ and set $L = L(E)$. Let E_0 be the subgraph of E with the same vertices but with all edges emitted by v_m removed; thus the nonsinks in E_0 are $v_1, \ldots, v_{m-1}$ (if $m = 1$ then E_0 has no sinks). Let $L_0 := L(E_0)$. By induction $M_{E_0} \simeq \mathrm{Proj}(L_0)$ via $v \mapsto [v]_0$. Consider the following fg projective right L_0-modules:

$$P := v_m L_0, \qquad Q := \bigoplus_{s(e)=v_m} r(e)L_0.$$

By Bergman's result there is an L_0-algebra $S = L_0\langle[P] = [Q]\rangle$ and a universal right S-module isomorphism $\iota : P \otimes_{L_0} S \to Q \otimes_{L_0} S$. On the other hand, we note that $L = L(E)$ is also an L_0-algebra via the natural map $L_0 \to L$ given by the universal property of $L(E_0)$.

Claim: $S \simeq L$ as rings.

To see this, we observe that since $v_m \sim_0 \bigoplus_{s(e)=v_m} r(e)$ in $L(E)$ we get an isomorphism $P \otimes_{L_0} L \simeq Q \otimes_{L_0} L$; in terms of idempotents we can see this isomorphism by letting x be the row $(e : s(e) = v_m)$ and y be the column $(e^* : s(e) = v_m)$; then $v_m = xy$ and $\bigoplus_{s(e)=v_m} r(e) = yx$. It can be verified that this is a universal isomorphism, so $S \simeq L$ by the uniqueness aspect of Bergman's result.

We thus conclude, by Bergman's Theorem, that $\mathrm{Proj}(L) \simeq \mathrm{Proj}(L_0)/\langle[P] = [Q]\rangle$. Now $\mathrm{Proj}(L_0) \simeq M_{E_0}$ by induction, and the relation $[P] = [Q]$ translates (via $v \mapsto [v]_0$) to the relation $v_m = \sum_{s(e)=v_m} r(e)$. It follows that we have monoid isomorphisms

$$\mathrm{Proj}(L) \simeq \mathrm{Proj}(L_0)/\langle[P] = [Q]\rangle \simeq M_{E_0}\Big/\Big\langle v_m = \sum_{s(e)=v_m} r(e)\Big\rangle = M_E$$

as required. This proves the result when E has $m \geq 1$ nonsinks, and so the total result for finite graphs follows inductively.

Now we deduce the result for an infinite row-finite graph E. By the results of 2.4.1 we see that there are complete graph inclusions $F_i \hookrightarrow E$ with F_i finite and $E = \bigcup F_i$, and that $L(E) = \varinjlim L(F_i)$. By the case of finite graphs we have $M_{F_i} \simeq \mathrm{Proj}(L(F_i))$.

Using the easy facts that $E \mapsto M_E$ and $R \mapsto \text{Proj}(R)$ commute with direct limits, we deduce

$$M_E = \varinjlim M_{F_i} \simeq \varinjlim \text{Proj}(L(F_i)) \simeq \text{Proj}(\varinjlim L(F_i)) = \text{Proj}(L(E))$$

as required. Moreover, following the maps carefully one can check that the above isomorphism is indeed the map Φ stipulated in the statement of the theorem. The particular statement involving K_0 is achieved immediately upon groupification. ∎

Since $v \mapsto [v]_0$ gives an isomorphism $M_E \simeq \text{Proj}(L(E))$ as just proven, we get the following corollary.

Corollary. *Let $L(E)$ be an Lpa and let $p \in \text{Idem}(L(E))$ be any nonzero idempotent matrix. Then $p \sim_0 v_1 + \cdots + v_k$ for some vertices v_i. In particular, if $L(E)$ is purely infinite then all idempotents in $L(E)$ are infinite.*

If A is a purely infinite C*-algebra then every projection in A is infinite (see Proposition 6.11.5 in [9]), so this corollary is another reflection of C*-algebras.

Proof. Let $\Phi : M_E \to \text{Proj}(L(E))$ be the isomorphism $v \mapsto [v]_0$ as in the theorem. Then $[p]_0 \in \text{Proj}(L(E))$, so surjectivity of Φ implies $[p]_0 = \Phi(m)$ for some $m \in M_E$. On the other hand m is a sum of vertices, say $m = v_1 + \cdots + v_k$, and so

$$[p]_0 = \Phi(m) = \Phi(v_1 + \cdots + v_k) = [v_1]_0 \oplus \cdots \oplus [v_k]_0 = [v_1 + \cdots + v_k]_0.$$

Here we note that since vertices are orthogonal idempotents, the block matrix $v_1 \oplus \cdots \oplus v_k \in M_k(L(E))$ is equivalent to the idempotent $v_1 + \cdots + v_k \in L(E)$.

Now suppose $L(E)$ is purely infinite. As in the proof of Theorem 3.4.2(2), all vertices are infinite idempotents in $L(E)$. But any nonzero idempotent p is equivalent to a sum of a vertices by the previous paragraph, so in particular $v \lesssim p$ for some $v \in E^0$. Since v is infinite so must be p. ∎

We can also give a matrix interpretation of $K_0(L(E))$. Recall that if E is a row-finite directed graph, then its **adjacency matrix** is the $E^0 \times E^0$ matrix $A = A_E$ whose (v, w) entry is the number of edges from v to w. We think of A as an endomorphism of the abelian group $\mathbf{Z}^{E^0} = \bigoplus_{v \in E^0} \mathbf{Z}v$ — this makes sense since E is row-finite, so A takes finite sums to finite sums. If A^t is the tranpose of A, then $A^t v = \sum_{s(e)=v} r(e)$, so the relation $v = \sum_{s(e)=v} r(e)$ in $\text{Grp}(M_E)$ can be interpreted as $(1 - A^t)v = 0$. Denote by $A^{\text{ns}} : \mathbf{Z}^{E^0} \to \mathbf{Z}^{E^0 \setminus \text{sinks}}$ the matrix obtained from A by removing the rows corresponding to sinks, called the **nonsingular adjacency matrix**; in other words, remove the rows of A which are all zeros. We deduce the following:

Corollary. *Let E be a row-finite graph with nonsingular adjacency matrix A^{ns}. Then*

$$K_0(L(E)) \simeq \text{coker}(1 - (A^{\text{ns}})^t).$$

We have to remove the rows corresponding to sinks to avoid introducing relations of the form $v = 0$ when v is a sink. Since the Leavitt path algebras under consideration will be purely infinite, every vertex in E will connect to a cycle so E will have no sinks anyway.

Example. Let E consist of one vertex and n edges:
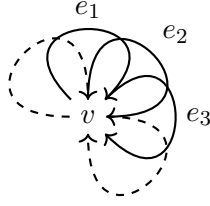
Then $L(E) \simeq L_n$ is the nth Leavitt algebra, which is purely infinite simple. By the theorem, $K_0(L_n)$ is the abelian group generated by v subject to the relation $v = nv$; thus

$$K_0(L_n) \simeq \langle v : (n-1)v = 0 \rangle \simeq \mathbf{Z}/(n-1)\mathbf{Z}.$$

We end off this section by stating the graphical interpretation of $K_0(C^*(E))$ when E has no sinks. A good reference for the proof is [23] (Theorem 7.16).

Theorem. *Let E be a countable row-finite graph with no sinks and let A be its adjacency matrix. Then*

$$K_0(C^*(E)) \simeq \operatorname{coker}(1 - A^t).$$

4.2 The algebraic Kirchberg–Phillips conjecture

The algebraic Kirchberg–Phillips question asks whether $K_0(L(E)) \simeq K_0(L(F))$ — in addition to one other mild piece of data — is enough to ensure $L(E) \simeq L(F)$, whenever $L(E)$ and $L(F)$ are purely infinite simple unital Leavitt path algebras. Although seemingly an obscure question to ask, the corresponding question for graph C*-algebras is well-known to have a *positive* solution: this is a special case of the far-reaching Kirchberg–Phillips theorem, a milestone in the classification of C*-algebras.

We begin this section by providing a general motivation for classification theorems in C*-algebras, and how to transfer them to Leavitt path algebras. Then we give an outline of some partial resolutions of the aforementioned unresolved question. Another natural question to ask is: if $L(E) \simeq L(F)$, how are E and F related? Very surprisingly, this question appears to be deeply connected to some theorems in symbolic dynamics on finite directed graphs; in later sections we will give an exposition of this connection.

4.2.1 Classification theorems in C*-algebras. If A and B are $*$-isomorphic C*-algebras, clearly $K_0(A) \simeq K_0(B)$ — but the converse is not always true. Take, for example, a finite-dimensional C*-algebra $A \simeq \bigoplus_{i=1}^n M_{d_i}(\mathbf{C})$. It's not hard to see that K_0 splits over direct sums, so

$$K_0(A) \simeq \bigoplus_{i=1}^n K_0(M_{d_i}(\mathbf{C})) \simeq \bigoplus_{i=1}^n \mathbf{Z} = \mathbf{Z}^n.$$

So $K_0(A)$ does not "see" the integers $d_1, \ldots, d_n$; rather, it only sees the number of summands. To encode the d_i's, recall that the isomorphism $K_0(M_d(\mathbf{C})) \simeq \mathbf{Z}$ is via the rank of projections, and the rank of the $d \times d$ identity is d: therefore $[1]_0 \mapsto d$ under this isomorphism. For $A \simeq \bigoplus_{i=1}^n M_{d_i}(\mathbf{C})$ we see that the isomorphism $K_0(A) \simeq \mathbf{Z}^n$ sends $[1_A]_0 \mapsto (d_1, \ldots, d_n)$ — so the particular element $[1_A]_0 \in K_0(A)$ encodes the

complete isomorphism type of A. This is a baby example of a "classification theorem" in C*-algebras.

The pair $(K_0(A), [1]_0)$ plays a key role in classification-type theorems. The element $[1]_0$ is called the **order unit** in K_0, and we use the shorthand $(K_0(A), [1_A]_0) \simeq (K_0(B), [1_B]_0)$ to mean that $K_0(A) \simeq K_0(B)$ via some isomorphism which sends $[1_A]_0 \mapsto [1_B]_0$. An example of the importance of the order unit is the following classical theorem of Elliott, generalizing the above remark on finite-dimensional algebras to unital af algebras; see Theorem IV.4.3 in [11] for a proof.

Theorem (Elliott, 1976). *Let A, B be two unital af C*-algebras with*

$$(K_0(A), [1_A]_0) \simeq (K_0(B), [1_B]_0).$$

Then $A \simeq B$ as C-algebras.*

With this theorem began a program to locate other classes of unital C*-algebras which can be classified by the pair $(K_0(A), [1]_0)$. In [22], Phillips (and independently Kirchberg, in a preprint) discovered that certain purely infinite simple C*-algebras can be K-theoretically classified in this manner, using the pair $(K_0(A), [1]_0)$ in addition to the group $K_1(A)$ — although the precise definition of K_1 is not required for this exposition, all that should be known is that $K_1(A)$ is another abelian group like $K_0(A)$ which acts as an isomorphism invariant of A.

Theorem (Kirchberg–Phillips, 2000). *Let A, B be two unital purely infinite simple UCT Kirchberg C*-algebras with*

$$(K_0(A), [1_A]_0) \simeq (K_0(B), [1_B]) \quad and \quad K_1(A) \simeq K_1(B).$$

Then $A \simeq B$.

The term "UCT Kirchberg algebra" has not been defined in these notes, but all that needs to be known here is that every purely infinite simple graph C*-algebra $C^*(E)$ is automatically one; see Remark 4.3 in [24].

The above two classification theorems have important implications for graph C*-algebras. Recall the Dichotomy Theorem 3.4.1, which states that every simple graph C*-algebra is either af or purely infinite. Combining this with the theorems of Elliott and Kirchberg–Phillips — in addition to the fact that all graph C*-algebras are UCT Kirchberg algebras — gives the following classification theorem for simple graph C*-algebras:

Corollary. *Let $C^*(E)$, $C^*(F)$ be two unital simple graph C*-algebras, and assume that*

$$(K_0(C^*(E)), [1]_0) \simeq (K_0(C^*(F)), [1]_0) \quad and \quad K_1(C^*(E)) \simeq K_1(C^*(F)).$$

Then $C^*(E) \simeq C^*(F)$ as C-algebras.*

Assuming that $C^*(E)$ is unital is a remarkable restriction, since it forces E to be a finite graph by Corollary 1.3.3(4)(b). Thus if $C^*(E)$ is a unital af algebra then it is necessarily finite-dimensional (by Theorem 3.3.5), so the "Elliott half" of the above classification theorem doesn't truly rely on Elliott's classification theorem. There is a

version of the above corollary which states that if E and F are infinite then the order unit can be ignored and isomorphism is still obtained.

Now we translate all this into Leavitt path algebras. For a unital ring R, the group $K_0(R)$ still contains a distinguished order unit $[1]_0$ — this can be thought of as either the class of the idempotent 1 or the class of the projective module R. Since the "Elliott half" of unital Leavitt path algebras again consists of matricial algebras $\bigoplus M_{d_i}(k)$, it's just as obvious that $(K_0(L(E)), [1]_0)$ is a complete isomorphism invariant for finite-dimensional Lpa's. Thus the major class of interest here consists of the Kirchberg–Phillips-type Lpa's: purely infinite simple.

Conjecture. *Let $L(E)$ and $L(F)$ be two purely infinite simple unital Leavitt path algebras such that*

$$(K_0(L(E)), [1]_0) \simeq (K_0(L(F)), [1]_0).$$

Then $L(E) \simeq L(F)$.

This is known as the **algebraic Kirchberg–Phillips conjecture**. In the remainder of these notes we will provide an exposition of the current research avenues towards a resolution of this problem, and some partial solutions which have arisen recently.

4.2.2 Morita equivalence. We will define here a notion of equivalence of rings which is weaker than isomorphism, but plays a key role in classification.

If R is a ring, let Mod_R denote the category of right R-modules. Two rings R, S are **Morita equivalent** if there exists an equivalence of categories $\Phi : \mathrm{Mod}_R \to \mathrm{Mod}_S$ — see Chapter 18 of [16] for details on categories and equivalences. We write $R \sim S$ to denote Morita equivalence; essentially this means that R and S have the same representation theory.

We first note that $R \mapsto M_n(R)$ is a Morita invariant functor.

Proposition 1. *Let R be a unital ring. Then $R \sim M_n(R)$ for any $n \geq 1$.*

Proof. If M is a right R-module, then $M^n = M \oplus \cdots \oplus M$ is naturally a right $M_n(R)$-module; moreover, if $f : M \to N$ is a homomorphism then there is a natural induced map $f^n : M^n \to N^n$. Thus we have a functor $\Phi : \mathrm{Mod}_R \to \mathrm{Mod}_{M_n(R)}$.

Conversely, let ε_{ij} be the $n \times n$ matrix with 1 in the (i, j) entry and 0's everywhere else. Given a right $M_n(R)$-module M, then the submodule $M\varepsilon_{11}$ is naturally an R-module by $xr := x\varepsilon_{11}r$ for $x \in M\varepsilon_{11}$, $r \in R$; given an $M_n(R)$-module homomorphism $f : M \to N$, there is an induced map $M\varepsilon_{11} \to N\varepsilon_{11}$ by restricting. Thus we have a functor $\Psi : \mathrm{Mod}_{M_n(R)} \to \mathrm{Mod}_R$.

One easily verifies that the functors Φ, Ψ defined in the above two paragraphs are "naturally inverse" to one another. ∎

Here now is a Morita invariant operation which will be pivotal in these notes. An idempotent $e \in R$ is **full** if it generates the unit ideal of R, *i.e.* $ReR = R$. If e is a full idempotent then the corner eRe is called a **full corner**. Notice that if R is simple — as will frequently be the case during our interest in classification — every corner is full.

Proposition 2. *Let R be a unital ring and let eRe be a full corner. Then $R \sim eRe$.*

This actually recovers Proposition 1, since ε_{11} is a full idempotent in $M_n(R)$ and $R \simeq \varepsilon_{11} M_n(R) \varepsilon_{11}$. In fact, Proposition 18.33 in [16] states that two unital rings R and

S are Morita equivalent and only if S is isomorphic to a full corner of some matrix ring of R.

Proof. Note that Re is a *right eRe*-module. One verifies that the functors

$$\begin{array}{ccc} \mathrm{Mod}_R & \to & \mathrm{Mod}_{eRe} \\ M & \mapsto & \mathrm{Hom}_R(eR, M) \\ N & \leftarrow\!\shortmid & \mathrm{Hom}_{eRe}(Re, N) \end{array}$$

are mutually inverse; see Example 18.30 and Proposition 18.33 in [16] for details. ∎

There are many ring-theoretic properties which are Morita invariant; to name a few: simplicity, semisimplicity, chain conditions, and pure infiniteness ([4]); the latter most property will be important in the coming sections. Another important Morita invariant is K_0.

Proposition 3. *Let R, S be Morita equivalent unital rings. Then $K_0(R) \simeq K_0(S)$.*

Proof. If $R \sim S$, then by Proposition 18.33 in [16] there is an *additive* Morita equivalence $\Phi : \mathrm{Mod}_R \to \mathrm{Mod}_S$, *i.e.* $\Phi(M \oplus N) \simeq \Phi(M) \oplus \Phi(N)$. So $\mathrm{Proj}(R) \to \mathrm{Proj}(S)$, $[P] \mapsto [\Phi(P)]$ is a monoid isomorphism, inducing a group isomorphism $K_0(R) \to K_0(S)$. ∎

4.2.3 Flow equivalence and Morita invariance. In this section we describe six graph transformations on finite graphs which preserve the Morita equivalence class of the associated Leavitt path algebra. We carry out this exposition following closely [3].

Let E be a finite graph.

(I) **Vertex expansion/contraction.** For any vertex $v \in E^0$ we construct a graph from E by adding a new vertex v_*, drawing an edge from v to v_*, and moving all edges emitted by v to v_*. Formally, let v_*, e_* be symbols. The **vertex expansion** of E at v is the graph E_v defined by

$$(E_v)^0 := E^0 \sqcup \{v_*\}, \qquad (E_v)^1 := E^1 \sqcup \{e_*\},$$

and with edges given by

$$s(e) := \begin{cases} v & \text{if } e = e_*, \\ v_* & \text{if } s_E(e) = v, \\ s_E(e) & \text{otherwise,} \end{cases} \qquad r(e) := \begin{cases} v_* & \text{if } e = e_*, \\ r_E(e) & \text{otherwise.} \end{cases}$$

For example,

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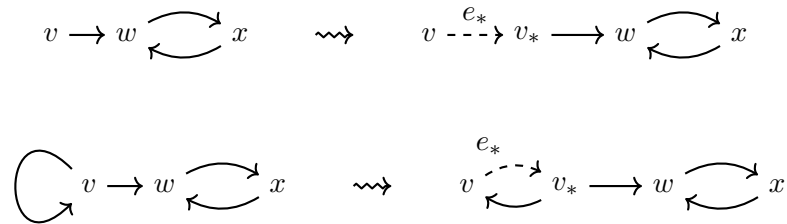
Conversely, E is called a **vertex contraction** of E_v.

(II) **In-splitting/amalgamation.** Let v be a nonsource and partition $r^{-1}(v) = \mathcal{E}_1 \sqcup \cdots \sqcup \mathcal{E}_d$ with $d \geq 1$. To in-split at v, replace v with d new vertices $v^1, \ldots, v^d$ and redirect the edges $e \in \mathcal{E}_i$ to v^i. For each edge emitted by v, draw d new edges emitted from each v^i.
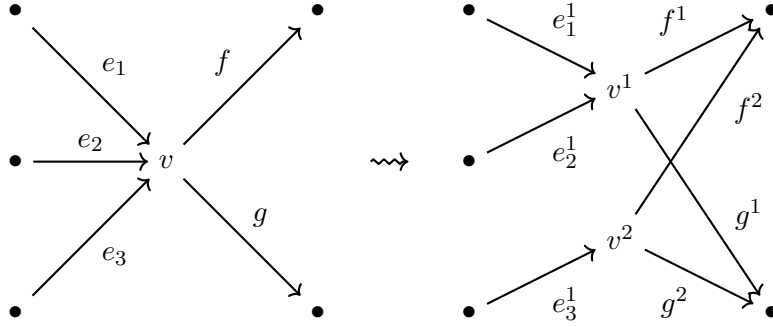
Formally let $\mathcal{P} = \{\mathcal{E}_1, \ldots, \mathcal{E}_d\}$. The **in-splitting** of E at v (with respect to $\mathcal{P}$) is the graph $E_{\text{in}}(\mathcal{P})$ defined by

$$E_{\text{in}}(\mathcal{P})^0 := \{w^1 : w \in E^0, w \neq v\} \sqcup \{v^1, \ldots, v^d\},$$
$$E_{\text{in}}(\mathcal{P})^1 := \{e^1 : e \in E^1, s(e) \neq v\} \sqcup \{e^1, \ldots, e^d : e \in E^1, s(e) = v\},$$

with the source and range of edges given by

$$s(e^i) := s(e)^i, \qquad r(e^i) := \begin{cases} v^j & \text{if } e \in \mathcal{E}_j, \\ r(e)^1 & \text{otherwise.} \end{cases}$$

For example, if $\mathcal{E}_1 = \{e_1, e_2\}$ and $\mathcal{E}_2 = \{e_3\}$ in the below graph, then

Conversely, E is called an **in-amalgamation** of $E_{\text{in}}(\mathcal{P})$ whenever $\mathcal{P}$ is a partition as described above.

(III) **Out-splitting/amalgamation.** This move is similar to in-splitting, except we partition $s^{-1}(v)$ instead of $r^{-1}(v)$. Let v be a nonsink and partition $s^{-1}(v) = \mathcal{E}_1 \sqcup \cdots \sqcup \mathcal{E}_d$ with $d \geq 1$. Let $\mathcal{P} = \{\mathcal{E}_1, \ldots, \mathcal{E}_d\}$. The **out-splitting** of E at v (with respect to $\mathcal{P}$) is the graph $E_{\text{out}}(\mathcal{P})$ defined by

$$E_{\text{out}}(\mathcal{P})^0 := \{w^1 : w \in E^0, w \neq v\} \sqcup \{v^1, \ldots, v^d\},$$
$$E_{\text{out}}(\mathcal{P})^1 := \{e^1 : e \in E^1, r(e) \neq v\} \sqcup \{e^1, \ldots, e^d : e \in E^1, r(e) = v\},$$

with the source and range of edges given by

$$s(e^i) := \begin{cases} v^j & \text{if } e \in \mathcal{E}_j, \\ s(e)^1 & \text{otherwise,} \end{cases} \qquad r(e^i) := r(e)^i.$$

For example, if $\mathcal{E}_1 = \{g_1, g_2\}$ and $\mathcal{E}_2 = \{g_3\}$ in the below graph, then
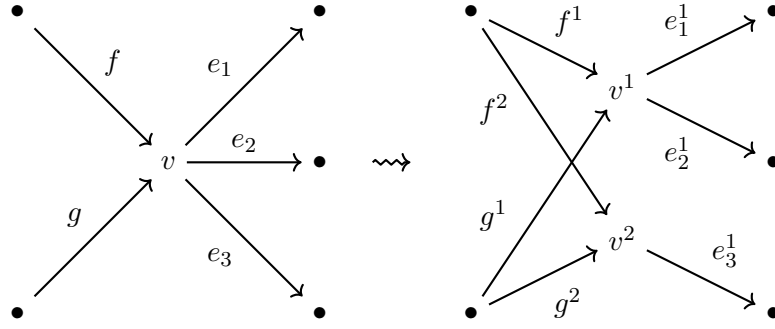
Conversely, E is called an **out-amalgamation** of $E_{\text{out}}(\mathcal{P})$ whenever $\mathcal{P}$ is a partition as described above.

The six operations described in (I), (II), and (III) are called **flow transformations**, and two finite graphs E, F are said to be **flow equivalent**, denoted $E \rightsquigarrow F$, if F is obtained from E by a finite sequence of flow transformations. Since $E \rightsquigarrow F$ implies $F \rightsquigarrow E$, this is indeed an equivalence relation.

The reason for introducing these moves is that they preserve Morita equivalence class. This result was first achieved in [3].

Proposition. *Let $L(E)$ be a simple unital Leavitt path algebra and suppose E has at least two vertices.*

(a) *Let v be a source in E and let $E \setminus v$ be the subgraph obtained by deleting v and all edges it emits. Then $L(E) \sim L(E \setminus v)$.*

(b) *Suppose further that E has no sinks or sources. If $E \rightsquigarrow F$ then $L(E) \sim L(F)$.*

Note that (a) implies — after repeatedly deleting sources — that $L(E)$ is Morita equivalent to $L(F)$ where F is finite and has no sources. If $L(E)$ is purely infinite then it also has no sinks (since every vertex connects to a cycle by Theorem 3.4.2(2)), and $L(F)$ is also purely infinite simple (since this is a Morita invariant property). Thus in the study of purely infinite simple unital Lpa's we can always arrange to be in the Morita equivalence class of situation (b) above. To summarize:

Corollary (flow invariance). *Let $L(E)$, $L(F)$ be purely infinite simple unital Leavitt path algebras. If $E \rightsquigarrow F$ then $L(E) \sim L(F)$.*

We now provide a partial proof of the proposition.

Proof of the proposition. (a) Evidently $E \setminus v \hookrightarrow E$ is a complete graph inclusion as defined in 1.2.4, so by 2.4.1 we have a natural algebra inclusion $L(E \setminus v) \hookrightarrow L(E)$. Let $p := \sum_{w \in E^0 \setminus \{v\}} w$ so that p is the identity element of $L(E \setminus v)$ and an idempotent of $L(E)$. Since $L(E)$ is simple, $pL(E)p$ is automatically a full corner and so $L(E) \sim pL(E)p$ by Proposition 4.2.2(2). We are done once we show that $L(E \setminus v) = pL(E)p$. To see this, observe that for a typical monomial $\mu\nu^*$ in $L(E)$ we have

$$p\mu\nu^*p = \sum_{w \in E^0 \setminus \{v\}} w\mu\nu^*w = \begin{cases} \mu\nu^* & \text{if } s(\mu) \neq v \neq s(\nu), \\ 0 & \text{otherwise.} \end{cases}$$

Assuming $s(\mu) \neq v \neq s(\nu)$, neither μ nor ν can pass through v since v is a source — therefore $\mu\nu^* \in L(E \setminus v)$. From this we see

$$pL(E)p = \text{span}\{\mu\nu^* : s(\mu) \neq v \neq s(\nu)\} = \text{span}\{\mu\nu^* : \mu, \nu \in (E \setminus v)^\bullet\} = L(E \setminus v)$$

as required.

(b) It must be shown that if $E_\sharp$ is obtained from E through a vertex expansion, an in-split, or an out-split, then $L(E) \sim L(E_\sharp)$. Here we prove this only for vertex expansions, since the proofs for in-splits and out-splits follow a similar sketch: one uses the universal property of $L(E)$ to establish a homomorphism $\varphi : L(E) \to L(E_\sharp)$ — automatically injective since $L(E)$ is simple — whose image will be a full corner of $L(E_\sharp)$ via the full idempotent $\varphi(1)$. Morita equivalence thus follows from Proposition 4.2.2(2). The assumption that E has no sinks or sources is only used in the proof for in-splits, which is omitted here. For details see Propositions 1.11 and 1.14 in [3].

To construct the edge expansion E_v recall that we fix symbols $v_* \in (E_v)^0$ and $e_* \in (E_v)^1$. Since $L(E)$ is simple, E is cofinal and satisfies Condition (L) by Corollary 3.1.5. It is easy to see that a vertex expansion E_v has the same properties, and therefore $L(E_v)$ is simple as well. Now the universal property of $L(E)$ can be used to establish a $*$-algebra homomorphism

$$\varphi : L(E) \to L(E_v) : \begin{cases} w \mapsto w, \\ f \mapsto e_* f, \end{cases}$$

which is injective since it is nonzero and $L(E)$ is simple. Consider the idempotent $p := \varphi(1) = \sum_{w \in E^0} w \in L(E_v)$, which is automatically full since $L(E_v)$ is simple: so $pL(E_v)p \sim L(E_v)$. We claim that $L(E) \simeq pL(E_v)p$ via the isomorphism φ. Indeed, it is easy to see that $pL(E_v)p$ is spanned by monomials $\mu\nu^*$ with $\mu, \nu \in (E_v)^\bullet$ and $s(\mu) \neq v' \neq s(\nu)$; if we let α, β be the paths obtained from μ, ν (resp.) by contracting all occurrences of e_*, then $\varphi(\alpha\beta^*) = \mu\nu^*$. So φ maps $L(E)$ onto $pL(E_v)p$. ∎

4.2.4 Franks's theorem and its consequences. Having seen that flow transformations preserve Morita equivalence class in the purely infinite simple case, we now turn to the completely graph-theoretic problem of determining tractable criteria for when two graphs are flow equivalent. At this point the theory comes full circle, returning to K-theory and the Kirchberg–Phillips problem: a theorem from symbolic dynamics, due to Franks, determines precisely when certain graphs are flow equivalent.

To state Franks's theorem we introduce some terminology. A finite graph E is **irreducible** if $v \to w$ for all vertices $v, w \in E^0$. A graph consisting of a single cycle is always irreducible; if E is irreducible and not a single cycle then it is termed **nontrivially irreducible**. Note that a single vertex with no edges counts as a trivial irreducible graph.

nontrivially irreducible   not irreducible   trivially irreducible
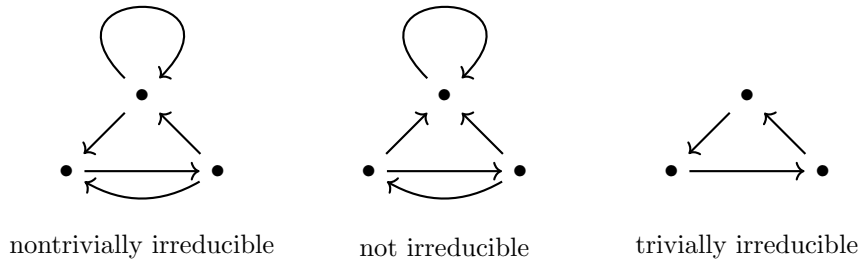
Nontrivial irreducibility is very strong: it implies Condition (L), cofinality, etc. (Note however that trivially irreducible graphs fail Condition (L).) So it is not difficult to use our characterizations of simplicity and pure infiniteness (Theorems and 3.1.5 and 3.4.2(2)) to show that E is a nontrivially irreducible graph if and only if $L(E)$ is purely infinite simple and E has no sources. Sourcelessness is a technicality, but we saw in Proposition 4.2.3(a) that we may frequently assume E is sourceless without much loss of generality — so nontrivially irreducible finite graphs will be prominent.

Franks originally proved his theorem in [14], and we state it as it appears in [3].

Theorem (Franks). *Let E, F be nontrivially irreducible graphs with respective adjacency matrices A, B. Then $E \rightsquigarrow F$ if and only if*

$$\operatorname{coker}(1 - A) \simeq \operatorname{coker}(1 - B) \qquad and \qquad \det(1 - A) = \det(1 - B).$$

Recall that we think of the adjacency matrix A as an endomorphism $A : \mathbf{Z}^{|E^0|} \to \mathbf{Z}^{|E^0|}$, and $\operatorname{coker}(1 - A)$ refers to the cokernel of the homomorphism $1 - A$. If E has no sinks, then Theorem 4.1.5 states that the K_0 of the Leavitt path algebra $L(E)$ is

$$K_0(L(E)) \simeq \operatorname{coker}(1 - A^t).$$

But one can verify, using the Smith normal form of an integer matrix (see Theorem 12.21 in of [13]), that $\operatorname{coker}(M) \simeq \operatorname{coker}(M^t)$ for any square integer matrix M — so the transposition is not apparent, hence $K_0(L(E))$ is truly the cokernel appearing in Franks's theorem. Franks's Theorem is therefore the indispensable middleman in deducing the following partial resolution of the algebraic Kirchberg–Phillips problem. The below corollary is still named Franks's theorem.

Corollary. *Let $L(E), L(F)$ be two purely infinite simple unital Leavitt path algebras and let A, B be the respective adjacency matrices of E, F. If*

$$K_0(L(E)) \simeq K_0(L(F)) \qquad and \qquad \det(1 - A) = \det(1 - B)$$

then $L(E)$ is Morita equivalent to $L(F)$.

Note that instead of using the order unit in K_0 we use the determinant condition appearing in Franks's theorem, and instead of isomorphism we obtain Morita equivalence. So this is indeed a weaker version of the algebraic Kirchberg–Phillips conjecture.

Proof. A source in E corresponds to a zero column in its adjacency matrix A_E, so using cofactor expansion we get $\det(1 - A_E) = \det(1 - A_{E \setminus v})$ for any source $v \in E^0$ — thus the determinant is invariant under deleting sources. If we let E' be the graph obtained by deleting all sources of E, then by Proposition 4.2.3(a) we have $L(E) \sim L(E')$, so the K_0 group does not change either.

Therefore, without loss of generality, we may assume that the graphs E, F given in the theorem are sourceless. But then since $L(E), L(F)$ are purely infinite simple, both E, F must be nontrivially irreducible graphs. The K_0 and determinant conditions are then exactly the invariants appearing in Franks's Theorem, so we can use it to conclude $E \rightsquigarrow F$ — this implies $L(E) \sim L(F)$ by Corollary 4.2.3. ∎

4.2.5 Huang's theorem and its consequences. Now we have sufficient conditions to determine Morita equivalence of purely infinite simple unital Lpa's. In this section the aim is to improve Morita equivalence to isomorphism: we will need the order unit in K_0. We continue to follow [3].

If $E \rightsquigarrow F$ is a flow transformation of nontrivially irreducible finite graphs, then by Franks's theorem there is an induced isomorphism $\varphi : \text{coker}(1 - A_E) \to \text{coker}(1 - A_F)$. Huang's result states that *every* automorphism of $\text{coker}(1 - A_E)$ is induced in this manner. This serves to strengthen the mysterious connection between Leavitt path algebras and symbolic dynamics.

Theorem (Huang). *Let E be a nontrivially irreducible graph with adjacency matrix A and let φ be a group automorphism of $\text{coker}(1 - A)$. Then φ is induced by a flow equivalence $E \rightsquigarrow E$.*

Translating Huang's theorem into Leavitt path algebras yields the extension of automorphisms on K_0 to Morita equivalences; we will also name this Huang's theorem.

Corollary 1. *Let $L(E)$ be a purely infinite simple unital Leavitt path algebra and let φ be a group automorphism of $K_0(L(E))$. Then there is a Morita equivalence $\Phi : \text{Mod}_{L(E)} \to \text{Mod}_{L(E)}$ such that $\varphi([P]) = [\Phi(P)]$ for every projective module P.*

Proof sketch. As done previously we may assume E is sourceless. Then E is nontrivially irreducible and φ is an automorphism of $K_0(L(E)) \simeq \text{coker}(1 - A)$. By Huang's theorem there is a flow equivalence $E \rightsquigarrow E$ inducing φ, and by Proposition 4.2.3(b) this flow equivalence gives rise to a Morita equivalence $\Phi : \text{Mod}_{L(E)} \to \text{Mod}_{L(E)}$. One verifies that this Φ is as required by tracking the Morita equivalences constructed in the proof of Proposition 4.2.3. ∎

From this we deduce a positive answer to the algebraic Kirchberg–Phillips conjecture *if* we assume the determinant condition. This is Corollary 2.7 in [3].

Corollary 2 (algebraic Kirchberg–Phillips theorem with determinant). *Let $L(E)$, $L(F)$ be purely infinite simple unital Leavitt path algebras and let A, B be the respective adjacency matrices. Then*

$$(K_0(L(E)), [1]_0) \simeq (K_0(L(F)), [1]_0) \quad and \quad \det(1 - A) = \det(1 - B)$$

implies $L(E) \simeq L(F)$ as rings.

Proof. If $\Phi : \text{Mod}_R \to \text{Mod}_S$ is a Morita equivalence such that $\Phi(R_R) = S_S$, then $R \simeq S$ as rings: the reason is that $\text{End}(M) \simeq \text{End}(\Phi(M))$ for any $M \in \text{Mod}_R$, so in particular

$$R \simeq \text{End}(R_R) \simeq \text{End}(S_S) \simeq S.$$

Our hypothesis implies, by Franks's theorem, that $L(E)$ and $L(F)$ are Morita equivalent — if we can tamper with this Morita equivalence to produce one that sends $L(E)$ to $L(F)$, then we can conclude $L(E) \simeq L(F)$ as above. This is the goal of the proof.

Let $\varphi : K_0(L(E)) \to K_0(L(F))$ be a unit-preserving isomorphism on K_0 — interpreting K_0 with projective modules, this means $\varphi([L(E)]) = [L(F)]$ in K_0. By Franks's theorem, the determinant condition implies that there is a Morita equivalence $\Phi : \text{Mod}_{L(E)} \to \text{Mod}_{L(F)}$. Then Φ preserves direct sums (see Theorem 18.33 in [16]), so

$$\psi := \Phi^{-1} \circ \varphi : K_0(L(E)) \to K_0(L(E))$$

is an automorphism of $K_0(L(E))$. By the preceding corollary, ψ extends to a Morita equivalence $\Psi : \text{Mod}_{L(E)} \to \text{Mod}_{L(E)}$, *i.e.* $[\Psi(P)] = \psi([P])$ for all projective $L(E)$-modules P. Finally, consider the Morita equivalence

$$\Gamma : \Phi \circ \Psi : \text{Mod}_{L(E)} \to \text{Mod}_{L(F)}.$$

Then Γ satisfies

$$\Gamma(L(E)) = \Phi(\Psi(L(E))) = \Phi(\psi(L(E))) = \varphi(L(E)) = L(F)$$

so we conclude $L(E) \simeq L(F)$ as in the first paragraph. ∎

Of course, it is natural to ask what happens if the determinant condition is removed: can isomorphism still be achieved? The following observation somewhat simplifies this gap: if $M, N \in M_n(\mathbf{Z})$ are square integer matrices such that $\text{coker}(X) \simeq \text{coker}(Y)$, then one may use Smith normal form to show that $Y = AXB$ for some integer matrices A, B which are invertible in $M_n(\mathbf{Z})$. Thus A and B must each have determinant ± 1, so $\det(X) = \pm \det(Y)$. To summarize: the isomorphism on K_0 *implies* the determinant condition up to sign. The above corollary deals with the case where the sign is positive, so we are left with the case where the sign is "mixed". This is called the **mixed sign case**.

Conjecture (mixed sign case). *Let $L(E)$, $L(F)$ be purely infinite simple unital Leavitt path algebras with adjacency matrices A, B. Suppose that*

$$(K_0(L(E)), [1]_0) \simeq (K_0(L(F)), [1]_0)$$

and that $\det(1 - A)$, $\det(1 - B)$ have different signs. Then $L(E) \simeq L(F)$.

4.2.6 The Cuntz splice. In this section we describe a seventh graph move $E \mapsto E_-$, known as the Cuntz splice, which changes the sign of the determinant while preserving K_0. It is currently unresolved whether $L(E)$ is Morita equivalent to $L(E_-)$, but a positive answer to this question would imply that K_0 is a complete Morita invariant.
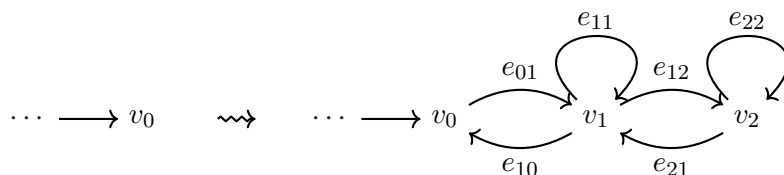
The **Cuntz graph** C is the finite graph

C is nontrivially irreducible, so its Leavitt path algebra $L(C)$ is purely infinite simple. Its adjacency matrix is

$$A_C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

which has $\det(1 - A_C) = -1$. For a finite graph E and $v_0 \in E^0$, the **Cuntz splice** of E at v_0, denoted $E_- = E_-(v_0)$, is the graph obtained by "attaching" the Cuntz graph to E at v_0.

More formally, E_- is given by the following data:

$$(E_-)^0 := E^0 \sqcup \{v_1, v_2\}, \qquad (E_-)^1 := E^1 \sqcup \{e_{01}, e_{10}, e_{11}, e_{12}, e_{21}, e_{22}\}$$

with source and range functions given by

$$s(e) := \begin{cases} v_i & \text{if } e = e_{ij}, \\ s_E(e) & \text{if } e \in E^1, \end{cases} \qquad r(e) := \begin{cases} v_j & \text{if } e = e_{ij} \\ r_E(e) & \text{if } e \in E^1. \end{cases}$$

Alternatively, if $A = (a_{ij}) \in M_{n \times n}(\mathbf{Z})$ the adjacency matrix of E, ordered so that its last row and column correspond to v_0, then E_- is defined to be the graph whose adjacency matrix is

$$A_- = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 1 & 0 \\ 0 & \cdots & 1 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

From this latter interpretation of the Cuntz splice, it is clear from a simple cofactor expansion that

$$\det(1 - A) = -\det(1 - A_-).$$

It is also clear that if $L(E)$ is purely infinite simple then so is $L(E_-)$, using Theorem 3.4.2(2). Despite the determinant changing sign, we can prove that E and E_- have isomorphic K_0.

Proposition. *Let $L(E)$ be a unital Leavitt path algebra where E has no sinks, and let E_- be a Cuntz splice of E. Then*

$$K_0(L(E)) \simeq K_0(L(E_-)).$$

Proof. Enumerate the vertices of E by $v_1, \ldots, v_n$ and let $A = (a_{ij})$ be the adjacency matrix of A, where a_{ij} is the number of edges from v_i to v_j. Then by Theorem 4.1.5, $K_0(L(E))$ is the abelian group generated by $v_1, \ldots, v_n$ subject to the relations

$$v_i = \sum_{j=1}^{n} a_{ij} v_j \qquad (\Diamond_i)$$

(this is because E has no sinks, so $A^{\mathrm{ns}} = A$). Assume that the Cuntz graph is attached at v_n and denote the new vertices by v_{n+1}, v_{n+2}.
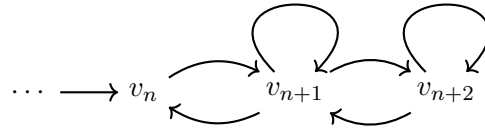
Then $K_0(L(E_-))$ is the abelian group generated by $v_1, \ldots, v_n, v_{n+1}, v_{n+2}$ subject to the relations $(\Diamond_1), \ldots, (\Diamond_{n-1})$, in addition to the three new relations

$$v_n = v_{n+1} + \sum_{j=1}^{n} a_{ij} v_j, \qquad (\Diamond_n')$$

$$v_{n+1} = v_n + v_{n+1} + v_{n+2}, \qquad (\Diamond_{n+1})$$

$$v_{n+2} = v_{n+1} + v_{n+2}. \qquad (\Diamond_{n+2})$$

But $(\diamondsuit_{n+2})$ reduces to $v_{n+1} = 0$, so $(\diamondsuit'_n)$ is the same as $(\diamondsuit_n)$. Then $(\diamondsuit_{n+1})$ implies $v_{n+2} = -v_n$, so v_{n+2} is not needed to generate the group. Therefore $K_0(L(E))$ and $K_0(L(E_-))$ have the same generators and relations, hence they are isomorphic. ∎

It's important to observe that the isomorphism in the above proof does not necessarily preserve the order unit.

Following from this calculation, the Cuntz splice allows us to construct purely infinite simple Leavitt path algebras which cannot be deduced to be isomorphic from Corollary 4.2.5(2). But it may still be true that $L(E)$ and $L(E_-)$ are Morita equivalent.

Conjecture. *Let $L(E)$ be a purely infinite simple unital Leavitt path algebra and let E_- be a Cuntz splice of E. Then*

$$L(E) \sim L(E_-).$$

The importance of this conjecture is as follows: if it is true, then it can deduced that K_0 — without order unit — is a complete Morita invariant for purely infinite simple unital Leavitt path algebras. To see this, we argue following Theorem 2.14 in [3]. Indeed if $L(E)$ and $L(F)$ are purely infinite simple unital Leavitt path algebras and $K_0(L(E)) \simeq K_0(L(F))$, then $\det(1 - A_E) = \pm \det(1 - A_E)$ as mentioned at the end of section 4.2.5. If the sign is positive then $L(E) \sim L(F)$ by Franks's theorem; if the sign is negative then the above proposition implies

$$K_0(L(F)) \simeq K_0(L(E)) \simeq K_0(L(E_-))$$

and

$$\det(1 - A_F) = -\det(1 - A_E) = \det(1 - A_{E_-})$$

which, again by Franks's theorem, implies $L(F) \sim L(E_-)$. If the above conjecture is true then we can conclude $L(E) \sim L(E_-) \sim L(F)$. However Huang's theorem cannot be used to imply isomorphism of rings, because the isomorphism $K_0(L(E)) \simeq K_0(L(E_-))$ established in the above proposition may not preserve the order unit.

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