Performance of Dynamic Hedging Strategies for Cash Balance Pension Plans

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Actuarial Science

Waterloo, Ontario, Canada, 2015

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Cash balance (CB) pension plans make up 25% of all defined benefit plans in the US. The benefits are accumulated at guaranteed crediting rates, the most popular choice is the yield on the 30-year Treasury bond. In this paper, we explore the pricing and hedging of the CB liability using financial theory and models. Due to the fact that crediting rates are often unmarketable, and motivated by the theory of replicating portfolios, we present the performance of a delta hedging strategy.

Our results suggest that the performance of the delta hedging strategy is related to the number of factors in the model rather than the number of hedging instruments. In particular, one-factor Hull White and two-factor Hull White model are not capable to construct an effective delta hedging portfolio.

Acknowledgements

I would like to thank my supervisors, Professor Mary Hardy and Professor David Saunders. I have received much help and guidance from them. I would like to thank my parents and my wife Feng, for their love, support and encouragement.

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Chapter 1

Introduction

Cash Balance (CB) pension plans are the fastest growing pension plan by far in the US. According to Kravitz [19], they have increased from less than 3% to 25% of all Defined Benefit (DB) plans over the past decade. The number of CB plans in the US was 9648 in 2014 and the number of participants reached 12.3 million.

The CB design originated in the 1980s, first established by the Bank of America. The motivation for this new design was to provide big companies with more predictable and less volatile costs. CB plans were supposed to preserve lower risk properties of Defined Contribution (DC) plans for pension sponsors, with relatively stable benefits for participants. CB plans remained rare for nearly a decade and started gaining popularity in the mid-1990s. Paulson [22] explained that, from 1995 to 2001, companies seeking ways to avoid the reversion tax applied to the excess pension assets for DB Plans frequently switched to CB plans. Coronado and Copeland [11] on the other hand, drew a different conclusion that the conversions from DB to CB were primarily driven by labor market conditions. When the Pension Protection Act came into effect in 2006, CB plans were officially approved by the Internal Revenue Service (IRS), and the new regulations solved legal issues such as age discrimination. In 2010, the IRS proposed further regulations that offered companies more options for crediting rates. The final version of this regulation will be in effect on or after January 1, 2016, and the new features will certainly attract more employers.

By strict classification, CB plans are regulated as DB plans but with better transparency and portability properties. Participants have individual accounts showing their up-to-date accrued benefits; however, those accounts are notional. The assets are not allocated to individuals and the amounts paid into the accounts need not be equal to the real contributions. Moreover, members have limited or no control over the choices of the investment options and their benefits are not directly affected by the performance of the investment portfolio. The crediting rates applied to the notional account are either a fixed rate, or tied to an index approved by the IRS. According to Kravitz [19], popular choices for the crediting rates are fixed rates of return, which account for 42.2% of all CB plans, 30-Year Treasury Rates with no floor, which account for 29.9% and 30-Year Treasury Rate with a Floor, which accounts for 23%. Another option gaining popularity recently is the actual rate of return of the investments (accounts for 4.4%). This option shifts the investment risks back to employees but with some protection such as the preservation of capital and the level of diversification for the portfolio. The choice of the 30-Year Treasury Rate as crediting rate dominated CB plans prior to the new regulation published in 2010 (around 95% of all CB plans), and it still accounts for more than half of the CB plans.

Traditional actuarial valuation of CB plans has been studied by Murphy [21], who concluded that traditional methods may understate the actual liability, and generate a loss on early termination of the plan. Brown et al. [9] have adopted the market value approach. They used a certainty model and Vasicek model to forecast future yields and evaluate the costs of CB liabilities. Hardy et al. [17] followed a similar approach, but with more sophisticated arbitrage-free interest rate models. Moreover, [17] derived explicit valuation formulae, and considered the sensitivity of valuation with respect to changes in the economic conditions, the interest rate models and the parameters. In [17], crediting rates are assumed to be paid continuously. Part of this work extend theirs to discrete settings, because in practice, crediting rate are usually accumulated annually or semi-annually.

Unlike the traditional actuarial approach, [17] viewed CB benefits as financial liabilities, specifically, they can be characterized as tradable assets, which means the maturity payoff amount is the same as that of the accumulation of the crediting rate. Their approach is also known as the market consistent valuation. Under a complete and arbitrage-free market, the price of transferring the pension liability to a third party would be same as the liability value if retained. Although this technique involves subjective selection of interest rate model and parameters, Hardy et al. [17] have shown that the results are insensitive to the choice of the interest rate model and the parameters, even using long time horizons and long-term crediting rates. The market consistent valuation technique has already been studied over other insurance and pension products. Some examples can be found in Marshall [20] and Chen and Hardy [10]. Despite the fact that the US public and private sector pension plans have not yet adopted market consistent valuation, much of the literature has discussed the benefit of using market values to assess the funding status of pension plans. Biggs [2] recalculated the funding ratio of US public pensions and concluded that traditional actuarial techniques significantly overstate the funding ratio. [3] outlined common misconceptions toward using long-term market rate as the discount rate for pension liabilities. In this paper, we reinforce the advantage of market consistent valuation from a risk management perspective, and insist that it provides a better measure of risks, and provides a natural way of hedging the maturity benefit.

Hedging the liability is an equally important topic as pricing and selection of models is a crucial criterion. In the broad area of interest rate derivatives, a frequently asked question is 'how many stochastic drivers are required to construct an effective hedging portfolio?' Fan et al. [14] have studied up to four factor models in the swaption market and conclude that low-dimensional models are capable of accurately pricing swaptions but are insufficient for hedging purpose. Gupta and Subrahmanyam [16] and Driessen et al. [13] draw a similar conclusion for the cap/floor market. Their work is closer to ours, as they used delta hedging strategies, whereas Fan et al. [14] focus on bucket hedging. In the case of CB plans, Hardy et al. [17] have shown that the difference in pricing using one and two factor Hull-White (HW) models is insignificant. However, the hedging of CB liability is still unexplored. Brown et al. [9] studied the duration of the CB plans, which leads to a model-independent duration hedging strategy. On the other hand, Harvey [18] mentioned the difficulties in calculating the duration of a CB plan, and outlined several practical investment strategies including credit default swaps, Treasury futures and swaptions, but without supporting theoretical or empirical analysis. Motivated by the results from the swaption and cap/floor market, and to match our pricing paradigm, we consider Delta hedging in one- and Two-Factor Hull White models.

This thesis continues from [17], by extending some formulae on the valuation of the liability and considering the risk management strategies on CB plans. The major contributions are:

- We develop general formulae for the CB liability valuation when the crediting rate is applied discretely.
- We consider the risk management of the CB liabilities by adopting the no-arbitrage framework to construct dynamic hedging strategies.
- We explore the effectiveness of different hedging strategies by increasing the number of hedging instruments and number of stochastic factors in the models.

This paper is arranged in the following way. Chapter 2 introduces the assumptions and notations we adopted in this thesis, and summarizes the main contributions from Hardy et al. [17]. Chapter 3 presents the liability valuation formulae for a cash balance pension plan, for both continuous crediting and discrete crediting, under One-Factor and Two-Factor Hull-White models. The impact of the starting term structure on the liability valuation is studied in the last section. Chapter 4 describes the Delta hedging strategies for the liabilities, and provides numerical illustrations on both simulations and real data. Chapter 5 gives conclusions and suggests future work.

Chapter 2

Assumptions, Notation and Literature Review

This chapter reviews the main results from Hardy et al. [17]. This thesis is a continuation from the work, thus, we adopt the same notation and parameters.

We use simplified assumptions in valuing the CB liabilities, by ignoring mortality and other demographic considerations. The price of the liability for the CB plan is calculated using the expected discounted value of the benefit at maturity with an arbitrage-free stochastic model for interest rates. This approach is market-based, and the lump sum benefit payment can be theoretically replicated with hedging instruments. Moreover, we use the accruals approach to update the participant's notional account and evaluate the liability as projected. Starting at time t, the individual notional account accumulates using the projected future crediting rate up to the retirement date T, where $T > t$, and we take the expectation of the discounted value of the payoff back to time t to get the liability value of this benefit. The future contributions and salary increases are ignored, as they are not yet included in the accrued benefits.

2.1 Notations for CB Benefit

Let F_t denote the notional account value at time t, and $i^c(t)$ denote the annually credited interest rate at time t. Since we are using the accruals approach, the account value is simply $F_t = F_0 \prod_{u=0}^{t-1}(1 + r^c(u))$ for the discrete case and $F_t = F_0 e^{\int_0^t r^c(u)du}$ for the continuous case. In this thesis, we will not separate the notation for the interest rate between continuous and discrete models, but we will clarify the usage.

The specification of the crediting rate changes according to the regulations. CB plans, which offer fixed crediting rates used to be allowed values between 3% and 8%, but the new proposed regulation restricts the maximum permitted fixed interest to be 6%. For the 30-year Treasury Rate with a floor, the new regulations cap the annual floor at 5%. Proposed regulations in 2010 allowed the crediting interest to be the actual rate of return based on a well-diversified portfolio. In 2014, the IRS has further relaxed the rules to allow the usage of return on subset of the assets in the portfolio. Other alternatives include the returns on regulated investment company, or a broad-based index, or any rates on the safe harbor list on IRS Report 96-08. In this paper we only consider crediting rates as the treasury bond yields without floor or cap.

2.2 The yield curve

We let $r(t)$ denote the continuously compounded short rate of interest at time t, where we model using an arbitrage free model. Denote $r_k(t)$ as the k-year continuously compounded spot rate at time t. At time 0, we have all the information about term structure, $r(0)$ and $r_k(0)$ for all k. Moreover, let $p(t,T)$ represent the price of the zero coupon bond with face value of \$1 at time t with maturity time T. The relationship between $r_k(t)$, $p(t,T)$ and $r(t)$ is as follows:

$$\begin{aligned} r(t) &= \lim_{k\to 0} r_k(t) \\ p(t,T) &= e^{-(T-t)r_{T-t}(t)} \\ p(t,T) &= E_t^Q\left[e^{-\int_t^T r(s)ds}\right] \end{aligned}$$

In the last equation, Q refers to a risk-neutral measure, where the discount factor at time t of a future payment due at T is $e^{-\int_t^T r(s)ds}$. Since we are using an arbitrage free model, the expected discounted value of the maturity payment under Q measure should match its market price. For a zero coupon bond, the payment at time T is \$1, and the market price of this payment at time t is $p(t,T)$. The price $p(t,T)$ should be equal to the expected discounted value of \$1 using the stochastic short rate r(s), which forms the third equality. The subscript t indicates that we are evaluating at time t with all the information up to that time.

Last but not least, forward rates play an important role in short-rate modelling. Here we denote $f(t, t+k)$ as the instantaneous forward rate at time $t+k$, contracted at t. The relation between the forward rate and the price of a zero coupon bond is:

$$p(t, t+k) = e^{-\int_t^{t+k} f(t,s)ds}$$
$$f(t, t+k) = \frac{-1}{p(t, t+k)} \frac{d}{dk} p(t, t+k)$$

and notice $r(t) = \lim_{k \to 0} f(t, t+k)$

2.3 The valuation formulae

Let $V(t, T)$ be the value of the pension liability at time t with final maturity at time T. The initial value (t=0) for the notional account is \$1. Hardy et al. [17] used the term "valuation factor" for $V(t, T)$. The price of this liability at time t is equal to the expected discounted value of the account value (benefit) at T, under the risk neutral measure. The formula for the valuation is

$$V(t, T) = E_t^Q \left[e^{\int_0^T r^c(s) + mds - \int_t^T r(s)ds} \right]$$

for crediting continuously and

$$V(t, T) = E_t^Q \left[e^{\sum_{s=1}^{nT} \frac{r^c(\frac{s}{n}) + m}{n} - \int_t^T r(s)ds} \right]$$

for crediting discretely (n times per year). In the rest of this paper, $r^c(t)$ is based on the k-year spot rate $r_k(t)$. Hardy et al. [17] has derived valuation formulae of V(0,T) under One-Factor and Two-Factor Hull-White models with continuous crediting rates. They also provide numerical results for valuation factors using par-yield rates as crediting rates.

2.4 Review of Hardy et al. [17]

Here we highlight the important findings from Hardy et al. [17] (for our purposes).

- The explicit formulae for V(0,T) using One-Factor Hull-White and Two-Factor Hull-White models.

- The difference between using One-Factor or Two-Factor Hull-White models in valuing the CB liabilities is not significant.
- The valuation factors using spot rates as the crediting rates are close to the ones using par-yield rates.
- The valuation factors are sensitive to the initial yield curve.
- The valuation factors are more volatile when using the fixed rate or the spot rate with longer duration as the crediting rate.
- The differences between CB and DC plans are not clearly understood by many participants and plan sponsors. CB plans do not share the transparency features as DC plans, and it preserves considerable investment risks.
- The traditional valuation approach with high discounting rate is problematic. CB liabilities calculated in this way will be underfunded, and may lead to problems of solvency, security and inequality.

In their conclusion, they suggested a further topic on CB plans, which is to explore the effective risk management strategies. As discussed in their results, using fixed rate or spot rate with long duration leads to volatile valuation factors. The risk management associated with a fixed rate CB liability is simple as the plan sponsors can easily match the future payments using the constant maturity bonds. In Chapter 4 of our thesis, we will look at the delta-hedging strategies to mitigate the risks of using a long-term spot rate as the crediting rate.

Chapter 3

Valuation Formulae

3.1 One-Factor Hull-White

We start by considering the One-Factor Hull-White model (also known as extended-Vasicek model) to model the future interest rates. This model has good tractability and allows a perfect match between the implied starting yield curve with the market. The model was first described in 1990 and since then has been used frequently in the area of asset pricing, and in a number of actuarial applications. Some examples are Boyle and Hardy [7], Chen and Hardy [10] and Marshall [20]. The instantaneous short rate under the risk-neutral measure Q has the following SDE:

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dW_t$$

where $a > 0$, $\sigma > 0$ are constant, $\theta(t)$ is a deterministic function of t chosen to match the market term structure at the starting date, and W_t is a standard Brownian motion under Q. The Hull-White model belongs to the affine model family (see Brigo and Mercurio [8]), as the price of a zero coupon bond has the following form

$$p(t, t+k) = E_t^Q\left[e^{-\int_t^{t+k} r(s)ds}\right] = e^{A(t,t+k)-B(t,t+k)r(t)}$$

where $A(t, t+k)$ and $B(t, t+k)$ are deterministic functions

$$B(t, t+k) = \frac{1 - e^{-ak}}{a}$$
$$A(t, t+k) = \log\frac{p(0, t+k)}{p(0, t)} + f(0, t)B(t, t+k) - \frac{\sigma^2}{4a}B(t, t+k)^2(1 - e^{-2at})$$

Notice that $B(t, t+k)$ does not depend on t. We use this notation to be consistent with the literature. The values of $p(0,t)$, $p(0,t+k)$, $f(0,t)$, for all t and k are input parameters of the model and are observed at the starting date. Under the Hull-White model, the log of the price for a zero coupon bond at time t is a linear function of the short rate at t, and requires no other information. Details about the derivation can be found in Björk [4].

As mentioned in Chapter 2, we choose $r^c(t)$ to be the treasury yield with a margin for short term notes or bills. Here we use the spot rate $r_k(t)$, stripped from the treasury bonds, as the crediting rate instead of the par-yields. $r_k(t)$ provides the advantage of tractability as we are able to derive explicit valuation formulae. In comparison, for par-yields, time-consuming simulation is required. Hardy et al. [17] have shown that valuation factors using the spot curve can be seen as an approximation to evaluation using par-yields since the results are similar.

Under the one-factor Hull White model, $r_k(t)$ can be written as a linear function of r(t)

$$\begin{aligned} r_k(t) &= \frac{-1}{k}\log(p(t,t+k)) \\ &= \frac{B(t,t+k)r(t) - A(t,t+k)}{k} \end{aligned}$$

We set the parameters as $a = 0.02$ and $\sigma = 0.006$ which imply the long term unconditional standard deviations for the short rate and the 30-year rate are 3% and 2.3%. The benchmark starting term structure is the market yield for US treasuries at 29 March 2013. Furthermore, we assume that the yield curve for maturities greater than 30 years is flat. The expected retirement time T is chosen to be 5 years, 10 years and 20 years. The margins associated with the choice of crediting rates are as suggested in IRS report 96-08 for safe harbor rates (See Table 3.1).

Standard Index	Associated Margin
Discount rate on 3-month Treasury Bills	175 basis points
Discount rate on 6-month Treasury Bills	150 basis points
Yield on 1-year Treasury Constant Maturities	100 basis points
Yield on 2-year or 3-year Treasury Constant Maturities	50 basis points
Yield on 5-year or 7-year Treasury Constant Maturities	25 basis points
Yield on 10-year or longer Treasury Constant Maturities	0 basis points

Table 3.1: Safe Harbor Rates

3.2 Continuous Accumulation of Crediting Rates

Hardy et al. [17] provided a valuation formula for V(t,T), where t=0. In some circumstances, for example the simulations in Chapter 4, it is necessary to generalize the valuation factors to any t in [0,T]. Let the crediting rate be $r_k(t) + m(k)$, then the valuation formula can be simplified to

$$\begin{aligned} V(t,T) &= E_t^Q\left[e^{\int_0^T r_k(s)ds - \int_t^T r(s)ds + m(k)T}\right] \\ &= e^{\int_0^t r_k(s)ds} e^{-\int_t^T \frac{A(s,s+k)}{k}ds} E_t^Q\left[e^{-\int_t^T \gamma r(s)ds}\right] e^{m(k)T} \end{aligned}$$

where $\gamma = (1 - \frac{B(t,t+k)}{k})$. The first term is the realized accumulation up to time t. The second term can be solved using numerical integration such as Simpson's method. The solution of the third term is given as

$$\begin{aligned} E_t^Q&\left[e^{-\int_t^T \gamma r(s)ds}\right] = \\ &P_\gamma(t,T) = \exp\{A_\gamma(t,T) - B_\gamma(t,T)r(t)\} \\ &A_\gamma(t,T) = \gamma \log\left\{\frac{P^M(0,T)}{P^M(0,t)}\right\} + \left\{\frac{\sigma^2\gamma}{4a}\{\frac{2}{a}(\gamma-1)(T-t) + (e^{-2at}-\gamma)B^2(t,T) \right. \\ &\qquad \left. + \frac{1}{a}(2-2\gamma)B(t,T)\} + \gamma f^M(0,t)B(t,T)\right\} \\ &B_\gamma(t,T) = \gamma B(t,T) \end{aligned}$$

Details can be found in Appendix B.

3.3 Discrete Accumulation of Crediting Rates

Valuing the liability in a discrete setting under the risk neutral measure is difficult. Fortunately, we can circumvent this problem by changing the measure. Since the maturity date T is fixed, we can use the T-year zero-coupon bond as the numeraire (this is often referred to as the forward-T measure) instead of the bank account. Details are given in Björk [4]. Let X(T) be any payment at time T, its current price (at time t) is

$$E_t^Q[X(T)e^{-\int_t^T r(s)ds}] = p(t,T)E_t^T[X(T)]$$

where superscript T represents the forward-T measure. Clearly, the advantage of using the T-measure is to eliminate the discount factor in the original expectation, which reduces the amount of integration required when short rate is stochastic and correlated to the payment $X(T)$. Using Girsanov's Theorem, the SDE for the short rate under the Hull-White model can be rewritten as

$$\begin{aligned} dr(t) &= (\theta(t) - ar(t) - \sigma^2 B(t,T))dt + \sigma dW_t^T \\ dW_t^T &= dW_t^Q - \sigma B(t,T)dt \end{aligned}$$

W^T is the standard Brownian motion under the forward-T measure. The derivation can be found in Björk [4].

Theorem 3.3.1. *Assume the evolution of the interest rate r(t) follows the One-Factor Hull-White model. Let n be the number of crediting periods per year, and assuming crediting rates are settled at the end of each period. The valuation factor starting at time t and maturing at time T is*

$$V(t,T) = e^{\sum_{s=1}^{tn} \frac{r^c(\frac{s}{n})}{n}} e^{A(t,T)-B(t,T)r(t)} e^{E_t^T[Z] + \frac{1}{2} Var_t^T[Z]} \tag{3.1}$$

where

$$\begin{aligned} E[Z] &= \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} \\ &\quad + \frac{B(0,k)}{nk} \sum_{i=tn+1}^{Tn} \left[r(t)e^{-a\left(\frac{i}{n}-t\right)} + \varphi\left(\frac{i}{n}\right) - \varphi(t)e^{-a\left(\frac{i}{n}-t\right)} \right. \\ &\quad \left. - \frac{\sigma^2}{a^2}\left(1 - \frac{1}{2}e^{-a\left(T-\frac{i}{n}\right)} - e^{-a\left(\frac{i}{n}-t\right)} + \frac{1}{2}e^{-a\left(T+\frac{i}{n}-2t\right)}\right)\right] \\ V[Z] &= \left(\frac{\sigma B(0,k)}{nk}\right)^2 \sum_{j=tn+1}^{Tn} \left(\frac{e^{-\frac{a}{n}(j-1)} - e^{-aT}}{e^{\frac{a}{n}} - 1}\right)^2 \frac{e^{2a\frac{j}{n}} - e^{2a\frac{j-1}{n}}}{2a} \\ \varphi(t) &= f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 \end{aligned}$$

Proof. From Section 2.3:

$$\begin{aligned}
V(t,T) &= E_t^Q\left[e^{\sum_{s=1}^{nT}\frac{r^c(\frac{s}{n})}{n}-\int_t^T r(s)ds}\right]\\
&= e^{\sum_{s=1}^{tn}\frac{r^c(\frac{s}{n})}{n}}E_t^Q\left[e^{\sum_{s=tn+1}^{Tn}\frac{r^c(\frac{s}{n})}{n}-\int_t^T r(s)ds}\right]\\
&= e^{\sum_{s=1}^{tn}\frac{r^c(\frac{s}{n})}{n}}P(t,T)E_t^T\left[e^{\sum_{s=tn+1}^{Tn}\frac{r^c(\frac{s}{n})}{n}}\right]\\
&= e^{\sum_{s=1}^{tn}\frac{r^c(\frac{s}{n})}{n}}e^{A(t,T)-B(t,T)r(t)}E_t^T\left[e^{\sum_{s=tn+1}^{Tn}\frac{r^c(\frac{s}{n})}{n}}\right]
\end{aligned}$$

The short rate's SDE under the T-measure is given in Brigo and Mercurio [8], which is

$$\begin{aligned}
r(t) &= r(s)e^{-a(t-s)}+\int_s^t e^{-a(t-u)}\Theta(u)du-\int_s^t e^{-a(t-u)}\sigma^2B(u,T)du+\sigma\int_s^t e^{-a(t-u)}dW^T(u)\\
&= r(s)e^{-a(t-s)}+\varphi(t)-\varphi(s)e^{-a(t-s)}-\int_s^t e^{-a(t-u)}\sigma^2B(u,T)du+\sigma\sum_{i=s+1}^{t}\int_{i-1}^{i}e^{-a(t-u)}dW^T(u)
\end{aligned}$$

where

$$\begin{aligned}
\varphi(t) &= f(0,t)+\frac{\sigma^2}{2a^2}(1-e^{-at})^2\\
\int_s^t e^{-a(t-u)}\sigma^2B(u,T)du &= \frac{\sigma^2}{a^2}\left(1-\frac{1}{2}e^{-a(T-t)}-e^{-a(t-s)}+\frac{1}{2}e^{-a(T+t-2s)}\right)
\end{aligned}$$

Set $Z = \sum_{s=tn+1}^{Tn} \frac{r^c\left(\frac{s}{n}\right)}{n}$.

$$
\begin{aligned}
Z &= \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} r\left(\frac{i}{n}\right) \\
&= \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B(k)}{nk} \sum_{i=tn+1}^{Tn} \left[r(t) e^{-a\left(\frac{i}{n}-t\right)} + \varphi\left(\frac{i}{n}\right) - \varphi(t) e^{-a\left(\frac{i}{n}-t\right)} \right. \\
&\qquad - \frac{\sigma^2}{a^2}\left(1 - \frac{1}{2}e^{-a\left(T-\frac{i}{n}\right)} - e^{-a\left(\frac{i}{n}-t\right)} + \frac{1}{2}e^{-a\left(T+\frac{i}{n}-2t\right)}\right) \\
&\qquad \left. + \sigma \sum_{j=tn+1}^{i} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{-a\left(\frac{i}{n}-u\right)} dW^T(u) \right] \\
&= \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B(k)}{nk} \sum_{i=tn+1}^{Tn} \left[r(t) e^{-a\left(\frac{i}{n}-t\right)} + \varphi\left(\frac{i}{n}\right) - \varphi(t) e^{-a\left(\frac{i}{n}-t\right)} \right. \\
&\qquad \left. - \frac{\sigma^2}{a^2}\left(1 - \frac{1}{2}e^{-a\left(T-\frac{i}{n}\right)} - e^{-a\left(\frac{i}{n}-t\right)} + \frac{1}{2}e^{-a\left(T+\frac{i}{n}-2t\right)}\right) \right] \\
&\qquad + \sigma \frac{B(k)}{nk} \sum_{j=tn+1}^{Tn} \frac{e^{-\frac{a}{n}(j-1)} - e^{-aT}}{e^{\frac{a}{n}} - 1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{au} dW^T(u)
\end{aligned}
$$

where second step to third step is because of:

$$
\begin{aligned}
\sum_{i=tn+1}^{Tn} \sum_{j=tn+1}^{i} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{-a\left(\frac{i}{n}-u\right)} dW^T(u) &= \sum_{j=tn+1}^{Tn} \sum_{i=j}^{Tn} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{-a\left(\frac{i}{n}-u\right)} dW^T(u) \\
&= \sum_{j=tn+1}^{Tn} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{au} \sum_{i=j}^{Tn} e^{-a\frac{i}{n}} dW^T(u) \\
&= \sum_{j=tn+1}^{Tn} \frac{e^{-\frac{a}{n}(j-1)} - e^{-aT}}{e^{\frac{a}{n}} - 1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{au} dW^T(u)
\end{aligned}
$$

Thus, the expectation and variance are

$$E^T[Z] = \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B(k)}{nk} \sum_{i=tn+1}^{Tn} \left[r(t)e^{-a\left(\frac{i}{n}-t\right)} + \varphi\left(\frac{i}{n}\right) - \varphi(t)e^{-a\left(\frac{i}{n}-t\right)} \right.$$
$$\left. -\frac{\sigma^2}{a^2}\left(1 - \frac{1}{2}e^{-a\left(T-\frac{i}{n}\right)} - e^{-a\left(\frac{i}{n}-t\right)} + \frac{1}{2}e^{-a\left(T+\frac{i}{n}-2t\right)}\right)\right]$$
$$\mathrm{Var}^T[Z] = \left(\frac{\sigma B(k)}{nk}\right)^2 \sum_{j=tn+1}^{Tn} \left(\frac{e^{-\frac{a}{n}(j-1)} - e^{-aT}}{e^{\frac{a}{n}} - 1}\right)^2 \frac{e^{2a\frac{j}{n}} - e^{2a\frac{j-1}{n}}}{2a}$$

That is

$$V(t,T) = e^{\sum_{i=1}^{tn} \frac{r_k\left(\frac{i}{n}\right)}{n}} P(t,T) E_t^T\left[e^{\sum_{i=tn+1}^{Tn} \frac{r_k\left(\frac{i}{n}\right)}{n}}\right]$$
$$= e^{\sum_{i=1}^{tn} \frac{r_k\left(\frac{i}{n}\right)}{n}} P(t,T) e^{E^T[Z] + \frac{V^T[Z]}{2}}$$

As required.

□

This formula is based on the assumption that crediting rates are set and accumulated at the end of each period. Quite often in practice, the crediting rates are settled at the beginning of the year (but also accumulated at the year end). In this case, the formula requires a slight modification. Details are provided in Appendix B.

Last but not least, the crediting periods are usually different from the hedging period, as the fund management may be updated over much shorter horizons. Thus, it is necessary to evaluate the liability between the crediting dates. The details are provide in Appendix B.

The valuation factors under a discrete setting are shown in Table 3.2. If the pension sponsor would like to transfer the obligation of future benefits to the market, these numbers are the fair market price for every dollar in the individual's notional account. Notice the valuation factors using annual crediting rates are fairly close to continuous crediting using the 30-year spot rate; however, the difference can be as high as 4% when using the 1-year spot rate, with margin. Since crediting rates with maturity other than 30-year do not have a large market share, the focus of this thesis is on crediting with the 30-year spot rate.

Crediting Rate (Spot Rates)	Time T to exit								
	Continuous			Year End			Year Begin		
	5-Yrs	10-Yrs	20-Yrs	5-Yrs	10-Yrs	20-Yrs	5-Yrs	10-Yrs	20-Yrs
30-yr rate	1.1709	1.2445	1.3928	1.1746	1.2499	1.4019	1.1672	1.2389	1.3835
20-yr rate	1.1401	1.2148	1.3956	1.1479	1.2272	1.4132	1.1321	1.2021	1.3778
10-yr rate	1.1020	1.1077	1.2760	1.1083	1.1169	1.3021	1.0951	1.0981	1.2494
5-yr rate +0.25%	1.0730	1.1083	1.2164	1.0863	1.1209	1.2504	1.0605	1.0944	1.1829
1-yr rate +1%	1.0612	1.1287	1.2610	1.0740	1.1517	1.3017	1.0513	1.1052	1.2214

Table 3.2: Valuation factors, V(0,T) per $1 of account balance at 29 March 2013, using the one-factor Hull-White model, $a = 0.02$, $\sigma = 0.006$

3.4 Two-Factor Hull-White

One drawback of the One-Factor Hull-White model is its inability to capture the correlation between spot rates with different maturities. The One-factor Hull-White model always assumes a linear relationship for all spot rates in the term structure. Thus, the correlation between the short rate and the crediting rate is equal to one. In this section, we consider an alternative model, called the Two-Factor Hull-White model, or Two-Additive-Factor Gaussian Model, where the instantaneous-short-rate process is modelled by a sum of two correlated processes and a deterministic function to fit the current term structure. Although the model has two underlying processes, it remains quite tractable for many applications such as interest rate caps and floors. One example is Arnaud [1]. Here we use the notation from Brigo and Mercurio [8]. The dynamics of the short rate process under the risk-neutral measure Q are:

$$\begin{aligned} r(t) &= x(t) + y(t) + \varphi(t), \qquad r(0) = r_0 \\ dx(t) &= -a_1 x(t)dt + \sigma_1 dW_1(t), \qquad x(0) = 0 \\ dy(t) &= -a_2 y(t)dt + \sigma_2 dW_2(t), \qquad y(0) = 0 \\ dW_1(t)dW_2(t) &= \rho dt, \end{aligned}$$

where $\varphi(t)$ serves the same function as $\theta(t)$ in the One-Factor Hull-White model to exactly fit the current term structure. Brigo and Mercurio [8] have provided the solution of the zero coupon bond price under the Two-Factor Hull-White model.

$$
\begin{aligned}
P(t,T) &= \exp\left\{A(t,T) - B(t,T,a_1)x(t) - B(t,T,a_2)y(t)\right\} \\
A(t,T) &= \log\frac{p(0,T)}{p(0,t)} + \frac{1}{2}(\nu(T-t) + \nu(t) - \nu(T)) \\
\nu(k) &= \frac{\sigma_1^2}{a_1^2}\left(k - 2B_k(a_1) + B_k(2a_1)\right) + \frac{\sigma_2^2}{a_2^2}\left(k - 2B_k(a_2) + B_k(2a_2)\right) \\
&\quad + \frac{2\rho\sigma_1\sigma_2}{a_1a_2}\left(k - B_k(a_1) - B_k(a_2) + B_k(a_1+a_2)\right) \\
B_k(a) &= B(0,k,a) = \frac{1-e^{-ak}}{a}
\end{aligned}
$$

Notice the solution is similar to One-Factor case, where the log price is a linear function of the two underlying processes x and y.

3.5 Continuous Accumulation of Crediting Rates

Here we provide valuation factor V(t,T), for $t \in [0,T]$, under the Two-Factor Hull-White model for the same reason as we use in the One-Factor case.

$$
\begin{aligned}
V(t,T) &= \exp\left\{\int_0^t r_k(s)ds\right\}\exp\left\{m(k)T\right\}\exp\left\{-\int_t^T \frac{A(s,s+k)}{k}ds\right\} \\
&\quad \exp\left(A^*(t,T) - \gamma_1 B(t,T,a_1)x(t) - \gamma_2 B(t,T,a_2)y(t)\right) \\
A^*(t,T) &= \log\frac{p(0,T)}{p(0,t)} + \frac{1}{2}(\nu^*(T-t) + \nu(t) - \nu(T)) \\
\nu^*(k) &= \frac{\gamma_1^2\sigma_1^2}{a_1^2}\left(k - 2B_k(a_1) + B_k(2a_1)\right) + \frac{\gamma_2^2\sigma_2^2}{a_2^2}\left(k - 2B_k(a_2) + B_k(2a_2)\right) \\
&\quad + \frac{2\rho\gamma_1\gamma_2\sigma_1\sigma_2}{a_1a_2}\left(k - B_k(a_1) - B_k(a_2) + B_k(a_1+a_2)\right) \\
\gamma_j &= 1 - \frac{B_k(a_j)}{k}
\end{aligned}
$$

The parameters $\nu(t)$ and $B_k(a)$ are defined in the last section. Details are provided in Appendix C. The values of the parameters are

$$
a_1 = 0.055 \qquad a_2 = 0.108 \qquad \sigma_1 = 0.032 \qquad \sigma_2 = 0.044 \qquad \rho = -0.9999
$$

which generate a 60% correlation between the short rate and the 30-year spot rate in the long run.

3.6 Discrete Accumulation of Crediting Rates

The valuation of a liability with discrete accumulation of crediting rates under the Two-Factor Hull-White model is similar to the One-Factor case.

Theorem 3.6.1. *Assume the evolution of the interest rate $r(t)$ follows the Two-Factor Hull-White model. Let n be the number of crediting periods per year, and assuming crediting rates are settled at the end of each period, the valuation factor starting at time t and maturing at time T is*

$$
\begin{aligned}
V(t,T) &= e^{\sum_{s=1}^{tn} \frac{r^c(\frac{s}{n})}{n}} e^{A(t,T)-B(t,T)r(t)} E_t^T \left[e^{\sum_{s=tn+1}^{Tn} \frac{r^c(\frac{s}{n})}{n}} \right] \\
&= e^{\sum_{s=1}^{tn} \frac{r^c(\frac{s}{n})}{n}} e^{A(t,T)-B(t,T)r(t)} e^{E_t^T[Z]+\frac{1}{2}Var_t^T[Z]}
\end{aligned}
$$

where

$$
\begin{aligned}
E[Z] = &\sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(0,k,a_1\right)}{nk}\left[x(t)e^{-a_1(\frac{i}{n}-t)} - M_x^T(t,\frac{i}{n})\right] \\
&+ \frac{B\left(0,k,a_2\right)}{nk}\left[y(t)e^{-a_2(\frac{i}{n}-t)} - M_y^T(t,\frac{i}{n})\right] \\
V[Z] = &\left(\frac{\sigma_1 B\left(0,k,a_1\right)}{nk}\right)^2 \sum_{j=tn+1}^{Tn} \left(\frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}}-1}\right)^2 \frac{e^{2a_1\frac{j}{n}} - e^{2a_1\frac{j-1}{n}}}{2a_1} \\
&+ \left(\frac{\sigma_2 B\left(0,k,a_2\right)}{nk}\right)^2 \sum_{j=tn+1}^{Tn} \left(\frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}}-1}\right)^2 \frac{e^{2a_2\frac{j}{n}} - e^{2a_2\frac{j-1}{n}}}{2a_2} \\
&2\rho \left(\frac{\sigma_1 B\left(0,k,a_1\right)}{nk}\right)\left(\frac{\sigma_2 B\left(0,k,a_2\right)}{nk}\right) \\
&\quad \sum_{j=tn+1}^{Tn} \left(\frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}}-1}\right)\left(\frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}}-1}\right) \frac{e^{(a_1+a_2)\frac{j}{n}} - e^{(a_1+a_2)\frac{j-1}{n}}}{a_1+a_2}
\end{aligned}
$$

Proof. Similar to single-factor Hull-White model, we price the discrete crediting liability using the Forward-T measure. Brigo and Mercurio [8] have provided the SDE and the distribution of the short rate under the T-measure as follows:

$$r(t) = x(t) + y(t) + \varphi(t)$$

$$dx(t) = \left[-a_1 x(t) - \frac{\sigma_1^2}{a_1}(1 - e^{-a_1(T-t)}) - \rho \frac{\sigma_1 \sigma_2}{a_2}(1 - e^{-a_2(T-t)}) \right] dt + \sigma_1 dW_1^T(t)$$

$$dy(t) = \left[-a_2 y(t) - \frac{\sigma_2^2}{a_2}(1 - e^{-a_2(T-t)}) - \rho \frac{\sigma_1 \sigma_2}{a_1}(1 - e^{-a_1(T-t)}) \right] dt + \sigma_2 dW_2^T(t)$$

$$dW_1^T(t) dW_2^T(t) = \rho dt$$

$$x(t) = x(s) e^{-a_1(t-s)} - M_x^T(s,t) + \sigma_1 \int_s^t e^{-a_1(t-u)} dW_1^T(u)$$

$$y(t) = y(s) e^{-a_2(t-s)} - M_y^T(s,t) + \sigma_2 \int_s^t e^{-a_2(t-u)} dW_2^T(u)$$

$$M_x^T(s,t) = \left(\frac{\sigma_1^2}{a_1^2} + \rho \frac{\sigma_1 \sigma_2}{a_1 a_2} \right) \left[1 - e^{-a_1(t-s)} \right] - \frac{\sigma_1^2}{2a_1^2} \left[e^{-a_1(T-t)} - e^{-a_1(T+t-2s)} \right] - \frac{\rho \sigma_1 \sigma_2}{a_2(a_1+a_2)} \left[e^{-a_2(T-t)} - e^{-a_2 T - a_1 t + (a_1+a_2)s} \right]$$

$$M_y^T(s,t) = \left(\frac{\sigma_2^2}{a_2^2} + \rho \frac{\sigma_1 \sigma_2}{a_1 a_2} \right) \left[1 - e^{-a_2(t-s)} \right] - \frac{\sigma_2^2}{2a_2^2} \left[e^{-a_2(T-t)} - e^{-a_2(T+t-2s)} \right] - \frac{\rho \sigma_1 \sigma_2}{a_1(a_1+a_2)} \left[e^{-a_1(T-t)} - e^{-a_1 T - a_2 t + (a_1+a_2)s} \right]$$

let $Z = \frac{1}{n} \sum_{i=tn+1}^{Tn} r_k \left(\frac{i}{n} \right)$, which is a normally distributed with mean and variance as follow:

$$Z = \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(\frac{i}{n}, \frac{i}{n}+k, a_1\right)}{nk} x\left(\frac{i}{n}\right) + \frac{B\left(\frac{i}{n}, \frac{i}{n}+k, a_2\right)}{nk} y\left(\frac{i}{n}\right)$$

$$= \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B(0,k,a_1)}{nk} \left[x(t) e^{-a_1(\frac{i}{n}-t)} - M_x^T(t, \frac{i}{n}) + \sigma_1 \int_t^{\frac{i}{n}} e^{-a_1(\frac{i}{n}-u)} dW_1^T(u) \right]$$

$$+ \frac{B(0,k,a_2)}{nk} \left[y(t) e^{-a_2(\frac{i}{n}-t)} - M_y^T(t, \frac{i}{n}) + \sigma_2 \int_t^{\frac{i}{n}} e^{-a_2(\frac{i}{n}-u)} dW_2^T(u) \right]$$

$$= \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(0,k,a_1\right)}{nk}\left[x(t)e^{-a_1(\frac{i}{n}-t)} - M_x^T(t,\frac{i}{n}) + \sigma_1 \sum_{j=tn+1}^{i} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{-a_1(\frac{i}{n}-u)} dW_1^T(u)\right]$$

$$+ \frac{B\left(0,k,a_2\right)}{nk}\left[y(t)e^{-a_2(\frac{i}{n}-t)} - M_y^T(t,\frac{i}{n}) + \sigma_2 \sum_{j=tn+1}^{i} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{-a_2(\frac{i}{n}-u)} dW_2^T(u)\right]$$

$$= \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(0,k,a_1\right)}{nk}\left\{\sum_{i=tn+1}^{Tn}\left[x(t)e^{-a_1(\frac{i}{n}-t)} - M_x^T(t,\frac{i}{n})\right]\right.$$

$$\left. +\sigma_1 \sum_{j=tn+1}^{Tn} \frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}}-1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{a_1 u} dW_1^T(u)\right\}$$

$$+ \frac{B\left(0,k,a_2\right)}{nk}\left\{\sum_{i=tn+1}^{Tn}\left[y(t)e^{-a_2(\frac{i}{n}-t)} - M_y^T(t,\frac{i}{n})\right]\right.$$

$$\left. +\sigma_2 \sum_{j=tn+1}^{Tn} \frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}}-1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{a_2 u} dW_2^T(u)\right\}$$

the expectation and variance are:

$$E^T[Z] = \sum_{i=tn+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(0,k,a_1\right)}{nk}\left[x(t)e^{-a_1(\frac{i}{n}-t)} - M_x^T(t,\frac{i}{n})\right]$$

$$+ \frac{B\left(0,k,a_2\right)}{nk}\left[y(t)e^{-a_2(\frac{i}{n}-t)} - M_y^T(t,\frac{i}{n})\right]$$

$$\mathrm{Var}^T[Z] = \left(\frac{\sigma_1 B\left(0,k,a_1\right)}{nk}\right)^2 \sum_{j=tn+1}^{Tn} \left(\frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}}-1}\right)^2 \frac{e^{2a_1\frac{j}{n}} - e^{2a_1\frac{j-1}{n}}}{2a_1}$$

$$+ \left(\frac{\sigma_2 B\left(0,k,a_2\right)}{nk}\right)^2 \sum_{j=tn+1}^{Tn} \left(\frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}}-1}\right)^2 \frac{e^{2a_2\frac{j}{n}} - e^{2a_2\frac{j-1}{n}}}{2a_2}$$

$$2\rho\left(\frac{\sigma_1 B\left(0,k,a_1\right)}{nk}\right)\left(\frac{\sigma_2 B\left(0,k,a_2\right)}{nk}\right)$$

$$\sum_{j=tn+1}^{Tn} \left(\frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}}-1}\right)\left(\frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}}-1}\right) \frac{e^{(a_1+a_2)\frac{j}{n}} - e^{(a_1+a_2)\frac{j-1}{n}}}{a_1+a_2}$$

Using the properties of the log-normal distribution, the liability is simply

$$V(t,T) = e^{\sum_{i=1}^{tn} \frac{r_k\left(\frac{i}{n}\right)}{n}} P(t,T) E_t^T \left[e^{\sum_{i=tn+1}^{Tn} \frac{r_k\left(\frac{i}{n}\right)}{n}} \right]$$
$$= e^{\sum_{i=1}^{tn} \frac{r_k\left(\frac{i}{n}\right)}{n}} P(t,T) e^{E^T[Z] + \frac{V^T[Z]}{2}}$$

As required. □

Similar to the One-Factor case, we have provided the derivation for the valuation of the liability when the crediting rate is settled at the beginning of the period, and for the valuation within the crediting periods in Appendix C.

The valuation factors under the Two-Factor Hull-White model are shown in Table 3.3. The effect of the crediting pattern is similar to the One-Factor Hull-White model. Comparing the valuation factor between these two models, the largest difference is around 2.5% for crediting with the 20-year spot rate during a 20-year horizon. For shorter periods, the differences are rather small.

Crediting Rate (Spot Rates)	Time T to exit								
	Continuous			Year End			Year Begin		
	5-Yrs	10-Yrs	20-Yrs	5-Yrs	10-Yrs	20-Yrs	5-Yrs	10-Yrs	20-Yrs
30-yr rate	1.1701	1.2468	1.4202	1.1735	1.2523	1.4307	1.1667	1.2412	1.4095
20-yr rate	1.1384	1.2133	1.4118	1.1458	1.2253	1.4304	1.1309	1.2010	1.3929
10-yr rate	1.1002	1.1027	1.2714	1.1059	1.1110	1.2968	1.0938	1.0941	1.2456
5-yr rate + 0.25%	1.0721	1.1045	1.2084	1.0851	1.1161	1.2404	1.0599	1.0915	1.1767
1-yr rate + 1%	1.0611	1.1280	1.2583	1.0739	1.1501	1.2963	1.0513	1.1052	1.2214

Table 3.3: Valuation factors, V(0,T) per $1 of account balance at 29 March 2013, using the Two-Factor Hull-White model, $a_1 = 0.055$, $a_2 = 0.108$, $\sigma_1 = 0.032$, $\sigma_2 = 0.044$, $\rho = -0.9999$

3.7 Impact of the initial yield curve and maturity of spot rates

In the continuous setting, the valuation factors are very sensitive to the initial yield curve, especially for longer horizons and longer durations of the crediting rate. In this section, we extend the analysis to the discrete crediting case. More importantly, we explore how the initial yield curve, maturity of the liability and the duration of crediting rates, affect the difference between the continuous valuation factors and the discrete valuation factors.

One-Factor HW

Figures 3.1, 3.2 and 3.3 plot the annually credited valuation factor with different initial yield curves and different maturities. Clearly, they have similar behavior to the continuous crediting. Thus, we can draw the similar conclusion as Hardy et al. [17] that valuation factors (regardless of the crediting rate) can be very sensitive to the initial yield curve, and the impact is larger for longer horizons and longer durations of the crediting rate.

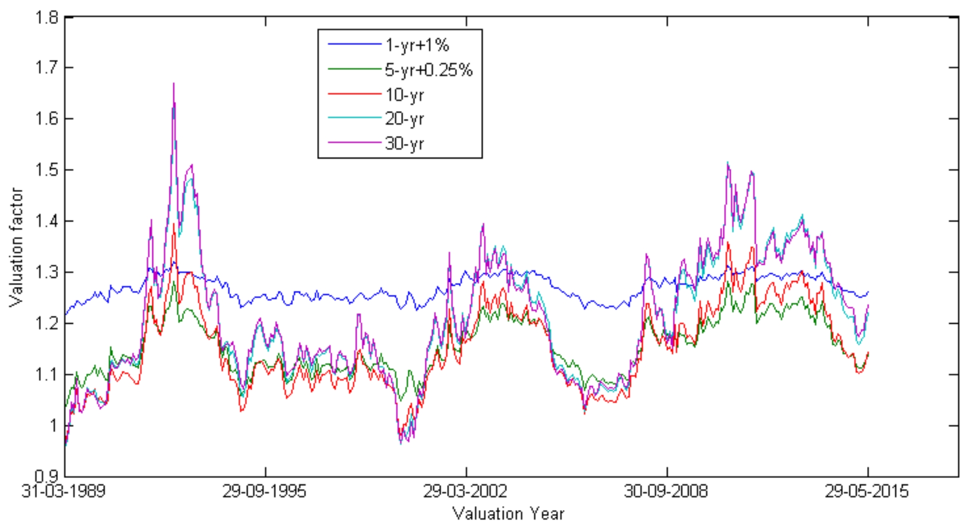

Figure 3.1: Valuation Factors for $T = 20$, from March 1989 to May 2015, crediting annually using the one-factor Hull-White model

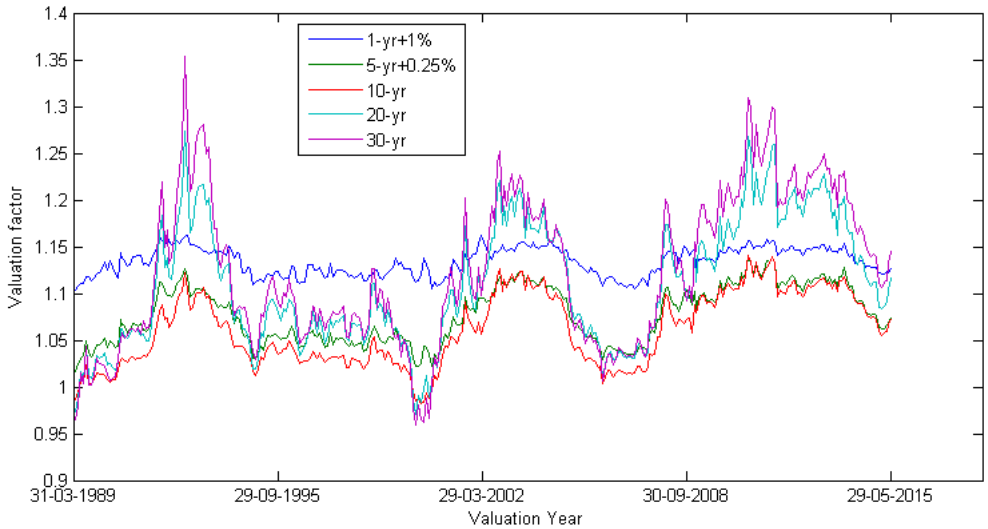

Figure 3.2: Valuation Factors for $T = 10$, from March 1989 to May 2015, crediting annually using the one-factor Hull-White model

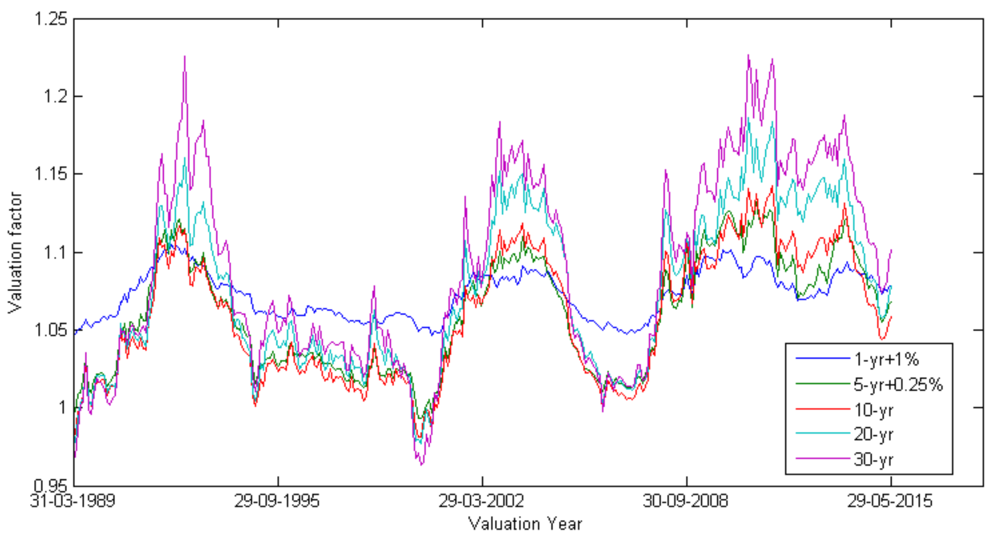

Figure 3.3: Valuation Factors for $T = 5$, from March 1989 to May 2015, crediting annually using the one-factor Hull-White model

Next, we look at the effect of the initial term structure on the difference between the valuation of liabilities with continuous crediting and annual crediting (with rates settled at the end of period). We start by looking at the valuation factors with initial term structure taken from 29 March 2013, with the horizons $T \in [5, 20]$ and the duration of spot rate $k \in [1, 30]$. The results are summarized in Figure 3.4. Notice that the longer time horizon leads to a larger difference as we expected. The differences increase as the maturity of the crediting rate shortens, which is also reasonable as the spot rates with shorter maturities tend to be more volatile.

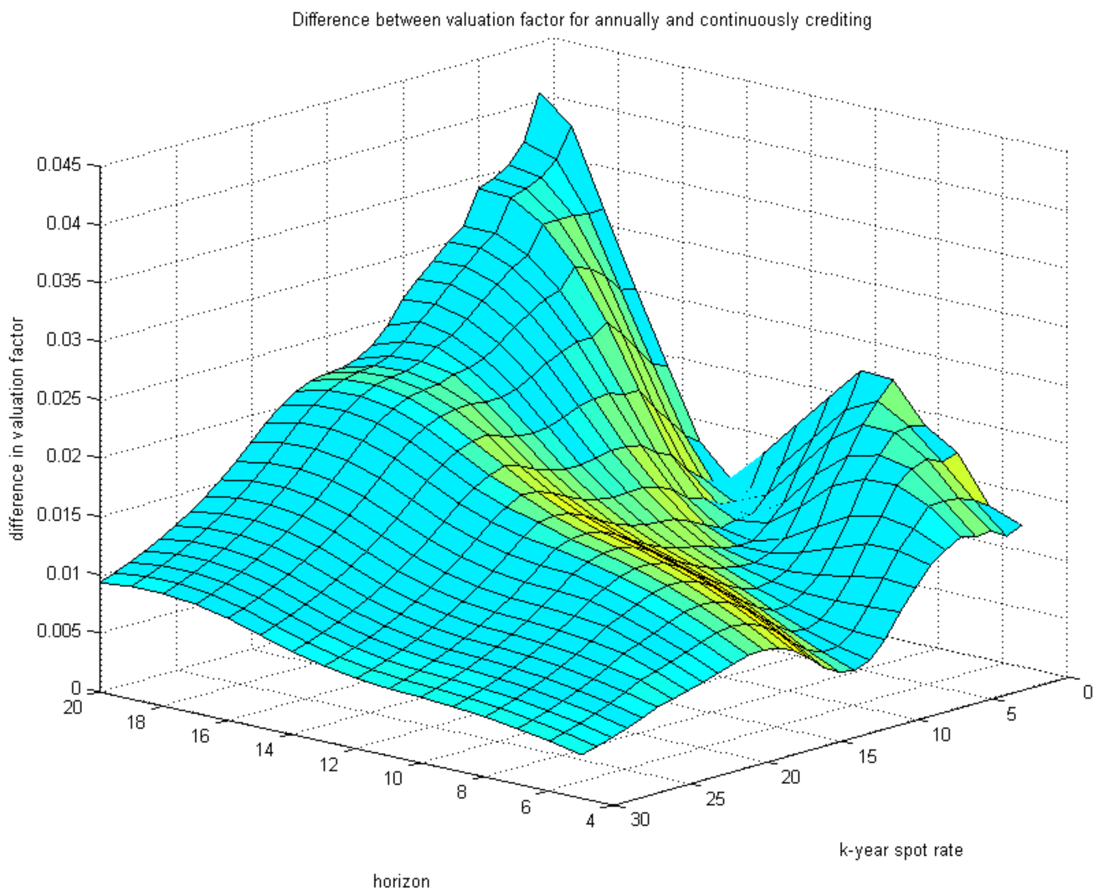

Figure 3.4: Difference between valuation factors crediting continuously and annually, for $T = 5 - 20$ year horizon, and $k = 1 - 30$ year spot rate, on 29 March, 2013, One-Factor Hull-White model

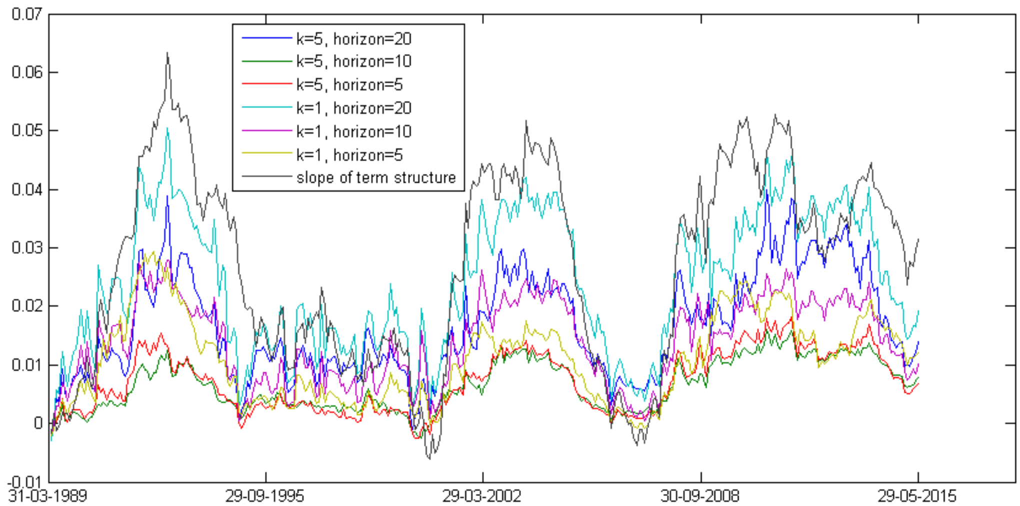

Figure 3.5: Difference between valuation factors crediting continuously and annually, from March 1989 to May 2015, One-Factor Hull-White model

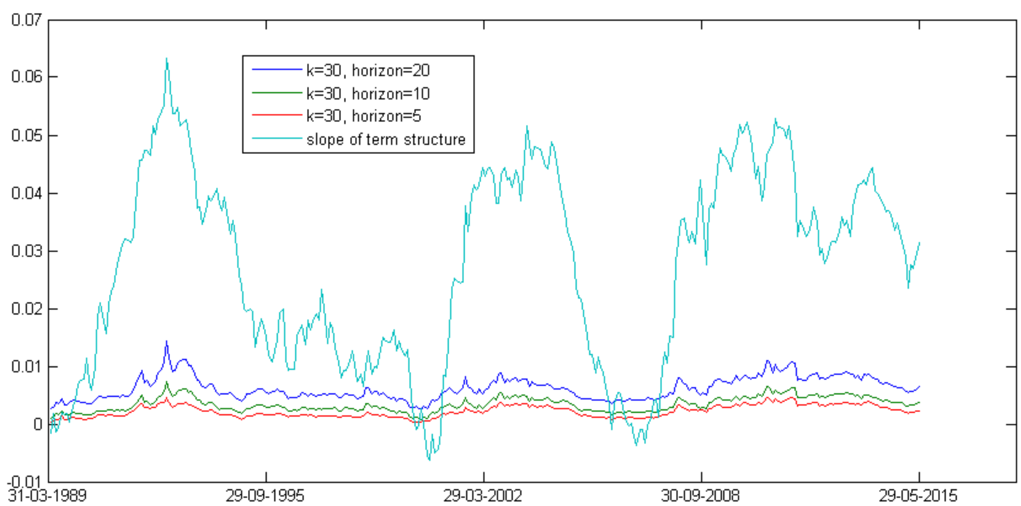

Figure 3.6: Difference between valuation factors crediting continuously and annually, using thirty year spot rate as crediting rate, from March 1989 to May 2015, One-Factor Hull-White model

To see the impact of the starting yield, we plot the difference between the continuous and the discrete valuation factors using different starting yield curves, with different crediting rates and different horizons (Figures 3.5 and 3.6). The evidence of the effect from the initial yield curve is clear, the differences between valuation factors change at different valuation dates and the shape of the difference lines are consistent for each k and T. Notice the black line, which has similar shape as the difference valuation factor line. It is the interest spreads between 30-year spot rates and 3-month spot rates, and it is often used as an approximation to the slope of the yield curve. This apparent coincidence can be explained by the long term behavior of the short rate under the one-factor Hull-White model.

$$
\begin{aligned}
E^Q[r(t)] &= f(0,t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 \\
\mathrm{Var}^Q[r(t)] &= \frac{\sigma^2 e^{-at}}{2a}\left(e^{2at} - 1\right) \\
E^Q[r_k(t)] &= \frac{-A(t,t+k)}{k} + \frac{B(t,t+k)}{k} E^Q[r(t)] \\
&= f(0,t,t+k) + \frac{B(t,t+k)\frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 + \frac{\sigma^2}{4a} B^2(t,t+k)(1 - e^{-2at})}{k} \\
\mathrm{Var}^Q[r_k(t)] &= \left(\frac{B(t,t+k)}{k}\right)^2 \frac{\sigma^2 e^{-at}}{2a}\left(e^{2at} - 1\right) \\
\lim_{t\to\infty} E^Q[r(t)] &= \frac{\sigma^2}{2a^2} + f(0,\infty) \\
\lim_{t\to\infty} \mathrm{Var}^Q[r(t)] &= \frac{\sigma^2}{2a} \\
\lim_{t\to\infty} E^Q[r_k(t)] &= \frac{\sigma^2}{4ka^3}\left((3 - e^{-ak})(1 - e^{-ak})\right) - \lim_{t\to\infty} \frac{\log \frac{p(0,t+k)}{p(0,t)}}{k} \\
\lim_{t\to\infty} \mathrm{Var}^Q[r(t)] &= \frac{(1 - e^{-ak})^2}{k^2 a^2} \frac{\sigma^2}{2a}
\end{aligned}
$$

Using the assumption of a flat yield curve for spot rates with maturities above 30-years, and $a = 0.02$ and $\sigma = 0.006$, we can rewrite the expectations as

$$
\begin{aligned}
\lim_{t\to\infty} E[r(t)] &= 0.045 + r_{30}(0) \\
\lim_{t\to\infty} E[r_k(t)] &= \frac{1.125}{k}\left((3 - e^{-0.02k})(1 - e^{-0.02k})\right) + r_{30}(0)
\end{aligned}
$$

Notice that the long term expected value of the crediting rate is directly related to the currently observed long rate. Thus, if the crediting rate is far below $r_{30}(0)$, it is expected to increase over time until it reaches its stationary state; therefore, the crediting rate observed within the year will be quite different from the one observed at the end, causing a large difference in the valuation factor. When the term structure of interest rates is relatively flat or if we use a longer maturity spot rate than the crediting rate, the rates become relatively stable across the year and the choice of crediting time becomes irrelevant. Observe that even when the crediting rate is r_{30}, its long term expectation is about 0.04147 above the observed $r_{30}(0)$; thus, the crediting rate across the whole term structure has an increasing trend, which explains why the valuation factor with annual crediting is generally greater than continuously crediting.

For relatively shorter horizons, the liability values under the Hull-White model are affected by the entire shape of the term structure, not only the difference between long rates and short rates. For numerical illustration, here we set k=30 and all other parameters as previously defined, Figure 3.7 shows the variance and the "model specified constant"

$$c_1 = E[r(t)] - f(0,t) = \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2$$
$$c_2 = E[r_k(t)] - f(0,t,t+k) = \frac{B(t,t+k)\frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 + \frac{\sigma^2}{4a}B^2(t,t+k)(1 - e^{-2at})}{k}$$

for short rates and crediting rates with respect to time t. Immediately from the graph, we observe what we expected, that short rate exhibits a larger variance, and slower increasing trend compared to the 30-yr spot rate, since $f(0,t,t+k) = \frac{1}{k}\int_t^{t+k} f(0,s)ds$, and $r(t)$ are very sensitive to the change of direction of the term structure, whereas $r_k(t)$ depends on the overall level of the entire term structure, especially when k is large. In Figure 3.6, where we set the duration of the crediting rate k=30, it is clear that although the difference between continuous and annually crediting is still affected by the initial yield curve (the small hump in 1992), the magnitude is relatively small. The largest differences between the two are 0.0146, 0.0076 and 0.0047 for the 20-year, 10-year and 5-year horizons, respectively.

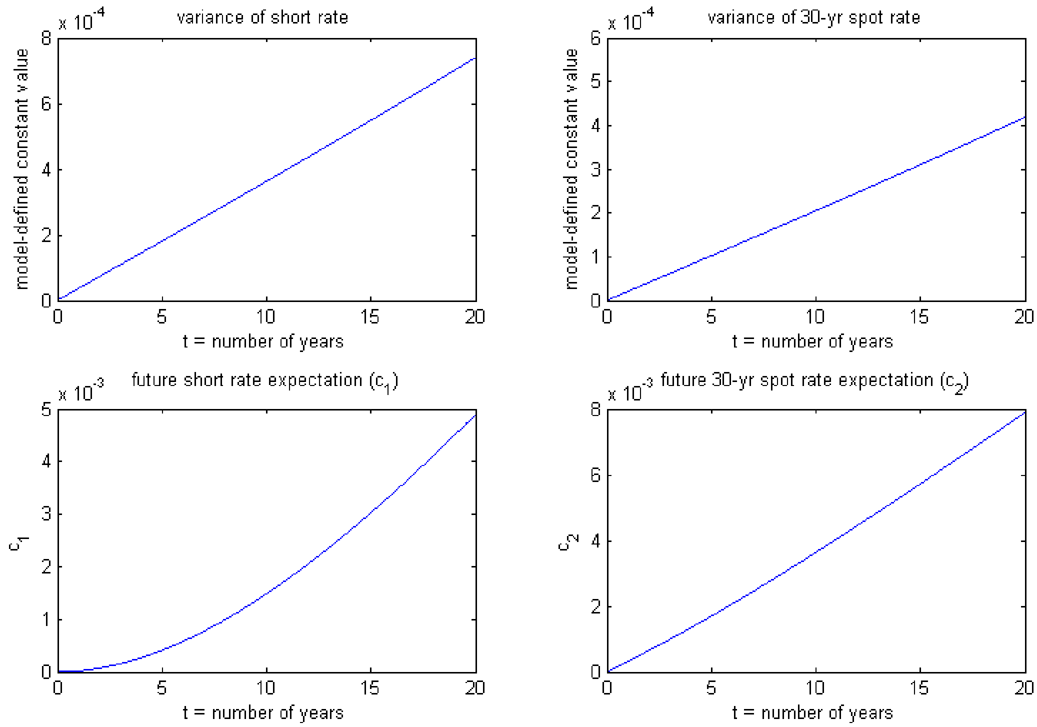

Figure 3.7: Variance and expectation($E[r(t)] - f(0,t)$ and $E[r_k(t)] - f(0,t,t+k)$) of future short rate and future 30-yr spot rate, using the One-Factor Hull-White model

Two-Factor Hull-White

The valuation factor graphs for the Two-Factor Hull-White model are given in Figures 3.8, 3.9 and 3.10. The shapes are very close to the One-Factor Hull-White model. The effect of the maturity of the liability and duration of the crediting rate is plotted in Figure 3.11, using the initial term structure from March 29, 2013. This graph has almost the same shape as the one for the One-Factor Hull-White model. The graphs of differences in valuation factors between continuous crediting and annual crediting are presented in Figures 3.12 and 3.13. The overall behavior of the differences are similar to one-factor case.

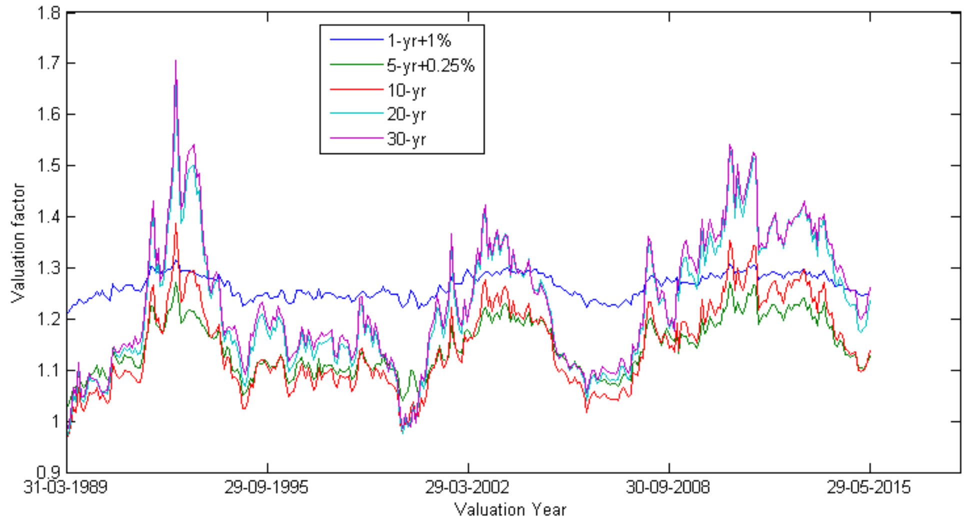

Figure 3.8: Valuation Factors for T=20, from March 1989 to May 2015, crediting annually using the Two-Factor Hull-White model

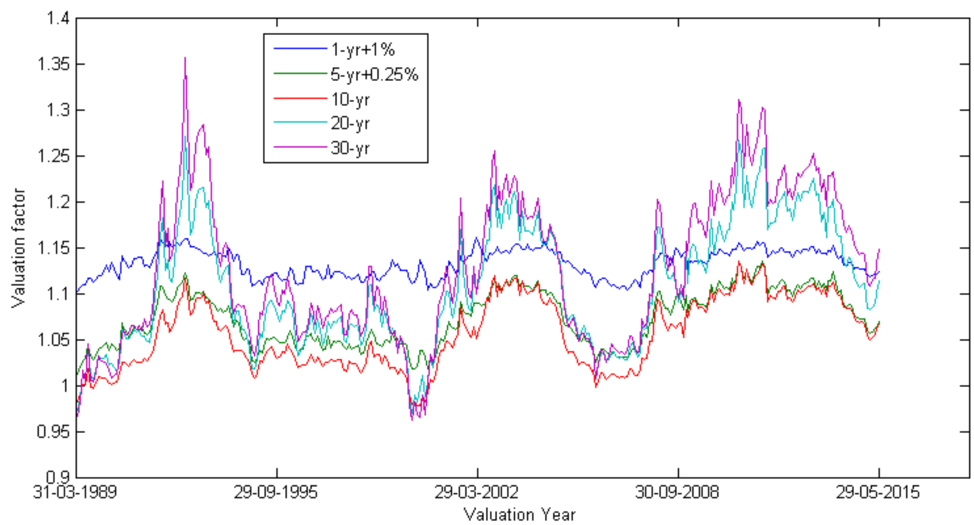

Figure 3.9: Valuation Factors for $T = 10$, from March 1989 to May 2015, crediting annually using Two-Factor Hull-White model

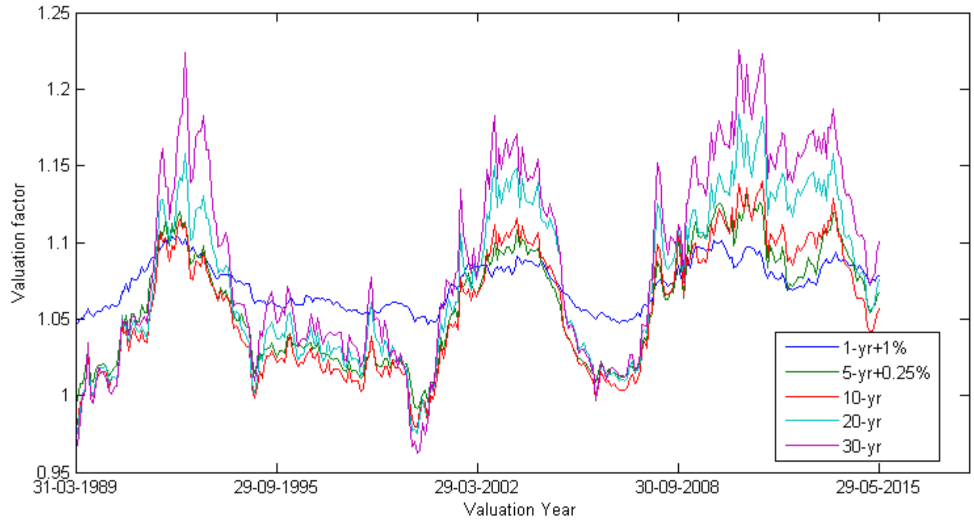

Figure 3.10: Valuation Factors for $T = 5$, from March 1989 to May 2015, crediting annually using Two-Factor Hull-White model

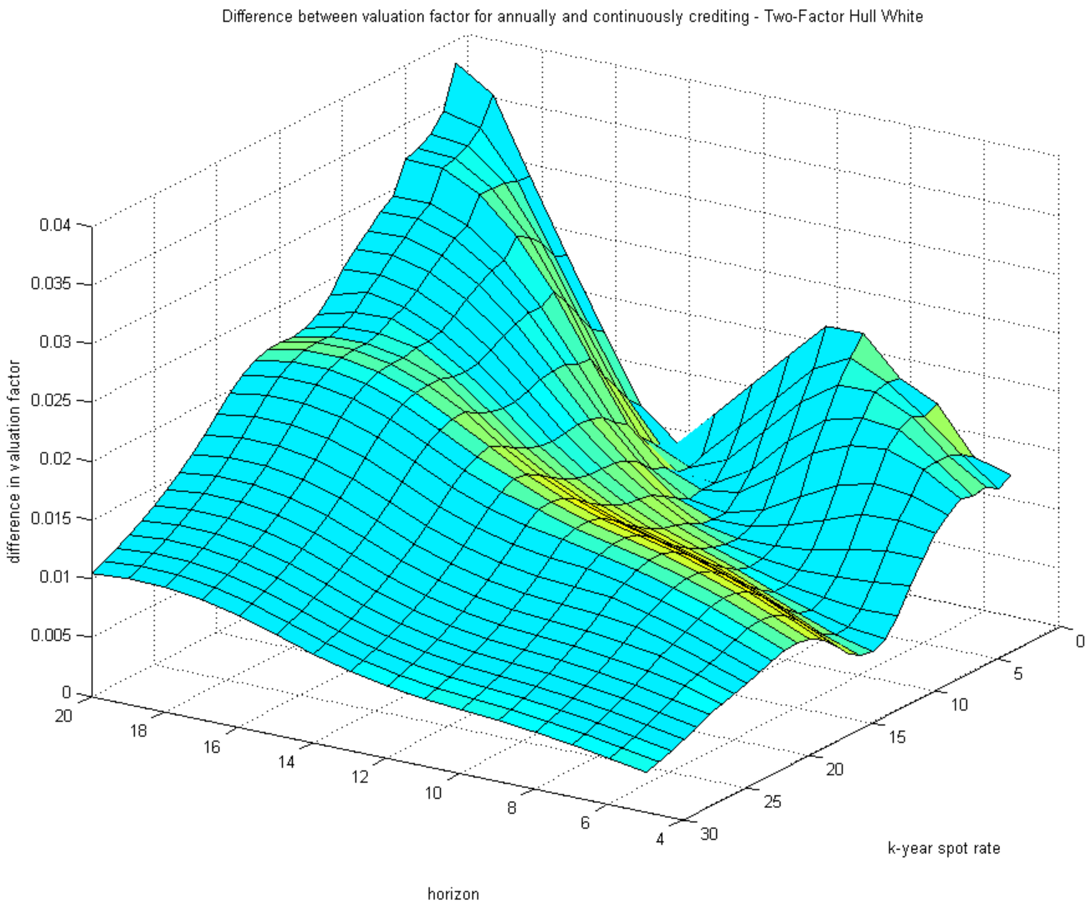

Figure 3.11: Difference between valuation factor crediting continuously and annually, for $T = 5 - 20$ year horizon, and k=1-30 year spot rate, on 29 March, 2013, Two-Factor Hull-White model

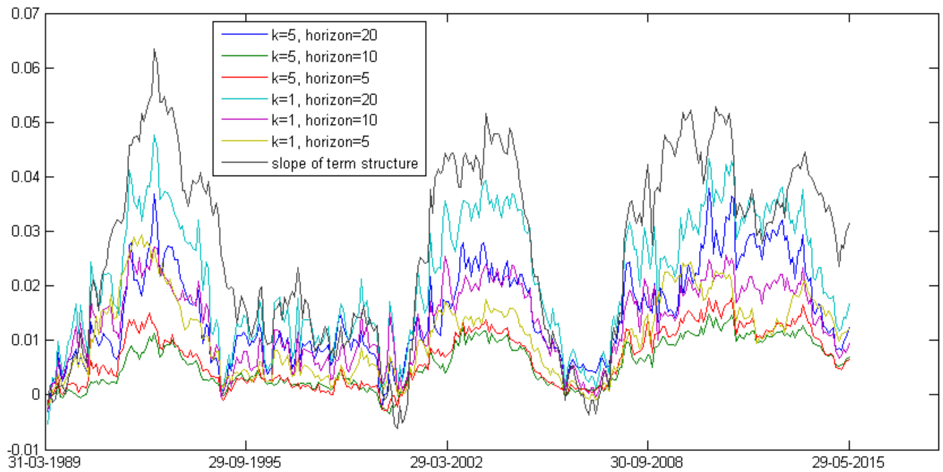

Figure 3.12: Difference between valuation factor crediting continuously and annually, from March 1989 to May 2015, Two-Factor Hull-White model

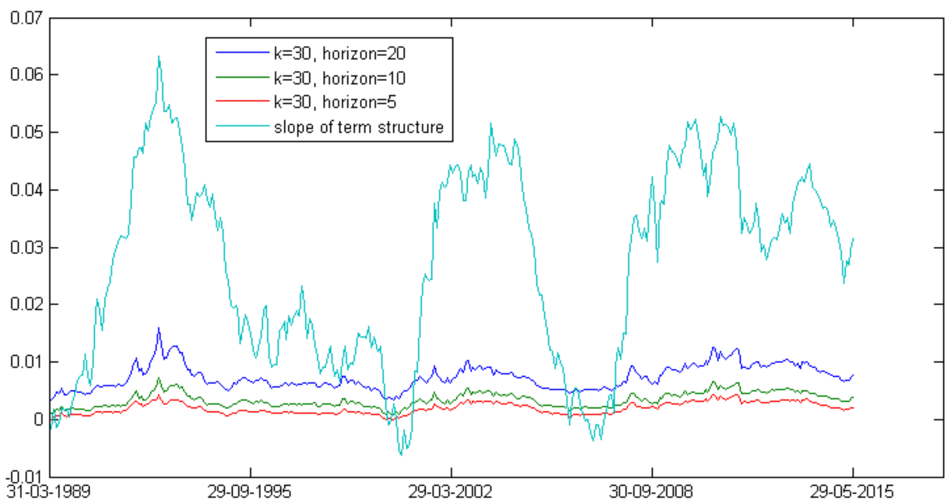

Figure 3.13: Difference between valuation factor crediting continuously and annually, using the thirty year spot rate as crediting rate, from March 1989 to May 2015, Two-Factor Hull-White model

Next, we explore the distributions of future short rates and spot rates.

$$
\begin{aligned}
E[r(t)] &= \varphi(t) \\
V[r(t)] &= \frac{\sigma_1^2}{2a_1}\left(1-e^{-2a_1 t}\right) + \frac{\sigma_2^2}{2a_2}\left(1-e^{-2a_2 t}\right) + \frac{2\rho\sigma_1\sigma_2}{a_1+a_2}\left(1-e^{-(a_1+a_2)t}\right) \\
E[r_k(t)] &= f(0,t,t+k) - \frac{1}{2k}(\nu(k)+\nu(t)-\nu(t+k)) \\
V[r_k(t)] &= \left(\frac{B_k(a_1)\sigma_1}{k}\right)^2 B_t(2a_1) + \left(\frac{B_k(a_2)\sigma_2}{k}\right)^2 B_t(2a_2) \\
&\qquad 2\rho\sigma_1\sigma_2 \frac{B_k(a_1)}{k}\frac{B_k(a_2)}{k} B_t(a_1+a_2) \\
\lim_{t\to\infty} E[r(t)] &= f(0,\infty) + 0.0305 \\
\lim_{t\to\infty} V[r(t)] &= 0.00099771 \\
\lim_{t\to\infty} E[r_k(t)] &= f(0,\infty) + 0.0146 \\
\lim_{t\to\infty} V[r_k(t)] &= 0.00051186
\end{aligned}
$$

We define $c_1 = E[r(t)] - f(0,t)$ and $c_2 = E[r_k(t)] - f(0,t,t+k)$ as in the One-Factor case. A few things here deserve some attention. First, the variances behave similarly to the One-Factor case. Second, due to the existence of two stochastic drivers, the short rate and the 30-year spot rate do not have similar movements. Third, the long run expectation of the short rate is around 1.5% higher than 30-year spot rate (regardless of the initial difference). Last but not least, the short rate expectation under the Two-Factor Hull-White is smaller than the One-Factor case, especially for years between 5 and 20, which generally implies a higher liability value calculated under the Two-Factor model. Table 3.3 supports this claim.

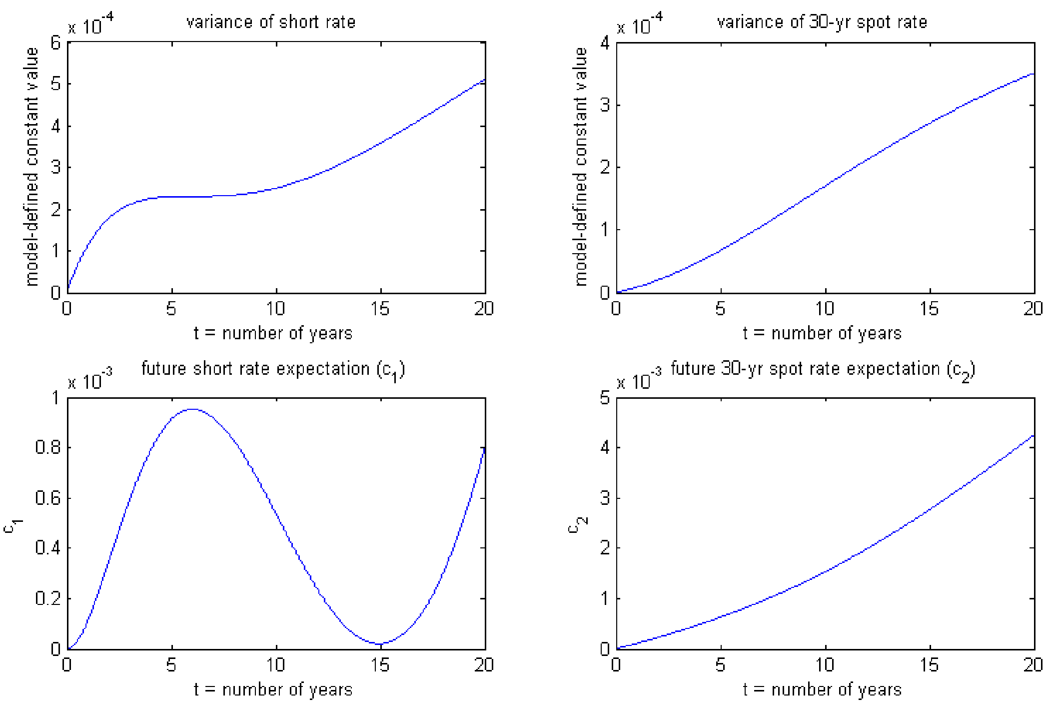

Figure 3.14: variance and expectation($E[r(t)]-f(0,t)$ and $E[r_k(t)]-f(0,t,t+k)$) of future short rate and future 30-yr spot rate, using Two-Factor Hull-White model

Lastly, in Figure 3.13, where we set the duration of the crediting rate to k=30, the result coincides with the One-Factor model. The difference between continuously crediting and discretely crediting is relatively small. This close relation motivates us to use the dynamic hedging strategies proposed in the continuous time setting as an approximation for discrete time.

Virtual Term Structures

To better demonstrate the effect of initial term structures, here we create five virtual term structures using the Vasicek Model shown in the graph 3.15. The instantaneous short rate under Vasicek Model has the following SDE:

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW(t)$$

and the parameters for the five virtual term structures are:

- Case 1: $\alpha = 0.3$, $\beta = 0.055$, $r(0) = 0.05$ and $\sigma = 0.03$
- Case 2: $\alpha = 0.3$, $\beta = 0.04$, $r(0) = 0.05$ and $\sigma = 0.03$
- Case 3: $\alpha = 0.3$, $\beta = 0.11$, $r(0) = 0.05$ and $\sigma = 0.03$
- Case 4: $\alpha = 0.3$, $\beta = 0.05$, $r(0) = 0.05$ and $\sigma = 0$
- Case 5: $\alpha = 0.3$, $\beta = 0.063$, $r(0) = 0.05$ and $\sigma = 0.08$

The corresponding liability value for each term structure are evaluated using the One-Factor Hull-White and the Two-Factor Hull-White models and are shown in Figure 3.16. The possible term structures we can achieve using different parameters within the Vasicek Model are upward, downward, hump and constant shapes. The parameters for the One-Factor Hull-White and Two-Factor Hull White are as specified before and the horizon is chosen to be 5-years, crediting with 30-yr spot rate. Notice all valuation factors are very close, and we present the numbers in Table 3.7. There is no significant difference in valuation using either the One-Factor Hull-White or the Two-Factor Hull-White, which results in less incentive for us to evaluate using extra factors. Crediting at year end often leads to a higher valuation factor as expected (vice versa crediting at the beginning of the year leads to lower valuation). However, this is not always true, as shown by a counterexample using the fifth virtual term structure. The reason behind this result is based on the steep downward trend in the term structure and the horizon being relatively short so that the crediting rate and short rate will not achieve their long term stationary state. On the contrary, if we change the horizon to 20 years, when the effect of the initial term structure on future rates has been reduced, crediting at year end will result in a larger valuation factor as we expected. See Table 3.7 and Figure 3.17.

	One-Factor Hull-White			Two-Factor Hull-White		
	Continuous	Year End	Year Begin	Continuous	Year End	Year Begin
Case 1	0.9995	1.0003	0.9988	0.9988	0.9994	0.9983
Case 2	0.9663	0.9666	0.9661	0.9656	0.9657	0.9656
Case 3	1.1316	1.1344	1.1285	1.1308	1.1333	1.1279
Case 4	1.0035	1.0043	1.0027	1.0028	1.0034	1.0022
Case 5	0.9275	0.9272	0.9279	0.9269	0.9263	0.9275

Table 3.4: liability value for virtual term structure using 30-yr spot rate and 5-yr horizon

	One-Factor Hull-White			Two-Factor Hull-White		
	Continuous	Year End	Year Begin	Continuous	Year End	Year Begin
Case 1	1.0365	1.0401	1.0330	1.0569	1.0615	1.0524
Case 2	1.0026	1.0059	0.9995	1.0223	1.0266	1.0183
Case 3	1.1708	1.1759	1.1655	1.1938	1.2001	1.1874
Case 4	1.0417	1.0454	1.0382	1.0622	1.0669	1.0577
Case 5	0.9559	0.9587	0.9535	0.9748	0.9784	0.9714

Table 3.5: liability value for virtual term structure using 30-yr spot rate and 20-yr horizon

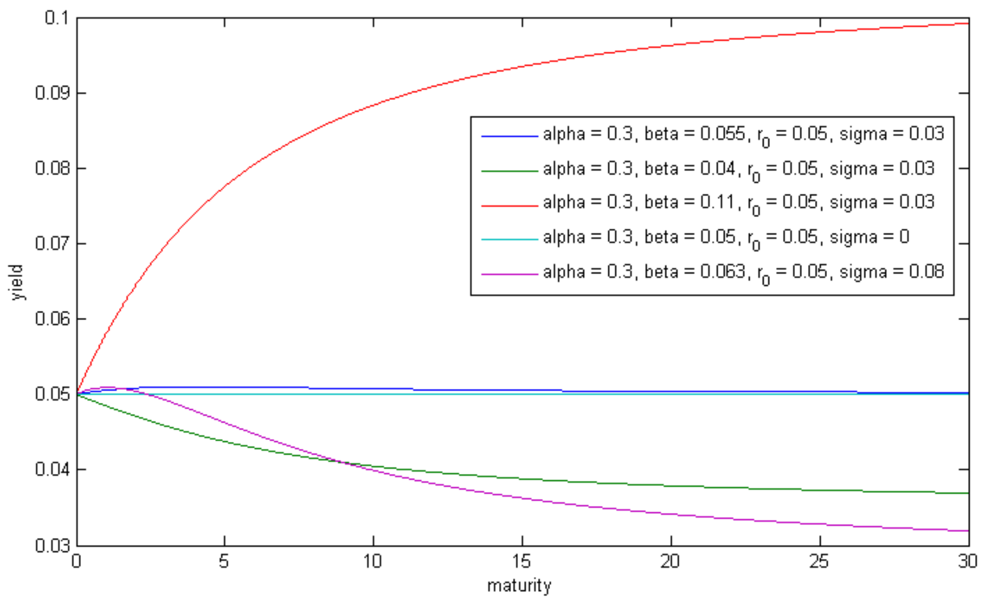

Figure 3.15: five virtual term structures

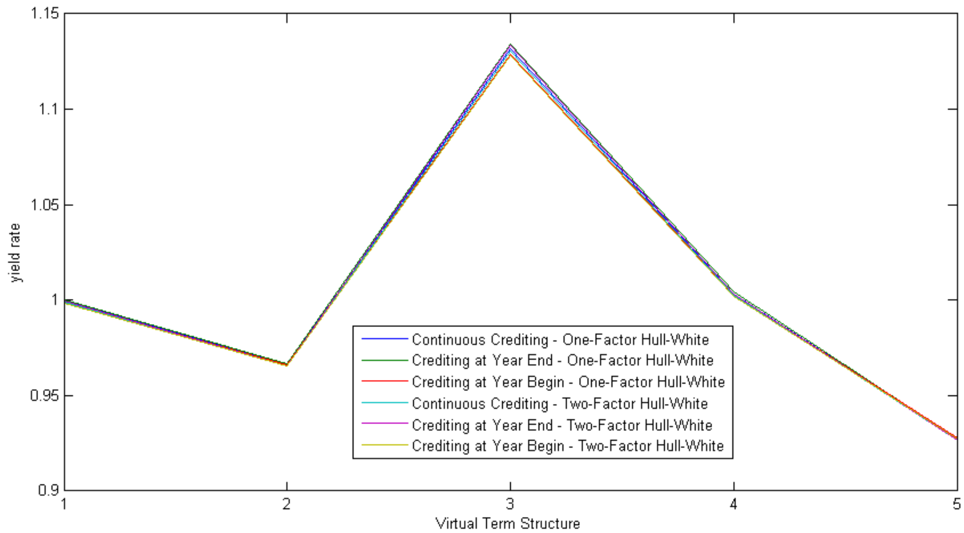

Figure 3.16: liability value for virtual term structure using 30-yr spot rate and 5-yr horizon

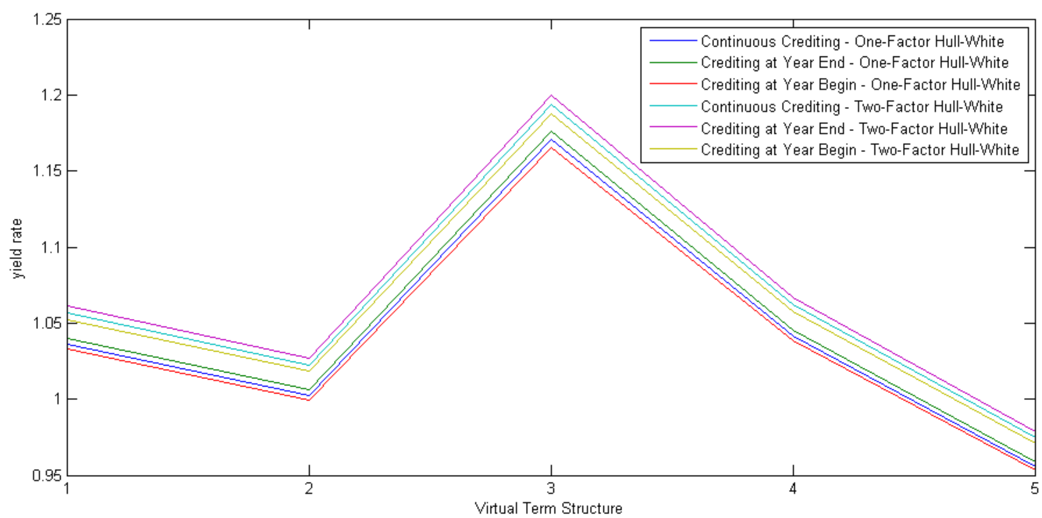

Figure 3.17: liability value for virtual term structure using 30-yr spot rate and 20-yr horizon

Chapter 4

Delta Hedging

4.1 Delta and Gamma - Black Scholes Model

The key assumption in market consistent valuation is that the price is obtained from the replicating portfolio instead of someone's subjective judgement. In the Black-Scholes framework, the pay-off of an European call/put option can be attained by a replicating portfolio consisting of only the underlying asset (stock) and the risk-free asset. The portfolio is immune to small changes in stock price, but the portfolio will need to be rebalanced continuously. The position in the stock has a special name "Delta" Δ_t, and is defined as the first order derivative of the price of the option with respect to the underlying stock value. Let $C(t, S_t)$ denote the price of the option at time t, and S_t as the stock price at t, then

$$\Delta_t = \frac{\partial C}{\partial S_t}(S_t, t)$$

The Delta of stock itself is 1. To accomplish Delta hedging, an investor holds the option and shorts Δ_t quantity of the underlying stock, with the rest invested in the risk free asset. The overall Delta of the portfolio is zero. As S and Δ vary continuously as time changes, the portfolio must remain at its "Delta neutral" position through dynamic hedging. Notice that investors are free to use other derivatives as the hedging instruments.

In practice where continuous hedging is impossible, Delta only tells us the rate of change in the option with respect to the price of the underlying asset, but we are also interested in the information on how fast Δ changes in order to determine how often we should re-balance our portfolio. Thus, we need another Greek, Γ_t

$$\Gamma_t = \frac{\partial^2 C}{\partial S_t^2}(S_t, t)$$

As shown, Γ_t tracks the changes in Δ_t. If $|\Gamma_t|$ is small, Δ changes slowly as stock prices change and the portfolio can be rebalanced infrequently. On the other hand, if Γ_t is large, Δ changes quickly and the position in stock needs to be rebalanced frequently, and investors may consider to expand their portfolios with another instrument to make the portfolio Gamma-neutral.

4.2 Delta Hedging - interest rate option, One-Factor Hull-White

As mentioned before, the CB liability studied in this paper can be viewed as a tradable asset with returns equal to the crediting rates. However, the Delta hedging method under the Black-Scholes framework cannot be applied directly to this asset because the underlying stochastic component, the interest rate, is not a tradable asset. Hence $\frac{\partial V(t,T)}{\partial r(t)}$ in this case is only a mathematically definition without any economic interpretation. Fortunately, this problem can be solved with an extra step. Here we consider zero coupon bonds as our hedging instruments due to the close relationship with the short rate. Theoretically, there are an infinite number of zero coupon bonds. The one with the same maturity as the pension liability (T) would be a natural choice. Since the Delta of the zero coupon bond price is not one, we need to find both the derivative of our liability function and zero coupon bond price with respect to the short rate. The ratio of these two derivatives will be our Delta in the hedge portfolio, or the number of zero coupon bonds to be held. We use the pricing formula presented in Chapter 2 for the zero coupon bonds and the CB liability to calculate their Deltas

$$\frac{\partial P(t,T)}{\partial r(t)} = -B(t,T)P(t,T)$$
$$\frac{\partial V(t,T)}{\partial r(t)} = -\gamma B(t,T)V(t,T)$$
$$\Delta_t = \frac{\partial V(t,T)}{\partial P(t,T)} = \frac{\gamma V(t,T)}{P(t,T)}$$

Thus, in order to hedge one unit of liability, we need Δ_t units of the zero coupon bond with maturity T and the remaining amount $(\Delta_t P(t,T) - V(t,T))$ in the bank account with continuous return r(t).

Gamma for the liability would be defined as the derivative of Δ_t with respect to the short rate. However, to access the sensitivity of Δ, it is more natural to consider the derivative of Δ_t with respect to the hedging instrument, P(t,T), which will be

$$\begin{aligned}\frac{\partial \Delta_t}{\partial P(t,T)} &= \frac{\partial \frac{\gamma V(t,T)}{P(t,T)}}{\partial P(t,T)} = \frac{\gamma \Delta_t}{P(t,T)} - \frac{\gamma V(t,T)}{P(t,T)^2} = \frac{\gamma^2 V(t,T) - \gamma V(t,T)}{P(t,T)^2} \\ &= \frac{(\gamma - 1)\Delta_t}{P(t,T)}\end{aligned}$$

In terms of Gamma-hedging, we will look at the original definition of Gamma, which is

$$\Gamma_t^V = \frac{\partial^2 V(t,T)}{\partial r^2} = \gamma^2 B(t,T)^2 V(t,T)$$

To construct the Gamma-hedging portfolio, we need two different hedging instruments. In our case, two zero coupon bonds with maturities T_1 and T_2 respectively. Their Γ's would be

$$\begin{aligned}\Gamma_t^{T_1} &= \frac{\partial^2 P(t,T_1)}{\partial r^2} = B(t,T_1)^2 P(t,T_1) \\ \Gamma_t^{T_2} &= \frac{\partial^2 P(t,T_2)}{\partial r^2} = B(t,T_2)^2 P(t,T_2)\end{aligned}$$

By matching Δ and Γ of the liability using these two zero coupon bonds (solving two linear equations), we derive the position m_1 and m_2 for each bond

$$\begin{aligned}m_1 &= \frac{\gamma B(t,T)(\gamma B(t,T) - B(t,T_2))V(t,T)}{B_1(B_1 - B_2)P(t,T_1)} \\ m_2 &= \frac{\gamma B(t,T)(\gamma B(t,T) - B(t,T_1))V(t,T)}{B_2(B_2 - B_1)P(t,T_2)}\end{aligned}$$

4.3 Delta Hedging - interest rate option, Two-Factor Hull-White

As studied by Fan et al. [14], Gupta and Subrahmanyam [16] and Driessen et al. [13] in swaption and cap/floor markets, multi-factor models with more hedging instruments may improve the hedging performance significantly. In CB liabilities, the underlying factors are the short rate and crediting rate. It is thus intuitive to include another factor to construct our hedging portfolio. Fortunately, due to the tractability of the Two-Factor Hull-White model, the positions for each bond can be explicitly derived. The price of the zero coupon bond and plan liability have the following stochastic differential equations:

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \sigma_1 B(t,T,a_1)dW_1(t) - \sigma_2 B(t,T,a_2)dW_2(t)$$
$$\frac{dV(t,T)}{V(t,T)} = r(t)dt - \sigma_1\gamma_1 B(t,T,a_1)dW_1(t) - \sigma_2\gamma_2 B(t,T,a_2)dW_2(t)$$

Let T_1 and T_2 be the maturities of the zero coupon bonds, and their positions Δ_1 and Δ_2 are derived by matching the Brownian motion terms for the liability:

$$m_1 = \frac{\left(\gamma_1 B(t,T,a_1)B(t,T_2,a_2) - \gamma_2 B(t,T,a_2)B(t,T_2,a_1)\right)V(t,T)}{\left(B(t,T_1,a_1)B(t,T_2,a_2) - B(t,T_1,a_2)B(t,T_2,a_1)\right)P(t,T_1)}$$
$$m_2 = \frac{\left(\gamma_1 B(t,T,a_1)B(t,T_1,a_2) - \gamma_2 B(t,T,a_2)B(t,T_1,a_1)\right)V(t,T)}{\left(B(t,T_2,a_1)B(t,T_1,a_2) - B(t,T_1,a_1)B(t,T_2,a_2)\right)P(t,T_2)}$$

4.4 Simulation

In a frictionless market with no transaction costs, Delta hedging with continuous readjustment will eliminate the effect of diffusion in the underlying stochastic process, and thus the maturity benefit will be risk-free. In the real world, however, continuously hedging is practically infeasible and transaction costs do exist. To analyze the performance of Delta hedging, it is necessary to set a rebalancing interval in discrete time. Here we use annually, monthly, and weekly (and set monthly as the benchmark) rebalancing. As suggested in the previous chapter, crediting using different horizons leads to similar results. Thus, we use continuous crediting in our simulations. It is both evident and intuitive that as we increase the number of rebalancing points, the transaction costs will increase and the hedge error will decrease. Thus, to make a comprehensive comparison, it is necessary to incorporate statistics from both transaction costs and hedging losses. Let c be the transaction costs, which are assumed to be proportional to the amount of bonds traded. Let h be the rebalancing horizon and n be the total number of periods such that $n \times h = T$. Table 4.1 provides the measures we are using to analyze the performance of our hedging portfolio. For each measure, we calculate the empirical mean, 0.005% quantile and 99.5% quantile.

Cumulative Transaction Costs	$\sum_{t=1}^{n-1} c(\Delta_{th} - \Delta_{(t-1)h})P(th,T)$
Present Value of Transaction Costs	$\sum_{t=1}^{n-1} ce^{\int_0^{th} -r(s)ds}(\Delta_{th} - \Delta_{(t-1)h})P(th,T)$
Hedging Error at time t (e_t)	$V(t+h,T) - \Delta_t P(t+h,T) - (V(t,T) - \Delta_t P(t,T))e^{\int_t^{t+h} r(s)ds}$
Cumulative Absolute Hedging Error	$\frac{1}{n}\sum_{t=1}^{n} \lvert e_t \rvert$
Present Value of Hedge Error	$\sum_{t=1}^{n} e^{\int_0^{th} -r(s)ds} e_{th}$

Table 4.1: Statistics for hedging performance

This section provides simulation results on the hedging performance of the delta-hedging/delta-gamma-hedging portfolios. The assumptions are outlined as follow

- Starting term structure is chosen at 29 March, 2013.
- The liability horizons are 5 years, 10 years and 20 years.
- Crediting rate is same as the 30-year spot rate.
- The crediting rate are continuously accumulated.
- Hedging intervals are set as annually, monthly and weekly.
- We run 10,000 simulations for each scenario.
- Parameters are the same as described in previous sections.
- Transaction cost c = 0.1%.
- The hedging instruments are the zero coupon bonds. The first has the same maturity as the liability, and the second one is 30-year zero coupon bond.

Delta and Gamma, One-Factor Hull-White

Before we closely examine the performance of hedging strategies, it is logical to investigate Δ and Γ to have a reasonable overview of the performance of our portfolio. Figure 4.1 provides some sample paths of Delta and Gamma through simulation, and also their distributions by Fan chart (from 0.5% to 99.5%). The Confidence Interval (C.I.) for Delta is increasing till the expiration date but with decreasing speed. The sample simulation clearly shows that the main source of variation comes from the beginning (especially around year 5 to year 10), and Delta becomes stable in the last few years. This is intuitively correct as time approaches to maturity, the price of the zero coupon bond converges to one, so the liability becomes less variable. The plot of Gamma again supports this observation. Gamma starts with a negative value and a downward trend in the early years, but eventually moves toward zero when t approaches maturity.

Figure 4.2 plots m_1 and m_2 for Delta-Gamma hedging, for maturity T=5 and 20 (monthly re-balancing). Notice, m_2, the position in the second zero coupon bond (30-year), is relatively small compared to m_1, the position in the first zero coupon bond (with the same maturity as the liability). For maturity T=5, the number of the second bond required is only one-tenth of the number of the first bond. This behavior is partially explained by the nature of the first instrument, since its value becomes less volatile when approaching the maturity. Moreover, it may also suggest that Delta hedging maybe sufficient and introducing a new hedging instrument may not be effective.

For the Two-Factor Hull-White model, Figure 4.3 plots m_1 and m_2 for Delta hedging, for maturity T=5 and 20 with monthly re-balancing. Notice the position of each hedging asset is greater in magnitude, directly suggesting a higher transaction costs needed. Although the position of the second instrument preserves the same behavior as the second instrument in Delta-Gamma hedging in the One-Factor case, it has a relatively higher weight when the horizon is short. For example, in the 5-year case, the initial position of the second instrument is only about one-tenth of the first instrument in the One-Factor case, but more than 40% in the Two-Factor case.

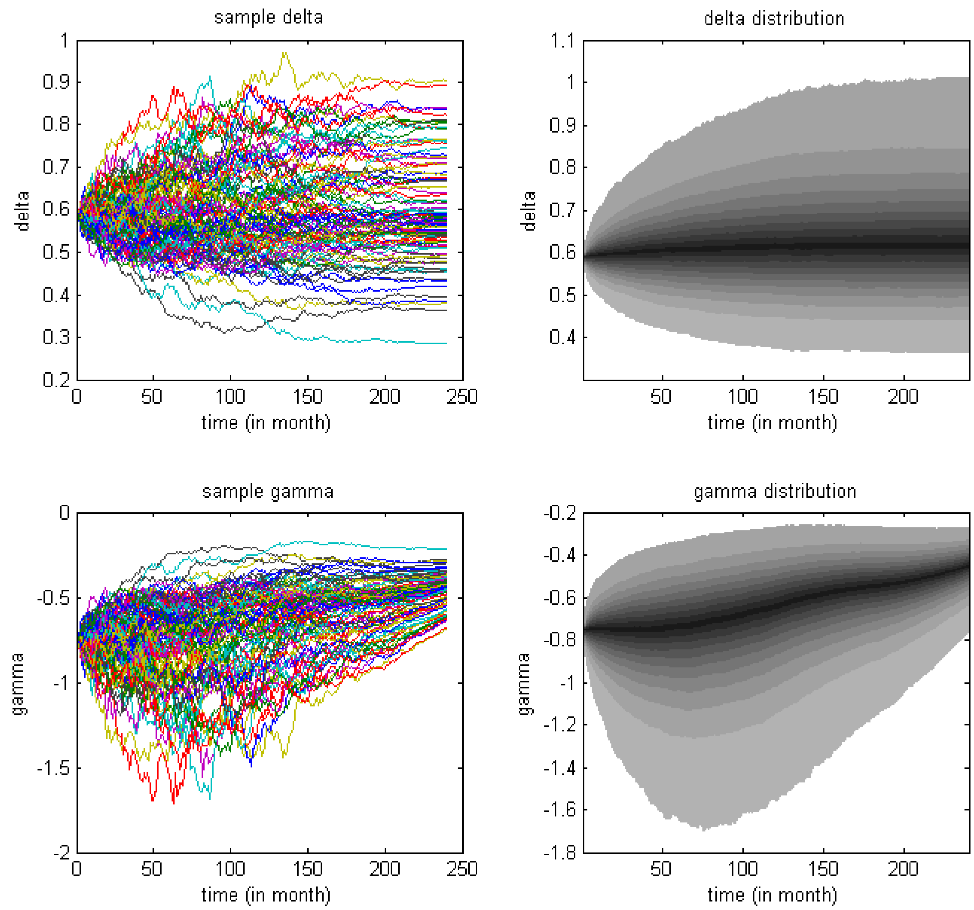

Figure 4.1: Delta and Gamma for the One-Factor Hull-White simulation, k=30, T=20, hedge freq.=monthly

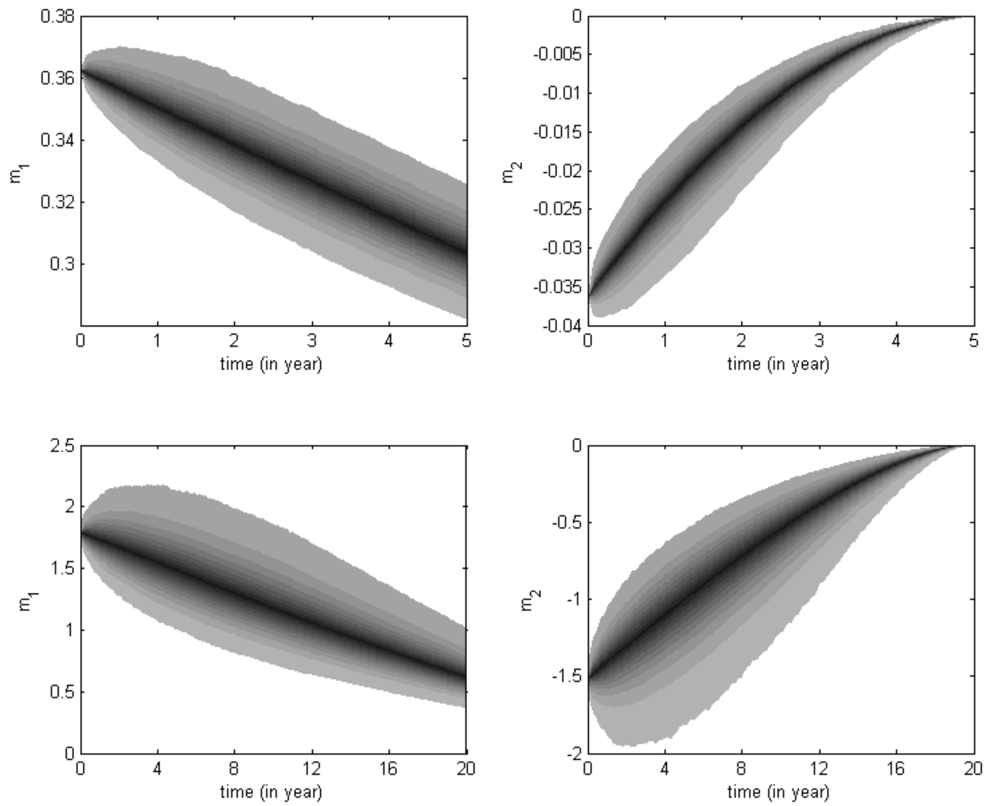

Figure 4.2: m_1 and m_2 for Delta-Gamma Hedging, the One-Factor Hull-White simulation, k=30, T=5 and 20, hedge freq.=monthly

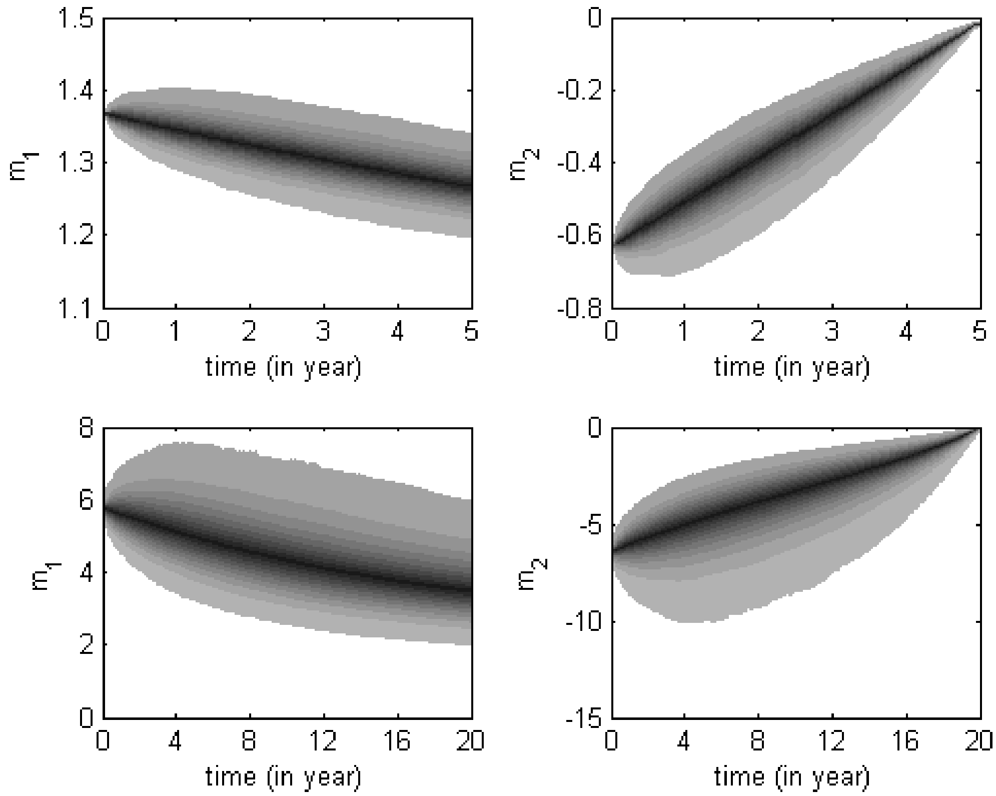

Figure 4.3: m_1 and m_2 for Delta Hedging, Two-Factor Hull-White simulation, k=30, T=5 and 20, hedge freq.=monthly

Hedge Loss and Transaction Cost - Delta Hedging with One-Factor HW

This section explores the distribution of hedge loss of different dynamic hedging strategies. The derivation follows Boyle and Emanuel [5]. Given the current time t and setting the rebalancing interval h to be small, we investigate the distribution of hedge loss (HL) at time t+h using the One-Factor Hull-White model by neglecting higher orders of time steps (ex. $h^{3/2}$).

$$HL = \frac{1}{2}B(t,T)^2V(t,T)\sigma^2h\left[(\gamma^2-\gamma)(u^2-1)\right]$$
$$u = \frac{W^Q(t+h)-W^Q(h)}{\sqrt{h}}$$

where B(t,T), σ, and γ are the same as before, and u is a standardized normal distribution. Hence u^2 is a chi-square distribution with degree 1. Details of the derivation are provided in Appendix C.

It is easy to show that $\gamma^2-\gamma$ is negative, and u^2, $B(t,T)^2$, $V(t,T)$ and σ^2 are always positive. As such, the distribution of HL will have a negative skewed shape with mean equal to zero, and the probability of being negative is roughly 32% ($Pr(|u|>1)$). The hedge will result in positive gain on relatively fewer occasions, but the magnitude of gain can be relatively large. In addition, it is clear as $t \to T$, $B(t,T) \to 0$ and the hedge loss eventually stabilizes as liabilities approaches to the maturity. Figure 4.4 shows the volatility of the hedge loss is decreasing as t increases (the jagged shape comes from the linear interpolation of the forward curve), and the large negative hedge loss implies the distribution is left skewed.

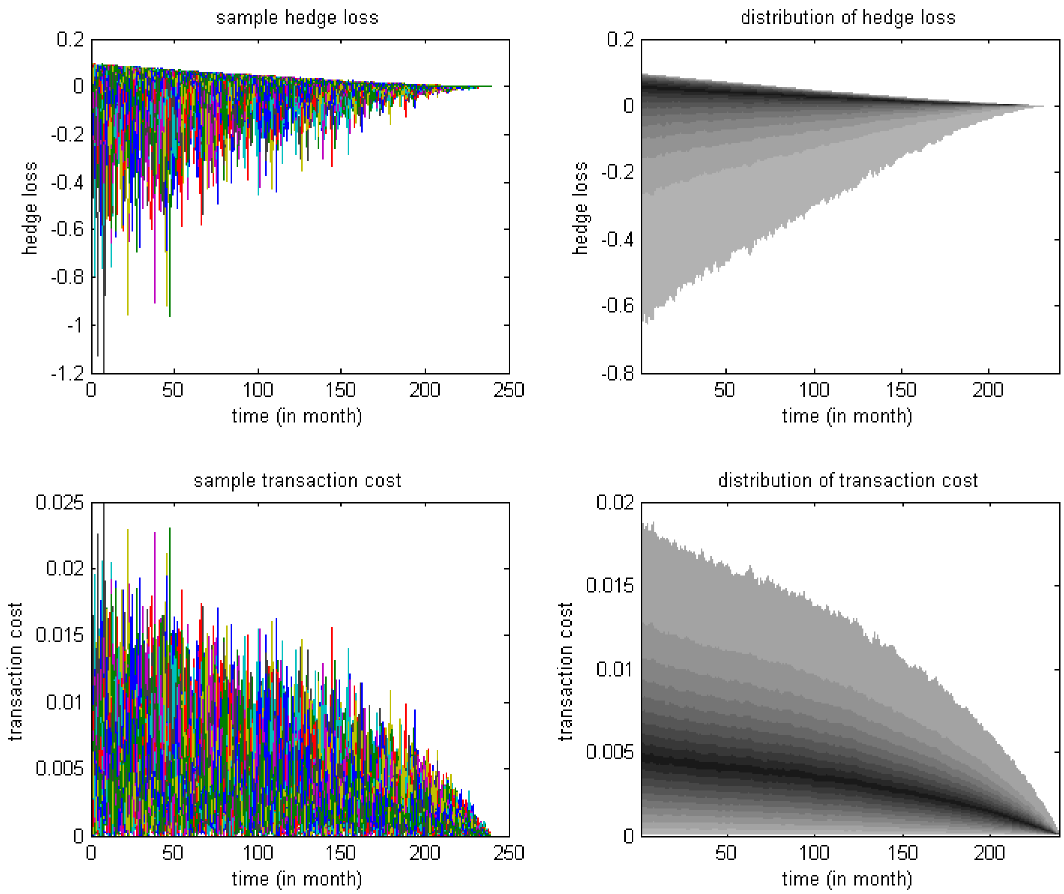

Figure 4.4: Hedge Loss and Transaction Costs, Delta Hedging, HW, T=20, k=30, starting 29 March, 2013

There is a direct relation between hedge loss and transaction cost ($s(t)$), as we ignore higher order terms of h. Let c denotes the transaction cost, which is a percentage of the transaction amount in the zero coupon bonds.

$$s(t+h) = c\gamma(|HL(t+h) + (V(t,T) - \Delta_t P(t,T)(1 + r(t)h)|)$$

The absolute value sign indicates that transaction cost is a right-skewed distribution. Figure 4.4 provides some samples and distributions of transaction cost plot for monthly rebalancing with a 20-year maturity. This is more strong empirical evidence supporting the aforementioned property as Delta approaches stability near the maturity. We have smaller rebalancing amounts, and thus lower transaction costs required.

Tables 4.2 and 4.3 provide the distribution statistics for the hedge loss and transaction costs with different hedging strategies. Keep in mind that our initial notional account is $1000. Since we simulate short rates using the same model as we used to construct the hedging portfolio, it is not surprising that the hedging errors are relatively small and have a decreasing trend as the hedging horizon becomes shorter, few observations are made:

- Mean HL are insignificant compared to the fund value.
- 99.5% percentile of HL can be as large as 6.4% of the initial fund value (20 year horizon with annual rebalance).
- HL is left-skewed (as absolute of 0.5% percentile is greater than 99.5% percentile)
- Cumulative HL increases as the horizon increases.
- Cumulative HL decreases as the hedge frequency increases.
- Transaction costs increase as the horizon increases or the hedge frequency increases.
- Transaction costs can be larger than the hedge loss when we rebalance the portfolio weekly.
- Transaction costs are less volatile.
- Rebalancing at weekly interval does not reduce the overall costs compare to rebalancing at monthly interval.

	Maturity	Hedge Freq.	Mean	Std.	0.5%	99.5%
Cum. HL	5	Annual	0.00012	0.11984	-0.51981	0.13744
		Month	-0.00001	0.03466	-0.10916	0.06937
		Week	0.00294	0.01672	-0.04412	0.04218
	10	Annual	-0.00127	0.66519	-2.60312	0.98309
		Month	0.00127	0.19008	-0.55602	0.42524
		Week	0.01900	0.09265	-0.23636	0.24335
	20	Annual	-0.02523	3.65921	-13.02972	6.39545
		Month	0.00870	1.04131	-2.93610	2.45166
		Week	0.05218	0.50811	-1.36102	1.29911
Abs. HL	5	Annual	0.13928	0.08855	0.03155	0.57082
		Month	0.13899	0.02580	0.09075	0.22865
		Week	10.05897	0.07961	9.85617	10.26459
	10	Annual	1.10774	0.49066	0.43527	3.33165
		Month	1.09938	0.14022	0.81106	1.54860
		Week	43.43580	0.99231	41.01238	46.17909
	20	Annual	8.94989	2.71630	4.65559	19.52744
		Month	8.84808	0.76433	7.13228	11.11104
		Week	85.83797	5.99196	72.70191	104.11773
PV HL	5	Annual	0.00012	0.11898	-0.51857	0.13594
		Month	-0.00002	0.03453	-0.10874	0.06898
		Week	0.00014	0.01663	-0.04703	0.03893
	10	Annual	-0.00107	0.65047	-2.58727	0.95006
		Month	0.00117	0.18695	-0.55144	0.41613
		Week	0.00093	0.09087	-0.24920	0.22089
	20	Annual	-0.02071	3.40922	-12.25593	5.84188
		Month	0.00755	0.97772	-2.73939	2.30034
		Week	0.00504	0.47747	-1.32965	1.18914

Table 4.2: Statistics for Hedging Loss, Delta Hedging - One-Factor Hull-White model

	Maturity	Hedge Freq.	Mean	std.	0.5%	99.5%
Cum. Trans	5	Annual	0.01165	0.00462	0.00264	0.02580
		Month	0.04167	0.00458	0.03083	0.05393
		Week	0.08674	0.00466	0.07504	0.09868
	10	Annual	0.05024	0.01361	0.02034	0.09047
		Month	0.17389	0.01351	0.14028	0.21012
		Week	0.36194	0.01392	0.32638	0.39911
	20	Annual	0.22711	0.04216	0.13173	0.34792
		Month	0.77852	0.04697	0.66592	0.90574
		Week	1.61969	0.06193	1.47185	1.79139
PV Trans	5	Annual	0.01145	0.00456	0.00259	0.02551
		Month	0.04111	0.00455	0.03032	0.05311
		Week	0.08561	0.00465	0.07392	0.09792
	10	Annual	0.04741	0.01311	0.01884	0.08688
		Month	0.16535	0.01331	0.13219	0.20110
		Week	0.34436	0.01442	0.30841	0.38327
	20	Annual	0.19308	0.03836	0.11021	0.30965
		Month	0.66900	0.04767	0.55526	0.80427
		Week	1.39250	0.07178	1.21877	1.59447

Table 4.3: Statistics for Transaction Costs, Delta Hedging - One-Factor Hull-White model

Hedge Loss and Transaction Cost - Delta-Gamma Hedging with One-Factor HW

Under the Delta-Gamma hedging strategy, when ignoring higher order terms for h, the hedge error in each period is zero (see Appendix D). This provides theoretical support that the Delta-Gamma strategy should have a better performance than Delta hedging. However, even when we set the time step very small, higher order terms do still have an effect on the hedging performance, and thus the hedge loss will never be zero. Figure 4.5 provides the distribution of hedge loss and transaction costs for the monthly hedge under the Delta-Gamma hedging strategy. Notice immediately that the hedge loss is relatively smaller in size and approximately symmetric in distribution. On the other hand, the transaction costs increased significantly, which may offset the benefit of reducing the hedge loss by introducing a new hedge instrument. Table 4.4 provides statistics for hedge loss and transaction costs for Delta-Gamma hedging using the One-Factor Hull-White model. Notice that Delta-Gamma hedging provides more stable hedging losses compared to the Delta hedging strategy but is closer in mean values. The improvement in hedging performance is significant on the tail when the hedging interval is small and the liability horizon is long. For shorter liability horizon or longer rebalancing time, the improvement is rather marginal. The rather small benefit of adopting the Gamma portfolio, in terms of hedging error, is offset by the increasing transaction costs due to the introduction of new instruments. One can easily see that transaction costs incurred by adopting the Gamma portfolio is constantly larger, and often exceeds the cumulative hedge loss. Together, we can conclude that when short rates dynamic follow the One-Factor Hull-White model, there is no incentive to use the more complicated Gamma-neutral portfolio.

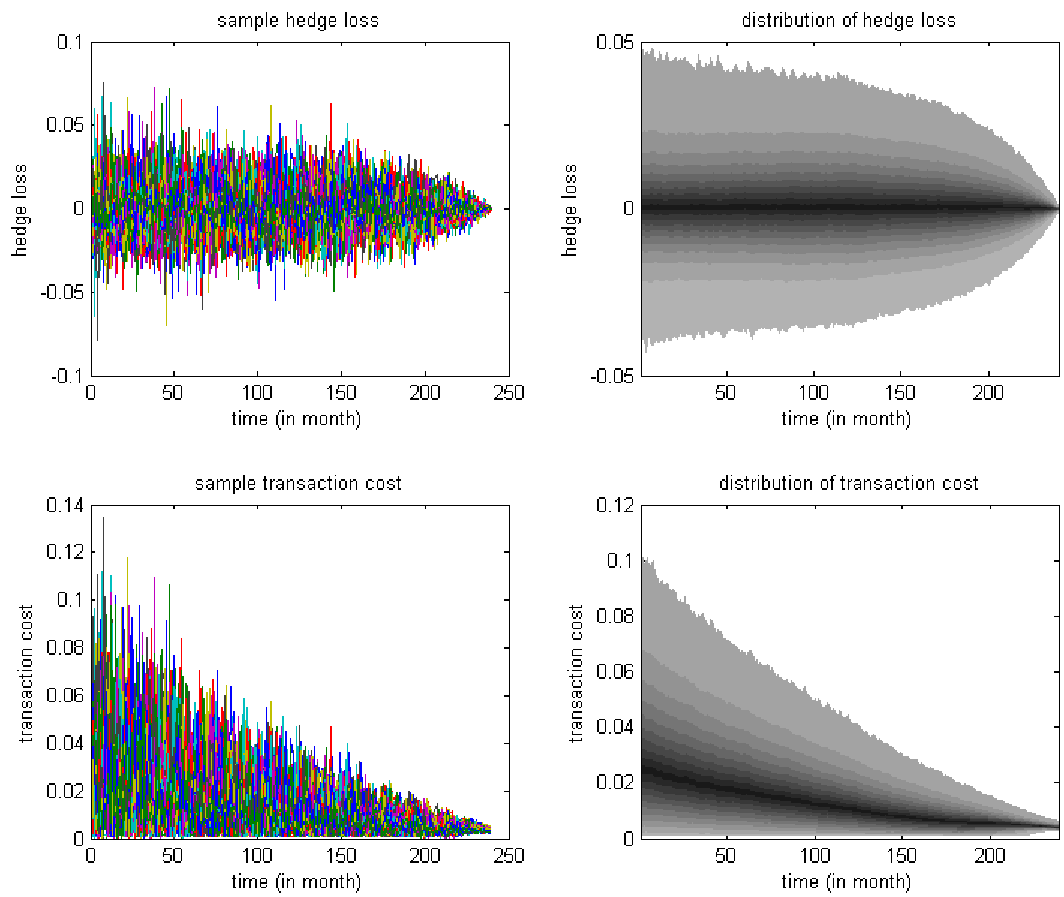

Figure 4.5: Hedge Loss and Transaction Costs, Delta-Gamma Hedging, HW, T=20, k=30, starting 29 March, 2013

	Maturity	Hedge Freq.	Mean	Std.	0.5%	99.5%
Cum. HL	5	Annual	-0.00143	0.23785	-0.54092	0.69206
		Month	-0.00002	0.01733	-0.04541	0.04459
		Week	0.00284	0.00305	-0.00482	0.01075
	10	Annual	0.00227	0.68544	-1.63112	1.97153
		Month	0.00048	0.05188	-0.13225	0.13453
		Week	0.01812	0.00718	0.00003	0.03657
	20	Annual	-0.00725	2.35775	-6.04921	6.24523
		Month	0.00151	0.18487	-0.49412	0.46006
		Week	0.04659	0.02821	-0.02569	0.11849
Abs. HL	5	Annual	0.38447	0.14892	0.09822	0.88456
		Month	0.09352	0.01059	0.06767	0.12267
		Week	10.05919	0.07764	9.86241	10.26044
	10	Annual	1.55867	0.43259	0.66174	2.97898
		Month	0.40649	0.03276	0.32378	0.49568
		Week	43.43683	0.98846	41.02134	46.17967
	20	Annual	7.87190	1.50552	4.47558	12.37762
		Month	2.13749	0.14296	1.80607	2.54658
		Week	85.46619	5.98326	72.41618	103.55399
PV HL	5	Annual	-0.00057	0.23431	-0.52338	0.69334
		Month	-0.00002	0.01717	-0.04429	0.04473
		Week	0.00004	0.00417	-0.01049	0.01098
	10	Annual	0.00495	0.65226	-1.47506	1.97676
		Month	0.00043	0.04998	-0.12451	0.13220
		Week	0.00009	0.01321	-0.03332	0.03465
	20	Annual	0.00949	1.99288	-4.46534	5.92345
		Month	0.00152	0.15846	-0.37571	0.43790
		Week	0.00014	0.04060	-0.09804	0.11392

Table 4.4: Statistics for Hedging Loss, Delta-Gamma Hedging - One-Factor Hull-White model

	Maturity	Hedge Freq.	Mean	std.	0.5%	99.5%
Cum. Trans	5	Annual	0.05663	0.01025	0.03175	0.08353
		Month	0.08367	0.00722	0.06555	0.10288
		Week	0.13146	0.00646	0.11545	0.14857
	10	Annual	0.20956	0.03468	0.12712	0.30601
		Month	0.36547	0.02492	0.30418	0.42977
		Week	0.65942	0.02458	0.59781	0.72460
	20	Annual	1.48362	0.18096	1.05800	2.00419
		Month	3.10679	0.16518	2.69866	3.54541
		Week	5.94743	0.18367	5.47909	6.45552
PV Trans	5	Annual	0.05598	0.01084	0.03016	0.08493
		Month	0.08206	0.00780	0.06257	0.10340
		Week	0.12943	0.00692	0.11247	0.14778
	10	Annual	0.19547	0.03974	0.10686	0.30811
		Month	0.34547	0.02882	0.27583	0.42313
		Week	0.62907	0.02867	0.55851	0.70738
	20	Annual	1.22951	0.25096	0.69179	2.00139
		Month	2.71256	0.21883	2.19213	3.33546
		Week	5.27531	0.27876	4.60347	6.05107

Table 4.5: Statistics for Transaction Costs, Delta Gamma Hedging - One-Factor Hull-White model

Hedge Loss and Transaction Cost - Delta Hedging with Two-Factor HW

The distribution of hedge loss under the Two-Factor Hull-White is less clear, Appendix D shows that the hedge loss is an affine function of three correlated noncentral chi-squared distributions, which suggests existence of skewness (but unclear on whether it is positive skewed or negative skewed). Figure 4.6 plots some sample hedge loss and transaction costs and their distributions for monthly rebalancing a 20-year CB liability. Hedge loss under this case is mostly right skewed, but with median below zero. Tables 4.6 and 4.7 provide statistics for hedge loss and transaction costs for Delta hedging using the Two-Factor Hull-White model. The overall trend is similar to the One-Factor Hull-White model (cumulative hedge losses increase as the horizon increases, decrease as hedge frequency increases, etc.). The hedge errors are, in general, much greater than the one-factor case, and less stable. Keep in mind that we use the same model for the simulation and the construction of the hedging portfolio. It may not be exact but intuitively speaking, delta hedging under Two-

Factor Hull-White can be seen as a linear approximation to a quadratic function. As shown in Appendix D, when ignoring higher powers of h, Delta hedging in Two-Factor Hull-White can not eliminate all its second-order partial derivatives against its two stochastic factors, which consist of four terms $\frac{\partial^2 V(t,T)}{\partial x(t)^2}$, $\frac{\partial^2 V(t,T)}{\partial y(t)^2}$, $\frac{\partial^2 V(t,T)}{\partial x(t)y(t)}$ and $\frac{\partial^2 V(t,T)}{\partial y(t)x(t)}$, whereas Delta hedging under One-Factor Hull-White ignores only one term $\frac{\partial^2 V(t,T)}{\partial r(t)^2}$. Lastly for the transaction cost, the results from Table 4.7 are the same as suggested in Figure 4.3, since the position of each hedging instrument is large, so is the transaction amount required at each rebalancing time. It is interesting to notice that in term of the overall costs, rebalancing at weekly interval does not provide advantage compare to rebalancing at monthly interval.

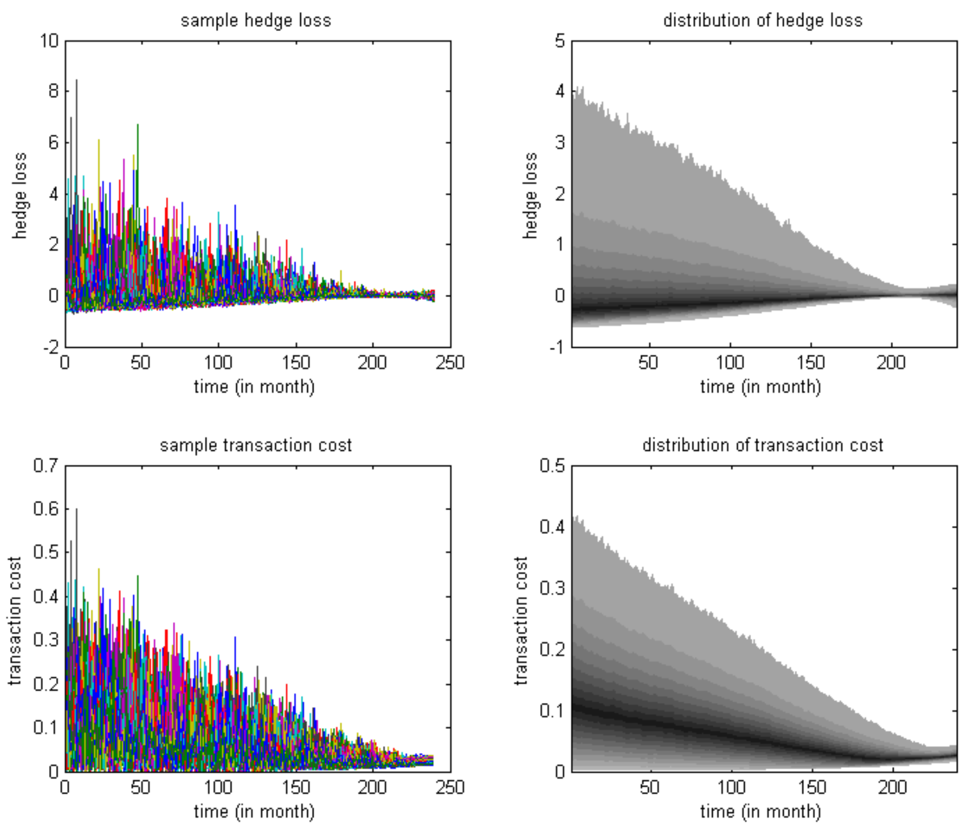

Figure 4.6: Hedge Loss and Transaction Costs, Delta Hedging, Two-Factor Hull-White model, T=20, k=30, starting 29 March, 2013

	Maturity	Hedge Freq.	Mean	Std.	0.5%	99.5%
Cum. HL	5	Annual	0.01330	4.55570	-9.72805	14.67625
		Month	-0.00001	0.68642	-1.52025	2.04911
		Week	0.00496	0.29713	-0.70186	0.83200
	10	Annual	0.14003	10.14166	-19.50107	35.53867
		Month	-0.01117	2.26032	-5.20216	6.56617
		Week	0.01271	1.07220	-2.59532	2.97290
	20	Annual	0.36486	26.75531	-52.17324	89.78127
		Month	-0.04606	6.85945	-16.43885	19.30821
		Week	0.04398	3.31399	-8.11122	9.13800
Abs. HL	5	Annual	7.51931	3.00878	2.18134	19.06372
		Month	3.36190	0.48768	2.35777	4.94431
		Week	28.20261	0.90796	25.95334	30.63143
	10	Annual	20.81922	7.03561	8.82254	49.47529
		Month	14.04736	1.75119	10.39289	19.49206
		Week	65.77260	4.33254	55.54290	78.11998
	20	Annual	70.45452	19.18196	36.39332	143.73593
		Month	59.23206	6.33607	45.65166	78.54302
		Week	186.58909	21.66104	139.25043	252.17424
PV HL	5	Annual	-0.01693	4.43692	-9.74818	14.24737
		Month	-0.00029	0.68110	-1.50862	2.04128
		Week	-0.00276	0.29470	-0.70422	0.82050
	10	Annual	0.05651	9.70589	-18.90431	33.77252
		Month	-0.01195	2.22034	-5.08972	6.52193
		Week	-0.01264	1.05335	-2.57316	2.92474
	20	Annual	0.20256	24.60599	-47.72093	84.37468
		Month	-0.04699	6.45338	-15.74675	18.57060
		Week	-0.04263	3.11845	-7.80343	8.45304

Table 4.6: Statistics for Hedging Loss , Delta Hedging - Two-Factor Hull-White model

	Maturity	Hedge Freq.	Mean	std.	0.5%	99.5%
Cum. Trans	5	Annual	0.24439	0.04718	0.13696	0.36984
		Month	0.42017	0.03307	0.34112	0.50798
		Week	0.69802	0.03555	0.61311	0.79613
	10	Annual	0.83386	0.13736	0.49860	1.20931
		Month	1.83933	0.14616	1.50978	2.27543
		Week	3.48580	0.22381	2.96583	4.14895
	20	Annual	5.25641	0.69842	3.69765	7.41245
		Month	13.74721	1.23407	11.01663	17.44236
		Week	27.19674	2.19390	22.24137	33.60348
PV Trans	5	Annual	0.28064	0.03967	0.18803	0.38480
		Month	0.41580	0.03175	0.34249	0.50562
		Week	0.68850	0.03918	0.59814	0.80066
	10	Annual	0.82759	0.11565	0.55364	1.19090
		Month	1.75117	0.16962	1.39750	2.28756
		Week	3.33747	0.28791	2.68799	4.19138
	20	Annual	4.55060	0.66676	3.18679	6.82941
		Month	12.09056	1.44037	9.05203	16.61837
		Week	24.18352	2.74660	18.23898	32.49188

Table 4.7: Statistics for Transaction Costs , Delta - Two-Factor Hull-White model

4.5 Hedging Performance with Real Data

When the simulation is generated from the same model we used to construct the hedging portfolio, the Delta hedging strategy works well to hedge the CB liabilities. However, it is necessary to test its performance using the historical data. Here we collected the data from 1989 to 2015 for Treasury Strips from Bloomberg, assuming the account is crediting with the 30-year spot rate. Limited by the length of our data set and the number of tradable treasury bonds, we only look at 5-year and 10-year horizons. We have tested the hedging performance with hedge horizon set as annually, monthly and weekly, and we set the crediting horizon equal to the hedge horizon to ease the computation. All other assumptions remain the same. Figures 4.7 and 4.8 plot the cumulative hedging loss and cumulative transaction cost for a plan with starting account balance of $1000.

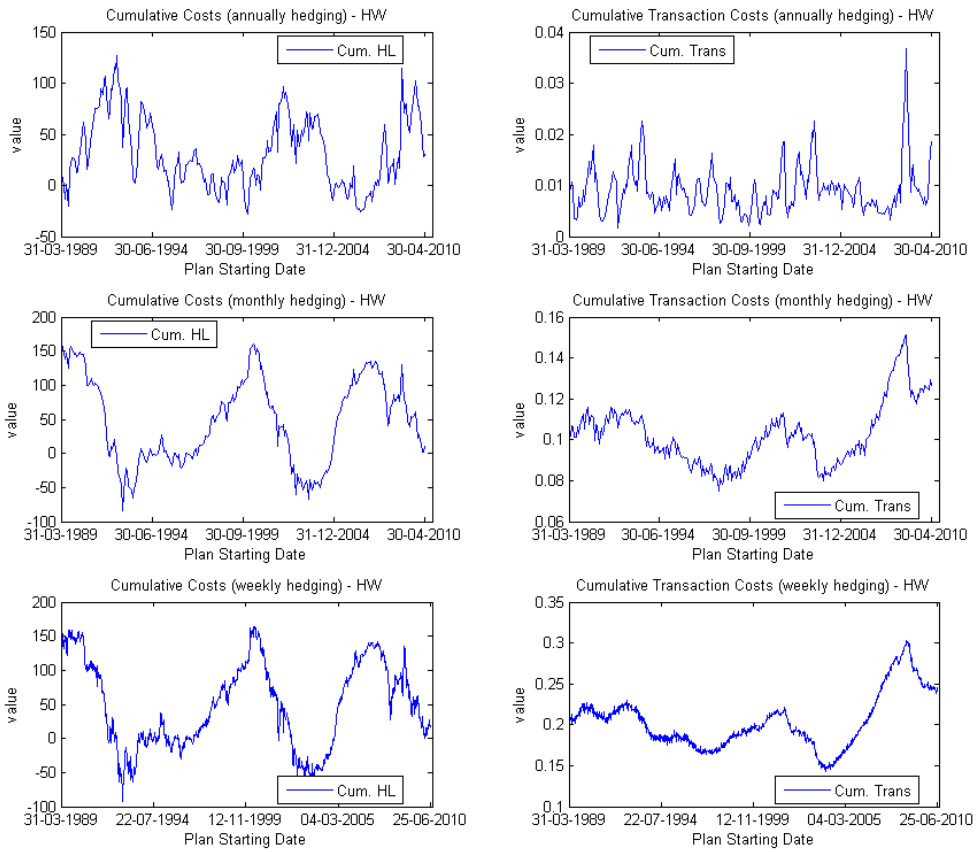

Figure 4.7: Performance of Delta hedging, 5-year horizon, using 30-year spot rate crediting annually, One-Factor Hull-White model

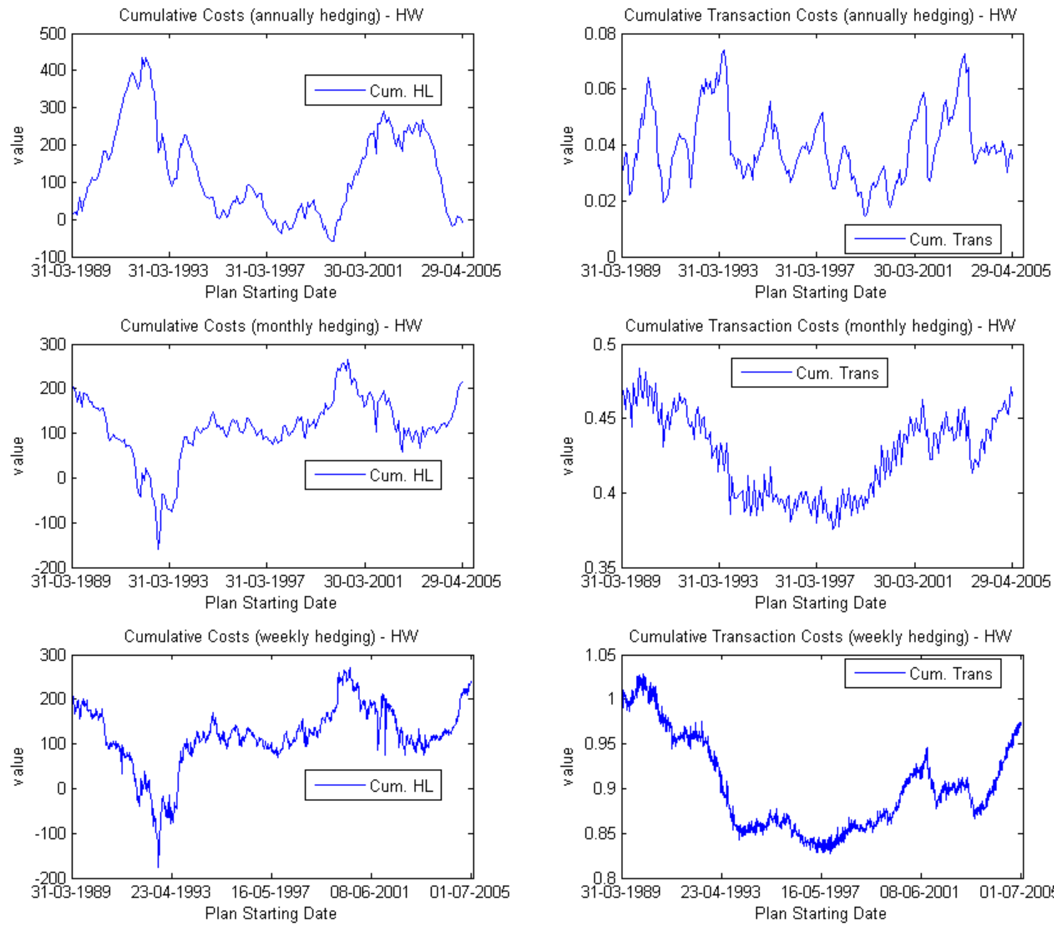

Figure 4.8: Performance of Delta hedging, 10-year horizon, using 30-year spot rate crediting annually, One-Factor Hull-White model

Clearly, the real performance of the Delta hedging strategy is not as promising as with simulations from the Hull-White model. In the One-Factor Hull-White, for a five year plan, the hedging loss can be as high as 15% of the plan's initial value and for a ten year plan, the hedging loss may reach 25% of initial value. When comparing the hedging loss with the slope of the yield curve, there is a significant negative relationship (more subtle for a ten year plan). Evidently, when the interest rate is high, using the One-Factor Hull-White model tends to overestimate the liabilities and vice versa. Where most of the plans ended with a loss, a few of them (starting at year 1992 and at 2002) actually make some gains. This poor performance of the delta hedge is partially explained by the fact that the One-Factor Hull-White model does not capture the real movements of the interest rate, as suggested by the fact that increasing hedge frequency does not lead to a better result. Notice that the transaction cost remains small in the simulations, it is negligible compared to the hedge errors. Figures 4.9 and 4.10 plot the cumulative hedging loss and cumulative transaction cost for the Delta-Gamma portfolio. Similar results are obtained from the simulation. Adding an extra instrument has little benefit, which confirms that the poor hedging strategies are due to the capability of our model. The transaction cost involved using the delta-gamma strategy is higher than using the delta strategy, which agrees with the simulation result. Looking at the hedging performance of the Two-Factor Hull-White model in Figures 4.11 and 4.12, although hedging performance is not improved in the annually rebalanced case, it does have a huge improvement when hedge the horizon becomes shorter. For the monthly and weekly hedge with 5-year horizon, the hedging errors range from 0 to 50 (5%), which is more accurate compared to the One-Factor Hull-White case. The transaction cost also increased substantially, but the overall improvement is still significant. Finally, similar to the result in the simulations, increasing the rebalancing frequency from monthly to weekly does not decrease the cost.

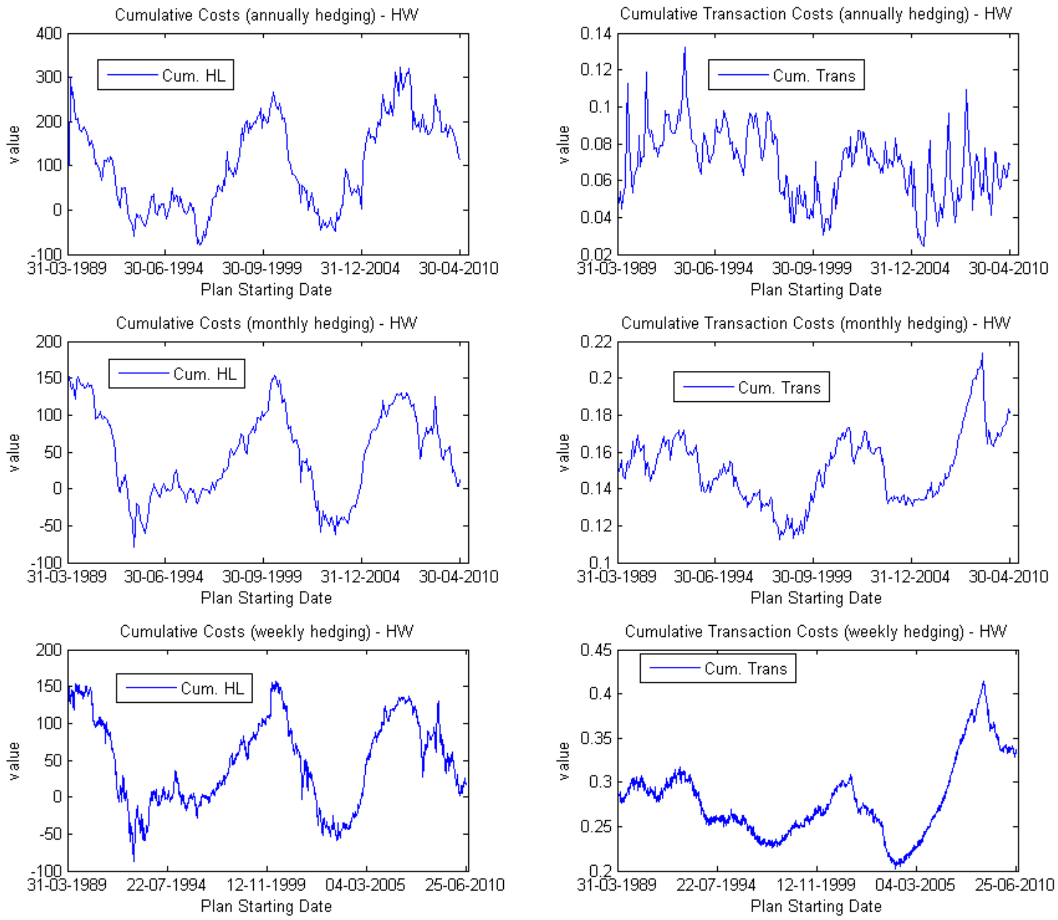

Figure 4.9: Performance of Delta-Gamma hedging, 5-year horizon, using 30-year spot rate crediting annually, One-Factor Hull-White model

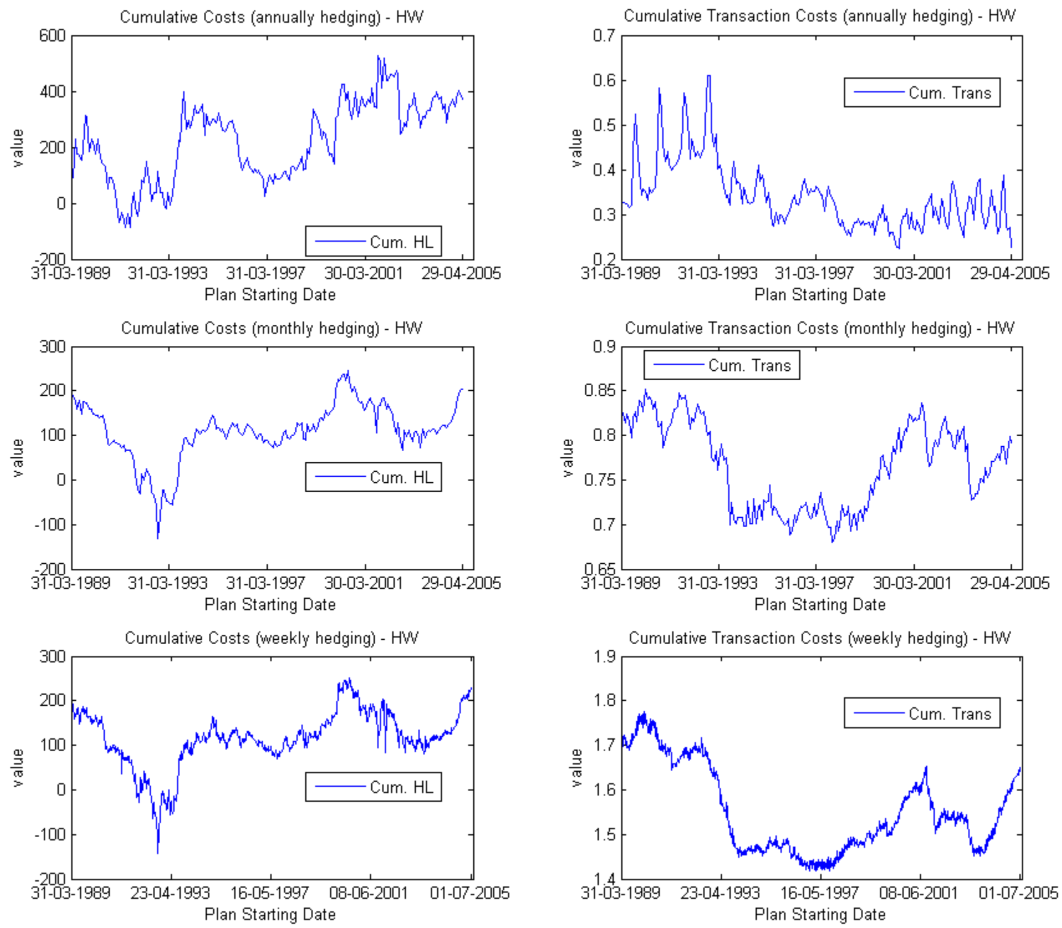

Figure 4.10: Performance of Delta-Gamma hedging, 10-year horizon, using 30-year spot rate crediting annually, One-Factor Hull-White model

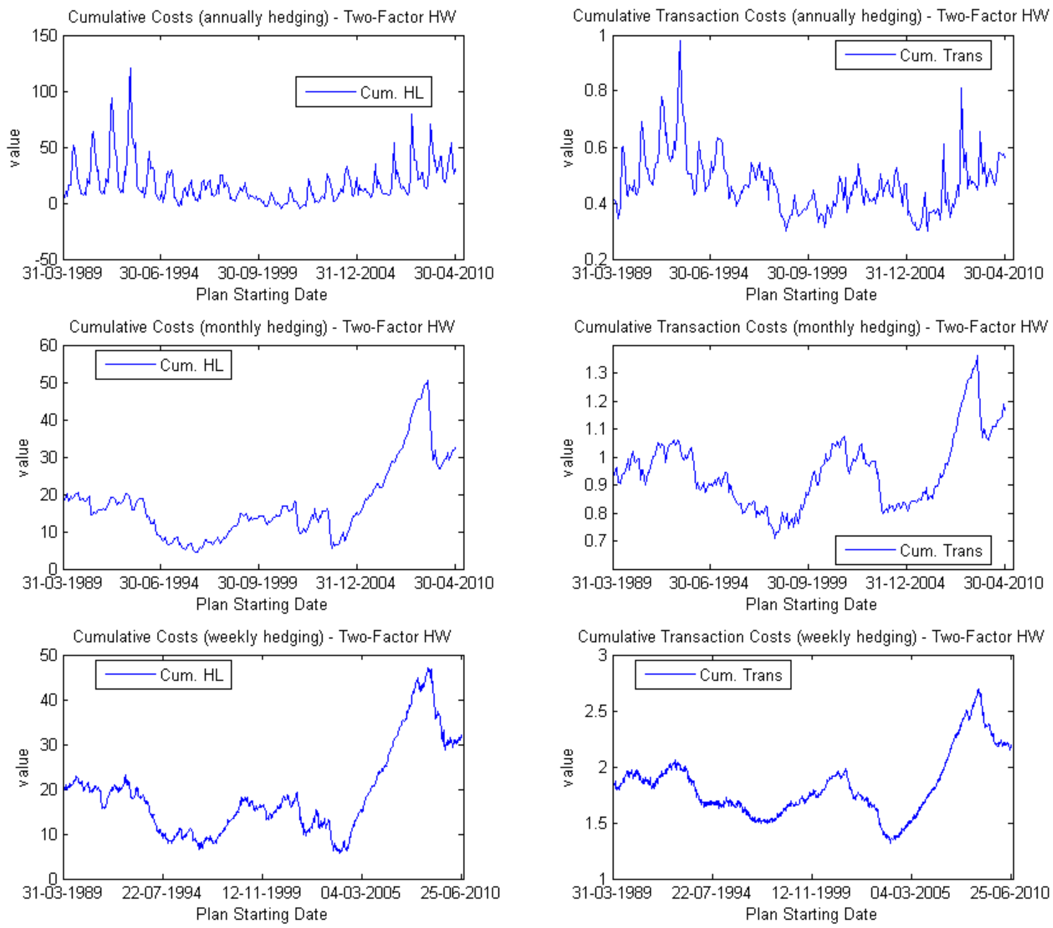

Figure 4.11: Performance of Delta hedging, 5-year horizon, using 30-year spot rate crediting annually, Two-Factor Hull-White model

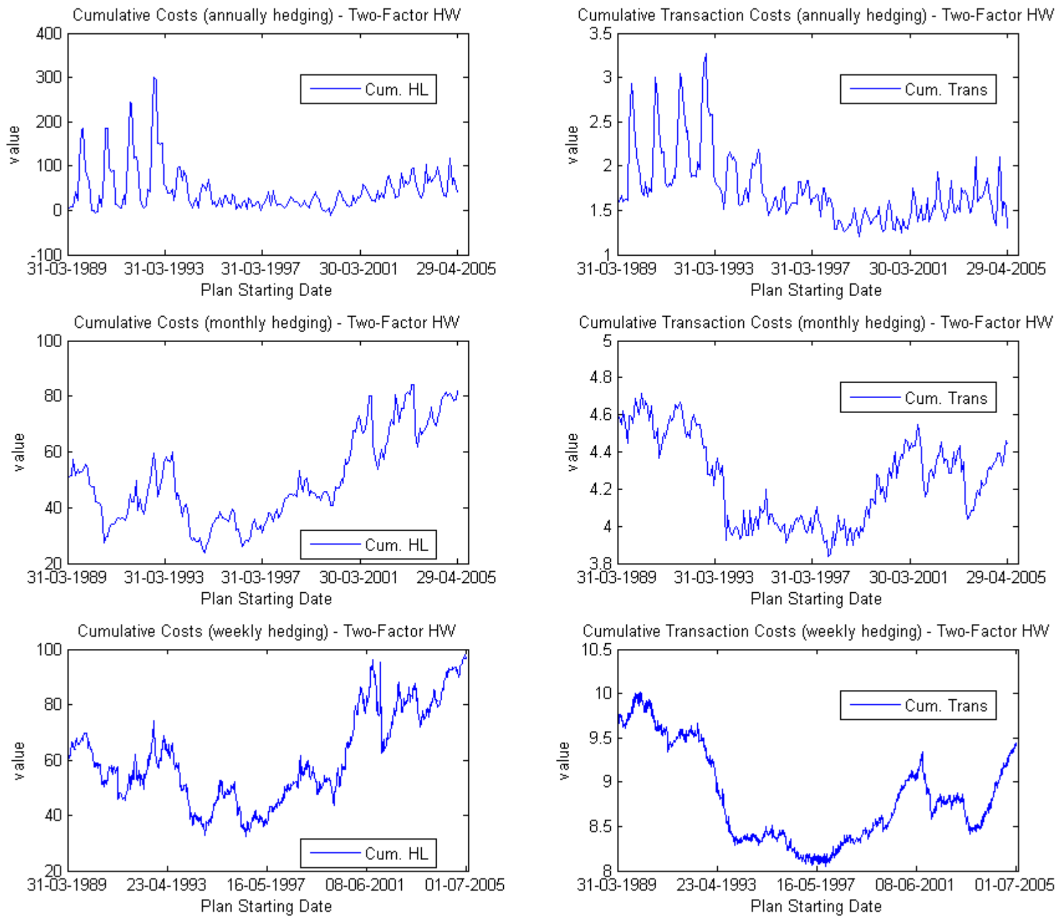

Figure 4.12: Performance of Delta hedging, 10-year horizon, using 30-year spot rate crediting annually, Two-Factor Hull-White model

Chapter 5

Conclusion

The main results of this thesis are:

- Liability valuation under continuous crediting is close to discrete crediting.
- Liability valuation using Two-Factor Hull-White model is close to One-Factor Hull-White model.
- The initial term structure has a significant effect on the valuation factor, and the difference between valuation factors using different crediting patterns.
- The maturity of the liability and the duration of the crediting rate have large impacts on the difference between valuation factors using different crediting patterns.
- Using One-Factor Hull-White model for simulation and portfolio construction, Delta hedging and Delta-Gamma hedging strategies have outstanding performance.
- Using Two-Factor Hull-White model for simulation and portfolio construction, Delta hedging portfolio is not sufficient to capture all the variations.
- Using real data, Delta hedging using One-Factor Hull-White model has poor performance. Including another instrument to form a Delta-Gamma hedging strategy does not improve the result.
- Using real data, Delta hedging with Two-Factor Hull-White model is not satisfactory but significantly improves the hedging performance.

- Using real data, transaction cost is relatively small compare to the hedge loss, but its importance may increases with the number of hedging instruments or the accuracy of the model.

Future work on Cash Balance Pension plans may include

- Exploring dynamic hedging strategies using multi-factor term structure models.
- Exploring static or semi-static hedging strategies.
- Pricing and Hedging for CB plans with caps and floors.

Appendix A

Fitting the Forward Curve

For the Hull-White model, in order to simulate the future short rate, we need the market instantaneous forward rate curve, at least on daily intervals for our purpose. However, we do not have the instantaneous forward rate data in the market, thus, we must derive them through the equality:

$$P(0,T) = e^{\int_0^T f(t)dt}$$

However, once again, the market does not have bond price for every maturity, the data we obtained are zero coupon bonds with 0.25,0.5,1,2,3,4,5,6,7,8,9,10,15,20 and 30 year maturities. Thus we need an interpolation method to fit the forward rate. Here we use the method presented in [12], which assumes piecewise linear function between each time interval (i.e. 0-0.25, 0.25-0.5,0.5-1,1-2,2-3,3-4,4-5,5-6,6-7,7-8,8-9,9-10,10-15,15-20,20-30).

$$f_i(t) = a_i t + b_i$$

Also, assume the short rate is the same as the three month treasury bill rate, and let the forward rates in first interval be constant, so that $b_1 = r(t)$ and $a_1 = 0$. Denoting t_i as the end point of the i^{th} interval, for example, $t_1 = 0.25$, we have:

$$\begin{aligned} f_{i+1}(t_i) &= f_i(t_i) \\ a_{i+1}t_i + b_{i+1} &= a_i t_i + b_i \\ b_{i+1} &= t_i(a_i - a_{i+1}) + b_i \end{aligned}$$

In the last line we have b_{i+1} in terms of a_i, a_{i+1} and b_i, substitute it into the expression for $P(0,T)$

$$
\begin{aligned}
P(0,t_{i+1}) &= \prod_{k=1}^{i} \exp\left\{-\int_{t_{k-1}}^{t_k} (a_k x + b_k) dx\right\} \exp\left\{-\int_{t_i}^{t_{i+1}} a_{i+1}x + b_{i+1}dx\right\} \\
&= P(0,t_i)\exp\left\{-\frac{a_{i+1}(t_{i+1}^2 - t_i^2)}{2} - b_{i+1}(t_{i+1} - t_i)\right\} \\
&= P(0,t_i)\exp\left\{-\frac{a_{i+1}(t_{i+1}^2 - t_i^2)}{2} - (t_i(a_i - a_{i+1}) + b_i)(t_{i+1} - t_i)\right\} \\
\frac{P(0,t_{i+1})}{P(0,t_i)} &= \exp\left\{\frac{-a_{i+1}(t_{i+1}^2 - t_i^2) + 2a_{i+1}t_i(t_{i+1} - t_i)}{2}\right\} \exp\left\{-b_i(t_{i+1} - ti)\right\} \exp\left\{-t_i a_i(t_{i+1} - ti)\right\} \\
a_{i+1} &= 2\frac{\log(\frac{P(0,t_{i+1})}{P(0,t_i)}) + b_i(t_{i+1} - ti) + t_i a_i(t_{i+1} - ti)}{2t_i t_{i+1} - t_i^2 - t_{i+1}^2} \\
b_{i+1} &= t_i(a_i - a_{i+1}) + b_i
\end{aligned}
$$

Thus, we can recursively calculate the forward curve.

Appendix B

One-Factor Hull-White Model Derivation

Continuously Crediting

Let $Z = \int_t^T r(u)du$. According to Brigo and Mercurio [8], (page 66)

$$Z \sim N\left(B(t,T)[r(t)-\varphi(t)] + \log\frac{P(0,t)}{P(0,T)} + \frac{1}{2}[\upsilon(0,T)-\upsilon(0,t)], \upsilon(t,T)\right)$$

$$\upsilon(t,T) = \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]$$

$$\varphi(t) = f(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2$$

let $Z_k = -\int_t^T \gamma r(u)du$ where $\gamma = 1 - \frac{B(t,t+k)}{k}$, since it is an affine function of Z, it is also normally distributed

$$Z_k \sim N\left(-\gamma\left\{B(t,T)[r(t)-\varphi(t)] + \log\frac{P(0,t)}{P(0,T)} + \frac{1}{2}[\upsilon(0,T)-\upsilon(0,t)]\right\}, \gamma^2\upsilon(t,T)\right)$$

and we can find the expectation of e^{Z_k}

$$\begin{aligned}
E\left[e^{-\int_t^T \gamma r(u)du}\right] &= exp\left\{-\gamma\left\{B(t,T)[r(t)-\varphi(t)] + \log\frac{P(0,t)}{P(0,T)} + \frac{1}{2}[\upsilon(0,T)-\upsilon(0,t)]\right\} + \frac{1}{2}\gamma^2\upsilon(t,T)\right\} \\
&= \exp\left\{-\gamma B(t,T)r(t) + \gamma f(0,t)B(t,T) + \gamma B(t,T)\frac{\sigma^2}{2a^2}(1-e^{-at})^2 - \gamma\log\frac{P(0,t)}{P(0,T)}\right. \\
&\qquad \left. -\gamma\frac{1}{2}[\upsilon(0,T)-\upsilon(0,t)-\gamma\upsilon(t,T)]\right\} \\
&= \exp\left\{-\gamma B(t,T)r(t) + \gamma f(0,t)B(t,T) - \gamma\log\frac{P(0,t)}{P(0,T)}\right. \\
&\qquad -\gamma\frac{\sigma^2}{2a^2}\left[T - 2\frac{1-e^{-aT}}{a} + \frac{1-e^{-2aT}}{2a} - t + 2\frac{1-e^{-at}}{a} - \frac{1-e^{-2at}}{2a}\right. \\
&\qquad \left.\left. -\gamma\left(T - t - 2\frac{1-e^{-a(T-t)}}{a} + \frac{1-e^{-2a(T-t)}}{2a}\right) - B(t,T)(1-2e^{-at}+e^{-2at})\right]\right\} \\
&= \exp\left\{-\gamma B(t,T)r(t) + \gamma f(0,t)B(t,T) - \gamma\log\frac{P(0,t)}{P(0,T)}\right. \\
&\qquad -\gamma\frac{\sigma^2}{2a^2}\left[(T-t)(1-\gamma) - 2e^{-at}\left(\frac{1-e^{-a(T-t)}}{a}\right) + e^{-2at}\left(\frac{1-e^{-2a(T-t)}}{2a}\right)\right. \\
&\qquad \left.\left. +\gamma\left(2\frac{1-e^{-a(T-t)}}{a} - \frac{1-e^{-2a(T-t)}}{2a}\right) - B(t,T)(1-2e^{-at}+e^{-2at})\right]\right\} \\
&= \exp\left\{-\gamma B(t,T)r(t) + \gamma f(0,t)B(t,T) - \gamma\log\frac{P(0,t)}{P(0,T)}\right. \\
&\qquad \left. -\gamma\frac{\sigma^2}{2a^2}\left[(T-t)(1-\gamma) - \frac{a\left(e^{-2at}-\gamma\right)}{2}B(t,T)^2 - (1-\gamma)B(t,T)\right]\right\}
\end{aligned}$$

As required.

Crediting using the spot rate at the beginning of the period

$Z = \frac{1}{n}\sum_{i=tn}^{Tn-1} r_k\left(\frac{i}{n}\right)$

$$
\begin{aligned}
Z &= \frac{r_k(t)}{n} + \sum_{i=tn+1}^{Tn-1} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} r\left(\frac{i}{n}\right) \\
&= \frac{r_k(t)}{n} + \sum_{i=tn+1}^{Tn-1} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B(k)}{nk} \sum_{i=tn+1}^{Tn-1} \left[r(t) e^{-a\left(\frac{i}{n}-t\right)} + \varphi\left(\frac{i}{n}\right) - \varphi(t) e^{-a\left(\frac{i}{n}-t\right)} \right. \\
&\quad - \frac{\sigma^2}{a^2}\left(1 - \frac{1}{2} e^{-a\left(T-\frac{i}{n}\right)} - e^{-a\left(\frac{i}{n}-t\right)} + \frac{1}{2} e^{-a\left(T+\frac{i}{n}-2t\right)}\right) \\
&\quad \left. + \sigma \sum_{j=tn+1}^{i} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{-a\left(\frac{i}{n}-u\right)} dW^T(u) \right] \\
&= \frac{r_k(t)}{n} + \sum_{i=tn+1}^{Tn-1} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B(k)}{nk} \sum_{i=tn+1}^{Tn-1} \left[r(t) e^{-a\left(\frac{i}{n}-t\right)} + \varphi\left(\frac{i}{n}\right) - \varphi(t) e^{-a\left(\frac{i}{n}-t\right)} \right. \\
&\quad \left. - \frac{\sigma^2}{a^2}\left(1 - \frac{1}{2} e^{-a\left(T-\frac{i}{n}\right)} - e^{-a\left(\frac{i}{n}-t\right)} + \frac{1}{2} e^{-a\left(T+\frac{i}{n}-2t\right)}\right) \right] \\
&\quad + \sigma \frac{B(k)}{nk} \sum_{j=tn+1}^{Tn-1} \frac{e^{-\frac{a}{n}j} - e^{-aT}}{1 - e^{-\frac{a}{n}}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{au} dW^T(u)
\end{aligned}
$$

$$
\begin{aligned}
E[Z] &= \frac{r_k(t)}{n} + \sum_{i=tn+1}^{Tn-1} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B(k)}{nk} \sum_{i=tn+1}^{Tn-1} \left[r(t) e^{-a\left(\frac{i}{n}-t\right)} + \varphi\left(\frac{i}{n}\right) - \varphi(t) e^{-a\left(\frac{i}{n}-t\right)} \right. \\
&\quad \left. - \frac{\sigma^2}{a^2}\left(1 - \frac{1}{2} e^{-a\left(T-\frac{i}{n}\right)} - e^{-a\left(\frac{i}{n}-t\right)} + \frac{1}{2} e^{-a\left(T+\frac{i}{n}-2t\right)}\right) \right] \\
V[Z] &= \left(\frac{\sigma B(k)}{nk}\right)^2 \sum_{j=tn+1}^{Tn-1} \left(\frac{e^{-\frac{a}{n}j} - e^{-aT}}{1 - e^{-\frac{a}{n}}}\right)^2 \frac{e^{2a\frac{j}{n}} - e^{2a\frac{j-1}{n}}}{2a}
\end{aligned}
$$

The rest are the same.

Valuation between crediting dates

Let $t_1 < t < t_1 + \frac{1}{n}$, where t_1 is the last payment time, and $t_1 + \frac{1}{n}$ will be the next payment time. Denote $Z = \frac{1}{n} \sum_{t_1 n+1}^{Tn} r_k(i)$, which is the distribution of the accumulation of the crediting rate (settled at the end of each period).

$$
\begin{aligned}
Z &= \sum_{i=t_1n+1}^{Tn} \frac{-A\left(\frac{i}{n},\frac{i}{n}+k\right)}{nk} + \frac{B\left(\frac{i}{n},\frac{i}{n}+k\right)}{nk} r\left(\frac{i}{n}\right) \\
&= \sum_{i=t_1n+1}^{Tn} \frac{-A\left(\frac{i}{n},\frac{i}{n}+k\right)}{nk} + \frac{B(k)}{nk}\sum_{i=t_1n+1}^{Tn}\Bigg[r(t)e^{-a\left(\frac{i}{n}-t\right)} + \varphi\left(\frac{i}{n}\right) - \varphi(t)e^{-a\left(\frac{i}{n}-t\right)} \\
&\qquad -\frac{\sigma^2}{a^2}\left(1-\frac{1}{2}e^{-a\left(T-\frac{i}{n}\right)} - e^{-a\left(\frac{i}{n}-t\right)} + \frac{1}{2}e^{-a\left(T+\frac{i}{n}-2t\right)}\right)\Bigg] \\
&\qquad + \sigma\frac{B(k)}{nk}\left[\sum_{i=t_1n+2}^{Tn}\sum_{j=t_1n+2}^{i}\int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{-a\left(\frac{i}{n}-u\right)}dW^T(u) + \sum_{i=t_1n+1}^{Tn}\int_t^{t_1+\frac{1}{n}} e^{-a(\frac{i}{n}-u)}dW^T(u)\right]
\end{aligned}
$$

Since $\int_{t_i}^{t_i+\frac{1}{n}} f(u)dW^T(u)$ and $\int_{t_j}^{t_j+\frac{1}{n}} f(u)dW^T(u)$ are independent for $i \neq j$, each with expectation zero, the variance of Z is

$$
\begin{aligned}
\mathrm{Var}^T\left[\sum_{i=t_1n+1}^{Tn}\int_t^{t_1+\frac{1}{n}} e^{-a(\frac{i}{n}-u)}dW^T(u)\right] &= \int_t^{t_1+\frac{1}{n}} \mathrm{Var}^T\left[\sum_{i=t_1n+1}^{Tn} e^{-a(\frac{i}{n}-u)}dW^T(u)\right] \\
&= \int_t^{t_1+\frac{1}{n}}\left(\sum_{i=t_1n+1}^{Tn} e^{-a(\frac{i}{n}-u)}\right)^2 du \\
&= \left(\frac{e^{-at_1}-e^{-aT}}{e^{\frac{a}{n}}-1}\right)^2\int_t^{t_1+\frac{1}{n}} e^{2au}du \\
&= \left(\frac{e^{-at_1}-e^{-aT}}{e^{\frac{a}{n}}-1}\right)^2\left(\frac{e^{2a(t_1+\frac{1}{n})}-e^{2at}}{2a}\right)
\end{aligned}
$$

Then, $E^T[Z]$ and $\mathrm{Var}^T[Z]$ can be easily derived from a modification of previous formulae.

$$E^T[Z] = \sum_{i=t_1n+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B(k)}{nk} \sum_{i=t_1n+1}^{Tn} \left[r(t)e^{-a\left(\frac{i}{n}-t\right)} + \varphi\left(\frac{i}{n}\right) - \varphi(t)e^{-a\left(\frac{i}{n}-t\right)} \right.$$
$$\left. -\frac{\sigma^2}{a^2}\left(1 - \frac{1}{2}e^{-a\left(T-\frac{i}{n}\right)} - e^{-a\left(\frac{i}{n}-t\right)} + \frac{1}{2}e^{-a\left(T+\frac{i}{n}-2t\right)}\right)\right]$$
$$\mathrm{Var}^T[Z] = \left(\frac{\sigma B(k)}{nk}\right)^2 \sum_{j=t_1n+2}^{Tn} \left(\frac{e^{-\frac{a}{n}(j-1)} - e^{-aT}}{e^{\frac{a}{n}} - 1}\right)^2 \frac{e^{2a\frac{j}{n}} - e^{2a\frac{j-1}{n}}}{2a}$$
$$+ \left(\frac{\sigma B(k)}{nk}\right)^2 \left(\frac{e^{-at_1} - e^{-aT}}{e^{\frac{a}{n}} - 1}\right)^2 \left(\frac{e^{2a\left(t_1+\frac{1}{n}\right)} - e^{2at}}{2a}\right)$$

Valuation with crediting settled at the beginning of the year would follow the exact same procedure.

Appendix C

Two-Factor Hull-White Model Derivation

Continuously Crediting

Let $Z_1 = \int_t^T x(u)du$ and $Z_2 = \int_t^T y(u)du$. According to Brigo and Mercurio [8], (page 66)

$$\begin{aligned}
E[Z_1] &= \frac{1 - e^{-a_1(T-t)}}{a_1} x(t) \\
E[Z_2] &= \frac{1 - e^{-a_2(T-t)}}{a_2} y(t) \\
\text{Var}[Z_1] &= \frac{\sigma_1^2}{a_1^2} \left[T - t + \frac{2}{a_1} e^{-a_1(T-t)} - \frac{1}{2a_1} e^{-2a_1(T-t)} - \frac{3}{2a_1} \right] \\
\text{Var}[Z_2] &= \frac{\sigma_2^2}{a_2^2} \left[T - t + \frac{2}{a_2} e^{-a_2(T-t)} - \frac{1}{2a_2} e^{-2a_2(T-t)} - \frac{3}{2a_2} \right] \\
\text{Cov}[Z_1, Z_2] &= \rho \frac{\sigma_1 \sigma_2}{a_1 a_2} \left[T - t + \frac{e^{-a_1(T-t)} - 1}{a_1} + \frac{e^{-a_2(T-t)} - 1}{a_2} - \frac{e^{-(a_1+a_2)(T-t)} - 1}{a_1 + a_2} \right]
\end{aligned}$$

Let $Z_k = \int_t^T r_k(u) - r(u)du = \int_t^T \frac{-A(u,u+k)}{k} - \gamma_1 x(u) - \gamma_2 y(u) - \varphi(u)du$

$$E[Z_k] = \int_t^T \frac{-A(u,u+k)}{k}du - \gamma_1 \frac{1-e^{-a_1(T-t)}}{a_1}x(t) - \gamma_2 \frac{1-e^{-a_2(T-t)}}{a_2}y(t) - \int_t^T \varphi(u)du$$

$$\begin{aligned}\text{Var}[Z_k] = \gamma_1^2 \frac{\sigma_1^2}{a_1^2}\left[T - t + \frac{2}{a_1}e^{-a_1(T-t)} - \frac{1}{2a_1}e^{-2a_1(T-t)} - \frac{3}{2a_1}\right] \\ + \gamma_2^2 \frac{\sigma_2^2}{a_2^2}\left[T - t + \frac{2}{a_2}e^{-a_2(T-t)} - \frac{1}{2a_2}e^{-2a_2(T-t)} - \frac{3}{2a_2}\right] \\ + 2\gamma_1\gamma_2\rho\frac{\sigma_1\sigma_2}{a_1a_2}\left[T - t + \frac{e^{-a_1(T-t)}-1}{a_1} + \frac{e^{-a_2(T-t)}-1}{a_2} - \frac{e^{-(a_1+a_2)(T-t)}-1}{a_1+a_2}\right]\end{aligned}$$

notice $\nu^*(T-t) = \text{Var}[Z_k]$, and from Brigo and Mercurio [8], (page 136)

$$\exp\left(\int_t^T -\varphi(u)du\right) = \frac{P(0,T)}{P(0,t)}\exp\left(\frac{1}{2}[\nu(0,t) - \nu(0,T)]\right)$$

Together with the margin term, and the accumulated crediting rates, we derived the valuation factor.

Crediting using the spot rate at the beginning of the period

Let $Z = \frac{1}{n}\sum_{i=tn}^{Tn-1} r_k\left(\frac{i}{n}\right)$

$$Z = \frac{r_k(t)}{n} + \sum_{i=tn+1}^{Tn-1} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(\frac{i}{n}, \frac{i}{n}+k, a_1\right)}{nk}x\left(\frac{i}{n}\right) + \frac{B\left(\frac{i}{n}, \frac{i}{n}+k, a_2\right)}{nk}y\left(\frac{i}{n}\right)$$

$$= \frac{r_k(t)}{n} + \sum_{i=tn+1}^{Tn-1} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(0,k,a_1\right)}{nk} \left\{ \sum_{i=tn+1}^{Tn-1} \left[x(t)e^{-a_1(\frac{i}{n}-t)} - M_x^T(t,\frac{i}{n}) \right] \right.$$

$$\left. +\sigma_1 \sum_{j=tn+1}^{Tn-1} \frac{e^{-\frac{a_1}{n}j} - e^{-a_1 T}}{1-e^{\frac{-a_1}{n}}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{a_1 u} dW_1^T(u) \right\}$$

$$+ \frac{B\left(0,k,a_2\right)}{nk} \left\{ \sum_{i=tn+1}^{Tn-1} \left[y(t)e^{-a_2(\frac{i}{n}-t)} - M_y^T(t,\frac{i}{n}) \right] \right.$$

$$\left. +\sigma_2 \sum_{j=tn+1}^{Tn-1} \frac{e^{-\frac{a_2}{n}j} - e^{-a_2 T}}{1-e^{\frac{-a_2}{n}}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{a_2 u} dW_2^T(u) \right\}$$

$$E^T[Z] = \frac{r_k(t)}{n} + \sum_{i=tn+1}^{Tn-1} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(0,k,a_1\right)}{nk} \left\{ \sum_{i=tn+1}^{Tn-1} \left[x(t)e^{-a_1(\frac{i}{n}-t)} - M_x^T(t,\frac{i}{n}) \right]) \right\}$$

$$+ \frac{B\left(0,k,a_2\right)}{nk} \left\{ \sum_{i=tn+1}^{Tn-1} \left[y(t)e^{-a_2(\frac{i}{n}-t)} - M_y^T(t,\frac{i}{n}) \right] \right\}$$

$$\mathrm{Var}^T[Z] = \left(\frac{\sigma_1 B\left(0,k,a_1\right)}{nk} \right)^2 \sum_{j=tn+1}^{Tn-1} \left(\frac{e^{-\frac{a_1}{n}j} - e^{-a_1 T}}{1-e^{\frac{-a_1}{n}}} \right)^2 \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{2a_1 u} du$$

$$\left(\frac{\sigma_2 B\left(0,k,a_2\right)}{nk} \right)^2 \sum_{j=tn+1}^{Tn-1} \left(\frac{e^{-\frac{a_2}{n}j} - e^{-a_2 T}}{1-e^{\frac{-a_2}{n}}} \right)^2 \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{2a_2 u} du$$

$$2\rho \frac{\sigma_1 B\left(0,k,a_1\right)}{nk} \frac{\sigma_2 B\left(0,k,a_2\right)}{nk} \sum_{j=tn+1}^{Tn-1} \frac{e^{-\frac{a_1}{n}j} - e^{-a_1 T}}{1-e^{\frac{-a_1}{n}}} \frac{e^{-\frac{a_2}{n}j} - e^{-a_2 T}}{1-e^{\frac{-a_2}{n}}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{(a_1+a_2)u} du$$

$$= \left(\frac{\sigma_1 B\left(0,k,a_1\right)}{nk} \right)^2 \sum_{j=tn+1}^{Tn-1} \left(\frac{e^{-\frac{a_1}{n}j} - e^{-a_1 T}}{1-e^{\frac{-a_1}{n}}} \right)^2 \frac{e^{2a_1\frac{j}{n}} - e^{2a_1\frac{j-1}{n}}}{2a_1}$$

$$\left(\frac{\sigma_2 B\left(0,k,a_2\right)}{nk} \right)^2 \sum_{j=tn+1}^{Tn-1} \left(\frac{e^{-\frac{a_2}{n}j} - e^{-a_2 T}}{1-e^{\frac{-a_2}{n}}} \right)^2 \frac{e^{2a_2\frac{j}{n}} - e^{2a_2\frac{j-1}{n}}}{2a_2}$$

$$2\rho \frac{\sigma_1 B\left(0,k,a_1\right)}{nk} \frac{\sigma_2 B\left(0,k,a_2\right)}{nk} \sum_{j=tn+1}^{Tn-1} \frac{e^{-\frac{a_1}{n}j} - e^{-a_1 T}}{1-e^{\frac{-a_1}{n}}} \frac{e^{-\frac{a_2}{n}j} - e^{-a_2 T}}{1-e^{\frac{-a_2}{n}}} \frac{e^{(a_1+a_2)\frac{j}{n}} - e^{(a_1+a_2)\frac{j-1}{n}}}{(a_1+a_2)}$$

Valuation between crediting dates

Similar to One-Factor HW, let $t_1 < t < t_1 + \frac{1}{n}$, where t_1 is the last payment time, and $t_1 + \frac{1}{n}$ will be the next payment time. Denote $Z = \frac{1}{n}\sum_{t_1 n+1}^{Tn} r_k(i)$, which is the distribution of the accumulation of the crediting rate (settled at the end of each period).

$$
\begin{aligned}
Z &= \sum_{i=t_1 n+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(\frac{i}{n}, \frac{i}{n}+k, a_1\right)}{nk} x\left(\frac{i}{n}\right) + \frac{B\left(\frac{i}{n}, \frac{i}{n}+k, a_2\right)}{nk} y\left(\frac{i}{n}\right) \\
&= \sum_{i=t_1 n+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} + \frac{B\left(0, k, a_1\right)}{nk} \left\{ \sum_{i=t_1 n+1}^{Tn} \left[x(t) e^{-a_1\left(\frac{i}{n}-t\right)} - M_x^T\left(t, \frac{i}{n}\right) \right] \right. \\
&\qquad \left. + \sigma_1 \sum_{j=t_1 n+2}^{Tn} \frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}} - 1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{a_1 u} dW_1^T(u) + \sigma_1 \int_t^{t_1+\frac{1}{n}} \sum_{i=t_1 n+1}^{Tn} e^{-a_1\left(\frac{i}{n}-u\right)} dW_1^T(u) \right\} \\
&\quad + \frac{B\left(0, k, a_2\right)}{nk} \left\{ \sum_{i=t_1 n+1}^{Tn} \left[y(t) e^{-a_2\left(\frac{i}{n}-t\right)} - M_y^T\left(t, \frac{i}{n}\right) \right] \right. \\
&\qquad \left. + \sigma_2 \sum_{j=t_1 n+2}^{Tn} \frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}} - 1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{a_2 u} dW_2^T(u) + \sigma_2 \int_t^{t_1+\frac{1}{n}} \sum_{i=t_1 n+1}^{Tn} e^{-a_2\left(\frac{i}{n}-u\right)} dW_2^T(u) \right\}
\end{aligned}
$$

And $E^T[Z]$ and $\mathrm{Var}^T[Z]$ can be easily modified from previous formulae.

$$
\begin{aligned}
E^T[Z] = \sum_{i=t_1 n+1}^{Tn} \frac{-A\left(\frac{i}{n}, \frac{i}{n}+k\right)}{nk} &+ \frac{B\left(0, k, a_1\right)}{nk} \left\{ \sum_{i=t_1 n+1}^{Tn} \left[x(t) e^{-a_1\left(\frac{i}{n}-t\right)} - M_x^T\left(t, \frac{i}{n}\right) \right] \right. \\
&+ \frac{B\left(0, k, a_2\right)}{nk} \left\{ \sum_{i=t_1 n+1}^{Tn} \left[y(t) e^{-a_2\left(\frac{i}{n}-t\right)} - M_y^T\left(t, \frac{i}{n}\right) \right] \right.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}^T[Z] = & \left(\frac{\sigma_1 B(0,k,a_1)}{nk}\right)^2 \sum_{j=t_1 n+2}^{Tn} \left(\frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}} - 1}\right)^2 \frac{e^{2a_1 \frac{j}{n}} - e^{2a_1 \frac{j-1}{n}}}{2a_1} \\
& + \left(\frac{\sigma_1 B(0,k,a_1)}{nk}\right)^2 \left(\frac{e^{-a_1 t_1} - e^{-a_1 T}}{e^{\frac{a_1}{n}} - 1}\right)^2 \left(\frac{e^{2a_1(t_1+\frac{1}{n})} - e^{2a_1 t}}{2a_1}\right) \\
& + \left(\frac{\sigma_2 B(0,k,a_2)}{nk}\right)^2 \sum_{j=t_1 n+2}^{Tn} \left(\frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}} - 1}\right)^2 \frac{e^{2a_2 \frac{j}{n}} - e^{2a_2 \frac{j-1}{n}}}{2a_2} \\
& + \left(\frac{\sigma_2 B(0,k,a_2)}{nk}\right)^2 \left(\frac{e^{-a_2 t_1} - e^{-a_2 T}}{e^{\frac{a_2}{n}} - 1}\right)^2 \left(\frac{e^{2a_2(t_1+\frac{1}{n})} - e^{2a_2 t}}{2a_2}\right) \\
& + 2\rho \left(\frac{\sigma_1 B(0,k,a_1)}{nk}\right) \left(\frac{\sigma_2 B(0,k,a_2)}{nk}\right) \Bigg\{ \\
& \sum_{j=t_1 n+2}^{Tn} \left(\frac{e^{-\frac{a_1}{n}(j-1)} - e^{-a_1 T}}{e^{\frac{a_1}{n}} - 1}\right) \left(\frac{e^{-\frac{a_2}{n}(j-1)} - e^{-a_2 T}}{e^{\frac{a_2}{n}} - 1}\right) \frac{e^{(a_1+a_2)\frac{j}{n}} - e^{(a_1+a_2)\frac{j-1}{n}}}{a_1+a_2} \\
& + \left(\frac{e^{-a_1 t_1} - e^{-a_1 T}}{e^{\frac{a_1}{n}} - 1}\right) \left(\frac{e^{-a_2 t_1} - e^{-a_2 T}}{e^{\frac{a_2}{n}} - 1}\right) \frac{e^{(a_1+a_2)(t_1+\frac{1}{n})} - e^{(a_1+a_2)t}}{a_1+a_2} \Bigg\}
\end{aligned}
$$

Valuation with the crediting rate settled at the beginning of the year follows the same procedure.

Appendix D

Hedge Error and Transaction Cost

One-Factor Hull White - Delta Hedging

Here we outline the analytical formula for hedging error, we neglect higher powers of h, where h represents the rebalancing period. Given V(t,T) and P(t,T), we can approximate V(t+h, T) and P(t+h, T) as

$$
\begin{aligned}
V(t+h,T) - V(t,T) &= \frac{\partial V(t,T)}{\partial t}h + \frac{\partial V(t,T)}{\partial r(t)}\partial r(t) + \frac{1}{2}\frac{\partial^2 V(t,T)}{\partial r(t)^2}(\partial r(t))^2 \\
P(t+h,T) - P(t,T) &= \frac{\partial P(t,T)}{\partial t}h + \frac{\partial P(t,T)}{\partial r(t)}\partial r(t) + \frac{1}{2}\frac{\partial^2 P(t,T)}{\partial r(t)^2}(r(t))^2
\end{aligned}
$$

Assuming we construct our replicating portfolio by holding the pension liability, Δ_t ($\Delta_t = \gamma \frac{V(t,T)}{P(t,T)}$) units of zero coupon bonds held with maturity T, and the rest in the bank account at time t, then the hedge loss (HL) at time t+h is approximately:

$$
\begin{aligned}
HL(t+h) &= V(t+h,T) - \Delta_t P(t+h,T) - (V(t,T) - \Delta_t P(t,T)(1+r(t)h) \\
&= V(t+h,T) - V(t,T) - \Delta_t (P(t+h,T) - P(t,T)) - (V(t,T) - \Delta_t P(t,T))r(t)h \\
&= \left[\frac{\partial A_\gamma(t,T)}{\partial t} - \frac{\partial B_\gamma(t,T)}{\partial t} r(t) + \frac{A(t,t+k)}{k} + r_k(t)\right] V(t,T)h \\
&\quad - B_\gamma(t,T)(\Theta(t) - ar(t))V(t,T)h - \sigma B_\gamma(t,T)V(t,T)u\sqrt{h} + \frac{1}{2}(B_\gamma(t,T))^2\sigma^2 V(t,T)u^2 h \\
&\quad - \Bigg(\left[\frac{\partial A(t,T)}{\partial t} - \frac{\partial B(t,T)}{\partial t} r(t)\right] \gamma V(t,T)h \\
&\quad - B(t,T)(\Theta(t) - ar(t))\gamma V(t,T)h - \sigma B(t,T)\gamma V(t,T)u\sqrt{h} + \frac{1}{2}(B(t,T))^2\sigma^2\gamma V(t,T)u^2 h\Bigg) \\
&\quad - (V(t,T) - \Delta_t P(t,T))r(t)h \\
&= \frac{\partial A_\gamma(t,T)}{\partial t} V(t,T)h + \frac{1}{2}(B_\gamma(t,T))^2\sigma^2 V(t,T)u^2 h + \frac{A(t,t+k)}{k} V(t,T)h + r_k(t)V(t,T)h \\
&\quad - \gamma\frac{\partial A(t,T)}{\partial t} V(t,T)h - \gamma\frac{1}{2}(B(t,T))^2\sigma^2 V(t,T)u^2 h - \frac{B(t,t+k)}{k} V(t,T)r(t)h \\
&= \frac{1}{2} B(t,T)^2 V(t,T)\sigma^2 h \left[(\gamma^2 - \gamma)(u^2 - 1)\right]
\end{aligned}
$$

as required.

Similar to Hedge Loss, we derive the distribution of the transaction costs at time t+h given information at time t. Let s(t) be the transaction at time t and c be transaction cost proportional to the transaction amount. Then:

$$
\begin{aligned}
s(t+h) &= cP(t+h,T)(|\Delta(t+h) - \Delta(t)|) \\
&= c\gamma(|V(t+h,T) - \Delta(t)P(t+h,T)|) \\
&= c\gamma(|HL(t+h) + (V(t,T) - \Delta_t P(t,T)(1+r(t)h)|)
\end{aligned}
$$

As required.

One-Factor Hull White - Delta-Gamma Hedging

Similar to Delta-Hedging, here we ignore higher order of error terms, and approximate the distribution of the hedge loss at time t+dt, given information at time t.

$$
\begin{aligned}
HL(t+dt) &\approx V(t+h,T) - m_1(t)P(t+dt,T_1) - m_2(t)P(t+dt,T_2) \\
&\qquad - (V(t,T) - m_1(t)P(t,T_1) - m_2(t)P(t,T_2))(1+r(t)dt) \\
&= V(t+dt,T) - V(t,T) - m_1(P(t+dt,T_1) - P(t,T_1)) - m_2(P(t+dt,T_2) - P(t,T_2)) \\
&\qquad - (V(t,T) - m_1(t)P(t,T_1) - m_2(t)P(t,T_2))r(t)dt \\
&\approx \frac{\partial V(t,T)}{\partial t}dt + \frac{\partial V(t,T)}{\partial r(t)}dr(t) + \frac{1}{2}\frac{\partial^2 V(t,T)}{\partial r(t)^2}(dr(t))^2 \\
&\qquad - m_1\left[\frac{\partial P(t,T_1)}{\partial t}dt + \frac{\partial P(t,T_1)}{\partial r(t)}dr(t) + \frac{1}{2}\frac{\partial^2 P(t,T_1)}{\partial r(t)^2}(dr(t))^2\right] \\
&\qquad - m_2\left[\frac{\partial P(t,T_2)}{\partial t}dt + \frac{\partial P(t,T_2)}{\partial r(t)}dr(t) + \frac{1}{2}\frac{\partial^2 P(t,T_2)}{\partial r(t)^2}(dr(t))^2\right] \\
&\qquad - [V(t,T) - m_1P(t,T_1) - m_2P(t,T_2)]\, r(t)dt
\end{aligned}
$$

Next, set dt = h, and to simplify the length, we use some shorthand notation.

- $A_\gamma = A_\gamma(t,T)$
- $B_\gamma = B_\gamma(t,T)$
- $V = V(t,T)$
- $P_1 = P(t,T_1)$
- $P_2 = P(t,T_2)$
- $A_1 = A(t,T_1)$
- $B_1 = B(t,T_1)$
- $A_2 = A(t,T_2)$
- $B_2 = B(t,T_2)$
- $A_k = A(t,t+k)$
- $B_k = B(t,t+k)$

and the construction of Delta-Gamma hedging portfolio implies the following equalities

$$B_\gamma(t,T)V(t,T) = m_1 B(t,T_1)P(t,T_1) + m_2 B(t,T_2)P(t,T_2)$$
$$B_\gamma(t,T)^2 V(t,T) = m_1 B(t,T_1)^2 P(t,T_1) + m_2 B(t,T_2)^2 P(t,T_2)$$

We replace dt by h, then we can expand and simplify the approximate hedge loss

$$\begin{aligned} H(t,t+h) = &\left[\frac{\partial A_\gamma}{\partial t} - \frac{\partial B_\gamma}{\partial t} r(t) + \frac{A_k}{k} + r_k(t)\right] Vh - B_\gamma V(r(t+h) - r(t)) + \frac{1}{2}\sigma^2 B_\gamma^2 V h u^2 \\ &- m_1 \left\{ \left[\frac{\partial A_1}{\partial t} - \frac{\partial B_1}{\partial t} r(t)\right] P_1 h - B_1 P_1 (r(t+h) - r(t)) + \frac{1}{2}\sigma^2 B_1^2 P_1 h u^2 \right\} \\ &- m_2 \left\{ \left[\frac{\partial A_2}{\partial t} - \frac{\partial B_2}{\partial t} r(t)\right] P_2 h - B_2 P_2 (r(t+h) - r(t)) + \frac{1}{2}\sigma^2 B_2^2 P_2 h u^2 \right\} \\ &- \left[V(t,T) - m_1 P(t,T_1) - m_2 P(t,T_2)\right] r(t) h \\ = &\left[\frac{\partial A_\gamma}{\partial t} - \frac{\partial B_\gamma}{\partial t} r(t) + \frac{A_k}{k} + r_k(t)\right] Vh - m_1 \left\{ \left[\frac{\partial A_1}{\partial t} - \frac{\partial B_1}{\partial t} r(t)\right] P_1 h \right\} \\ &- m_2 \left\{ \left[\frac{\partial A_2}{\partial t} - \frac{\partial B_2}{\partial t} r(t)\right] P_2 h \right\} - \left[V(t,T) - m_1 P(t,T_1) - m_2 P(t,T_2)\right] r(t) h \end{aligned}$$

where

$$\begin{aligned} \frac{\partial B(t,T)}{\partial t} &= -e^{-a(T-t)} = aB(t,T) - 1 \\ \frac{\partial B_\gamma(t,T)}{\partial t} &= aB_\gamma(t,T) - \gamma \\ r_k(t) &= \frac{-A(t,t+k)}{k} + \frac{B_(t,t+k)}{k} r(t) \\ \gamma &= 1 - \frac{B(t,t+k)}{k} \end{aligned}$$

Then we can eliminate all stochastic terms.

$$H(t,t+h) = \frac{\partial A_\gamma}{\partial t} Vh - m_1 \frac{\partial A_1}{\partial t} P_1 h - m_2 \frac{\partial A_2}{\partial t} P_2 h$$

According to Björk [4], (page 384)

$$\frac{\partial A(t,T)}{\partial t} = \Theta B(t,T) + \frac{\sigma^2}{2} B(t,T)^2$$

and similarly, we have

$$\frac{\partial A_\gamma(t,T)}{\partial t} = \Theta B_\gamma(t,T) + \frac{\sigma^2}{2} B_\gamma(t,T)^2$$

Substituting these results back into the equation, we simply have $H(t, t+h) = 0$. Of course, hedge loss here is calculated by ignoring higher order error terms. As simulations are based on discrete intervals, even for weekly rebalancing periods, there will still be small errors present; however, compared to Delta hedging, Delta-Gamma hedging is theoretically more effective.

Two-Factor Hull White - Delta Hedging

The approximate hedge loss under Delta hedging strategies using the Two-Factor Hull-White model involves the same procedure as the One-Factor case, with tedious calculations. Given V(t,T), the distribution of V(t+h, T) can be approximated by ignore higher order error terms

$$\begin{aligned} V(t+h,T) - V(t,T) \approx &\frac{\partial V(t,T)}{\partial t} h + \frac{\partial V(t,T)}{\partial x(t)}(x(t+h) - x(t)) + \frac{\partial V(t,T)}{\partial y(t)}(y(t+h) - y(t)) \\ &+ \frac{1}{2}\left[\frac{\partial^2 V(t,T)}{\partial x(t)^2}(x(t+h) - x(t))^2 + \frac{\partial^2 V(t,T)}{\partial y(t)^2}(y(t+h) - y(t))^2 \right. \\ &\left. + 2\frac{\partial^2 V(t,T)}{\partial x(t)\partial y(t)}(x(t+h) - x(t))(y(t+h) - y(t))\right] \end{aligned}$$

Changing V to P will give us the approximate distribution for P(t+h,T) given P(t,T). The hedge loss is approximately distributed as:

$$
\begin{aligned}
HL(t+h) \approx\ & V(t+h,T) - V(t,T) - m_1(P(t+h,T_1) - P(t,T_1)) - m_2(P(t+h,T_2) - P(t,T_2)) \\
& - (V(t,T) - m_1P(t,T_1) - m_2P(t,T_2))r(t)h \\
\approx\ & \left[\frac{\partial A^*(t,T)}{\partial t} - \frac{\partial \gamma_1 B(t,T,a_1)}{\partial t}x(t) - \frac{\partial \gamma_2 B(t,T,a_2)}{\partial t}y(t) + r_k(t) + \frac{A(t,t+k)}{k}\right] V(t,T)h \\
& + \frac{1}{2}\left[\sigma_1\gamma_1 B(t,T,a_1)u_1 + \sigma_2\gamma_2 B(t,T,a_2)u_2\right]^2 V(t,T)h \\
& - m_1\left\{\left[\frac{\partial A(t,T_1)}{\partial t} - \frac{\partial B(t,T_1,a_1)}{\partial t}x(t) - \frac{\partial B(t,T_1,a_2)}{\partial t}y(t)\right] P(t,T_1)h \right.\\
& \left. +\frac{1}{2}\left[\sigma_1 B(t,T_1,a_1)u_1 + \sigma_2 B(t,T_1,a_2)u_2\right]^2 P(t,T_1)h\right\} \\
& - m_2\left\{\left[\frac{\partial A(t,T_2)}{\partial t} - \frac{\partial B(t,T_2,a_1)}{\partial t}x(t) - \frac{\partial B(t,T_2,a_2)}{\partial t}y(t)\right] P(t,T_2)h \right.\\
& \left. +\frac{1}{2}\left[\sigma_1 B(t,T_2,a_1)u_1 + \sigma_2 B(t,T_2,a_2)u_2\right]^2 P(t,T_2)h\right\} \\
& - (V(t,T) - m_1P(t,T_1) - m_2P(t,T_2))r(t)h
\end{aligned}
$$

Notice the first order derivatives against stochastic drivers, for example $\frac{\partial V(t,T)}{\partial x(t)}$, are canceled out due to the equalities:

$$
\begin{aligned}
\gamma_1 B(t,T,a_1)V(t,T) &= m_1 B(t,T_1,a_1)P(t,T_1) + m_2 B(t,T_2,a_1)P(t,T_2) \\
\gamma_2 B(t,T,a_2)V(t,T) &= m_1 B(t,T_1,a_2)P(t,T_1) + m_2 B(t,T_2,a_2)P(t,T_2)
\end{aligned}
$$

Also consider the following equalities:

$$
\begin{aligned}
\frac{\partial B(t,T,a)}{\partial t} &= aB(t,T,a) - 1 \\
r_k(t) &= \frac{-A(t,t+k)}{k} + \frac{B(t,t+k,a_1)}{k}x(t) + \frac{B(t,t+k,a_2)}{k}y(t) \\
r(t) &= x(t) + y(t) + \varphi(t) \\
\gamma_j &= 1 - \frac{B(t,t+k,a_j)}{k}
\end{aligned}
$$

we can further reduce our equation for the hedge loss distribution to

$$\begin{aligned}
HL(t+h) \approx\; & \frac{\partial A^*(t,T)}{\partial t} V(t,T)h - m_1 \frac{\partial A(t,T_1)}{\partial t} P(t,T_1)h - m_2 \frac{\partial A(t,T_2)}{\partial t} P(t,T_2)h \\
& + \frac{1}{2} \left[\sigma_1\gamma_1 B(t,T,a_1)u_1 + \sigma_2\gamma_2 B(t,T,a_2)u_2\right]^2 V(t,T)h \\
& - \frac{m_1}{2} \left[\sigma_1 B(t,T_1,a_1)u_1 + \sigma_2 B(t,T_1,a_2)u_2\right]^2 P(t,T_1)h \\
& - \frac{m_2}{2} \left[\sigma_1 B(t,T_2,a_1)u_1 + \sigma_2 B(t,T_2,a_2)u_2\right]^2 P(t,T_2)h \\
& - (V(t,T) - m_1 P(t,T_1) - m_2 P(t,T_2))h\varphi(t)
\end{aligned}$$

and it is straightforward to derive

$$\begin{aligned}
\frac{\partial A^*(t,T)}{\partial t} &= f(0,t) + \frac{1}{2} \big[\sigma_1^2 B(0,t,a_1)^2 + \sigma_2^2 B(0,t,a_2)^2 + 2\rho\sigma_1\sigma_2 B(0,t,a_1)B(0,t,a_2) \\
&\quad - \big(\gamma_1^2\sigma_1^2 B(t,T,a_1)^2 + \gamma_2^2\sigma_2^2 B(t,T,a_2)^2 + 2\rho\sigma_1\sigma_2\gamma_1\gamma_2 B(t,T,a_1)B(t,T,a_2)\big)\big] \\
\frac{\partial A(t,T)}{\partial t} &= f(0,t) + \frac{1}{2} \big[\sigma_1^2 B(0,t,a_1)^2 + \sigma_2^2 B(0,t,a_2)^2 + 2\rho\sigma_1\sigma_2 B(0,t,a_1)B(0,t,a_2) \\
&\quad - \big(\sigma_1^2 B(t,T,a_1)^2 + \sigma_2^2 B(t,T,a_2)^2 + 2\rho\sigma_1\sigma_2 B(t,T,a_1)B(t,T,a_2)\big)\big]
\end{aligned}$$

Moreover, from Brigo and Mercurio [8] (page 135)

$$\varphi(t) = f(0,t) + \frac{1}{2} \big[\sigma_1^2 B(0,t,a_1)^2 + \sigma_2^2 B(0,t,a_2)^2 + 2\rho\sigma_1\sigma_2 B(0,t,a_1)B(0,t,a_2)\big]$$

which leads to

$$\begin{aligned}
HL(t+h) \approx\; & \frac{1}{2} \big\{\left[\sigma_1\gamma_1 B(t,T,a_1)u_1 + \sigma_2\gamma_2 B(t,T,a_2)u_2\right]^2 V(t,T)h \\
& \quad - \big([\sigma_1\gamma_1 B(t,T,a_1)]^2 + [\sigma_2\gamma_2 B(t,T,a_2)]^2 + 2\rho\sigma_1\gamma_1 B(t,T,a_1)\sigma_2\gamma_2 B(t,T,a_2)\big)\big\} V(t,T)h \\
& - \frac{m_1}{2} \big\{\left[\sigma_1 B(t,T_1,a_1)u_1 + \sigma_2 B(t,T_1,a_2)u_2\right]^2 \\
& \quad - \big([\sigma_1 B(t,T_1,a_1)]^2 + [\sigma_2 B(t,T_1,a_2)]^2 + 2\rho\sigma_1 B(t,T_1,a_1)\sigma_2 B(t,T_1,a_2)\big)\big\} P(t,T_1)h \\
& - \frac{m_2}{2} \big\{\left[\sigma_1 B(t,T_2,a_1)u_1 + \sigma_2 B(t,T_2,a_2)u_2\right]^2 \\
& \quad - \big([\sigma_1 B(t,T_2,a_1)]^2 + [\sigma_2 B(t,T_2,a_2)]^2 + 2\rho\sigma_1 B(t,T_2,a_1)\sigma_2 B(t,T_2,a_2))\big)\big\} P(t,T_2)h
\end{aligned}$$

Appendix E

Exact Simulation

One-Factor Hull-White

Instead of approximating $\int_s^t r(u)du$ and $\int_s^t r_k(u)du$ using numerical integration methods such as Trapezoid method or Simpson Method, to avoid discretization errors, here we simulate $\int_s^t r(u)du$ and $\int_s^t r_k(u)du$ jointly with $r(u)$ and $r_k(u)$ (similar to Glasserman [15], Section 3.3). The procedure is simple, we calculate the covariances between each two of these four random variables, and draw their samples randomly. Since $r_k(u)$ and $r(u)$ have a linear relationship, so $\int_s^t r(u)du$ and $\int_s^t r_k(u)du$, their correlations are simply one, and this reduces our covariance matrix from 4-by-4 to 2-by-2.

$$
\begin{aligned}
E[r(t)] &= r(s)e^{-a(t-s)} + f(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 \\
&\quad - \left(f(0,s) + \frac{\sigma^2}{2a^2}(1-e^{-as})^2\right) e^{-a(t-s)} \\
\mathrm{Var}[r(t)] &= \sigma^2 B(2a, t-s)
\end{aligned}
$$

$$
\begin{aligned}
\text{Cov}[r(t), \int_s^t r(u)du|r(s)] &= \int_s^t Cov[r(t), r(u)]du \\
&= \int_s^t e^{-a(t-u)} Var[r(u)|r(s)]du \\
&= \frac{\sigma^2}{2} B(s,t,a)^2 \\
E[\int_s^t r(u)du|r(s)] &= \int_s^t r(s)e^{-a(u-s)} + \varphi(u) - \varphi(s)e^{-a(u-s)} du \\
&= r(s)B(s,t,a) + \log \frac{P(0,s)}{P(0,t)} + \frac{\sigma^2}{2a^2} \left[t - s - 2e^{-as}B(s,t,a) + e^{-2as}B(s,t,2a) \right] \\
&\quad - \left[f(0,s) + \frac{\sigma^2}{2} B(0,s,a)^2 \right] B(s,t,a) \\
\text{Var}[\int_s^t r(u)du|r(s)] &= \frac{\sigma^2}{a^2} \left[t - s - 2B(s,t,a) + B(s,t,2a) \right]
\end{aligned}
$$

Since $\int_s^t r_k(u)du$ is an affine function of $\int_s^t r(u)du$, thus, we do not need to generate a new variable for $\int_s^t r_k(u)du$ and $r_k(t)$.

Two-Factor Hull-White

Simulating of $\int_s^t r(u)$ and $\int_s^t r_k(u)$ using Two-Factor model is similar to the One-Factor case, except now we have two stochastic drivers for $r(t)$ and $\int_s^t r_k(u)$ does not have a linear relationship with $\int_s^t r(u)$ (but is a linear combination of $\int_s^t x(u)du$ and $\int_s^t y(u)du$). Thus, we need to derive a 4-by-4 covariance matrix.

$$
\begin{aligned}
\text{Var}[x(t)] &= \sigma_1^2 B(2a_1, t-s) \\
E[x(t)] &= x(s)e^{-a_1(t-s)} \\
\text{Var}[y(t)] &= \sigma_2^2 B(2a_2, t-s) \\
E[y(t)] &= y(s)e^{-a_2(t-s)} \\
\text{Cov}(x(t), y(t)) &= \rho\sigma_1\sigma_2 B(a_1 + a_2, t-s) \\
r(t) &= x(t) + y(t) + \varphi(t) \\
\varphi(t) &= f(0,t) + \frac{1}{2}[\sigma_1^2 B^2(a_1,t) + \sigma_2^2 B^2(a_2,t) + 2\rho\sigma_1\sigma_2 B(a_1,t)B(a_2,t)]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}[\int_s^t r(u)du] &= \mathrm{Var}\left\{\int_s^t \sigma_1 \int_s^u e^{-a_1(u-j)} dW_1(j)du + \int_s^t \sigma_2 \int_s^u e^{-a_2(u-j)} dW_2(j)du\right\} \\
&= \mathrm{Var}\left(\sigma_1 \int_s^t \int_j^t e^{-a_1(u-j)} dudW_1(j)\right) + \mathrm{Var}\left(\sigma_2 \int_s^t \int_j^t e^{-a_2(u-j)} dudW_2(j)\right) \\
&\qquad + 2\mathrm{Cov}\left(\sigma_1 \int_s^t \int_j^t e^{-a_1(u-j)} dudW_1(j), \sigma_2 \int_s^t \int_j^t e^{-a_2(u-j)} dudW_2(j)\right) \\
&= \frac{\sigma_1^2}{a_1^2}\left(t-s-2B(a_1,t-s)+B(2a_1,t-s)\right) + \frac{\sigma_2^2}{a_2^2}\left(t-s-2B(a_2,t-s)+B(2a_2,t-s)\right) \\
&\qquad 2\rho\frac{\sigma_1\sigma_2}{a_1a_2}\left\{t-s+B(a_1+a_2,t-s)-B(a_1,t-s)-B(a_2,t-s)\right\} \\
E[\int_s^t r(u)du] &= B(a_1,t-s)x(s) + B(a_2,t-s)y(s) + \log\frac{P(0,s)}{P(0,t)} - \frac{1}{2}\nu(0,s) + \frac{1}{2}\nu(0,t)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}(x(t), \int_s^t r(u)du) &= \int_s^t \mathrm{Cov}(x(t), r(u))du \\
&= \int_s^t \mathrm{Cov}(x(u)e^{-a_1(t-u)}, x(u)+y(u))du \\
&= \int_s^t e^{-a_1(t-u)} \frac{\sigma_1^2}{2a_1}\left(1-e^{-2a_1(u-s)}\right) du \\
&\qquad \int_s^t \left(e^{-a_1(t-u)}\right) \rho\frac{\sigma_1\sigma_2}{a_1+a_2}\left[1-e^{-(a_1+a_2)(u-s)}\right] du \\
&= \frac{\sigma_1^2}{2} B(a_1,t-s)^2 + \frac{\rho\sigma_1\sigma_2}{a_1+a_2}\left[B(a_1,t-s) - e^{-a_1(t-s)}B(a_2,t-s)\right] \\
\mathrm{Cov}(y(t), \int_s^t r(u)du) &= \frac{\sigma_2^2}{2} B(a_2,t-s)^2 + \frac{\rho\sigma_1\sigma_2}{a_1+a_2}\left[B(a_2,t-s) - e^{-a_2(t-s)}B(a_1,t-s)\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}(x(t), \int_s^t r_k(u)du) &= \int_s^t \mathrm{Cov}\left(x(u)e^{-a_1(t-u)}, \frac{B(a_1,k)x(u)+B(a_2,k)y(u)}{k}\right) du \\
&= \frac{B(a_1,k)}{k}\int_s^t e^{-a_1(t-u)}\frac{\sigma_1^2}{2a_1}\left(1-e^{-2a_1(u-s)}\right) du \\
&\quad + \int_s^t \left(e^{-a_1(t-u)}\frac{B(a_2,k)}{k}\right)\rho\frac{\sigma_1\sigma_2}{a_1+a_2}\left[1-e^{-(a_1+a_2)(u-s)}\right] du \\
&= \frac{B(a_1,k)}{k}\frac{\sigma_1^2}{2}B(a_1,t-s)^2 \\
&\quad + \frac{\rho\sigma_1\sigma_2}{a_1+a_2}\left(\frac{B(a_2,k)}{k}B(a_1,t-s) - \frac{B(a_2,k)}{k}e^{-a_1(t-s)}B(a_2,t-s)\right) \\
\mathrm{Cov}(y(t), \int_s^t r_k(u)du) &= \frac{B(a_2,k)}{k}\frac{\sigma_2^2}{2}B(a_2,t-s)^2 \\
&\quad + \frac{\rho\sigma_1\sigma_2}{a_1+a_2}\left(\frac{B(a_1,k)}{k}B(a_2,t-s) - \frac{B(a_1,k)}{k}e^{-a_2(t-s)}B(a_1,t-s)\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}[\int_s^t r_k(u)du] &= \left(\frac{B(a_1,k)}{k}\right)^2\frac{\sigma_1^2}{a_1^2}\left(t-s-2B(a_1,t-s)+B(2a_1,t-s)\right) \\
&\quad + \left(\frac{B(a_2,k)}{k}\right)^2\frac{\sigma_2^2}{a_2^2}\left(t-s-2B(a_2,t-s)+B(2a_2,t-s)\right) \\
&\quad + \left(\frac{B(a_1,k)}{k}\right)\left(\frac{B(a_2,k)}{k}\right)\frac{2\rho\sigma_1\sigma_2}{a_1a_2} \\
&\qquad \{t-s+B(a_1+a_2,t-s)-B(a_1,t-s)-B(a_2,t-s)\} \\
E[\int_s^t r_k(u)du] &= \frac{B(a_1,k)}{k}B(a_1,t-s)x(s) + \frac{B(a_2,k)}{k}B(a_2,t-s)y(s) - \int_s^t \frac{A(u,u+k)}{k}du
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}[\int_s^t r(u)du, \int_s^t r_k(u)du] &= \frac{B(a_1,k)}{k}\mathrm{Var}\left(\int_s^t x(u)du\right) + \frac{B(a_2,k)}{k}\mathrm{Var}\left(\int_s^t y(u)du\right) \\
&\quad + (\frac{B(a_1,k)}{k} + \frac{B(a_2,k)}{k})\mathrm{Cov}\left(\int_s^t x(u), \int_s^t y(u)\right) \\
&= \frac{B(a_1,k)}{k}\frac{\sigma_1^2}{a_1^2}\left(t-s-2B(a_1,t-s)+B(2a_1,t-s)\right) \\
&\quad + \frac{B(a_2,k)}{k}\frac{\sigma_2^2}{a_2^2}\left(t-s-2B(a_2,t-s)+B(2a_2,t-s)\right) \\
&\quad + \left(\frac{B(a_1,k)}{k} + \frac{B(a_2,k)}{k}\right)\frac{\rho\sigma_1\sigma_2}{a_1 a_2} \\
&\qquad \{t-s+B(a_1+a_2,t-s)-B(a_1,t-s)-B(a_2,t-s)\}
\end{aligned}
$$

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