A Copula-based Quantile Risk

Measure Approach to Hedging under

Regime Switching

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, our work builds on the future hedging strategy presented by Barbi and Romagnoli (2014). The authors propose the optimal hedge ratio as the minimizer of a generic quantile risk measure (QRM), which includes Value-at-Risk (VaR) and Expected Shortfall (ES). Moreover, the quantiles of the hedged portfolio can be represented in terms of a copula function, so that the dependence structure between the spot and futures could be better captured and hedging performance improved. In that paper, it has been shown that the empirical performance of the model is in general superior compared to some of the existing future hedging models that only consider limited risk measures or discard copula method. However, the model suggests that we use the static copula to fit observations during a previous long period and represent the spot-futures dependence structure. It may result in a poor representation as the dependence between the spot and futures is always characterized as time-varying. Moreover, as a consequence, it may yield a less accurate optimal hedge ratio and inefficient hedging performance. Motivated by this drawback, this thesis starts with the discussion of the robustness of the model in Barbi and Romagnoli (2014), where we use simulated data to conduct sensitivity analysis and performance test. Then an extension is proposed in which we allow the copula parameter to be dynamic and switch between different regimes. We consider two regimes and they correspond to relatively strong and weak dependence between the spot and futures return series. With such extension, we propose an hedging strategy to calculate the approximate optimal hedge ratio, which we call the extended regime-switching hedging strategy or the extended model. Monte Carlo simulations are followed to compare its new hedging performance with that

of the original model without regime switching. The extended regime-switching model shows good advantage in capturing the dynamic dependence, but it dominates the original model in hedging effectiveness only when there are significant regime shifts in the spot-futures dependence and the difference of dependence level in two regimes is more dramatic. Finally, our proposed extended model methodology is applied to empirical data, where we use FTSE 100 stock index and its corresponding futures contract. The empirical results reconfirm our conclusions getting from simulated data.

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Chapter 1

Introduction

Recently, with the rapid expansion of futures markets, hedging has been widely studied and used by academics and practitioners as a useful risk management tool. Investors are allowed to reduce their risks of loss in the cash market by taking an opposite position in the futures market. One of the main challenges in such approaches is the problem of proper selection of an optimal hedge ratio (OHR), which is the size of the position to be held in the futures market in order to hedge each unit in the spot market.

There have been plenty of studies concerned with the determination of the optimal hedge ratio. Basically, the optimal hedge ratio largely depends on the particular objective function to be optimized. From the view of risk management, the optimal hedge ratio aims to minimize the specific criterion of risk measure for the portfolio, which could be the variance, value-at-risk or expected shortfall. All of these criteria of risk measure have their own advantages and drawbacks. Taking the most widely-used hedging strategy for instance,

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minimum variance of the hedged portfolio might be the simplest model to understand and implement (see, Ederington (1979), Johnson (1960), Myers and Thompson (1989)). However, variance is a proper measure of portfolio risk only when investors have either quadratic preferences or returns are drawn from an elliptical probability distribution, which seems quite unrealistic (Barbi and Romagnoli, 2014). Because of this, later other risk measures, like VaR and ES, became popular in hedging. VaR is defined as the largest loss on a portfolio that can be expected with a particular probability over a certain horizon, while ES means the expected loss on the portfolio, conditional on the loss being less than or equal to the portfolio VaR. To some extent, ES seems more satisfying as it takes into account the expected size of a loss and has property of additivity therefore coherent. However, with high popularity and great development, VaR still works well through the long history of risk management. Harris and Shen (2006) and Cao et al. (2010) respectively proposed non-parametric and semi-parametric approaches to estimate optimal hedge ratios by minimizing VaR and ES as the objective functions, and the hedging performances got improved compared with that of minimizing variance method, especially for returns containing leptokurtosis and skewness.

However, the development of risk measure criteria does more than that. In Acerbi (2002), a general class of quantile risk measures (QRM) has been proposed. A QRM can be represented as a weighted average of quantiles of the return probability distribution. It has the form of $\rho_\phi = -\int_0^1 \phi(s)q_s ds$, where q_s is the s-quantile of the return probability distribution and $\phi(s)$ represents a weighting function[1]. Choosing different weighting functions $\phi(s)$, QRMs nest VaR and ES as special cases, and also contain other risk measures, such

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[1]We will define the weighting function and describe the detailed requirement in next chapter

as Exponential Risk Measure: $ERM(R^h) = -\int_0^1 \frac{ke^{-ks}}{1-e^{-k}}q_s(R^h)ds$. The latter could include risk aversion coefficient into the utility function (Cotter and D (2006)). By making use of this class of QRMs, Barbi and Romagnoli (2014) proposed a new model to determine the optimal hedge ratio so that the QRM of the hedged portfolio is minimized. This method seems flexible as it nests the VaR or ES minimization hedging methods. Moreover, the quantiles of the hedged portfolio were represented in terms of a copula function, and the dependence structure between the spot and futures could be better captured. The authors demonstrated that the empirical performance of the method is better than that of some alternative hedging approaches which only consider limited risk measures or discard copula method. However, in Barbi and Romagnoli (2014), the model to fit the copula is based on the assumption that the dependence structure between spot and futures returns does not change in a quite long period. The authors estimated the copula parameter by measuring the average dependence level during a previous long period, which seems impractical in the real world. According to many theoretical and empirical studies before (see for instance, Patton (2002), Alizadeh and Nomikos (2004), Hsu et al. (2008)), we know that the dependence pattern between the spot and futures is always time-varying. The static copula calibration by observations in a long period may result in a poor representation of the spot-futures dependence, and as a consequence, may yield a less accurate optimal hedge ratio and inefficient hedging performance. These facts motivate our extensions and improvements of the approach proposed by Barbi and Romagnoli (2014). In particular, we include a hidden Markov chain in the copula parameter describing the dependence dynamics. The dependence between returns of the spot and futures is allowed to switch between different regimes, which correspond to relatively strong and weak dependence.

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The main objective of the thesis is to investigate whether our extended hedging strategy based on a regime-switching copula could improve hedging effectiveness compared with the original hedging strategy based on a static copula. We make contributions in the following aspects:

1. By analysing robustness and sensitivity of the original minimum copula-based QRM hedging model presented in Barbi and Romagnoli (2014), we have identified some factors that may significantly affect the hedging effectiveness.

2. We have relaxed some assumptions of the original model by allowing the copula parameter to follow a two-state regime switching model. For this model, we have proposed a hedging strategy that is easier to calculate than the optimal one. Using simulated data, we have verified capability of this extended regime switching strategy to capture a time-varying spot-futures dependence and to improve hedging effectiveness.

3. By conducting Monte Carlo simulations and an empirical application, we have identified the conditions under which the extended model based on a regime-switching copula leads to more efficient hedging strategies when compared with those based on a static copula.

The rest of the thesis is organized as follows. In Chapter 2, we present the minimum copula-based quantile risk measure approach to estimate the optimal hedge ratio as proposed by Barbi and Romagnoli (2014), and conduct robustness tests of the method based on simulated data. In Chapter 3, we extend the model by introducing a hidden Markov

process and run Monte Carlo simulations to compare the performance of regime switching and non-switching modelling, suggesting the appropriateness of our extended approach. Empirical data is applied to the model in Chapter 4 to test its applicability. Finally in Chapter 5, conclusions are made and potential future work is proposed.

Chapter 2

Minimum Copula-based QRM Approach to Hedging and Its Robustness

In this chapter, we first state the optimal hedging problem, and then present the method of finding the optimal hedge ratio proposed by Barbi and Romagnoli (2014). To examine robustness and suitability of this approach, we discuss how the optimal hedge ratio or hedging effectiveness is influenced by the parameter estimation method.

2.1 The Optimal Hedging Problem and Minimum Copula-based QRM Hedging Strategy

When investors use futures to hedge in the market, the key issue they face is the determination of the optimal hedge ratio, that is, the position in futures contracts needed to be held to hedge a given spot position at the current time. Generally, as Chen et al. (2003) points out, the determination of the OHR can be traced to the choice of the objective function to be optimized, which can be either minimization of a risk measure or maximization of the agent's specific utility function. From the perspective of risk management, the optimal hedging problem can be expressed as minimization of a measure of risk of the hedged portfolio.

Formally, if we define R^S and R^F as the per-period return of the spot and the futures position respectively, the per-period return on the hedged portfolio is given by

$$R^h = R^S - hR^F, \tag{2.1}$$

where h is the hedging size of the futures position relative to one unit spot position. We denote the risk measure as $\mathrm{RM}(R^h)$, which depends on the hedging size and distribution of hedged portfolio. Then our ultimate objective in this hedging problem is to find the optimal h^* as

$$h^* = \arg\min_h \mathrm{RM}(R^h). \tag{2.2}$$

Optimized under different risk measures chosen by investors, the optimal hedge ratios

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may yield quite different results. Therefore, choosing an appropriate risk measure seems crucial to this optimal hedging problem. As we know, variance, as the most common risk measure criterion, has been adopted in hedging and risk management for a long time. However, VaR and ES are getting more popularity in both academia and industry, as they emphasize more on extreme loss and tail risk. As a result, such approaches lead to more robust risk management. For these reasons, Quantile Risk Measures (QRMs) have been introduced as the more powerful risk measures that not only nest VaR and ES, but also contain many other risk measures, such as Exponential Risk Measure (ERM). Barbi and Romagnoli (2014) innovatively apply this class of risk measures in future hedging, and the optimal hedge ratio is determined by minimizing a QRM of the hedged portfolio. The QRM can be specified as different risk measures to meet investors' preference and needs in risk management, so this hedging strategy is a quite generic approach. The model proved to be superior in hedging compared to some of the existing future hedging models that only consider limited risk measures. In the following, we make further analysis of QRMs, as well as the implementation of minimum QRM hedging strategy presented in Barbi and Romagnoli (2014).

According to Acerbi (2002), QRMs is a general class of quantile risk measures whose elements can be written as a weighted average of quantiles of the return probability distribution. It has the form of

$$\text{QRM}z = -\int_0^1 \phi(s) q_s ds, \tag{2.3}$$

where q_s is the quantile of the return probability distribution and $\phi(s)$ represents a weighting function defined for all $s \in [0,1]$ such that (i) $\phi(s) \geqslant 0$, (ii) $\int_0^1 \phi(s) ds = 1$, and (iii)

$d\phi(s)/ds \leqslant 0$. Different choices for weighting function result in different risk measures. For example, it could be VaR with confidence level $1 - \alpha$ (denoting as $\mathrm{VaR}_{1-\alpha}$) if the weighting function is a degenerate function corresponding to the Dirac delta at the point $s = \alpha$, while it equals ES with confidence level $1 - \alpha$ (denoting as $\mathrm{ES}_{1-\alpha}$) when the weighting function is set as $1/\alpha$ for all tail quantiles with $s \leqslant \alpha$, and 0 otherwise. Or it could have the form of ERM if we adopt an exponential function of risk-aversion coefficient as weighting function. For simplicity, in this thesis, we mainly focus on three representative QRMs: VaR, ES and ERM. Other risk measures could also be applied in this model similarly by changing the weighting function in Equation (2.3).

Specifically, some well-known examples of QRMs can be represented with regard to quantiles of hedged returns as follows:

$$\mathrm{VaR}_{1-\alpha}(R^h) = -q_\alpha(R^h), \tag{2.4}$$

$$\mathrm{ES}_{1-\alpha}(R^h) = -\frac{1}{\alpha} \int_0^\alpha q_s(R^h) ds, \tag{2.5}$$

$$\mathrm{ERM}(R^h) = -\int_0^1 \frac{ke^{-ks}}{1 - e^{-k}} q_s(R^h) ds, \tag{2.6}$$

where $1 - \alpha$ and k are confidence level and coefficient of absolute risk aversion respectively. From these equations, we can observe that once the quantiles of the hedged portfolio are given, the risk measure can be obtained easily. Replacing the $\mathrm{RM}(R^h)$ in the general optimization Equation (2.2) with these chosen risk measures, leads to the corresponding

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hedging problems:

$$h^* = \arg\min_h \text{VaR}_{1-\alpha}(R^h)$$
$$= \arg\min_h \left[-q_\alpha(R^h) \right] \tag{2.7}$$

$$h^* = \arg\min_h \text{ES}_{1-\alpha}(R^h)$$
$$= \arg\min_h \left[-\frac{1}{\alpha} \int_0^\alpha q_s(R^h)ds \right] \tag{2.8}$$

$$h^* = \arg\min_h \text{ERM}(R^h)$$
$$= \arg\min_h \left[-\int_0^1 \frac{ke^{-ks}}{1-e^{-k}} q_s(R^h)ds \right] \tag{2.9}$$

In order to solve Equations (2.7) - (2.9), we need to determine the s-quantiles of the hedged portfolio R^h for each s and h. The quantile $q_s(R^h)$ can be regarded as a function $g(s, h)$, and we aim to recover it for each s and h. Using the C-convolution operator introduced in Cherubini et al. (2011), Barbi and Romagnoli (2014) proposed a practical approach to this problem. They first modelled the dependence structure between the spot and futures return series by a copula[1] function $C^t_{R^S,R^F}$.[2] Then the distribution function of R^h can be represented in the form of partial derivative of the copula function $C^t_{R^S,R^F}$ by means of the C-convolution operator. After obtaining the hedged portfolio distribution function $F_{R^h}(s)$, the quantile $q_s(R^h)$ is just an inverse transformation, which exactly settles the question. Equation (2.10) below shows a more detailed solution. The s-quantile of the

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[1] To put it simply, a copula is a function that links univariate marginals to their full multivariate distribution. For a detailed overview of copulas and application in finance, we can refer to Nelsen (2007) and Genest et al. (2009).

[2] The superscript t here refers to the fact that the dependence structure is generally time-dependent. It will be dynamically estimated in the application in the model.

hedged portfolio, i.e. $q_s(R^h)$, solves the following:

$$1 - \int_0^1 D_1 C_{R^S,R^F}^t(\omega, 1 - F_{R^F}(\frac{q_s(R^h) - q_\omega(R^S)}{h}))d\omega = s. \qquad (2.10)$$

Here $D_1 C(u,v) = \frac{\partial C}{\partial u}(u,v)$ is the partial derivative of the copula function $C_{R^S,R^F}^t(u,v)$, and F_{R^S} and F_{R^F} respectively represent the continuous marginal distribution of the spot return R^S and the futures R^F.

Apparently from Equation (2.10), there is no explicit analytical solution to the quantile $q_s(R^h)$, and hence a numerical method must be used. First, we need to determine the ranges in which the hedge ratio h and probability value s may happen[3]. For specific values of s and h, the corresponding value of $q_s(R^h)$ can be derived from Equation (2.10). For example, if we set $s = 5\%$ and $h = 1$, we can get the value of 5%-quantile of the hedged portfolio where the futures position is of the same size as the spot position but in different directions. All the quantiles $q_s(R^h)$ with respect to different s and h can be obtained in the same way. Substituting these quantile values into the optimization Equations (2.7), (2.8) and (2.9), we are able to get the optimal hedge ratios h^* correspondingly.

The general idea of solving the optimal hedging problem has been explained. However, to use the method in practice we need to estimate the F_{R^F}, $q_\omega(R^S)$ and C_{R^S,R^F}^t in Equation (2.10). This is equivalent to estimate cumulative distribution functions of the spot and futures returns (i.e. $\hat{F}_{R^S}$ and $\hat{F}_{R^F}$), as well as the copula function that determines the dependence structure between the spot and futures return series. According to the

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[3]The complete domain of s is [0,1]. However, the calculation amount differs with different risk measure as some risk measures only consider the very left tail quantiles. We limit h to the interval [0,2] for computational reasons. The spot and futures are usually positively correlated so that h cannot be negative. And in practice, futures position that is twice as large as the spot position is rarely observed.

implementation that uses empirical spot and futures return series in Barbi and Romagnoli (2014), we follow a dynamic hedging strategy with a rolling window approach. Figure 2.1 depicts the methodology in terms of estimation windows and hedging decisions.
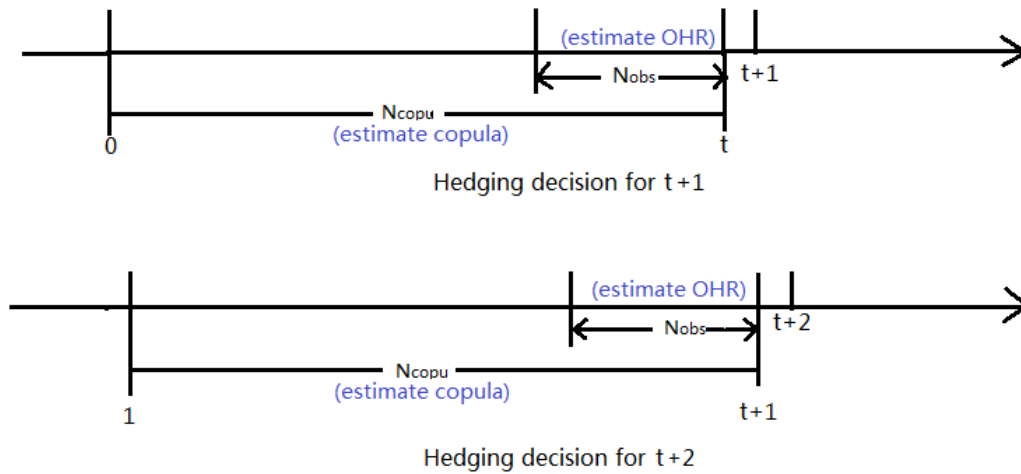
Figure 2.1: Illustration for hedging strategy in terms of windows to fit the copula and estimate the OHR. This figure depicts how the hedging decision is made using the historical data. These two sub-graphs respectively illustrate the methodology of making hedging decisions for different time points. The observations for estimating copula and OHR are always kept up-to-date and the OHR is continuously re-estimated using a day-by-day roller.

The observations used for estimating the copula and OHR are continuously kept up-to-date. Note that here not only the optimal hedge ratio but also the copula parameter is re-estimated using a day-by-day roller. However in Barbi and Romagnoli (2014), the authors only re-estimate the copula parameter periodically. They assume that the copula parameter does not change frequently. Therefore, once they get an estimated copula parameter, the

copula parameter is used to determine the optimal hedge ratios for the following long period. We modify this approach and let the copula parameter be re-estimated every day using the updated observations. We believe that this is a better approach, since the spot-futures dependence is regarded as time-varying in our analysis and a dynamic copula parameter might better capture the dependence structure between the spot and futures.

In order to make the method more explicit, below we describe it step-by-step. Suppose that we are standing at the time point t, and daily observations over a sufficiently long time interval are available. We first determine the hedge ratio for the time $t + 1$.

Step 1, Use the previous N_{copu} observations (including the observations at time t) to estimate the copula function representing the spot-futures dependence.

N_{copu} should not be too small so that we can estimate reasonably accurately the dependence. For example, $N_{copu} = 1250$ might be a reasonable choice, which corresponds to around 5 years' historical observations. To fit a copula, we need two steps: identifying the form of the copula function from the given copulas and estimating the copula parameter. For this, the method proposed by Genest and Rivest (1993) is followed. More details and performance of the method are discussed in the next section.

Step 2, Use the previous N_{obs} observations to estimate the marginal distributions of the spot and futures returns, and further obtain the quantiles of the hedged portfolio by solving Equation (2.10).

When solving Equation (2.10), empirical cumulative distribution functions of the spot and futures returns are calculated using the previous N_{obs} observations. Hence we are able to

derive all the quantiles of the hedged portfolio for each probability value s and hedging size h. Normally, N_{obs} is much less than N_{copu} because return data seem more volatile than the spot-futures dependence and the more recent observations are needed to provide more accurate prediction of return movements. The optimal hedge ratio for the time $t+1$ is the hedging size h which minimizes the risk of the hedged portfolio estimated with the previous N_{obs} observations until time t. Therefore, the choice of N_{obs} might affect the estimation error as well as hedging effectiveness, which merits close attention. We make further analysis on the choice of N_{copu} and N_{obs} in the next section. Specifically, for daily return series, if N_{obs} is set as 250, it means nearly previous one year daily returns of spot and futures constitute the estimation window and determine the optimal hedge ratio for the next day.

Step 3, Use the chosen risk measure to get the optimal hedge ratio by solving the corresponding optimization Equations (2.8) **-** (2.10)**.**

The choice of risk measure depends on the investors' needs and preference in risk management. Different risk measures correspond to different optimization functions, such as VaR for Equation (2.7) and ES for Equation (2.8) and so on. In the optimization equation we use all the quantiles of the hedged portfolio we obtained in Step 2. Then the optimal hedge ratio which minimizes the risk of the hedged portfolio is obtained.

Steps 1 - 3 above describe the procedure of finding the optimal hedge ratio for the time $t+1$. If we continue this process by updating the samples for estimation, we can get the time-varying optimal hedge ratios. For instance, if we want to get the optimal hedge ratio for the time $t+2$, we could first delete the first observation in the original sample and add the observation at the time $t+1$, keeping the size of the estimation samples unchanged.

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Then, we repeat the steps above, and the OHR for the time $t + 2$ is obtained. We use the same procedure at any other time point t'. The estimation data are always kept up-to-date. Both the copula parameter and the optimal hedge ratio are re-estimated using a day-by-day roller, which makes full use of the new information and seems more effective.

2.2   Robustness of the Hedging Model

In this section we study sensitivity of the above method to the selection of the estimation procedure.

2.2.1   Choice of estimation method to fit copulas

Recalling the procedures outlined in Section 2.1, the first step is to use the last N_{copu} observations to fit a copula. Accordingly, the estimation error stemming from copula fit might have a significant influence on the hedging results, which is worthy of further discussion. In the following, we discuss estimation of the copula.

First of all, the copulas used in the implementation are limited to three Archimedean copulas: Clayton, Gumbel and Frank. In Appendix A, we briefly introduce these three copulas, including their forms and features. We choose these Archimedean copulas due to their advantages, where there is only one parameter θ needed to be estimated. Based on different generator functions, these three copulas account for different characteristics so that they cover a wide range of situations. Figure 2.2 draws the scatter-plots of these three copulas. For comparison, we also show the Gaussian copula.
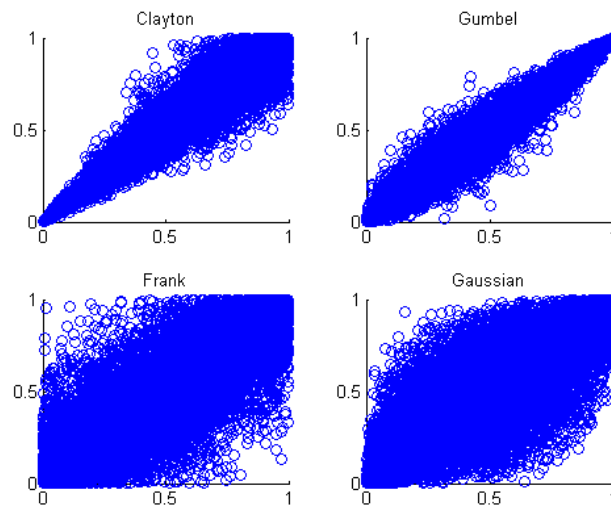
Figure 2.2: Scatter-plots of three Archimedean copulas and the Gaussian copula. Both Frank copula and Gaussian copula show symmetric patterns without significant tail dependence. However, the Frank copula can better capture weak dependence in the tails than the Gaussian copula.

From the scatter-plots, it is apparent that Clayton copula shows strong lower tail dependence and left skew, while Gumbel copula exhibits upper tail dependence and right skew. The dependence in Frank copula is tail symmetry, which is similar with that in the Gaussian copula. However, the Frank copula can capture weak dependence in the tails better than the Gaussian copula and it can also allow for negative dependence.

Since we have determined the class of copulas for implementation, we follow the method presented in Genest and Rivest (1993) and Frees and Valdez (1998) to estimate the copula function. The method is based on two steps:

- Identify the form of a copula that fits data best;

- Estimate the copula parameter

In the first step, the methodology introduced in Genest and Rivest (1993) is followed to identify the best copula function among these three copulas. Appendix B describes the algorithm in details. In the empirical analysis in Chapter 4, we will apply this method to real data and illustrate the process we use to identify the most appropriate Archimedean copula.

Now we focus on the copula parameter estimation. Suppose from the first step, we know that the copula is of the form $C(:, \theta)$. Then, the parameter θ can be estimated by Maximum Likelihood Estimation (MLE) method. Recalling the Sklar's theorem, if we have two random variables X and Y, and H is the two-dimensional distribution function together with marginal distribution functions $F(x)$ and $G(y)$, then there exists a copula C such that

$$H(x, y) = C(F(x), G(y)). \tag{2.11}$$

Thus, by differentiating the equation above, the density function is given by

$$h(x, y) = c(F(x), G(y))f(x)g(y), \tag{2.12}$$

where h is the density function associated to H, f and g are respectively the density functions for each marginal distribution, and the copula density c is obtained by differentiating the copula function as $c(u, v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$. Based on Equation (2.12), estimation of a parametric copula density involves two steps:

- Estimation of the data's marginal distributions;

- Estimation of the copula parameter via Maximum Likelihood Estimation method.

As noted in Joe and Xu (1996), there are three likelihood methods available for implementation:

(1) The Exact Maximum Likelihood (EML) method, where parameters of the copula and marginal distributions are estimated simultaneously. Suppose the parameters in the marginal distributions of F and G are respectively denoted as θ_1 and θ_2. We use N_{copu} observations to estimate the copula parameter θ. We can write the log-likelihood function as follows:

$$\ln L(\theta_1, \theta_2, \theta) = \sum_{t=1}^{N_{copu}} \ln c(F(x_t, \theta_1), G(y_t, \theta_2)) + \sum_{t=1}^{N_{copu}} \ln f(x_t, \theta_1) + \sum_{t=1}^{N_{copu}} \ln g(y_t, \theta_2),$$
(2.13)

and all the parameter estimates in the model are given by:

$$\begin{aligned} (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}) &= \arg\max_{(\theta_1, \theta_2, \theta)} \ln L(\theta_1, \theta_2, \theta) \\ &= \arg\min_{(\theta_1, \theta_2, \theta)} -\ln L(\theta_1, \theta_2, \theta) \end{aligned}.$$
(2.14)

(2) The Inference Functions for Margins (IFM) method, which splits the whole process into two steps: firstly, parameters of the marginal distributions (θ_1 and θ_2) are estimated by maximizing corresponding log-likelihood function; secondly, MLE is applied to estimate the dependence parameter of the copula. Different from the

Equations (2.13) and (2.14), the process is as follows:

$$\hat{\theta}_1 = \arg\max_{\theta_1} \ln L_1(\theta_1) = \sum_{t=1}^{N_{copu}} \ln f(x_t, \theta_1),$$

$$\hat{\theta}_2 = \arg\max_{\theta_2} \ln L_2(\theta_2) = \sum_{t=1}^{N_{copu}} \ln g(y_t, \theta_2), \qquad (2.15)$$

$$\hat{\theta} = \arg\max_{\theta} \ln L_c(\theta) = \sum_{t=1}^{N_{copu}} \ln c(F(x_t, \hat{\theta}_1), G(y_t, \hat{\theta}_2)).$$

(3) The Canonical Maximum Likelihood (CML) method, where only the parameter of the copula is estimated. The empirical cumulative distribution functions (cdfs) are obtained by mapping variables to uniforms (From X and Y with observation N_{copu} into U_x and U_y). After that, the parameter of the copula is obtained by maximizing the log-likelihood of cumulative density function of the copula:

$$\ln L(\theta; U_x, U_y) = \sum_{t=1}^{N_{copu}} \ln c(U_{xt}, U_{yt}, \theta), \qquad (2.16)$$

$$\hat{\theta} = \arg\max_{\theta} \ln L(\theta; U_x, U_y) = \arg\min_{\theta} \left[-\ln L(\theta; U_x, U_y) \right]. \qquad (2.17)$$

Among these three estimation methods, estimating with EML can be computationally very intensive as the marginal distribution may contain several unknown parameters. In addition, studies such as Frees and Valdez (1998) have shown that the difference between estimates from EML and IFM could be acceptable. Therefore, IFM and CML should be more practical. To some extent, choosing from these two estimation methods to fit a copula is equivalent to choosing the estimation method to fit the marginal distributions: **para-**

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metric estimation method or ***non-parametric*** one. Parametric estimation method means that using a specific class of distributions to fit the data while non-parametric estimation method could adopt empirical cdf or kernel density estimation. Both methods have their own strength and weakness, however, which one is more appropriate largely depends on the information about the marginal distributions. It is hard to figure out the distribution by just a glance at the simple scatter-plot of the return data. Therefore, the use of parametric marginal distributions instead of non-parametric marginal distributions bears a model risk and may lead to significantly wrong interpretations of the dependence structure. On the other hand, parametric estimation methods are efficient if the distribution model under consideration is true. In Frahm et al. (2005), in order to illustrate this conclusion, the author simulated data to compare the empirical copula densities which were either obtained via empirical distributions or via fitted parametric marginal distribution. As the data were simulated with t marginal distribution, it showed quite a good fit when the parametric marginal distribution was fitted by t distribution, otherwise the estimates might largely depart from the true distribution, such as using normal distribution to fit data. Therefore, to avoid the misspecification, unless otherwise stated, hereafter we use the empirical cumulative distribution function to estimate the marginal distributions and transform the original data into uniform distributions. Then we follow the CML method to estimate the copula parameter.

2.2.2 Choice of the sample size to fit copulas

In our simulation study we also aim to investigate how the sample size for copula estimation affects the estimation performance. Generally, limited by the size of the available sample, the estimation may be significantly biased or inaccurate, which will lead to unreliable results. According to our dynamic hedging strategy described in Section 2.1, on each day, N_{copu} pairs of observations are used to estimate the copula parameter representing the spot-futures dependence. Thus, we want to find an appropriate value of N_{copu} that can lead to reasonably accurate results.

In the following, we conduct different Monte Carlo simulations to investigate performance of estimation methods based on different sample sizes. To draw more general conclusions, simulations from different distributions are considered. We have chosen the most common Normal distribution and Student's t distribution for the marginal distributions respectively. They are combined with three different Archimedean copula functions (Clayton, Gumbel and Frank), which represent the data's dependence. In this way, we could investigate six cases. For example, in one case, one simulation run could produce N_{copu} pairs of random samples $(X_1, Y_1), \ldots, (X_{N_{copu}}, Y_{N_{copu}})$ generated from a distribution with normal marginal distributions ($\mu 1 = \mu 2 = 0, \sigma 1 = \sigma 2 = 0.5$) and Clayton copula ($\theta = 3$). In another case, N_{copu} pairs of random samples $(X_1, Y_1), \ldots, (X_{N_{copu}}, Y_{N_{copu}})$ could be generated from a distribution with Student's t marginal distributions ($\nu 1 = \nu 2 = 2$) and Gumbel copula ($\theta = 3$). In each of the six cases, 1000 independent simulation runs are conducted for each different N_{copu} value. We set the sample size N_{copu} ranging from 250, 500, 750, 1000, 1250 to 1500, and compare the estimation performance under such different

sample sizes in each case. Given the simulated observations, only the copula parameter θ needs to be estimated by using CML estimation method. Table 2.1 lists the average value and corrsponding mean-square error (MSE) of estimates $\hat{\theta}$ for different sample size N_{copu} in different distribution cases.

As we can observe from the table, different distributions of the underlying returns seem to have no significant impact on the estimates. Panel A and B compare performance of the estimation procedure for data with different marginal distributions. They show similar patterns:

- First of all, for larger sample size, the copula parameter estimate $\hat{\theta}$ converges to the true value of the parameter.

- Moreover, the copula estimation error becomes smaller and smaller when the sample size N_{copu} increases. The estimation error here is defined as the mean-square error (MSE) that measures the average of squares of the difference between the parameter estimate and its true value.

All the results are reasonable and intuitive, since the larger sample size of observations tends to result in more stable and convergent estimates. According to Table 2.1, when the sample size is set to be 250, the MSE values of estimates can be as large as 38%. This value is hard to accept as the error of this size may influence significantly the value of OHR and hedging effectiveness. With the sample size increasing, the estimation error gets lower. We can infer that if the sample size is set to be equal or larger than 1000, all the MSE values of estimates are lower than 5%. It means nearly 4 years of historical observations

or even longer are necessary to provide reliable estimates of the dependence. Therefore in our model, we set the sample size for estimating copulas as 1250, which means nearly 5 years of daily observations. These results are also consistent with the original setting in Barbi and Romagnoli (2014) and could allow us to compare the two models.

Table 2.1: Estimation performance of the copula parameter with different sample size for estimating copula

Panel A

Simulated data generated from:

Marginal distributions: Normal Distribution ($\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 0.5$)

Dependence:	Clayton Copula: $\theta = 3$		Gumbel Copula: $\theta = 3$		Frank Copula: $\theta = 3$	
Sample Size	Average value of estimate $\hat{\theta}$	Mean-square error of $\hat{\theta}$	Average value of estimate $\hat{\theta}$	Mean-square error of $\hat{\theta}$	Average value of estimate $\hat{\theta}$	Mean-square error of $\hat{\theta}$
N_{copu}=250	2.9975	10.89%	2.6534	37.47%	2.9954	17.56%
N_{copu}=500	2.9917	5.79%	2.7900	13.84%	2.9920	9.40%
N_{copu}=750	3.0119	3.77%	2.8585	6.96%	2.9937	6.13%
N_{copu}=1000	2.9995	2.78%	2.8829	4.50%	3.0051	4.26%
N_{copu}=1250	3.0010	2.21%	2.9076	3.33%	3.0047	3.73%
N_{copu}=1500	3.0009	1.83%	2.9198	2.32%	2.9996	3.00%

Panel B

Simulated data generated from:

Marginal distributions: Student's t Distribution ($\nu_1 = \nu_2 = 2$)

Dependence:	Clayton Copula: $\theta = 3$		Gumbel Copula: $\theta = 3$		Frank Copula: $\theta = 3$	
Sample Size	Average value of estimate $\hat{\theta}$	Mean-square error of $\hat{\theta}$	Average value of estimate $\hat{\theta}$	Mean-square error of $\hat{\theta}$	Average value of estimate $\hat{\theta}$	Mean-square error of $\hat{\theta}$
N_{copu}=250	3.0164	10.71%	2.6510	38.07%	2.9889	18.00%
N_{copu}=500	3.0089	5.49%	2.7935	13.71%	2.9850	9.28%
N_{copu}=750	3.0076	3.78%	2.8487	7.35%	3.0124	5.91%
N_{copu}=1000	3.0004	2.78%	2.8845	4.62%	2.9995	4.24%
N_{copu}=1250	2.9990	2.04%	2.9053	3.19%	2.9983	3.52%
N_{copu}=1500	2.9940	1.88%	2.9203	2.47%	3.0034	2.82%

In Panel A, the simulated data are generated from a distribution with normal marginal distributions and different Archimedean copulas.

In Panel B, the simulated data are generated from a distribution with t marginal distributions and different Archimedean copulas.

2.2.3 Choice of the sample size for calculating the OHR

As shown in Figure 2.1, finding an efficient hedging strategy not only includes selection of the proper sample size for estimation of the copula, but also involves the choice of the sample size for calculating OHRs, which is labelled as N_{obs}. The optimal hedge ratio is re-estimated using a day-by-day roller, keeping the number of observations in estimation window unchanged. If N_{obs} is quite large, then the estimates may not reflect the newest information in the market. Otherwise, if N_{obs} is too small, the estimation error will be larger. Therefore, it is important to choose an appropriate sample size for calculating the OHR. In the following part, we use simulated data to explore the impact of N_{obs} on hedging effectiveness.

To ease the computations, we simulate a set of bivariate data using normal marginal distributions ($\mu1 = \mu2 = 0$, $\sigma1 = \sigma2 = 0.5$) and Clayton copula ($\theta = 10$).[4] This set of simulated data consists of 1300 pairs of observations, and the last 50 pairs of observations are employed as the evaluation period to test the hedging effectiveness of the strategy. On each day during that period, the optimal hedge ratio is determined by minimizing the risk of the hedged portfolio using the previous N_{obs} observations. We conduct the hedging on each day and evaluate the hedging performance over the evaluation period. The objective of our study is to compare effectiveness of the hedging strategies when we use different sizes of observations to calculate the OHR.

Specifically, three kinds of hedging strategies are conducted using simulated data, which

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[4]Based on the one-to-one correspondence between Archimedean copula parameter and Kendall's τ correlation, if Clayton copula parameter θ is set as 10, then the corresponding Kendall's τ correlation is about 83.3%, which is relatively reasonable. We have also used the simulated data with $\theta = 3$, and we found the OHR and hedging effectiveness were very low. So we discard that.

respectively minimize different risk measures (VaR, ES and ERM). Under each kind of strategy, we choose different values of N_{obs} to calculate the optimal hedge ratios, and then calculate their corresponding hedging effectivenesses. The calculation of hedging effectiveness differs under strategies using different risk measures, but it is always measured as the percentage reduction of portfolio risk due to hedging. For instance, when we take the minimum of VaR with confidence level $1 - \alpha$ as the objective function to estimate OHR, the hedging effectiveness (HE) is represented as the reduction percentage of $\mathrm{VaR}_{1-\alpha}$ of the portfolio after and before hedging:

$$HE = \mathrm{VaR\ Reduction} = 1 - \frac{\mathrm{VaR}_{1-\alpha}(R^h)}{\mathrm{VaR}_{1-\alpha}(R^S)}. \tag{2.18}$$

The $\mathrm{VaR}_{1-\alpha}(R^h)$ and $\mathrm{VaR}_{1-\alpha}(R^S)$ are respectively the value-at-risk of the portfolio after and before hedging. Similarly, the measures of hedging effectiveness for the hedging strategies of minimizing ES (with confidence level $1 - \alpha$) and ERM (with coefficient of risk aversion k) can be represented as:

$$HE = \mathrm{ES\ Reduction} = 1 - \frac{\mathrm{ES}_{1-\alpha}(R^h)}{\mathrm{ES}_{1-\alpha}(R^S)}, \tag{2.19}$$

$$HE = \mathrm{ERM\ Reduction} = 1 - \frac{\mathrm{ERM}_k(R^h)}{\mathrm{ERM}_k(R^S)}. \tag{2.20}$$

Tables 2.2 - 2.4 depict the comparison results. Under each hedging strategy, the hedging effectiveness is compared when different sizes of observations are chosen to estimate the OHR. Since it is common in the industry to use historical data ranging from 3 months to 2 years, we set the value of N_{obs} as 125, 250 and 500, and compare performance of

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Table 2.2: Hedging effectiveness of minimizing VaR with different sample sizes for calculating the OHR.

Sample Size	Minimizing VaR$(1 - \alpha = 90\%)$		
	OHR		VaR reduction
	mean	median	
$N_{obs} = 125$	0.8792	0.8850	74.96%
$N_{obs} = 250$	0.8744	0.9000	73.76%
$N_{obs} = 500$	0.9531	0.9450	71.61%
Sample Size	Minimizing VaR$(1 - \alpha = 95\%)$		
	OHR		VaR reduction
	mean	median	
$N_{obs} = 125$	0.8709	0.8400	80.42%
$N_{obs} = 250$	0.8617	0.8550	80.07%
$N_{obs} = 500$	0.8933	0.8850	79.05%
Sample Size	Minimizing VaR$(1 - \alpha = 99\%)$		
	OHR		VaR reduction
	mean	median	
$N_{obs} = 125$	0.7637	0.7400	74.10%
$N_{obs} = 250$	0.7587	0.7500	75.10%
$N_{obs} = 500$	0.7804	0.7750	76.15%

The table compares the hedging effectiveness of strategies that minimize VaR of the hedged portfolio with different sample sizes for estimating the OHR. The hedging effectiveness is evaluated by the VaR reduction of the portfolio due to hedging over the evaluation period. The VaR with different confidence levels are also considered: VaR$_{90\%}$, VaR$_{95\%}$ and VaR$_{99\%}$.

Table 2.3: Hedging effectiveness of minimizing ES with different sample sizes for calculating the OHR.

Sample Size	Minimizing ES($1 - \alpha = 90\%$)		
	OHR		ES reduction
	mean	median	
$N_{obs} = 125$	0.8199	0.8075	76.80%
$N_{obs} = 250$	0.8025	0.8050	75.75%
$N_{obs} = 500$	0.8595	0.8550	78.16%
Sample Size	Minimizing ES($1 - \alpha = 95\%$)		
	OHR		ES reduction
	mean	median	
$N_{obs} = 125$	0.7953	0.7775	76.78%
$N_{obs} = 250$	0.7929	0.7900	75.83%
$N_{obs} = 500$	0.8165	0.8100	78.57%
Sample Size	Minimizing ES($1 - \alpha = 99\%$)		
	OHR		ES reduction
	mean	median	
$N_{obs} = 125$	0.7110	0.6900	69.10%
$N_{obs} = 250$	0.7083	0.7000	70.10%
$N_{obs} = 500$	0.7228	0.7150	71.60%

The table compares the hedging effectiveness of strategies that minimize ES of the hedged portfolio with different sample sizes for estimating the OHR. The hedging effectiveness is evaluated by the ES reduction of the portfolio due to hedging over the evaluation period. The ES with different confidence levels are also considered: $ES_{90\%}$, $ES_{95\%}$ and $ES_{99\%}$.

Table 2.4: Hedging effectiveness of minimizing ERM with different sample sizes for calculating the OHR.

Sample Size	Minimizing ERM(k=5)		ERM reduction
	OHR		
	mean	median	
$N_{obs} = 125$	0.8488	0.8375	75.65%
$N_{obs} = 250$	0.8371	0.8350	74.92%
$N_{obs} = 500$	0.8981	0.8950	75.73%
Sample Size	Minimizing ERM(k=10)		ERM reduction
	OHR		
	mean	median	
$N_{obs} = 125$	0.8188	0.8050	76.09%
$N_{obs} = 250$	0.8068	0.8100	75.35%
$N_{obs} = 500$	0.8598	0.8600	77.05%
Sample Size	Minimizing ERM(k=100)		ERM reduction
	OHR		
	mean	median	
$N_{obs} = 125$	0.7329	0.7100	71.85%
$N_{obs} = 250$	0.7292	0.7250	72.25%
$N_{obs} = 500$	0.7512	0.7450	74.81%

The table compares the hedging effectiveness of strategies that minimize ERM of the hedged portfolio with different sample sizes for estimating the OHR. The hedging effectiveness is evaluated by the ERM reduction of the portfolio due to hedging over the evaluation period. The different risk aversion coefficients are considered: 5, 10 and 100, which correspond to the increasing degrees of investors' risk aversion.

the resulting hedging strategies. By investigating the tables, it is difficult to draw a clear conclusion about the effect the sample size N_{obs} has on hedging. Under each hedging strategy, trivial difference of hedging effectiveness can be observed between the hedging strategies using different N_{obs}. Moreover, the results are quite mixed. Taking the case of minimizing VaR as example, we can see the comparison of hedging effectiveness using different sample sizes in Table 2.2. When we choose minimizing VaR at 90% confidence level as the hedging objective, the hedging effectiveness weakens with the larger N_{obs} as the VaR reduction is becoming lower and lower. However, the hedging effectiveness gets improved with larger N_{obs} if the hedging objective changes into minimum VaR at 99% confidence level, although the improvement is just around 2% even from $N_{obs} = 125$ to $N_{obs} = 500$. Similar patterns can be seen from other tables, and it seems that the sample size for calculating the OHR does not have a significant effect on the hedging effectiveness. These results are quite consistent with those obtained by Malliaris and Urrutia (1991). The authors use OLS regression approach to hedge and find that the length of the estimation period used for computing the hedge ratio does not appear to have an impact on the effectiveness of the hedge. Moreover, Harris and Shen (2006) also mentions that different window lengths of observations for getting OHR are qualitatively very similar. Therefore, considering the data frequency, we choose N_{obs}=250 in our further studies, meaning that almost one year of daily returns are applied to determine the OHR for the next day.

Although we find that N_{obs} has no significant effect on the hedging effectiveness, there are still some other findings worthy of attention:

- Firstly, other things being equal, we are expecting to get lower values of the optimal

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hedge ratio when we choose minimization of VaR or ES at a higher confidence level as the hedging objective. VaR or ES at a higher confidence level means that we concentrate more on the very left tail risk. The very left tail results in the reduced accordance between the spot and futures, so it makes sense that we obtain a hedging size farther away from 1.

- Secondly, other things being equal, we are expecting to get lower values of the optimal hedge ratio when we choose a higher risk-aversion coefficient for ERM in the hedging objective function. This fact is similar with the first finding above. To some extend, risk measures at a higher confidence level represent greater risk aversion of the investors. Therefore, if we set a higher risk-aversion coefficient for ERM when we hedge, it is better we adopt a smaller value of the hedging size.

- Lastly, when we still use simulated data generated from the same distribution but with a small value of the copula parameter, it turns out that all the hedging effectiveness decreases dramatically compared to the case with a larger value of the copula parameter. We reached this conclusion after conducting additional simulation with $\theta = 3$ instead of $\theta = 10$. We found that the hedging effectiveness is very low, being only around 30%. It is known that for the copulas that we used larger values of the parameter lead to stronger dependence. At the same time, stronger dependence between the spot and futures returns could lower the basis risk of hedging and further improve the hedging effectiveness.

In our analysis, we conduct the out-of-sample test of hedging effectiveness during the evaluation period. We call it the "out-of-sample test" because the period to estimate hedge

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ratios is different from the period to conduct and evaluate the hedge. During our hedging evaluation period, the optimal hedge ratio on each day is determined by minimizing the risk of the hedged portfolio using the last N_{obs} observations. We conduct the hedging on each day and then measure the hedging effectiveness by evaluating the risk reduction over the evaluation period. In that case, the out-of-sample test is more trustworthy for the forecast performance of the model. Moreover, the small size of the evaluation period for effectiveness test may have an important limitation. For further understanding of the factors influencing the optimal hedge ratio and hedging effectiveness, in the next section we discuss sensitivity of the OHR and hedging effectiveness. We aim to confirm the conclusions we have reached in this section and use them to improve the hedging effectiveness.

2.3   Sensitivity Analysis of the Hedge Ratio and Hedging Effectiveness

In the last section we identified factors that may have some impact on the optimal hedge ratio and hedging effectiveness, such as different risk measures, investors' risk aversion and the spot-futures dependence. In this section, we use simulated data to investigate how the optimal hedge ratio and hedging effectiveness are affected by these factors.

Different risk measures and investors' risk aversion

First we study how hedging strategies based on different risk measures impact the OHR and corresponding hedging effectiveness. We use the simulated data to conduct an analysis

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and the procedures are processed as follows:

(a) 250 pairs of returns on the spot and futures position are generated with normal marginal distributions ($\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 0.5$) and Clayton copula ($\theta = 10$).

(b) Based on the simulated 250 pairs of return observations, we apply our model to get the optimal hedge ratio by minimizing the risk measure of the hedged portfolio. VaR, ES and ERM are employed as the measures of risk, and their corresponding OHRs solve Equations (2.7) - (2.9) respectively. In addition, different confidence levels ranging from 90%, 95% to 99% are considered for VaR and ES, and three risk aversion coefficients are chosen for ERM: k=5, k=10 and k=100. As the most common hedging strategy uses the variance as the risk measure, we also calculate the optimal hedge ratio by minimizing the variance of the hedged portfolio. This yields 10 cases of hedging strategies that use different risk measures or take different degrees of risk aversion.

(c) In each case, we construct the hedged portfolio and calculate the corresponding hedging effectiveness. The simulated 250 pairs of return observations form the evaluation period. On each day, we construct the hedged portfolio using the optimal hedge ratio obtained in step (b). Then the hedging effectiveness is measured as the percentage of the risk reduction due to hedging over the period.[5] For risk measures VaR,

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[5]Here it is different from the out-of-sample hedging effectiveness test, which we mentioned in Section 2.2.3. The period to estimate the OHR is consistent with the period to conduct and evaluate the hedge. We call it the in-sample hedging effectiveness test. Compared to the out-of-sample effectiveness test, it technically will provide us a better result as we conduct and evaluate the hedge using the hedge ratio which minimizes the risk during the evaluation period. Both the in-sample and out-of-sample test can reflect the hedging effectiveness of the model. However, the former one is easier to understand and implement while

ES and ERM, the hedging effectiveness is obtained from Equations (2.18) - (2.20) respectively. For variance, the hedging effectiveness is calculated as follows:

$$HE = 1 - \frac{\text{variance}(R^h)}{\text{variance}(R^S)},$$

(2.21)

where variance(R^h) and variance(R^S) are respectively the variance of the portfolio after and before hedging.

(d) We repeat steps from (a) to (c) 1000 times, and 1000 estimated OHRs and corresponding hedging effectiveness for each hedging strategy are obtained. We get the average value and standard deviation of these OHRs and hedging effectiveness.

All the results are shown in Table 2.5. From that, we can observe:

- Firstly, the strategy of minimizing the portfolio variance gets the largest values of OHR and hedging effectiveness.

- Moreover, the lower the degree of confidence level ($1 - \alpha$ for VaR and ES) or risk aversion coefficient (k for ERM), the closer the obtained OHR to the optimal hedge ratio obtained from the OHR obtained from the minimizing portfolio variance.

- Lastly, with the higher confidence level or risk aversion coefficient, the hedging strategies that minimize VaR, ES and ERM of the hedged portfolio tend to show lower hedging effectiveness.

the latter is more trustworthy for the forecast performance. We use the in-sample test here just for the ease of implementation. It is more common to use the out-of-sample hedging effectiveness test in empirical analysis.

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Table 2.5: Sensitivity of the OHR and hedging effectiveness to different risk measures and investors' risk aversion.

	Variance	VaR($1-\alpha=90\%$)	VaR($1-\alpha=95\%$)	VaR($1-\alpha=99\%$)
OHR	0.9326	0.9051	0.8480	0.7690
	(0.0308)	(0.0565)	(0.0565)	(0.0604)
Hedging Effectiveness	87.4881%	74.8326%	72.4516%	66.6673%
	(0.0219)	(0.0387)	(0.0403)	(0.0638)
		ES($1-\alpha=90\%$)	ES($1-\alpha=95\%$)	ES($1-\alpha=99\%$)
OHR		0.8203	0.7800	0.7166
		(0.0466)	(0.0480)	(0.0582)
Hedging Effectiveness		69.8691%	68.0585%	64.9577%
		(0.0388)	(0.0420)	(0.0628)
		ERM(k=5)	ERM(k=20)	ERM(k=100)
OHR		0.8644	0.7960	0.7379
		(0.0588)	(0.0465)	(0.0605)
Hedging Effectiveness		69.3420%	68.6000%	66.0576%
		(0.0500)	(0.0411)	(0.0550)

Simulated data are employed to compare the optimal hedge ratio obtained by minimizing different risk measures. Moreover, the different risk measures take different confidence levels or risk aversion coefficients. The simulation runs for 1000 times and the table shows the average OHR and corresponding hedging effectiveness for each hedging strategy. Numbers in brackets are the corresponding standard deviation of the values.

These conclusions are similar to the results obtained in the empirical test in Barbi and Romagnoli (2012), which also show the significant effect of investors' risk aversion on the hedge ratios. It is evident that the confidence level is quite consistent with the risk aversion coefficient. Both of these parameters represent the investors' risk aversion in future hedging. According to the formula of ERM in Equation (2.6) : $ERM(R^h) = -\int_0^1 \frac{ke^{-ks}}{1-e^{-k}} q_s(R^h) ds$, we can draw the plot of the exponential weighting function $\phi(s) = ke^{-ks}/(1-e^{-k})$ against the probability levels, for different values of the risk aversion coefficients, which is shown

in the Figure 2.3.
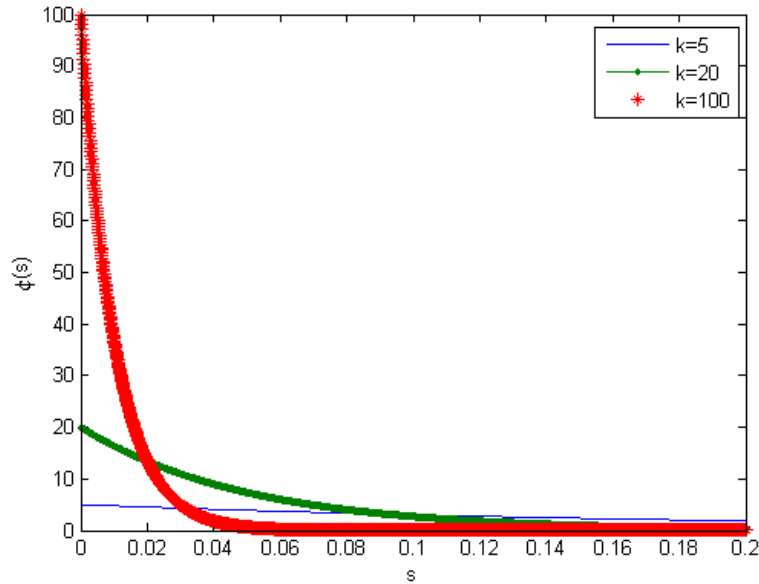
Figure 2.3: Exponential weighting function of ERM, i.e. $\phi(s) = ke^{-ks}/(1 - e^{-k})$, against probability level s for different risk aversion coefficients. Here k is set as 5, 20 and 100.

According to the Figure 2.3, when the risk aversion coefficient becomes higher, ERM emphasizes more the left tail of the probability distribution. In the case of ERM with k=100, the exponential weighting function appropriately assigns 99 percent of its unitary mass to the left 5 percent of the hedged portfolio probability distribution. In contrast, when we integrate the exponential weighting function with k=5, we find that the left 5 percentage of the probability distribution only accounts for 22% of the unitary weighting mass while only 5 percent of the unitary mass is concentrated on the left 1 percent tail of the probability distribution. Similarly, for VaR and ES, a higher confidence level means that the investor aims to control extreme left-tail risk. If the objective function depends more

on the left-tail risk in minimizing VaR, ES and ERM, the optimal hedge ratio should depart more significantly from the OHR based on the minimization of the portfolio variance. That is why we observe that for lower confidence levels or risk aversion coefficients, the OHR obtained from minimization of VaR, ES and ERM are closer to the OHR based on variance. Moreover, when we set the confidence level or risk aversion coefficient to larger values, then the effectiveness of the corresponding hedging strategies will be lower as we manage only more extreme events. This is consistent with the results presented in Table 2.5. Therefore, it is important to consider investors' risk aversion when we compute the hedge ratios for risk management purposes.

The spot-futures dependence

Another issue worth studying is how the dependence between spot and futures return series influences the OHR and hedging effectiveness. In Section 2.2.3, we mention that data simulated from the same distribution but with a smaller value of the copula parameter may lead to lower hedging effectiveness. The copula parameter is closely related with the dependence between data, where larger values of copula parameter mean stronger dependence. A large number of related studies have suggested that hedging with a futures contract that has strong dependence with the spot can improve dramatically hedging effectiveness. In Moosa et al. (2003), the authors have pointed out that "Although the theoretical arguments for why model specification does matter are elegant, the difference model specification makes for hedging performance seems to be negligible. What matters for the success or failure of a hedge is the correlation between the prices of the un-hedged position and the hedging instrument. Low correlation invariably produces insignificant results and ineffective hedges,

whereas high correlation produces effective hedges irrespective of how the hedge ratio is measured". To further understand the relationship between hedging effectiveness and the spot-futures dependence, as well as the the effect of the spot-futures dependence on the OHR, in the following, we conduct some simple tests using simulated data.

Similarly to the simulation procedures we used for testing sensitivity of OHR to different risk measures, in each simulation run, 250 pairs of observations are generated with normal marginal distributions ($\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 0.5$) and Clayton copula. Now, however, we consider different values of the copula parameter. For the Archimedean copulas that we use, the copula parameter has a one-to-one correspondence with the Kendall's τ correlation, which represents the dependence between spot and futures returns. Appendix A lists the detailed formulae describing such relationship for the three copulas. Taking the Clayton copula as an example, the relationship between Kendall's τ correlation and the Clayton copula parameter θ for a bivariate data set is given by:

$$\tau = \frac{\theta}{\theta + 2}.$$
(2.22)

A larger value of the copula parameter corresponds to a larger Kendall's τ correlation, and the latter means the stronger dependence between the spot and futures return series. In the simulation, we allow the Clayton copula parameter to range from 3, 10 to 18. This is equivalent to using the data with the Kendall's τ correlation being 60%, 83% and 90% respectively. For simplicity, we only apply the hedging strategy based on minimization of the portfolio's VaR. For each case we run 1000 times of simulation and calculate the average values of the OHR and hedging effectiveness, as well as their standard deviation.

All the results are shown in Table 2.6. We can draw the following conclusions:

- The results proved to be consistent with the common conclusion that stronger spot-futures dependence could largely improve hedging effectiveness. In the hedging strategy case of minimizing the portfolio VaR, which is presented in Table 2.6, we compare the hedging performance when the futures returns have different dependence with the spot returns. If the Kendall's τ correlation between the spot and futures return series is 60%, then the hedging strategy could only achieve hedging effectiveness of nearly 45%. However, when the Kendall's τ correlation becomes 83.33% and 90%, the hedging strategy could reach around 70% and 80% of risk reduction respectively.

- Moreover, when the spot-futures dependence is relatively weak, compared to the spot-futures dependence, the confidence level of VaR seems to have no significant effect on the hedging effectiveness. For example, in the second row of Table 2.6, the weakest spot-futures dependence has led to very low hedging effectiveness, which is around 45%. In this case, even changing the confidence level from 99% to 90% does not improve the risk reduction greatly. Therefore, the strategy that uses the futures contract with stronger dependence with the spots to lower the basis risk seems more efficient in improving hedging effectiveness. However, when the spot-futures dependence becomes stronger, the confidence level of VaR tends to show more significant effect on the hedging effectiveness. From either Table 2.5 or Table 2.6, we can observe that if the spot-futures dependence is relatively strong with the corresponding Kendall's τ correlation being not less than 80%, changing the confidence level of VaR from 99% to 90% could at least increase the hedging effectiveness by around

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8%. At that time, it is also important for us to take the investors' risk aversion into consideration when making hedging decisions.

- Meanwhile, Table 2.6 also suggests that the stronger the dependence between spot and futures returns, the higher the optimal hedging size. For example, in the case of minimizing $\text{VaR}_{90\%}$ of the hedged portfolio, the average OHR obtained increases from 0.76 to 0.94 when the Kendall's τ correlation between the spot and futures return series changes from 60% to 90%. A similar conclusion can be drawn for other cases. Stronger spot-futures dependence means less basis risk, and the spot and futures returns are more likely to converge by the time when the hedging expires. That means the future hedging is more reliable to prevent losses in the spot market. It also makes sense that the hedging size is closer to 1 when the spot-futures dependence increases.

Lastly, we consider another test where we change marginal distributions from normal to Student's t ($\nu 1 = \nu 2 = 5$). All things being equal, we aim to investigate whether different marginal distributions would impact the OHR and hedging effectiveness. The results are presented in Table 2.7. It turns out that marginal distributions of spot and futures returns have no visible influence on the hedging performance. Table 2.6 and Table 2.7 exhibit quite similar patterns and confirm that stronger spot-futures dependence could improve hedging effectiveness and the OHR value significantly.

In conclusion, the spot-futures dependence shows pronounced influence on the hedging effectiveness and hedge ratio. If the futures contract has strong dependence with spots, it could greatly improve the hedging effectiveness, as well as the optimal hedge ratio value.

By measuring the dependence between the spot and futures, we are able to roughly tell whether hedging is going to be effective. In addition, it is also important to take investors' risk aversion into consideration, especially when the dependence between spot and futures returns is relatively weak. Generally, when the confidence level (for VaR and ES) and the risk aversion coefficient (for ERM) are lower, then the investor puts less emphasis on the left-tail risk. In that case, hedging effectiveness could be improved and the obtained OHR is closer to the OHR obtained from the minimization of the portfolio variance.

Table 2.6: The influence of different spot-futures dependence on hedging effectiveness (simulated data generated from normal marginal distributions)

Copua Parameter	Corresponding Kendall's τ	VaR($1-\alpha=90\%$)		VaR($1-\alpha=95\%$)		VaR($1-\alpha=99\%$)	
		OHR	Effectiveness	OHR	Effectiveness	OHR	Effectiveness
$\theta = 3$	60.00%	0.7641	46.4102%	0.7002	46.1766%	0.6595	45.4925%
		(0.0524)	(0.0621)	(0.0455)	(0.0628)	(0.0265)	(0.0822)
$\theta = 10$	83.33%	0.9040	74.6065%	0.8468	72.5433%	0.7669	66.8945%
		(0.0531)	(0.0388)	(0.0548)	(0.0396)	(0.0585)	(0.0630)
$\theta = 18$	90.00%	0.9423	83.8040%	0.9040	81.2065%	0.8315	74.0617%
		(0.0469)	(0.0302)	(0.0525)	(0.0330)	(0.0584)	(0.0620)

Simulated data are generated from normal marginal distributions ($\mu1 = \mu2 = 0, \sigma1 = \sigma2 = 0.5$) and Clayton copula but with different parameters ($\theta = 3, 10, 18$). Minimizing VaR of the portfolio is applied to get the OHR and corresponding hedging effectiveness. In each case, 1000 times of simulation are conducted and the table shows the average values of the obtained OHR and hedging effectiveness. The numbers in brackets are the standard deviation of the value.

Table 2.7: The influence of different spot-futures dependence on hedging effectiveness (simulated data generated from t marginal distributions)

Copua Parameter	Corresponding Kendall's τ	VaR($1-\alpha=90\%$) OHR	Effectiveness	VaR($1-\alpha=95\%$) OHR	Effectiveness	VaR($1-\alpha=97\%$) OHR	Effectiveness
$\theta=3$	60.00%	0.7471 (0.0708)	42.07% (0.0705)	0.6746 (0.0852)	43.34% (0.0761)	0.6312 (0.0694)	43.60% (0.0728)
$\theta=10$	83.33%	0.9069 (0.0663)	72.50% (0.0494)	0.8402 (0.0756)	68.89% (0.0574)	0.7992 (0.0813)	66.2283% (0.0674)
$\theta=18$	90.00%	0.9404 (0.0580)	82.0879% (0.0395)	0.9015 (0.0646)	78.3296% (0.0509)	0.8684 (0.0734)	74.9087% (0.0663)

Simulated data are generated from t marginal distributions ($\nu_1 = \nu_2 = 5$) and Clayton copula but with different parameters ($\theta = 3, 10, 18$). Minimizing VaR of the portfolio is applied to get the OHR and corresponding hedging effectiveness. In each case, 1000 times of simulation are conducted and the table shows the average values of the obtained OHR and hedging effectiveness. The numbers in brackets are the standard deviation of the values. Here we use the VaR with confidence level of 97% instead of 99% because the latter will result in lots of NaN value under simulated data with t marginal distributions.

Chapter 3

Hedging under Regime Switching Model

It is evident that the spot-futures dependence plays a key role in improving hedging effectiveness. Therefore, an accurate description of the dependence structure between the spot and futures is crucial when making hedging decisions. According to Barbi and Romagnoli (2014), the minimum copula-based QRM approach introduced in previous chapters proves superior in hedging effectiveness when compared to some of the existing models that only consider limited risk measures or discard copula method. However, this model of determining the optimal hedge ratio may not be realistic due to its implicit assumption that the dependence between spot and futures return series is very time-insensitive. The authors obtain the copula parameter by measuring the average dependence level during a previous long period. These assumptions might be inconsistent with the empirical findings that the dependence structure between the spot and futures is always characterized as time-varying

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(see King et al. (1994); Longin and Solnik (1995); Forbes and Rigobon (2002); Sarno and Valente (2000), Salvador and Aragó (2014); Guidolin and Timmermann (2006)). Therefore, in order to further improve hedging strategies, we extend this relatively static copula parameter approach by incorporating a Markov regime switching copula model to capture a dynamic spot-futures dependence. With such extension, we propose an hedging strategy to calculate the approximate[1] optimal hedge ratio, which we call the extended regime-switching hedging strategy or the extended model. We aim to investigate whether a model like this could yield more efficient optimal hedge ratio and accordingly improve hedging effectiveness.

In this chapter, we first provide a brief introduction to regime switching models. Then we extend our original copula-based QRM hedging strategy by applying a two-state regime switching copula to capture the dynamic characteristic of the spot-futures dependence, where the copula parameter is allowed to be time-varying. Simulated observations are used to investigate the accuracy of the extended model to estimate the time-varying dependence between spot and futures return series, as well as for a comparison of hedging effectiveness between the extended model and the original model. In the test of hedging effectiveness, for simplicity, we take the minimum VaR of the hedged portfolio as the hedging objective and compare hedging performance of the two models in that case.

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[1]Here we call "approximate optimal" as our method calculates the optimal hedge ratio with an approximation in order to ease the implementation. In the following Section 3.1, we will explain it in detail.

3.1 Regime Switching Model and the Extended Hedging Strategy

A regime switching model was first introduced by Hamilton (1989) and has since been widely applied in finance. It provides an intuitive and effective way to capture market behaviours under different economic conditions. In the regime switching framework, the observed variable is allowed to follow different process in different state, and hence the model is more suitable to capture a real dynamic process of financial data. As explained in Chapter 2, our minimum copula-based QRM hedging strategy uses a copula model to describe the dependence between spot and futures return series. However, the copula parameter at any time point is determined by the average level of the spot-futures dependence during a previous long period, which is not consistent with some of the empirical findings. Recently, several authors have recognized that the time-varying characteristic of the spot-futures dependence and regime switching models might provide a suitable approach to settle this problem. In Sarno and Valente (2000), the authors employed regime shifts to model spot and futures price movements, which turned out to capture well the time-varying dynamic of the time series. Alizadeh and Nomikos (2004), Lee et al. (2006, 2007, 2009, 2010), Salvador and Aragó (2014) applied the regime switching model to take the state-dependency property between spot and futures series into account when developing hedging strategies. In all of these studies, the authors demonstrate that the hedging effectiveness got improved compared to state-independent strategies. Motivated by all these findings, we extend our original model by using a regime switching copula to capture the dynamic dependence between spot and futures return series. Our objective is to verify

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that such an extension indeed leads to more efficient hedge ratio and improved hedging effectiveness.

In our setting, the regime only affects the dependence structure, which we achieve by allowing the copula parameter to follow a two-state regime switching process.[2] We first follow the method presented in Genest and Rivest (1993) to identify the specific copula among the three common Archimedean copulas: Clayton, Gumbel and Frank. Then we estimate the dynamic copula parameter by the MLE method. Given the specific copula $C(;\theta)$, the dynamic process for the copula parameter is characterized by regime shifts to reflect the spot-futures dependence movement, where the two states correspond to relatively strong and weak dependence. For simplicity and ease of implementation, we assume that the copula parameter θ is constant in each state ($S_t = 1$ or $S_t = 2$).

$$\theta_{S_t} = \begin{cases} \theta_1, & S_t = 1, \text{ in State 1} \\ \theta_2, & S_t = 2, \text{ in State 2}. \end{cases} \tag{3.1}$$

The unobserved latent state variable S_t follows a Markov chain with the transition probabilities

$$\mathbf{P} = \begin{pmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{pmatrix}, \tag{3.2}$$

where p_{ij} represents the probability of moving from state i at time t to state j at time $t+1$. In our model, the transition probability matrix is time-invariant, that is, p_{11} and p_{22} will

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[2]In this thesis, for simplifying our model, only the copula parameter is allowed to follow the regime-switching process. This might be not considered completed as the univariate marginal distributions of the spot and futures returns might also contain the regime shifts. However, we ignored such possibilities just for the ease of the implementation.

not change with time t. We impose this limitation just for convenience, although there has been some applications of time-varying transition probabilities (see Ding (2012)). Copula parameters in each state and the transition probabilities can be estimated using Maximum Likelihood Estimation method.

Having set the model to capture a dynamic dependence between spot and futures return series, our extended regime-switching hedging strategy is recalculated day-by-day. On each day, the optimal hedge ratio is determined by finding the minimum QRM of the hedged portfolio. The latter is clearly affected by the dependence between the spot and futures on that day. Under the regime switching copula framework, the copula parameter is state dependent and hence unobservable. Therefore, it is necessary to take the probabilities of being in each state into consideration. Formally, at any time point t, we define the probabilities of the spot-futures dependence being in state 1 or in state 2 as $\pi_{1,t}$ and $\pi_{2,t} = 1 - \pi_{1,t}$. These are probabilities of being in each state conditional on the information we have up to time t. We use $\hat{\theta}_1$ and $\hat{\theta}_2$ to represent the estimated copula parameter in state 1 and in state 2 respectively. Then we could get the corresponding copula function at time t, as

$$C_t(u, v) = \pi_{1,t} C_t(u, v; \hat{\theta}_1) + \pi_{2,t} C_t(u, v; \hat{\theta}_2). \tag{3.3}$$

According to our minimum QRM hedging approach, the optimal hedge ratio at time t thus can be obtained by minimizing risk of the hedged portfolio. However in practice, this approach may result in heavy calculations as the copula function changes on each day, and it is very time-consuming to solve such an optimization problem. Therefore, in order to simplify calculations, hereafter we use an alternative method as an approximation to

the optimal hedge ratio under the regime switching model. First, we still use a two-state regime switching copula model to capture the spot-futures dependence. Different state corresponds to different estimated copula parameter. We calculate the optimal hedge ratio in each state using the corresponding estimated copula parameter. Then the ultimate hedge ratio is approximated as the weighted average of these two optimal hedge ratios, weighted by the probability of being in each state. We regard such obtained hedge ratio as the approximate optimal hedge ratio under the extended hedging strategy, which allows for regime shifts in the dependence.

Specifically, suppose that we are standing at the time point t, and daily observations over a sufficiently long time interval are available. In order to apply the extended hedging strategy and get the approximate optimal hedge ratio for the time $t+1$, we need two steps:

1. **The first step is to model a regime switching copula to capture the time-varying dependence between spot and futures return series.**
   We first identify the appropriate copula that best fits the observations. After that, the probabilities of being in each state at time $t + 1$ as well as the corresponding copula parameters in each state are estimated by MLE method.

2. **Then in the second step, we calculate the optimal hedge ratio by minimizing risk of the hedged portfolio in each state. The optimal hedge ratio at time $t + 1$ is approximated as the weighted average of the two hedge ratios in each state.**
   At time $t+1$, the optimal hedge ratio in each state is solved using the corresponding estimated copula parameter, where the minimum QRM of the hedged portfolio is

achieved. We denote them as $h_{1,t+1}$ and $h_{2,t+1}$. Then we use the following formula as an approximation to the optimal hedge, which is represented by:

$$h_{t+1} = \pi_{1,t+1} h_{1,t+1} + \pi_{2,t+1} h_{2,t+1}. \tag{3.4}$$

3.1.1 Modelling a regime-switching copula

In order to use a copula to capture the spot-futures dependence, we have to identify the form of the copula function, as well as the copula parameter. This task is similar to modelling a non-switching copula, which we described in Chapter 2. However, under the extended regime switching copula approach, we allow for regime shifts in the copula parameter, and the estimation of the dynamic copula parameter is more complicated. In the following, we explain the steps in details.

Identification of an appropriate copula

We use the method in Genest and Rivest (1993) to identify the appropriate copula that best fits the observations. We detail the algorithm in Appendix B.

As we explained in Chapter 2, we still consider the three Archimedean copulas: Clayton, Gumbel and Frank. Suppose we have a bivariate data set: $(X_{11}, X_{21}),...(X_{1n}, X_{2n})$, and the distribution function F has an associated copula C_ψ. Then we aim to identify the appropriate copula form among the three given Archimedean copulas.

An intermediate (unobserved) random variable $Z_i = F(X_{1i}, X_{2i})$ is incorporated with the distribution function $K(z) = Prob(Z_i \leq z)$. According to Genest and Rivest (1993),

there also exists the relation between the distribution function and the generator of an Archimedean copula, expressed as $K_\psi(z) = z - \frac{\psi(z)}{\psi'(z)}$, where ψ is the generator function of the copula. The rationale of this method to identify the copula is to find out ψ that can make parametric estimate $\hat{K}_\psi(z)$ most closely resemble the non-parametric estimate $\hat{K}(z)$. The closeness measure can be done by minimizing a distance such as $\int [\hat{K}_\psi(z) - \hat{K}(z)]^2 d\hat{K}(z)$ or comparing the plot of $\hat{K}_\psi(z)$ and $\hat{K}(z)$ versus z. In Barbi and Romagnoli (2014), the authors also specify this problem of choosing the appropriate copula generator ψ when applying minimum QRM approach to hedge. When we employ a QRM that requires all the quantiles of the underlying probability distribution, such as exponential risk measure, the copula form ψ can be obtained by minimizing

$$\text{MSE} = \int_0^1 [\hat{K}_\psi(z) - \hat{K}(z)]^2 d\hat{K}(z). \tag{3.5}$$

The MSE can be regarded as the mean square error of the estimated copula parameters. Otherwise, if the QRM emphasizes more the left tail of the underlying probability distribution, such as VaR in our analysis or ES, the copula form ψ should be determined by minimizing the tail mean square error (T-MSE), that is

$$\text{T-MSE} = \frac{1}{\alpha} \int_0^1 [\hat{K}_\psi(z) - \hat{K}(z)]^2 1_{z \le \alpha} d\hat{K}(z), \tag{3.6}$$

where 1-α is the chosen confidence level.

In the next chapter of empirical analysis, we will apply this method to determine the most appropriate copula for our empirical data.

Estimation of the copula parameter using Maximum Likelihood

Given the copula type, the copula parameters along with the probabilities of being in each state are estimated by maximizing the log-likelihood of cumulative density function of the copula as we did in the non-switching copula case in Chapter 2. The Canonical Maximum Likelihood method is applied, where the data of returns are transformed into uniforms first and then plugged into the copula log-likelihood function. However, what is different here is that the copula parameter θ also depends on a non-observed discrete state variable S_t, which follows a Markov chain. In Chapter 2, our original hedging strategy does not consider regime shifts of the copula parameter, and the log-likelihood function to estimate the copula parameter is specified as

$$\ln L(\theta; U_x, U_y) = \sum_{t=1}^{T} \ln c(U_{xt}, U_{yt}, \theta | \omega_{t-1}). \tag{3.7}$$

In this formula, U_x and U_y are uniforms transformed from return series using empirical cumulative distribution, ω_{t-1} represents the information we have up to time $t-1$, θ is the constant copula parameter to be estimated and T is the total number of observations. Compared with it, the extended model with a regime-switching copula has the log-likelihood function represented by:

$$\ln L(\theta_{S_t}; U_x, U_y) = \sum_{t=1}^{T} \ln c(U_{xt}, U_{yt}, \theta_{S_t} | \omega_{t-1}). \tag{3.8}$$

Here, the state variable S_t follows a Markov chain with the transition probability matrix P. There are two states considered for the copula parameter to reflect the change of the

spot-futures dependence. θ_1 and θ_2 respectively stand for the copula parameter in state 1 and state 2. Taking into account the probabilities of state variable S_t being in state 1 or 2, we can rewrite Equation (3.8) as

$$\ln L(\theta_{S_t}; U_x, U_y) = \sum_{t=1}^{T} \ln(\sum_{j=1}^{2} c(U_{xt}, U_{yt}, \theta_{S_t} | S_t = j, \omega_{t-1}) Pr(S_t = j | \omega_{t-1})). \qquad (3.9)$$

As the state variable S_t is unobservable, it is necessary to incorporate Kim's filter (Kim and Nelson, 1999) to estimate the conditional probabilities of being in each state: $Pr(S_t = 1 | \omega_{t-1})$ and $Pr(S_t = 2 | \omega_{t-1})$. It consists of two main parts: i) **The Filter process** and ii) **Smoothing process**. The filter refers to an estimate of the state variable based on information available up to time t, $Pr(S_t | \omega_t)$, while smoothing refers to the estimation of the state variable based on all available information through time T, which we denote by $Pr(S_t | \omega_T)$. During the whole estimation process, three types of probabilities merit attention:

(1) **Predicted Probability**, which is the predicted probability of each state at time t based on the information set until time $t - 1$: $Pr(S_t = j | \omega_{t-1})$

(2) **Filtered Probability**, which is the updated probability of each state at time t based on the information up to time t: $Pr(S_t = j | \omega_t)$

(3) **Smoothed Probability**, which is the estimated probability of each state using all T observations: $Pr(S_t = j | \omega_T)$

In our estimation, to make full use of all available observations, both the filter and smoothing processes are involved. Borrowing the idea in da Silva Filho et al. (2012), the estimation

follows a forward-filtering-backward-smoothing process, which is depicted below. We aim to get the predicted probabilities and the copula parameters of each state at time t, by using all available information.

(a) **Set the starting probabilities ($t = 0$) of each state:** $Pr(S_0 = j|\omega_0)$ for $j = 1, 2$. They could be any naive guesses, say $Pr(S_0 = j|\omega_0) = 0.5$. However, we normally use the unconditional probabilities of S_t, which are shown in Equations (3.10) and (3.11). The constant transition probabilities p_{11} and p_{22} are initialized with any arbitrary values.

$$Pr(S_0 = 1|\omega_0) = \frac{1 - p_{11}}{2 - p_{11} - p_{22}}, \tag{3.10}$$

$$Pr(S_0 = 2|\omega_0) = \frac{1 - p_{22}}{2 - p_{11} - p_{22}}. \tag{3.11}$$

(b) **Predict the probabilities of each state at $t = 1$:** $Pr(S_1 = j|\omega_0)$ for $j = 1, 2$. In the following equation, we set $t = 1$, and get the probabilities of each state based on the information up to time $t - 1$:

$$\Pr(S_t = j|\omega_{t-1}) = \sum_{i=1}^{2} p_{ij}(Pr(S_{t-1} = i|\omega_{t-1}))). \tag{3.12}$$

(c) **Update filtered probabilities of each state at $t = 1$ with new information at time t:** $Pr(S_1 = j|\omega_1)$ for j=1, 2.

$$Pr(S_t = j|\omega_t) = \frac{c(U_{x_t}, U_{y_t}, \theta_{S_t}|S_t = j, \omega_{t-1})Pr(S_t = j|\omega_{t-1})}{\sum_{k=1}^{2} c(U_{x_t}, U_{y_t}, \theta_{S_t}|S_t = k, \omega_{t-1})Pr(S_t = k|\omega_{t-1})}, \tag{3.13}$$

where U_x and U_y are uniforms transformed from return series using empirical cu-

mulative distributions, and the density function $c(x, y, \theta)$ is specified by the copula function and the parameter θ.

(d) **Continue to update the predicted probabilities and filtered probabilities at time t ($t = 2$ to T): $Pr(S_t = j|\omega_{t-1})$ and $Pr(S_t = j|\omega_t)$ for j=1, 2.**

Set $t = t + 1$ and repeat steps (b)-(c) until time $t = T$. At each time t ($t = 2$ to T), the predicted probabilities of each state are first estimated by Equation (3.12). Then Equation (3.13) in step (c) provides the filtered probabilities of each state at time t. Therefore, till time T, all the filtered probabilities can be obtained: $Pr(S_t = j|\omega_t)$ for $j = 1, 2$ and $t = 1, \ldots, T$.

(e) **Initialize the smoothing process in $t = T$: $Pr(S_T = j|\omega_T)$, and go backwards recursively.**

The smoothed probabilities at time T are equal to the filtered probabilities at time T for each state, which we get from step (d).

(f) **Calculate the smoothed probabilities of each state from time $t = T - 1$ to 0.**

The smoothing process is conducted backward. For each t ranging from $T - 1$ to 0, the smoothed probabilities of each state are given as:

$$Pr(S_t = j|\omega_T) = \sum_{k=1}^{2} \frac{p_{jk}Pr(S_t = j|\omega_t)Pr(S_{t+1} = k|\omega_T)}{\sum_{i=1}^{2} p_{ik}Pr(S_t = i|\omega_t)}. \quad (3.14)$$

Equation (3.14) comes from the fact that $Pr(S_t = j|\omega_T) = \sum_{k=1}^{2} Pr(S_t = j, S_T = k|\omega_T)$. At time $t = 0$, we get the smoothed probability $Pr(S_0 = j|\omega_T)$, and it could be used as the renewed starting probabilities in step (a). In this way, all the

information could be used in the estimation process, and the effect of the arbitrary initial probabilities on the log likelihood value could be minimized.

(g) **Use the smoothed probabilities at time $t = 0$, $Pr(S_0 = j|\omega_T)$, to replace the starting probabilties in step (a), and repeat steps (b)-(d) to get all the updated predicted probabilities and filtered probabilities.**

In this step, we regard the smoothed probabilities $Pr(S_0 = j|\omega_T)$ as the more reliable probabilities at time $t = 0$, which take all the T observations into account. Thus, by repeating the filter process, we have updated the predicted probabilities and filtered probabilities of each state from time $t = 1$ to T: $Pr(S_t = j|\omega_{t-1})$ and $Pr(S_t = j|\omega_t)$ for $j = 1, 2$.

(h) **Maximize the log-likelihood function in Equation 3.9, and get estimated state transition probabilites $(\hat{p_{11}}, \hat{p_{22}})$ and copula parameters $(\hat{\theta}_1, \hat{\theta}_2)$ in each state .**

$$\ln L(\theta_{S_t}; U_x, U_y) = \sum_{t=1}^{T} \ln(\sum_{j=1}^{2} c(U_{xt}, U_{yt}, \theta_{S_t}|S_t = j, \omega_{t-1})Pr(S_t = j|\omega_{t-1})), \quad (3.15)$$

$$(\hat{\theta}_1, \hat{\theta}_2, \hat{p_{11}}, \hat{p_{22}}) = \arg \max_{\theta_1, \theta_2, p_{11}, p_{22}} \ln L(\theta_{S_t}; U_x, U_y). \quad (3.16)$$

The parameters are obtained by solving Equation (3.16) and this is an optimization problem. It is equivalent to minimization of a negative log-likelihood function.

Until now, we have completed the whole process of estimation of a regime switching copula model. In the next subsection, we explain how a hedging strategy can be used based on this dynamic copula model.

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3.1.2 Determination of the approximate OHR under a regime-switching copula

In empirical hedging, we use historical observations to forecast the situation in the future. Then, based on the estimated model, we determine the hedging strategy with the objective of reducing the potential risk in the future. Thus, if we stand at time t and have T available previous observations, then our objective is to find an optimal hedge ratio for the next time $t + 1$.

The approximate optimal hedge ratio at time $t + 1$ could be obtained using a two-step procedure. As we can see from Equation (3.4), the first step is to get the predicted probabilities and corresponding copula parameters in each state at time $t + 1$. They are attainable by following the procedure described in Section 3.1.1, which models the regime switching copula on the available T observations prior to time $t + 1$. The updated filtered probabilities of being in each state at time t are obtained in step (g), and we denote them as $Pr(S_t = 1|\omega_t)$ and $Pr(S_t = 2|\omega_t)$, respectively. Moreover, by means of conducting estimation using Maximum likelihood method in step (h), we could get the estimated state transition probabilities $(\hat{p_{11}}, \hat{p_{22}})$ and the copula parameters $(\hat{\theta}_1, \hat{\theta}_2)$ in each state. Therefore, the predicted probabilities of being in each state at time $t + 1$ can be represented as:

$$
\begin{pmatrix} \pi_{1,t+1} & \pi_{2,t+1} \end{pmatrix} = \begin{pmatrix} Pr(S_t = 1|\omega_t) & Pr(S_t = 2|\omega_t) \end{pmatrix} \begin{pmatrix} \hat{p_{11}} & 1 - \hat{p_{11}} \\ 1 - \hat{p_{22}} & \hat{p_{22}} \end{pmatrix}. \tag{3.17}
$$

In Equation (3.17), $\pi_{1,t+1}$ and $\pi_{2,t+1}$ respectively give us the predicted probabilities of being

in state 1 or in state 2 at time $t + 1$ based on all information till time t.

Next, we calculate the optimal hedge ratios under each state using the corresponding estimated copula parameter. The procedure of finding the optimal hedge ratio is similar to the one we used in the non-switching model. The optimal hedge ratios give the minimum QRM of the hedged portfolio in each state. We denote these two hedge ratios as $h_{1,t+1}$ and $h_{2,t+1}$. They represent, respectively, the optimal hedge ratios at time $t + 1$ if the copula parameter is in state 1 or in state 2. The choice of the quantile risk measure in the objective function depends on the investors' preference and needs for risk management. Finally, considering the probabilities of being in each state at time $t+1$, we get the ultimate approximate optimal hedge ratio h_{t+1}^*:

$$h_{t+1}^* = \begin{pmatrix} \pi_{1,t+1} & \pi_{2,t+1} \end{pmatrix} \begin{pmatrix} h_{1,t+1} \\ h_{2,t+1} \end{pmatrix}. \tag{3.18}$$

The approximate optimal hedge ratio for time other than $t+1$ can be determined in the same way. For instance, take one time $t + i$, our objective is to determine the approximate optimal hedge ratio on that day. Firstly, all available observations prior to time $t + i$ are used to capture the spot-futures dependence. We could get the predicted probabilities of being in each state at time $t + i$, as well as the copula parameters in each state. In the second step, we find the optimal hedge ratios in each state using the corresponding copula parameter. They are obtained by minimization of the hedged portfolio risk using the N_{obs} observations prior to time $t + i$. Finally, according to our method, the approximated optimal hedge ratio at time $t + i$ is the weighted average of these two hedge ratios of each

state, weighted by the predicted probabilities of being in the corresponding state.

3.2 Adequacy of Regime Switching Model to Capture Time-Varying Dependence Structure

In this section, we conduct Monte Carlo simulations to test whether the extended regime switching model could estimate accurately the dependence structure between spot and futures return series. Two models are considered for comparison:

Model 1: Without regime switching in dependence (the copula parameter does not change over time, and it is specified as a constant value θ)

Model 2: With regime switching in dependence (the copula parameter shows regime shifts over time, and it follows a two-state regime switching model)

First of all, we simulate data sets from both of the two models. Then, they are used to be fitted by each of the two models. Thus, we consider four cases:

Case 1. **Source data come from Model 1, and are used to calibrate Model 1;**

Case 2. **Source data come from Model 1, and are used to calibrate Model 2;**

Case 3. **Source data come from Model 2, and are used to calibrate Model 1;**

Case 4. **Source data come from Model 2, and are used to calibrate Model 2.**

We measure the model performance by comparing the estimated copula parameter dynamics with the true parameter value in the simulation settings. In this way, we could figure

out whether our regime switching model works well for the data containing regime shifts in dependence or not. The performance of the regime-switching model is also compared to that of the non-switching model.

First we run preliminary simulation, for the sake of providing a first look at the comparison between different cases. Two sets of data are simulated as the source data: One is a bivariate time series of size T=1250 generated from Model 1. It is specified as Clayton copula with non-switching parameter $\theta = 10$ and normal marginal distributions with $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 0.5$. Another data set is simulated from Model 2. It is also specified as Clayton copula with normal marginal distributions ($\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 0.5$). However, the copula parameter follows a two-state regime switching model:

$$\theta_{S_t} = \begin{cases} 3, & S_t = 1, \text{ in State 1} \\ 18, & S_t = 2, \text{ in State 2,} \end{cases} \tag{3.19}$$

with a constant state transition probability matrix $\mathbf{P}$. We use p_{ij} to represent the probability of moving from state i at time t to state j at time t+1, and they are set as $p_{11} = 0.95$, $p_{12} = 0.05$, $p_{21} = 0.1$ and $p_{22} = 0.9$. Having simulated data, we respectively use Model 1 and Model 2 to fit each set of data. The estimation results are explained below.

When the simulated data come from Model 1 (non-switching dependence), the estimation results that fit Model 1 and Model 2 are exhibited in Figure 3.1. In the simulation, the copula parameter θ was fixed and equal to 10. The results suggest that both models can estimate this parameter quite accurately. The estimated copula parameter in Model 1 is 9.5304, which is very close to the true parameter value. This result is expected as

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observations were simulated from this model. In Model 2, there are two regime states specified, and we get the estimated copula parameter in each state with $\hat{\theta}_1 = 9.3096$ and $\hat{\theta}_2 = 12.9532$. The hidden Markov process is supposed to move between two states. However, as we can see from Figure 3.2, the filtered probability of being in state 1 is always much larger than the probability of being in state 2. This explains why the estimated copula parameter in Model 2 keeps staying in state 1 with the value of 9.3096 in Figure 3.1. The estimated copula parameter in Model 2 is close to the true value of 10, and hence the estimated Model 2 is not far from the true one.

When the simulated data come from Model 2 (with regime shifts in dependence), the estimation results that fit Model 1 and Model 2 are depicted in Figure 3.3. Comparing with Figure 3.1, Figure 3.3 suggests that Model 2 outperforms Model 1 significantly in this situation. The upper sub-figure in Figure 3.3 compares the dependence dynamics in the simulation setting, Model 1 fitting and Model 2 fitting. The copula parameter in the two states are specified as 3 and 18 in the simulation, corresponding to relatively weak and strong dependence. However, the constant copula parameter of 4.52 is estimated from Model 1. Apparently, it is far from the true value and may result in significant mistakes in dependence estimation. In contrast, the performance estimated from Model 2 seems much better, which can be seen from the overlapped graphs in the lower sub-figure in Figure 3.3. Estimated copula parameters in the two states are respectively equal to 3.0861 and 18.0425, which are very close to the true values of 3 and 18. Moreover, the estimated state transition probabilities are $\hat{p_{11}} = 0.9423$ and $\hat{p_{22}} = 0.9012$, while the true values were $p_{11} = 0.95$ and $p_{22} = 0.9$. In conclusion, the estimated Model 2 captures well the true dependence dynamic. We have calculated that 137 observations among the whole 1250

Figure 3.1: Comparison between the two models to fit data without regime shifts in the dependence. The source data is simulated from Clayton copula and normal marginal distributions.
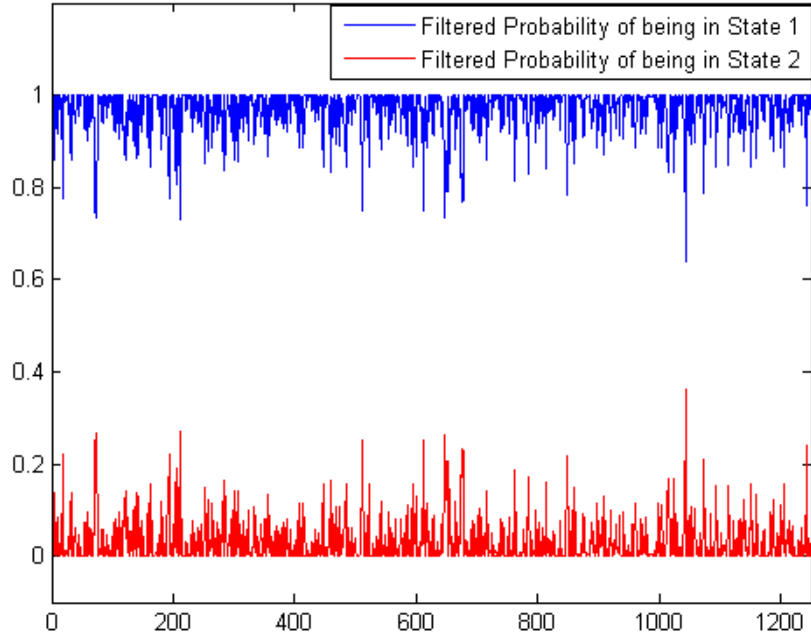
Figure 3.2: Filtered probabilities of being in state 1 and state 2 when using Model 2 to fit simulated data without regime-switching dependence

observations get the wrong estimated state, accounting for around 13.84%.

The above results suggest that Model 1 and Model 2 can both provide quite accurate estimation of the dependence dynamics for data without regime shifts in dependence. However, for data with regime shifts in the dependence structure, the estimated Model 1 can depart quite dramatically from the true model, especially when the copula parameters in two states are very different. Although there exists some estimation error by fitting Model 2 to the data, our estimation procedure provides reasonably accurate results. Since in practice it is difficult to judge from a plot of a time series whether the spot and futures contain regime shifts in the dependence or not, Model 2 provides us a better choice to
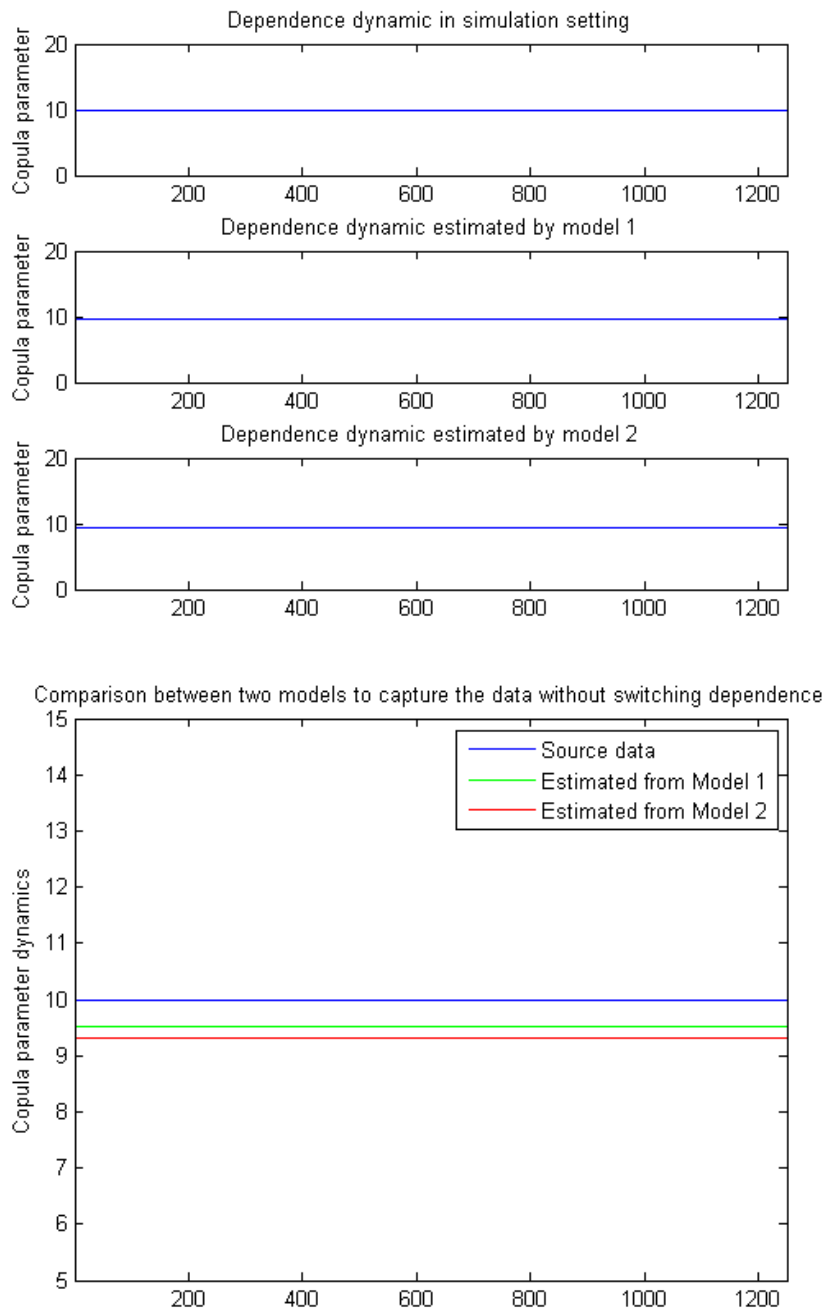
63
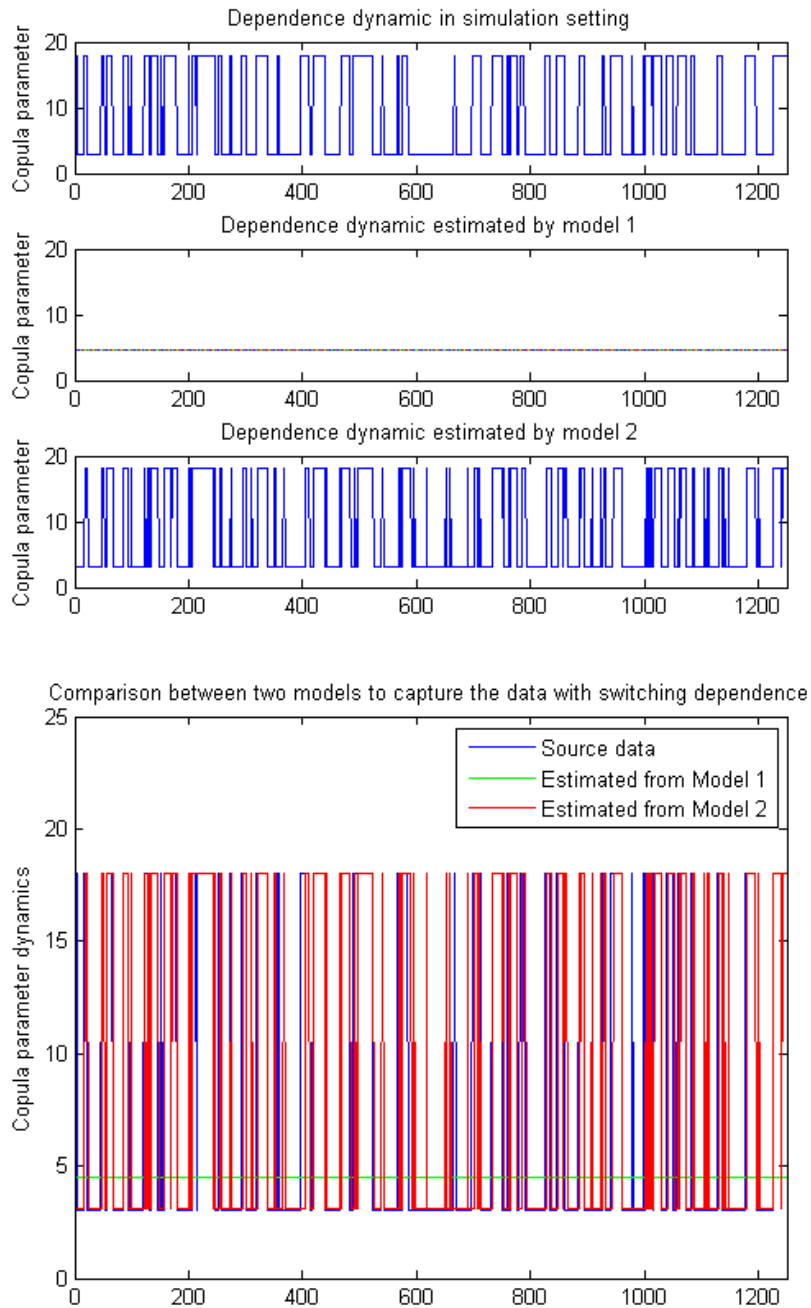
Figure 3.3: Comparison between the two models to fit data with switching dependence. The source data is simulated from Clayton copula and normal marginal distributions, and the copula parameter follows a two-state regime switching model.

capture the dependence between the spot and futures.

In the remainder of this section, we extend our previous tests by considering Model 2 with different copula functions. We also investigate how the sample size impacts accuracy of our estimation procedure.

Specifically, except data simulated from Clayton copula, data characterized by Gumbel copula and Frank copula are added to test the accuracy of the estimation procedure using Model 2. Table 3.1 lists estimation results when Model 2 is fitted to data simulated from these three different copulas and normal marginal distributions ($\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 0.5$), which are denoted as Cases 1-3 respectively. Given the type of a copula, the copula parameter follows a two-regime switching model to reflect the dynamic dependence. In each case, the bivariate data sets of sizes T=1000, 2000 and 5000 are simulated and we apply Model 2 with correctly specified copula and marginal distributions to fit these data one by one, so that we could get the corresponding estimated parameters. We replicate this process 1000 times and obtain the mean square errors (MSE) of each estimated parameter. The results are shown in Table 3.1.

In the table, we also show estimation results when Model 1 is fitted. As we can see from the numbers in the second column in Table 3.1, the estimated copula parameter departs dramatically from the true values. For example in Case 1, the Clayton copula parameter is set as 3 in state 1 and 18 in state 2. However, when we use Model 1 to fit such simulated data with size T=1000, we get estimated constant copula parameter as 4.2288. The simulated data are generated with two-regime shifts in the copula parameter, but Model 1 assumes that the copula parameter does not change over time. Therefore it is

Table 3.1: Estimation performance of Model 2 to capture the dependence dynamics

Case 1: Data simulated with Clayton copula and normal marginal distributions
True values of parameters: $\theta_1 = 3$, $\theta_2 = 18$, $p_{11} = 0.95$, $p_{22} = 0.9$

Sample	Model 1	Model 2				
Size	mean($\hat{\theta}$)	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)	MSE($\hat{p_{11}}$)	MSE($\hat{p_{22}}$)	Error(%)
T=1000	4.2288	2.7502	9.9108	0.0009	0.0198	13.7674
T=2000	4.2360	1.9089	12.4995	0.0020	0.1136	15.3765
T=5000	4.2321	0.7811	1.8711	0.0002	0.0066	12.0889

Case 2: Data simulated with Gumbel copula and normal marginal distributions
True values of parameters: $\theta_1 = 2$, $\theta_2 = 15$, $p_{11} = 0.95$, $p_{22} = 0.9$

Sample	Model 1	Model 2				
Size	mean($\hat{\theta}$)	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)	MSE($\hat{p_{11}}$)	MSE($\hat{p_{22}}$)	Error(%)
T=1000	2.6552	0.0059	3.3793	0.0001	0.0005	8.3416
T=2000	2.6475	0.0032	1.3544	0.0001	0.0002	7.9060
T=5000	2.6365	0.0012	0.3714	0.00002	0.0001	7.7103

Case 3: Data simulated with Frank copula and normal marginal distributions
True values of parameters: $\theta_1 = 2$, $\theta_2 = 15$, $p_{11} = 0.95$, $p_{22} = 0.9$

Sample	Model 1	Model 2				
Size	mean($\hat{\theta}$)	MSE($\hat{\theta}_1$)	MSE($\hat{\theta}_2$)	MSE($\hat{p_{11}}$)	MSE($\hat{p_{22}}$)	Error(%)
T=1000	3.8833	0.0786	1.4708	0.0002	0.0006	14.8660
T=2000	3.9683	0.0420	0.8851	0.0001	0.0003	14.5170
T=5000	3.8668	0.0180	0.3412	0.00004	0.0001	14.3640

Simulated data are generated from different types of copula and with different sample sizes. Given the copula type, the copula parameter follows a two-state regime switching model to reflect the dynamic dependence.

not surprising to get the estimated copula parameter dramatically departing from the true values. Such results are also consistent with what we found above in this section, which suggests that Model 1 is not appropriate to capture the dependence for data containing dependence shifts.

Columns 3-6 list the mean square errors of the copula parameters and the state transition probabilities estimated when using Model 2. In all cases, we can see that the state transition probabilities achieve quite good estimation performance with very low errors. A bit larger estimation errors are shown in copula parameters, but they still can be regarded as acceptable. Also, the estimated copula parameter in state 1 is relatively closer to the true value compared to that in state 2, reflecting the better estimation.

In the last column, the 'Error(%)' means the average percentage of observations that are estimated to be in a wrong state. These values are obtained by the following steps: the states of the dependence through all the observations are first recorded in each run of simulation, then we compare them with the estimated dependence states from fitting Model 2, and count the number of observations estimated in the wrong state. In each case, the simulation process is repeated by 1000 times, and we average all the results and finally get the listed percentage values to represent the estimation performance of Model 2. For an additional explanation, let us consider Case 1 with T=1000. The number of 13.7674% means that nearly 138 among 1000 observations are estimated as being in wrong states. All values of 'Error(%)' are less than 16%, and they do not change so much when sample size varies. This proves the good estimation performance when using Model 2 to capture dependence even for data with different sizes.

Meanwhile, by comparing these three cases with different copula-featured data, we can also see some differences in terms of estimation performance. Overall, Model 2 provides a better fit for data simulated from Gumbel copula, although the superiority is not obvious. First of all, the MSEs of parameters are a little lower in Cases 2 and 3 than that in Case 1, illustrating better parameter estimation to data simulated from Gumbel copula and

Frank copula. Moreover, the values of 'Error(%)' (state error percentage) are quite smaller in Gumbel case than in the other two cases. Only around 8% of observations generated by Gumbel copula are estimated with wrong states, while for other two types of copulas, the percentage could reach 14%. However, we believe all these errors are in an acceptable range. In conclusion, in all cases that we have considered, the hidden state, which describes the current value of the copula parameter, can be filtered reasonably accurately.

Last but not least, most cases show that the estimation errors are decreasing when the sample size increases, implying that more accurate estimation results might be obtained by increasing the size of observations. But the improvement is quite modest, even by increasing the observations from 1000 to 5000. Moreover, there also exist cases when estimation performance deteriorates for larger sample sizes, which is a bit strange.

3.3 Hedging Effectiveness of the extended Regime-Switching Hedging Strategy

In the last section, we have demonstrated that a regime switching model for time-varying dependence can be reasonably accurately estimated. In this section, we apply the model to our extended hedging strategy, which we described in Section 3.1. Comparing to the original hedging strategy, the extended strategy allows for regime shifts in the spot-futures dependence, where we assume that the copula parameter follows a two-state regime switching model.

In this section we investigate, using simulated data, whether the extended hedging

strategy improves hedging effectiveness when compared with the original non-switching strategy. The hedging objective is to minimize risk of the hedged portfolio. Among different types of quantile risk measure, we choose VaR to investigate whether the extended hedging strategy could improve hedging effectiveness by employing the regime switching model. The hedging effectiveness of the strategy is measured by the percentage the VaR of the portfolio is reduced after hedging. It can be represented as

$$HE = \text{VaR Reduction} = 1 - \frac{\text{VaR}_{1-\alpha}(R^h)}{\text{VaR}_{1-\alpha}(R^S)}, \tag{3.20}$$

where 1-α is the confidence level of VaR, and $\text{VaR}_{1-\alpha}(R^h)$, $\text{VaR}_{1-\alpha}(R^S)$ stand for value-at-risk of the portfolio after and before hedging.

A set of bivariate data is simulated with Clayton copula and normal marginal distributions ($\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 0.012$)[3]. The copula parameter follows a two-state regime switching model, where the parameter value is specified as 10 and 18 respectively in state 1 and state 2. According to the relationship between Kendall's τ correlation and Clayton copula parameter, the copula parameter of 10 and 18 means the Kendall's τ correlation between the two coordinates can reach 83.3% and 90% accordingly. They are basically in line with the actual situation where the coordinates are futures and spot returns, respectively. The set of simulated data consists of 2250 pairs of observations, and the last 1000 pairs of observations are employed as evaluation period to test hedging effectiveness of the strategy. For determining the approximate optimal hedge ratio for each day, all previous

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[3]Here, we set the mean and standard deviation of returns as 0 and 0.012 based on our sample data in empirical study. In next chapter, FTSE index and its corresponding future contracts are applied into our model in empirical study, so we would like to make our simulation more targeted.

available observations are used to estimate the regime switching model to capture dependence dynamics, so that we could predict the dependence or specific copula parameter on that day. According to Section 3.1.2, the approximate optimal hedge ratio on each day is represented as the weighted average of the two optimal hedge ratios in each state, which can be obtained by the minimization of VaR of the hedged portfolio using the previous $N_{obs} = 250$ observations and corresponding estimated copula parameter. We conduct this dynamic hedging strategy using a day-by-day roller, and finally calculate hedging effectiveness of the strategy. VaR with different confidence levels are also considered: $\text{VaR}_{90\%}$, $\text{VaR}_{95\%}$ and $\text{VaR}_{99\%}$. The results are presented in Table 3.2.

The table compares hedging effectiveness of the extended regime-switching strategy with that of the original non-switching strategy when we respectively use minimum $\text{VaR}_{90\%}$, $\text{VaR}_{95\%}$ and $\text{VaR}_{99\%}$ of the hedged portfolio as the objective function. Except for the case of minimum $\text{VaR}_{90\%}$, the other two cases show that the extended hedging strategy with regimes-switching dependence does outperform the original strategy, although the improvement is rather modest. This raises up the question as to why the more accurate model of the time-varying dependence does not improve more significantly the effectiveness of hedging.

To figure out this problem, we further investigate the case that minimizes $\text{VaR}_{90\%}$ as the hedging objective. In Figure 3.4, we show the estimated time-varying copula parameter and the hedge ratio for the two strategies.

As we can see from the graph, the estimated hedge ratios for the two strategies are quite similar, with both values ranging from 0.85 to 1.05. This might be largely due to the

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Table 3.2: Comparison of hedging effectiveness between extended regime-switching strategy and the original non-switching strategy (Case 1)[1]

Data simulated with Clayton copula and normal marginal distributions						
Parameter setting: $\theta_1 = 10$, $\theta_2 = 18$, $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 0.012$						

Minimum $VaR_{90\%}$

Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	$VaR_{90\%}$	Mean(OHR)	$VaR_{90\%}$	HE	Mean(OHR)	$VaR_{90\%}$	HE
0	0.01491	0.9332	0.00355	76.1683%	0.9356	0.00361	75.8001%

Minimum $VaR_{95\%}$

Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	$VaR_{95\%}$	Mean(OHR)	$VaR_{95\%}$	HE	Mean(OHR)	$VaR_{95\%}$	HE
0	0.01943	0.8754	0.00532	72.6211%	0.8799	0.00527	72.8856%

Minimum VaR$_{99\%}$

Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	$VaR_{99\%}$	Mean(OHR)	$VaR_{99\%}$	HE	Mean(OHR)	$VaR_{99\%}$	HE
0	0.02734	0.7626	0.00893	67.3347%	0.7688	0.00861	68.4870%

[1] Simulated data are generated from Clayton copula and normal marginal distributions. The copula parameter follows a two-state regime switching model with quite close values in two states, specified as 10 at state 1 and 18 at state 2.

fact that the copula parameters in the two states are set very close to each other. $\theta_1 = 10$ represents the Kendall's τ correlation of 83.33% between data while $\theta_2 = 18$ means the correlation goes to 90%. So the small gap between the dependence levels in the two states might result in the minor improvement of hedging effectiveness when using the extended regime switching strategy.

To confirm this conjecture, we employ a different set of parameters. The simulated data

Figure 3.4: Estimated time-varying copula parameter and OHR for the two strategies. In the simulated data, there exist regimes shifts in the dependence but the dependence levels in two states are quite close.

are still generated with normal marginal distributions ($\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 0.012$) and Clayton copula, where the copula parameter follows a two-state regime switching model. However, instead of using two close values as the copula parameters in the two states, we set the parameter in the two states as 1 and 38. Thus, in state 1 the Kendall's τ correlation is just 33.33% while it can reach 95% in state 2. The dependence between spot and futures return series varies more drastically.

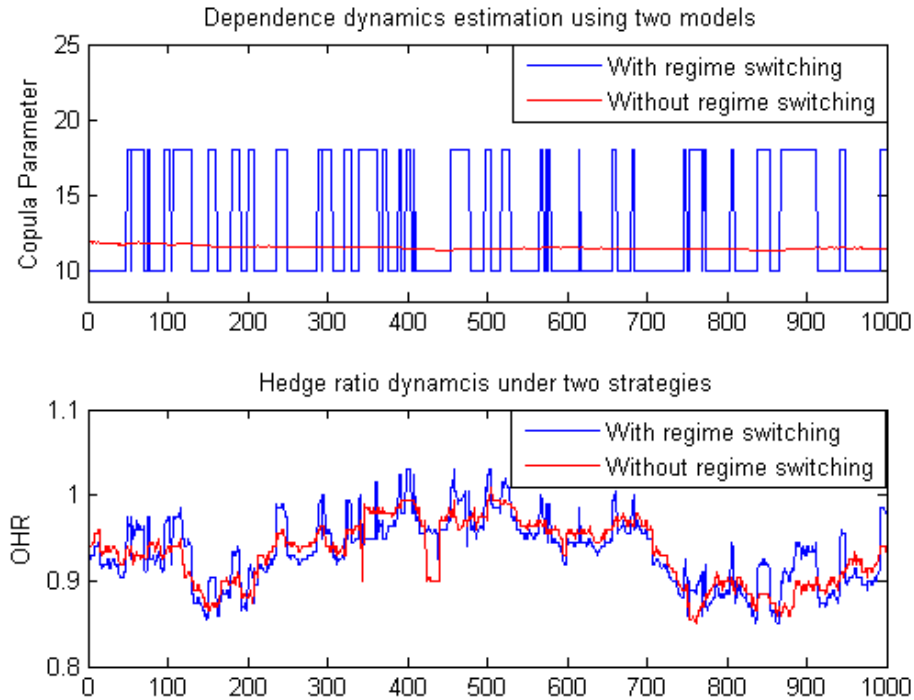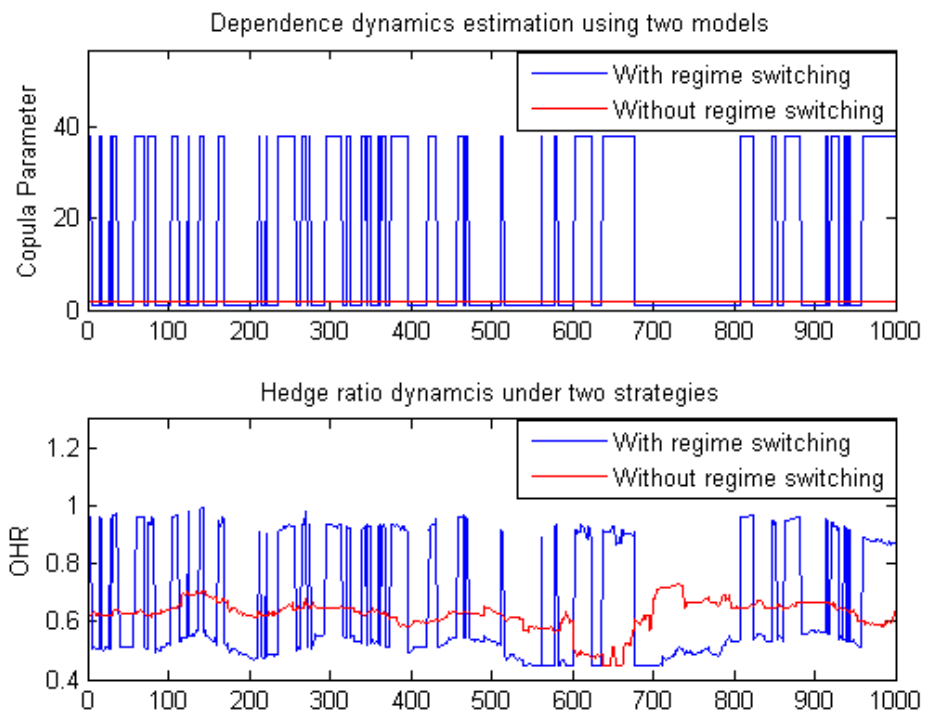Figure 3.5 exhibits the new estimated time-varying copula parameter and the hedge

Figure 3.5: Estimated time-varying copula parameter and OHR for the two strategies. In the simulated data, there exist regimes shifts in the dependence and the dependence levels in the two states are significantly different.

ratio for the two strategies, using new simulated data. The results are quite different from Figure 3.4, since now the two hedging strategies exhibit quite different performance. The estimated copula parameter under the original non-switching hedging strategy keeps moving around 1.75, which is close to the true copula parameter value at state 1, but largely departs from the true copula parameter value at state 2. Therefore, if the dependence is supposed to be in state 2 with the copula parameter of 38, this leads to a large estimation error when using the original non-switching hedging strategy. Comparison results of hedging effectiveness between the extended regime-switching strategy and the original non-switching strategy using the new simulated data are listed in Table 3.3.

All the results are consistent with our analysis above. Applied to the new simulated data, the extended hedging strategy has shown better performance in hedging when compared with the original non-switching hedging strategy. For example, when we use minimum $VaR_{90\%}$ as the hedging objective, the hedging effectiveness of the extended strategy can reach 30.4457% while for the original hedging strategy is only 26.0891%. The cases that use minimizing $VaR_{95\%}$ and $VaR_{99\%}$ lead to similar conclusion. We can summarize our finding as follows:

1. For data containing more significant regime shifts in the dependence, meaning that the dependence levels in different regimes are quite different, our extended hedging strategy based on a regime-switching model proves to be more efficient than the one based on a fixed copula.

2. On the other hand, when the gap between strong dependence and weak dependence is not significant, the extended strategy shows no advantage in improving hedging ef-

fectiveness compared with the original non-switching strategy. This is also suggested by the results in Table 3.2. Therefore, in such cases, it is not necessary to consider the regime switching in the dependence, and the original hedging strategy also works well for time-varying copula parameters.

Table 3.3: Comparison of hedging effectiveness between extended regime-switching strategy and the original non-switching strategy (Case 2)[1]

Data simulated with Clayton copula and normal marginal distributions						
Parameter setting: $\theta_1 = 1$, $\theta_2 = 38$, $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 0.012$						

Minimum VaR$_{90\%}$						
Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy	
OHR	VaR$_{90\%}$	Mean(OHR)	VaR$_{90\%}$	HE	Mean(OHR)	VaR$_{90\%}$	HE
0	0.01491	0.6274	0.01102	26.0891%	0.6575	0.01037	30.4475%

Minimum VaR$_{95\%}$						
Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy	
OHR	VaR$_{95\%}$	Mean(OHR)	VaR$_{95\%}$	HE	Mean(OHR)	VaR$_{95\%}$	HE
0	0.01917	0.5668	0.01461	23.8144%	0.6253	0.01418	26.0243%

Minimum VaR$_{99\%}$						
Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy	
OHR	VaR$_{99\%}$	Mean(OHR)	VaR$_{99\%}$	HE	Mean(OHR)	VaR$_{99\%}$	HE
0	0.02731	0.5459	0.02151	21.2399%	0.6342	0.02035	25.4692%

[1] Simulated data are generated from Clayton copula and normal marginal distributions. The copula parameter follows a two-state regime switching model with very different values in two states, specified as 1 at state 1 and 38 at state 2.

In order to provide a more comprehensive analysis, we now consider the more extreme condition where the dependence between data even does not show any regime shifts. That

is, the dependence level in simulated data is fixed. For this, we generate bivariate data using normal marginal distributions ($\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 0.012$) and Clayton copula with a constant parameter 10. Using the simulated data, we can compare hedging effectiveness between the extended regime-switching strategy and the original non-switching hedging strategy. The results are shown in Table 3.4.

The numbers suggest that there is no significant difference between two hedging strategies in terms of hedging effectiveness. All the differences in hedging effectiveness between these two strategies are less than 0.3%, while the extended switching strategy only dominates the original non-switching model in the cases of minimizing $\text{VaR}_{90\%}$ and $\text{VaR}_{95\%}$. These results reaffirms our conclusion that the extended hedging strategy shows superior only when the data exhibit significant regime shifts in the dependence. Moreover, the results show that it is safe to use regime switching models even in cases without regime switching.

A large number of researches have discussed whether the more complicated model, such as regime switching model, could definitely improve the hedging effectiveness. However, the results vary and suggest that the more complicated model does not always work and the characteristic of sample data should first be taken into consideration. Just as our case shows, using a more accurate model to capture the dependence between data series should lead to more efficient hedge ratio and higher hedging effectiveness. However, if the dependence itself does not show big difference in different regimes, then the hedging effectiveness could not be significant improved by using the extended model. Moreover, using regime switching model also takes more efforts, such as computation time. It is quite necessary and important to first study the characteristic of data before applying a model.

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Table 3.4: Comparison of hedging effectiveness between extended regime-switching strategy and the original non-switching strategy (Case 3)[1]

Data simulated with Clayton copula and normal marginal distributions							
Parameter setting: $\theta_1 = \theta_2 = 10$, $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 0.012$							

Minimum VaR$_{90\%}$

Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	VaR$_{90\%}$	Mean(OHR)	VaR$_{90\%}$	HE	Mean(OHR)	VaR$_{90\%}$	HE
0	0.01590	0.8893	0.00350	77.9925%	0.8956	0.00348	78.1337%

Minimum VaR$_{95\%}$

Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	VaR$_{95\%}$	Mean(OHR)	VaR$_{95\%}$	HE	Mean(OHR)	VaR$_{95\%}$	HE
0	0.01920	0.8352	0.00495	74.2076%	0.8435	0.00490	74.4614%

Minimum VaR$_{99\%}$

Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	VaR$_{99\%}$	Mean(OHR)	VaR$_{99\%}$	HE	Mean(OHR)	VaR$_{99\%}$	HE
0	0.02888	0.7941	0.00970	66.3933%	0.8023	0.00980	66.0700%

[1] Simulated data are generated from Clayton copula and normal marginal distributions. The copula parameter no long follows a regime switching model, but is set as a constant value of 10.

Finally, in addition to the main findings regarding hedging effectiveness of the extended hedging strategy, we can draw other conclusions from Tables 3.2 - 3.4. For example, we may consider the question as to how the approximate optimal hedge ratio and hedging effectiveness change when we use VaR as risk measure but at different confidence levels in hedging objective function. Our results suggest answers that seem consistent with our findings in Chapter 2, where we looked at factors influencing the optimal hedge ratio and hedging effectiveness:

1. The hedging effectiveness is a decreasing function of the confidence level. That is to say, other things being equal, if we use VaR with a higher value of confidence level as the risk measure in hedging objective function, $e.g.$ changing from $VaR_{90\%}$ to $VaR_{95\%}$, it would yield lower hedging effectiveness. The increase of confidence level means that we aim to place more emphasis on the extreme tail risk management, so it makes sense that the hedging performance will gradually decline.

2. The hedging effectiveness is a positive function of the degree of dependence between the spot and futures position. This holds regardless of which risk measure is chosen for risk criteria in hedging objective function. In Table 3.2, the copula parameters in the two states are set as 10 and 18, and they are fixed as 10 in Table 3.4. Thus it is apparent that the copula parameter or dependence level in the case of Table 3.2 is never lower than that in the case of Table 3.4. Consequently, the hedging effectiveness in Table 3.2 always stays a little higher than that in Table 3.4 with the same hedging strategy. The dependence between the spot and futures always plays the most important role in determining the efficiency of the hedging strategy.

Chapter 4

Results of Empirical Analysis

In this chapter, we use the empirical data of FTSE 100 (Financial Times and London Stock Exchange Index) and its corresponding futures contracts to perform an out-of-sample test of hedging performance. Both the extended minimum QRM hedging strategy that considers regime shifts in the dependence and the original non-switching hedging strategy are applied, whose methodology we described in Chapters 2 and 3. We aim to compare their hedging effectiveness in the empirical application. In our implementation, we employ VaR as the criterion for optimal hedging, but other quantile risk measures can be applied in the same way.

4.1   Data Description

We collect daily closing prices for the FTSE 100 index and settlement prices for the corresponding futures contract, spanning from February 28, 2006 to January 26, 2015. Futures

prices are treated as a continuous series by rolling over maturity on the first day of the delivery month. All data are obtained from Bloomberg, and they account for nearly 10 years of observations. We take returns of the spot and futures for analysis in our model, therefore the price data are changed into log-return data by using the following formulae:

$$s_t = \ln(1 + \frac{S_t - S_{t-1}}{S_{t-1}}), \tag{4.1}$$

$$f_t = \ln(1 + \frac{F_t - F_{t-1}}{F_{t-1}}), \tag{4.2}$$

where S_t and F_t are the spot and futures prices at time t respectively. Consequently, we get a total number of 2250 log-return observations. The returns are depicted in Figure 4.1.
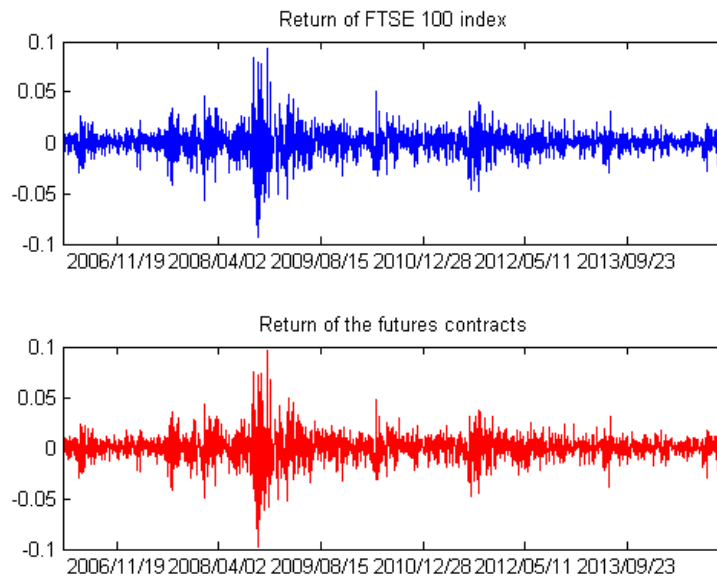

Figure 4.1: Log-returns of FTSE 100 index and its corresponding futures contract. The data range from March 1, 2006 to January 26, 2015, with a total number of 2250 observations.

As we can see from the graphs of returns, the spot and futures returns almost coincide with each other, suggesting a high positive correlation. High correlation between the spot and futures lowers the basis risk. As a result, we are expecting to get quite good hedging performance.

Table 4.1 presents descriptive statistics of our empirical spot and futures returns, as well as the spot-futures dependence measures. We can see from Panel A that both the spot and futures returns show the characteristic features of non-normality. There exhibit excess kurtosis and non-positive skewness in the spot and futures returns. According to the Jarque-Bera test, we have verified that the normality hypothesis is rejected for all cases, at every confidence level. This implies that an assumption of normality of the marginal distributions is not suitable, as it may lead to incorrect inference. In the following implementation of our models, we still use empirical distribution to estimate the marginal distributions, which we explained in Section 2.2.1.

In Panel B, we measure the spot-futures dependence for different periods in our sample using Pearson's ρ and Kendall's τ correlation. Such dependence measures are listed in the table. The results suggest that there still exist varying levels of dependence through time, while Figure 4.1 indicates strong dependence between spot and futures returns overall. For example, the Pearson's ρ could reach around 0.98 in the period from 1/3/2006 to 16/2/2009. However, the corresponding value of Pearson's ρ is just 0.86 for the time interval from 4/2/2013 to 29/1/2014. Thus, it seems reasonable that we apply dynamic model to capture the time-varying dependence between spot and futures returns.

Table 4.1: Summary statistics and dependence measures for FTSE 100 Index and its futures return series

Panel A. Summary Statistics		
	Spot	Futures
mean(%)	0.0067	0.0064
Standard deviation(%)	1.2768	1.2715
skewness	-0.1326	-0.2074
kurtosis (excess)	10.6379	10.6346
JB test	1	1
Panel B. Dependence Measures		
	Pearson's ρ	Kendall's τ
01/3/2006—23/2/2007	0.9833	0.8834
26/2/2007—20/2/2008	0.9869	0.8854
21/2/2008—16/2/2009	0.9896	0.8896
17/2/2009—11/2/2010	0.9678	0.8619
12/2/2010—08/2/2011	0.9591	0.8376
09/2/2011—06/2/2012	0.9865	0.8997
07/2/2012—01/2/2013	0.9715	0.8433
04/2/2013—29/1/2014	0.8618	0.8063
30/1/2014—26/1/2015	0.9713	0.8621

The empirical data represent the log-returns of FTSE 100 index and its corresponding futures contract. They range from March 1, 2006 to January 26, 2015, with a total number of 2250 observations. We conduct the hedging on the last 1000 observations as the out-of-sample hedging effectiveness test, in order to compare the hedging performance with the two models. In the Jarque-Bera test, the value of 1 indicates the rejection of the null hypothesis that data come from a normal distribution.

4.2 Hedging with Empirical Data

For our empirical data, after estimating the model, we apply the hedge on the last 1000 observations, which corresponds to the time period from February 9, 2011 to January 26, 2015. The hedging performance is then measured over this period. As the main objective of our analysis is a comparison of hedging effectiveness between the extended regime-switching hedging strategy and the original non-switching strategy, both of these two models are conducted and we can find the detailed procedures in Chapters 2 and 3. On each day t, we determine the hedge ratio for the next day $t + 1$ by following two steps:

1. **Use a dynamic copula model to fit the previous $N_{copu} = 1250$ observations (including the observations at time t), and then predict the spot-futures dependence at time $t + 1$.**

   According to Chapter 3, the extended hedging strategy uses a two-state regime switching model for the copula parameter to capture regime shifts in the dependence. In contrast, the original hedging strategy assumes a constant copula parameter.

2. **Determine the optimal or approximate optimal hedge ratio for time $t + 1$ using the previous $N_{obs} = 250$ observations.**

   Under the non-switching hedging strategy, the optimal hedge ratio is simply determined by minimizing the risk of the hedged portfolio. However, when we apply the extended strategy, first we need to calculate the optimal hedge ratio in each state using the corresponding estimated copula parameter of the state. Then we use the weighted average of these obtained hedge ratios as the approximate optimal hedge ratio at time $t + 1$.

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In contrast to what we did in Chapter 3, in Step 1 we use the dynamic copula model to fit the previous 1250 observations instead of all the available observations. We have decided not to use more observations since only recent ones are relevant for predictions. In addition, as our simulation results in Chapter 2 suggest, this number of observations is sufficient to achieve an acceptable level of accuracy. On each day, we keep updating our information set to estimate the spot-futures dependence and to find the new optimal hedge ratio, which is used to construct the hedging portfolio on the subsequent day. Finally, the hedging effectiveness is measured by the risk reduction of the hedged portfolio due to hedging. In the following, we compare the hedging process under the two models and show the results in detail.

4.2.1 Estimation of the spot-futures dependence

Recalling the procedure outlined in Section 3.1.1, to determine a copula that captures properly the dependence structure between spot and futures returns, we first need to identify the appropriate copula that best fits the observations, and then estimate the copula parameter. As the copula parameters under the two hedging strategies follow different processes, the estimation of the copula parameters differs under the two models. However, the method is the same when we identify the appropriate form of the copula function.

Identify the appropriate copula

In this step, we still follow the method in Genest and Rivest (1993) to identify the appropriate copula from three given Archmedean copulas: Clayton, Gumbel and Frank. We have illustrated the complete identification process in Section 3.1.1 and the detailed algorithm can be found in Appendix B. According to Equations (3.5) and (3.6), the inspection of the MSE and T-MSE allows us to identify the optimal copula function when we apply the minimum QRM hedging strategy. MSE denotes the mean square error of the estimated copula parameters and measures the goodness of fit over all the quantiles of the empirical probability distribution. T-MSE denotes the tail-mean square error of the estimated copula parameters and measures the goodness of fit over the tail quantiles (at α probability level) of the empirical probability distribution.

In our empirical analysis with FTSE 100 index and its futures contracts, we employ VaR as a risk measure. We compare both the estimated MSE and T-MSE for different copulas, which are shown in Table 4.2.

Table 4.2: Comparison of performance based on different copulas

	MSE	T-MSE $(1-\alpha=90\%)$	T-MSE $(1-\alpha=95\%)$	T-MSE $(1-\alpha=99\%)$
Clayton	0.3685×10^{-3}	3.5053×10^{-3}	0.8930×10^{-3}	0.0387×10^{-3}
Gumbel	0.3007×10^{-3}	4.4327×10^{-3}	1.2332×10^{-3}	0.0638×10^{-3}
Frank	1.2471×10^{-3}	10.8559×10^{-3}	4.2002×10^{-3}	0.4225×10^{-3}

From the comparisons of the fitting performance for different copulas, it is apparent that Clayton copula outperforms the other two copulas in every case as its corresponding

values of MSE and T-MSE are always the lowest. Therefore, Clayton copula seems to be a good fit to our empirical data and we will use it in the following implementation.

Estimation of the copula parameter using Maximum Likelihood

After identifying the form of the copula, Maximum Likelihood Estimation (MLE) method can be applied to estimate the copula parameter, as illustrated in Section 3.1.1. On each day during the hedging evaluation period, we estimate the dynamic copula parameter with both the extended regime-switching model and the original non-switching model. The estimated time-varying copula parameter for these two models is exhibited in Figure 4.2.

From the graph, we can observe the significant difference between the estimated copula parameter under the two models. Under the original non-switching model, the estimated copula parameter keeps moving between 8 and 10 throughout the time. Thus, under this model, the estimated spot-futures dependence exhibits quite a stable status with minor variation. In contrast, the extended regime-switching model allows for regime shifts in the dependence, and thus the estimated copula parameter switches between the relative strong and weak dependence states. At one time, it can be in the strong dependence state with the copula parameter of 15. However, at another time, it could jump to the weak dependence state with the parameter of 2. The allowance of regime shifts in dependence could capture the sudden change in dependence, therefore providing more accurate estimation of the dependence dynamics. However in the graph, except for some points showing very low values of the estimated copula parameter, most values of the estimated copula parameter still cluster either in the range between 9 and 10 or 12 and 15. This fact suggests that
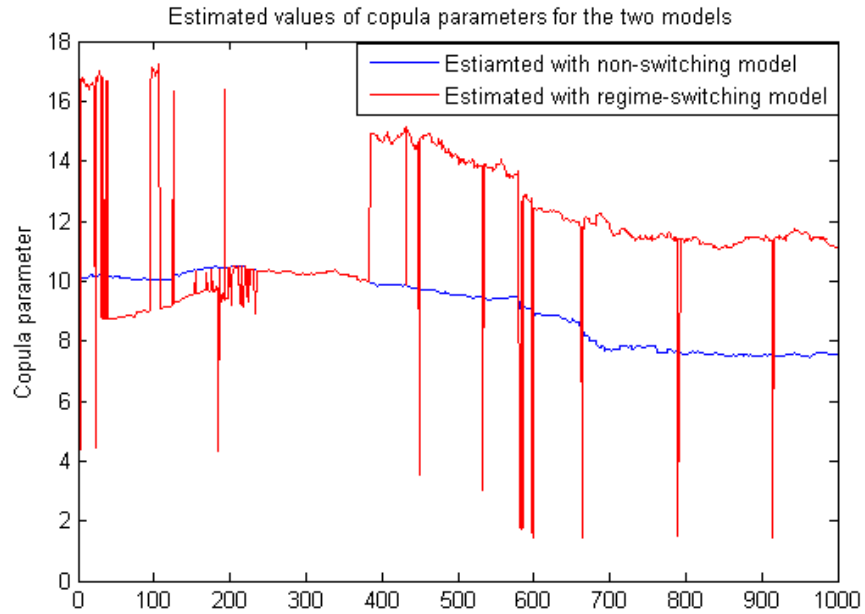
Figure 4.2: Estimated values of the copula parameters for the two models. On each day during the hedging evaluation period, the extended regime-switching model and the non-switching model are respectively used to fit the previous 1250 observations. The estimated copula parameter on each day under each model is reflected in the graph, representing the dependence dynamics.

even if there is a difference in the estimated copula parameter dynamics under the two models, it may not be significant for the purpose of hedging. In the following subsection, we investigate more closely hedging strategies based on estimated hedge ratios under the two models.

4.2.2 Estimation of the optimal hedge ratios

Given the estimated time-varying copula parameter, the determination of the optimal hedge ratio on each day is different under the two different models. We have explained the

details in Section 3.1.2. Under the original non-switching model, the copula parameter is regarded as a relative average dependence value over a period and can directly represent the spot-futures dependence at time t. The optimal hedge ratio is determined by the minimization of the risk of the hedged portfolio. However, under the extended regime-switching model, it is uncertain that the copula parameter on each day is in the strong dependence state or weak dependence state. Each dependence state has its corresponding estimated copula parameter value and that leads to two different hedge ratios. In this case, the optimal hedge ratio is approximated as the weighted average value of these two different hedge ratios, weighted by the predicted probabilities of being in each state.

In our implementation, we choose the hedging objective as minimum VaR of the hedged portfolio respectively at the confidence levels of 90%, 95% and 99%. In each case, the estimated hedge ratios under the non-switching and regime switching models are compared in Figure 4.3. We are trying to see whether these two models result in significantly different hedging ratios. Apparently, the estimated optimal hedge ratio dynamics does not suggest significant difference under the two models. In Figure 4.2, we can still observe the non-negligible difference of the estimated copula parameters under the two models. However, as seen from the graph of the hedge ratios, the estimated time-varying optimal hedge ratios under the two models are quite close to each other, irrespective of the confidence level we choose for VaR to conduct the hedging.

This result is consistent with our analysis based on simulated data in Section 3.3, where we reached the conclusion that in the case when there are no significant regime shifts in the dependence or the gap between strong and weak dependence is not large enough, the extended regime switching strategy will not show big difference from the original non-
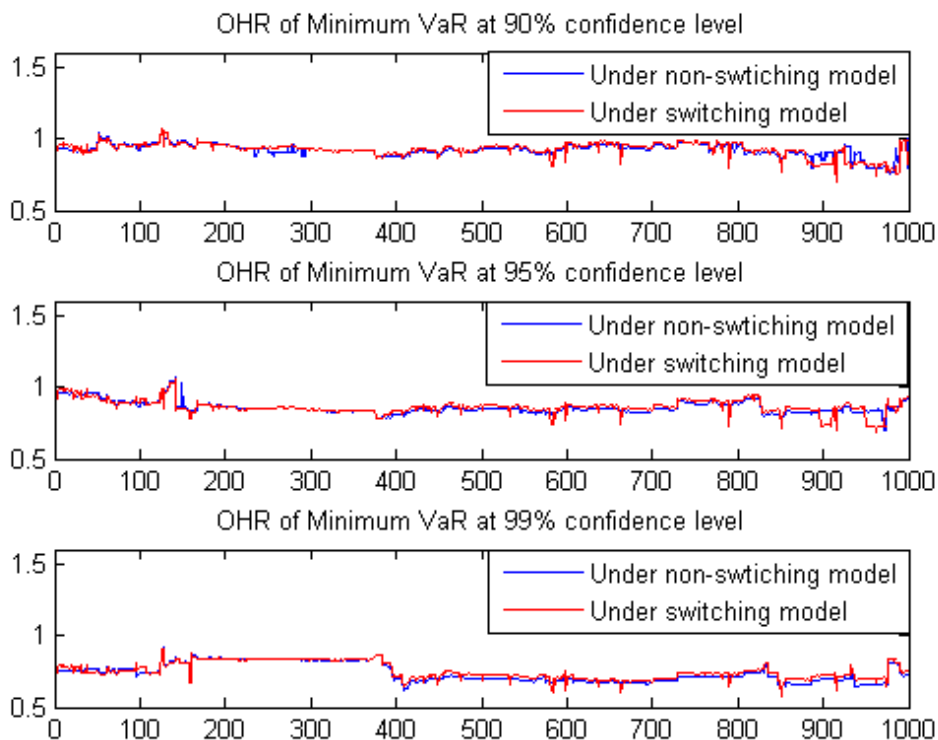
Figure 4.3: Comparison of estimated OHRs under the two models. We take the minimum VaR of the hedged portfolio as the hedging objective, respectively at confidence level of 90%, 95% and 99%.

switching strategy in hedge ratios and hedging effectiveness. The presented results suggest that this case applied to our data. Just as we mentioned before, although we can observe different patterns of estimated copula parameters under the two models in Figure 4.2, the difference is not significant enough to result in big differences in hedge ratios. Under the extended regime-switching model, only a small number of points show very low values of the estimated copula parameter, while most of the estimated copula parameters fluctuate either between 9 and 10 or between 12 and 15. This is very close to the estimated copula

parameter interval under the non-switching model, which is between 8 and 10. According to the one-to-one relationship between Clayton copula parameter and Kendall's τ correlation measure, the copula parameter of 8 and 15 respectively mean the Kendall's τ correlation being around 80.00% and 88.24%. The gap between these two levels of dependence is not too large. Therefore, it is reasonable to expect that the estimated time-varying hedge ratio shows a quite similar pattern under the two different models.

4.3   Comparison of Hedging Effectiveness between the Two Strategies

In this section, we further discuss hedging effectiveness of the two models. In particular, we investigate whether the extended regime-switching model achieves better performance when compared to the original non-switching model.

We employ the minimum VaR of the hedged portfolio as the hedging objective, where we use levels equal to 90%, 95% and 99%. The hedging effectiveness is measured by the VaR reduction of the portfolio due to hedging. All the comparisons are shown in Table 4.3. Both models have achieved quite good performance with the hedging effectiveness being above 65%. However, comparing these two models, we do not see a significant difference in terms of the hedging effectiveness. The two models perform quite similarly, since the difference between their hedging effectiveness is always less than 1%, irrespective of the level we use for VaR. When we minimize VaR at the levels of 90% and 95%, the extended regime-switching model just produces a little bit higher value of hedging effectiveness.

Table 4.3: Comparison of hedging effectiveness between extended regime-switching strategy and the original non-switching strategy for FTSE100 Index (using the last 1000 observations as evaluation period for hedging effectiveness test)

Minimum VaR at confidence level of 90%						
Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	$\text{VaR}_{90\%}$	Mean(OHR)	$\text{VaR}_{90\%}$	HE	Mean(OHR)	$\text{VaR}_{90\%}$	HE
0	0.01096	0.9242	0.00228	79.1623%	0.9265	0.00228	79.2022%

Minimum VaR at confidence level of 95%						
Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	$\text{VaR}_{95\%}$	Mean(OHR)	$\text{VaR}_{95\%}$	HE	Mean(OHR)	$\text{VaR}_{95\%}$	HE
0	0.01594	0.8593	0.00360	77.4186%	0.8638	0.00355	77.7002%

Minimum VaR at confidence level of 99%						
Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	$\text{VaR}_{99\%}$	Mean(OHR)	$\text{VaR}_{99\%}$	HE	Mean(OHR)	$\text{VaR}_{99\%}$	HE
0	0.02700	0.7400	0.00819	69.6722%	0.7528	0.00838	68.9754%

The empirical data comes from the log-return data ranging from March 1, 2006 to January 26, 2015, a total of 2250 observations. We conduct the hedging during the hedging evaluation period to test the hedging effectiveness of the models.

However, the original model even outperforms the extended regime-switching model when we minimize $\text{VaR}_{99\%}$ of the hedged portfolio, although the difference is hardly noticeable.

For a more reliable conclusion, we also change the hedging evaluation period to compare hedging effectiveness of the two strategies. Table 4.4 lists the comparison results when we use the last 500 observations as the evaluation period to conduct the hedging and test hedging effectiveness. This new hedging evaluation period is just the second half of the

Table 4.4: Comparison of hedging effectiveness between extended regime-switching strategy and the original non-switching strategy for FTSE100 Index (using the last 500 observations as evaluation period for hedging effectiveness test)

Minimum VaR at confidence level of 90%							
Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	$VaR_{90\%}$	Mean(OHR)	$VaR_{90\%}$	HE	Mean(OHR)	$VaR_{90\%}$	HE
0	0.00952	0.9164	0.00214	77.5239%	0.9159	0.00220	76.8590%

Minimum VaR at confidence level of 95%							
Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	$VaR_{95\%}$	Mean(OHR)	$VaR_{95\%}$	HE	Mean(OHR)	$VaR_{95\%}$	HE
0	0.01308	0.8498	0.00323	75.2936%	0.8570	0.00316	75.8216%

Minimum VaR at confidence level of 99%							
Unhedged Portfolio		Hedged portfolio with original non-switching strategy			Hedged portfolio with extended regime-switching strategy		
OHR	$VaR_{99\%}$	Mean(OHR)	$VaR_{99\%}$	HE	Mean(OHR)	$VaR_{99\%}$	HE
0	0.02154	0.6957	0.00819	61.9909%	0.7163	0.00847	60.6812%

The empirical data comes from the log-return data ranging from March 1, 2006 to January 26, 2015, a total of 2250 observations. We conduct the hedging during the hedging evaluation period to test the hedging effectiveness of the models.

original evaluation period we used in Table 4.3. In Figure 4.2, which depicts the estimated copula parameter for the two models, we find that our estimation procedure performs better in the second half of the hedging evaluation period. In that period, we could observe a clearer pattern of how the estimated copula parameter differs under the two models. Therefore, in order to minimize the effect of possible estimation error, we use the second half of the original evaluation period to conduct the hedging and to compare the

hedging performance of the two models. The results in Table 4.4 are consistent with those in Table 4.3, where the two hedging strategies perform quite similarly in terms of hedging effectiveness.

Thus, we are able to make the following conclusion: for the real data we used, the extended regime-switching model does not show any significant advantage in improving hedging effectiveness when compared to the original model. This is largely because there is no significant regime shifts in the dependence and the gap between the two states is not large enough. The hedging ratios under the two models are quite similar, and so are the corresponding hedging effectiveness.

Chapter 5

Concluding Remarks and Future Research

In this thesis, our work starts with introducing a futures hedging strategy presented by Barbi and Romagnoli (2014). The authors propose a model to determine the optimal hedge ratio by minimizing a quantile risk measure of the hedged portfolio. As QRMs nest many commonly used risk measures, such as VaR, ES and ERM, this approach provides a generic hedging strategy applicable to different choices of the risk measure. Moreover, in this method, the quantiles of the hedged portfolio can be represented in terms of a copula function, so that the spot-futures dependence can be better captured. According to Barbi and Romagnoli (2014), this hedging strategy has shown great advantages in improving hedging effectiveness compared to some of the existing methods. However, this model is based on the assumption that the dependence structure between spot and futures returns does not change over time, which may not be satisfied in reality. As a consequence, the

model may fail to capture properly the true dependence between spot and futures returns, and thus may lead to a sub-optimal hedging strategy. These facts motivate us to propose some extensions and improvement of the approach proposed by Barbi and Romagnoli (2014). In our extended hedging strategy, we extend the static copula parameter approach into a dynamic one and incorporate a Markov regime switching model to capture the dynamic spot-futures dependence by allowing for regime shifts in the copula parameter. In this thesis, we aim to investigate whether our extended regime-switching hedging strategy could yield more efficient hedge ratio and accordingly improve hedging effectiveness when compared to the original hedging strategy.

Since our extended regime-switching hedging strategy is based on the original minimum QRM hedging strategy presented in Barbi and Romagnoli (2014), in Chapter 2, we conducted some robustness and sensitivity analysis of the original model. We discussed whether different estimation methods could impact the results, and also examined how some factors affect the hedge ratio and hedging effectiveness. Monte Carlo simulations were conducted, from which we can draw the following conclusions:

a The dependence between the spot and futures has a significant impact on the hedge ratio and hedging effectiveness. In particular, a stronger spot-futures dependence may largely improve hedging effectiveness.

b It is important to take investors' risk aversion into consideration when deciding about optimal hedging strategy. We found, for example, that lower values of the confidence level (for VaR and ES) or the risk aversion coefficient (for ERM) lead to improved effectiveness of the corresponding hedging strategies. In additions, in such cases, the

OHR is closer to the OHR obtained from minimization of the portfolio variance.

In Chapter 3, we extended the original minimum QRM hedging strategy by adding a more accurate copula fitting method, where the time-varying copula parameter follows a two-state regime switching model. The extended regime-switching hedging strategy proved to be able to provide a quite accurate estimation of the time-varying spot-futures dependence. Then we conducted Monte Carlo simulations to compare its hedging effectiveness with that of the original non-switching hedging strategy. The results suggest that:

a For data containing more significant regime shifts in the dependence, meaning that the dependence levels in different regimes are quite different, our extended hedging strategy based on a regime-switching model proves to be more efficient than the original one based on a fixed copula.

b On the other hand, when the gap between strong dependence and weak dependence is not significant, the extended strategy shows no advantage in improving hedging effectiveness compared with the original non-switching strategy. Therefore, in such cases, it is not necessary to consider the regime switching in the dependence, and the original hedging strategy also works well for time-varying copula parameters.

Finally in Chapter 4, empirical application to FTSE 100 Index and its corresponding futures was presented. We found that the extended regime switching model does capture the dependence dynamics well. At the same time, however, the model does not show significant advantage in improving hedging effectiveness when compared to the original model. This is largely because there is no significant regime shifts in the dependence and

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the gap between the two states is not large enough. We can still observe high values of hedging effectiveness under both models, which shows a good hedging performance of the minimum QRM method. In terms of the choice of a hedging model, it is necessary to discover some relevant characteristics of the data first and then apply the one that is most suitable.

In summary, our study makes contributions in several aspects. We have analysed robustness and sensitivity of the original minimum copula-based QRM hedging model and, as a result, have identified factors that may significantly influence hedging effectiveness. We have also relaxed the assumption of the original model by allowing the copula parameter to follow a regime-switching model. We check the capability of the extended regime-switching strategy in terms of improving hedging effectiveness. Both Monte Carlo simulation and an empirical application confirm that the approximate optimal hedging strategies based on different risk measures can be effectively determined within our extended model. We have also identified the conditions under which the extended model leads to more efficient hedging strategies when compared with those based on a static copula. There are several aspects of the proposed model worthy of further study:

- More complicated regime switching models could be applied in the extended hedging strategy. For example, we may increase the number of regimes, or allow some parameters of the marginal distributions to change in each regime.

- In our implementation, we only employ the value-at-risk as the risk measure to compare the hedging effectiveness of the two models. Other quantile risk measures could be implemented in a similar way and possibly lead to new findings.

- In empirical analysis, we present an example where we use FTSE 100 index. However, different future markets have different features, which may lead to different conclusions. For instance, we might see a different degree of the spot-futures dependence in commodity market and it would be interesting to check how our hedging strategy performs then.

APPENDICES

Appendix A

Copula Families Used in Our Implementation

In this thesis, we consider three commonly used Archimedean copulas in implementation: Clayton, Gumbel and Frank. These choices cover a wide range of situations as the Clayton and Gumbel copulas are known to exhibit strong left and upper tail dependence respectively, while the Frank copulas is symmetric and shows no significant tail dependence. Moreover, these three copulas own the advantage of Archimedean copulas, where the explicit form of the copula function is determined by the corresponding unique generator function ψ. Given a decreasing and convex function ψ: $(0,1] \to [0,+\infty)$ such that $\psi(1) = 0$, the function

$$C^{(\psi)}(u,v) = \psi^{-1}(\psi(u) + \psi(v)), \quad \forall u, v \in (0,1]$$

defines a bivariate Archimedean copula with generator ψ. Different choice of ψ identifies different Archimedean copulas.

In the following, we briefly introduce these three copulas one by one, including their forms and features. There is only one parameter θ in their functions to be estimated.

Clayton Copula. This family of copulas has the copula function

$$C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, \quad \theta \in (0, \infty)$$

The Clayton copula does not allow for negative dependence and it shows strong lower tail dependence. Its corresponding Kendall's τ correlation can be represented as $\tau = \frac{\theta}{\theta+2}$.

Gumbel Copula. This family of copulas has the copula function of

$$C(u_1, u_2; \theta) = exp(-[(-\ln(u_1))^\theta + (-\ln(u_2))^\theta]^{1/\theta}), \quad \theta \in [1, \infty)$$

The Gumbel copula shows strong upper tail dependence and its corresponding Kendall's τ correlation can be represented as $\tau = 1 - \frac{1}{\theta}$.

Frank Copula. This family of copulas has the copula function of

$$C(u_1, u_2; \theta) = \frac{-1}{\theta} \ln(1 + ((e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1))/(e^{-\theta} - 1)), \quad \theta \in (-\infty, \infty)/\{0\}$$

The Frank copula allows for negative dependence and the dependence is tail symmetry, akin to the Gaussian and Student-t copulas. However, the Frank copula can capture weak dependence in the tails better than the Gaussian copula and has simpler form than

t copula. The Frank copula accounts for strong positive or negative dependence. Its corresponding Kendall's τ correlation can be represented as $\tau = 1 + \frac{4}{\theta}(D_1(\theta) - 1)$, where $D_1(\theta) = \frac{1}{\theta}\int_0^\theta \frac{t}{e^t - 1}dt$ is the Debye function.

Appendix B

Algorithm for Identifying an Archimedean Copula

In Genest and Rivest (1993), the authors provide us the procedure to identify an Archimedean copula in empirical application. It begins by assuming that we have a set of n bivariate observations $(X_{11}, X_{21}), \ldots, (X_{1n}, X_{2n})$, and the distribution function F has an associated Archimedean copula C_ψ. We aim to identify the generator ψ, whose corresponding copula C_ψ best fits the distribution function of (X, Y).

An intermediate (unobserved) random variable $Z_i = F(X_{1i}, X_{2i})$ is incorporated with distribution function $K(z) = Prob(Z_i \leq z)$. For Archimedean copulas, it is known that the distribution function is also related with the generator ψ of the Archimedean copula as

$$K_\psi(z) = z - \frac{\psi(z)}{\psi'(z)}.$$

The rationale of the method to identify the most appropriate copula is just to find out the ψ that can make parametric estimate $\hat{K}_\psi(z)$ most closely resemble the non-parametric estimate $\hat{K}(z)$. The detailed steps are illustrated as follows:

1 Construct the non-parametric estimate $\hat{K}(z)$:

   (a) First, use the observations $(X_{11}, X_{21}), \ldots, (X_{1n}, X_{2n})$ to get Z_i (i=1,...,n) that
   Z_i={number of (X_{1j}, X_{2j}) such that $X_{1j} < X_{1i}$ and $X_{2j} < X_{2i}$}/(n-1).

   (b) Second, the non-parametric estimate $\hat{K}(z)$ can be calculated by the formula:
   $\hat{K}(z)$=proportion of Z_i's that are equal to or less than z.

2 Construct the parametric estimate $\hat{K}_\psi(z)$ for each kind of Archimedean copula:

   (a) First, calculate the Kendall's τ correlation coefficient using the observations:
   $$\tau = \binom{n}{2}^{-1} \sum_{i<j} sign[(X_{1i} - X_{1j})(X_{2i} - X_{2j})].$$

   (b) Second, for each Archimedean copula we consider, the copula parameter has a one-to-one relationship with the Kendall's τ correlation. For example, we denote the Clayton copula parameter as θ, then there is $\tau = \frac{\theta}{\theta+2}$. Therefore, we are able to specify the copula generator function ψ for each copula, using its corresponding copula parameter. Finally, the parametric estimate $\hat{K}_\psi(z)$ can be obtained for each kind of copula, represented as $\hat{K}_\psi(z) = z - \frac{\psi(z)}{\psi'(z)}$.

3 Compare the non-parametric estimate $\hat{K}(z)$ with the parametric estimate $\hat{K}_\psi(z)$ for each kind of copula, then choose the one that makes $\hat{K}(z)$ and $\hat{K}_\psi(z)$ most close to

each other. The closeness can be measured by the value of $\int[\hat{K}_\psi(z) - \hat{K}(z)]^2 d\hat{K}(z)$ and the most appropriate copula should reach the minimum of this value.

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